1. (1 point) Honor code

Concept: Projection

2. (2 points) Consider a matrix A and a vector $\mathbf{b}$ which does not lie in the column space of A. Let $\mathbf{p}$ be the projection of $\mathbf{b}$ on to the column space of A. If $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 7 \\ 4 \\ 2 \\ 5 \end{bmatrix}$, find $\mathbf{b}$.

Solution:

Solution: Let **p** be the projection of **b** on to the column space of A

A = $\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$, p= $\begin{bmatrix} 7 \\ 4 \\ 2 \\ 5 \end{bmatrix}$

we know that error vector(e)

e= b - p

p = A$\hat{x}$

e= b -A$\hat{x}$

$a_1 and a_2$are columns of A

e vector is perpendicular to columns of A

a$_1^T e = 0$

$a_2^T e = 0$

$A^T(b - A\hat{x}) = 0$

so

$p = A(A^TA)^{-1}A^Tb$

$p = Pb$

P be the projection matrix

$P = A(A^T A)^{-1} A^T$

first we need to check invertibility $of A^T A$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$perform$ gaussian $elimination$

$$\begin{matrix} R_3 <= R_3 - 2 * R_1 \\ \sim \\ R_4 = R_4 - 3 * R_1 \end{matrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & -6 \\ 0 & -8 \end{bmatrix}$$

$$\begin{matrix} R_3 <= R_3 + 3 * R_2 \\ \sim \\ R_4 = R_4 + 4 * R_2 \end{matrix} \begin{bmatrix} [1] & 3 \\ 0 & [2] \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

no of nonzero $pivots = no$ of $columns = 2$

so it is invertible A has independent columns so $\mathrm{A}^T A$ is invertible

$$\mathrm{A}^T A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 14 \end{bmatrix}$$

$using$ gaussian $elimination$

$$(A^T A)^{-1} = \begin{bmatrix} 14 & 6 \\ 6 & 14 \end{bmatrix}^{-1} = \tfrac{1}{80} \begin{bmatrix} 7 & -3 \\ -3 & 7 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \tfrac{1}{80} \begin{bmatrix} 7 & -3 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 2 & 0 & 1 \end{bmatrix}$$

$$P = \tfrac{1}{20} \begin{bmatrix} 13 & 9 & -1 & 3 \\ 9 & 7 & -3 & -1 \\ -1 & -3 & 7 & 9 \\ 3 & -1 & 9 & 13 \end{bmatrix}$$

$Pb = p$

solve this system of equation

$$\begin{matrix} R_1 <=> R_3 \\ \sim \end{matrix} \quad \tfrac{1}{20} \begin{bmatrix} -1 & -3 & 7 & 9 \\ 9 & 7 & -3 & -1 \\ 13 & 9 & -1 & 3 \\ 3 & -1 & 9 & 13 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \\ 5 \end{bmatrix}$$

$$R_2 <= R_2 + 9 * R_1$$

$$R_3 <= R_3 + 13 * R_1 \sim R_4 <= R_4 + 3 * R_1$$

$$\frac{1}{20}\begin{bmatrix} -1 & -3 & 7 & 9 \\ 0 & -20 & 60 & 80 \\ 0 & -30 & 90 & 120 \\ 0 & -10 & 30 & 40 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 22 \\ 33 \\ 11 \end{bmatrix}$$

$R_3 <= R_3 - \frac{30}{20} * R_2$

$\sim$

$R_4 <= R_4 - \frac{10}{30} * R_1$

$$\frac{1}{20}\begin{bmatrix} -1 & -3 & 7 & 9 \\ 0 & -20 & 60 & 80 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 22 \\ 0 \\ 0 \end{bmatrix}$$

b_3 and b_4 are free varriable

it leads to infinite $solution$

let $b_3 = 0, b_4 = 0$

$b_2 = -22, b_1 = 26$

$$b_{particular} = \begin{bmatrix} 26 \\ -22 \\ 0 \\ 0 \end{bmatrix}$$

$$b_{nullspace} = x * \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + y * \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

$$b_{total} = \begin{bmatrix} 26 \\ -22 \\ 0 \\ 0 \end{bmatrix} + x * \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + y * \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 26 \\ -22 \\ 0 \\ 0 \end{bmatrix} ...Ans$$

3. (2 points) Consider the following statement: Two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ cannot have the same projection $\mathbf{p}$ on the column space of A.

   (a) Give one example where the above statement is True.

