Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: YES

2. You have two identical fair coins. You toss the first coin and if the output is heads then you stay with this coin and toss it again. If the the output is tails then you switch to the other coin and repeat the same process. This process can be summarized as follows:

Step 1: Select coin 1
Step 2: Toss the coin
Step 3: If result = Heads, go to step 2
Step 4: If result = Tails, switch to the other coin and go to step 2

(a) (½ point) What is the probability that after 99 tosses you end up with the same coin that you started with?

Solution: This means that you never switched the coin or switched it multiple times so that you again have coin 1.
if number of tails are odd then you are not using same coin otherwise you are using same coin, since events are independent and coins are fair, you will end up with prob (1/2)
when you toss n times, prob. of tails even is
$\binom{n}{0}(0.5)^0(0.5)^n + \binom{n}{2}(0.5)^2(0.5)^{n-2} + ... + \binom{n}{n}(0.5)^n(0.5)^0$

$= 2^{n-1}(0.5)^n = \frac{1}{2}(2^n)(0.5)^n = \frac{1}{2}$, so for any value of n, 0.5 will be answer

(b) (½ point) What is the probability that after 100 tosses you end up with the same coin that you started with?

Solution: This means that you switched the coin once or thrice or... so that you get coin 2.
when you toss n times, prob. of tails odd is
$\binom{n}{1}(0.5)^1(0.5)^{n-1} + \binom{n}{3}(0.5)^3(0.5)^{n-3} + ... + \binom{n}{n}(0.5)^n(0.5)^0$

$= 2^{n-1}(0.5)^n = \frac{1}{2}(2^n)(0.5)^n = \frac{1}{2}$, so for any value of n, 0.5 will be answer

(c) (1 point) What if instead of fair coins you have identical biased coins with probability of heads $= p$ ($p \neq \frac{1}{2}$)?

Solution: for n number of tosses, you will end up with same coin if even time tail comes.
Either 0 or 2 or 4 or ...
Summing up all this, we get A = prob that we will end up with same coin(n=100 for this example)
A $= \binom{n}{0}(1-p)^0(p)^n + \binom{n}{2}(1-p)^2(p)^{n-2} + ... + \binom{n}{n}(1-p)^n(p)^0$(if n is even)

A $= \binom{n}{0}(1-p)^0(p)^n + \binom{n}{2}(1-p)^2(p)^{n-2} + ... + \binom{n}{n-1}(1-p)^{n-1}(p)^1$(if n is odd)

(1 - A) is probability that we will end up with different coin

3. You are dealt a hand of 5 cards from a standard deck of 52 cards which contains 13 cards of each suite (hearts, diamonds, spades and clubs).

(a) (½ point) What is the probability that you get an ace, a king, a queen, a joker and a 10 of the same suite? Let us call such a hand as the King's hand.

Solution: Hint: it is very small :-)
4/52 for ace lets say
then (1/51)*(1/50)*(1/49)*(1/48), So $4/\binom{52}{5}$
= 0.00000153907

(b) (½ point) Let n be the number of times you play this game. What is the minimum value of n so that the probability of having no King's hand in these n turns is less than $\frac{1}{e}$?

Solution: Hint: it is very large :-)
1/e=0.36787944117
king hand=0.00000153907
no king hand=0.99999846093
$(0.99999846093)^n < 0.36787944117$
taking log both the sides
n > -0.4342944819/-6.68410123e-7
n >= 649741

4. (1 point) A spacecraft explodes while entering the earth's atmosphere and disintegrates into 10000 pieces. These pieces then fall on your town which contains 1600 houses. Each piece is equally likely to fall on every house. What is the probability that no piece falls on your house (assume you have only one house in the town and all houses are of the same size and equally spaced - for example you can assume that the houses are arranged in a 40×40 grid).

Solution: ASSUMING MORE THAN ONE PIECE CAN FALL ON ONE HOUSE:
total pieces = 10000
prob of piece fall on particular house = 1/1600
Prob. that House is safe from i^{th} piece = $P(s_i)$ = 1 - 1/1600 = 1599/1600
Since they are independent, for all 10000 pieces it will be
$= P(s_1) * P(s_2) * ... * P(s_{10000})$
$= (1599/1600)^{10000}$
= 0.00192668582

5. What's in a name?

   (a) (½ point) Why is the hypergeometric distribution called so? (We understand what is geometric but what is "hyper"?)

   Solution:
   Hint 1: Find the ratio of $p_X(j)$ to $p_X(j-1)$ for a hypergeometric random variable.

   $$\text{p(j)}=\frac{\binom{a}{j}\binom{n-a}{n-j}}{\binom{N}{n}}$$

   $$\text{p(j-1)}=\frac{\binom{a}{j-1}\binom{n-a}{n-j+1}}{\binom{N}{n}}$$

   $$\frac{p(j)}{p(j-1)} = \frac{\binom{a}{j}\binom{n-a}{n-j}}{\binom{a}{j-1}\binom{n-a}{n-j+1}}$$

   $$= \frac{(j-1)!(a-j+1)!(n-j-1)!(N-a-n+j+1)!}{j!(a-j)!(n-j)!(N-a-n+j)!}$$
   $$= \frac{(a-j+1)(a-j+1)}{j(N-a-n+j)}$$

   Hint 2: Why is the geometric distribution called the geometric distribution?

for any geometric distribution random variable y=j, $p_Y(j) = (1-p)^{j-1}p$, all probabilities are like geometric series, every next step is multiplied by some number, (1-p) here.

In hypergeometric distribution, ratio of every 2 consecutive element j and j-1 is $\dfrac{(a-j+1)(a-j+1)}{j(N-a-n+j)}$, it is like dynamic multiplication, so it it called hypergeometric distribution.

(b) (½ point) Why is the negative binomial distribution called so?

Solution: if there are f failures and s sucesses, then we get s^{th} sucess in last trial if total trails is s+f,
$p_x(f) = \binom{s+f-1}{s-1}p^s(1-p)^f$
In Binomial distribution, we consider how many successes in n trails, while in -ve binomial, we consider how many trials we need to get s successes OR how many failures are there untill we get s successes. We normally define it by no. of failures. We can see that question is like exact opposite than binomial que. So we can say it negative binomial .

6. Consider a binomial random variable whose distribution $p_X(x)$ is fully specified by the parameters n and p.

(a) (½ point) What is the ratio of $p_X(j)$ to $p_X(j-1)$?

