UNIT-3

MAGNETOSTATIC FIELDS

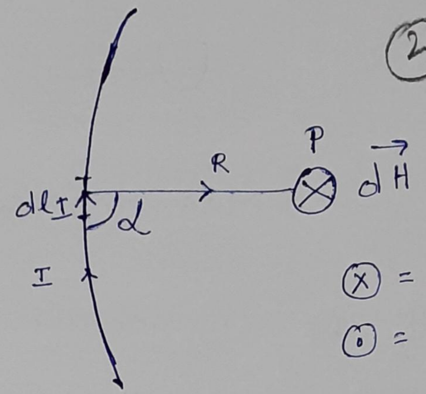
If the charges are moving with a constant velocity, a static magnetic field or a magnetostatic field is produced. A magnetostatic field is produced by a constant current flow (or direct current).

There are two major laws in magnetostatics they are (i) Biot-Savart's law
(ii) Ampere's law

Biot Savart's law

Biot Savart's law states that the differential magnetic field intensity "dH" produced at a point P as shown in figure, by the differential current element "Idl" is proportional to the product "Idl" and sine of the angle "α" between the element and the line joining P to the element and inversly proportional to the square of the distance "R" between "P" ~~to~~ and the element.

$$\vec{dH} \propto \frac{I\,dl\,\sin\alpha}{R^2}\,\hat{a}_n$$



$$\vec{dH} = \frac{I\,dl\,\sin\alpha\,\hat{a}_n}{4\pi R^2}$$

$\otimes$ = Into the page

$\odot$ = out of the page

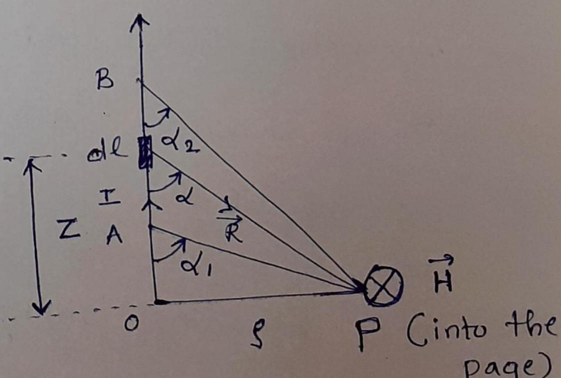
$$\vec{dH} = \frac{I}{4\pi R^2}\,\frac{\vec{dl}\times\vec{R}}{R}$$

$$\vec{dH} = \frac{I\,\vec{dl}\times\vec{R}}{4\pi R^3}$$

where $\vec{R}$ = Vector along the line joining the differential current element and the point P.

Magnetic Field due to straight current carrying filamentary conductors of finite length

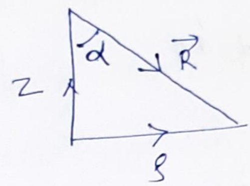
Consider a current carrying straight conductor of finite length AB as shown in figure. We assume that the conductor is along the z-axis with its lower and upper ends, respectively, subtending angles α_1 and α_2 respectively. At point P, $\vec{H}$ is to be determined.



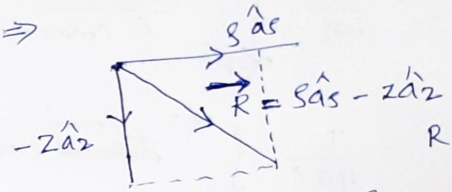
$\vec{dH}$ at point P due to an element $\vec{dl}$ at $(0,0,z)$

$$\vec{dH} = \frac{I\,\vec{dl} \times \vec{R}}{4\pi R^3}$$

$$\vec{dl} = dz\,\hat{a}_z \qquad \vec{R} = \rho\,\hat{a}_\rho - z\,\hat{a}_z$$



$\Rightarrow$



$$R = |\vec{R}| = \sqrt{\rho^2 + z^2}$$

$$\vec{dH} = \frac{I\,dz\,\hat{a}_z \times [\rho\hat{a}_\rho - z\hat{a}_z]}{4\pi\left(\sqrt{\rho^2+z^2}\right)^3}$$

$$dz\,\hat{a}_z \times (\rho\hat{a}_\rho - z\hat{a}_z) = \begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi & \hat{a}_z \\ 0 & 0 & dz \\ \rho & 0 & -z \end{vmatrix}$$

$$= \hat{a}_\rho(0-0) - \hat{a}_\phi[0 - \rho\,dz] + \hat{a}_z[0-0]$$

$$= \rho\,dz\,\hat{a}_\phi$$

$$\vec{dH} = \frac{I\rho\,dz\,\hat{a}_\phi}{4\pi\left(\sqrt{\rho^2+z^2}\right)^3}$$

$$\vec{H} = \int_A^B \frac{I\rho\,dz\,\hat{a}_\phi}{4\pi(\rho^2+z^2)^{3/2}}$$

Put $z = \rho\cot\alpha \Rightarrow dz = -\rho\,\mathrm{cosec}^2\alpha\,d\alpha$

At $z = A$, $\alpha = \alpha_1$ $\quad \frac{A}{\rho} = \cot\alpha_1 \Rightarrow \alpha_1 = \cot^{-1}(A/\rho)$

At $z = B$, $\alpha = \alpha_2$ $\quad \frac{B}{\rho} = \cot\alpha_2 \Rightarrow \alpha_2 = \cot^{-1}(B/\rho)$

$$\vec{H} = \int_{\alpha_1}^{\alpha_2} \frac{I\, \rho(-\rho \operatorname{cosec}^2\alpha)\, d\alpha}{4\pi \left[\rho^2 + \rho^2 \cot^2\alpha\right]^{3/2}}\, \hat{a}_\phi$$

(4)

$$\vec{H} = -\frac{I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \operatorname{cosec}^2\alpha\, d\alpha}{\rho^3 \operatorname{cosec}^3\alpha}\, \hat{a}_\phi$$

$$\vec{H} = -\frac{I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{1}{\rho \operatorname{cosec}\alpha}\, d\alpha\, \hat{a}_\phi$$

$$\vec{H} = -\frac{I}{4\pi\rho} \int_{\alpha_1}^{\alpha_2} \sin\alpha\, d\alpha\, \hat{a}_\phi$$

$$\vec{H} = -\frac{I}{4\pi\rho} \left[-\cos\alpha \Big|_{\alpha_1}^{\alpha_2}\right] \hat{a}_\phi$$

$$\boxed{\vec{H} = \frac{I}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right] \hat{a}_\phi}$$

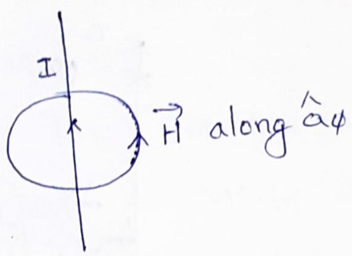
where ρ = shortest distance or perpendicular distance between the line conductor and the point "P"

$\hat{a}_\phi = \hat{a}_l \times \hat{a}_\rho$

$\hat{a}_l$ = Unit Vector along the line carrying current I

$\hat{a}_\rho$ = Unit Vector along the perpendicular line from the line current to the field point

