1. (1 point) Honor code.

Concept: System of linear equations

2. (2 points) This question has two parts as mentioned below:
   (a) Find a 2 x 3 system Ax = b whose complete solution is

$$x = \begin{bmatrix}1\\1\\1\end{bmatrix} + w\begin{bmatrix}1\\2\\1\end{bmatrix}$$

Solution: Solution : Yashasvi Mahajan (cs21m076@smail.iitm.ac.in)

Solution:

We know that ,
Complete Solution = Particular Solution + Null space
Particular solution is the solution to Ax=b.
Null space consist of linear combination of the basis of Null space. Null space is all values of x for which Ax=0 satisfies.

Given :

Particular Solution , $\mathbf{x}=\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$
Basis for the NULL space , $\mathbf{x}=\begin{bmatrix}1 & 2 & 1\end{bmatrix}^\top$

Let,A $\in \mathbb{R}^2\mathbf{x}\mathbb{R}^3, \mathbf{x} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^2$

$$A\mathbf{x} = b \implies \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \mathbf{x} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Particular solution satisfies A**x**=b

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} y1 \\ y2 \end{bmatrix}$$

$$a + b + c = y_1 \tag{1}$$

$$d + e + f = y_2 \tag{2}$$

Basis for NULL space satisfies A**x**=0,

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a + 2b + c = 0 \tag{3}$$

$$d + 2e + f = 0 \tag{4}$$

Subtracting $eq^n(3)$-$eq^n(1)$

$$b = -y_1 \tag{5}$$

Substituting the value of b in $eq^n(3)$

$$a - 2y_1 + c = 0$$

$$a = 2y_1 - c \tag{6}$$

Let c=1,y_1=3.Substitute this values in $eq^n(5)$ and $eq^n(6)$,we get value of b and a,

$$a = 5, b = -3, c = 1, y_1 = 3 \tag{7}$$

Subtracting $eq^n(4)$-$eq^n(2)$

$$e = -y_2 \tag{8}$$

Substituting the value of e in $eq^n(4)$

$$e - 2y_2 + f = 0$$

$$d = 2y_2 - f \tag{9}$$

Let f=1,y_2=4.Substitute this values in $eq^n(8)$ and $eq^n(9)$,we get value of e and d

$$d = 7, e = -4, f = 1, y_2 = 4 \tag{10}$$

Thus , matrix will be

$$\begin{bmatrix} 5 & -3 & 1 \\ 7 & -4 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \mathbf{x_3} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \tag{11}$$

System of equation which has the given complete solution is ,

$$5\mathbf{x_1} - 3\mathbf{x_2} + 1\mathbf{x_3} = 3 \tag{Ans}$$

$$7\mathbf{x_1} - 4\mathbf{x_2} + 1\mathbf{x_3} = 4$$

(b) Now find a 3 x 3 system which has these solutions exactly when $b_1 + b_2 + b_3 = 0$. (Note: $b = [b_1\ b_2\ b_3]^T$.)

Solution:
Given :
Particular Solution , $\mathbf{x}=\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$
Basis for the NULL space , $\mathbf{x}=\begin{bmatrix}1 & 2 & 1\end{bmatrix}^\top$

Let, A $\in \mathbb{R}^3\mathbf{x}\mathbb{R}^3, \mathbf{x}$ and $\mathbf{b} \in \mathbb{R}^3$

$$A\mathbf{x} = b \implies \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \mathbf{x} = \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$$

Particular solution satisfies A$\mathbf{x}$=b

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$a + b + c = \mathbf{b_1} \tag{1}$$

$$d + e + f = \mathbf{b_2} \tag{2}$$

$$g + h + i = \mathbf{b_3} \tag{3}$$

Basis for NULL space satisfies A$\mathbf{x}$=0,

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solution:

$$a + 2b + c = 0 \tag{4}$$
$$d + 2e + f = 0 \tag{5}$$
$$g + 2h + i = 0 \tag{6}$$

Solving above equations we get ,

$$b = -\mathbf{b_1} \tag{7}$$
$$a = 2\mathbf{b_1} - c \tag{8}$$
$$e = -\mathbf{b_2} \tag{9}$$
$$d = 2\mathbf{b_2} - f \tag{10}$$
$$h = -\mathbf{b_3} \tag{11}$$
$$g = 2\mathbf{b_3} - i \tag{12}$$