Solution: $\dfrac{p_X(j)}{p_X(j-1)} = \dfrac{\binom{n}{j}p^j(1-p)^{n-j}}{\binom{n}{j-1}p^{j-1}(1-p)^{n-j+1}}$

$$= \frac{\frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}}{\frac{n!}{(n-j+1)!(j-1)!}p^{j-1}(1-p)^{n-j+1}}$$

$$= \frac{(n-j+1)(p)}{(j)(1-p)}$$

(b) (½ point) Based on the above ratio can you find the value(s) of j for which $p_X(j)$ will be maximum ?

Solution: $\dfrac{p_X(j)}{p_X(j-1)} = \dfrac{(n-j+1)(p)}{(j)(1-p)}$

if (n-j+1)(p) < (j)(1-p), then surely $p_x(j)$ is not maximum. So possible case is (n-j+1)(p) ≥ (j)(1-p)

means j≤p(n-1)

p(n+1) may or may not be an integer

if it is not an integer, then only value of j such that $p_x(j)$ is maximum is ceil(p(n+1)-1)floor(p(n+1))

if it is integer, then there are two values possible, p(n+1) and p(n+1)-1

for every j > p(n+1),
$p_x(j) > p_x(j+1) > p_x(j+2) > ...\ p_x(n)$

for every j < p(n+1),
$p_x(j) > p_x(j-1) > p_x(j-2) > ...\ p_x(0)$

So, $p_x(j+k) < p_x(j) = p_x(j+1) > p_x(j-k)$ for all k

7. Consider a Poisson random variable whose distribution $p_X(x)$ is fully specified by the parameter λ.

(a) (½ point) What is the ratio of $p_X(j)$ to $p_X(j-1)$?

Solution: $p(j) = \dfrac{e^{-\lambda}\lambda^j}{j!}$

$p(j-1) = \dfrac{e^{-\lambda}\lambda^{(j-1)}}{(j-1)!}$

$$\frac{p(j)}{p(j-1)} = \frac{\dfrac{e^{-\lambda}\lambda^j}{j!}}{\dfrac{e^{-\lambda}\lambda^{(j-1)}}{(j-1)!}}$$

$$= \frac{\lambda}{j}$$

(b) (½ point) Based on the above ratio can you find the value(s) of j for which $p_X(j)$ will be maximum ?

Solution: based on λ and j's values, we can have 3 cases possible.
if $\lambda >$ j, then $p_x(j) > p_x(j-1)$
In this case, $p_x(j)$ may or may not be the maximum, because it dependds on value of $p_x(j+1)$

if $\lambda =$ j, then $p_x(j) < p_x(j-1)$
means, $p_x(j)$ is not maximum

if $\lambda >$ j, then $p_x(j) = p_x(j-1)$
for $\dfrac{p_x(j-1)}{p_x(j-2)} = \dfrac{\lambda}{j-1} = \dfrac{\lambda}{\lambda-1} > 1$
means $p_x(j-1) > p_x(j-2)$, same for any $k \geq 1$, $p_x(j-k) > p_x(j-k-1)$

for $\dfrac{p_x(j+1)}{p_x(j)} = \dfrac{\lambda}{j+1} = \dfrac{\lambda}{\lambda+1} < 1$
means $p_x(j+1) < p_x(j)$, same for any $k \geq 1$, $p_x(j-k) > p_x(j-k-1)$

So, whenever $j = \lambda, p_x(j-k-1) < p_x(j-1) = p_x(j) > p_x(j+k)$ for all k$\geq$1

if λ is not integer, then we can set j=floor(λ)

8. For each of the following random variables show that the sum of the probabilities of all the values that the random variable can take is 1?
   (a) (½ point) A negative binomial random variable whose distribution is fully specified by p (probability of success) and r (fixed number of desired successes)

Solution: $\binom{x-1}{r-1} = \dfrac{(x-1)(x-2)...(r)}{(x-r)!}$

$= (-1)^{x-r}\binom{r}{x-r}$

We know that $(p)^{-r} = (1-q)^{-r} = \Sigma_{x-r=0}^{\infty}\binom{-r}{x-r}(-q)^{x-r}$

$= \Sigma_{x-r=0}^{\infty}\binom{-r}{x-r}(1-p)^{x-r}$

$\Sigma_{x-r=0}^{\infty}\binom{-r}{x-r}(1-p)^{x-r}p^r$

$= p^{-r}p^r = 1$

(b) (½ point) A hypergeometric random variable whose distribution is fully specified by N (number of objects in the given source), a (number of favorable objects in the source) and n (size of the sample that you want to select)

Solution: $\Sigma_{r=0}^{m+n}\binom{m+n}{r}x^r = (1+x)^{m+n}$

$= (1+x)^m(1+x)^n$

$= (\Sigma_{i=0}^{m}\binom{m}{i}x^i)(\Sigma_{j=0}^{n}\binom{n}{j}x^j)$

$= \Sigma_{r=0}^{m+n}(\Sigma_{k=0}^{r}\binom{m}{k}\binom{n}{r-k})x^r$

Distribution:

$= \Sigma_0^N \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}}$

$= \frac{1}{\binom{N}{n}}\Sigma_0^N \binom{a}{x}\binom{N-a}{n-x}$

$= \frac{\binom{N}{n}}{\binom{N}{n}}$

$= 1$

(c) (½ point) A Poisson random variable whose distribution is fully specified by λ (i.e., arrival rate in unit time)

Solution: $\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}$
$= e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$
$= e^{-\lambda}e^{\lambda}$
$= 1$

9. There are 100 seats in a movie theatre. Customers can buy tickets online. Based on past data, the theatre owner knows that 5% of the people that book tickets do not show up (of course, he gets to keep the money they paid for the ticket). To make more money he decides to sell more tickets than the number of seats. For example, if he sells 102 tickets, then as long as at least 2 customers don't show up, he will be able to make extra

money while not dissatisfying any customers.

(a) (½ point) If he sells 105 tickets what is the probability that no customer would be denied a seat on arrival.

> **Solution:** No customer is denied, means at least 5 people didn't come
> p=0.05, n=105 here, so $\lambda = 5.25$
> = total - 0 or 1 or 2 or 3 or 4 people didn't come
> $= 1 - e^{-\lambda}(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!})$
> $= 1 - e^{-5.25}(\frac{\lambda^0}{1} + \frac{\lambda^1}{1} + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \frac{\lambda^4}{24})$
> $= 1 - e^{-5.25}(72.80125)$
> = 1 - 0.397773682
> = 0.602226318

(b) (½ point) What is the maximum number of seats that he can sell so that there is at least a 90% chance that every customer will get a seat on arrival?

