Unit-4 - MAXWELL'S EQUATIONS

In Electrostatics we have studied $\vec{E}$ as a function of x, y, z, i,e $\vec{E}(x,y,z)$. In Magnetostatics we got $\vec{H}$ as a function of x, y, z i,e $\vec{H}(x,y,z)$

Now we have to examine the situation where Electric and Magnetic fields are dynamic or time Varying. Dynamic EM field is one in which Electric and Magnetic fields are interdependent. In other words, a time varying electric field necessarily involves a corresponding time-varying magnetic field.

Electrostatics fields are produced by static electric charges, Magnetostatics fields are due to motion of electric charges, whereas time varying fields or Waves due to accelerated charges or time Varying currents

Now we have to understand

1) Electro motive force (EMF) based on Faraday's experiment
2) Displacement current from Maxwell's equation

FARADAY'S LAW

Faraday's law states that the induced electromotive force (emf), V_{emf} (in volts), in any closed circuit is equal to rate of change of the magnetic flux linkage by the circuit

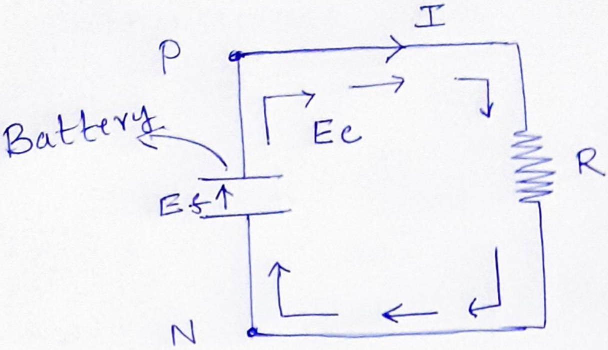
$$V_{emf} = -\frac{d\lambda}{dt} = -\frac{d}{dt}N\psi = -N\frac{d\psi}{dt}$$

Where $\lambda = N\psi$ = Flux linkage

N = Number of turns in the circuit

ψ = Flux through each turn.

Negative sign shows that the induced Voltage acts in such a way as to oppose the flux producing it. This behavior is known as Lenz's law, which emphasizes that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the change in original magnetic field.



The electrochemical action of the battery results in an emf - produced field E_f.

Due to the accumulation of charges at the battery terminals, an electrostatic field $E_e (= -\nabla V)$ also exists

We have following observations

1. An electrostatic field $\vec{E_e}$ cannot maintain a steady current in a closed circuit, since $\oint_L \vec{E_e} \cdot \vec{dl} = 0$ [$\vec{E_e}$ is conservative]

2. An emf - produced field $\vec{E_f}$ is non conservative.

TRANSFORMER AND MOTIONAL ELECTROMOTIVE FORCE

We know $V_{emf} = -\frac{d\psi}{dt} \rightarrow (1)$ [Faraday's law]

where ψ = Magnetic flux

We know $V_{emf} = \oint_L \vec{E} . \vec{dl} \rightarrow (2)$ [In terms of $\vec{E}$]

$$\cancel{V_{emf} = \oint_S \frac{d\vec{B}}{dt} . ds}$$

$$V_{emf} = -\frac{d}{dt}\psi = -\frac{d}{dt}\int_S \vec{B} . \vec{ds} \rightarrow (3)$$

$$\therefore V_{emf} = \oint_L \vec{E} . \vec{dl} = -\frac{d}{dt}\int_S \vec{B} . \vec{ds} \rightarrow (4)$$

It is clear from equation (4) in time varying situation both electric and magnetic fields are present. The variation of flux with time ~~in~~

$$\left[\cancel{V_{emf} = -\frac{d\psi}{dt}} \text{ or } \cancel{V_{emf} = \oint_L \vec{E} . \vec{dl} = -\frac{d}{dt}\int_S \vec{B} . \vec{ds}}\right]$$

~~Can be~~ may be caused in three ways. They are

1. By having a stationary loop in a time varying $\vec{B}$-field
2. By having a time varying loop area in a static B-field
3. By having a time varying loop area in time varying B-field.

1 Stationary loop in Time Varying B-field

(Transformer emf)

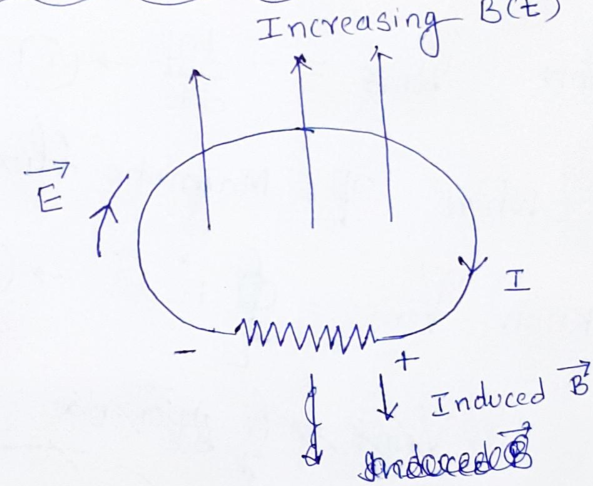


Figure shows a stationary conducting loop placed in a time-varying magnetic field $\vec{B}$.

The induced emf is given by

$$V_{emf} = \oint \vec{E} \cdot \vec{dl} \longrightarrow (1)$$

$$V_{emf} = -N \frac{d\psi}{dt}$$

$N = 1$

$$V_{emf} = -\frac{d}{dt}\psi = -\frac{d}{dt}\int_S \vec{B} \cdot \vec{dS}$$

$$V_{emf} = \int_S -\frac{d\vec{B}}{dt} \cdot \vec{dS} \longrightarrow (2)$$

1 = 2

$$\oint_L \vec{E} \cdot \vec{dl} = \int_S -\frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

By applying stroke's theorem

$$\int_S \nabla \times \vec{E} \cdot \vec{ds} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

$$\boxed{\therefore \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

This is one of the Maxwell's equation for time varying fields..

