Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

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Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: Yes

Concept: Linear Combinations

2. (2 points) Consider the vectors $[x, y]$, $[a, b]$ and $[c, d]$.

(a) Express $[x, y]$ as a linear combination of $[a, b]$ and $[c, d]$.

Solution:

Let p, q $\in R$ such that,
$[x, y] = p[a, b] + q[c, d] = [pa, pb] + [qc, qd] = [pa + qc, pb + qd]$
Therefore, $[x, y] = [pa + qc, pb + qd]$,
solving for p and q,
Equation 1) $pa + qc = x$ Equation 2) $pb + qd = y$
Performing, $(Eq1) * b) - (Eq2) * a)$
$qbc - qad = bx - ay => q = \frac{bx-ay}{bc-ad}$
Substituting q in $Eq1$, $p = \frac{x-c(\frac{bx-ay}{bc-ad})}{a} = \frac{cy-dx}{bc-ad}$
Therefore, $[x, y] = \frac{cy-dx}{bc-ad}[a, b] + \frac{bx-ay}{bc-ad}[c, d]$

(b) Based on the expression that you have derived above, write down the condition under which $[x, y]$ cannot be expressed as a linear combination of $[a, b]$ and $[c, d]$. (Must: the condition should talk about some relation between the scalars a, b, c, d, x and y)

Solution:

Based on the derived expression,

Clearly if the denominator of the weights are 0, the linear combination does not make sense, i.e. $[x, y]$ cannot be expressed as linear combination of $[a, b]$ and $[c, d]$.

Therefore the condition is $bc - ad = 0 => bc = ad$

Hence if $bc = ad$, we cant express as linear combination.

Concept: Elementary matrices

3. (1 point) Compute L and U for the matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get A = LU with 4 pivots

Solution:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ 0 & 0 & c-b & c \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Therefore, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ $U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$

To get $A = LU$ with 4 pivots, U should have non zero pivots in all 4 rows.

I.e. $a \neq 0$, $b - a \neq 0$, $c - b \neq 0$ and $d - c \neq 0$.

The 4 conditions are,

1)$a \neq 0$

2)$a \neq b$

3)$b \neq c$

4)$c \neq d$

4. (1 point) Let $E_1, E_2, E_3, \ldots, E_n$ be n lower triangular elementary matrices. Let $(i_1, j_1), (i_2, j_2), \ldots (i_n, j_n)$ be the position of the non-zero off-diagonal element in each of these elementary matrices. Further, $if\ k \neq m\ then\ (i_k, j_k) \neq (i_m, j_m)$ (*i.e.*, no two elementary matrices in the sequence have a non-zero off-diagonal element in the same position). Prove that the product of these n elementary matrices will have all diagonal entries as 1. (Proving this will help you understand why the diagonal elements of L are always equal to 1.)

Solution:

For a elementary matrix E_p in E_1 to E_n, position of off-diagonal element is (i_p, j_p).

This is same as the row operation, $R_{i_p} = R_{i_p} + E_{i_p j_p} R_{j_p}$.

When this is applied on a matrix M, since E is lower triangle matrix, any row in M can be updated using its above rows only. i.e. if $R_a = R_a + xR_b$, then $b < a$.

Now, since we are looking at product of n elementary matrices, it is, $(E_1E_2..)E_{n-1}E_n$, and since E_n has diagonal 1 and elements above diagonal 0,

when E_{n-1} is applied on E_n, its diagonal elements are unaffected as each row can only be added with a previous row and all elements above a diagonal element in E_n are 0 (Lower triangle matrix). Further result will be a lower triangle matrix as all values above diagonal in result will be linear combination of 0s and hence will be 0.

So, $E_{n-1}E_n$ is also a lower triangle matrix with diagonals 1.

Similarly if we apply E_{n-2} to $E_{n-1}E_n$, we get a lower triangle matrix with diagonals 1 with same logic as above.

Similarly we can apply all the elementary matrices one by one to arrive at the product, $(E_1E_2..)E_{n-1}E_n$ which will also be a lower triangle matrix with diagonals 1.

Therefore, product of these n elementary matrices will have all diagonal entries as 1.

Concept: Inverse

5. (½ point) Show that the matrix $B^\top AB$ is symmetric if A is symmetric.

Solution:

Given that A is symmetric, therefore, $A^\top = A$

Now, $(B^\top(AB))^\top = (AB)^\top(B^\top)^\top = B^\top A^\top B$

But as $A^\top = A$, $=> B^\top A^\top B = B^\top AB$

Therefore, $B^\top AB = (B^\top AB)^\top$

I.e. $B^\top AB$ is a symmetric matrix.

6. (2 points) Prove that a $n \times n$ matrix A is invertible if and only if Gaussian Elimination of A produces n non-zero pivots.