> **Solution:** If you can find an analytical solution, then great! If you can't then you can write a program to find the answer. In that case, paste the code here.

```
import math
sum=0
i=0
while 1-sum > 0.9:
  sum=0
  for j in range(i):
    lambdaa=(100+i)*0.05
    sum=sum+math.e**(-lambdaa)*((lambdaa**j)/math.factorial(j))
  print(100+i,1-sum)
  i=i+1
i=i-2
print("ANSWER IS : ",100+i)
Output:
100 1
101 0.9935906665537436
102 0.9628098459503547
103 0.8874263049960854
ANSWER IS :  102
```

> Explanation :
>
> $1 - e^{-\lambda}(\frac{\lambda^0}{0!}) \geq 0.9$ This hold true

1 - $e^{-\lambda}(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!}) \geq 0.9$ This hold true, means we can allow to sell one more ticket and still prob. of no one is denied is ≥ 0.9, we go till this continues till prob is ≥ 0.9

ANSWER IS 102 for this problem.

(c) (1 point) Suppose he makes a profit of 5 INR for every satisfied customer and a loss of 50 INR (as penalty) for every dissatisfied customer (i.e., a customer who does not get a seat). What is his expected gain/loss if he sells 105 tickets?

Solution: if satisfied customers are ≤ 100, no deduction is there, all those customer who show up will make 5rs.
For values >100, for each dissatisfied customer we will fined by 50 rs, analytically this is large expression, so writing in code.

```
import numpy as np
from math import factorial
lambdaa=105*0.05
total=0
a=105
for i in range(0,a+1):
  if i>=5:
    temp=(a-i)*5
  else:
    temp=500-(a-i-100)*50
  total = total + temp*np.exp(-lambdaa)*(lambdaa**i)/factorial(i)
print(total)
```

OUTPUT : 456.25687

P.S.: This is what many international airlines do. They often sell more tickets than the number of available seats thereby profiting twice from the same seat!

10. In recently conducted elections, there were a total of 100 counting centres. The losing party claims that some of the counting machines were rigged by a hacker. To verify these allegations, the Election Commission decides to manually recount the votes in some centres (obviously, manual recounting in all centres would be prohibitively expensive so it can only do so in some centres).

(a) (½ point) If 5% of the machines were rigged then in how many centres should recounting be ordered so that there is a 50% chance that rigging would be detected (i.e., in at least one of the selected centres the number of votes counted manually will not match the number of votes counted by the machine)

Solution: If you can find an analytical solution, then great! If you can't then you can write a program to find the answer. In that case, paste the code here.
At least 1 means (total - exact 0)
total prob must be 1
prob of 0 machine out of x machines found being rigged is $\binom{x}{0}(0.95)^x(0.05)^0$

So, 1 - $\binom{x}{0}(0.95)^x(0.05)^0 = 0.5$

$(0.95)^x = 0.5$
x $log(.95)$ = -0.30102999566

$$x = \frac{-0.30102999566}{-0.02227639471}$$

x = 13.5134073345, So they should detect 14 machines(because 13.51 is more closer to 14 than 13)

(b) (½ point) If the hacker knows that the Election commission can only afford to do a recounting in 10 randomly sampled centres then what is the maximum number of machines he/she can rig so that there is less than 50% chance that the rigging will get detected.

Solution: If you can find an analytical solution, then great! If you can't then you can write a program to find the answer. In that case, paste the code here.
let $x_1 = \dfrac{x}{100}$
At least 1 means (total - exact 0)
total prob must be 1
prob of 0 machine out of 10 machines found being rigged is $\binom{10}{0}(1-x_1)^{10}(x_1)^0$

So, 1 - $\binom{10}{0}(1-x_1)^{10}(x_1)^0 < 0.5$

$(1-x_1)^{10} < 0.5$
$1-x_1 < 0.93303299153$
$x_1 < 0.06696700846$
x < 6.696700846
So maximum of 6 machines should be rigged :)

11. (1 point) Amar and Bala are two insurance agents. Their manager has given them a list of 40 potential customers and a target of selling a total of 5 policies by the end of the day. They decide to split the list in half and each one of them talks to 20 people on the list. Amar is a better salesman and has a probability p_1 of selling a policy when he talks to customer. On the other hand, Bala has a probability p_2 ($< p_1$) of selling a policy when he talks to a customer. The customers do not know each other and hence

one customer does not influence another. What is the probability that they will be able to meet their target by the end of the day? (it doesn't matter if Amar sells more policies than Bala or the other way round - the only thing that matters is that the total should be **exactly** 5).

Solution: I like this problem. So easy to state. So easy to relate to. But not easy to solve :-) The beauty is that if $p_1 = p_2$ then the problem becomes trivial but when you make $p_1 \neq p_2$ then it becomes hard! Further, imagine that instead of Amar and Bala what would happen if we had Amar, Bala and Chandu!
First we need to choose 20 people that will be checked by Amar.
$\binom{40}{20}$
There are 6 possibilities :
Amar sells 5 , Bala sells 0
Amar sells 4 , Bala sells 1
Amar sells 3 , Bala sells 2
Amar sells 2 , Bala sells 3
Amar sells 1 , Bala sells 4
Amar sells 0 , Bala sells 5
For Amar sells 5 , Bala sells 0 ,

$\binom{20}{5}p_1^5(1-p_1)^{15} + \binom{20}{0}p_2^0(1-p_2)^{20}$

Generalizing for all cases, Answer will be :

$\binom{40}{20}[\Sigma_{i=0}^{5}[\binom{20}{i}p_1^i(1-p_1)^{20-i} + \binom{20}{5-i}p_2^{5-i}(1-p_2)^{15+i}]]$

12. Find the expectation and variance of the following discrete random variables
    (a) (½ point) A binomial random variable whose distribution is fully specified by p (probability of success) and n (number of trials)

Solution: Expectation $= \Sigma_{x=0}^{n} x\binom{n}{x}p^x(1-p)^{n-x}$

$= \Sigma_{x=1}^{n}\dfrac{n!}{(n-x)!(x-1)!}p^x(1-p)^{n-x}$

Let a=x-1, b=n-1
E[X] = (a+1)p$\Sigma_{a=0}^{b}\dfrac{b!}{a!(a-b)!}p^a(1-p)^{b-a}$

= np$\Sigma_{a=0}^{b}\dfrac{b!}{a!(a-b)!}p^a(1-p)^{b-a}$

Let q=p and r=(1-p), then

$\frac{E[X]}{np} = \Sigma_{a=0}^{b} \frac{b!}{a!(a-b)!} q^a r^{b-a}$

$= (q+r)^a$(binomial thm)
$= (p+1-p)^a$
$= 1$
E[X] = np

Variance:

$E[X^2] = E[X(X-1)+X]$
$E[X(X-1)] = \Sigma_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!(x)!} p^x (1-p)^{n-x}$

$E[X(X-1)] = n(n-1)p^2 \Sigma_{x=2}^{n} \frac{(n-2)!}{(n-x)!(x-2)!} p^(x-2)(1-p)^{n-x}$

Let a=x-2, b=n-2
$E[X(X-1)] = n(n-1)p^2 \Sigma_{x=2}^{n} \frac{(b)!}{b-a)!(a)!} p^(a)(1-p)^{b-a}$

$E[X(X-1)] = n(n-1)p^2$

$E[X^2] = E[X(X-1)] + E[X]$
Variance = $E[X^2]$ - $E[X]^2$
$= n(n-1)p^2 + np - np^2$
= np(1-p)

(b) (½ point) A negative binomial random variable whose distribution is fully specified by p (probability of success) and r (fixed number of desired successes)

Solution: let's take X as -ve binomial RV

$p_X(n) = \binom{n-1}{r-1}\ p^r(1-p)^{n-r}$, n is # of trials

# of failure = j, then , r+j =n

$p_X(j) = \binom{r+j-1}{j}\ p^r(1-p)^j$

E(X) = $\sum_{j=0}^{\infty} j\ \binom{r+j-1}{j}\ p^r(1-p)^j$

$= \sum_{j=1}^{\infty}\ r \frac{(r+j-1)!}{(r(r-1)!(j-1)!)}\ p^r(1-p)^j$

$= (rp^r) \sum_{j=1}^{\infty} \binom{r+j-1}{j-1} (1-p)^j$

taking t=j-1

$= rp^r(1-p)(\sum_{t=0}^{\infty} \binom{r+t}{t} (1-p)^t)$

$= \frac{(1-p)}{p}r$

E(X^2)

$= \sum_{j=1}^{\infty} \frac{(r+j-1)!}{(r(r-1)!(j-1)!)} p^r(1-p)^j rj$

$= \sum_{j=1}^{\infty} rj \frac{(r+j-1)!}{(r!(j-1)!)} p^r(1-p)^j rj$

taking s = j-1, t = r+1

$= \sum_{s=0}^{\infty} r\ (s+1)(s+t-1)!/((t-1)!(s)!)\ p^{t-1}(1-p)^{1+s}$

$= \frac{r(1-p)}{p}\ (\sum_{s=0}^{\infty} s \binom{s+t-1}{s} p^t(1-p)^s + \sum_{s=0}^{\infty} \binom{s+t-1}{s} p^t(1-p)^s)$

$= \frac{r(1-p)}{p}\ (\frac{t(1-p)}{p} + 1)$

= (rt$\frac{(1-p)^2}{p^2} + \frac{r(1-p)}{p}$)

= (r(r+1)$\frac{(1-p)^2}{p^2} + \frac{r(1-p)}{p}$)

Var(X) $= E(X^2) - (E(X))^2$

= (r(r+1)$\frac{(1-p)^2}{p^2} + \frac{r(1-p)}{p}$) - $\frac{r^2(1-p)^2}{p^2}$

$= \frac{r(1-p)}{p^2}$

(c) (½ point) A hypergeometric random variable whose distribution is fully specified by N (number of objects in the given source), a (number of favorable objects in the source) and n (size of the sample that you want to select)

Solution: let X be hypergeometric random variable, N is number of objects, n is size of sample, a is favourable objects, X=j gives prob of j successes in n trails.

$p_X(j) = \frac{\binom{a}{j}}{\binom{N-a}{n-j}}\binom{N}{n}$

E(X) $= \Sigma_{j=0}^{a}\frac{\binom{a}{j}}{\binom{N-a}{n-j}}\binom{N}{n}j$

$= \Sigma_{j=1}^{a}\frac{\binom{a}{j}}{\binom{N-a}{n-j}}\binom{N}{n}j$

$=\frac{an}{N}\binom{N-a}{n-j}\frac{(n-1)!(N-n)!}{(N-1)!}\Sigma_{j=1}^{a}\frac{(a-1)!}{(a-j)!(j-1)!}$

$= \frac{an}{N}\Sigma_{j=1}^{a}\frac{\binom{a-1}{j-1}\binom{N-a}{n-j}}{\binom{N-1}{n-1}}$

$= \frac{an}{N}\Sigma_{y=0}^{a-1}\frac{\binom{a-1}{y}\binom{N-a}{n-y-1}}{\binom{N-1}{n-1}}$,j=y+1

$= \frac{an}{N}$ because $\binom{N-1}{n-1} = \Sigma_{y=0}^{a-1}\binom{a-1}{y}\binom{N-a}{n-y-1}$

We know that E[X^2] = E[X(X-1)]+E[X]

for j=0 and 1, answer will be zero.

$\Sigma_{j=2}^{a}j(j-1)\frac{\binom{a}{j}}{\binom{N-a}{n-j}}\binom{N}{n}$

$=\frac{an(a-1)(n-1)}{N(N-1)}\binom{N-a}{n-j}\frac{(n-2)!(N-n)!}{(N-2)!}\Sigma_{j=2}^{a}\frac{(a-2)!}{(a-j)!(j-2)!}$

$=\frac{an(a-1)(n-1)}{N(N-1)}\Sigma_{j=2}^{a}\frac{\binom{a-2}{j-2}\binom{N-a}{n-j}}{\binom{N-2}{n-2}}\Sigma_{j=2}^{a}\frac{(a-2)!}{(a-j)!(j-2)!}$

$=\frac{an(a-1)(n-1)}{N(N-1)}\Sigma_{y=0}^{a-2}\frac{\binom{a-2}{y}\binom{N-a}{n-y-2}}{\binom{N-2}{n-2}}\Sigma_{j=2}^{a}\frac{(a-2)!}{(a-j)!(j-2)!}$

$$=\frac{an(a-1)(n-1)}{N(N-1)}$$

$$\text{Var(X)} = \frac{an(a-1)(n-1)}{N(N-1)} + (\frac{an}{N})^2 + \frac{an}{N}$$

$$= \frac{an(N-a)(N-n)}{N^2(N-1)}$$

(d) (½ point) A Poisson random variable whose distribution is fully specified by λ (i.e., arrival rate in unit time)