$\vec{H} = \frac{I}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right]\hat{a}_\phi$ is generally applicable for any straight filamentary conductor of finite length. The conductor need not lie on z-axis, but it must be straight. The $\vec{H}$ is always along the unit vector $\hat{a}_\phi$ i,e along concentric circular paths irrespective of the length of the wire or the point of interest P

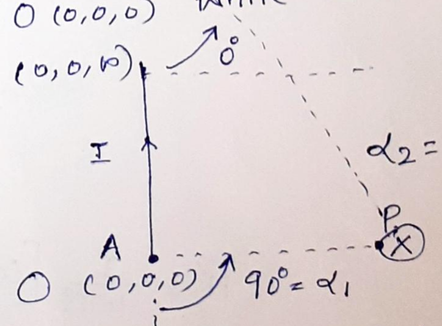


Secial case 1 : When the conductor is Semi infinite, so that point A is now at O (0,0,0) while B (0,0,∞)

$\alpha_1 = 90°$. $\alpha_2 = 0°$

$$\vec{H} = \frac{I}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right]\hat{a}_\phi$$

$$\vec{H} = \frac{I}{4\pi\rho}\left[1 - 0\right]\hat{a}_\phi$$



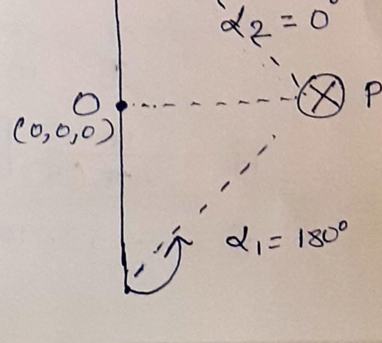
Special case 2 : When the conductor is infinite in length, point B is at (0,0,∞) and point A is at (0,0,−∞)

So that $\alpha_1 = 180°$, $\alpha_2 = 0°$

$$\vec{H} = \frac{I}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right]\hat{a}_\phi$$

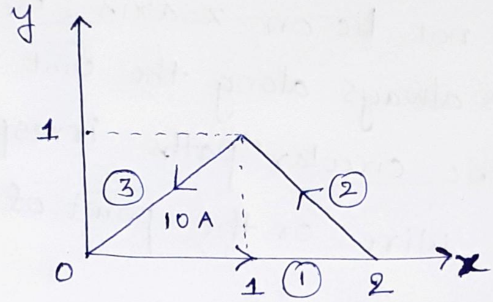
$$\vec{H} = \frac{I}{4\pi\rho}\left[1 - (-1)\right]\hat{a}_\phi$$

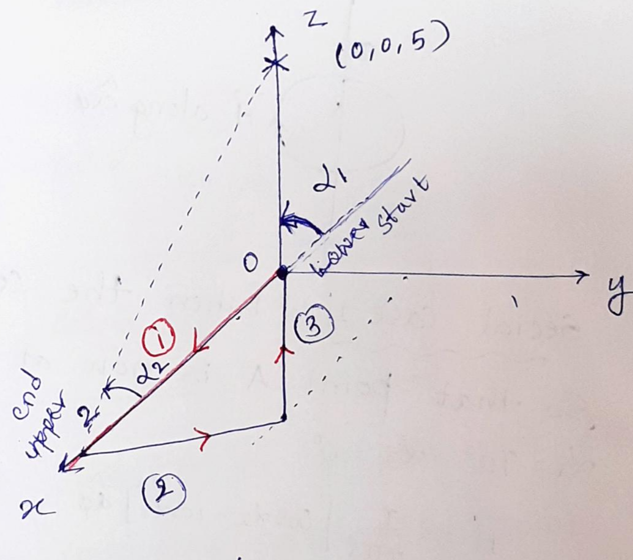
$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$



7.1) The conducting triangular loop shown in figure carries a current of 10A. Find $\vec{H}$ at (0, 0, 5) due to side (1) of the loop.

(6)





$$\vec{H} = \frac{I}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right]\hat{a}_\phi$$

$\alpha_1 = 90^\circ$ $\qquad \cos\alpha_2 = \dfrac{2}{\sqrt{2^2+5^2}} = \dfrac{2}{\sqrt{29}}$

$\cos\alpha_1 = 0^\circ$

$$\hat{a}_\phi = \hat{a}_l \times \hat{a}_\rho = \hat{a}_x \times \hat{a}_z$$

$$= -\hat{a}_y$$

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\hat{a}_y(1) = -\hat{a}_y$$

$I = 10$, $\rho = 5$

$$\vec{H} = \frac{10}{4\pi(5)}\left[\overset{0.3715}{\frac{2}{\sqrt{29}}} - 0\right](-\hat{a}_y)$$

$$\vec{H} = -59.108 \times 10^{-3}\,\hat{a}_y \text{ A/m}$$

Ampere's circuit law
Ampere's law or Ampere's - Maxwell's Equation

Ampere's law or Ampere's circuit law states that the line integral of $\vec{H}$ around a closed path is same as the net current I_{enc} enclosed by the path.

$$\boxed{\oint_L \vec{H}\cdot\vec{dl} = I_{enc}}$$

Ampere's law is similar to Gauss's law, since Ampere's law is easily applied to determine $\vec{H}$ when the current distribution is symmetrical.

$$I_{enc} = \oint_L \vec{H}\cdot\vec{dl} \rightarrow (1)$$

Applying stroke's theorem

$$\oint_L \vec{H}\cdot\vec{dl} = \int_S (\nabla\times\vec{H})\cdot\vec{ds}$$

$$I_{enc} = \int_S (\nabla\times\vec{H})\cdot\vec{ds} \rightarrow (2)$$

$$I_{enc} = \int_S \vec{J}\cdot\vec{ds} \longrightarrow (3)$$

$$(2) = (3)$$

$$\therefore \boxed{\nabla\times\vec{H} = \vec{J}}$$

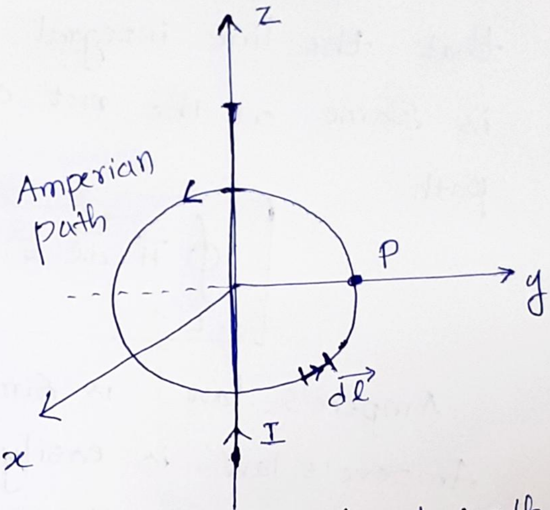
This is the third Maxwell's Equation

$\nabla\times\vec{H} = \vec{J} \neq 0$, that is, a magnetostatic field is not conservative

Applications of Ampere's law

A) Ampere's law applied to Infinite line Current

Consider an infinitely long filamentary current I along the z-axis as shown in figure.



