Instructions

1. All answers are to be written on one side of a single A4 sheet (could be a single page from your notebook also). We will refer to this as the **answer sheet**.
2. All Qs are either MCQs or fill-in-the-blanks.
3. Each Q carries 2 marks.
4. For MCQ Qs, write down the Q number followed by the option (A,B,C,D)
5. For fill-in-the-blanks Qs just write down the Q number followed by the answer (the answer could be a scalar, a vector or a matrix)
6. On the **answer sheet**, we only expect you to write the answer and not provide any explanations
7. You can write the explanations separately in rough sheets
8. By 9:50 am, you will take a photo of your **answer sheet** and upload them on gradescope.
9. By 10:00 am, you need to upload the **rough sheets**.

Questions

1. Write down a 4×4 singular matrix whose null space is orthogonal to its column space. (Or explain why such a matrix can never exist.)

> **Solution:** For any symmetric matrix the null space would be orthogonal to the column space. If A is symmetric,
>
> 1. rowspace(A) = columnspace(A)
> 2. rowspace(A) $\perp$ nullspace(A) (always)
>
> From 1 and 2, it follows that columnspace(A) $\perp$ nullspace(A)
>
> So you could have just written any singular symmetric matrix!

2. Consider a matrix $A \in \mathbb{R}^{n\times 2}$ such that $\frac{n}{2}$ is an even number. The first $\frac{n}{2}$ entries of the first column of A contains alternating 1s and -1s and the last $\frac{n}{2}$ entries of the first column of A contains alternating -1s and 1s. The second column of A contains the numbers 1 to n. For example, if $n = 8$ then

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 3 \\ -1 & 4 \\ -1 & 5 \\ 1 & 6 \\ -1 & 7 \\ 1 & 8 \end{bmatrix}$$

Now consider the vector $\mathbf{b} \in \mathbb{R}^n$ which contains all 1s, i.e., $b^T = [111....1]$. What will be the projection of $\mathbf{b}$ onto the column space of A? (Please note that we expect a solution for the generic case and not the specific case on $n = 8$.)

Solution: I will start with the formula:

$p = A(A^T A)^{-1} A^T b$

I would then realise that the instructor would expect me to do this complex computation under 5 minutes only if some parts of it were easy to compute (you get only 4.5 minutes per Q).

What would make the computation simple? Think. Think. What do I worry the most about? Inverse. When would computing the inverse be easy? When $A^T A$ is a diagonal matrix? When would $A^T A$ be a diagonal matrix? If the columns of A are orthogonal. Is that the case here? Yes (check the dot product: the sum of the first and the last element cancels out the sum of the second and the second last element and so on).

So $A^T A$ would indeed be a diagonal matrix but what would the diagonal elements be? n and $\frac{n(n+1)(2n+1)}{6}$ (sum of squares of the numbers 1 to n). What would the inverse be? Just the reciprocal of these elements. (Still solving it mentally?).

Now let's focus on the last part $A^T b$. Remember A^T is $\mathbb{R}^{2\times n}$ (two rows) and b is $\mathbb{R}^{n\times 1}$ so the answer would be $\mathbb{R}^{2\times 1}$. This should be easy. Why? because the product will only contain 2 elements (no matter what the value of n is). I just need to find these 2 elements. What happens when you multiply a matrix by a vector of all 1s. You just get the sum of the elements of the rows. So the first row of the product would be 0 and the second row would be $\frac{n(n+1)}{2}$. (I like the 0 here because I have a feeling that this 0 will produce more 0s going forward)

Almost there!

$(A^T A)^{-1} = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & \frac{6}{n(n+1)(2n+1)} \end{bmatrix}$, $A^T b = \begin{bmatrix} 0 \\ \frac{n(n+1)}{2} \end{bmatrix}$ (I love all the 0s that show up here)

$(A^TA)^{-1}A^Tb = \begin{bmatrix} 0 \\ \frac{3}{2n+1} \end{bmatrix}$

The above vector when multiplied by A, would give the answer as (0 times column 1 of A) + ($\frac{3}{2n+1}$ times column 2 of A). Hence,

$$p = A(A^TA)^{-1}A^Tb = \frac{3}{2n+1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ \cdots \\ n \end{bmatrix}$$

3. Find the eigenvalues of the following matrix.

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \sqrt{2} & 1 & \frac{1}{\sqrt{2}} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Solution: What do I notice here? The off-diagonal elements are messy but the diagonal elements are clean. Why is this good? Because I know that the sum of the eigenvalues is equal to the trace (sum of the diagonal elements). So even though I see some messy values outside the diagonal, I know that the eigenvalues would be clean.