Solution:

Proof (the if part):

Given that Gaussian elimination of A produces n non-zero pivots.

Therefore, $A = (E_1E_2...E_n)U$ where $E_1, E_2, ...E_n$ are elementary matrices and U is a upper triangular matrix with n non-zero pivots which we got using gaussian elimination.

Since Elementary matrices are invertible, $E_1, ...E_n$ are invertible and hence their matrix mulitplication $(E_1E_2...E_n)$ is also invertible.

U is a $n \times n$ upper triangular matrix and has n non-zero pivots, and therefore it has number of rows and non-zero pivots equal. Therefore it is also invertible.

Therefore $(E_1E_2...E_n)U$ is invertible, => A is invertible.

Hence, A is invertible if Gaussian Elimination of A produces n non-zero pivots.

Proof (the only if part):

Given that A is invertible.

If we perform gaussian elimination on A, we can write,

$(E_n...E_1)A = U$ where $E_1, ...E_n$ are the elementary matrices corresponding to the elementary operations performed in gaussian elimination to arrive at a upper triangular matrix U.

$A = (E_n...E_1)^{-1}U$

Since Elementary matrices are invertible, inverse exists for $E_1 to E_n$ and hence $(E_1...E_n)$ is also invertible as it is matrix multiplication of invertible matrices. Therefore, $(E_n...E_1)^{-1}$ exists and is invertible.

Since A and $(E_n...E_1)^{-1}$ are invertible, U must be invertible.

Here, U is a upper triangular matrix which is also invertible. If U of dimensions $n \times n$ has less than n non-zero pivots, it cant be invertible.

Therefore, U must have n non-zero pivots.

Hence, A is invertible only if Gaussian Elimination of A produces n non-zero pivots.

7. (1 ½ points) If A and B are $n \times n$ and $n \times m$ matrices respectively and a and b are $n \times 1$ and $m \times 1$ vectors respectively, then what is the cost of:

(a) Computing AB

Solution:

Since A is $n \times n$ and B is $n \times m$, AB will be $n \times m$.
To get every value in AB, we need to perform 'n' multiplications and 'n-1' additions.
Since there are $n \times m$ values in AB, number of operations performed are,
# of Multiplications $= n \times m \times (n) = mn^2$
# of Additions $= n \times m \times (n-1) = mn(n-1)$
Cost of computing AB is $O(mn^2)$

(b) Computing $B^T a$

Solution:

Since B is $n \times m$, therefore $B^\top$ is $m \times n$, a is $n \times 1$, $B^\top a$ will be $m \times 1$.
To get every value in $B^\top a$, we need to perform 'n' multiplications and 'n-1' additions.
Since there are $m \times 1$ values in $B^\top a$, number of operations performed are,
# of Multiplications $= m \times 1 \times (n) = mn$
# of Additions $= m \times 1 \times (n-1) = m(n-1)$
Cost of computing $B^\top a$ is $O(mn)$

(c) Computing A^{-1}

Solution:

Since we can compute A^{-1} using Gaussian elimination.
Gaussian Elimination has a cost of $O(n^3)$.
Hence, to find A^{-1}, cost is $O(n^3)$.

Concept: LU factorisation

8. (1 ½ points) (a) Under what conditions is the would A have a full set of pivots ?

$$A=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

Since it is given in format, $A = LDU$,

For A to have full set of pivots, diagnol elements in D must not be 0 as even if one of the diagnols in D is 0, then the corresponding row will have a zero pivot.

Therefore, for A to have a full set of pivots, $d_1 \neq 0$, $d_2 \neq 0$, $d_3 \neq 0$

(b) Solve as two triangular systems, without multiplying LU to find A:

$$\text{LUx}=\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}2&4&4\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\0\\2\end{bmatrix}$$

Solution:

Let us split $LUx = b$ as 2 equations,

$1) Ly = b$, $2) Ux = y$

Solving Equation 1),

$$\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}2\\0\\2\end{bmatrix} => y_1 = 2, y_1 + y_2 = 0 => y_2 = -y_1 = -2$$

$=> y_1 + y_3 = 2 => y_3 = 2 - y_1 = 2 - 2 = 0$

$$=> y = \begin{bmatrix}2\\-2\\0\end{bmatrix}$$

Solving Equation 2),

$$\begin{bmatrix}2&4&4\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-2\\0\end{bmatrix} => \text{in row 3, } x_3 = 0,$$

in row 2, $x_2 + 2x_3 = -2 => x_2 = -2$

in row 1, $2x_1 + 4x_2 + 4x_3 = 2 => 2x_1 - 8 + 0 = 2 => x_1 = 5$

Therefore solution to given system is, $\begin{bmatrix}5 & -2 & 0\end{bmatrix}^\top$

9. (2 points) Consider the following system of linear equations. Find the LU factorisation of the matrix A corresponding to this system of linear equations. Show all the steps involved. (this is where you will see what happens when you have to do more than 1 permutations).

$$
\begin{aligned}
x + y &= -3 \\
w - x - y &= +2 \\
3w - 3x - 3y - z &= -19 \\
-5x - 3y - 3z &= -2
\end{aligned}
$$

Solution:

Let A = $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 3 & -3 & -3 & -1 \\ 0 & -5 & -3 & -3 \end{bmatrix}$, $X = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$, $b = \begin{bmatrix} -3 \\ 2 \\ -19 \\ -2 \end{bmatrix}$

We need to solve $AX = b$.