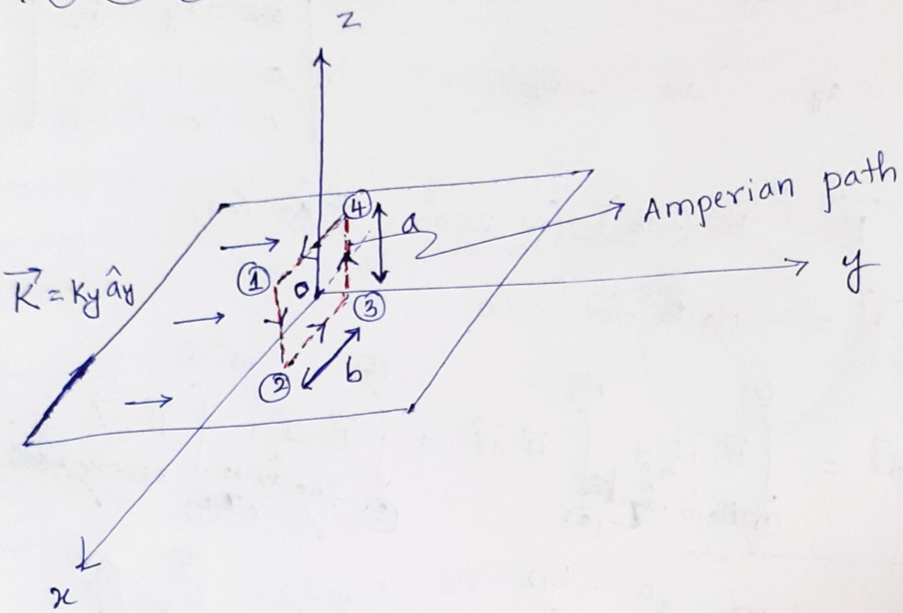
To determine $\vec{H}$ at an observation point P, we allow a closed path to pass through "P". This path on which Ampere's law is applied, is known as an Amperian path.

$$\therefore \quad I = \oint H_\phi \hat{a}_\phi \cdot \rho \, d\phi \, \hat{a}_\phi = H_\phi \int_0^{2\pi} \rho \, d\phi$$

$$I = H_\phi \, 2\pi\rho$$

$$\boxed{H = \frac{I}{2\pi\rho} \hat{a}_\phi}$$

B. Ampere's law applied to Infinite Sheet of Current



Consider an infinite current sheet in the z=0 plane. If the sheet has a uniform current density $\vec{K} = K_y \hat{a}_y$ A/m as shown in figure. Applying Ampere's law to the rectangular closed path 1-2-3-4-1

$$\oint \vec{H} \cdot \vec{dl} = I_{enc} = K_y b \rightarrow (1)$$

$$\oint \vec{H} \cdot \vec{dl} = \int_1^2 \vec{H} \cdot \vec{dl} + \int_2^3 \vec{H} \cdot \vec{dl} + \int_3^4 \vec{H} \cdot \vec{dl} + \int_4^1 \vec{H} \cdot \vec{dl}$$

$$\int_{+a/2}^{-a/2}$$

$\vec{H}$ above the sheet is along $a_\phi = \hat{a}_l \times \hat{a}_s$

$\hat{a}_\phi = \hat{a}_y \times \hat{a}_z = \hat{a}_x$

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{a}_x$$

$H = H_0 \hat{a}_x$ for $z > 0$

$\vec{H}$ below the sheet is along

$\hat{a}_\phi = \hat{a}_\ell \times \hat{a}_\rho$

$\hat{a}_\ell = \hat{a}_y \qquad \hat{a}_\rho = -\hat{a}_z$

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

$\therefore \vec{H} = H_o(-\hat{a}_x)$ for $z<0$ $\quad = \hat{a}_x(-1)$

$\vec{H} = -H_o\hat{a}_x$ for $z<0$

$$\oint \vec{H}\cdot d\vec{l} = \int_{1}^{2} \vec{H}\cdot d\vec{l} + \int_{2}^{3} \vec{H}\cdot d\vec{l} + \int_{3}^{4} \vec{H}\cdot d\vec{l} + \int_{4}^{1} \vec{H}\cdot d\vec{l}$$

$+H_o\hat{a}_x\cdot\hat{a}_z\,dz \qquad \pm H_o\hat{a}_x\cdot\hat{a}_x\,dx$

$\int H_o\hat{a}_x\cdot dz\,\hat{a}_z + \int_0^{-b/2} H_o\hat{a}_x\cdot dz\,\hat{a}_z$

$$= \int_{1}^{2} \cancel{\vec{H}\cdot d\vec{l}}^{\,0} + \int_{+b/2}^{-b/2} (-H_o\hat{a}_x)\cdot(+dx\,\hat{a}_x) + \int_{3}^{4} \cancel{\vec{H}\cdot d\vec{l}}^{\,0}$$

$$+ \int_{-b/2}^{b/2} H_o\hat{a}_x\cdot dx\,\hat{a}_x$$

$$= -H_o\left[-\frac{b}{2} - \frac{b}{2}\right] + H_o\left[\frac{b}{2} + \frac{b}{2}\right]$$

$\oint \vec{H}\cdot d\vec{l} = H_o b + H_o b = 2H_o b \to (2)$

$K_y\cancel{b} = 2H_o\cancel{b}$

$H_o = \dfrac{K_y}{2}$

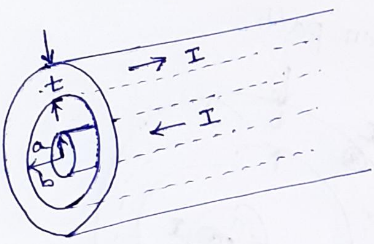
$\therefore \vec{H} = H_o\hat{a}_x = \dfrac{K_y}{2}\hat{a}_x$ for $z>0$

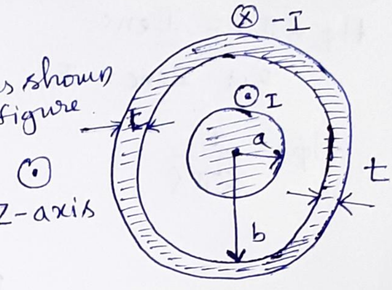
$\vec{H} = -H_o\hat{a}_x = -\dfrac{K_y}{2}\hat{a}_x$ for $z<0$

In general $\boxed{\vec{H} = \frac{1}{2}\vec{K}\times\hat{a}_n}$ where $\hat{a}_n$ is a unit ~~vector~~ normal vector directed from the current sheet to the point of interest

C. ~~Infinitely~~ Ampere's law applied to Infinitely long Coaxial Transmission line

Consider an infinitely long Coaxial transmission line consisting of two cylinders as shown in figure