Putting values of $c = 1$, $f = 1$, $i = 1$, $b_1 = -1$, $b_2 = -1$, $b_3 = 2$ such that $b_1 + b_2 + b_3 = 0$ we get ,

$$A\mathbf{x} = b \implies \begin{bmatrix} -3 & 1 & 1 \\ -3 & 1 & 1 \\ 3 & -2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Thus system of equation is ,

$$-3x1 + x2 + x3 = -1$$
$$-3x1 + x2 + x3 = -1$$
$$3x1 - 2x2 + x3 = 2$$

3. (2 points) Consider the matrices A and B below

$$\text{(i) A} = \begin{bmatrix} 1 & 6 & -3 & 4 \\ 0 & 2 & 4 & 0 \\ 2 & 12 & -6 & 8 \end{bmatrix} \text{ (ii) B} = \begin{bmatrix} 3 & 1 & 2 \\ 12 & 4 & 8 \\ 6 & 2 & 4 \end{bmatrix}$$

(a) Write down the row reduced echelon form of matrices A and B (also mention the steps involved).

Solution: Given,

$$\text{A} = \begin{bmatrix} 1 & 6 & -3 & 4 \\ 0 & 2 & 4 & 0 \\ 2 & 12 & -6 & 8 \end{bmatrix}$$

$R_3 \longleftarrow R_3 - 2R_1$

$$\begin{bmatrix} \boxed{1} & 6 & -3 & 4 \\ 0 & \boxed{2} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we have achieved Row echelon form. Lets extend further

$R_1 \longleftarrow R_1 - 3R_2$

$$\begin{bmatrix} \boxed{1} & 0 & -15 & 4 \\ 0 & \boxed{2} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \longleftarrow R_2/2$

$$\begin{bmatrix} \boxed{1} & 0 & -15 & 4 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore row reduced echelon form of A is $\begin{bmatrix} 1 & 0 & -15 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Similarly,

B = $\begin{bmatrix} 3 & 1 & 2 \\ 12 & 4 & 8 \\ 6 & 2 & 4 \end{bmatrix}$

$R_2 \longleftarrow R_2 - 4R_1$
$R_3 \longleftarrow R_3 - 2R_1$

$$\begin{bmatrix} \boxed{3} & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \longleftarrow R_1/3$

$$\begin{bmatrix} \boxed{1} & 1/3 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the row reduced echelon form of B is $\begin{bmatrix} 1 & 1/3 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) Find all solutions to $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$.

Solution: $\mathbf{Ax = 0}$

$$\Longrightarrow \begin{bmatrix} 1 & 6 & -3 & 4 \\ 0 & 2 & 4 & 0 \\ 2 & 12 & -6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

After performing Gaussian elimination (previous problem) we get

$$\Longrightarrow \begin{bmatrix} \boxed{1} & 6 & -3 & 4 \\ 0 & \boxed{2} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Forming equations

$x_1 + 6x_2 - 3x_3 + 4x_4 = 0$
$2x_2 + 4x_3 = 0$

Since we have two free variables i.e., x_3 and x_4, we choose below combinations

put $x_3 = 0$, $x_4 = 1$

$$\Longrightarrow \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

put $x_3 = 1$, $x_4 = 0$

$$\Longrightarrow \begin{bmatrix} 15 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Hence, the solution of Ax = 0 is

$$a \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 15 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$\mathbf{Bx = 0}$

$$\begin{bmatrix} 3 & 1 & 2 \\ 12 & 4 & 8 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Performing Gaussian elimination we get

$$\begin{bmatrix} \boxed{3} & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, the two free variables are x_2 and x_3

put $x_2 = 0$ and $x_3 = 1$

$\implies \begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix}$

put $x_2 = 1$ and $x_3 = 0$

$\implies \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$

Hence the solution of Bx = 0 is

$$a \begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$$

(c) Write down the basis for the four fundamental subspaces of A.

