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Unit-4 - MAXWELL'S EQUATIONS

In Electrostatics we have studied \vec{E} as a function of x, y, z , i.e. $\vec{E}(x, y, z)$. In Magnetostatics we got \vec{H} as a function of x, y, z , i.e. $\vec{H}(x, y, z)$

Now we have to examine the situation where Electric and Magnetic fields ^{are} dynamic or time varying. Dynamic EM field is one in which Electric and Magnetic fields are interdependent. In other words, a time varying electric field necessarily involves a corresponding time-varying magnetic field.

Electrostatic fields are produced by static electric charges, Magnetostatic fields are due to motion of electric charges, whereas time varying fields or waves due to accelerated charges or time varying currents

Now we have to understand

1) Electro motive force (EMF) based on Faraday's experiment

2) Displacement current from Maxwell's equation

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FARADAY'S LAW

Faraday's law states that the induced electromotive force (emf), V_{emf} (in volts), in any closed circuit is equal to rate of change of the magnetic flux linkage by the circuit

$$V_{emf} = -\frac{d\lambda}{dt} = -\frac{d}{dt} N\varphi = -N \frac{d\varphi}{dt}$$

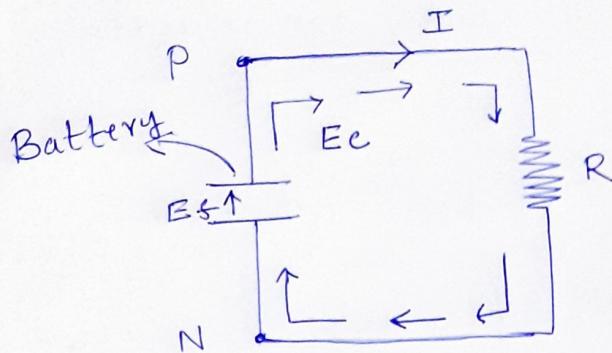
Where $\lambda = N\varphi$ = Flux linkage

N = Number of turns in the circuit

φ = Flux through each turn.

Negative sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This behavior is known as Lenz's law, which emphasizes that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the change in original magnetic field.

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The electrochemical action of the battery results in an emf-produced field E_s . Due to the accumulation of charges at the battery terminals, an electrostatic field $E_e (-\nabla V)$ also exists.

We have following observations

- ① An electrostatic field \vec{E}_e cannot maintain a steady current in a closed circuit, since $\oint \vec{E}_e \cdot d\vec{l} = 0$ [\vec{E}_e is conservative]
- ② An emf-produced field \vec{E}_s is nonconservative.

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TRANSFORMER AND MOTIONAL ELECTRO MOTIVE FORCE

We know $V_{emf} = -\frac{d\psi}{dt} \rightarrow (1)$ [Faraday's law]

where ψ = Magnetic flux

We know $V_{emf} = \oint_L \vec{E} \cdot d\vec{l} \rightarrow (2)$ [In terms of \vec{E}]

$$V_{emf} \approx \oint_L \vec{B} \cdot d\vec{s}$$

$$V_{emf} = -\frac{d}{dt} \psi = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{s} \rightarrow (3)$$

$$\therefore V_{emf} = \oint_L \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{s} \rightarrow (4)$$

It is clear from equation (4) in time varying situations both electric and magnetic fields are present. The variation of flux with time is

$$\boxed{V_{emf} = \frac{d\psi}{dt} \text{ or } V_{emf} = \oint_L \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{s}}$$

- Causes may be caused in three ways. They are
- ① By having a stationary loop in a time varying \vec{B} -field
 - ② By having a time varying loop area in a static \vec{B} -field
 - ③ By having a time varying loop area in time varying \vec{B} -field.

(1) Stationary loop in Time Varying B-field

(Transformer emf)

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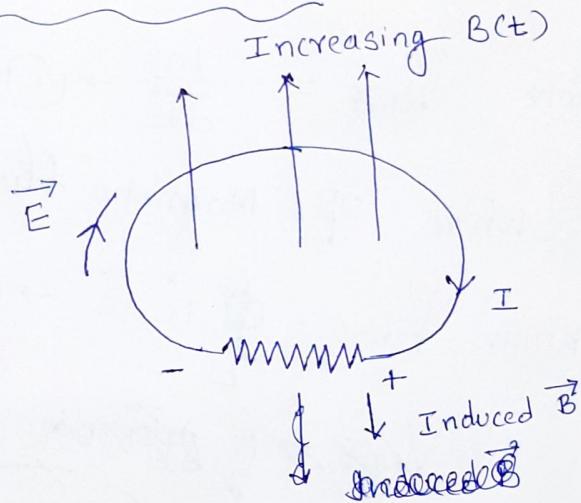


Figure shows a stationary conducting loop placed in a time-varying magnetic field \vec{B} .

The induced emf is given by

$$V_{\text{emf}} = \oint \vec{E} \cdot d\vec{l} \quad \rightarrow (1)$$

$$V_{\text{emf}} = -N \frac{d\psi}{dt}$$

$$V_{\text{emf}} = -\frac{d\psi}{dt} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{S}$$

$$V_{\text{emf}} = \int_S -\frac{d\vec{B}}{dt} \cdot d\vec{S} \quad \rightarrow (2)$$

~~The electric field is produced by the battery~~

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$$\textcircled{1} = \textcircled{2}$$

$$\oint \vec{E} \cdot d\vec{l} = \int_S -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

By applying stroke's theorem

$$\int_S \nabla \times \vec{E} \cdot d\vec{s} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

$$\therefore \nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

This is one of the Maxwell's equation

for time varying fields.

(2) Moving Loop in static \vec{B} -field (Motional emf)

When a conducting loop is moving in a static \vec{B} -field, an emf is induced in the loop. We recall that force on a charge moving with uniform velocity \vec{u} in a magnetic field \vec{B} is

$$\vec{F}_m = Q(\vec{u} \times \vec{B})$$

Motional Electric field is given by

$$\vec{E}_m = \frac{\vec{F}_m}{Q} = \vec{u} \times \vec{B}$$

If we consider conducting loop, moving with uniform velocity \vec{u} as consisting of large number of electrons free electrons, the emf induced in the loop is

$$V_{emf} = \oint_L \vec{E}_m \cdot d\vec{l} = \oint_L (\vec{u} \times \vec{B}) \cdot d\vec{l} \quad (7)$$

This type of emf is called ^{as} motional emf or flux-cutting emf because it is due to motional action.