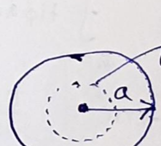




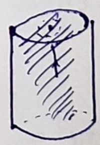
Case (i): $0 \le \rho < a$ or $0 \le \rho \le a$

$$\oint_{L_1} \vec{H}\cdot\vec{dl} = I_{enc} \Rightarrow \int_{\phi=0}^{2\pi} H_\phi \hat{a}_\phi \cdot \rho\, d\phi\, \hat{a}_\phi = H_\phi (2\pi\rho) = I_{enc}$$

$$H_\phi 2\pi\rho = I_{enc} \rightarrow (1)$$



$$I_{enc} = \int_S \vec{J}\cdot\vec{ds}$$



~~$\frac{I}{\pi a^2}$~~

$$\vec{J} = \frac{I}{\pi a^2}\hat{a}_z \qquad ds = \rho\, d\phi\, d\rho\, \hat{a}_z$$

$$I_{enc} = \int_S \frac{I}{\pi a^2}\hat{a}_z \cdot \rho\, d\phi\, d\rho\, \hat{a}_z$$

$$= \frac{I}{\pi a^2}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\rho} \rho\, d\phi\, d\rho\, \hat{a}_z$$

$$= \frac{I}{\cancel{\pi} a^2}\cancel{\pi}\rho^2 = \frac{I\rho^2}{a^2} \rightarrow (2)$$

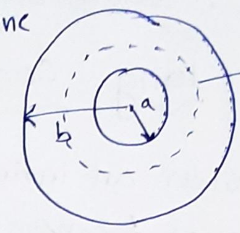
$(1) = (2)$

$$H_\phi 2\pi\rho = \frac{I\rho^2}{a^2} \qquad \boxed{H_\phi = \frac{I\rho}{2\pi a^2}}$$

(12)

Case (ii): ~~(detete)~~ $a \leq \rho \leq b$

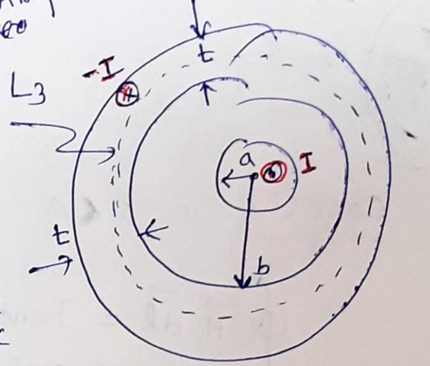
$$\oint_{L_2} \vec{H} \cdot \vec{dl} = I_{enc} \Rightarrow \int_{\phi=0}^{2\pi} H_\phi a_\phi \cdot \rho \, d\phi \, \hat{a}_\phi = I_{enc}$$



$$H_\phi \, 2\pi\rho = I_{enc}$$

But $I_{enc} = I$

$$H_\phi = \frac{I}{2\pi\rho}$$



Case (iii): $b \leq \rho \leq (b+t)$

$$\oint_{L_3} \vec{H} \cdot \vec{dl} = I_{enc}$$

$$\int_{\phi=0}^{2\pi} H_\phi \hat{a}_\phi \cdot \rho \, d\phi \, \hat{a}_\phi = I_{enc}$$

$$H_\phi \, 2\pi\rho = I_{enc}$$

$$I_{enc} = I + \int \vec{J} \cdot \vec{dS}$$

$$I_{enc} = I + \int_{\phi=0}^{2\pi} \int_{\rho=b}^{\rho} \left(\frac{-I}{\pi((b+t)^2 - b^2)} \right) \hat{a}_z \cdot (\rho \, d\phi \, d\rho \, \hat{a}_z) \qquad J = -\frac{I}{\pi[(b+t)^2 - b^2]}$$

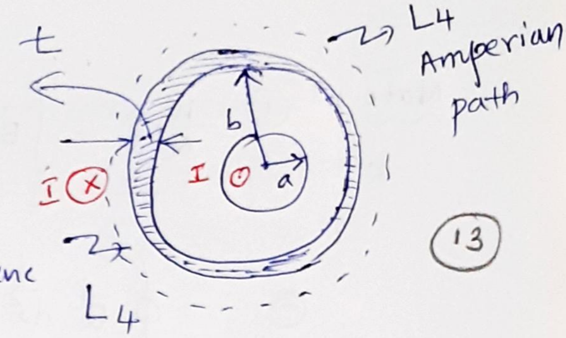
$$I_{enc} = I - \frac{I}{\cancel{\pi}[(b+t)^2 - b^2]} \times \cancel{\pi} \rho^2 \Big|_{\rho=b}^{\rho}$$

$$= I - \frac{(\rho^2 - b^2) I}{(b+t)^2 - b^2}$$

$$H_\phi (2\pi\rho) = I \left[1 - \frac{(\rho^2 - b^2)}{t^2 + 2bt} \right]$$

$$H_\phi = \frac{I}{2\pi\rho} \left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right]$$

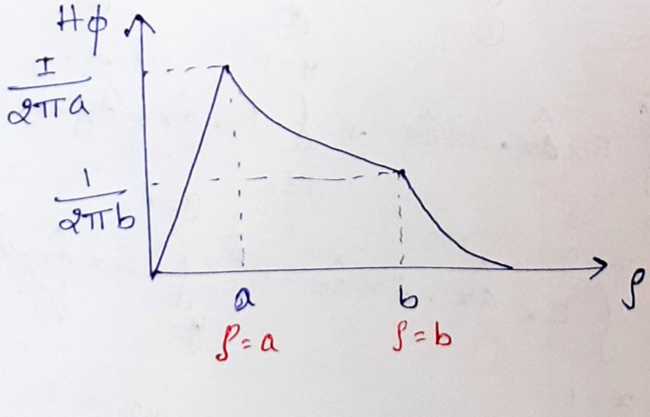
Case (iv) : $\rho \geq b+t$



(13)

$$\oint_{L_4} \vec{H} \cdot \vec{dl} = I - I = 0 = I_{enc}$$

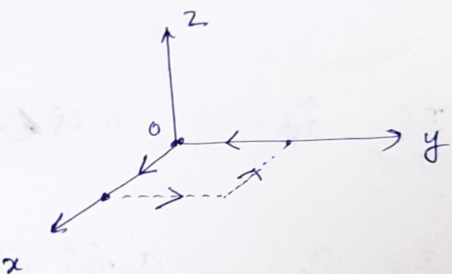
$$\vec{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\,\hat{a}_\phi & 0 \leq \rho < a \text{ or } 0 \leq \rho \leq a \\[2ex] \dfrac{I}{2\pi\rho}\,\hat{a}_\phi & a \leq \rho \leq b \\[2ex] \dfrac{I}{2\pi\rho}\left[1 - \dfrac{(\rho^2 - b^2)}{t^2 + bt}\right]\hat{a}_\phi & b \leq \rho \leq (b+t) \\[2ex] 0 & \rho \geq (b+t) \end{cases}$$

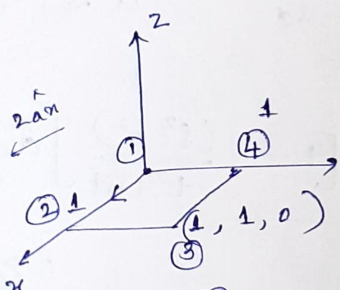


Note: ① $\frac{W}{Q} = -\int_L \vec{E}\cdot\vec{dl}$

② $\oint_L \vec{E}\cdot\vec{dl} = 0$ $\qquad \oint_L \vec{E}\cdot\vec{dl} = V_{AB} + V_{BA}$