2 Moving Loop in Static B-field (Motional emf)

When a conducting loop is moving in a static $\vec{B}$-field, an emf is induced in the loop. We recall that force on a charge moving with uniform velocity $\vec{u}$ in a magnetic field $\vec{B}$ is

$$\vec{F_m} = Q(\vec{u} \times \vec{B})$$

Motional Electric field is given by

$$\vec{E_m} = \frac{\vec{F_m}}{Q} = \vec{u} \times \vec{B}$$

If we consider conducting loop, moving with uniform velocity $\vec{u}$ as consiting of large number of ~~electrons~~ free electrons, the emf induced in the loop is

$$V_{emf} = \oint_L \vec{E_m} \cdot \vec{dl} = \oint_L (\vec{u} \times \vec{B}) \cdot \vec{dl} \quad (7)$$

This type of emf is called as motional emf or flux-cutting emf because it is due to motional action.

It is the kind of emf found in electrical machines such as motors, generators and alternators.

$$V_{emf} = \oint_L (\vec{u} \times \vec{B}) \cdot \vec{dl} = \int_S \nabla \times (\vec{u} \times \vec{B}) \cdot \vec{ds} \rightarrow (1)$$

$$V_{emf} = \oint_L \vec{E_m} \cdot \vec{dl} = \int_S (\nabla \times \vec{E_m}) \cdot \vec{ds} \rightarrow (2)$$

where $\vec{E_m} = \frac{F_m}{Q} = \vec{u} \times \vec{B}$

(1) = (2)

$$\boxed{\nabla \times \vec{E_m} = \nabla \times (\vec{u} \times \vec{B})}$$

3. Moving Loop in Time Varying Field

In the general case, a moving conducting loop in a time varying magnetic field. Both transformer emf and motional emf are present.

$$V_{emf} = \oint_L \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} + \oint_L (\vec{u}\times\vec{B})\cdot d\vec{l}$$

$$V_{emf} = \int_S (\nabla\times\vec{E})\cdot d\vec{S}$$

$$V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} + \int_S (\nabla\times\vec{u}\times\vec{B})\cdot d\vec{S}$$

$$\therefore \quad \nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} + \nabla\times\vec{u}\times\vec{B}$$

Note:

1. In moving loop in static B-field, the integral of the equation $\oint_L \vec{u}\times\vec{B}\cdot d\vec{l}$ is zero along the portion of the loop where $\vec{u}=0$. Thus $d\vec{l}$ is taken along the portion of the loop that is cutting the field where $\vec{u}$ ~~is non zero~~ has non-zero value.
2. The direction of the induced current is same as that Em or $\vec{u}\times\vec{B}$. The limits of the integration in eqn $\oint_L \vec{u}\times\vec{B}\cdot d\vec{l}$ ~~is~~ are selected in the direction opposite to the induced current.

Displacement Current

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} \rightarrow (1)$$

$$\nabla \cdot (\nabla \times \vec{H}) = 0 \rightarrow (2)$$

But from (1) $\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \rightarrow (3)$

(2) $\neq$ (3)

$$\nabla \times \vec{H} = \vec{J} + \vec{J}d$$

$\vec{J}$ → Conduction current density

$\vec{J}d$ → displacement current density

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \nabla \cdot \vec{J}d$$

$$0 = \nabla \cdot \vec{J} + \nabla \cdot \vec{J}d$$

$$\nabla \cdot \vec{J}d = -\nabla \cdot \vec{J}, \quad \nabla \cdot \vec{J}d = +\frac{\partial \rho_v}{\partial t}$$

$$\nabla \cdot \vec{J}d = \frac{\partial\, \nabla \cdot \vec{D}}{\partial t}$$

$$\nabla \cdot \vec{J}d = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\boxed{\vec{J}d = \frac{\partial \vec{D}}{\partial t}} \rightarrow (4)$$

(4) gives displacement current density

$$\therefore \nabla \times H = \vec{J} + \vec{J}d$$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$ is one of the time varying Maxwell's Equation.

Displacement current $I_d = \int_s \vec{J}d \cdot \vec{ds}$

Displacement current density $J_d = \frac{I_d}{\text{Area}}$

Generalized Forms of Maxwell's Equations

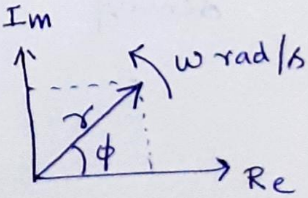
Maxwell's Equations in Final forms

Maxwell's equations for Time Varying fields are

Differential Form	Integral form	Remarks
$\nabla\cdot\vec{D} = \rho_v$	$\oint_S \vec{D}\cdot ds = \int_V \rho_v\, dv$	Gauss's law
$\nabla\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot ds = 0$	Non-existance of isolated magnetic charge
$\nabla\times\vec{E} = -\frac{d\vec{B}}{dt}$	$\oint_L \vec{E}\cdot dl = -\frac{d}{dt}\int_S \vec{B}\cdot ds$	Faraday's law
$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{dt}$	$\oint_L \vec{H}\cdot dl = \int_S \left(\vec{J} + \frac{\partial\vec{D}}{dt}\right)\cdot ds$	Ampere's Circuit law

Time Harmonic Fields

A time-Harmonic field is one that varies periodically or sinusoidally with time. Sinusoids are easily expressed in phasors. A phasor is a complex number that contains the amplitude and phase of a sinusoidal oscillations.



$$z = x + jy$$

$$z = r\angle\phi$$

$$z_1 = x_1 + jy_1 = r_1\angle\phi_1$$

$$z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

$$z_1 z_2 = r_1 r_2 \angle\phi_1 + \phi_2$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle\phi_1 - \phi_2$$

Time Harmonic Fields

A time harmonic field is one that varies periodically or sinusoidally with time.

Sinusoidal variation is selected because it can be expressed in phasors. The concept ~~is~~ of phasor is as follows.