It is the kind of emf found in electrical machines such as motors, generators and alternators.

$$V_{emf} = \oint_L (\vec{u} \times \vec{B}) \cdot d\vec{l} = \int_S \nabla \times (\vec{u} \times \vec{B}) \cdot d\vec{S} \rightarrow (1)$$

$$V_{emf} = \oint_L \vec{E}_m \cdot d\vec{l} = \int_S (\nabla \times \vec{E}_m) \cdot d\vec{S} \rightarrow (2)$$

$$(1) = (2)$$

$$\text{where } \vec{E}_m = \frac{F_m}{Q} = \vec{u} \times \vec{B}$$

$$\boxed{\nabla \times \vec{E}_m = \nabla \times (\vec{u} \times \vec{B})}$$

③. Moving Loop in Time Varying Field

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In the general case, a moving conducting loop in a time varying magnetic field. Both transformer emf and motional emf are present.

$$V_{\text{emf}} = \oint_L \vec{E} \cdot d\vec{l} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_L (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

$$V_{\text{emf}} = \int_S (\nabla \times \vec{E}) \cdot d\vec{s}$$

$$V_{\text{emf}} = - \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \int_S (\nabla \times \vec{u} \times \vec{B}) \cdot d\vec{s}$$

$$\therefore \nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{u} \times \vec{B}$$

Note:

- ① In moving loop in static B-field, the integral of the equation $\oint_L \vec{u} \times \vec{B} \cdot d\vec{l}$ is zero along the portion of the loop where $\vec{u} = 0$. Thus $d\vec{l}$ is taken along the portion of the loop that is cutting the field where \vec{u} is non-zero has non-zero value.
- ② The direction of the induced current is same as E_m or $\vec{u} \times \vec{B}$. The limits of the integration in eqn $\oint_L \vec{u} \times \vec{B} \cdot d\vec{l}$ are selected in the direction opposite to the induced current.

Displacement Current

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} \rightarrow ①$$

$$\nabla \cdot (\nabla \times \vec{H}) = 0 \rightarrow ②$$

$$\text{But from } ① \quad \nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} = -\frac{\partial \phi}{\partial t} \rightarrow ③$$

$$② \neq ③$$

$$\nabla \times \vec{H} = \vec{J} + \vec{J}_d$$

Conduction
current density

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \nabla \cdot \vec{J}_d$$

$$0 = \nabla \cdot \vec{J} + \nabla \cdot \vec{J}_d$$

$$\nabla \cdot \vec{J}_d = -\nabla \cdot \vec{J}, \quad \nabla \cdot \vec{J}_d = -\frac{\partial \phi}{\partial t}$$

$$\text{Inductance} \frac{\partial \phi}{\partial t}$$

$$\text{Inductance} \vec{J}_d$$

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

$$\vec{J}_d = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\boxed{\vec{J}_d = \frac{\partial \vec{D}}{\partial t}} \rightarrow ④$$

④ gives displacement current

density

$$\therefore \nabla \times \vec{H} = \vec{J} + \vec{J}_d$$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

is one of the

time varying Maxwell's Equations.

$$\text{Displacement current} \quad I_d = \int_s \vec{J}_d \cdot d\vec{s}$$

$$\text{Displacement current density} \quad \vec{J}_d = \frac{I_d}{\text{Area}}$$

Generalized Forms of Maxwell's Equations

Maxwell's Equations in Final forms

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Maxwell's equations for Time Varying fields are

Differential Form

$$\nabla \cdot \vec{D} = \rho_v$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$-\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Integral form

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho_v dv$$

$$\oint_S \vec{B} \cdot d\vec{s} = 0$$

$$\oint_L \vec{E} \cdot d\vec{l} = \frac{\partial \phi_B}{\partial t} - \frac{1}{c} \int_S \vec{B} \cdot d\vec{s}$$

$$\oint_L \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{s}$$

Remarks

Gauss's law

Non-existence of isolated magnetic charge

Faraday's law

Ampere's circuit law

Time Harmonic Fields

A time-Harmonic field is one that varies periodically or sinusoidally with time. Sinusoids are easily expressed in phasors. A phasor is a complex number that contains the amplitude and phase of a sinusoidal oscillations.

$$z = x + jy$$

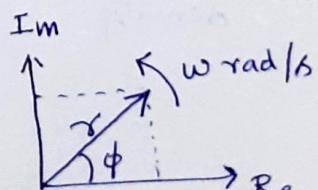
$$z = r \angle \phi$$

$$z_1 = x_1 + jy_1 = r_1 \angle \phi_1$$

$$z_2 = x_2 + jy_2 = r_2 \angle \phi_2$$

$$z_1 z_2 = r_1 r_2 \angle (\phi_1 + \phi_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle (\phi_1 - \phi_2)$$



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Time Harmonic Fields

A time harmonic field is one that varies periodically or sinusoidally with time

Sinusoidal variation is selected because it can be expressed in phasors. The concept of phasor is as follows.

A phasor is a complex number that contains the amplitude and phase of the sinusoidal variation

$$z = x + jy = \rho \angle \phi$$

$$z = \rho e^{j\phi}$$

$$z = \rho (\cos \phi + j \sin \phi)$$

$$\rho = |z| = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1} (y/x)$$

$$z_1 = x_1 + jy_1 = \rho_1 \angle \phi_1$$

$$z_2 = x_2 + jy_2 = \rho_2 \angle \phi_2$$

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

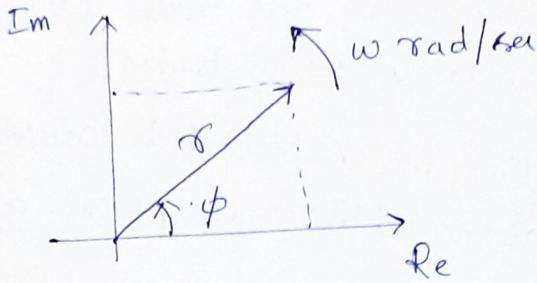
$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

$$z_1 z_2 = \rho_1 \rho_2 \angle (\phi_1 + \phi_2)$$

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} \angle (\phi_1 - \phi_2)$$

