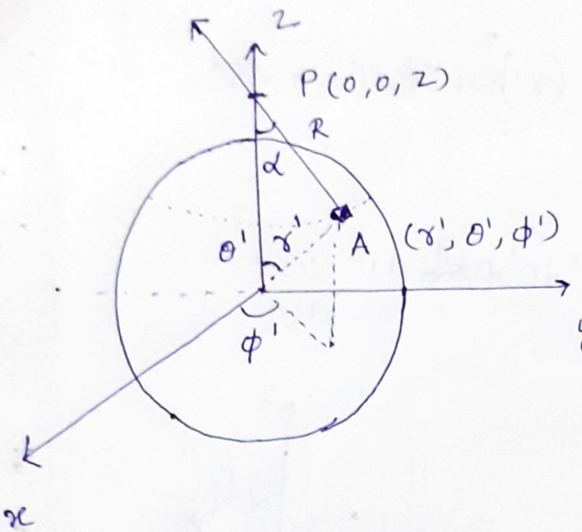
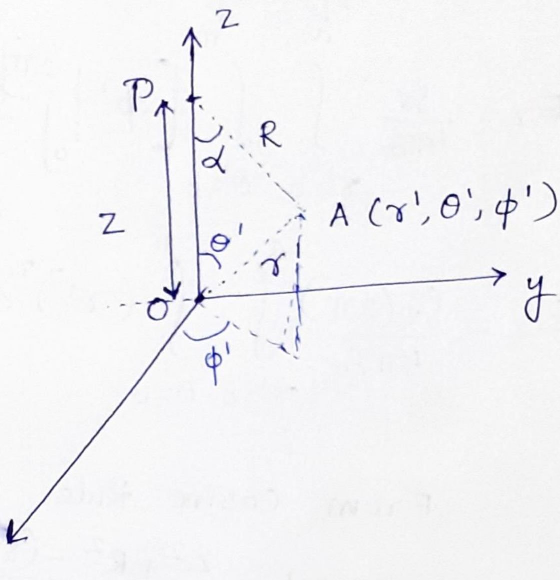
Electric Field Intensity due to Volume charge





Consider a sphere of radius "a". Let the volume of the sphere be filled with a uniform volume charge density ρ_v C/m³

The elemental electric field is given by $d\vec{E}$

$$d\vec{E} = \frac{\rho_v dv \, \hat{a}_R}{4\pi \epsilon_0 R^2}$$

Since it is a spherical coordinate system, Symmetry exists in x-y plane. Due to Symmetry E_x and E_y add upto zero and only z-component exists.

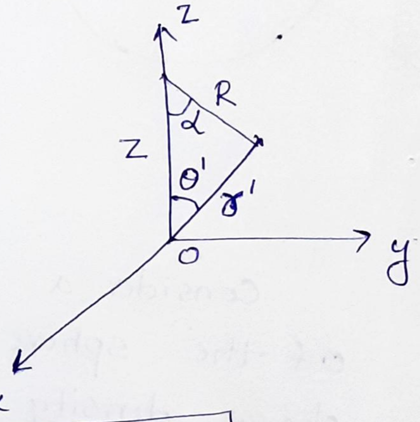
$$dE_z = \frac{\rho_v dv}{4\pi \epsilon_0 R^2} \cos\alpha$$

$$E_z = \int_{\phi'=0}^{2\pi} \int_{r'=0}^{a} \int_{\theta'=0}^{\pi} \frac{\rho_v \cos\alpha}{4\pi\varepsilon_0 R^2} \overbrace{(r')^2 \sin\theta' \, d\theta' \, dr' \, d\phi'}^{dv}$$

Integrating w.r.to ϕ'

$$E_z = \frac{\rho_v}{4\pi\varepsilon_0} \int_{r'=0}^{a} \int_{\theta'=0}^{\pi} \left[\phi' \Big|_0^{2\pi} \right] \frac{\cos\alpha}{R^2} (r')^2 \sin\theta' \, d\theta' \, dr' \, \cancel{d\phi'}$$

$$E_z = \frac{\rho_v (2\pi)}{4\pi\varepsilon_0} \int_{r'=0}^{a} \int_{\theta'=0}^{\pi} (r')^2 \sin\theta' \, d\theta' \, \cancel{d\phi'} \, dr' \, \frac{\cos\alpha}{R^2}$$