A phasor is a complex number that contains the amplitude and phase of the sinusoidal variation.

$$z = x + jy = r\angle\phi$$

$$z = r.e^{j\phi}$$

$$z = r(\cos\phi + j\sin\phi)$$

$$r = |z| = \sqrt{x^2+y^2}$$

$$\phi = \tan^{-1}(y/x)$$

$$z_1 = x_1 + jy_1 = r_1\angle\phi_1$$

$$z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

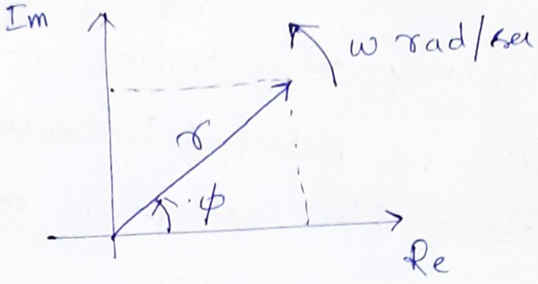
$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

$$z_1 z_2 = r_1 r_2 \angle\phi_1 + \phi_2$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle\phi_1 - \phi_2$$

Representation of a phasor



$$\sqrt{z} = \sqrt{r} \angle \phi/2$$

$$z^* = x - jy = r \angle -\phi$$

If $\phi = \omega t + \theta$

$$re^{j\phi} = re^{j(\omega t + \theta)} = re^{j\omega t} \cdot e^{j\theta}$$

$$Re[re^{j\phi}] = Re[re^{j(\omega t + \theta)}]$$

$$= r\cos(\omega t + \theta)$$

$$Im[re^{j\phi}] = Im[re^{j(\omega t + \theta)}]$$

$$= r\sin(\omega t + \theta)$$

<u>Ex</u>:

$$I(t) = Re[I_0 e^{j\theta} \cdot e^{j\omega t}]$$

$$I(t) = Re[I_0 e^{j(\omega t + \theta)}]$$

$$I(t) = Re[I_0 e^{j(\omega t + \theta)}]$$

$$I(t) = I_0 \cos(\omega t + \theta)$$

$$I'(t) = I_m\left[I_o e^{j\theta} \cdot e^{j\omega t}\right]$$

$$I'(t) = I_o \sin(\omega t + \theta)$$

$$I'(t) = Re\left[I_o e^{j\theta} \cdot e^{j\omega t} \cdot e^{-j90^\circ}\right]$$

$$I'(t) = I_o \cos(\omega t + \theta - 90^\circ)$$

Now we introduce a phasor $I_s = I_o e^{j\theta}$

$$I_s = I_o e^{j\theta} = I_o \angle\theta$$

$$\therefore \quad I(t) = I_o \cos(\omega t + \theta)$$

$$I(t) = Re\left[I_o e^{j\omega t} \cdot e^{j\theta}\right]$$

$$I(t) = Re\left[\underset{\sim}{I_o e^{j\theta}} \cdot e^{j\omega t}\right]$$

$$I(t) = Re\left[I_s e^{j\omega t}\right]$$

In general, a phasor is a complex quantity and it could be a scalar or vector.

If a vector $\vec{A}(x, y, z, t)$ is a time hormonic field, the vector $\vec{A}(x, y, z, t)$ can be represented as

$$\vec{A}(x, y, z, t) = Re\left[\vec{A_s}(x, y, z) e^{j\omega t}\right] \rightarrow (1)$$

$\vec{A_s}(x, y, z)$ is a phasor.

In general, Phasor is a function of position, not a function of time.

For example if $\vec{A} = A_o \cos(\omega t - \beta x)\hat{a}_y$

$$\vec{A} = Re\left[A_o e^{-j\beta x} \hat{a}_y e^{j\omega t}\right]$$

The phasor form of $\vec{A}$ is $\vec{A_s}$

$$\vec{A_s} = A_o e^{-j\beta x} \hat{a}_y$$

$$\frac{d\vec{A}}{dt} = \frac{d}{dt}\left[Re\left(\vec{A_s}e^{j\omega t}\right)\right]$$

$$= \cancel{A_s}\ Re\left[\vec{A_s}\frac{d}{dt}e^{j\omega t}\right]$$

$$= Re\left[\vec{A_s}\,(j\omega)\,e^{j\omega t}\right]$$

$$\frac{d\vec{A}}{dt} = Re\left[j\omega\vec{A_s}\,e^{j\omega t}\right]$$

$\frac{d\vec{A}}{dt} = Re\left[j\omega\vec{A_s}e^{j\omega t}\right]$ is showing that taking the time derivative of the instantaneous quantity is equivalent to multiplying its phasor form by $j\omega$.

$$\frac{d\vec{A}}{dt} \rightarrow j\omega\vec{A_s}$$

Because $\vec{A} = Re\left[\vec{A_s}\,e^{j\omega t}\right]$

$$\frac{d\vec{A}}{dt} = Re\left[j\omega\vec{A_s}\,e^{j\omega t}\right]$$

$$\therefore \frac{d\vec{A}}{dt} \rightarrow j\omega\vec{A_s}$$

i.e Replace $\vec{A_s}$ by $j\omega\vec{A_s}$

Similarly for $\int \vec{A}\,dt \rightarrow \frac{\vec{A_s}}{j\omega}$

i.e Replace $\vec{A_s}$ by $\frac{\vec{A_s}}{j\omega}$

Similarly $\int \vec{A}\, dt \rightarrow \frac{\vec{A_s}}{j\omega}$ if it is $e^{j\omega t}$ in ①

For $\int \vec{A}\, dt$ implies replace $\vec{A_s}$ by $\frac{\vec{A_s}}{j\omega}$ if it is

Note: $\vec{A}(x,y,z,t)$ can also be $\vec{A}(x,y,z,t) = Im[\vec{A_s}(x,y,z) e^{j\omega t}]$