Representation of a phasor

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$$\sqrt{Z} = \sqrt{\gamma} \angle \frac{\phi}{2}$$

$$Z^* = x - jy = \gamma \angle -\phi$$

$$\text{If } \phi = \omega t + \theta$$

$$\gamma e^{j\phi} = \gamma e^{j(\omega t + \theta)} = \gamma e^{j\omega t} e^{j\theta}$$

$$\operatorname{Re}[\gamma e^{j\phi}] = \operatorname{Re}[e^{j(\omega t + \theta)}]$$

$$= \gamma \cos(\omega t + \theta)$$

$$\operatorname{Im}[\gamma e^{j\phi}] = \operatorname{Im}[e^{j(\omega t + \theta)}]$$

$$= \gamma \sin(\omega t + \theta)$$

Ex:

$$I(t) = I_0 [e^{j\theta} e^{j\omega t}]$$

$$I(t) = I_0 [e^{j(\omega t + \theta)}]$$

$$I(t) = \operatorname{Re}[I_0 e^{j(\omega t + \theta)}]$$

$$I(t) = I_0 \cos(\omega t + \theta)$$

$$I'(t) = \text{Im} [I_0 e^{j\theta} e^{j\omega t}]$$

$$I'(t) = I_0 \sin(\omega t + \theta)$$

$$I'(t) = \text{Re} [I_0 e^{j\theta} \cdot e^{j\omega t} e^{-j90^\circ}]$$

$$I'(t) = I_0 \cos(\omega t + \theta - 90^\circ)$$

Now we introduce a phasor $I_s = I_0 e^{j\theta}$

$$I_s = I_0 e^{j\theta} = I_0 \underline{\theta}$$

$$\therefore I(t) = I_0 \cos(\omega t + \theta)$$

$$I(t) = \text{Re} [I_0 e^{j\omega t} e^{j\theta}]$$

$$I(t) = \text{Re} [I_0 e^{j\theta} \cdot e^{j\omega t}]$$

$$I(t) = \text{Re} [I_s e^{j\omega t}]$$

In general, a phasor is a complex quantity

and it could be a scalar or vector.

If a vector $\vec{A}(x, y, z, t)$ is a time harmonic field, the vector $\vec{A}(x, y, z, t)$ can be represented as

$$\vec{A}(x, y, z, t) = \text{Re} [\vec{A}_s(x, y, z) e^{j\omega t}] \rightarrow ①$$

Point ~~Ex. If $\vec{A}(x, y, z, t) = A_s(x, y, z) \sin(\omega t + \phi)$ then $\vec{A}_s(x, y, z)$ is a phasor.~~ Point ② ~~Ex. If $\vec{A}(x, y, z, t) = A_s(x, y, z) \cos(\omega t + \phi)$ then $\vec{A}_s(x, y, z)$ is a phasor.~~

On general, Phasor is a function of position, not a function of time.

For example if $\vec{A} = A_0 \cos(\omega t - \beta x) \hat{a}_y$

$$\vec{A} = \text{Re} [A_0 e^{j\omega t} e^{-j\beta x} \hat{a}_y e^{j\omega t}]$$

The phasor form of \vec{A} is \vec{A}_s

$$\vec{A}_s = A_0 e^{-j\beta x} \hat{a}_y$$

$$\begin{aligned}\frac{d\vec{A}}{dt} &= \frac{d}{dt} \left[\operatorname{Re} (\vec{A}_s e^{j\omega t}) \right] \\ &= A_s \operatorname{Re} \left[\vec{A}_s \frac{d}{dt} e^{j\omega t} \right] \\ &= \operatorname{Re} \left[\vec{A}_s \cancel{(j\omega)} e^{j\omega t} \right] \\ \frac{d\vec{A}}{dt} &= \operatorname{Re} \left[\cancel{j\omega} \vec{A}_s e^{j\omega t} \right]\end{aligned}$$

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$\frac{d\vec{A}}{dt} = \operatorname{Re} [j\omega \vec{A}_s e^{j\omega t}]$ is showing that taking the time derivative of the instantaneous quantity is equivalent to multiplying its phasor form by $j\omega$.

$$\begin{aligned}\therefore \frac{d\vec{A}}{dt} &\stackrel{\text{cancel } j\omega}{=} \operatorname{Re} [j\omega \vec{A}_s e^{j\omega t}] \\ &= \operatorname{Re} [j\omega \vec{A}_s e^{j(\omega t + 90^\circ)}] \\ &= \cancel{j\omega} \operatorname{Re} \left[\vec{A}_s e^{-j\frac{\pi}{2}} e^{j\omega t} \right] \\ &\stackrel{\text{cancel } \vec{A}_s}{=}\end{aligned}$$

$$\frac{d\vec{A}}{dt} \rightarrow \cancel{j\omega} \vec{A}_s$$

Because $\vec{A} = \operatorname{Re} [\vec{A}_s e^{j\omega t}]$

$$\frac{d\vec{A}}{dt} = \operatorname{Re} [j\omega \vec{A}_s e^{j\omega t}]$$

$\therefore \frac{d\vec{A}}{dt} \rightarrow j\omega \vec{A}_s$ ~~is obtained from \vec{A}_s~~
ie Replace \vec{A}_s by $j\omega \vec{A}_s$

Similarly, for $\vec{A} = \operatorname{Re} \frac{\vec{A}_s}{j\omega}$

be Replace \vec{A}_s by $\cancel{\frac{\vec{A}_s}{j\omega}}$

Similarly $\int \vec{A} dt \rightarrow \frac{\vec{A}_s}{j\omega}$ if it is ejnt in ①

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For $\int \vec{A} dt$ implies replace \vec{A}_s by $\frac{\vec{A}_s}{j\omega}$ if it is ejnt

Note: $\vec{A}(x, y, z, t)$ can also be $\vec{A}(x, y, z, t) = \text{Im}[\vec{A}(x, y, z) e^{j\omega t}]$