Solution: Expectation:

$$\Sigma_{i=1}^{\infty}\frac{ie^{-\lambda}\lambda^i}{i!}$$
$$= e^{-\lambda}\Sigma_{i=1}^{\infty}\frac{i\lambda^i}{i!}$$
$$= e^{-\lambda}\Sigma_{i=1}^{\infty}\frac{\lambda^i}{(i-1)!}$$
$$= e^{-\lambda}\lambda\Sigma_{j=0}^{\infty}\frac{\lambda^j}{j!}\text{(j=i-1)}$$
$$= e^{-\lambda}\lambda e^{\lambda}$$
$$= \lambda$$

Variance:

$$E[X^2] = E[X(X-1)+X]$$
$$E[X(X-1)] = \Sigma_{i=1}^{\infty}\frac{i(i-1)e^{-\lambda}\lambda^i}{i!}$$
$$= e^{-\lambda}\Sigma_{i=1}^{\infty}\frac{i(i-1)\lambda^i}{i!}$$
$$= e^{-\lambda}\Sigma_{i=1}^{\infty}\frac{\lambda^i}{(i-2)!}$$
$$= e^{-\lambda}\lambda^2\Sigma_{j=0}^{\infty}\frac{\lambda^j}{j!}\text{(j=i-2)}$$
$$= e^{-\lambda}\lambda^2 e^{\lambda}$$
$$= \lambda^2$$
$$E[X^2] = E[X(X-1)] + E[X]$$

Variance = $E[X^2]$ - $E[X]^2$
$= \lambda^2 + \lambda - \lambda^2$
$= \lambda$

13. (1 point) Consider a language which has only 5 words w_1, w_2, w_3, w_4, w_5. The way you construct a sentence in this language is by selecting one of the 5 words with probabilities p_1, p_2, p_3, p_4, p_5 respectively $(\sum_{i=1}^{5} p_i = 1)$. This word will be the first word in the sentence. You will then repeat the same process for the second word and continue to form a sentence of arbitrary length. As should be obvious, the i-th word in the sentence is independent of all words which appear before it (and after it). What is the expected position at which the word w_2 will appear for the first time?

Solution: p(p_2=1) = p_2 occurs at first place = p_2
p(p_2=2) = p_2 occurs at second place = $(1 - p_2) * p_2$
p(p_2=3) = p_2 occurs at third place = $(1 - p_2)^2 * p_2$
...
p(p_2=n) = p_2 occurs at nth place = $(1 - p_2)^{n-1} * p_2$

E[X] = x*p(x)
= 1*p(1) + 2*p(2) + 3*p(3) + ... + n*p(n)
= 1*p_2 + 2*$(1 - p_2) * p_2$ + ... + n*$(1 - p_2)^n * p_2$

(1-p_2)*E[X] = 1*$(1 - p_2) * p_2$ + 2*$(1 - p_2)^2 * p_2$ + ... + n*$(1 - p_2)^n * p_2$

E[X]-(1-p_2)*E[X] = p_2 + $p_2(1 - p_2)$ + ... + $p_2(1 - p_2)^n$
p_2*E[X] = p_2 + $p_2 \dfrac{1 - p_2}{1 - 1 + p_2}$
E[X] = $\dfrac{1}{p_2}$ (you can replace p_2 by $p_1 + p_3 + p_4 + p_5$)

14. Two fair dice are rolled. Let X be the sum of the two numbers that show up and let Y be the difference between the two numbers that show up (number on first dice minus number on second dice).

(a) (½ point) Show that $E[XY] = E[X]E[Y]$

Solution:

	11	12	13	14	15	16
x	2	3	4	5	6	7
y	0	-1	-2	-3	-4	-5
xy	0	-3	-8	-15	-24	-35
	21	22	23	24	25	26
x	3	4	5	6	7	8
y	1	0	-1	-2	-3	-4
xy	3	0	-5	-12	-21	-32
	31	32	33	34	35	36
x	4	5	6	7	8	9
y	2	1	0	-1	-2	-3
xy	8	5	0	-7	-16	-27
	41	42	43	44	45	46
x	5	6	7	8	9	10
y	3	2	1	0	-1	-2
xy	15	12	7	0	-9	-20
	51	52	53	54	55	56
x	6	7	8	9	10	11
y	4	3	2	1	0	-1
xy	24	21	16	9	0	-11
	61	62	63	64	65	66
x	7	8	9	10	11	12
y	5	4	3	2	1	0
xy	35	32	27	20	11	0

P(X=2):1/36
P(X=3):2/36
P(X=4):3/36
P(X=5):4/36
P(X=6):5/36
P(X=7):6/36
P(X=8):5/36
P(X=9):4/36
P(X=10):3/36
P(X=11):2/36
P(X=12):1/36

P(Y=-5):1/36
P(Y=-4):2/36
P(Y=-3):3/36
P(Y=-2):4/36
P(Y=-1):5/36
P(Y=0):6/36
P(Y=1):5/36
P(Y=2):4/36
P(Y=3):3/36

P(Y=4):2/36
P(Y=5):1/36

E(X):((2*1)+(3*2)+(4*3)+(5*4)+(6*5)+(7*6)+(8*5)+(9*4)+(10*3)+(11*2)+(12*1))/36
=252/36

E(Y):((-5*1)+(-4*2)+(-3*3)+(-2*4)+(-1*5)+(0*36)+(5*1)+(4*2)+(3*3)+(2*4)+(1*5))/36
=0

E[XY]=(every xy is unique and -xy is there for every xy, so it sums up to 0, not calculating here)
So, E[XY]=E[X]*E[Y]

(b) (½ point) Are X and Y independent? Explain your answer.

Solution: $P_{X=2,Y=0} = \frac{1}{36}$
$P(X=2)P(Y=0) = \frac{1}{36}\frac{1}{6}! = \frac{1}{36}$
Means they are not independent by definition, because for every value of x and y, $p_{x=i,y=j} = p_{x=i}p_{y=j}$ should satisfy.

15. The *martingale doubling system* is a betting strategy in which a player doubles his bet each time he loses. Suppose that you are playing roulette in a fair casino where the roulette contains only 36 numbers (no 0 or 00). You bet on red each time and hence your probability of winning each time is 1/2. Assume that you enter the casino with 100 rupees, start with a 1-rupee bet and employ the martingale system. Your strategy is to stop as soon as you have won one bet or you do not have enough money to double the previous bet.

(a) (½ point) Under what condition will you not have enough money to double your previous bet?