   Solution: You have to show that the above statement is true for any two vectors. I have given zero marks,if you have shown it for particular $\mathbf{b}_1$ and b_2

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$P = A(A^\top A)^{-1}A^\top$$

Since A=I,replacing A with I in above equation:

$$P = I(I^\top I)^{-1}I^\top \quad \therefore P = I$$

Projection of b_1 on column space of A will be=Pb_1=Ib_1=b_1
Projection of b_2 on column space of A will be=Pb_2=Ib_2=b_2
$\because b_1 \; and \; b_2$ are different vectors ,the projection will be different for any two vectors.

(b) Give one example where the above statement is False.

Solution: Let A=$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}$

$A^\top A = \begin{bmatrix} 10 & 9 \\ 9 & 10 \end{bmatrix}$

$(A^\top A)^{-1} = \frac{1}{19}\begin{bmatrix} 10 & -9 \\ -9 & 10 \end{bmatrix}$

$(A^\top A)^{-1}A^\top = \frac{1}{19}\begin{bmatrix} 1 & 11 & -8 & 2 \\ 1 & -8 & 11 & 2 \end{bmatrix}$

$A(A^\top A)^{-1}A^\top = \frac{1}{19}\begin{bmatrix} 2 & 3 & 3 & 4 \\ 3 & 14 & -5 & 6 \\ 3 & -5 & 14 & 6 \\ 4 & 6 & 6 & 8 \end{bmatrix}$

$\therefore P = A(A^\top A)^{-1}A^\top = \frac{1}{19}\begin{bmatrix} 2 & 3 & 3 & 4 \\ 3 & 14 & -5 & 6 \\ 3 & -5 & 14 & 6 \\ 4 & 6 & 6 & 8 \end{bmatrix}$

Let $b_1 = \begin{bmatrix} 50 \\ -9 \\ 1 \\ 0 \end{bmatrix}$

Let $b_2 = \begin{bmatrix} 51 \\ -10 \\ 0 \\ 1 \end{bmatrix}$

Projection of b_1 on column space of A=Pb_1=$\frac{1}{19}\begin{bmatrix}2 & 3 & 3 & 4\\3 & 14 & -5 & 6\\3 & -5 & 14 & 6\\4 & 6 & 6 & 8\end{bmatrix}\begin{bmatrix}50\\-9\\1\\0\end{bmatrix}=\begin{bmatrix}4\\1\\11\\8\end{bmatrix}$

Projection of b_2 on column space of A=Pb_2=$\frac{1}{19}\begin{bmatrix}2 & 3 & 3 & 4\\3 & 14 & -5 & 6\\3 & -5 & 14 & 6\\4 & 6 & 6 & 8\end{bmatrix}\begin{bmatrix}51\\-10\\0\\1\end{bmatrix}=\begin{bmatrix}4\\1\\11\\8\end{bmatrix}$

Thus,b_1 *and* b_2 have same projection p on the column space of A.

(c) Based on the above examples, state the generic condition under which the above statement will be True or False.

Solution: The condition is False except when matrix A is invertible.
If matrix A is invertible we can apply $(A^\top A)^{-1} = A^{-1}(A^\top)^{-1}$
$\therefore P = A(A^\top A)^{-1}A^\top = AA^{-1}(A^\top)^{-1}A^\top = (AA^{-1})((A^\top)^{-1}A^\top) = I.$

Thus,projection of any vector(say b) will be =Pb=Ib=b.$\therefore$ if two vectors will be different,their projections will also be different.

Partial Marks given for answers:"condition is false except when matrix P is invertible ", "condition is false except when $N(A^T)$ is zero-dimensional"

4. (2 points) (a) Find the projection matrix P_1 that projects onto the line through $\mathbf{a} = \begin{bmatrix}1\\2\end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to $\mathbf{a}$.

Solution: Of Vardhman Anand Uikey (cs21m069@smail.iitm.ac.in)

Solution:
as we know that projection matrix $P_1 = \frac{aa^\top}{a^\top a}$

so $aa^\top = \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$

$a^\top a = \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = 5$

$P_1 = \frac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$

if a line is perpendicular to a then dot product of this two line is 0.

so $a^\top x = 0$

$\begin{bmatrix}1 & 2\end{bmatrix} x = 0$

$x = \begin{bmatrix}-2\\1\end{bmatrix}$

so $P_2 = \frac{xx^\top}{x^\top x}$

$xx^\top = \begin{bmatrix}-2\\1\end{bmatrix}\begin{bmatrix}-2 & 1\end{bmatrix} = \begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$

$x^\top x = \begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = 5$

so $P_2 = \frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$

(b) Compute $P_1 + P_2$ and $P_1 P_2$ and explain the result.