③





$\vec{E} = 2\hat{a}_x$

$$\oint \vec{E}\cdot\vec{dl} = \int_{①}^{②} \vec{E}\cdot\vec{dl} + \int_{②}^{③} \vec{E}\cdot\vec{dl} + \int_{③}^{④} \vec{E}\cdot\vec{dl} + \int_{④}^{①} \vec{E}\cdot\vec{dl}$$

$$= \int_{x=0}^{1} E_x \hat{a}_x \cdot dx\,\hat{a}_x + \int_{y=0}^{y=1} \cancelto{0}{E_x \hat{a}_x \cdot dy\,a_y}$$

$$+ \int_{x=1}^{x=0} E_x \hat{a}_x \cdot dx\,\hat{a}_x + \int_{y=1}^{y=0} \cancelto{0}{E_x \hat{a}_x \cdot dy\,\hat{a}_y}$$

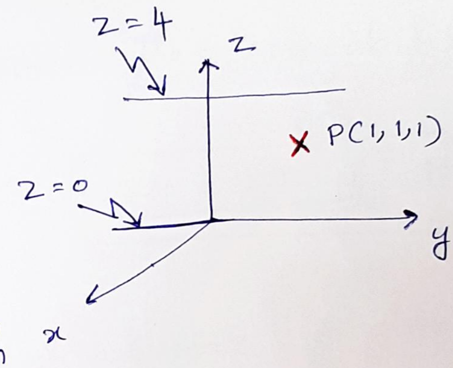
$$= E_x\, x \Big|_{x=0}^{x=1} + E_x\, x \Big|_{x=1}^{x=0}$$

$$= E_x[1] + E_x[-1] = 0$$

Example 7.5) Planes $z=0$ and $z=4$ carry current $\vec{K} = -10\hat{a}_x$ A/m and $\vec{K} = 10\hat{a}_x$ A/m, respectively. Determine $\vec{H}$ at (a) $(1,1,1)$ (b) $(0,-3,10)$

Solution:

(a) Let $\vec{H}$ at $(1,1,1)$



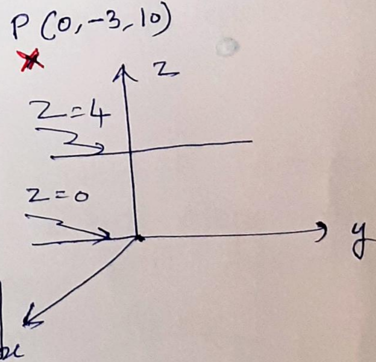
$$\vec{H} = \vec{H_0} + \vec{H_4}$$

$$= \frac{1}{2}\vec{K}\times\hat{a}_n + \frac{1}{2}\vec{K}\times\vec{a}_n$$

$$= \frac{1}{2}(-10\hat{a}_x)\times\hat{a}_z + \frac{1}{2}(10\hat{a}_x\times(-\hat{a}_z))$$

$$= \frac{1}{2}(10)\hat{a}_y + \frac{1}{2}(10)\hat{a}_y$$

$$\vec{H} = 10\hat{a}_y \text{ A/m}$$



(b) $\vec{H} = \vec{H_0} + \vec{H_4}$

$$\vec{H} = \frac{1}{2}\left[\vec{K}\times\hat{a}_n\right] + \frac{1}{2}\left[\vec{K}\times\vec{a}_n\right]$$

$$= \frac{1}{2}\left[-10\hat{a}_x\times\hat{a}_z\right] + \frac{1}{2}\left[10\hat{a}_x\times\hat{a}_z\right]$$

$$= +5\hat{a}_y - 5\hat{a}_y = 0$$

$$\vec{H} = 0 \text{ A/m}$$

MAGNETIC FLUX DENSITY - MAXWELL'S EQUATION

The magnetic flux density $\vec{B}$ is similar to electric flux density $\vec{D}$.

We know $\vec{D} = \varepsilon_0 \vec{E}$ But $\boxed{\vec{B} = \mu_0 \vec{H}}$

where μ_0 = Permeability of free space

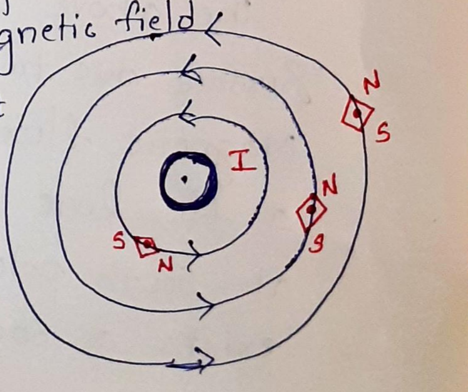
$$\boxed{\mu_0 = 4\pi \times 10^{-7} \ H/m}$$

The magnetic flux through a surface S is given by

$$\psi = \int_S \vec{B} \cdot d\vec{S}.$$

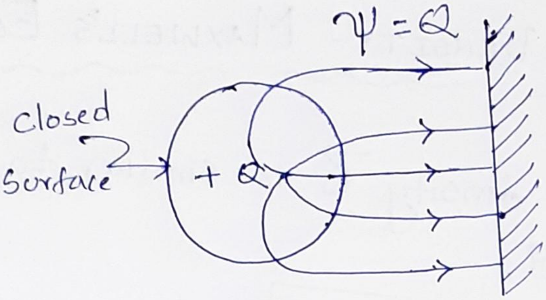
Where ψ is in Webers (Wb) and the Magnetic flux density $\vec{B}$ is in Webers per square meter (Wb/m^2) or tesla (T).

A magnetic flux line is a path to which $\vec{B}$ is tangential at every point on the line. It is a line along which the needle of a magnetic compass will orient itself if placed in the presence of magnetic field.



The direction of $\vec{B}$ is taken as that indicated as "north" by the needle of the magnetic compass.

Notice that each flux line is closed and has no beginning or end.

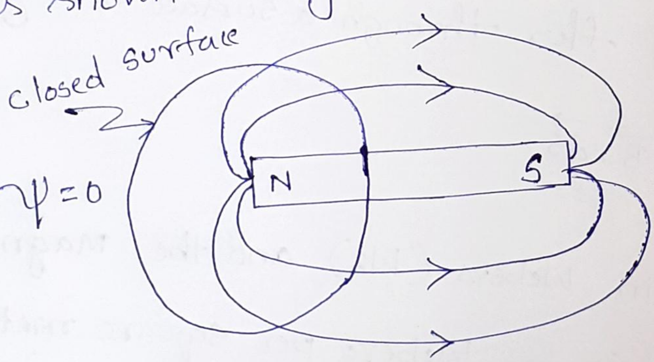


Flux leaving a closed surface due to isolated electric charge ψ ~~Electrostatic field~~

The above diagram shows flux leaving a closed surface due to Electrostatic field.

$\therefore$ Flux $= \psi = \oint \vec{D}.\vec{ds} = Q$

Thus it is possible to have an isolated electric charge as shown in figure



Flux leaving a closed surface due to magnetic charge

$$\psi = \oint_S \vec{B}.\vec{ds} = 0$$

The above diagram shows flux leaving a closed surface due to magnetic charge $\psi = \oint_S \vec{B}.\vec{ds} = 0$ Magnetic flux lines close upon themselves. That is a flux line started at North pole 'N' will end at South pole 'S'. For every North pole 'N', there exists a south pole 'S'. There fore it is not possible to have isolated magnetic poles or isolated magnetic charges.