Solution: you want to bet but you dont have money, so it means that you have not won till now.
First lose : 1 INR $----->$ 100-1 = 99
Second lose : 2 INR $----->$ 99-2 = 97
Third lose : 4 INR $----->$ 97-4 = 93
Fourth lose : 8 INR $----->$ 93-8 = 85
Fifth lose : 16 INR $----->$ 85-16 = 69
Sixth lose : 32 INR $----->$ 69-32 = 37
Now you don't have 64 INR :(
Since all events are independent and prob of lose is 0.5, prob. of you not have

enough money to double your previous bet is $(0.5)^6$.
Means if you lose 6 times consecutively, then i have to stop.

(b) (½ point) What would your expected winnings be under this system? (for every 1 INR you bet you get 2 INR if you win)

Solution: First win : 1 INR $---->$ 100+1 = 101
Second win : 2 INR $---->$ 99+2 = 101(prob of winning in second game)
Third win : 4 INR $---->$ 97+4 = 101
Fourth win : 8 INR $---->$ 93+8 = 101
Fifth win : 16 INR $---->$ 85+16 = 101
Sixth win : 32 INR $---->$ 69+32 = 101
You stand up when you lose sixth time, and you lose amount is -63 INR.
P(Firstwin) : +1 winnings : 0.5
P(Secondwin) : +1 winnings : $(0.5)^2$
P(Thirdwin) : +1 winnings : $(0.5)^3$
P(Fourthwin) : +1 winnings : $(0.5)^4$
P(Fifthwin) : +1 winnings : $(0.5)^5$
P(Sixthwin) : +1 winnings : $(0.5)^6$
P(SixthLose) : -63 winnings : $(0.5)^6$
E[Winnings] = $1((0.5) + (0.5)^2 + (0.5)^3 + (0.5)^4 + (0.5)^5 + (0.5)^6) - 63(0.5)^6$
= 0

16. (1 point) You have 800 rupees and you play the following game. A box contains two green hats and two red hats. You pull out the hats out one at a time without replacement until all the hats are removed. Each time you pull out a hat you bet half of your present fortune that the pulled hat will be a green hat. What is your expected final fortune?

Solution: all possible outcomes are GGRR,GRGR,GRRG,RGGR,RGRG,RRGG

GGRR=1200,1800,900,450(means if we get green first, then we get 800+400=1200, then again green comes, 1800, then two times red comes, so 900 and then 450.)
GRGR=1200,600,900,450
GRRG=1200,600,300,450
RGGR=400,600,900,450
RGRG=400,600,300,450
RRGG=400,200,300,450
P(GGRR)=$P(G_1)P(G_2|G_1)P(R_1|G_1G_2)P(R_2|G_1G_2R_1) = \frac{1}{2}\frac{1}{3}\frac{1}{1}\frac{1}{1} = \frac{1}{6}$
P(GRGR)=$\frac{1}{2}\frac{2}{3}\frac{1}{2}\frac{1}{1} = \frac{1}{6}$

P(GRRG)=$\frac{1}{2}\frac{2}{3}\frac{1}{2}\frac{1}{1} = \frac{1}{6}$
P(RGGR)=$\frac{1}{2}\frac{2}{3}\frac{1}{2}\frac{1}{1} = \frac{1}{6}$
P(RGRG)=$\frac{1}{2}\frac{2}{3}\frac{1}{2}\frac{1}{1} = \frac{1}{6}$
P(RRGG)=$\frac{1}{2}\frac{1}{3}\frac{1}{1}\frac{1}{1} = \frac{1}{6}$
Expected value = $\Sigma x P(x)$
$= \frac{1}{6}(6 * 450) = 450$

17. There are 6 dice. Each dice has 0 on five sides and on the 6th side it has a number between 1 and 6 such that no two dice have the same number (i.e, if dice 2 has the number 3 on the 6th side then no other dice can have the number 3 on the sixth side). All the 6 dice are rolled and let X be the sum of the numbers on the faces which show up.

    (a) Find $E[X]$ and $Var(X)$

    Solution: X can take values from 0 to 21
    let X_i be RV representing number on i^{th} die.(0,0,0,0,0,i)
    X=$\Sigma_{i=1}^{6} X_i$

    E(X)=E($\Sigma_{i=1}^{6} X_i$)

    $=E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) + E(X_6)$

    $=(0(\frac{5}{6}) + 1(\frac{1}{6})) + (0(\frac{5}{6}) + 2(\frac{1}{6})) + (0(\frac{5}{6}) + 3(\frac{1}{6})) + (0(\frac{5}{6}) + 4(\frac{1}{6})) + (0(\frac{5}{6}) + 5(\frac{1}{6})) + (0(\frac{5}{6}) + 6(\frac{1}{6}))$ =3.5
    Var(X) = $E(X^2) - [E(X)]^2$
    E[X^2] = $\Sigma_{i=0}^{6} E(X_i^2) + \Sigma_{k=0}^{6}\Sigma_{j=1,j!=k}^{6} E(X_k X_j) = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) + 2(1(2 + 3 + 4 + 5 + 6) + \frac{1}{36}(2)(3 + 4 + 5 + 6) + 3(4 + 5 + 6) + 4(5 + 6) + 56)$

    $= 24.89$

    $var(X) = 24.89 - 3.5^2 = 12.64$

    (b) Suppose you are the owner of a casino which has a game involving these 6 dice. The players can bet on the sum of the numbers that will show up. If I bet on the number 21 what should the payoff be so that the bet looks as attractive as possible to me but in the long run the casino will not lose (e.g., in the game of roulette a payoff of 1:35 looks attractive while still protecting the interests of the casino).

> **Solution:** taking minimum transfer of currency as 1rs, then we will take 1rs from player, and will give x otherwise.
> prob. of sum being 21 is $(\frac{1}{6})^6$
> Then if we make profit 0, expected value should be:
>
> $$(\frac{1}{6})^6 x + (1 - \frac{1}{6})^6 = 0$$
>
> We want to make profit such that it looks almost equal to 0, so minimum x will be:
>
> $$(\frac{1}{6})^6 x + (1 - \frac{1}{6})^6 < 0$$
>
> x<46656, so 46655rs is best possible maximum winning value possible for x here.

18. (1 point) A friend invites you to play the following game. He will toss a fair coin till the first heads appears. If the first head appears on the k-th toss then he will give you 2^k rupees. His condition is that you should first pay him a 100 million rupees to get a chance to play this game. Would you be willing to pay this amount to get a chance to play this game? Explain your answer.