Solution: **Column Space** - pivot columns of A. In our case since only two non zero pivots, our basis for column space has only two vectors.

i.e., $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 2 \\ 12 \end{bmatrix}$

Null Space - The basis for null space are the special solutions of Ax = 0.

i.e., $\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 15 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

Row Space - The basis of row space are the independent rows of A

$$\begin{bmatrix} 1 \\ 6 \\ -3 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 2 \\ 4 \\ 0 \end{bmatrix}$$

Alternatively, a more cleaner basis will be the corresponding rows of the R matrix (i.e., row reduced echelon form)

i.e., $\begin{bmatrix} 1 \\ 0 \\ -15 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

Left Null Space - Here basis will be the vectors that make the rows zero i.e, we first apply the row operations on I

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Applying $R_3 \longleftarrow R_3 - 2R_1$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ Applying $R_1 \longleftarrow R_1 - 3R_2$

$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ Applying $R_2 \longleftarrow R_2/2$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1/2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Here the last row gives us the zero row in R

i.e., $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ is the basis of left null space

(d) Write down the basis for the four fundamental subspaces of B.

Solution: **Column Space** - pivot columns of B. In our case since only one non zero pivot, our basis for column space has only one vector.

i.e., $\begin{bmatrix} 3 \\ 12 \\ 6 \end{bmatrix}$

Null Space - The basis for null space are the special solutions of Bx = 0.

i.e., $\begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}$

Row Space - The basis of row space are the independent rows of R

$\begin{bmatrix} 1 \\ 1/3 \\ 2/3 \end{bmatrix}$

Left Null Space - Here basis will be the vectors that make the rows zero i.e, we first apply the row operations on I

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Applying $R_2 \longleftarrow R_2 - 4R_1$ and $R_3 \longleftarrow R_3 - 2R_1$

$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$
Applying $R_1 \longleftarrow R_1/3$

$\begin{bmatrix} 1/3 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Here the last two rows gives us the zero rows in R

i.e., $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$ is the basis of left null space

Concept: Rank

4. (1 ½ points) Consider the matrices A and B as given below:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -6 & -4 & -2 \\ 3 & 2 & x \end{bmatrix} \text{ and B} = \begin{bmatrix} 7 & 2 & 7 \\ y & 2 & y \end{bmatrix}$$

Give the values for entries x and y such that the ranks of the matrices A and B are

(a) 1

Solution:

Solution: For matrix A, if $x = 1$, both the rows R_2 and R_3 are dependent on R_1. So, the rank of A is 1 for $x = 1$.

For matrix B, if $y = 7$, both the rows of B are same and thus there is only one independent row which makes the rank of B equals 1.

(b) 2

Solution:
Solution: For any value of x such that $x \neq 1$ will make R_3 an independent row. So, since R_2 is already dependent on R_1, the rank of A is 2 for two independent rows.

For $y = 0$, B becomes:

$$B = \begin{bmatrix} 7 & 2 & 7 \\ 0 & 2 & 0 \end{bmatrix}$$

Since this matrix has two pivot elements, there are two independent rows and the rank is 2.

(c) 3

Solution:
Solution: No value of x for A can yield a matrix with rank 3. Since R_2 is always dependent on R_1, there can be at most two pivots for matrix A.

Similarly, there is no value of y that can yield a matrix with rank 3. This is because, there are only two rows in matrix B. If we suppose that there are 3 independent columns of B, then there also must be 3 independent rows in B. But B has only two rows so the rank of 3 is not possible.

Concept: Nullspace and column space

5. (½ point) State True or False and explain you answer: The nullspace of R is the same as the nullspace of U (where R is the row reduced echelon form of A and U is the matrix in LU decomposition of A).

Solution: True because each step involved in getting R from U corresponds to multiplication with an elementary matrix E.
Therefore, we can write $R = EU$
Let x be in the null space of U, then
$Ux = 0$
Also, $Rx = (EU)x = E(Ux) = E(0) = 0$
$\Rightarrow$ x lies in the null space of R.