Maxwell's equations in Phasor form

$$\nabla \times \vec{E}(x, y, z, t) = - \frac{d}{dt} \vec{B}(x, y, z, t) \rightarrow ①$$

$$\vec{E}(x, y, z, t) = \text{Re}[\vec{E}_s(x, y, z) e^{j\omega t}]$$

$$\vec{B}(x, y, z, t) = \text{Re}[\vec{B}_s(x, y, z) e^{j\omega t}]$$

$$\rightarrow ②$$

$$\text{Consider } \nabla \times \vec{E}(x, y, z, t) = \nabla \times [\text{Re}[\vec{E}_s e^{j\omega t}]] \rightarrow ③$$

Curl operator ∇ operated only on (x, y, z)

$\therefore ②$ can be written as

$$\nabla \times \vec{E}(x, y, z, t) = \text{Re} \left\{ [\nabla \times \vec{E}_s] e^{j\omega t} \right\} \rightarrow ③$$

$$\nabla \times \vec{E}(x, y, z, t) = \text{Re}[\vec{B}_s e^{j\omega t}]$$

$$\vec{B}(x, y, z, t) = \text{Re}[\vec{B}_s e^{j\omega t}]$$

$$\text{Consider } - \frac{d}{dt} \vec{B}(x, y, z, t) = - \frac{d}{dt} \text{Re}[\vec{B}_s e^{j\omega t}]$$

$$= - \text{Re} \left[\vec{B}_s \frac{d}{dt} e^{j\omega t} \right]$$

$$- \frac{d \vec{B}}{dt} = - \text{Re}[j\omega \vec{B}_s e^{j\omega t}]$$

$$- \frac{d \vec{B}}{dt} = \text{Re}[-j\omega \vec{B}_s e^{j\omega t}] \rightarrow ④$$

$$\therefore ③ = ④ \Rightarrow \boxed{\nabla \times \vec{E}_s = -j\omega \vec{B}_s} \rightarrow ⑤$$

Eqn $\nabla \times \vec{E}_s = -jw\vec{B}_s$ is the phasor form

$$\text{of } \nabla \times \vec{E}(x, y, z, t) = -\frac{\partial}{\partial t} \vec{B}(x, y, z, t) \quad (22)$$

where \vec{E}_s may be like $\vec{E}_s = E_0 e^{-jBx} \hat{ay}$

$$\text{Similarly, } \nabla \times \vec{H}(x, y, z, t) = \vec{J}_s + \frac{\partial}{\partial t} \vec{D}(x, y, z, t) \rightarrow (1)$$

$$\vec{H}(x, y, z, t) = \operatorname{Re} [\vec{H}_s(x, y, z) e^{jwt}] \rightarrow (2)$$

$$\nabla \times \vec{H}(x, y, z, t) = \nabla \times \operatorname{Re} [\vec{H}_s(x, y, z) e^{jwt}]$$

$$= \operatorname{Re} [\nabla \times \vec{H}_s e^{jwt}] \rightarrow (3)$$

$$\begin{aligned} & \text{Consider } \vec{J}_s + \frac{\partial}{\partial t} \vec{D}(x, y, z, t) \\ &= \operatorname{Re} [\vec{J}_s e^{j\omega t}] + \frac{\partial}{\partial t} \operatorname{Re} [\vec{D}_s e^{j\omega t}] \left. \begin{array}{l} \vec{D} = \operatorname{Re} [\vec{D}_s e^{j\omega t}] \\ \end{array} \right\} \\ &= \operatorname{Re} [\vec{J}_s e^{j\omega t}] + \operatorname{Re} [\vec{D}_s \frac{\partial}{\partial t} e^{j\omega t}] \\ &= \operatorname{Re} [\vec{J}_s e^{j\omega t}] + \operatorname{Re} [j\omega \vec{D}_s e^{j\omega t}] \left. \begin{array}{l} \\ \end{array} \right\} \rightarrow (4) \end{aligned}$$

$$(3) = (4)$$

$$\nabla \times \vec{H}_s = \vec{J}_s + j\omega \vec{D}_s$$

$\nabla \times \vec{H}_s = \vec{J}_s + j\omega \vec{D}_s$ is the phasor form

$$\text{of } \nabla \times \vec{H}(x, y, z, t) = \vec{J}_s + \frac{\partial}{\partial t} \vec{D}(x, y, z, t)$$

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Similarly all Maxwell's Equations can be
expressed in phasor form
Time - Harmonic Maxwell's equations Assuming Time factor e^{jwt}

Point form

$$\nabla \cdot \vec{D}_s = S_s$$

$$\nabla \cdot \vec{B}_s = 0$$

$$\nabla \times \vec{E}_s = -j\omega \vec{B}_s$$

$$\nabla \times \vec{H}_s = \vec{J}_s + j\omega \vec{D}_s$$

Integral form

$$\oint \vec{D}_s \cdot d\vec{s} = \int \epsilon v_s dV$$

$$\oint \vec{B}_s \cdot d\vec{s} = 0$$

$$\oint \vec{E}_s \cdot d\vec{l} = -j \int \vec{B}_s \cdot d\vec{s}$$

$$\oint \vec{H}_s \cdot d\vec{l} = \left(\vec{J}_s + j\omega \vec{D}_s \right) \cdot \vec{d}s$$

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9.1) Example (9.1) [From Sadiku Textbook]

A conducting bar can slide freely over two conducting rails as shown in figure. Calculate the induced voltage in the bar.

(a) If the bar is stationed at $y=8\text{cm}$ and

$$\vec{B} = 4 \cos 10^6 t \hat{a}_z \text{ m Wb/m}^2$$

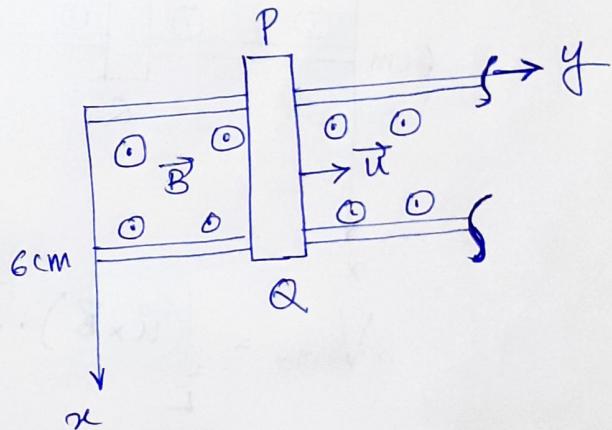
(b) If the bar slides at a velocity $u = 20 \hat{a}_y \text{ m/s}$

$$\text{and } \vec{B} = 4 \hat{a}_z \text{ m Wb/m}^2$$

(c) If the bar slides at a velocity $u = 20 \hat{a}_y \text{ m/s}$

$$\text{and } \vec{B} = 4 \cos (10^6 t - y) \hat{a}_z \text{ m Wb/m}^2$$

Solution:



(a) If the bar is stationed at $y=8\text{cm}$ and

$$\vec{B} = 4 \cos 10^6 t \hat{a}_z \text{ m Wb/m}^2$$

conducting loop is stationary and \vec{B} is time varying

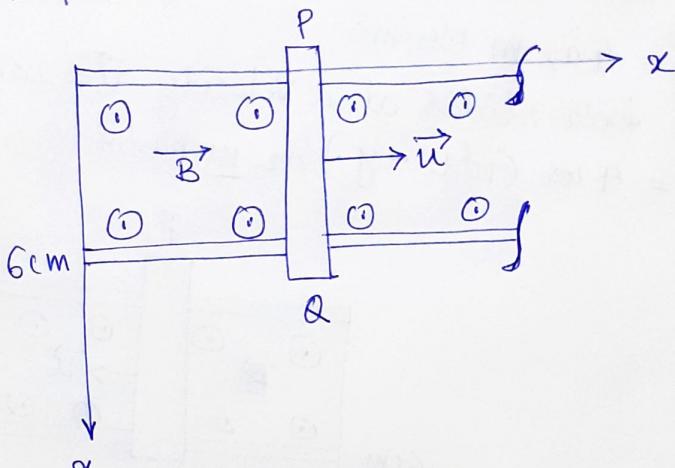
$$V_{emf} = - \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = - \int_{y=0}^{0.08} \int_{x=0}^{0.06} \frac{\partial}{\partial t} (4 \cos 10^6 t \times 10^{-3}) \hat{a}_z \cdot dx dy \hat{a}_z$$

$$= - \int_{y=0}^{0.08} \int_{x=0}^{0.06} 4 \times 10^6 \times 10^{-3} 8 \sin 10^6 t dx dy$$

$$= 4 \times 10^3 8 \sin 10^6 t \times 0.08 \times 0.06 = 19.2 \sin 10^6 t \text{ V}$$

The polarity of the induced voltage is such that 'P' on the bar is at lower potential than 'Q' when \vec{B} is increasing. [According to Lenz's law]

(b) If the bar slides at a velocity $\vec{u} = 20 \hat{y} \text{ m/s}$ and $\vec{B} = 4az \text{ mWb/m}^2$



$$V_{\text{emf}} = \int (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

$$= \int_0^L (\vec{u} \hat{y} \times \vec{B} \hat{z}) \cdot d\vec{x} \hat{z}$$

The induced current will be in the direction \hat{x}

$$\hat{y} \times \hat{z} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{a}_x \text{ direction}$$

\therefore The limits of integration will be in a direction opposite to \hat{a}_x

(26)

$$V_{emf} = \int_{x=0.06}^0 u \hat{a}_y \times B \hat{a}_z \cdot dx \hat{a}_x$$

$$= \int_{x=0.06}^0 (20x) \hat{a}_x \cdot dx \hat{a}_x \times 10^{-3}$$

$$= -20x \times 0.06 V = -4.8 mV$$

c) If the bar slides at a velocity $\vec{u} = 20 \hat{a}_y \text{ m/s}$
and $\vec{B} = 4 \cos(10^6 t - y) \hat{a}_z \text{ m Wb/m}^2$

$$V_{emf} = - \int_S \frac{d\vec{B}}{dt} \cdot d\vec{s} + \int_L (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

$$V_{emf} = - \int_{y=0}^{0.06} \int_{x=0}^y \frac{d}{dt} (4 \cos(10^6 t - y)) \hat{a}_z dy dx \hat{a}_x \times 10^{-3}$$

$$+ \int_{0.06}^0 (20 \hat{a}_y) \times (4 \times 10^{-3} \cos(10^6 t - y) \hat{a}_z) \cdot dx \hat{a}_x$$

$$= 240 \cos(10^6 t - y) \Big|_0^y - 80 \times 10^{-3} \times 0.06 \cos(10^6 t - y)$$

$$= 240 \cos(10^6 t - y) - 240 \cos 10^6 t - 4.8 \times 10^{-3} \cos(10^6 t - y)$$

$$\approx 240 \cos(10^6 t - y) - 240 \cos 10^6 t$$

Because Motional emf is negligible compared to transformer emf
 $\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$

$$\approx -480 \sin\left(\frac{10^6 t - y + 10^6 t}{2}\right) \sin\left(\frac{10^6 t - y - 10^6 t}{2}\right)$$

$$= +480 \sin\left(\frac{10^6 t - y}{2}\right) \sin\left(\frac{y}{2}\right) V$$

9.2) Example 9.2

The loop shown in figure is inside a uniform magnetic field $\vec{B} = 50\hat{a}_x \text{ mWb/m}^2$. If side DC of the loop rotates cuts the flux lines at the frequency of 50 Hz and the loop yz -plane at time $t=0$, find

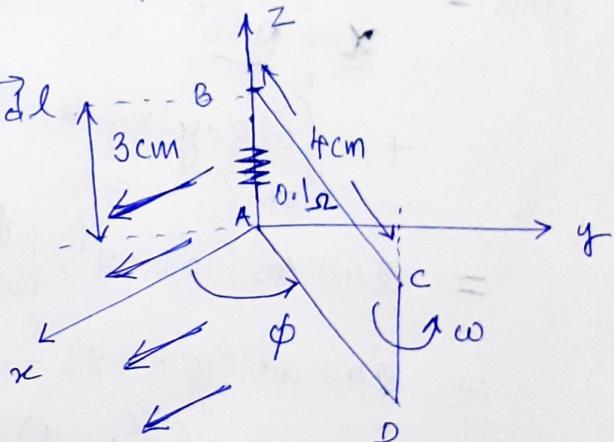
(a) The induced emf at $t=1 \text{ ms} = 1 \text{ ms}$

