Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

N. Kw

N Kausik
Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: Yes

Concept: Linear Transformation

2. (1 point) Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Suppose $T(\begin{bmatrix} 2 & 3 & -6 \end{bmatrix}^\top) = \begin{bmatrix} 2 & 5 & 1 \end{bmatrix}^\top$ and $T(\begin{bmatrix} 4 & -2 & 8 \end{bmatrix}^\top) = \begin{bmatrix} 7 & -2 & -2 \end{bmatrix}^\top$. Find $T(\begin{bmatrix} -2 & 13 & -34 \end{bmatrix}^\top)$

Solution:

Let us consider,

A = $([2 \quad 3 \quad -6]^\top)$

B = $([4 \quad -2 \quad 8]^\top)$

C = $([-2 \quad 13 \quad -34]^\top)$

Using property of Linear Transformations,

Suppose, $C = p(A) + q(B)$ where p, q $\in \mathbb{R}$

then, $T(C) = T(p(A) + q(B)) = p(T(A)) + q(T(B))$

Solving for p and q, (using $C = p(A) + q(B)$)

$2p + 4q = -2$

$3p - 2q = 13$

upon solving, we get $p = 3$, $q = -2$

Therefore, $T([-2 \quad 13 \quad -34]^\top) = T(C) = p(T(A)) + q(T(B))$

$= 3 * ([2 \quad 5 \quad 1]^\top) - 2 * ([7 \quad -2 \quad -2]^\top)$

$= ([6 \quad 15 \quad 3]^\top) - ([14 \quad -4 \quad -4]^\top) = ([-8 \quad 19 \quad 7]^\top)$

Therefore, $T([-2 \quad 13 \quad -34]^\top) = [-8 \quad 19 \quad 7]^\top$

3. (1 point) Prove that if $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(a\mathbf{x}) = aT(\mathbf{x})$ then $T(b\mathbf{x}+c\mathbf{y}) = bT(\mathbf{x}) + cT(\mathbf{y})$.

Solution:

Given that,

Equation 1) $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$

Equation 2) $T(a\mathbf{x}) = aT(\mathbf{x})$

To prove, $T(b\mathbf{x}+c\mathbf{y}) = bT(\mathbf{x}) + cT(\mathbf{y})$

From Equation 1, we have LHS = $T(b\mathbf{x}+c\mathbf{y}) = T(b\mathbf{x}) + T(c\mathbf{x})$

Now, using Equation 2, $T(b\mathbf{x}) = bT(\mathbf{x})$ and $T(c\mathbf{x}) = cT(\mathbf{x})$

Therefore, $T(b\mathbf{x}) + T(c\mathbf{x}) = bT(\mathbf{x}) + cT(\mathbf{x})$ = RHS

Hence proved.

4. (2 points) Let T be a transformation defined from $\mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{X}) = \mathbf{X} + 1$ where $\mathbf{X} \in \mathbf{R}^n$
   Formally argue if T is a linear transformation or not. If Yes, then give the matrix representing that transformation.

Solution:

Since, $T(\mathbf{X}) = \mathbf{X} + 1_n$, where 1_n is a vector of size n with all values 1.

Using definition of Linear Transformation,

T is a linear transformation if $T(\mathbf{X} + \mathbf{Y}) = T(\mathbf{X}) + T(\mathbf{Y})$

LHS $= T(\mathbf{X} + \mathbf{Y}) = (\mathbf{X} + \mathbf{Y}) + 1_n = a\mathbf{X} + b\mathbf{Y} + 1_n$

RHS $= T(\mathbf{X}) + T(\mathbf{Y}) = (\mathbf{X} + 1_n) + (\mathbf{Y} + 1_n) = \mathbf{X} + 1_n + \mathbf{X} + 1_n$

$= \mathbf{X} + \mathbf{Y} + 2_n$

Since $2_n \neq 1_n$, LHS $\neq$ RHS

Therefore, given T is NOT a linear transformation.

5. (2 points) Suppose $A \in \mathbf{R}^{3\times 3}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3 (\mathbf{x} \neq \mathbf{0}, \mathbf{y} \neq \mathbf{0})$. Further, suppose $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\top$. If $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$ is one solution for $A\mathbf{x} = b$, write down at least one more solution (you are welcome to write down all the infinite solutions if you want :-)).