6. (1 point) Construct a matrix whose column space contains $[2, 5, 3]^\top$ and $[0, 3, 1]^\top$ and whose null space contains $[1, 3, 2]^\top$

Solution: RAHUL KUMAR (cs21m049@smail.iitm.ac.in)

Solution: Let A is $m \times n$ matrix.
Let $v_1 = [2, 5, 3]^T$, $v_2 = [0, 3, 1]^T$ and $u_1 = [1, 3, 2]^T$
$\because \mathcal{C}(A) \in \mathbb{R}^m$ and $v_1, v_2 \in C(A)$
$\therefore m = 3$ as v_1 and v_2 are points of a $3D$ space.
Similarly $\mathcal{N}(A) \in \mathbb{R}^n$ and $u_1 \in N(A)$ so $n = 3$
Now we know that A is a 3×3 matrix.
vector v_1 and v_2 are pointing in different direction so we can say that
$dim(\mathcal{C}(A)) >= 2$
but there is also a non-zero null space so $dim(\mathcal{C}(A)) = 2$ and $dim(\mathcal{N}(A)) = 1$
$\Longrightarrow$ we have 2 pivot column in A and 1 free column in A.
$\because v_1, v_2 \in \mathcal{C}(A)$
$\therefore$ we can take these two vector as two independent columns of A and let the 3^{rd} column of A is $[a, b, c]^T$
From the above discussion:

$$A = \begin{bmatrix} 2 & 0 & a \\ 5 & 3 & b \\ 3 & 1 & c \end{bmatrix}$$

$\because u_1 \in \mathcal{N}(A)$
$\therefore Au_1 = 0$

$$\begin{bmatrix} 2 & 0 & a \\ 5 & 3 & b \\ 3 & 1 & c \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$
$2 \times 1 + 0 \times 3 + a \times 2 = 0 \implies a = -1$
$5 \times 1 + 3 \times 3 + b \times 2 = 0 \implies b = -7$
$3 \times 1 + 1 \times 3 + c \times 2 = 0 \implies c = -3$

$$\text{final } A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 3 & -7 \\ 3 & 1 & -3 \end{bmatrix}$$

7. (2 points) Consider the matrix $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 9 \end{bmatrix}$. The column space of this matrix is a 2 dimensional plane. What is the equation of this plane? (You need to write down the steps you took to arrive at the equation)

Solution:

Given that $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 9 \end{bmatrix}$

Performing (1) $Row_2 = Row_2 - \frac{2}{3}.Row_1$ and (2) $Row_3 = Row_3 - \frac{1}{3}.Row_1$ on A, we get:

$$A^{'} = E_1.A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-2}{3} & 1 & 0 \\ \frac{-1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 9 \end{bmatrix}$$

Again, performing $Row_3 = Row_3 - 9.Row_2$ on $A^{'}$, we get:

$$A^{''} = E_2.A^{'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -9 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The pivot columns of A are the two columns of A. Hence, the basis for the Column space of A would be $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$.

Any vector in the column space of A can be generated using a linear combination of the basis vectors of C(A).

$\therefore \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 10 \end{bmatrix}$ also belongs in the column space of A.

Now, we have three points in the column space of A and using them, we obtain the following set of equations:

$$\begin{aligned} a(3) + b(2) + c(1) + d &= 0 \\ a(0) + b(1) + c(9) + d &= 0 \\ a(3) + b(3) + c(10) + d &= 0 \end{aligned}$$

These equations can be written in matrix form as follows:

$$\begin{bmatrix} 3 & 2 & 1 & 1 \\ 0 & 1 & 9 & 1 \\ 3 & 3 & 10 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Performing (1) $Row_3 = Row_3 - Row_1$ on the above matrix, we get:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 1 \\ 0 & 1 & 9 & 1 \\ 3 & 3 & 10 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 0 & 1 & 9 & 1 \\ 0 & 1 & 9 & 0 \end{bmatrix}$$

Again, performing $Row_3 = Row_3 - Row_2$ on the above matrix, we get:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 1 \\ 0 & 1 & 9 & 1 \\ 0 & 1 & 9 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & 1 \\ 0 & 1 & 9 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Thus, we are left with:

$$\begin{aligned} 3a + 2b + c + d &= 0 \\ b + 9c + d &= 0 \\ -d &= 0 \end{aligned}$$