(b) The induced current at $t=3 \text{ ms} = 3 \text{ ms}$

Solution:

(a) Since \vec{B} field is static, the induced emf is motional, that is

$$V_{\text{emf}} = (\vec{u} \times \vec{B}) \cdot \vec{dl}$$



$$\vec{dl} = \vec{dl}_{DC} = dz \hat{a}_z$$

$$\vec{u} = \frac{d\vec{l}}{dt} = \frac{dz}{dt} \hat{a}_z = \frac{\delta d\phi}{dt} \hat{a}_\phi$$

$$\vec{u} = 4S \left(\frac{d\phi}{dt} \right) \hat{a}_\phi$$

$$\vec{u} = S\omega \hat{a}_\phi \quad \text{where } S = AD = 4 \text{ cm}$$

$$\omega = 2\pi f$$

$$\vec{B} = B_0 \hat{ax}$$

\vec{B} is in rectangular coordinate system, but we have to convert it into cylindrical coordinate system

$$\begin{bmatrix} B_s \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} B_s \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} B_0 \cos\phi \\ -B_0 \sin\phi \\ 0 \end{bmatrix} \quad \vec{B} = B_0 \cos\phi \hat{as} - B_0 \sin\phi \hat{a\phi} + 0 \hat{az}$$

$$\vec{u} = sw \hat{a\phi} \quad \vec{B} = B_0 \cos\phi \hat{as} - B_0 \sin\phi \hat{a\phi} + 0 \hat{az}$$

$$\vec{u} \times \vec{B} = \begin{vmatrix} \hat{as} & \hat{a\phi} & \hat{az} \\ 0 & sw & 0 \\ B_0 \cos\phi & -B_0 \sin\phi & 0 \end{vmatrix}$$

$$\begin{aligned} &= sw B_0 \cos\phi \hat{az} - sw B_0 \cos\phi \hat{az} \\ (\vec{u} \times \vec{B}) \cdot \vec{dl} &= (-sw B_0 \cos\phi) \hat{az} \cdot dz \hat{az} \\ &= -sw B_0 \cos\phi \\ &= -4 \times 10^3 \times (2\pi \times 50) \times 50 \times \cos\phi \\ &= -4 \times 10^3 \times 2\pi \times 5 \times 5 \times 10^2 \times \cos\phi dz \\ &= -4 \times 10^3 \times 2\pi \times 5 \times 5 \times 10^2 \times 0.2\pi \cos\phi dz \\ &= \int_{0.03}^{0.03} (-0.2\pi \cos\phi) dz = -6\pi \cos\phi mV \end{aligned}$$

$$\omega = \frac{d\phi}{dt} \Rightarrow \phi = \omega t + C_0$$

$$\text{At } t = 0, \text{ At } \phi = \frac{\pi}{2}$$

$$\frac{\pi}{2} = 0 + C_0$$

$$C_0 = \frac{\pi}{2}$$

(21)

$$V_{emf} = -6\pi \cos(\omega t + \frac{\pi}{2}) \text{ mV}$$

$$= 6\pi \sin(\omega t) \text{ mV}$$

At $t = 1 \text{ ms}$

$$V_{emf} = 6\pi \sin(2\pi \times 50 \times 1 \times 10^{-3}) \text{ V}$$

$$= 5.825 \text{ mV}$$

(b)

$$i = \frac{V_{emf}}{R} = \frac{6\pi \sin(2\pi \times 50 \times 3 \times 10^{-3})}{0.1}$$

$$i = 60\pi \sin(0.3\pi) \text{ mA}$$

$$i = 0.1525 \text{ A}$$

Good Explanation

(1)

8.6) The electric field and magnetic field in free space are given by

$$\vec{E} = \frac{50}{s} \cos(10^6 t + \beta z) \hat{\alpha}_\phi \text{ V/m}$$

$$\vec{H} = \frac{H_0}{s} \cos(10^6 t + \beta z) \hat{\alpha}_s \text{ A/m}$$

Express these in phasor form and determine the constants H_0 and β such that the fields satisfy Maxwell's equation.

Solution: $\vec{E} = \frac{50}{s} \cos(10^6 t + \beta z) \hat{\alpha}_\phi \text{ V/m}$

$$\vec{E} = \frac{50}{s} \cos(\omega t + \beta z) \hat{\alpha}_\phi \text{ V/m}$$

$$\vec{E} = \operatorname{Re} \left[\frac{50}{s} e^{j(\omega t + \beta z)} \hat{\alpha}_\phi \right]$$

$$\vec{E} = \operatorname{Re} \left[\frac{50}{s} e^{j\beta z} \hat{\alpha}_\phi e^{j\omega t} \right]$$

$$\vec{E}_s = \frac{(50/s)}{s} e^{j\beta z} \hat{\alpha}_\phi \rightarrow (1)$$

$$\vec{H} = \frac{H_0}{s} \cos(10^6 t + \beta z) \hat{\alpha}_s$$

$$\vec{H} = \operatorname{Re} \left[\frac{H_0}{s} e^{j(10^6 t + \beta z)} \hat{\alpha}_s \right]$$

$$\vec{H} = \operatorname{Re} \left[\frac{H_0}{s} e^{j\beta z} \hat{\alpha}_s e^{j10^6 t} \right]$$

$$\vec{H}_s = \frac{H_0}{s} e^{j\beta z} \hat{\alpha}_s \rightarrow (2)$$

Since ~~satisfy~~ Maxwell's it is in free Space

$$\sigma = 0, \alpha = 0, \epsilon = \epsilon_0, \mu = \mu_0$$

Since it ~~satisfy~~ Maxwell's Equation,

Consider Maxwell's equation in Phasor form

$$\nabla \cdot \vec{B}_s = \epsilon_0 \nabla \cdot \vec{E}_s = 0 \rightarrow \nabla \cdot \vec{E}_s = 0 \quad \left. \begin{array}{l} \nabla \times \vec{H}_s = J + \vec{D} \\ \nabla \cdot \vec{B}_s = \mu_0 \nabla \cdot \vec{H}_s = 0 \rightarrow \nabla \cdot \vec{H}_s = 0 \end{array} \right\} \nabla \times \vec{E}_s = -j\omega \mu_0 H_s$$