Solution:

We have,

Equation 1) $Ax = b$

Equation 2) $Ay = 0$

Now, by adding Equation 1 and Equation 2,

$Ax + Ay = b + 0$

Using properties of Linear Transformation (A is a matrix => Linear Transformation)
$Ax + Ay = A(x + y)$

$=> A(x + y) = b$

Since $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$ is a solution, we can put x = $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$,

$=> A(\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top + y) = b$

$=> Az = b$, where $z = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top + y$.

Therefore, $z = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top + y$ is also a solution to $Ax = b$.

If we take $y = \begin{bmatrix}1 & 2 & 3\end{bmatrix}^\top$,

$\begin{bmatrix}2 & 3 & 4\end{bmatrix}^\top$ is a solution.

Concept: Matrix multiplication

6. (1 point) Statement: If A and B are matrices such that A is not a Null or Identity matrix and B is not a Null matrix,
   if $AB = A^2$ then A = B.
   options:
   a) always true,
   b) always false,
   c) sometimes can be true , sometimes can be false also
   Explain your answer based on the option you have chosen.

   Solution:

CORRECT OPTION: c) sometimes can be true , sometimes can be false also

Option b) is wrong as,

For $A = B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

In $AB = A^2$, $AB = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}$

Hence, $AB = A^2$ is true and also $A = B$ is also true. Therefore, option b) which says that if $AB = A^2$ is true, then A cant be equal to B always cannot be correct.

Option a) is wrong as,

For $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

In $AB = A^2$, $AB = AI = A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $A^2 = AA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Clearly from above, $AB = A^2$, but $A \neq B$.

This violates the condition: if $AB = A^2$ then A = B.

Therefore option a) which says that if $AB = A^2$ is true, then A must be equal to B always cannot be correct.

Hence the correct option is c).

7.

$$A = \begin{bmatrix} 1 & 2 & 5 & 6 \\ -1 & 3 & -2 & 1 \\ 3 & 0 & 0 & 1 \\ 1 & 5 & 4 & -14 \end{bmatrix}$$

For each of the equations below, find $\mathbf{x}$

(a) ($1/2$ point) $A\mathbf{x} = \begin{bmatrix} 1 & 10 & 4 & -11 \end{bmatrix}^\top$

Solution:

Suppose $x = \begin{bmatrix} a & b & c & d \end{bmatrix}^\top$

If we view Ax as linear combination of columns of A with x as weights,

$Ax = a(\begin{bmatrix} 1 & -1 & 3 & 1 \end{bmatrix}^\top) + b(\begin{bmatrix} 2 & 3 & 0 & 5 \end{bmatrix}^\top) + c(\begin{bmatrix} 5 & -2 & 0 & 4 \end{bmatrix}^\top)$

$+d(\begin{bmatrix} 6 & 1 & 1 & -14 \end{bmatrix}^\top) = \begin{bmatrix} 1 & 10 & 4 & -11 \end{bmatrix}^\top$

Here, $1 = (1 + 4 + 6) - (10) = 1 + 2 * (2) - 2 * (5) + 6$

Therefore, if we consider weights $1, 2, -2, 1$,

$10 = -1 + 2 * (3) - 2 * (-2) + 1 = -1 + 6 + 4 + 1$

$4 = 3 + 2 * (0) - 2 * (0) + 1 = 3 + 0 + 0 + 1$

$-11 = 1 + 2 * (5) - 2 * (4) - 14 = 1 + 10 - 8 - 14$

$=> x = \begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix}^\top$

(b) (½ point) $A\mathbf{x} = \begin{bmatrix} 16 & 4 & -30 & 81 \end{bmatrix}^\top$

Solution:

Suppose $x = \begin{bmatrix} a & b & c & d \end{bmatrix}^\top$

If we view Ax as linear combination of columns of A with x as weights,

$Ax = a(\begin{bmatrix} 1 & -1 & 3 & 1 \end{bmatrix}^\top) + b(\begin{bmatrix} 2 & 3 & 0 & 5 \end{bmatrix}^\top) + c(\begin{bmatrix} 5 & -2 & 0 & 4 \end{bmatrix}^\top)$

$+d(\begin{bmatrix} 6 & 1 & 1 & -14 \end{bmatrix}^\top) = \begin{bmatrix} 16 & 4 & -30 & 81 \end{bmatrix}^\top$