I would take this as a hint and realise that the instructor does not really want me to use the characteristic equation and do all the messy computation that comes with it.

Then how do I approach this? What kind of matrices have easy to compute eigenvalues? Identity, permutation, projection, symmetric, singular, markov, orthogonal. Which ones from the above list can I rule out easily? All except singular and projection (i.e., I can easily see that the matrix is not identity, permutation, markov, etc). Of these two, which one is easier to check? Singular, because I just need to see if the columns are dependent. Let me start by checking that. Now, you know what you are looking for. Once you know what you are looking for, you will see it! You will see that column 2 is $\frac{1}{\sqrt{2}}$ times column 1 and column 3 is $\frac{1}{2}$ times column 1. This is not easy to see when you first read the matrix. But now that you know what you are looking for (dependent columns) you will see this easily!

The last 2 columns are dependent on the first column. So you know that this is a rank-1 matrix which means this is a singular matris. That, in turn, means it will have 0 eigen values. How many? 2. So what would the third eigen value be? 3 (1 +

$1 + 1 = 3 + 0 + 0$).

Hence, 0, 0, 3.

4. Consider 4 independent vectors $a_1, a_2, a_3, a_4 \in \mathbb{R}^n$. These 4 vectors form the basis of a 4 dimensional subspace in $\mathbb{R}^n$. If one of these vectors is the standard basis vector (i.e., its i-th element is 1 and all other elements are 0, $0 < i < n$) then will using the Gram-Schmidth process to find an orthonormal basis always result in the 4 standard basis vectors. Explain your answer.

Solution: You can do this by actually taking 4 vectors such that one of them is a standard basis vector whereas the others are some random vectors. With a few steps of Gram-Schmidth you will realise that the process will result in the 4 standard basis vectors only if:

1. (1.5 marks) If you start with a_1 as the standard basis vector (remember, you can choose any vector as the first vector)
2. (0.5 marks) If all the vectors (a_1, a_2, a_3, a_4) have only non-negative elements.

5. Consider the following matrix

$$A = \begin{bmatrix} 0.18 & 0 & 0 & 0.82 & 0 \\ 0.10 & 0.20 & 0.30 & 0.40 & 0 \\ 0.15 & 0.20 & 0.30 & 0.25 & 0.10 \\ 0.50 & 0 & 0 & 0.50 & 0 \\ 0.18 & 0.22 & 0.30 & 0.10 & 0.20 \end{bmatrix}$$

What number should you subtract from the diagonal so that the determinant of the resulting matrix would be 0? (There may be multiple answers. You can write any one.)

Solution: What does subtracting an element from the diagonal mean? It means I am looking for a matrix $A - kI$ and I want $|A - kI| = 0$. Wait! I know this! I really know this! k must be an eigenvalue of the matrix A for this condition to be true. Ok, let me compute the eigenvalues of the matrix and then I would be done. Alas! Easier said than done! Computing the eigenvalues of a 5×5 matrix using the characteristic equation would easily take me 20 minutes. Not worth the time. Not what the instructor would expect me to do!

So, what do I do? Think of matrices which have easy to compute eigenvalues. Identity, permutation, projection, markov. Markov! Markov! The first row sums to 1. The second row sums to 1. All rows sum to 1! This is a Markov matrix! I know 1 eigenvalue of any Markov matrix. It is 1!

Hence, the answer is 1! Nicely done.

6. Suppose $A = BCB^{-1}$ where $A, B, C \in \mathbb{R}^n$ and none of them is a diagonal matrix. If $\mathbf{x} = [1, 1, 1]$ is an eigenvector of C and $B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ then write down at least one eigenvector of A.

Solution: It should be obvious that there should be some relation between the eigenvectors of A and C (otherwise what is the point of mentioning the eigenvector of C in the equation). So, I need to find this relation.