$$\nabla \cdot \vec{D}_S = \nabla \cdot \epsilon \vec{E}_S = \frac{\epsilon_0}{s} \frac{d}{ds} (\tilde{E}_{\phi s}) = 0, \quad \nabla \cdot \vec{H}_S = \frac{1}{s} \frac{d}{ds} (s H_{\phi s}) = 0$$

$$\nabla \cdot \vec{D}_S = 0 \quad [\text{free space}]$$

$$\nabla \times \vec{H}_S = \vec{J}_S + j\omega \vec{P}_S$$

$$\nabla \times \vec{H}_S = \frac{1}{s} \frac{d}{ds} [s \times \frac{H_0}{s} e^{jBz} \hat{a}_s] = 0$$

$$= 0$$
(2)

$$\nabla \times \vec{H}_S = j\omega \epsilon_0 \vec{E}_S = j\omega \epsilon_0 \vec{E}_S \rightarrow (1) \quad \vec{H}_S = \frac{H_0}{s} e^{jBz} \hat{a}_s$$

$$\nabla \times \vec{H}_S = \frac{1}{s} \begin{vmatrix} \hat{a}_s & s \hat{a}_\phi & \hat{a}_z \\ \frac{d}{ds} & \frac{d}{ds} & \frac{d}{ds} \\ \frac{H_0}{s} e^{jBz} & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{s} \left[-s \hat{a}_\phi \left(0 - \frac{H_0}{s} \frac{d}{dz} (e^{jBz}) \right) \right]$$

$$= \frac{H_0}{s} jB e^{jBz} \hat{a}_\phi$$

$$= \frac{jH_0 B}{s} e^{jBz} \hat{a}_\phi \rightarrow (2)$$

$$\nabla \times \vec{H}_S = j\omega \epsilon_0 \vec{E}_S = j\omega \epsilon_0 \frac{50}{s} e^{jBz} \hat{a}_\phi \rightarrow (3)$$

$$(2) = (3)$$
~~$$j\omega \epsilon_0 \frac{50}{s} e^{jBz} \hat{a}_\phi = \frac{jH_0 B}{s} e^{jBz} \hat{a}_\phi$$~~

$$H_0 B = \omega \epsilon_0 \times 50$$

$$\frac{H_0 B}{50} = \frac{\omega \epsilon_0 \times 50}{B^2}$$

$$\frac{H_0}{50} = \frac{\omega}{B} \frac{\epsilon_0 \times 50}{B}$$

~~$$\frac{H_0}{50} = \frac{\omega}{B} \frac{\epsilon_0 \times 50}{B}$$~~

$$H_0 = \frac{\omega}{\sqrt{\frac{H_0}{\epsilon_0}}} \times 50$$

$$\left. \begin{array}{l} (2) + (3) \\ \omega = 20 \\ (3) \\ \frac{\omega}{B} = \frac{1}{\sqrt{\frac{H_0}{\epsilon_0}}} \end{array} \right\}$$

(3)

$$H_0 \beta = 50 \omega \epsilon_0 \rightarrow (3)$$

$$\nabla \times \vec{E}_S = -j\omega \vec{B}_S$$

$$\nabla \times \vec{A} = \frac{1}{s} \begin{vmatrix} \hat{a}_s & s\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_s & sA_\phi & A_z \end{vmatrix}$$

$$\nabla \times \vec{E}_S = -j\omega \vec{B}_S \rightarrow (4)$$

$$\nabla \times \vec{E}_S = -j\omega \mu_0 \vec{H}_S \rightarrow (4) \text{ But } \vec{E}_S = \frac{50}{s} e^{j\beta z} \hat{a}_s$$

$$\nabla \times \vec{E}_S = \frac{1}{s} \begin{vmatrix} \hat{a}_s & s\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \cancel{\frac{50}{s} e^{j\beta z}} & 0 \end{vmatrix}$$

$$= \frac{1}{s} \left[\hat{a}_s \left(0 - \frac{50}{s} j\beta e^{j\beta z} \right) \right] = -\frac{50}{s} j\beta e^{j\beta z} \hat{a}_s$$

$$= -\frac{50}{s} j\beta e^{j\beta z} \hat{a}_s \rightarrow (5)$$

$$\nabla \times \vec{E}_S = -j\omega \mu_0 \vec{H}_S$$

$$-j\omega \mu_0 \vec{H}_S = -\frac{50}{s} j\beta e^{j\beta z} \hat{a}_s$$

$$j\omega \mu_0 \frac{H_0}{s} e^{j\beta z} \hat{a}_s = \frac{50}{s} j\beta e^{j\beta z} \hat{a}_s$$

$$\omega \mu_0 H_0 = 50 \beta$$

$$\frac{H_0}{\beta} = \frac{50}{\omega \mu_0} \rightarrow (4)$$

$$(3) \times (4) \quad H_0 \phi \times \frac{H_0}{\beta} = 50 \mu_0 \epsilon_0 \times \frac{50}{\omega \mu_0}$$

$$H_0^2 = 50^2 \cdot \frac{\epsilon_0}{\mu_0}$$

$$H_0 = \pm \sqrt{\frac{1}{\frac{\mu_0}{\epsilon_0}}} = \pm \frac{50}{120\pi}$$

$$H_0 = \pm 0.1326$$

(4)

$$H_0 = \pm \frac{50}{120\pi}$$

$$H_0 \beta = 50\omega \epsilon_0$$

$$\beta = \frac{50\omega \epsilon_0}{H_0} = \pm \frac{50\omega \epsilon_0}{\frac{50}{120\pi}}$$

$$\beta = \pm 120\pi \omega \epsilon_0$$

$$\beta = \pm \sqrt{\frac{\mu_0}{\epsilon_0}} \times \omega \times \epsilon_0$$

$$\beta = \pm \sqrt{\mu_0 \epsilon_0} \times \omega$$

$$\beta = \pm \sqrt{\cancel{4\pi \times 10^{-7}} \times \frac{10^9}{\cancel{36\pi} 9}} \times 10^6$$

$$= \pm \frac{10^6}{3 \times 10^8} = \pm \frac{1}{3 \times 10^2} = \underline{\underline{\pm 3.33 \times 10^{-3}}}$$

$$\beta = \pm 3.33 \times 10^{-3}$$

(5)