Solution: Of VARUN GUMMA (cs21m070@smail.iitm.ac.in)

Solution: $P_1 + P_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix} + \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This is because for any $\mathbf{v}$, $(P_1 + P_2)\mathbf{v} = P_1\mathbf{v} + P_2\mathbf{v}$, i.e. projection onto the line through $\begin{bmatrix} 1 & 2 \end{bmatrix}^\top$ and line through $\begin{bmatrix} -2 & 1 \end{bmatrix}^\top$ (which are orthogonal). This is the same as projecting a vector on to any basis. The components of the vector along the basis (i.e. $P_1\mathbf{v}, P_2\mathbf{v}$) add up to the vector itself (parallelogram law of addition). $\therefore$ for every vector $\mathbf{v}$, $P_1\mathbf{v_1} + P_2\mathbf{v_2} = \mathbf{v} = (P_1 + P_2)\mathbf{v}$, and hence $P_1 + P_2 = I$.

$P_1P_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For any vector $\mathbf{v}$, $P_1P_2\mathbf{v} = P_1\mathbf{w}$ ($\mathbf{w} = P_2\mathbf{v}$). Here $\mathbf{w}$ is along P_2 (as it is projected onto it). $P_2\mathbf{w}$ now tries to project $\mathbf{w}$ onto the line through $\begin{bmatrix} 1 & 2 \end{bmatrix}^\top$ which will be zero ($\mathbf{w}$ is along a line perpendicular to it, and projection onto a perpendicular line results in 0 as initially there was no component along that direction). As this holds for every vector in that space, i.e. $P_1P_2\mathbf{v} = 0$, P_1P_2 must be $\mathbf{0}$.

Concept: Dot product of vectors

5. (1 point) For all the vectors u, v $\in$ R^n, u^{T}v $\leq ||u||_2||v||_2$.
Prove the statement if true, or give counterexample if false.

Solution:

Solution: $u^Tv = ||u||_2||v||_2 cos\theta$

θ is angle between u and v. Since cosθ ranges from -1 to 1.we get u^{T}v $\leq ||u||_2||v||_2$

Concept: Vector norms

6. (1 point) The L_p-norm of a vector $\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ is defined as:

$$||\mathbf{x}||_p = (|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

(a) Prove that $||\mathbf{x}||_\infty = max_{1\le i\le n}|x_i|$

Solution:

Solution: Let $x_j = max_{1\le i\le n}|x_i|$ then,

$$\begin{aligned}||\mathbf{x}||_p &= (|x_1|^p + |x_2|^p + \cdots + |x_j|^p + \cdots + |x_j|^p)^{\frac{1}{p}}\\ &= |x_j|((\frac{|x_1|}{|x_j|})^p + (\frac{|x_2|}{|x_j|})^p + \cdots + (\frac{|x_j|}{|x_j|})^p + \cdots + (\frac{|x_j|}{|x_j|})^p)^{\frac{1}{p}}\end{aligned}$$

We know that for every $i \neq j$,

$$|x_i| \le |x_j|$$
$$\frac{|x_i|}{|x_j|} \le 1$$

For $\frac{|x_i|}{|x_j|}$ strictly less than 1, as p $\to \infty$

$$\frac{|x_i|}{|x_j|}^p = 0$$

For $\frac{|x_i|}{|x_j|} = 1$,

$$\begin{aligned}\frac{|x_i|}{|x_j|} &= 1\\ \frac{|x_i|}{|x_j|}^p &= 1^p\\ &= 1\end{aligned}$$

Let's suppose there are k such terms for which $\frac{|x_i|}{|x_j|} = 1$ then,

$$\begin{aligned}||\mathbf{x}||_p &= |x_j|((\frac{|x_1|}{|x_j|})^p + (\frac{|x_2|}{|x_j|})^p + \cdots + (\frac{|x_j|}{|x_j|})^p + \cdots + (\frac{|x_j|}{|x_j|})^p)^{\frac{1}{p}}\\ &= |x_j|(0+k)^{\frac{1}{p}}\\ &= |x_j|k^{\frac{1}{p}}\end{aligned}$$

As p $\to \infty$, $\frac{1}{p} \to 0$ and $k^{\frac{1}{p}} \to 1$ then,

$$\begin{aligned}||\mathbf{x}||_\infty &= |x_j|\\ &= max_{1\le i\le n}|x_i|\end{aligned}$$

(b) True or False (explain with reason): $||\mathbf{x}||_0$ is a norm.