Maxwell's equations in Phasor form

$$\nabla \times \vec{E}(x,y,z,t) = -\frac{d}{dt}\vec{B}(x,y,z,t) \rightarrow ①$$

$$\vec{E}(x,y,z,t) = Re[\vec{E_s}(x,y,z) e^{j\omega t}]$$

$$\vec{B}(x,y,z,t) = Re[\vec{B_s}(x,y,z) e^{j\omega t}]$$

Consider $\nabla \times \vec{E}(x,y,z,t) = \nabla \times [Re\{\vec{E_s} e^{j\omega t}\}] \rightarrow ②$

Curl operator ∇ operates only on (x,y,z)

$\therefore$ ② can be written as

$$\nabla \times \vec{E}(x,y,z,t) = Re\{[\nabla \times \vec{E_s}] e^{j\omega t}\} \rightarrow ③$$

$$\vec{B}(x,y,z,t) = Re[\vec{B_s} e^{j\omega t}]$$

Consider $-\frac{d}{dt}\vec{B}(x,y,z,t) = -\frac{d}{dt}\left[Re[\vec{B_s} e^{j\omega t}]\right]$

$$= -Re\left[\vec{B_s}\frac{d}{dt}e^{j\omega t}\right]$$

$$-\frac{d\vec{B}}{dt} = -Re[j\omega \vec{B_s} e^{j\omega t}]$$

$$-\frac{d\vec{B}}{dt} = Re[-j\omega \vec{B_s} e^{j\omega t}] \rightarrow ④$$

$\therefore$ ③ = ④ $\Rightarrow$ $\boxed{\nabla \times E_s = -j\omega \vec{B_s}} \rightarrow ⑤$

Eqn $\nabla \times \vec{E_s} = -j\omega \vec{B_s}$ is the phasor form

of $\nabla \times \vec{E}(x,y,z,t) = -\frac{\partial}{\partial t}\vec{B}(x,y,z,t)$ (22)

where $\vec{E_s}$ may be like $\vec{E_s} = E_0 e^{-j\beta x}\hat{a_y}$

Similarly, $\nabla \times \vec{H}(x,y,z,t) = \vec{J} + \frac{\partial}{\partial t}\vec{D}(x,y,z,t) \rightarrow (1)$

$\vec{H}(x,y,z,t) = Re\left[\vec{H_s}(x,y,z)e^{j\omega t}\right] \rightarrow (2)$

$\nabla \times \vec{H}(x,y,z,t) = \nabla \times Re\left[\vec{H_s}(x,y,z)e^{j\omega t}\right]$

$= Re\left[\nabla \times \vec{H_s} e^{j\omega t}\right] \rightarrow (3)$

Consider $J + \frac{\partial \vec{D}}{\partial t}(x,y,z,t)$

$\vec{D} = Re\left[\vec{D_s} e^{j\omega t}\right]$

$= Re\left[\vec{J_s} e^{j\omega t}\right] + \frac{\partial}{\partial t} Re\left[\vec{D_s} e^{j\omega t}\right]$

$= Re\left[\vec{J_s} e^{j\omega t}\right] + Re\left[\vec{D_s}\frac{\partial}{\partial t} e^{j\omega t}\right]$

$= Re\left[\vec{J_s} e^{j\omega t}\right] + Re\left[j\omega \vec{D_s} e^{j\omega t}\right] \rightarrow (4)$

$(3) = (4)$

$\nabla \times \vec{H_s} = \vec{J_s} + j\omega \vec{D_s}$

$\nabla \times \vec{H_s} = \vec{J_s} + j\omega \vec{D_s}$ is the phasor form

of $\nabla \times \vec{H}(x,y,z,t) = \vec{J} + \frac{\partial}{\partial t}\vec{D}(x,y,z,t)$

Similarly all Maxwell's Equations can be expressed in phasor form

Time - Harmonic Maxwell's equations Assuming Time factor $e^{j\omega t}$

Point form	Integral form
$\nabla \cdot \vec{D_s} = \rho_s$	$\oint \vec{D_s} \cdot \vec{ds} = \int \rho_{vs}\, dv$
$\nabla \cdot \vec{B_s} = 0$	$\oint \vec{B_s} \cdot \vec{ds} = 0$
$\nabla \times \vec{E_s} = -j\omega \vec{B_s}$	$\oint_L \vec{E_s} \cdot \vec{dl} = -j\omega \int_S \vec{B_s} \cdot \vec{ds}$
$\nabla \times \vec{H_s} = \vec{J_s} + j\omega \vec{D_s}$	$\oint \vec{H_s} \cdot \vec{dl} = \int (\vec{J_s} + j\omega \vec{D_s}) \cdot \vec{ds}$

13/9/2022

9.1) Example (9.1) [From Sadiku Textbook]

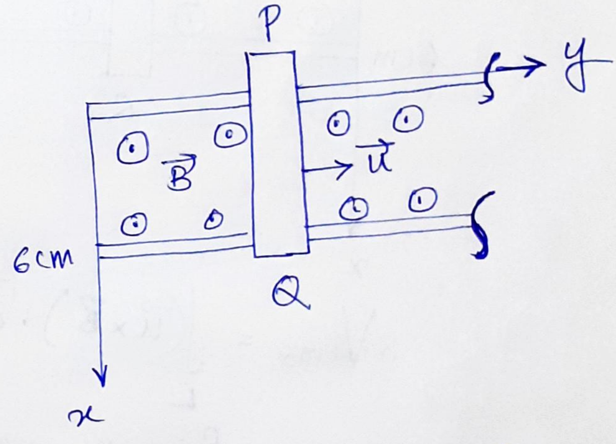
A conducting bar can slide freely over two conducting rails as shown in figure. Calculate the induced voltage in the bar

(a) If the bar is stationed at $y = 8cm$ and $\vec{B} = 4\cos 10^6 t\ \hat{a}_z$ m Wb/m²

(b) If the bar slides at a velocity $u = 20\hat{a}_y$ m/s and $\vec{B} = 4\hat{a}_z$ m Wb/m²