Performing LU factorisation for A,

1) R3 = R3 - 3*R2, (Therefore let E_1 be multiplied to obtained matrix after performing the operation to get back A. E_1 should perform the operation,

R3 = R3 + 3*R2)

$$A => \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -5 & -3 & -3 \end{bmatrix}$$

2) R4 = R4 + 5*R1, (Therefore let E_2 be multiplied to obtained matrix after performing the operation to get back A. E_2 should perform the operation,

R4 = R4 - 5*R1)

$$A => \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -3 \end{bmatrix}$$

3) Interchange R1 and R2, also R3 and R4.

$$A => \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\text{Simplifying, } => \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & -5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

10. (1 point) For a square matrix A, prove that LDU factorisation is unique.

Solution:

To prove that LDU factorisation is unique,

Suppose there are two LDU factorisations of A, $A = LDU$ and $A = L_1D_1U_1$.

Then we must prove that $L = L_1$, $D = D_1$,$U = U_1$.

Since, $A = LDU = L_1D_1U_1$, premultiply both sides by L^{-1} and postmultiply by U_1^{-1}.

$=> L^{-1}LDUU_1^{-1} = L^{-1}L_1D_1U_1U_1^{-1} => DUU_1^{-1} = L^{-1}L_1D_1$

Now, since U, U_1^{-1} are upper triangle matrices with diagonals 1, UU_1^{-1} also has diagonals 1. Similarly $L^{-1}L_1$ is multiplication of two lower triangle matrices with diagonal 1, so it also has diagonals 1.

Therefore for $DUU_1^{-1} = L^{-1}L_1D_1$, D must be equal to D_1. $=> D = D_1$.

Therefore, $UU_1^{-1} = L^{-1}L_1$.

Now, since U, U_1^{-1} are upper triangle matrices, they have all elements below diagonal as 0. Therefore their product also has all elements below diagonal as 0.

Similarly, since L^{-1}, L_1 are lower triangle matrices, they have all elements above diagonal as 0. Therefore their product also has all elements above diagonal as 0.

But as $UU_1^{-1} = L^{-1}L_1$, both LHS and RHS must have all elements above and below diagonal as 0 and diagonal elements as 1. i.e. $UU_1^{-1} = L^{-1}L_1 = I$.

Therefore, $U = (U_1^{-1})^{-1} = U_1$ and $L = (L_1^{-1})^{-1} = L_1$.

I.e. proven that $L = L_1$, $D = D_1$,$U = U_1$.

Therefore LDU factorisation is unique.

11. (1 ½ points) Consider the matrix A which factorises as:

$$\begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Without computing A or A^{-1} argue that

(a) A is invertible (I am looking for an argument which relies on a fact about elementary matrices)

Solution:

If $A = LU$,

here L is $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$ which is equivalent to the following row operations,

$R1 = R1$

$R2 = R2 + 7R1$

$R3 = R3 + 5R2$

Hence L is invertible as it is a product of elementary matrices which are invertible.

Also, U is $L^\top$.

Since, for a square matrix M, if inverse exists, then $M^{-1}M = I$.

For $M^\top$, $(MM^{-1})^\top = I^\top = I$

=> $(M^{-1})^\top M^\top = ((M^\top)^{-1})M^\top = I$. Inverse of $M^\top$ is $(M^{-1})^\top$ and since M^{-1} exists, $(M^{-1})^\top$ also exists. Therefore $M^\top$ is invertible.

Therefore, since L is invertible, U is also invertible.

Therefore LU is also invertible and hence,

A is invertible.

(b) A is symmetric (convince me that $A_{ij} = A_{ji}$ without computing A)

Solution:

If $A = LU$,

from observation, $U = L^\top$.

Therefore $A = LL^\top$.

Product of a matrix with its transpose produces a square symmetric matrix and hence A is symmetric.

(c) A is tridiagonal (again, without computing A convince me that all elements except along the 3 diagonals will be 0.)

Solution:

For A to be a tridiagnol matrix, as A is $3x3$, A_{13} and A_{31} must be 0.

Since $A = LU$, therefore A will have columns which are linear combinations of columns of L with weights as the elements of a column of U.