Solution:

Solution: False

$\|x\|_0$ is defined such that,

$$\|x\|_0 = \sum_{i=1}^{n} |x_i|^0, \quad |x_i| \neq 0$$

And any norm, p has a property that,

$$p(a\mathbf{u}) = |a|p(\mathbf{u})$$

Now, let us take an vector $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and a scalar a $= 5$ then,

$$\begin{aligned} p(a\mathbf{u}) &= p(\begin{bmatrix} 5 \\ 10 \end{bmatrix}) \\ &= (|5|^0 + |10|^0) \\ &= 2 \end{aligned}$$

$$\begin{aligned} ap(\mathbf{u}) &= 5 * p(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) \\ &= 5 * (|1|^0 + |2|^0) \\ &= 5 * 2 \\ &= 10 \end{aligned}$$

Here we can see,

$$p(a\mathbf{u}) \neq |a|p(\mathbf{u})$$

Therefore $\|x\|_0$ is not a norm.

Concept: Orthogonal/Orthornormal vectors and matrices

7. (1 point) Consider the following questions:

(a) Construct a 2×2 orthogonal matrix, such that none of its entries are real.

Solution: $\frac{1}{\sqrt{2}} \begin{bmatrix} i & i \\ -i & i \end{bmatrix}$

(b) Now, construct a 4×4 matrix, such that all its columns are orthogonal and all its entries are +1, -1, +2 or -2.

Solution: $\begin{bmatrix} 2 & -2 & 1 & -1 \\ 2 & 2 & 1 & 1 \\ 1 & -1 & -2 & 2 \\ 1 & 1 & -2 & -2 \end{bmatrix}$

8. (1 point) Consider the vectors $\mathbf{a} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(a) What multiple of $\mathbf{a}$ is closest to $\mathbf{b}$?

Solution:
The closest multiple of a to b, is actually the projection of b on a. Let p be the projection of b on a, hence :- p = $\frac{a^Tb}{a^Ta}$a

$$a^Tb = \begin{bmatrix} 4 & 6 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 9$$

also $a^Ta = \begin{bmatrix} 4 & 6 & 2 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \end{bmatrix} = 81$

Hence , p = $\frac{a^Tb}{a^Ta}$a = $\frac{9}{81}a = \frac{1}{9}a$

Thus , $\frac{1}{9}a$,ie, $\frac{1}{9} \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \end{bmatrix}$ is the multiple of a , closest to b.

(b) Find orthonormal vectors $\mathbf{q_1}$ and $\mathbf{q_2}$ that lie in the plane formed by $\mathbf{a}$ and $\mathbf{b}$?

Solution: Let $\vec{a_1} = a_1 = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \end{bmatrix}$

$$\vec{a_2} = a_2 - \frac{\vec{a_1}^T a_2}{\vec{a_1}^T a_1}\vec{a_1}$$

After Solving, we get

$$\vec{a_2} = \frac{1}{9}\begin{bmatrix} 5 \\ -6 \\ -2 \\ 4 \end{bmatrix}$$

$$q_1 = \frac{\vec{a_1}}{||\vec{a_1}||} = \frac{1}{9}\begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \end{bmatrix}$$

$$q_2 = \frac{\vec{a_2}}{||\vec{a_2}||} = \frac{1}{9}\begin{bmatrix} 5 \\ -6 \\ -2 \\ 4 \end{bmatrix}$$

Solution:

9. (1 point) True or False : If A is Unitary matrix then A^2 must be an Unitary matrix. Prove the statement if True, or give counterexample if false.

Solution: Statement is True.