From Cosine Rule

$$\cos\alpha = \frac{z^2 + R^2 - (r')^2}{2zR}$$

$$\cos\theta' = \frac{z^2 + (r')^2 - R^2}{2zr'}$$

differentiating $\cos\theta'$ w.r.to "R"

$$-\sin\theta' \frac{d\theta'}{dR} = \frac{0 + 0 - 2R}{2zr'} \Rightarrow \boxed{\sin\theta' \, d\theta' = \frac{R}{zr'} dR}$$

Consider $\cos\theta' = \dfrac{z^2 + (r')^2 - R^2}{2zr'}$

At $\theta' = 0$ $\quad 1 = \dfrac{z^2 + (r')^2 - R^2}{2zr'}$, $\quad 2zr' = z^2 + (r')^2 - R^2$

$R^2 = z^2 + (r')^2 - 2zr \Rightarrow R = z - r'$

At $\theta' = \pi$ $\quad -1 = \dfrac{z^2 + (r')^2 - R^2}{2zr'}$, $\quad -2zr' = z^2 + (r')^2 - R^2$

$R^2 = z^2 + (r')^2 + 2zr' \Rightarrow R = z + r'$

$$E_z = \frac{\rho_v (2\pi)}{4\pi\varepsilon_0} \int_{r'=0}^{a} \int_{R=z-r'}^{R=z+r'} (r')^{\cancel{2}} dr' \frac{\cancel{R}}{z r'} dR \times \frac{1}{R^2} \times \left[\frac{z^2 + R^2 - (r')^2}{2z \cancel{R}}\right]$$

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$$E_z = \frac{\rho_v (\cancel{2}\pi)}{4\pi\varepsilon_0 (\cancel{2} z^2)} \int_{r'=0}^{a} \int_{R=z-r'}^{R=z+r'} \left(\frac{(r')^{\cancel{2}}}{\cancel{r'}}\right) dr' \times \left[\frac{z^2 + R^2 - (r')^2}{R^2}\right] dR\, dr'$$

$$E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_{r'=0}^{a} \int_{R=z-r'}^{R=z+r'} (r') dr' \left[\frac{\cancel{R^2}}{\cancel{R^2}} + \frac{(z^2 - (r')^2)}{R^2}\right] dR$$

$$E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_{r'=0}^{a} \int_{R=z-r'}^{R=z+r'} (r') dr' \left[1 + \frac{z^2 - (r')^2}{R^2}\right] dR$$

Integrating w.r.to. dR

$$E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_{r'=0}^{a} r' dr' \left[R - \frac{(z^2 - (r')^2)}{R}\right] \Bigg|_{R=z-r'}^{R=z+r'}$$

$$E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_{r'=0}^{a} r' dr' \left[z + r' - \frac{[z^2 - (r')^2]}{z + r'} - \left(z - r' - \frac{(z^2 - (r')^2)}{z - r'}\right)\right]$$

$$E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_{r'=0}^{a} r' dr' \left[z + r' - \frac{(z - r')\cancel{(z + r')}}{\cancel{(z + r')}} - (z - r') + \frac{\cancel{(z - r')}(z + r')}{\cancel{(z - r')}}\right]$$

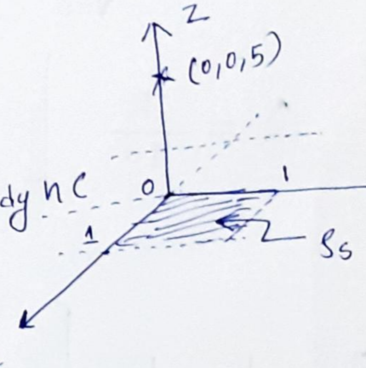
$$= \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_{r'=0}^{a} 4 (r')^2 dr' = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \times 4 \frac{(r')^3}{3} \Bigg|_0^a = \frac{\rho_v 4\pi a^3}{4\pi\varepsilon_0 z^2 3}$$

$$\boxed{E_z = \frac{\rho_v 4\pi a^3}{4\pi\varepsilon_0 z^2 3} = \frac{\rho_v}{4\pi\varepsilon_0 z^2} \left(\frac{4\pi a^3}{3}\right)}$$