> **Solution:** Hint: What do you "expect" from your friends?
> let k be the toss number at which first head appears
> k=1 has prob H = 0.5
> k=2 has prob TH = $(0.5)^2$
> k=3 has prob TTH = $(0.5)^3$
> ...
> k=n has prob $T^{(n-1)}H = (0.5)^n$
> $E[K] = \Sigma(k * p(k))$
> = 1(0.5) + 2*$(0.5)^2$ + ... + n*$(0.5)^n$
> 0.5*$E[K]$ = 1*$(0.5)^2$ + $2 * (0.5)^3$ + ... + n*$(0.5)^{n+1}$
> $E[K]$-0.5*$E[K]$ = (0.5) + $(0.5)^2$ + $(0.5)^3$ + ... + $(0.5)^n$ = 1
> $E[K] = 2$
> So expected value I will get is $2^2 = 4 << 100$ million
> So I will NOT play this game.

19. Suppose you are playing with a deck of 20 cards which contain 10 red cards and 10 black

cards. The dealer opens the cards one by one but you cannot see a card before he opens it. Before he opens a card you are supposed to guess the color of the card.

(a) (½ point) If you are guessing randomly then what is the expected number of correct guesses that you will make?

> **Solution:** since both black and red has equal probability, and we are guessing randomly, P(card=red)=0.5, P(card=black=0.5)
> Now P(guessR | cardR) = $\dfrac{P(guessR \cap cardR)}{P(cardR)}$ = 0.25 , same for black
> prob. of correct guess is 2*(0.25) = 0.5, incorrect guess is 1-0.5 = 0.5
> Prob of 0 correct guess is : $\binom{20}{0}(0.5)^{20}(0.5)^0$
> Prob of 1 correct guess is : $\binom{20}{1}(0.5)^{19}(0.5)^1$
> ...
> Prob of 20 correct guess is : $\binom{20}{20}(0.5)^{0}(0.5)^{20}$
> Expected value = as proved in Q12, expectation of Bernoulli random variable with trial n and prob p is np, so 0.5*20 = 10

(b) (½ point) Can you think of a better strategy than random guessing?

> **Solution:**
> Choose card that is less opened till now, since this is without replacement case, you will always have more prob. on cards which are not came out more frequently.

(c) (1 point) What is the expected number of correct guesses under this intelligent strategy? It is hard (but possible) to come up with an analytical solution. However, it is easy to do a simulation. Write a program to play this game a 1000 times and note down the number of correct guesses each time. Based on this simulation calculate the estimate number of correct guesses.

> **Solution:** First paste your code here and then write down the expected number of correct guesses. I am not looking for the analytical solution.

```
#Q19c
import random
total=0
for j in range(1000):
  red=10
  black=10
  a=[0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1] #0 is red, 1 is black
  correct=0
  for i in range(20,0,-1):
```

```
    if red>black:
      guess=0
    else:
      guess=1
    #guess=random.randint(0,1)
    q=random.randint(0,i-1)
    if a[q] is 0:
      red=red-1;
    else:
      black=black-1;
    if guess is a[q]:
      correct=correct+1
    del a[q]
  total=total+correct
total=total/1000
print(total)
```

12.332 is expected correct guesses value by my program

20. Suppose every morning the front page of Chennai Times contains a photo of exactly one of the n celebrities of Kollywood. You are a movie buff and collect these photos. Of course, on some days the paper may publish the photograph of a celebrity which is already in your collection. Suppose you have already obtained photos of $k-1$ celebrities. Let X_k be the random variable indicating that number of days you have to wait before you obtain the next new picture (after obtaining the first $k-1$ pictures).

    (a) (½ point) Show that X_k has a geometric distribution with $p = (n-k+1)/n$

Solution: prob of a celebrity's photo is $\frac{1}{n}$, because n celebrities are there

prob. that photo is in your collection is $\frac{k-1}{n}$ since you have (k-1) photos

So prob. that photo is not in your collection is 1-$\frac{k-1}{n} = \frac{n-k+1}{n}$

you have to wait i number of days to get new photo means for (i-1) times you got old photo and at i^{th} time you got new one.

$= (\frac{k-1}{n})^{i-1}\frac{n-k+1}{n}$

$= (1-p)^{i-1}p$

So here p is $\dfrac{n-k+1}{n}$

(b) (½ point) Simulate this experiment with 50 celebrities. Carry out a large number of simulations and estimate the expected number of days required to get the photos of the first 25 celebrities and the next 25 celebrities. Paste your code and estimates of the two expected values below.

Solution:

```
import random
a=[]
for i in range(1,51):
  a.append(i)
s1=0
s2=0
for i in range(100000):
  collection=[]
  collection=set(collection)
  count=0
  while len(collection)!=25:
    p=random.randint(1,50)
    collection.add(p)
    count=count+1
  s1=s1+count
  count=0
  while len(collection)!=50:
    p=random.randint(1,50)
    collection.add(p)
    count=count+1
  s2=s2+count
print(s1/100000,s2/100000)
OUTPUT:34.16824 190.85779
```

21. You want to test a large population of N people for COVID19. The probability that a person may be infected is p and it is the same for every person in the population. Instead of independently testing each person you decide to do pool testing wherein you collect the blood samples of k people and test them together (N is divisible by k). If the test is negative then you conclude that all are negative and no further tests are required for these k people. However, if the test is positive then you do k more tests (one for each person).

(a) (½ point) What is the probability that the test for a given pool of k people will be positive?

Solution: Means at least one person is positive in k people = 1 - (0 person positive), since they are independent, k people being negative will be $(1-p)^k$
So overall prob is 1 - $(1-p)^k$

(b) (1 point) What is the expected number of tests required under this strategy to conclusively test the entire population?

Solution:
let X be random variable denoting number of tests required
we need at least $\frac{N}{k}$ tests, plus we test k people again if group tests positive
so X = $\frac{N}{k} + k * P$, P is number of groups where blood sample tested positive.

E(X) = $\frac{N}{k} + k * E[P]$

P is RV with binomial distribution.

there are $\frac{N}{k}$ groups with 1 - $(1-p)^k$ probability.

Expectation of binomial RV = np
$= \frac{N}{k}(1-(1-p)^k)$

E(X) = $\frac{N}{k} + N(1-(1-p)^k)$

(c) (½ point) When would such a pooling strategy be beneficial?

Solution: its beneficial when we can judge with less than N tests

$\frac{N}{k} + N(1-(1-p)^k) <$ N
$\frac{1}{k} <$ (1-p)^k

22. A family decides to have children until they have a girl or until there are 3 children, whichever happens first. Let X be the random variable indicating the number of girls in the family and let Y be the random variable indicating the number of boys in the family. Assume that the probability of having a girl child is the same as that of having a boy child.