Consider A as unitary matrix
$\implies AA^H =$ I (by definition)

$A^2 = AA$
Finding
$(AA)(AA)^H$
by using conjugate transpose property i.e, ($(AB)^H = B^H A^H$)
$= (AA)(A^H A^H)$
$= A(AA^H)A^H$
$= AIA^H$
$= AA^H$
$=$ I

Therefore, A^2 is a unitary matrix

10. (1 point) If Q is an orthogonal matrix ,show that for any two vectors x and y of the proper dimension :

$$||Qx - Qy|| = ||x - y||$$

Solution:

Given : Q is an orthogonal matrix and x,y are any two vectors
So, we can write the norm as:

$$||Qx - Qy|| = (Qx - Qy)^T (Qx - Qy)$$

Taking the transpose inside bracket

$$||Qx - Qy|| = [(Qx)^T - (Qy)^T](Qx - Qy)$$

Using property $(AB)^T = B^T A^T$

$$||Qx - Qy|| = (x^T Q^T - y^T Q^T)(Qx - Qy)$$

$$||Qx - Qy|| = (x^T Q^T)(Qx) - (x^T Q^T)(Qy) - (y^T Q^T)(Qx) + (y^T Q^T)(Qy)$$

Using property of othogonal matrices $Q^T Q = QQ^T = I$

$$||Qx - Qy|| = x^T x - x^T y - y^T x + y^T y$$

$$||Qx - Qy|| = x^T (x - y) - y^T (x - y)$$

$$||Qx - Qy|| = (x^T - y^T)(x - y)$$

Taking the transpose outside bracket

$$||Qx - Qy|| = (x - y)^T (x - y)$$

$$\therefore ||Qx - Qy|| = ||x - y||$$

Concept: Determinants

11. (2 points) Let A be a n $\times$ n matrix such that $A[i][j] = \begin{cases} 1 & i - j = 1 \text{ OR } i = j \\ -1 & j - i = 1 \\ 0 & otherwise \end{cases}$

Prove $|A_n| = |A_{n-1}| + |A_{n-2}|$.

Solution: Given, $A[i][j] = \begin{cases} 1 & i-j=1 \text{ OR } i=j \\ -1 & j-i=1 \\ 0 & otherwise \end{cases}$

i.e., the matrix is a tridiagonal matrix with principal diagonal having 1, diagonal above principal diagonal is -1 and diagonal below principal diagonal is 1

$$|A_n| = \begin{bmatrix} 1 & -1 & 0 & 0 & . & . & 0 \\ 1 & 1 & -1 & 0 & . & . & 0 \\ 0 & 1 & 1 & -1 & . & . & 0 \\ 0 & 0 & 1 & 1 & . & . & 0 \\ . \\ . \\ 0 & 0 & 0 & 0 & . & 1 & 1 \end{bmatrix}$$

Here, observe that
$|A_1| = [\ 1\ ] = 1$

$|A_2| = \det \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2$

$|A_3| = \det \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 2 + 1 = 3$

Therefore, here $|A_3| = |A_2| + |A_1|$

Now, extending above to the general case

Find the determinent by expanding wrt 1st column.

$$|A_n| = 1.\det \begin{vmatrix} 1 & -1 & 0 & 0 & . & . & 0 \\ 1 & 1 & -1 & 0 & . & . & 0 \\ 0 & 1 & 1 & -1 & . & . & 0 \\ 0 & 0 & 1 & 1 & . & . & 0 \\ . \\ . \\ 0 & 0 & 0 & 0 & . & 1 & 1 \end{vmatrix}_{(n-1)} \quad - 1.\det \begin{vmatrix} -1 & 0 & 0 & 0 & . & . & 0 \\ 1 & 1 & -1 & 0 & . & . & 0 \\ 0 & 1 & 1 & -1 & . & . & 0 \\ 0 & 0 & 1 & 1 & . & . & 0 \\ . \\ . \\ 0 & 0 & 0 & 0 & . & 1 & 1 \end{vmatrix}_{(n-2)}$$

expanding the second term with respect to 1st column again i.e., det $\begin{vmatrix} -1 & 0 & 0 & 0 & . & . & 0 \\ 1 & 1 & -1 & 0 & . & . & 0 \\ 0 & 1 & 1 & -1 & . & . & 0 \\ 0 & 0 & 1 & 1 & . & . & 0 \\ . \\ . \\ 0 & 0 & 0 & 0 & . & 1 & 1 \end{vmatrix}_{(n-2)}$

$= $ (-1).det $\begin{vmatrix} 1 & -1 & 0 & 0 & . & . & 0 \\ 1 & 1 & -1 & 0 & . & . & 0 \\ 0 & 1 & 1 & -1 & . & . & 0 \\ 0 & 0 & 1 & 1 & . & . & 0 \\ . \\ . \\ 0 & 0 & 0 & 0 & . & 1 & 1 \end{vmatrix}_{(n-2)}$ - (1).det $\begin{vmatrix} 0 & 0 & 0 & 0 & . & . & 0 \\ 1 & 1 & -1 & 0 & . & . & 0 \\ 0 & 1 & 1 & -1 & . & . & 0 \\ 0 & 0 & 1 & 1 & . & . & 0 \\ . \\ . \\ 0 & 0 & 0 & 0 & . & 1 & 1 \end{vmatrix}_{(n-2)}$

The second term here is 0 since a 0 row.