$\therefore d = 0, b = -9c, a = \frac{17}{3}c.$
Assuming $c = 3$, we get $a = 17, b = -27, c = 3, d = 0.$

The equation of the required plane is therefore $17x - 27y + 3z = 0$

8. (1 point) True or false? (If true give logical, valid reasoning or give a counterexample if false)
a. If the row space equals the column space then $A^T = A$

Solution: False
For example, let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
Here, row space is equal to column space. (Both are equal to whole of $\mathbf{R^2}$)
But $A \neq A^\top$

b. If $A^T = -A$ then the row space of A equals the column space.

Solution: True
We know that row space of A equals the column space of $A^\top$.
Here, row space of A equals the column space of -A.
But column space of -A is same as column space of A because if a vector b lies in column space of A, then we can write:
$Ax = b$
We can also write:
$(-A)(-x) = b$
$\Rightarrow$ b lies in the column space of -A.
$\therefore$ Row space of A equals the column space of A.

9. (1 point) What are the dimensions of the four subspaces for **A, B,** and **C,** if I is the 3×3 identity matrix and 0 is the 3×2 zero matrix?
A=$\begin{bmatrix} I & 0 \end{bmatrix}$ and B $= \begin{bmatrix} I & I \\ 0^\top & 0^\top \end{bmatrix}$ and C=$\begin{bmatrix} 0 \end{bmatrix}$

Solution:

$$A_{3\times5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$\dim(\mathcal{C}(A)) = 3$ (rank = 3 as 3 non-zero pivots after GE)
$\dim(\mathcal{N}(A)) = 5 - 3 = 2$ (rank nullity theorem)
$\dim(\mathcal{C}(A^\top)) = 3$ (row rank = column rank)
$\dim(\mathcal{N}(A^\top)) = 3 - 3 = 0$ (rank nullity theorem)

$$B_{5\times6} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(\mathcal{C}(B)) = 3$ (rank = 3 as 3 non-zero pivots after GE)
$\dim(\mathcal{N}(B)) = 6 - 3 = 3$ (rank nullity theorem)

$\dim(\mathcal{C}(B^\top)) = 3$ (row rank = column rank)
$\dim(\mathcal{N}(B^\top)) = 5 - 3 = 0$ (rank nullity theorem)

$$C_{3\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\dim(\mathcal{C}(C)) = 0$ (rank = 3 as 3 non-zero pivots after GE)
$\dim(\mathcal{N}(C)) = 2 - 0 = 2$ (rank nullity theorem)
$\dim(\mathcal{C}(C^\top)) = 0$ (row rank = column rank)
$\dim(\mathcal{N}(C^\top)) = 3 - 0 = 3$ (rank nullity theorem)

10. (2 points) Solve the following questions.

(a) If A is an m×n matrix, find $\dim(\mathcal{R}(A)) + \dim(\mathcal{C}(A)) + \dim(\mathcal{N}(A)) + \dim(\mathcal{N}(A^\top))$. (in terms of n & m)

Solution: $\dim(\mathcal{R}(A)) + \dim(\mathcal{C}(A)) + \dim(\mathcal{N}(A)) + \dim(\mathcal{N}(A^\top))$

We know that dim(C(A)) = r (where r is the rank of the matrix)

Then dim(N(A)) = n - r (by rank nullity theorem, where n is the number of columns)

Similarly, dim(R(A)) = dim(C(A^T)) = r (because row rank = column rank)

and, dim((A^T)) = m - r (by rank nullity theorem, where m = number of rows in A)

Therefore, r + (n - r) + r + (m -r)

= n + m

Hence, $\dim(\mathcal{R}(A)) + \dim(\mathcal{C}(A)) + \dim(\mathcal{N}(A)) + \dim(\mathcal{N}(A^\top))$ = n + m

(b) Let A and B be two $n \times n$ matrices such that $AB = 0$. Show that the row space of A is contained in the left null space of B.