For 3rd column of A, its first value, i.e.A_{13} will be 0 as weights for the linear combination are 0, 5 and 1 from 3rd column of U and since L is lower triangle matrix, in its first row, all elements expect the first are 0. Therefore in the linear combination all values are multiplied with 0 except first value which also results in 0 since first weight is 0. Therefore A_{13} is simply sum of zeroes.

Therefore, $A_{13} = 0$. Since we proved A is symmetric, therefore $A_{31} = A_{13} = 0$.

Therefore A is a tridiagnol matrix.

Concept: Lines and planes

12. (1 ½ points) Consider the following system of linear equations

$$\begin{aligned} a_1x_1 + b_1y_1 + c_1z_1 &= 1 \\ a_2x_2 + b_2y_2 + c_2z_2 &= 2 \\ a_3x_3 + b_3y_3 + c_3z_3 &= 3 \end{aligned}$$

Each equation represents a plane, so find out the values for the coefficients such that the following conditions are satisfied:

1. All planes intersect at a line
2. All planes intersect at a point
3. Every pair of planes intersects at a different line.

Solution:

If we write the equations as matrices,

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1) For all planes intersecting at a line, one of the equations should be linear combination of other 2.

Therefore, if we take p and q as the weights, in b, $p + 2q = 3 => p = 3 - 2q$

$$\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} pa_1 + qa_2 \\ pb_1 + qb_2 \\ pc_1 + qc_2 \end{bmatrix} = \begin{bmatrix} (3-2q)a_1 + qa_2 \\ (3-2q)b_1 + qb_2 \\ (3-2q)c_1 + qc_2 \end{bmatrix}$$

Example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

2) For all planes intersecting at a point, None of the equations should be linear combination of other 2.

Example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3) For every pair of planes intersecting at a different line, Any one of the equations coeffs should be linear combination of coeffs of other two equations but their values in b should not match the linear combination of values in b for other two equations.

Therefore, if we take p and q as the weights, in b, $p + 2q = 3 => p = 3 - 2q$

But if coeffs are linear combination with weights which dont satisfy $p = 3 - 2q$, then they will fall under this category.

Example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

13. (1 ½ points) Starting with a first plane $u - v - w = -1$, find the equation for

(a) the parallel plane through the origin.

Solution:

For a parallell plane, it should have same coefficients as given plane and as it needs to pass through origin,
$=> u - v - w = 0$
Since coeffs are same, it is parallell to given plane and since rhs is 0, it passes through the point $(0, 0, 0)$ as $=> 0 - 0 - 0 = 0 = RHS$.
Plane is $u - v - w = 0$.

(b) a second plane that also contains the points (-1,-1,1) and (-7,-5,-1).

Solution:

Given plane $u - v - w = 1$ passes through those points.
If we consider the plane, $3u - 4v - w = 0$,
for $(-1, -1, 1)$, $=> -3 + 4 - 1 = 0 = RHS$
for $(-7, -5, -1)$, $=> -21 + 20 + 1 = 0 = RHS$
Plane is $3u - 4v - w = 0$.

(c) a third plane that meets the first and second in the point (2, 1, 2).

Solution:

If we consider the plane, $y = 1$,
for $(2, 1, 2)$, $=> 0 + 1 + 0 = 1 = RHS$.
It intersects the other 2 planes at $(2, 1, 2)$.
Plane is $y = 1$.

Concept: Transpose

14. (2 points) Consider the transpose operation.

(a) Show that it is a linear transformation.

Solution:

Consider two mxn matrices A, B.
Let M_{ij} be the element of M at j^{th} column of i^{th} row.
To prove that transpose is a linear transformation, we need to prove that,
$T(pA + qB) = pT(A) + qT(B)$
$LHS = (pA + qB)^\top$
Now, as transpose operation interchanges the row and column positions of each value, $pA_{ij} + qB_{ij}$ gets assigned to LHS_{ji} on performing transpose.
Therefore, $LHS_{ij} = pA_{ji} + qB_{ji}$
$RHS = pA^\top + qB^\top$
Now, as transpose operation interchanges the row and column positions of each value, $RHS_{ij} = pA^\top_{ij} + qB^\top_{ij} = pA_{ji} + qB_{ji} = LHS_{ij}$
Therefore LHS = RHS and hence Transpose is a linear transformation.

(b) Find the matrix corresponding to this linear transformation.

Solution:

Suppose we consider a m x 1 matrix A.
Its transpose should be of dimensions 1 x m.
However for any matrix M of dimensions p x m, if we multiply to A,
MA will have dimensions p x 1.
Since if $m \neq 1$, dimensions of MA for any possible M and $A^\top$ cannot match, we cannot form a matrix corresponding to transpose operation such that $A^\top = MA$ where M is the corresponding matrix for the transpose operation.