Therefore, $=$ (-1).det $\begin{vmatrix} 1 & -1 & 0 & 0 & . & . & 0 \\ 1 & 1 & -1 & 0 & . & . & 0 \\ 0 & 1 & 1 & -1 & . & . & 0 \\ 0 & 0 & 1 & 1 & . & . & 0 \\ . \\ . \\ 0 & 0 & 0 & 0 & . & 1 & 1 \end{vmatrix}_{(n-2)}$

substituting back into the main equation

$|A_n| = $ 1.det $\begin{vmatrix} 1 & -1 & 0 & 0 & . & . & 0 \\ 1 & 1 & -1 & 0 & . & . & 0 \\ 0 & 1 & 1 & -1 & . & . & 0 \\ 0 & 0 & 1 & 1 & . & . & 0 \\ . \\ . \\ 0 & 0 & 0 & 0 & . & 1 & 1 \end{vmatrix}_{(n-1)}$ + 1.det $\begin{vmatrix} 1 & -1 & 0 & 0 & . & . & 0 \\ 1 & 1 & -1 & 0 & . & . & 0 \\ 0 & 1 & 1 & -1 & . & . & 0 \\ 0 & 0 & 1 & 1 & . & . & 0 \\ . \\ . \\ 0 & 0 & 0 & 0 & . & 1 & 1 \end{vmatrix}_{(n-2)}$

Which is nothing but, $|A_{n-1}| + |A_{n-2}|$

Therefore, $|A_n| = |A_{n-1}| + |A_{n-2}|$
(recursively extend)

12. (1 point) What is the least number of zeros in a $n \times n$ matrix that will guarantee $det(A) = 0$. Construct such matrix for n = 4.

On the other hand, what is the maximum numbers of zeros in a $n \times n$ matrix that will guarantee $det(A) \neq 0$. Construct such matrix for n = 4.

Solution:

Given $n \times n$ matrix $A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$

Matrix A has total n^2 entries.

From property 6 of determinants, we say that if row or column of matrix is zero then $det(A)$=0.

We are having n^2 entries, and to have zero row or column there should be $n^2 - n + 1$ zero entries in matrix. To ensure that the row or column is zero.

For n=4 there should be 13 zero entries. Matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Number of zero entries in matrix can not guarantee that $det(A) \neq 0$.

We can confirm this by taking number of zero entries equal to 0. And there is dependent row or column in matrix then $det(A) = 0$.

On other hand if A is identity matrix then A has $n^2 - n$ zero entries and still $det(A) \neq 0$.Matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Hence number of zeros in matrix can not guarantee $det(A) \neq 0$.

solution of JAYESH HARISHCHANDRA MAHAJAN

13. (1 point) This question is about properties 9 and 10 of determinants.

(a) Prove that $det(AB) = det(A)det(B)$

Solution: We have to prove that $det(AB) = det(A)det(B)$ We can denote $det(AB), det(A), det(B)$ as $|AB|, |A|, |B|$

While proving this, we are having two cases regarding invertibility of A as following

1. A is not invertible i.e. Rank of $A = \rho(A) < n$
   Since A is not invertible, we have $|A| = 0$
   Since we know that rank of product of given two matrix is less than equal to rank of individual matrices, hence we have

$$\rho(AB) \leq \rho(A)$$

$$\rho(AB) \leq \rho(B)$$

Hence AB is also not invertible and $|AB| = 0$
Since $|A| = 0$ and $|AB| = 0$, irrespective of what B is we will be having $|AB| = |A||B|$

2. A is invertible i.e. Rank of $A = \rho(A) = n$
   Since A is invertible, there exist elementary matrices $E_1, E_2, \dots E_k$ such that
   $$A = E_1 E_2 \dots E_k$$
   Now if we prove that for each elementary matrix E, $|EB| = |E||B|$, then we could use this to prove $|AB| = |A||B|$ as below
   $$|AB| = |E_1 E_2 \dots E_k B| = |E_1||E_2||E_k||B| = |A||B|$$
   Hence now we will show that $|EB| = |E||B|$.
   Now there can three kinds of elementary matrices as below