How, do I do that? Think. I can't think of anything. Ok, then let me Write. What do I write? Well, what can I write? Hmm, I can write Cx because I know the eigenvector of C. Hmm, that does not help much! Okay, what else can I write. $A = BCB^{-1}$. Can I squeeze in Cx somewhere in this equation. No. There is a B^{-1} sitting in front of the C but I want an x to sit in front of C. Can I get rid of B^{-1}? Yes, I can by multiplying both sides by B to get,

$AB = BC$

Now let me multiply both sides by x,

$ABx = BCx$
$ABx = \lambda Bx$
Wait! let me read that again, but this time with appropriate punctuation!
$A\underbrace{Bx} = \lambda\underbrace{Bx}$
Oh, Merciful Tehlu! Bx is an eigenvector of A. Yay!

7. Consider a square symmetric matrix S such that $x^T S x \geq 0 \; \forall x \in \mathbb{R}^n$. Which of the following cannot be an eigenvalue of any such matrix S: $\{-2, -1, 0, 1, 2\}$. Explain your answer.

Solution: Why is this Q asking me about eigenvalues? The Q only mentions $x^TSx \geq 0 \; \forall x \in \mathbb{R}^n$. From here, how do I get a connection to eigenvalues. The given condition only contains a matrix and vectors. Wait, the given condition contains vectors! If I want a connection to eigenvalues then I should focus on eigenvectors. That is, I should focus on the case when x is an eigenvector. What would happen in that case?

$x^TSx \geq 0$
$x^T \underbrace{\lambda x} \geq 0$
$\lambda x^Tx \geq 0$
But x^Tx is always positive (sum of squares of real numbers). So for the product to be ≥ 0, λ must be ≥ 0. Hence, it cannot be -1, -2.

8. Consider the following matrix:
$A = \begin{bmatrix} 0.64 & 0.36 \\ 0.32 & 0.68 \end{bmatrix}$
Write down the eigenvalues and eigenvectors of A^∞.

Solution: Though Process: We are dealing with powers of A. Eigenvalues of the powers of A are the powers of the eigenvalues of A. Eigenvectors of the powers of A are the same as the eigenvectors of A. So, I just need to find the eigenvalues and eigenvectors of A.

That should be easy! Well, it is if you realise that the A is a Markov matrix (Come on now, you can't miss that!). The eigen values are 1 and (0.64 + 0.68 - 1 = 0.32).

So, eigenvalues of A would be 1, 0.32. I will let you take over from here!

9. Write down a 3×3 matrix whose eigenvalues are 0, 1, 1. The matrix should not be a diagonal matrix.

Solution: This one was tricky but I like it because it combines multiple concepts!

Here is how I would think about it. I want the eigenvalues to be 0,1,1. Hmm, 0 and 1. Hmm, 0 and 1. Wait! that looks like a special case! Or those look like the eigenvalues of a special matrix. Which special matrix? Projection matrix. How do I construct a projection matrix? I know that $P = A(A^TA)^{-1}A^T$. So I can start with some A and just do this computation.

So far so good! But will any A be okay? What kind of A am I looking for? What do the values 0,1,1 tell me about the column space of A? It is 2-dimensional. Why? Because any vector perpendicular to this 2D plane will have the projection 0 (and hence, will be an eigenvector of P with eigenvalue 0). And any vector which lies in this 2D plane will not change after projection (and hence will be an eigenvector of P with value 1).

Okay, so I am looking for a matrix A whose columnspace is 2 dimensional. What is the simplest such matrix that I can think of?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

And, I will stop there and let you figure out the rest!

10. A class contains 10 students named A, B, C, D, E, F, G, H, I, J. In how many ways can you arrange these students such that F and G are not adjacent to each other in the queue.

Solution: The first rule of counting: focus on the restrictions first. So, let's start by putting F in the sequence first. If we put F in the first or last position then there are 8 spots available for G. However, if we put F in any of the middle 8 positions then there are only 7 spots available for G (as the two spots on either side of F) are not available. Thus we have 3 cases

Case 1: F in the first position

There is only 1 way of putting F in the first position. There are 8 ways of putting G in one of the valid spots. After this, the remaining 8 students can be arranged in the available spots in 8! ways

Case 2: F in the last position

Once again, there is only 1 way of putting F in the last position. There are 8 ways of putting G in one of the valid spots. After this the remaining 8 students can be arranged in the available spots in 8! ways.

Case 3: F is in one of the 8 middle positions

There are 8 ways of placing F in one of the middle positions. After this, there are 7 ways of putting G in one of the valid spots. Having done that the remaining 8 students can be arranged in the available spots in 8! ways

Hence, the total number of ways of doing this 1*8*8! + 1*8*8! + 8*7*8! = 9*8*8! = 29,03,040.