Here, $16 = (8 + 35) - (9 + 18) = -9 * (1) + 4 * (2) + 7 * (5) + -3 * (6)$

Therefore, if we consider weights $-9, 4, 7, -3$,

$4 = -9 * (-1) + 4 * (3) + 7 * (-2) - 3 * (1) = 9 + 12 - 14 - 3$

$-30 = -9 * (3) + 4 * (0) + 7 * (0) - 3 * (1) = -27 + 0 + 0 - 3$

$81 = -9 * (1) + 4 * (5) + 7 * (4) - 3 * (-14) = -9 + 20 + 28 + 42$

$=> x = \begin{bmatrix} -9 & 4 & 7 & -3 \end{bmatrix}^\top$

8. (1 point) Give two matrices A and B (of appropriate dimensions) such that $A \neq B$ and,

(a) (½ point) $AB = BA$

Solution:

$A = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}$ and $B = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}$ and $BA = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}$

Here, $AB = BA$

(b) (½ point) $AB \neq BA$

Solution:

$A = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}$ and $B = I = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 7 & 19 \\ 18 & 49 \end{bmatrix}$ and $BA = \begin{bmatrix} 25 & 43 \\ 18 & 31 \end{bmatrix}$

Here, $AB \neq BA$

9. If A, B & C are matrices (assume appropriate dimensions) prove that,

(a) (½ point) $(A + B)^T = A^T + B^T$

Solution:

Let A and B be two (mxn) matrices.

Let C be another (mxn) matrix such that $C = A + B$

Let elements of A be (a_{ij}), elements of B be (b_{ij}) and elements of C be (c_{ij}), where $1 \leq i \leq m$ and $1 \leq j \leq n$.

Therefore, since $C = A + B, c_{ij} = a_{ij} + b_{ij}$

Also, $(a_{ij})^\top = a_{ji}$, similarly for elements of B and C.

Now, LHS $= (A+B)^\top = (a_{ij}+b_{ij})^\top = (c_{ij})^\top = c_{ji} = (a_{ji}+b_{ji}) = ((a_{ij})^\top+(b_{ij})^\top)$ $= A^\top + B^\top =$ RHS

Hence proved.

(b) (½ point) $(AB)^T = B^T A^T$

Solution:

Let A be a (mxn) matrix and B be a (nxk) matrix.
Let C be another (mxk) matrix such that $C = AB$
Let elements of A be (a_{ij}), elements of B be (b_{ij}) and elements of C be (c_{ij}), where $1 \leq i \leq m$ and $1 \leq j \leq n$.
Therefore, since $C = AB, c_{ij} = \sum_{p=1}^{n}(a_{pj})(b_{ip})$
Also, $(a_{ij})^\top = a_{ji}$, similarly for elements of B and C.
Now, LHS $= (AB)^\top = (C)^\top = (c_{ij})^\top = c_{ji} = \sum_{p=1}^{n}(a_{jp})(b_{pi})$
RHS $= B^\top A^\top = \sum_{p=1}^{n}(a_{pj})^\top(b_{ip})^\top = \sum_{p=1}^{n}(a_{jp})(b_{pi}) =$ LHS
Hence proved.

10. (1 point) Let A be any matrix. In the lecture we saw that $A^\top A$ is a square symmetric matrix. Is $AA^\top$ also a square symmetric matrix? (Hint: The answer is either "Yes, except when ..." or "No, except when ...".)

Solution:

Here, let $B = AA^\top$,
Using $(XY)^\top = Y^\top X^\top$, $B^\top = (AA^\top)^\top = (A^\top)^\top A^\top$
Using $(X^\top)^\top = X$, $(A^\top)^\top A^\top = AA^\top = B$
Hence, $B = B^\top$, which describes a square symmetric matrix.
Therefore, $AA^\top$ is a square symmetric matrix for any matrix A.
Answer: Yes, for any matrix A, $AA^\top$ is a square symmetric matrix.

Concept: Inverse

11. (1 point) If $(A+B)^2 = A^2+2AB+B^2$. Show that $AB = BA$. (assume AB, BA exists.)

Solution:

For A^2 and B^2 to exist, both A and B must be square matrices. Also, since AB and BA exist, both A and B must be of same dimensions.

Therefore, let us assume A and B are two (nxn) matrices.