(c) If the bar slides at a velocity $\vec{u} = 20\hat{a}_y$ m/s and $\vec{B} = 4\cos(10^6 t - y)\hat{a}_z$ m Wb/m²

Solution:



(a) If the bar is stationed at $y = 8cm$ and $\vec{B} = 4\cos 10^6 t\ \hat{a}_z$ m Wb/m² and $\vec{B}$ is time varying

Conducting loop is stationary and

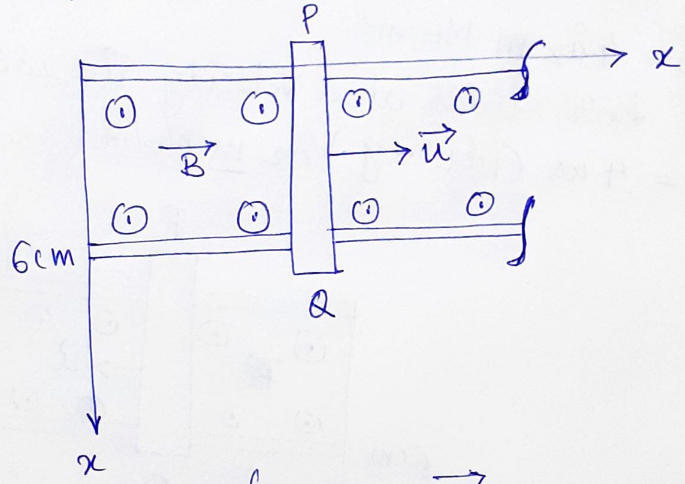
$$V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} = -\int_{y=0}^{0.08}\int_{x=0}^{0.06} \frac{\partial}{\partial t}(4\cos 10^6 t \times 10^{-3})\hat{a}_z \cdot dx\,dy\,\hat{a}_z$$

$$= -\int_{y=0}^{0.08}\int_{x=0}^{0.06} 4\times 10^6 \times 10^{-3} \sin 10^6 t\, dx\, dy$$

$$= 4\times 10^3 \sin 10^6 t \times 0.08 \times 0.06 = 19.2 \sin 10^6 t\ V$$

The polarity of the induced voltage is such that 'P' on the bar is at lower potential ~~and~~ than 'Q' when $\vec{B}$ is increasing. [According to Lenz's law]

(b) If the bar slides at a velocity $\vec{u} = 20\,\hat{a}_y$ m/s and $\vec{B} = 4\,a_z$ mWb/m^2



$$V_{emf} = \int_L (\vec{u} \times \vec{B}) \cdot \vec{dl}$$

$$= \int_{x=l}^{0} (u\hat{a}_y \times B\hat{a}_z) \cdot dx\,\hat{a}_x$$

The induced current will be in the direction

$$\hat{a}_y \times \hat{a}_z = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{a}_x \text{ direction}$$

$\therefore$ The limits of integration will be in a direction opposite to $\hat{a}_x$

$$V_{emf} = \int_{x=0.06}^{0} u\,\hat{a}_y \times B\,\hat{a}_z \cdot dx\,\hat{a}_x$$

$$= \int_{x=0.06}^{0} (20\times4)\,\hat{a}_x \cdot dx\,\hat{a}_x \times 10^{-3}$$

$$= -20\times4\times0.06\ V = -4.8\,mV$$

(c) If the bar slides at a velocity $\vec{u} = 20\,\hat{a}_y$ m/s and $\vec{B} = 4\cos(10^6 t - y)\,\hat{a}_z$ m Wb/m2

$$V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot \vec{ds} + \int_L (\vec{u}\times\vec{B})\cdot\vec{dl}$$

$$V_{emf} = -\int_{\cancel{x}=0}^{0.06}\int_{y=0}^{y'} \frac{\partial}{\partial t}(4\cos(10^6 t - y'))\,\hat{a}_z\, dy'\,dx\,\hat{a}_z \times 10^{-3}$$

$$+ \int_{0.06}^{0} (20\,\hat{a}_y)\times(4\times10^{-3}\cos(10^6 t - y)\hat{a}_z)\cdot dx\,\hat{a}_x$$

$$(\hat{a}_y\times\hat{a}_z) = \hat{a}_x \cdot dx\,\hat{a}_x$$

$$= 240\cos(10^6 t - y')\Big|_0^y - 80\times10^{-3}\times0.06\cos(10^6 t - y)$$

$$= 240\cos(10^6 t - y) - 240\cos 10^6 t - \cancel{4.8\times10^{-3}\cos(10^6 t - y)} \to 0$$

neglected

$$\simeq 240\cos(10^6 t - y) - 240\cos 10^6 t$$

Because Motional emf is negligeble compared to transformer emf

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\simeq -480\sin\left(\frac{10^6 t - y + 10^6 t}{2}\right)\sin\left(\frac{\cancel{10^6 t} - y - \cancel{10^6 t}}{2}\right)$$

$$= +480\sin\left(10^6 t - \frac{y}{2}\right)\sin\left(\frac{y}{2}\right)\ V$$

9.2) Example 9.2

The loop shown in figure is inside a uniform magnetic field $\vec{B} = 50\,\hat{a}_x$ mWb/m². If side DC of the loop cuts the flux lines at the frequency of 50 Hz and the loop yz-plane at time $t=0$, find

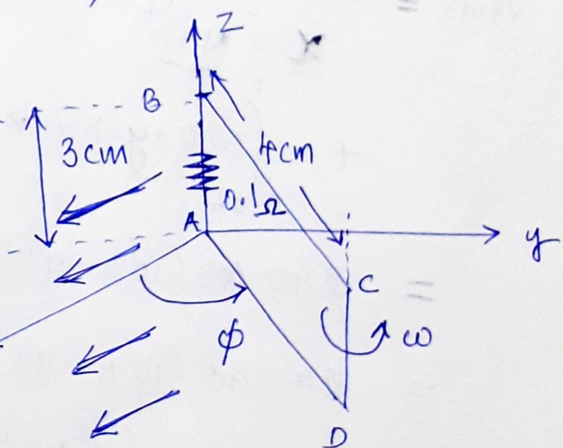
(a) The induced emf at $t = 1$ msec $= 1$ms