4) The finite sheet $0 \le x \le 1$, $0 \le y \le 1$ on the $z=0$ plane has a charge density $\rho_s = xy(x^2+y^2+25)^{3/2}$ nC/m^2,
Find (a) The total charge on the sheet.
(b) The electric field at $(0,0,5)$
(c) The force experienced by a $-1mC$ charge located at $(0,0,5)$

Solution:

(a) The total charge $= Q$



$$Q = \int_S \rho_s \, ds = \int_{x=0}^{1}\int_{y=0}^{1} xy[x^2+y^2+25]^{3/2} \, dx\,dy \text{ nC}$$

$$Q = \int_{x=0}^{1}\int_{y=0}^{1} xy[x^2+y^2+25]^{3/2} \, dx\,dy$$

put $x^2 = u \Rightarrow 2x\,dx = du \Rightarrow x\,dx = \frac{du}{2}$

At $x=0$, $0 = u$
$x = 1$, $1 = u$

$$Q = \int_{y=0}^{1}\int_{u=0}^{1} y\,[u+y^2+25]^{3/2} \overset{x\,dx}{\left(\frac{du}{2}\right)} dy$$

$\int u^n du = \frac{u^{n+1}}{n+1}$ for $n \neq -1$

$$Q = \int_{y=0}^{1} y\,\frac{[u+y^2+25]^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)}\Bigg|_{u=0}^{1} \frac{dy}{2}$$

$$Q = \int_{y=0}^{1} \frac{y\,[y^2+u+25]^{5/2}\Big|_{u=0}^{1}}{\frac{5}{\cancel{2}}} \times \frac{dy}{\cancel{2}} = \frac{1}{5}\int_{y=0}^{1} \left[(y^2+26)^{5/2} - (y^2+25)^{5/2}\right] y\,dy$$

$$Q = \frac{1}{5}\int_{y=0}^{y=1}\left[(y^2+26)^{5/2} - (y^2+25)^{5/2}\right] y\,dy$$

put $y^2 = V$

$$2y\,dy = dV \Rightarrow y\,dy = \frac{dV}{2} \quad \begin{cases} \text{At } y=0, & V=0 \\ \text{At } y=1, & V=1 \end{cases}$$

$$Q = \frac{1}{5}\int_{V=0}^{1}\left[(V+26)^{5/2} - (V+25)^{5/2}\right]\frac{dV}{2}$$

$$Q = \frac{1}{10}\left[\left.\frac{(V+26)^{\frac{5}{2}+1}}{\left(\frac{5}{2}+1\right)}\right|_{V=0}^{V=1} - \left.\frac{(V+25)^{\frac{5}{2}+1}}{\left(\frac{5}{2}+1\right)}\right|_{V=0}^{V=1}\right]$$

$$Q = \frac{1}{\underset{5}{\cancel{10}}}\times\frac{1}{\frac{7}{\cancel{2}}}\left[(27)^{\frac{7}{2}} - (26)^{7/2} - \left((26)^{7/2} - (25)^{7/2}\right)\right]$$

$$= \frac{1}{35}\left[(27)^{7/2} - 2(26)^{7/2} + (25)^{7/2}\right]\times 10^{-9}$$

$$\boxed{Q = 33.15\times 10^{-3}\,C}$$

(b) $$\vec{E} = \int_S \frac{\rho_s\,ds}{4\pi\varepsilon_0 r^2}\,\hat{a}_R = \int_S \frac{\rho_s\,ds}{4\pi\varepsilon_0}\times\frac{(r-r')}{|r-r'|^3}$$