Solution:
A = n x n, B = n x n
Also, AB = 0

Row space of A = column space of A^T
i.e., A^Tx = b
b $\in$ R(A)
Hence, row space of $A^T \in \mathbb{R}^n$

Left null space of B = N(B^T)
B^Ty = 0
i.e., y $\in$ N(B^T)
Hence Left null space $\in \mathbb{R}^n$

Say, b $\in$ R(A) then
A^Tx = b ——1

Consider AB = 0
(Taking transpose on both sides)
$\implies (AB)^T = 0$
$\implies B^T A^T = 0$
(Multiply by x on both sides)
$\implies B^T A^T x = 0$
(Now using 1)
$\implies B^T$ p = 0

Here, by definition of Left null space, p $\in$ N(B^T)
i.e., p = A^Tx = row space of A is in the left null space of B

Hence, the row space of A is contained in the left null space of B

11. (1 point) True or false? If A is a $n \times n$ square matrix then $\mathcal{N}(A) = \mathcal{N}(AA^T)$ (If true give logical, valid reasoning or give a counterexample if false)

Solution:
Solution: False, as we can find the following counter example.
Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ which upon Guassian elimination yields $U_A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, which implies the null space is of the form $\alpha \begin{bmatrix} -3 & 1 \end{bmatrix}^\top$. $AA^T = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ which upon Gaussian Elimination yields $U_{AA^T} = \begin{bmatrix} 10 & 20 \\ 0 & 0 \end{bmatrix}$, which implies the null space is of the form $\alpha \begin{bmatrix} -2 & 1 \end{bmatrix}^\top$. Hence, we can observe that $\mathcal{N}(A) \neq \mathcal{N}(AA^T)$.

12. (2 points) Without explicitly computing the product of given two matrices, find bases for each of its four sub-spaces.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And also explain the four sub-spaces along with the method you followed to compute them.

Solution: 1 Mark for bases and 1 mark for explaining the bases in terms of 2D-plane or line. I have also given marks even if you have given linear combination of these bases vectors. Just giving bases vectors will only fetch 1 mark.

> In this system,
> C(A) is a 2D plane in $\mathbb{R}^3$
>
> N(A) is a 3D space in $\mathbb{R}^5$
>
> C(A^T) is a 2D plane in $\mathbb{R}^5$
>
> N(A^T) is a 1D line in $\mathbb{R}^3$

Part of Solution of MOHAMMED SAFI UR RAHMAN KHAN (cs21m035@smail.iitm.ac.in)

Entire Solution:

The above matrices can be written as $A = BC$,

where $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Thus, we need to find the four-fundamental subspaces for A.

1. Finding the Nullspace for A ie. $\mathcal{N}(\mathbf{A})$:

Consider $Ax = BCx = 0$ where x belongs to $\mathcal{N}(A)$. Here B is an Invertible matrix, and using thus fact we can say that B does not affect the Nullspace of BC ie. $\mathcal{N}(A) = \mathcal{N}(BC) = \mathcal{N}(C)$

Reason: Let x be in the nullspace of C ie $Cx = 0$. Now, we can see that after multiplying both sides on the left by matrix B, we get $BCx = B0$ ie. $BCx = 0$ which means that x is also in the nullspace of BC.

Also, let y be in the nullspace of BC ie $BCy = 0$. Thus $B^{-1}BCy = B^{-1}0$ ie. $Cy = 0$. Thus we can say that y is also in the nullspace of C.

Thus B does not change the nullspace of BC and it is same as nullspace of C.

Now, to find $\mathcal{N}(C)$, we can observe that C has 2 pivot columns, ie. column 2 and column 4 in C. Hence $rank(C) = 2$.

Thus, using rank-nullity theorem, the $\dim(\mathcal{N}(C))$ = Number of columns of C - rank(C) = 5 - 2 = 3

Now, to find the basis for $\mathcal{N}(C)$, we need to alternately put 1 in place of a free variable position in x, keep the remaining 2 free variables positions in x as 0, and modify the pivot variables positions in x accordingly to get result 0 in RHS.

Example: Keeping entry corresponding to row 1 as 1, entries for row 3 and row 5 as 0, we need to keep row 2 and row 4 as 0 in x, so that we get:

$$Cx = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is one of the vectors belonging to $\mathcal{N}(C)$. Similarly, we can obtain $\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ as the other vectors belonging to $\mathcal{N}(C)$.