(b) The induced current at $t = 3$ msec $= 3$ms

Solution :

(a) Since $\vec{B}$ field is static, the induced emf is motional, that is

$$V_{emf} = \int_L (\vec{u} \times \vec{B}) \cdot \vec{dl}$$



$$\vec{dl} = \vec{dl}_{DC} = dz\,\hat{a}_z$$

$$\vec{u} = \frac{\vec{dl'}}{dt} = \frac{\rho\, d\phi}{dt}\,\hat{a}_\phi$$

$$\vec{u} = \rho \left(\frac{d\phi}{dt}\right) \hat{a}_\phi$$

$$\vec{u} = \rho\,\omega\,\hat{a}_\phi \quad \text{where } \rho = AD = 4\text{cm}$$

$$\omega = 2\pi f$$

$\vec{B} = B_0 \hat{a}_x$

$\vec{B}$ is in rectangular coordinate System, but we have to convert it into cylindrical coordinate System

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} B_0\cos\phi \\ -B_0\sin\phi \\ 0 \end{bmatrix} \quad \vec{B} = B_0\cos\phi\,\hat{a}_\rho - B_0\sin\phi\,\hat{a}_\phi + 0\,\hat{a}_z$$

$\vec{u} = \rho\omega\,\hat{a}_\phi \qquad \vec{B} = B_0\cos\phi\,\hat{a}_\rho - B_0\sin\phi\,\hat{a}_\phi + 0\,\hat{a}_z$

$$\vec{u}\times\vec{B} = \begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi & \hat{a}_z \\ 0 & \rho\omega & 0 \\ B_0\cos\phi & -B_0\sin\phi & 0 \end{vmatrix}$$

$= - \rho\omega B_0\cos\phi\,\hat{a}_z$

$(\vec{u}\times\vec{B})\cdot\vec{dl} = (-\rho\omega B_0\cos\phi)\,\hat{a}_z \cdot dz\,\hat{a}_z$

$= -\rho\omega B_0\cos\phi$

$= -4\times10^{-3}\times(2\pi\times50)\times 50\times\cos\phi$

$= -4\times10^{-3}\times 2\pi\times5\times5\times10^{2}\times\cos\phi\, dz$

$=$

$= -0.2\pi\cos\phi\, dz$

$$V_{emf} = \int_{z=0}^{0.03} (-0.2\pi\cos\phi)\,dz = -6\pi\cos\phi \text{ mV}$$

$\omega = \frac{d\phi}{dt} \Rightarrow \phi = \omega t + C_0$

At $t = 0$, $\phi = \frac{\pi}{2}$

$\frac{\pi}{2} = 0 + C_0$

$C_0 = \frac{\pi}{2}$

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$V_{emf} = -6\pi \cos\left(\omega t + \frac{\pi}{2}\right) mV$

$= 6\pi \sin(\omega t) mV$

At $t = 1 msec$

$V_{emf} = 6\pi \sin\left(2\pi \times 50 \times 1 \times 10^{-3}\right) V$

$= 5.825 \, mV$

(b) At $t = 3 msec$

$i = \frac{V_{emf}}{R} = \frac{6\pi \sin\left(2\pi \times 50 \times 3 \times 10^{-3}\right)}{0.1}$

$i = 60\pi \sin(0.3\pi) mA$

$i = 0.1525 A$

Good Explanation

8.6) The electric field and magnetic field in free space are given by

$$\vec{E} = \frac{50}{\rho}\cos(10^6 t + \beta z)\,\hat{a}_\phi \text{ V/m}$$

$$\vec{H} = \frac{H_o}{\rho}\cos(10^6 t + \beta z)\,a_\rho \text{ A/m}$$

Express these in phasor form and determine the constants H_o and β such that the fields satisfy Maxwell's equation.

Solution: $\vec{E} = \frac{50}{\rho}\cos(10^6 t + \beta z)\,\hat{a}_\phi$ V/m

$$\vec{E} = \frac{50}{\rho}\cos(\omega t + \beta z)\,\hat{a}_\phi \text{ V/m}$$

$$\vec{E} = Re\left[\frac{50}{\rho} e^{j(\omega t + \beta z)}\,\hat{a}_\phi\right]$$

$$\vec{E} = Re\left[\frac{50}{\rho} e^{j\beta z}\,\hat{a}_\phi\, e^{j\omega t}\right]$$

$$\vec{E_s} = \left(\frac{50}{\rho}\right) e^{j\beta z}\,\hat{a}_\phi \rightarrow (1)$$

$$\vec{H} = \frac{H_o}{\rho}\cos(10^6 t + \beta z)\,\hat{a}_\rho$$

$$\vec{H} = Re\left[\frac{H_o}{\rho} e^{j(10^6 t + \beta z)}\,\hat{a}_\rho\right]$$

$$\vec{H} = Re\left[\frac{H_o}{\rho} e^{j\beta z}\,\hat{a}_\rho\, e^{j10^6 t}\right]$$

$$\vec{H_s} = \frac{H_o}{\rho} e^{j\beta z}\,\hat{a}_\rho \longrightarrow (2)$$

Since ~~satisfy Maxwell's~~ it is in free space

$\rho_v = 0$, $\sigma = 0$ $\quad \varepsilon = \varepsilon_0$, $\mu = \mu_0$

Since it satisfy Maxwell's Equation,

consider Maxwell's equation in Phasor form

$\nabla \cdot \vec{D_s} = \varepsilon_0 \nabla \cdot \vec{E_s} = 0 \rightarrow \nabla \cdot \vec{E_s} = 0$

$\nabla \cdot \vec{B_s} = \mu_0 \nabla \cdot \vec{H_s} = 0 \rightarrow \nabla \cdot \vec{H_s} = 0$