$$r - r' = (0,0,5) - (x,y,0) = -x\hat{a}_x - y\hat{a}_y + 5\hat{a}_z$$

$$|r-r'| = (x^2+y^2+25)^{1/2}$$

$$\vec{E} = \int_{x=0}^{1}\int_{y=0}^{1}\frac{10^{-9}\,xy\,(x^2+y^2+25)^{3/2}\,(-x\hat{a}_x - y\hat{a}_y + 5\hat{a}_z)\,dx\,dy}{4\pi\varepsilon_0\,(x^2+y^2+25)^{3/2}}$$

$$\vec{E} = \int_{x=0}^{1} \int_{y=0}^{1} \frac{1\times10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi}} \left[-x^2y\,\hat{a}_x - xy^2\,\hat{a}_y + 5xy\,\hat{a}_z\right] dx\,dy$$

$$\vec{E} = \int_{x=0}^{1} \int_{y=0}^{1} 9 \left[-x^2y\,\hat{a}_x\,dx\,dy - xy^2\,\hat{a}_y\,dx\,dy + 5xy\,\hat{a}_z\,dx\,dy\right]$$

$$\vec{E} = -9 \int_{x=0}^{1} \int_{y=0}^{1} x^2y\,dx\,dy\,\hat{a}_x$$

$$- 9 \int_{x=0}^{1} \int_{y=0}^{1} xy^2\,dx\,dy\,\hat{a}_y$$

$$+ 5\times9 \int_{x=0}^{1} \int_{y=0}^{1} xy\,dx\,dy\,\hat{a}_z$$

$$\vec{E} = -9\left[\int_{x=0}^{1} x^2\,dx \int_{y=0}^{1} y\,dy\right] \hat{a}_x$$

$$- 9\left[\int_{x=0}^{1} x\,dx \int_{y=0}^{1} y^2\,dy\right] \hat{a}_y$$

$$+ 5\times9\left[\int_{x=0}^{1} x\,dx \int_{y=0}^{1} y\,dy\right] \hat{a}_z$$

$$\vec{E} = -9\left[\frac{x^3}{3}\Big|_0^1 \times \frac{y^2}{2}\Big|_0^1\right]\hat{a}_x$$

$$-9\left[\frac{x^2}{2}\Big|_0^1 \times \frac{y^3}{3}\Big|_0^1\right]\hat{a}_y$$

$$+5\times 9\left[\frac{x^2}{2}\Big|_0^1 \times \frac{y^2}{2}\Big|_0^1\right]\hat{a}_z$$

$$\vec{E} = -9\left[\frac{1}{6}\right]\hat{a}_x - 9\left[\frac{1}{6}\right]\hat{a}_y + \frac{5\times 9}{4}\hat{a}_z$$

$$\vec{E} = \underline{-1.5}\,\hat{a}_x - \underline{1.5}\,\hat{a}_y + \underline{11.25}\,\hat{a}_z$$

(c) $\vec{F} = q\vec{E}$

$$= \left[-1\times 10^{-3}\right]\left[-1.5\hat{a}_x - 1.5\hat{a}_y + 11.25\hat{a}_z\right]$$

$$= \underline{1.5\times 10^{-3}}\,\hat{a}_x + \underline{1.5\times 10^{-3}}\,\hat{a}_y - \underline{11.25\times 10^{-3}}\,\hat{a}_z$$

Electric Flux Density

Electric flux density 'D' is given by

$$\vec{D} = \varepsilon_0 \vec{E} \quad [\text{In free space}]$$

In General $\vec{D} = \varepsilon E$ where $\varepsilon = \varepsilon_0 \varepsilon_r$

Electric Flux Density due to a point charge

is given by $\vec{D} = \varepsilon_0 \dfrac{Q}{4\pi\varepsilon_0 R^2} \hat{a}_R = \dfrac{Q}{4\pi R^2} \hat{a}_R$ [In free space]