$\therefore$ An isolated magnetic charge does not exist.

Thus the ~~flux~~ total flux through a closed surface in a magnetic field must be zero.

$$\boxed{\therefore \oint_S \vec{B}.\vec{ds} = 0}$$

This equation is referred to as the law of conservation of magnetic flux or Gauss's law for magnetostatic fields.

Although the magnetostatic field is not conservative, magnetic flux is conserved.

consider $\oint_S \vec{B}.\vec{ds} = 0$

Apply divergence theorem

$$\oint_S \vec{B}.\vec{ds} = \int_V \nabla.\vec{B}\, dV = 0$$

$$\boxed{\nabla.\vec{B} = 0}$$ This is the Fouth Maxwell's equation

MAXWELL'S EQUATION FOR STATIC FIELDS

Maxwell's equation for static electric and Magnetic fields

Differential (or Point) form	Integral form	Remarks
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot \vec{ds} = \int_V \rho_v \, dv$	Gauss's law
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot \vec{ds} = 0$	Non Existence of of magnetic monopole
$\nabla \times \vec{E} = 0$	$\oint_L \vec{E} \cdot \vec{dl} = 0$	Conservative nature of electrostatic field
$\nabla \times \vec{H} = \vec{J}$	$\oint_L \vec{H} \cdot \vec{dl} = \int_S \vec{J} \cdot \vec{ds}$	Ampere's law

MAGNETIC SCALAR AND VECTOR POTENTIALS

We know that for Electrostatic fields $\vec{E} = -\nabla V$
where V is potential.

Similarly we can define a potential associated with magnetostatic field $\vec{B}$. In fact, the magnetic potential could be scalar V_m or Vector $\vec{A}$.

Consider $\quad \vec{H} = -\nabla V_m \quad$ if $\vec{J} = 0$

$$\nabla \times \vec{H} = \nabla \times (-\nabla V_m)$$
$$= -\nabla \times (\nabla V_m)$$
$$= -0 \quad [\text{Because } \nabla \times (\nabla V_m) = 0]$$
$$= 0 \quad [\text{From vector identity}]$$

$\therefore \quad \vec{J} = \nabla \times \vec{H} = 0$

Thus the magnetic scalar potential V_m is defined only in a region where $\vec{J} = 0$

$$\nabla \times (-\nabla V_m) = 0$$
$$\nabla^2 V_m = 0 \quad \text{if } \vec{J} = 0$$

$\Rightarrow$

$\therefore$ V_m satisfies Laplace's equation.

Vector magnetic potential $\vec{A}$ is defined as

$$\vec{B} = \nabla \times \vec{A}$$

where $\quad \vec{A} = \int_L \frac{\mu_0 I \, d\vec{l}}{4\pi R} \quad$ for line current

$\cancel{\vec{A} = \int_S \mu_0 \vec{K} \, d\vec{S}}$

$$\vec{A} = \int_S \frac{\mu_0 \vec{K}\, ds}{4\pi R} \quad \text{for surface current}$$

$$\vec{A} = \int_V \frac{\mu_0 \vec{J}\, dv}{4\pi R} \quad \text{for volume current}$$

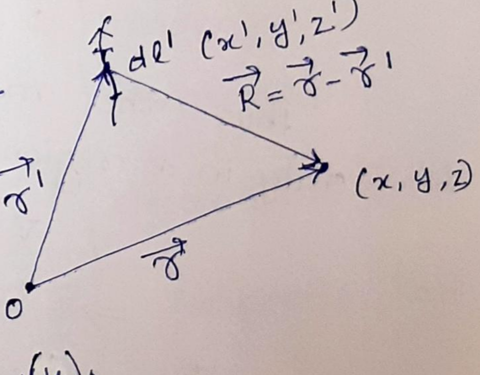
We can derive $\vec{A} = \int_L \frac{\mu_0 I\, \vec{dl}}{4\pi R}$ from $\vec{B}$

We know $\vec{B} = \frac{\mu_0}{4\pi} \int_L \frac{I\, \vec{dl'} \times \vec{R}}{R^3} \rightarrow (1)$

We $\vec{R}$ = distance vector from the line element $\vec{dl'}$ at the source point (x', y', z') to the field point (x, y, z) as shown in figure

$$R = |r - r'|$$

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$



$$\nabla\left(\frac{1}{R}\right) = \frac{d(1/R)}{dx}\hat{a}_x + \frac{d(1/R)}{dy}\hat{a}_y + \frac{d(1/R)}{dz}\hat{a}_z$$

$$\nabla\left(\frac{1}{R}\right) = \frac{d}{dx}\left(\frac{1}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}}\right)\hat{a}_x$$

$$+ \frac{d}{dy}\left(\frac{1}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}}\right)\hat{a}_y$$

$$+ \frac{d}{dz}\left(\frac{1}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}}\right)\hat{a}_z$$

$$= -\frac{1}{\cancel{2}} \times \frac{\cancel{2}(x-x')}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} \hat{a}_x$$

$$- \frac{1}{\cancel{2}} \times \frac{\cancel{2}(y-y')}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} \hat{a}_y$$

$$- \frac{1}{\cancel{2}} \times \frac{\cancel{2}(z-z')}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} \hat{a}_z$$

$$= \frac{-\left[(x-x')\hat{a}_x + (y-y')\hat{a}_y + (z-z')\hat{a}_z\right]}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}}$$

$$= -\frac{\vec{R}}{R^3}$$

$$\therefore -\frac{\vec{R}}{R^3} = \nabla\left(\frac{1}{R}\right) \Rightarrow \frac{\vec{R}}{R^3} = -\nabla\left(\frac{1}{R}\right) \rightarrow (2)$$

using (2) in (1)

$$\vec{B} = \frac{\mu_0}{4\pi} \int_L I\,\vec{dl'} \times \left[-\nabla\left(\frac{1}{R}\right)\right]$$

$$\vec{B} = -\frac{\mu_0}{4\pi} \int_L I\,\vec{dl'} \times \nabla\left(\frac{1}{R}\right)$$

We know from Vector identity

$$\vec{F} \times \nabla f = f\,\nabla \times \vec{F} - \nabla \times f\vec{F}$$

$$\text{III}^{ly}\quad \vec{dl'} \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R}\,\nabla \times \vec{dl'} - \nabla \times \frac{1}{R}\,\vec{dl'}$$

∇ operates with respect to (x, y, z) while $\vec{dl'}$ is a function of (x', y', z'), $\nabla \times \vec{dl'} = 0$

$$\therefore \vec{dl'} \times \nabla\left(\frac{1}{R}\right) = -\nabla \times \frac{\vec{dl'}}{R} \rightarrow (3)$$

$$\vec{B} = -\frac{\mu_0}{4\pi}\int_L I\left(-\nabla\times\frac{\vec{dl'}}{R}\right)$$

$$\vec{B} = +\nabla\times\int_L \frac{\mu_0 I \vec{dl'}}{4\pi R}$$

$$\vec{B} = \nabla\times\int_L \frac{\mu_0 I \vec{dl'}}{4\pi R} \rightarrow (4)$$

But $\vec{B} = \nabla\times\vec{A} \longrightarrow (5)$

(4) = (5)

Vector Magnetic potential

$$\boxed{\vec{A} = \int_L \frac{\mu_0 I \vec{dl'}}{4\pi R}} \rightarrow (6)$$

$$\psi = \int_S \vec{B}\cdot\vec{ds}$$

$$\psi = \int_S (\nabla\times\vec{A})\cdot\vec{ds}$$

Applying Stroke's theorem

$$\psi = \int_S (\nabla\times\vec{A})\cdot\vec{ds} = \oint_L \vec{A}\cdot\vec{dl}$$

Magnetic flux in terms of $\vec{A}$.