$\nabla \times \vec{H_s} = J + \partial D$

$\nabla \times E_s = -j\omega\mu_0 H_s$

$\nabla\cdot\vec{D_s} = \nabla\cdot\varepsilon\vec{E_s} = \frac{\varepsilon}{\rho}\frac{d}{d\phi}(\vec{E}_{\phi s}) = 0$, $\nabla\cdot\vec{H_s} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_{\phi s})$

$\nabla\cdot\vec{D_s} = 0$ [free space]

$\nabla\cdot\vec{H_s} = \frac{1}{\rho}\frac{d}{d\rho}\left[\rho\times\frac{H_0}{\rho}e^{-j\beta z}\right] = 0$

$= 0$

$\nabla\times H_s = \cancel{J_s}^{\,0} + j\omega D_s$

$\nabla\times H_s = j\omega\varepsilon\vec{E_s} = j\omega\varepsilon_0\vec{E_s} \rightarrow (1)$

$\vec{H_s} = \frac{H_0}{\rho}e^{j\beta z}\hat{a}_\rho$

$$\nabla\times\vec{H_s} = \frac{1}{\rho}\begin{vmatrix}\hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{d}{d\rho} & \frac{d}{d\phi} & \frac{d}{dz} \\ \frac{H_0}{\rho}e^{j\beta z} & 0 & 0\end{vmatrix}$$

$$= \cancel{\frac{1}{\rho}}\left[-\cancel{\rho}\hat{a}_\phi\left(0 - \frac{H_0}{\rho}\frac{d}{dz}(e^{j\beta z})\right)\right]$$

$$= \frac{H_0}{\rho}j\beta e^{j\beta z}\hat{a}_\phi$$

$$= \frac{jH_0\beta}{\rho}e^{j\beta z}\hat{a}_\phi \rightarrow (2)$$

$$\nabla\times\vec{H_s} = j\omega\varepsilon_0\vec{E_s} = j\omega\varepsilon_0\frac{50}{\rho}e^{j\beta z}\hat{a}_\phi \rightarrow (3)$$

$(2) = (3)$

$$\cancel{j}\omega\varepsilon_0\frac{50}{\cancel{\rho}}\cancel{e^{j\beta z}}\cancel{\hat{a}_\phi} = \frac{\cancel{j}H_0\beta}{\cancel{\rho}}\cancel{e^{j\beta z}}\cancel{\hat{a}_\phi}$$

$$H_0\beta = \omega\varepsilon_0\times 50$$

$$\frac{H_0\beta}{\beta^2} = \frac{\omega\varepsilon_0\times 50}{\beta^2}$$

$$\frac{H_0}{\beta} = \frac{\omega}{\beta}\times\frac{\varepsilon_0\times 50}{\beta}$$

$\omega = \beta v$

$\frac{\omega}{\beta} = v$

$\frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

$$\frac{H_0}{\cancel{\beta}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\times\frac{\varepsilon_0\times 50}{\cancel{\beta}}$$

$$H_0 = \frac{1}{\sqrt{\frac{\mu_0}{\varepsilon_0}}}\times 50$$

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$$H_0 \beta = 50 \omega \varepsilon_0 \rightarrow (3)$$

$$\nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\nabla \times \vec{E_s} = -j\omega \vec{B_s}$$

$$\nabla \times \vec{E_s} = -j\omega \vec{B_s} \rightarrow (4)$$

$$\nabla \times \vec{E_s} = -j\omega\mu_0 \vec{H_s} \rightarrow (4)$$

But $\vec{E_s} = \frac{50}{\rho} e^{j\beta z} \hat{a}_\phi$

$$\nabla \times \vec{E_s} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \cancel{\rho}\frac{50}{\cancel{\rho}} e^{j\beta z} & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\hat{a}_\rho \left(0 - \frac{50}{\cancel{\rho}} j\beta e^{j\beta z}\right)\right] = -\frac{50}{\rho} j\beta e^{j\beta z} \hat{a}_\rho$$

$$= -\frac{50}{\rho} j\beta e^{j\beta z} \hat{a}_\rho \rightarrow (5)$$

$$\nabla \times \vec{E_s} = -j\omega\mu_0 \vec{H_s}$$

$$-j\omega\mu_0 \vec{H_s} = -\frac{50}{\rho} j\beta e^{j\beta z} \hat{a}_\rho$$

$$\cancel{-}\cancel{j}\omega\mu_0 \frac{H_0}{\cancel{\rho}} \cancel{e^{j\beta z}} \cancel{\hat{a}_\rho} = \cancel{-}\frac{50}{\cancel{\rho}} \cancel{j}\beta \cancel{e^{j\beta z}} \cancel{\hat{a}_\rho}$$

$$\omega\mu_0 H_0 = 50\beta$$

$$\frac{H_0}{\beta} = \frac{50}{\omega\mu_0} \rightarrow (4)$$

(3) × (4)

$$H_0 \cancel{\beta} \times \frac{H_0}{\cancel{\beta}} = 50 \cancel{\omega} \varepsilon_0 \times \frac{50}{\cancel{\omega}\mu_0}$$

$$H_0^2 = 50^2 \cdot \frac{\varepsilon_0}{\mu_0}$$

$$H_0 = \pm\, 50 \times \frac{1}{\sqrt{\frac{\mu_0}{\varepsilon_0}}} = \pm \frac{50}{120\pi}$$

$$H_0 = \pm\, 0.1326$$

(4)

$$H_0 = \pm \frac{50}{120\pi}$$

$$H_0 \beta = 50\omega\varepsilon_0$$

$$\beta = \frac{50\omega\varepsilon_0}{H_0} = \frac{\pm \cancel{50}\,\omega\,\varepsilon_0}{\frac{\cancel{50}}{120\pi}}$$

$$\beta = \pm\, 120\pi\,\omega\,\varepsilon_0$$

$$\beta = \pm \sqrt{\frac{\mu_0}{\varepsilon_0}} \times \omega \times \varepsilon_0$$

$$\beta = \pm \sqrt{\mu_0 \varepsilon_0} \times \omega$$

$$\beta = \pm \sqrt{\cancel{4\pi} \times 10^{-7} \times \frac{10^{-9}}{\cancel{36\pi}\ 9}} \times 10^{6}$$

$$= \pm \frac{10^{6}}{3 \times 10^{8}} = \pm \frac{1}{3 \times 10^{2}} = \pm\, 3.33 \times 10^{-3}$$

$$\beta = \pm\, 3.33 \times 10^{-3}$$

8.7) In a medium characterized by $\sigma = 0$, $\mu = \mu_0$, ε_0 and $\vec{E} = 20 \sin(10^8 t - \beta z)\hat{a}_y$ V/m