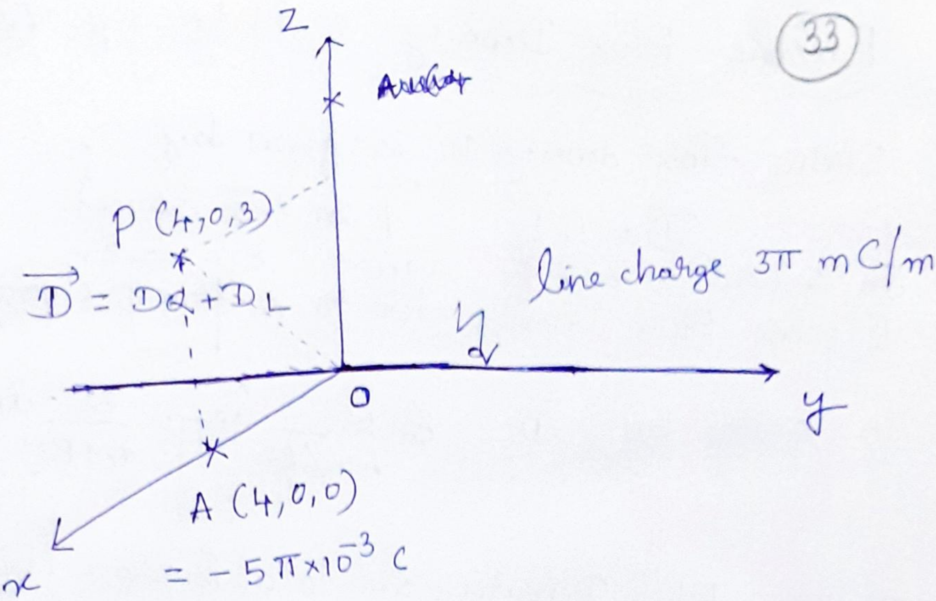
Electric Flux Density due to a Surface charge

is given by $\vec{D} = \varepsilon_0 \dfrac{\rho_s}{2\varepsilon_0} \hat{a}_n = \dfrac{\rho_s}{2} \hat{a}_n$ [In free space]

Electric Flux Density due to a volume charge distribution

is given by $\vec{D} = \varepsilon_0 \displaystyle\int_V \dfrac{\rho_v dv}{4\pi\varepsilon_0 R^2} \hat{a}_R = \int_V \dfrac{\rho_v dv}{4\pi R^2} \hat{a}_R$ [In free space]

1) Determine $\vec{D}$ at $(4,0,3)$ if there is a point charge -5π mC at $(4,0,0)$ and a line charge 3π mC/m along the y-axis



$$\vec{D} = \vec{D_Q} + \vec{D_L}$$

$$\vec{D_Q} = \frac{Q}{4\pi R^2}\hat{a}_R = \frac{(-5\pi\times10^{-3})}{4\pi\,|r-r'|^3}\times(r-r')$$

$$r - r' = (4,0,3) - (4,0,0) = (0,0,3)$$

$$\vec{D_Q} = \frac{-5\cancel{\pi}\times10^{-3}}{4\cancel{\pi}}\times\frac{(3\hat{a}_z)}{(3^2)^{3/2}} = -0.139\,\hat{a}_z\times10^{-3}$$

$$\vec{D_Q} = -0.139\times10^{-3}\,\hat{a}_z$$

$$\vec{D_L} = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$$

$$\hat{a}_\rho = \frac{(4,0,3)-(0,0,0)}{\sqrt{4^2+0^2+3^2}}$$

$$\vec{D_L} = \frac{3\cancel{\pi}\times10^{-3}}{2\cancel{\pi}\times5}\times\frac{(4\hat{a}_x+3\hat{a}_z)}{5}$$

$$\hat{a}_\rho = \frac{(4,0,3)}{5}$$

$$\rho = \sqrt{4^2+0^2+3^2} = 5$$

$$\vec{D_L} = 0.24\times10^{-3}\,\hat{a}_x + 0.18\times10^{-3}\,\hat{a}_z$$

$$\vec{D} = \vec{D_Q} + \vec{D_L}$$

$$\vec{D} = 0.24\times10^{-3}\,\hat{a}_x + 0.041\times10^{-3}\,\hat{a}_z \ \ C/m^2$$

or

$$\vec{D} = 0.240\,\hat{a}_x + 0.041\,\hat{a}_z \ \ mC/m^2$$

or

$$\vec{D} = 240\,\hat{a}_x + 41.0\,\hat{a}_z \ \ \mu C/m^2$$