In $(A+B)^2 = A^2 + 2AB + B^2$,

LHS $= (A+B).(A+B) = AA + AB + BA + BB = (A^2 + B^2) + (AB + BA)$

RHS $= (A^2 + B^2) + (2AB) = (A^2 + B^2) + (AB + AB)$

Since given that LHS = RHS,

$(A^2 + B^2) + (AB + BA) = (A^2 + B^2) + (AB + AB)$

$=> AB + BA = AB + AB$

$=> AB = BA$

Hence shown that if $(A+B)^2 = A^2 + 2AB + B^2$, then $AB = BA$.

12. What is the inverse of the following two matrices? (Hint: I don't want you to compute the inverse using some method. Instead think of the linear transformation that these matrices do and think how you would reverse that transformation. **You will have to explain your answer in words clearly stating the linear transformations being performed.**)

(a) (½ point)

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Solution:

Here, $A = (\frac{1}{2})(I)$, i.e. A is half of identity matrix of (4 x 4) dimensions.

Hence, when multiplied with any matrix B, $AB = (\frac{1}{2})(I)B = (\frac{1}{2})B$

I.e. it halves all elements of B.

Therefore, its inverse would be doubling all elements of B.

Hence, $=> A^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(b) (½ point)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

Here, $A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + I$

Hence, when multiplied with any matrix B (3 x n), $AB = (\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + I)B$

$= \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} B + IB = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} B + B$

Here, $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} B$ can be viewed as linear combinations of columns of the matrix with rows of B as weights.

As for each column in B, $\begin{bmatrix} b_{1c} & b_{2c} & b_{3c} \end{bmatrix}^\top$ where c is column number,

$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{1c} & b_{2c} & b_{3c} \end{bmatrix}^\top = b_{1c}(\begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^\top) + b_{2c}(\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top) + b_{3c}(\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top)$

$= (\begin{bmatrix} 0 & 2b_{1c} & 0 \end{bmatrix}^\top)$

Similarly doing for all columns of B, we get 2*(1st row of B) as 2nd row of multiplication output which will be of (3 x n) size with all 0 except 2nd row.

Hence, $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} B + B = \begin{bmatrix} 0 & 0 & 0 & ... \\ 2(b_{11}) & 2(b_{12}) & 2(b_{13}) & ... \\ 0 & 0 & 0 & ... \end{bmatrix} + B$

This is same as adding 2 times row 1 of B with row 2 of B. I.e. make R2 as R2 + 2(R1)

Hence, inverse would be to subtract the added 2(row 1) which is achieved with the same matrix but instead of 2 in row 2 column 1, we give a -2 which does the operation, make R2 as R2 - 2(R1)

=> $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) (1 point)

$$A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Solution:

Here, we can see that when we matrix multiply A with a matrix B (2 x n), the resulting matrix has columns which are obtained by rotating the column vectors of B by an angle θ.

Eg. If we take $\theta = 0^o$, A becomes $\begin{bmatrix} cos0^o & -sin0^o \\ sin0^o & cos0^o \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Therefore, AB = IB = B, which is expected as rotating by 0^o causes no change.

Hence to find inverse, it is simply rotating the resultant matrix with $-\theta$.

Hence inverse of A will be the same matrix with $-\theta = \begin{bmatrix} cos(-\theta) & -sin(-\theta) \\ sin(-\theta) & cos(-\theta) \end{bmatrix}$

Using $cos(-\theta) = cos\theta$ and $sin(-\theta) = -sin\theta$,

$=> A^{-1} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$

Concept: System of linear equations

13. (1 point) Argue why the following system of linear equations will not have any solutions.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -7 & -7 & -7 & -7 \\ 2 & 4 & 6 & 9 \\ 1 & 2 & 3 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ -35 \\ 10 \\ 6 \end{bmatrix}$$

Solution:

In the given system of linear equations, the 1st and 4th equations from the matrices are,

If x is $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top$,

$x_1 + 2x_2 + 3x_3 + 4x_4 = 5$

$x_1 + 2x_2 + 3x_3 + 4x_4 = 6$

Since both the equations have same coefficients for x_1, x_2, x_3, x_4, but have different values in RHS, they are parallel hyper-planes.