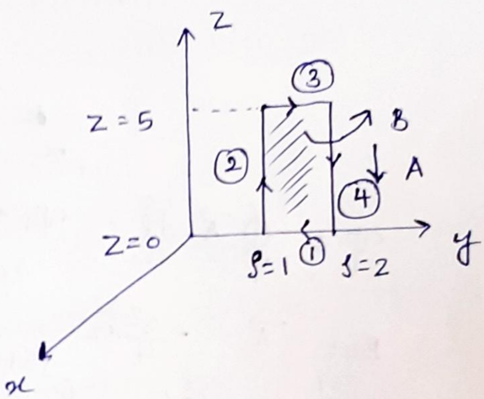
$$\boxed{\psi = \oint_L \vec{A}\cdot\vec{dl}}$$

Example 7.7) Given the magnetic Vector potential $\vec{A} = -\frac{\rho^2}{4}\hat{a}_z$ Wb/m Calculate the total magnetic flux crossing the surface

$\phi = \pi/2$, $1 \le \rho \le 2$ m , $0 \le z \le 5$ m

Solution :

Method-1 :



$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{B} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

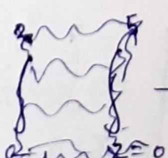
$$\vec{B} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{\rho^2}{4} \end{vmatrix}$$

$$\vec{B} = \frac{1}{\cancel{\rho}}\left[-\cancel{\rho}\hat{a}_\phi\right]\left[\frac{\partial}{\partial \rho}\left(-\frac{\rho^2}{4}\right) - 0\right] + \hat{a}_z \cancelto{0}{(0)} + \hat{a}_\rho \cancelto{0}{(0)}$$

$$= +\frac{2\rho}{4}\hat{a}_\phi = \frac{\rho}{2}\hat{a}_\phi$$

$$\vec{ds} = d\rho\, dz\, \hat{a}_\phi$$

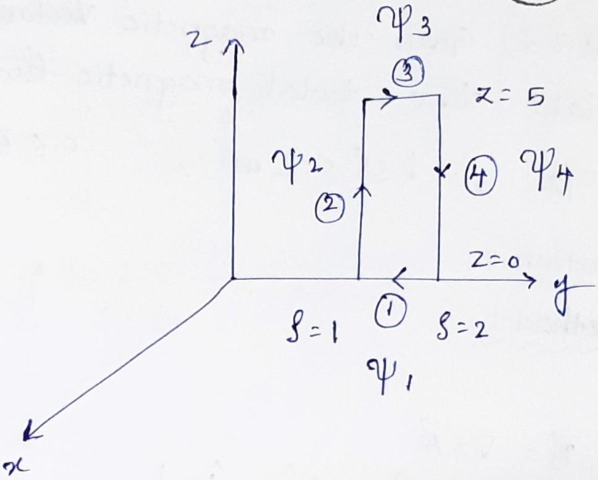
$$\psi = \int_S \vec{B}\cdot\vec{ds} =$$



$$= \int_{\rho=1}^{2}\int_{z=0}^{5} \frac{\rho}{2}\hat{a}_\phi \cdot d\rho\, dz\, \hat{a}_\phi = \left.\frac{\rho^2}{2\times 2}\right|_{\rho=1}^{2} \times \left. z \right|_{z=0}^{5}$$

$$= \frac{1}{4}\left[4-1\right]\times 5 = 3.75 \text{ Wb}$$

Method-2 :



$$\psi = \oint_L \vec{A}\cdot\vec{dl} = \psi_1 + \psi_2 + \psi_3 + \psi_4$$

But $\vec{A} = -\frac{\rho^2}{4}\hat{a}_z$

$$\psi_1 = \int_{\rho=2}^{\rho=1} \vec{A}\cdot d\rho\,\hat{a}_\rho = \int_{\rho=2}^{\rho=1} -\frac{\rho^2}{4}\hat{a}_z\cdot d\rho\,\hat{a}_\rho = 0 \quad \because \hat{a}_z\cdot\hat{a}_\rho = 0$$

$$\psi_3 = \int_{\rho=1}^{2} \vec{A}\cdot d\rho\,\hat{a}_\rho = \int_{\rho=1}^{\rho=2} -\frac{\rho^2}{4}\hat{a}_z\cdot d\rho\,\hat{a}_\rho = 0 \quad \because \hat{a}_z\cdot\hat{a}_\rho = 0$$

$$\psi_2 = \left.\int_{z=0}^{5} \vec{A}\cdot dz\,\hat{a}_z\right|_{\rho=1} = \left.\int_{z=0}^{5} -\frac{\rho^2}{4}\hat{a}_z\cdot dz\,\hat{a}_z\right|_{\rho=1} = \left.-\frac{\rho^2}{4}\left[z\Big|_0^5\right]\right|_{\rho=1}$$

$$\psi_2 = -\frac{1}{4}[5]$$

$$\psi_4 = \left.\int_{z=5}^{z=0} \vec{A}\cdot dz\,\hat{a}_z\right|_{\rho=2} = \left.\int_{z=5}^{z=0} -\frac{\rho^2}{4}\hat{a}_z\cdot dz\,\hat{a}_z\right|_{\rho=2} = \left.-\frac{\rho^2}{4}\left[z\Big|_5^0\right]\right|_{\rho=2}$$

$$\psi_4 = -\frac{2^2}{4}[0-5] = -\frac{4}{4}[0-5] = -1(-5) = 5$$

$$\psi = \psi_1 + \psi_2 + \psi_3 + \psi_4 = 0 - \frac{5}{4} + 0 + 5 = 3.75\ \text{Wb}$$

FORCES DUE TO MAGNETIC FIELDS

There are three types of forces due to magnetic fields they are

(i) Force on a charged particle
(ii) Force on a current element
(iii) Force between two current elements

(i) Force on a charged particle

For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by

$$\vec{F} = \vec{F_e} + \vec{F_m}$$

$$\vec{F} = Q\vec{E} + Q\,\vec{u}\times\vec{B}$$

$$\vec{F} = Q(\vec{E} + \vec{u}\times\vec{B}) \rightarrow (1)$$

Equation (1) is known as "Lorentz force equation" If the mass of the charged particle moving in $\vec{E}$ and $\vec{B}$ fields is m Equation (1) can also be written as

$$\vec{F} = m\vec{a} = Q(\vec{E} + \vec{u}\times\vec{B})$$

[From Newton's 2nd law of motion $\vec{F} = m\vec{a} = m\frac{d\vec{u}}{dt}$]

$$m\frac{d\vec{u}}{dt} = Q(\vec{E} + \vec{u}\times\vec{B})$$

Where $\vec{u}$ = Velocity of the charge 'Q'