Since these vectors are linearly independent, we can take these 3 vectors as the basis vectors for $\mathcal{N}(C)$, and as deduced above, these will also be the basis for $\mathcal{N}(BC)$ ie. $\mathcal{N}(A)$ since both are the same.

Thus below is our basis for $\mathcal{N}(\mathbf{A})$

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Thus we can see that $\mathcal{N}(A)$ spans a 3D plane in a 5D space.

2. Finding the Left Nullspace for A ie. $\mathcal{N}(\mathbf{A^T})$:

Since A is product of $3 * 3$ dimensional and $3 * 5$ dimensional matrices, A will be a $3 * 5$ dimensional matrix.

Also since rank(A) = rank$(C) = 2$, the dim$(\mathcal{N}(A^T))$ = Number of rows of A - rank(A) = 3 - 2 = 1.

Hence the dim$(\mathcal{N}(A^T)) = 1$

We can observe that B is a product of Elementary matrices

$$\text{ie. B} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}$$

Let $E_1 = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ and $E_2 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix}$. Thus $B = E_1E_2$

Since $A = BC$, we can rewrite it as $A = E_1E_2C$

Also, since every Elementary matrix is Invertible, we can again rewrite the above as $E_2^{-1}E_1^{-1}A = C$

$$\text{ie. } E_2^{-1}E_1^{-1}A = \begin{bmatrix}0 & 1 & 2 & 3 & 4\\0 & 0 & 0 & 1 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

We can compute the Inverse of the Elementary matrices ie. E_2^{-1} and E_1^{-1} by simply reversing the elementary operations performed, ie.:

a] For E_1, Operation performed is $R_2 \to R_2 + R_1$. Hence corresponding Inverse operation would be $R_2 \to R_2 - R_1$. Hence $E_1^{-1} = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

b] For E_2, Operation performed is $R_3 \to R_3 + R_2$. Hence corresponding Inverse operation would be $R_3 \to R_3 - R_2$. Hence $E_2^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{bmatrix}$

$$\text{Thus, we have: } \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{bmatrix}A = \begin{bmatrix}0 & 1 & 2 & 3 & 4\\0 & 0 & 0 & 1 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\text{ie. } \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\1 & -1 & 1\end{bmatrix}A = \begin{bmatrix}0 & 1 & 2 & 3 & 4\\0 & 0 & 0 & 1 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

We can observe that, upon multiplying Row 3 of the matrix on the left with A, we get a row full of 0's (zeros) in the RHS matrix

$$\text{ie. } \begin{bmatrix}1 & -1 & 1\end{bmatrix}A = \begin{bmatrix}0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Thus, we can observe from above that $\begin{bmatrix}1 & -1 & 1\end{bmatrix}$ lies in the Left Nullspace of A. Since, $\dim(\mathcal{N}(A^T)) = 1$, this can be taken as the basis for the Left Nullspace of A.

Thus below is our basis for $\mathcal{N}(\mathbf{A^T})$

$$\mathcal{N}(A^T) = \left\{\begin{bmatrix}1 & -1 & 1\end{bmatrix}\right\}$$

It can be observed that $\mathcal{N}(A^T)$ spans a 1D line in a 3D space.

3. Finding the Rowspace for A ie. $\mathcal{R}(\mathbf{A})$:

Since we saw above that $B = E_1E_2$ where E_1 and E_2 were Elementary matrices performing row transformation operations, and since linear combinations of vectors which are in the same subspace, also remains in the same subspace, we can make one observation as follows:

Elementary row operations do not change the rowspace of a matrix, hence left multiplying B with C also similarly will not change the rowspace of C

Hence $\mathcal{R}(A) = \mathcal{R}(C)$ ie. the rows with the pivot entries in C. The row 1 and row 2 in matrix C satisfy this criteria, ie. $\begin{bmatrix}0 & 1 & 2 & 3 & 4\end{bmatrix}$ and $\begin{bmatrix}0 & 0 & 0 & 1 & 2\end{bmatrix}$

Thus below is our basis for $\mathcal{R}(\mathbf{A})$

$$\mathcal{R}(A) = \left\{\begin{bmatrix}0 & 1 & 2 & 3 & 4\end{bmatrix}, \begin{bmatrix}0 & 0 & 0 & 1 & 2\end{bmatrix}\right\}$$

Thus we can see that $\mathcal{R}(A)$ spans a 2D plane in a 5D space.