   (a) E is formed by interchanging two rows of the identity matrix I
   Since E is formed by interchanging two rows of the identity matrix I, hence $|E| = -1$
   Since we apply E to B, hence EB is formed by interchanging two rows of B, hence $|EB| = -|B|$
   Hence $|EB| = -|B| = |E||B|$
   Hence $|EB| = |E||B|$

   (b) E is formed by multiplying a row of identity matrix I by c where $c \in R$
   Since E is formed by multiplying a row of identity matrix I by scalar c, hence $|E| = c$
   Since we apply E to B, hence EB is formed multiplying a row of B by scalar c, hence $|EB| = c|B|$
   Hence $|EB| = c|B| = |E||B|$
   Hence $|EB| = |E||B|$

   (c) E is formed by adding a multiple of one row of identity matrix I to another Since E is formed by adding a multiple of one row of identity matrix I to another, hence $|E| = 1$
   Since we apply E to B, hence EB is formed by adding some multiple of one row of B to other row, hence $|EB| = |B|$
   Hence $|EB| = |B| = |E||B|$
   Hence $|EB| = |E||B|$

   Hence for each case of elementary matrices, we have shown that $|EB| = |E||B|$

Hence we can use this further to prove $|AB| = |A||B|$ as specified earlier

$$|AB| = |E_1E_2 \dots E_kB| = |E_1||E_2||E_k||B| = |A||B|$$

Hence for both the cases we shown that $|AB| = |A||B|$

(b) Prove that $det(A^\top) = det(A)$

Solution: We have to prove that $det(A^\top) = det(A)$
While proving this, we are having two cases regarding invertibility of A as following

1. A is not invertible i.e. Rank of $A = \rho(A) < n$
   Since A is not invertible, we have $\rho(A) < n$, hence $|A| = 0$
   As we know that rank of A is equal to number of independent columns of A which is equal to number of independent rows of A.
   Since rows of A are nothing but columns of A, hence number of independent columns of A are equal to number of independent columns of A^T
   Hence A and A^T have same rank
   So we have $\rho(A^\top) = \rho(A)$
   Hence $\rho(A^\top) < n$
   Hence $|A^\top| = 0$

2. A is invertible i.e. Rank of $A = \rho(A) = n$
   Since A is invertible, there exist elementary matrices $E_1, E_2, \dots E_k$ such that
   $$A = E_1E_2 \dots E_k$$
   Now if we prove that for each elementary matrix E, $|E| = |E^T|$, then we could use this to prove $|A| = |A^T|$
   Since any elementary matrix E represents an elementary operation hence it can be observed that it is either upper triangular or lower triangular matrix. Since E is upper triangular or lower triangular matrix, hence its transpose E^T is also upper triangular or lower triangular matrix
   We know that the determinant of upper triangular or lower triangular matrix is nothing but the product of diagonal entries.
   Since the diagonal entries of any elementary matrix E are same as that of its transpose E^T. Hence determinant of elementary matrix and its transpose are same. i.e. $|E| = |E^T|$ Since we shown that $|E| = |E^T|$, let us prove $|A| = |A^T|$
   $$|A| = |E_1E_2 \dots E_k|$$

From proof in part(a) of question, we know $|E_1 E_2 \dots E_k| = |E_1||E_2| \dots |E_k|$

$$|A| = |E_1||E_2| \dots |E_k|$$

$$|A| = |E_k| \dots |E_2||E_1|$$
$$|A| = |E_k^T| \dots |E_2^T||E_1^T|$$
$$|A| = |E_k^T \dots E_2^T E_1^T|$$
$$|A| = |(E_1 E_2 \dots E_k)^T|$$
$$|A| = |A^T|$$

Hence for both the cases we shown that $|AB| = |A||B|$ **Case 1**. when $\rho(A) < n$ then $|A| = 0$.
But , $\rho(A) = \rho(A^\top)$
So , $\rho(A^\top) < n \implies |A^\top| = 0$.