Calculate β and $\vec{H}$

Solution: $\vec{E} = 20 \sin(10^8 t - \beta z)\hat{a}_y$ $\Big\}$ $\nabla \cdot \vec{D_s} = 0$

$$\vec{E} = \mathrm{Im}\left[20 e^{j(\omega t - \beta z)} \hat{a}_y\right]$$

$$\vec{E} = \mathrm{Im}\left[20 e^{-j\beta z}\hat{a}_y \cdot e^{j\omega t}\right] \rightarrow (1)$$

$$\vec{E} = \mathrm{Im}\left[\vec{E_s} e^{j\omega t}\right] \rightarrow (2)$$

$\vec{E_s} = 20 e^{-j\beta z}\hat{a}_y$ is a phasor.

$$\nabla \times \vec{E_s} = -j\omega \vec{B_s} \rightarrow (3)$$

$$\nabla \times \vec{E_s} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 20 e^{-j\beta z} & 0 \end{vmatrix}$$

$$\nabla \times \vec{E_s} = \hat{a}_x\left[0 - \frac{\partial}{\partial z} 20 e^{-j\beta z}\right]$$

$$= \hat{a}_x\left[-20 e^{-j\beta z}(-j\beta)\right]$$

$$= \hat{a}_x\left[20 j\beta e^{-j\beta z}\right] \rightarrow (4)$$

$$\nabla \times \vec{E_s} = -j\omega \vec{B_s} = -j\omega \mu_0 \vec{H_s} \rightarrow (5)$$

$$(4) = (5) \quad -\cancel{j}\omega\mu_0 \vec{H_s} = 20\cancel{j}\beta e^{-j\beta z}\hat{a}_x$$

$$\vec{H_s} = -\frac{20\beta}{\omega\mu_0} e^{-j\beta z}\hat{a}_x$$

$$\vec{H_s} = -\frac{20\beta}{\omega\mu_0} e^{-j\beta z}\hat{a}_x$$ (6)

$\nabla \cdot \vec{B_s} = 0$

$\nabla \cdot \vec{H_s} = 0$

$$\nabla \times \vec{H_s} = \vec{J_s} + j\omega\vec{D_s}$$

Since $\sigma = 0$ $\vec{J_s} = 0$

$$\therefore \nabla \times \vec{H_s} = j\omega\vec{D_s} \rightarrow (6)$$

$$\nabla \times \vec{H_s} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{20\beta}{\omega\mu_0} e^{-j\beta z} & 0 & 0 \end{vmatrix}$$

$$\nabla \times \vec{H_s} = -\hat{a}_y\left[0 - \frac{\partial}{\partial z}\left(-\frac{20\beta}{\omega\mu_0} e^{-j\beta z}\right)\right]$$

$$= -\frac{20\beta}{\omega\mu_0}(-j\beta)e^{-j\beta z}\hat{a}_y$$

$$\nabla \times \vec{H_s} = j\frac{20\beta^2}{\omega\mu_0} e^{-j\beta z}\hat{a}_y \rightarrow (7)$$

$$\nabla \times \vec{H_s} = j\omega\varepsilon_0\vec{E_s} \rightarrow (8)$$

$$(7) = (8) \quad \cancel{j}\omega\varepsilon_0\vec{E_s} = \cancel{j}\frac{20\beta^2}{\omega\mu_0} e^{-j\beta z}\hat{a}_y$$

$$\vec{E_s} = \frac{20\beta^2}{\omega^2\mu_0\varepsilon_0} e^{-j\beta z}\hat{a}_y \rightarrow (9)$$

$$\vec{E_s} = 20\, e^{-j\beta z}\hat{a}_y$$

$$\cancel{20} = \frac{\cancel{20}\beta^2}{\omega^2\mu_0\varepsilon_0}$$

$$\beta^2 = \omega^2\mu_0\varepsilon_0$$

$$\beta = \pm\omega\sqrt{\mu_0\varepsilon_0} = \pm\frac{10^8}{3\times10^8} = \pm 1/3$$

Continued

$$\vec{H_s} = -\frac{20\,\beta}{\omega\mu_0} e^{-j\beta z}\,\hat{a}_x = -\frac{20 \times \left(\pm\frac{1}{3}\right)}{10^8 \times 4\pi \times 10^{-7}} e^{-j\beta z}\,\hat{a}_x \quad (7)$$

$$\vec{H_s} = \pm\,\frac{\cancel{20}}{3} \times \frac{1}{{}_2\cancel{4}\pi \times 1\cancel{0}} e^{-j\beta z}\,\hat{a}_x$$

$$\vec{H_s} = \pm\,\frac{1}{6\pi} e^{-j\beta z}\,\hat{a}_x$$

$$\vec{H} = \mathrm{Im}\left[\vec{H_s}\, e^{j\omega t}\right]$$

$$= \mathrm{Im}\left[\pm\,\frac{1}{6\pi} e^{-j\beta z} e^{j\omega t}\,\hat{a}_x\right]$$

$$\vec{H} = \pm\,\frac{1}{6\pi} \sin(\omega t \pm \beta z)\,\hat{a}_x$$

$$= \pm\,\frac{1}{6\pi} \sin\left(10^8 t \pm \frac{1}{3} z\right)\hat{a}_x \quad A/m$$