4. Finding the Columnspace for A ie. $\mathcal{C}(\mathbf{A})$:

We know that $rank(A) = 2$. Hence $\dim(\mathcal{C}(A)) = rank(A) = 2$.

Also, since $A = BC$ where B is a Lower triangular product of Elementary matrices, we can fetch the required column vectors from B and use it as the basis for the $\mathcal{C}(A)$.

Reason: Multiplying C to the right side of B will not change the Columnspace of the resultant, ie., BC, which will still be in the Columnspace of B.

However, the columnspace of C wont be preserved after this multiplication. Since we want columnspace of A using the columns of BC, hence we use B to get these basis column vectors(since only the columnspace of B is the same before and after the multiplication BC).

Hence columns of A will also be in the Columnspace of B, as $A = BC$.

Now, $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, and since $rank(A) = 2$, we can simply take the first 2 columns of B, ie., $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ for the basis of $\mathcal{C}(A)$.

Thus below is our basis for $\mathcal{C}(\mathbf{A})$

$$\mathcal{C}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Thus we can see that $\mathcal{C}(A)$ spans a 2D plane in a 3D space.

Solution of SIDDESH RAMESH HEGDE (cs21m064@smail.iitm.ac.in)

or Column Space:

Solution: Let A = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

If we observe, it is of the form A = LU

Where, L = E^{-1}
And, U is obtained after applying E on A (i.e., essentially doing Gaussian Elimination)

So, U = $\begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

We have two pivots here and hence, the basis of the Column Space of A will be the pivot columns of A.
Since we are not allowed to directly multiply the two matrices, we can interpret L as elementary row operations. Then we can apply those row operations directly on the two pivot columns and get the corresponding rows of A.

The two row operations that L performs are as below

$$R_3 \longleftarrow R_3 + R_2$$
$$R_2 \longleftarrow R_2 + R_1$$

Applying these two operations one after another on the two pivot columns of U, we get the two columns of A as $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

Hence, the column space of A is given by these two basis vectors and can be written as

$$c.\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d.\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

Part of Solution of MOHAMMED SAFI UR RAHMAN KHAN (cs21m035@smail.iitm.ac.in)

Concept: Free variables

13. (2 ½ points) True or False (with reason if true or example to show it is false).
    (a) An matrix $m\times$n can have zero pivots.

Solution: True.
Zero matrix is an example.

(b) A real-symmetric matrix $m \times$m has no free variables.

Solution: False.
A 2x2 matrix with all 1s is a counter example.

(c) If A & B be are two $m \times$n matrices with non-zero pivots, then a matrix C = A + B can have zero pivots

Solution: True when A = -B.

(d) A free variable in a matrix always implies that there is either a zero-row or zero-column in the matrix.

Solution: False.
A 2x2 matrix with all 1s is a counter example.

(e) For any matrix A, does A^T and A^{-1} have the same number of pivots.

Solution: True
Since A^{-1} exists, the rank of A will be same as A^{-1} as rank(A) = rank(A^{-1}); and rank of A and A^T is anyway same.

Concept: Reduced Echelon Form

14. (½ point) Suppose R is $m \times n$ matrix of rank r, with pivot columns first:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(a) Find a right-inverse B with $RB = I$ if $r = m$.

Solution: If $r = m$, then there are no zero rows in R.
$\Rightarrow R = \begin{bmatrix} I & F \end{bmatrix}$
Here, I is an $m \times m$ identity matrix and F will be an $m \times (n-m)$ matrix.
The right-inverse B should be such that $\begin{bmatrix} I & F \end{bmatrix} B = I$
So, $B = \begin{bmatrix} I \\ 0 \end{bmatrix}$, where I is an $m \times m$ identity matrix and 0 is an $(n-m) \times m$ zero matrix so that:
$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} I \end{bmatrix}$