Case 2. when $\rho(A) = n$.
Since A is invertible we can represent A as a product of elementary matrices as

$$A = E_1 E_2 E_k$$

If we knew for each elementary matrix E that $|E^T| = |E|$, then it would follow that

$$|A| = |E_1 E_2 E_k|$$

We know from above proof that,

$$|E_1 E_2 E_k| = |E_1||E_2||E_k|$$

$$|A| = |E_1||E_2||E_k|$$
$$|A| = |E_1^T||E_2^T||E_k^T|$$
$$|A| = |E_1^T E_2^T E_k^T|$$
$$|A| = |A^T|$$

Thus, if we can prove that $|E| = |E^\top| \implies |A| = |A^\top|$.Now we focus on proving that,$|E| = |E^\top|$

Since, Elementary matrix or its transpose is either Upper triangular or Lower triangular matrix. Also, Determinant of Upper Triangular or Lower triangular matrix is product of diagonal entries. Diagonal entries of transpose of Elementary matrix is same as that of Elementary matrix. Thus , determinant of Elementary matrix is same as that of its transpose determinant. Hence , we proved that $|E| = |E^\top|$ and so $|A| = |A^\top|$.

14. (1 point) Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

(a) Find the area of the triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}$

> **Solution:** Given that $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.
>
> $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$
>
> It can be observed that $\mathbf{v} = 2\mathbf{u}$ and $\mathbf{u} + \mathbf{v} = 3\mathbf{u}$
>
> $\therefore \mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}$ all lie on the same line.
>
> Thus, the triangle formed by $\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}$ would have area $= 0$.

(b) Suppose you rotate these vectors along the origin such that the heads of vectors $\mathbf{u}$ and $\mathbf{v}$ trace two concentric circles, then find the area of figure trapped between circles

> **Solution:** Rotating the vectors along the origin in the suggested way would result in two concentric circles.
>
> The Radius of the inner circle would be the distance of the head of $\mathbf{u}$ from the origin i.e. $||u||$. $\therefore$ Inner Radius $= \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13}$
> The area of the inner circle would be $\pi(InnerRadius)^2 = \pi * 13$.
>
> The Radius of the outer circle would be the distance of the head of $\mathbf{v}$ from the origin i.e. $||v||$. $\therefore$ Outer Radius $= \sqrt{(6-0)^2 + (4-0)^2} = \sqrt{52}$
> The area of the outer circle would be $\pi(OuterRadius)^2 = \pi * 52$.
>
> $\therefore$ Area trapped between Outer and Inner Circle $= 52\pi - 13\pi = 39\pi = 122.46$ sq units (approximately).

15. (2 points) The determinant of the following matrix can be computed as a sum of 120 (5!) terms.

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

State true or false with an appropriate explanation: All the 120 terms in the determinant will be 0.

Solution:

Solution: True

The determinant can be computed by expanding along any one of the rows in A.

Expanding along Row_3:

$Det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34} + a_{35}C_{35}$

Each of these cofactors are themselves determinants of 4×4 matrices which can be computed as a sum of 24 (4!) terms.

Since a_{31}, a_{32} and a_{33} are zeros, therefore, $3 \times 24 = 72$ terms in the determinant will be zero.

For the remaining terms,

$$C_{34} = -\begin{vmatrix} x & x & x & x \\ x & x & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \end{vmatrix} \text{ and } C_{35} = \begin{vmatrix} x & x & x & x \\ x & x & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \end{vmatrix}$$

Expanding C_{34} along Row_3:

$C_{34} = a_{31}C'_{31} + a_{32}C'_{32} + a_{33}C'_{33} + a_{34}C'_{34}$

Here, each cofactor on RHS is a determinant of a 3×3 matrix which can be computed as a sum of 6 (3!) terms.

Since a_{31}, a_{32} and a_{33} are zeros, therefore, $3 \times 6 = 18$ terms in the determinant will be zero.

Similarly, 18 terms in the expansion of C_{35} will also be zero.

For the remaining terms in C_{34},

$$C'_{34} = -\begin{vmatrix} x & x & x \\ x & x & x \\ 0 & 0 & 0 \end{vmatrix}$$

Since it contains a zero row, therefore the determinant will be zero. To see if each term is zero, expand along Row_3

$C'_{34} = a_{31}C''_{31} + a_{32}C''_{32} + a_{33}C''_{33}$

Since each cofactor on RHS is a determinant of a 2×2 matrix which can be computed as a sum of 2 terms and since a_{31}, a_{32} and a_{33} are all zeros, therefore, $3 \times 2 = 6$ terms in the determinant will be zero.

Similarly, the remaining 6 terms in C_{35} will also be zeros.

Therefore, we get $72 + 18 + 18 + 6 + 6 = 120$ terms in the determinant of A as zeros.