~~Maxwell~~

8.7) In a medium characterized by $\alpha = 0$, $\mu = \mu_0$, ϵ_0 and $\vec{E} = 20 \sin(10^8 t - \beta z) \hat{y} \text{ V/m}$
 calculate β and \vec{H}

Solution: $\vec{E} = 20 \sin(10^8 t - \beta z) \hat{y} \quad \left\{ \nabla \cdot \vec{D}_S = 0 \right.$

$$\vec{E} = \text{Im} \left[20 e^{j(\omega t - \beta z)} \hat{y} \right]$$

$$\vec{E} = \text{Im} \left[20 e^{-j\beta z} \hat{y} \cdot e^{j\omega t} \right] \rightarrow (1)$$

$$\vec{E} = \text{Im} \left[\vec{E}_S e^{j\omega t} \right] \rightarrow (2)$$

$\vec{E}_S = 20 e^{-j\beta z} \hat{y}$ is a phasor.

$$\nabla \times \vec{E}_S = -j\omega \vec{B}_S \rightarrow (3)$$

$$\nabla \times \vec{E}_S = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 20 e^{-j\beta z} & 0 \end{vmatrix}$$

$$\nabla \times \vec{E}_S = \hat{x} \left[0 - \frac{\partial}{\partial z} 20 e^{-j\beta z} \right]$$

$$= \hat{x} \left[-20 e^{-j\beta z} (-j\beta) \right]$$

$$= \hat{x} \left[20 j \beta e^{-j\beta z} \right] \rightarrow (4)$$

$$\nabla \times \vec{E}_S = -j\omega \vec{B}_S = -j\omega \mu_0 \vec{H}_S \rightarrow (5)$$

$$(4) = (5) \quad -j\omega \mu_0 \vec{H}_S = 20 j \beta e^{-j\beta z} \hat{x}$$

$$\vec{H}_S = -\frac{20 \beta}{\omega \mu_0} e^{-j\beta z} \hat{x}$$

$$\vec{H}_S = -\frac{20 \beta}{\omega \mu_0} e^{-j\beta z} \hat{a}_x$$

$$\nabla \times \vec{H}_S = \vec{J}_S + j\omega \vec{D}_S$$

$$\nabla \cdot \vec{B}_S = 0$$

$$\nabla \cdot \vec{H}_S = 0$$
(6)

Since $\nabla \cdot \vec{a}_v = 0$ $\vec{J}_S = 0$

$$\therefore \nabla \times \vec{H}_S = j\omega \vec{D}_S \rightarrow (6)$$

$$\nabla \times \vec{H}_S = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{j}{\beta x} & \frac{j}{\beta y} & \frac{j}{\beta z} \\ -\frac{20 \beta}{\omega \mu_0} e^{-j\beta z} & 0 & 0 \end{vmatrix}$$

$$\nabla \times \vec{H}_S = -\hat{a}_y \left[0 - \frac{j}{\beta z} \left(\frac{20 \beta}{\omega \mu_0} e^{-j\beta z} \right) \right]$$

$$= -\frac{20 \beta}{\omega \mu_0} (-j\beta) e^{-j\beta z} \hat{a}_y$$

$$\nabla \times \vec{H}_S = j \frac{20 \beta^2}{\omega \mu_0} e^{-j\beta z} \hat{a}_y \rightarrow (7)$$

$$\nabla \times \vec{H}_S = j\omega \epsilon_0 \vec{E}_S \rightarrow (8)$$

$$(7) = (8) \quad j\omega \epsilon_0 \vec{E}_S = j \frac{20 \beta^2}{\omega \mu_0} e^{-j\beta z} \hat{a}_y$$

$$\vec{E}_S = \frac{20 \beta^2}{\omega^2 \mu_0 \epsilon_0} e^{-j\beta z} \hat{a}_y \rightarrow (9)$$

$$\vec{E}_S = 20 e^{-j\beta z} \hat{a}_y$$

$$+j0 = \frac{20 \beta^2}{\omega^2 \mu_0 \epsilon_0}$$

$$\beta^2 = \frac{\omega^2 \mu_0 \epsilon_0}{}$$

$$\beta = \pm \sqrt{\mu_0 \epsilon_0} = \pm \frac{10^8}{3 \times 10^8} = \pm \frac{1}{\sqrt{3}}$$

(7)

$$\vec{H}_S = -\frac{20}{\omega \mu_0} B e^{-jBz} \hat{a}_n$$

$$\vec{H}_S = \pm \frac{1}{6\pi} e^{-jBz} \hat{a}_n$$

$$= \pm \frac{20}{6\pi \times 10^8 \times 4\pi \times 10^{-7}} e^{-jBz} \hat{a}_n$$

$$= \pm \frac{20}{3 \times 10^8} e^{-jBz} \hat{a}_n$$

$$= \pm \frac{20}{3 \times 10^8} e^{-jBz} \hat{a}_n$$

~~Then in here~~

~~Then in here~~

~~Then in here~~

~~Re Im [Re Im (wt ± Bz) \hat{a}_n]~~

$$\vec{H} = \text{Im} \left[\pm \frac{1}{6\pi} \sin(wt \pm Bz) \hat{a}_n \right] A/m$$

Continued

$$\vec{H}_S = -\frac{20}{\omega \mu_0} B e^{-jBz} \hat{a}_n = -\frac{20 \times (\pm \frac{1}{6\pi})}{10^8 \times 4\pi \times 10^{-7}} e^{-jBz} \hat{a}_n \quad (7)$$

$$\vec{H}_S = \pm \frac{60}{3 \times 10^8 \times 4\pi \times 10^{-7}} e^{-jBz} \hat{a}_n$$

$$\vec{H}_S = \pm \frac{1}{6\pi} e^{-jBz} \hat{a}_n$$

$$\vec{H} = \text{Im} [\vec{H}_S e^{jw t}]$$

$$= \text{Im} \left[\pm \frac{1}{6\pi} e^{-jBz} e^{jw t} \hat{a}_n \right]$$

$$\vec{H} = \pm \frac{1}{6\pi} \sin(wt \pm Bz) \hat{a}_n$$

$$= \pm \frac{1}{6\pi} \sin(10^8 t \pm \frac{1}{3} z) \hat{a}_n \quad A/m$$