$\vec{B}$ = Magnetic Field or Magnetic Flux Density

$\vec{E}$ = Electric field Intensity

In such situation energy can only be transformed by the electric field

Force on a charged Particle

State of Particle	E-field	B-field	Combined $\vec{E}$ and $\vec{B}$ fields
Stationary	$Q\vec{E}$	—	$Q\vec{E}$
Moving	$Q\vec{E}$	$Q(\vec{u}\times\vec{B})$	$Q(\vec{E}+\vec{u}\times\vec{B})$

2. Force on a current element

To determine the force on a current element $I\vec{dl}$ of a current carrying conductor due to the magnetic field $\vec{B}$, we modify the equation for

$$\vec{F_m} = Q\vec{u}\times\vec{B}$$

using the fact that

$$\Delta I = \underbrace{\frac{\Delta Q}{\Delta t}}_{\text{convection current}}$$

$$= \rho_v \Delta S \underbrace{\frac{\Delta Y}{\Delta t}}_{\text{volume}}$$

$$= \rho_v \Delta S\, u_y$$

$$J_y = \frac{\Delta I}{\Delta S} = \rho_v u_y$$

$$\vec{J} = \rho_v \vec{u}$$

Convection current

$$\vec{J} = \rho_v \vec{u} \Rightarrow$$

But we know $\quad I\vec{dl} = \vec{K}\,dS = \vec{J}\,dV$

$$\therefore \quad I\,\vec{dl} = \underbrace{\rho_v \vec{u}}_{\vec{J}}\,dV \rightarrow (1)$$

$$I\,\vec{dl} = \frac{dQ}{dt}\vec{dl} = dQ\frac{\vec{dl}}{dt} = dQ\,\vec{u} \rightarrow (2)$$

$$\simeq Q\vec{u}$$

$$\boxed{d\vec{F} = \underbrace{I\,\vec{dl}}_{Q\vec{u}}\times\vec{B}} \rightarrow (3)$$

If the current 'I' is through a closed path "L" or Circuit, the force on the circuit is given by

$$\vec{F} = \oint_L I\, d\vec{l} \times \vec{B} \rightarrow (4)$$

Remarks:

*) An elemental charge dQ moving with a velocity $\vec{u}$ (thereby producing convection current $dQ\vec{u}$) is ~~equiv~~ equivalent to the conduction current $I d\vec{l}$

*) The B-field that exerts force on $I d\vec{l}$ must be due to another element. In other words $\vec{B}$-field is external to the current element $I d\vec{l}$.

If instead of the line current $I d\vec{l}$, we have surface currents $\vec{K} ds$ or a volume current $\vec{J} dv$

Then

$$d\vec{F} = \vec{K} ds \times \vec{B}$$

$$d\vec{F} = \vec{J} dv \times \vec{B}$$

$$\vec{F} = \int_S \vec{K} ds \times \vec{B} \quad \text{and} \quad \vec{F} = \int_V \vec{J} dv \times \vec{B}$$

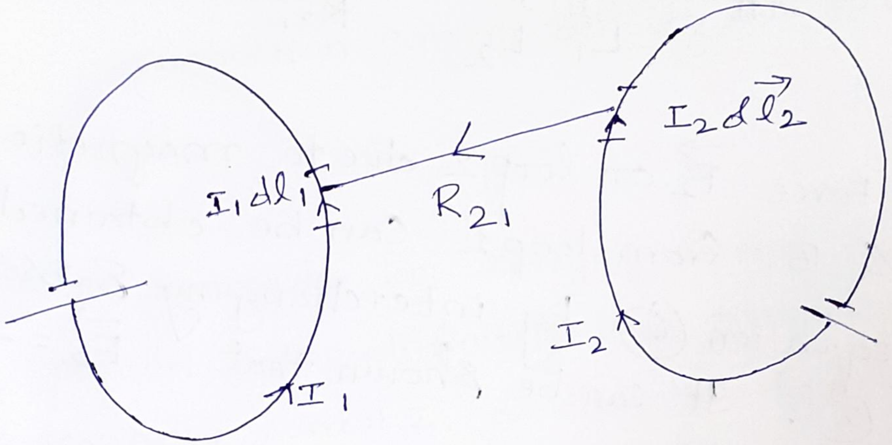
The magnetic field $\vec{B}$ is defined as the force per unit current element.

Alternatively, $\vec{B}$ may be defined as the vector that satisfies $\frac{F_m}{q} = \vec{u} \times \vec{B}$ or $\frac{F_e}{q} = \vec{E}$

$\vec{F_m}$ = Force due to magnetic field

$\vec{F_e}$ = Force due to Electric charge

Force between Two Current Elements



Let us consider the force between two elements $I_1 d\vec{l}_1$ and $I_2 d\vec{l}_2$. According to Biot Savart's law, both current elements produce magnetic fields.

Let $d(dF_1)$ = Force on element $I_1 d\vec{l}_1$ due to field $d\vec{B}_2$

$$d(dF_1) = I_1 d\vec{l}_1 \times d\vec{B}_2$$

But from Biot - Savart's law

$$d\vec{B}_2 = \frac{\mu_0 I_2 d\vec{l}_2 \times \hat{a}_{R_{21}}}{4\pi R_{21}^2}$$

$$d(dF_1) = \frac{\mu_0 I_1 d\vec{l}_1 \times (I_2 d\vec{l}_2 \times \hat{a}_{R21})}{4\pi R_{21}^2} \quad (5)$$

The equation (5) is essentially the law of force between two current elements and is anologous to coulomb's law which expresses force between stationary charges.

From (5) We can obtain the total force $\vec{F_1}$ on current loop 1 due to loop 2 is (30)

$$\vec{F_1} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{\vec{dl_1} \times (\vec{dl_2} \times a_{R21})}{R_{21}^2} \rightarrow (6)$$

The Force $\vec{F_2}$ on loop 2 due to magnetic field B_1 from loop 1 can be obtained from equation (6) by interchanging subscripts (1) and (2). It can be shown that $\vec{F_2} = -\vec{F_1}$

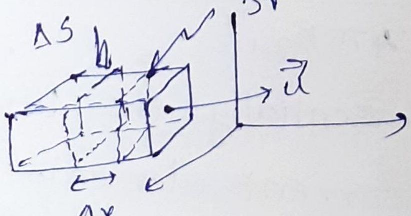
$\vec{F_2} = -\vec{F_1}$

$\therefore$ $\vec{F_2}$ and $\vec{F_1}$ obeys Newton's third law.

Note:

Convection Current

Convection current is different from Conduction current, does not involve conductors and consequently does not satisfy ohm's law. It occures when current flows through an insulating medium such as liquid, ~~rarefied~~ or Vacuum.



$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\rho_v \Delta S \Delta Y}{\Delta t}$$

$$= \rho_v \Delta S \frac{\Delta Y}{\Delta t}$$

$$= \rho_v \Delta S\, u_y$$

Convection Current $\vec{J_c} = \frac{\Delta I}{\Delta S} = \rho_v \vec{u}$