(a) (½ point) Find $E[X]$ and $Var[X]$.

Solution: sample space = $[g(0.5), bg(0.5)(0.5), bbg(0.5)(0.5)(0.5), bbb(0.5)(0.5)(0.5)]$
X can be 1 or 0
prob. of X = 1 is (0.5 + 0.25 + 0.125) / 1 = 0.8725
$E[X]$ = 1*P(X=1) + 0*P(X=0)
= 0.8725
$E[X^2] = 1^2 * P(X = 1) + 0^2 * P(X = 0)$
= 0.8725
$Var[X] = E[X^2]$ - $E[X]^2$
= 0.8725 - 0.7656 = 0.109375

(b) (½ point) Find $E[Y]$ and $Var[Y]$.

Solution: sample space = [g(0.5) , bg(0.5)(0.5) , bbg(0.5)(0.5)(0.5) , bbb(0.5)(0.5)(0.5)]
Y can be 0,1,2,3
prob. of Y = 3 $---->$ (0.125)
prob. of Y = 2 $---->$ (0.125)
prob. of Y = 1 $---->$ (0.25)
prob. of Y = 0 $---->$ (0.5)
$E[Y]$ = 3*P(Y=3) + 2*P(Y=2) + 1*P(Y=1) + 0*P(Y=0)
= 0.875
$E[Y^2] = 3^2 * P(Y = 3) + 2^2 * P(Y = 2) + 1^2 * P(Y = 1) + 0^2 * P(Y = 0)$
= 1.875
$Var[Y] = E[Y^2]$ - $E[Y]^2$
= 1.875 - 0.7656 = 1.109

23. Suppose n people bring their umbrellas to a meeting. While returning back each one randomly picks up an umbrella and walks out.

Let X_i be the random variable indicating whether the i-th person walked out with his own umbrella (i.e., the umbrella that he walks out with is the same as the umbrella that he walked in with).

(a) (½ point) Find $E[X_i^2]$

Solution: i can take values 0(does not takes own umbrella) or 1(takes own umbrella)
$E[X_i^2] = 0^2P(0) + 1^2P(1)$
$= 0^2P(0) + 1^2\dfrac{(n-1)!}{n!}$
$= \dfrac{1}{n}$

(b) (½ point) Find $E[X_iX_j]$ (for $i \neq j$)

Solution: $X_i = 1$ means i^{th} person took his own umbrella, 0 means he didn't
$E[X_iX_j] = 0^2P_{X_i,X_j}(0,0)+(0)(1)P_{X_i,X_j}(0,1)+(1)(0)P_{X_i,X_j}(1,0)+1^2P_{X_i,X_j}(1,1)$
$= \dfrac{1}{n(n-1)}$

Let S be the random variable indicating the number of people who walk out with their own umbrella.

(c) (½ point) Find E[S]

Solution: S=$\Sigma_{i=0}^{n} X_i$, $X_i = 1$ if person take his own umbrella
E[S] = E($\Sigma_{i=1}^{n} X_i$)
$= \Sigma_{i=0}^{n} E[X_i]$
$= \Sigma_{i=0}^{n}(0P(0) + 1P(1))$

$= \Sigma_{i=0}^{n}(0P(0) + \dfrac{(n-1)!}{n!})$

$= \Sigma_{i=0}^{n}\dfrac{1}{n}$

$= 1$

(d) (½ point) Var[S]

Solution: Var(S) = $E(S^2) - [E(S)]^2$
E[X^2] = $\Sigma_{i=0}^{6} E(X_i^2) + \Sigma_{k=0}^{6}\Sigma_{j=1,j!=k}^{6} E(X_kX_j)$

$=n(\dfrac{1}{n}) + 2\binom{n}{2}\dfrac{(n-2)!}{n!}$

=2
Var(S) = 2 - 1 = 1

24. (½ point) The covariance of two random variables X and Y is defined as $Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$. Show that if X and Y are independent then $Cov(X,Y) = 0$.

Solution: $E[(X - E[X])(Y - E[Y])]$
$= E[XY - YE[X] - XE[Y] - E[X]E[Y]]$
$= E[XY] - E[YE[X]] - E[XE[Y]] + E[E[X]E[Y]]$
$= E[XY] - E[YE[X]] - E[XE[Y]] + E[E[XY]]$
$= E[XY] - E[YE[X]] - E[XE[Y]] + E[XY]$
$= E[XY] - E[YE[X]] - E[XE[Y]] + E[XY]$

$= 2E[XY] - E[YE[X]] - E[XE[Y]]$
$= 2E[XY] - E[Y]E[X] - E[X]E[Y]$
$= 0$

25. Consider a die which is loaded such that the probability of a face is proportional to the number on the face. Let X be the random variable indicating the outcome of the die.

(a) (½ point) Find $E[X]$ and $Var(X)$

Solution: let p(1)=prob of digit "1" appears = x
p(2) = 2x
p(3) = 3x
p(4) = 4x
p(5) = 5x
p(6) = 6x
they must sum up to 1, so x+2x+3x+4x+5x+6x = 21x = 1 − − − > x=$\frac{1}{21}$
$E[X] = \Sigma x * p(x)$
$= (1(x) + 2(2x) + 3(3x) + 4(4x) + 5(5x) + 6(6x))$
= 91/21 = 4.33

$Var(X) = E[X^2]$ - $(E[X])^2$
$= (1^2(x) + 2^2(2x) + 3^2(3x) + 4^2(4x) + 5^2(5x) + 6^2(6x))$ - $(E[X])^2$
= 441/21 - $(91/21)^2$
= 21 - 18.77
= 2.23

(b) (½ point) Write code to simulate such a die. Run 1000 simulations and note down the number of times each face shows. Calculate the empirical expectation and see if it matches the theoretical expectation you have computed above.

Solution:

```
import random
p=[0,0,0,0,0,0]
for i in range(1000):
  a=random.randint(1,21)
  if a in [1]:
    a=1
  elif a in [2,3]:
    a=2
  elif a in [4,5,6]:
    a=3
```

```
  elif a in [7,8,9,10]:
    a=4
  elif a in [11,12,13,14,15]:
    a=5
  else:
    a=6
  p[a-1]=p[a-1]+1
for i in range(6):
  p[i]=p[i]/1000
Ex=0
for i in range(6):
  Ex=Ex+(i+1)*p[i]
print(Ex)
```

Answer is 4.34, near to calculated.