Since for any system of linear equations where 2 equations represent parallel hyper-planes have no solution which can satisfy both,

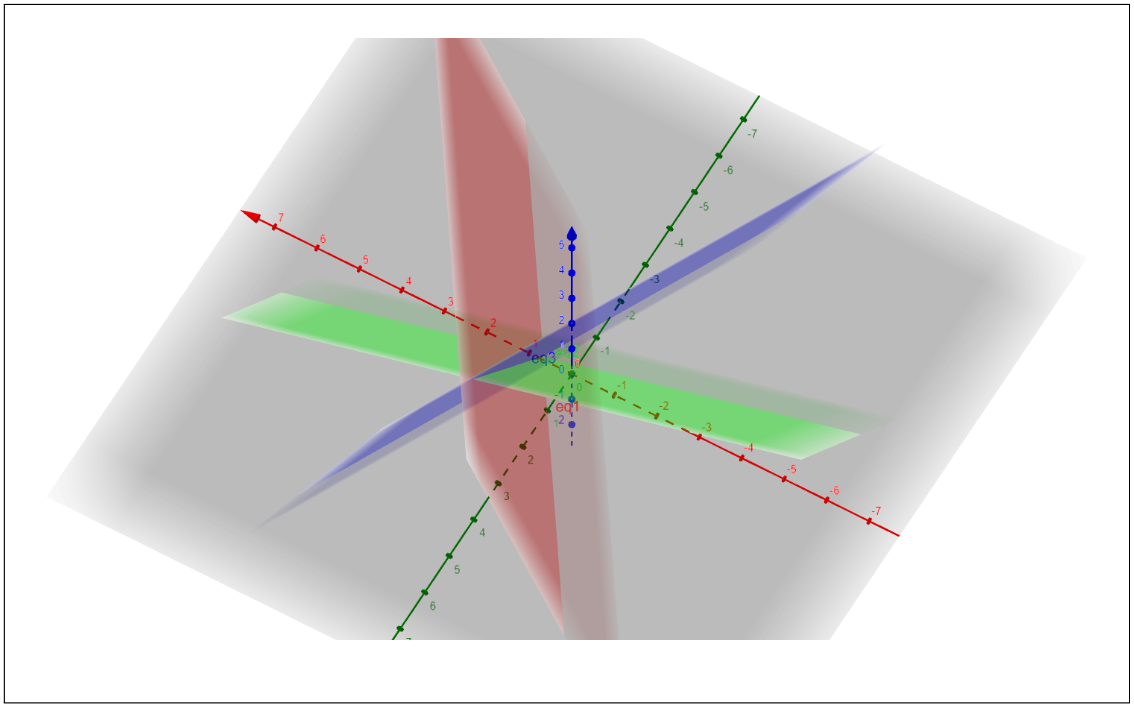
This system of linear equations has 0 solutions.

14. Consider the following 3 planes

$$\begin{aligned} 3x + 2y - z &= 2 \\ x - 4y + 3z &= 1 \\ 4x - 2y + 2z &= 3 \end{aligned}$$

(a) (½ point) Plot these planes in geogebra and paste the resulting figure here (you can download the figure as .png and paste it here)

Solution:



(b) (½ point) How many solutions does the above system of linear equations have? (based on visual inspection in geogebra)

Solution:

From inspection in geogebra, the 3 planes intersect in a line.

Therefore there are **infinite** solutions to the given system of linear equations.

(c) (1 point) Notice that the third equation can be obtained by adding the first two equations. Based on this observation, can you explain your answer for the number of solutions in the previous part of the question. (Note that I am looking for an answer in plain English which does not include terms like "linear independence" or "dependence of columns/rows". In other words, your answer should be based only on concepts/ideas which have already been discussed in the class)

Solution:

Since the 3rd equation can be obtained by simply adding the above 2 equations, we can infer that it does not provide any unique information to the system of linear equations.

The information about the solution to the system is provided only by the first two equations and hence it is similar to solving a system of linear equations with 3 variables using just 2 equations.

Since for 3 variables, we require at least 3 equations which provide unique information to solving it, therefore we get infinite solutions for this system.

15. Consider the following system of linear equations:

$$x + 2y + 4z = 1$$
$$x + 5y - 2z = 2$$

Add one more equation to the above system such that the resulting system of 3 linear equations has

(a) (½ point) 0 solutions

Solution:

For 0 solutions, we can add a equation which is parallel to any of the above 2 equations.

$x + 2y + 4z = 2$

This is parallel to equation 1 and so it causes the system to have 0 solutions.

(b) (1 point) exactly 1 solution

Solution:

For exactly 1 solution, we can add,

$x + 0y + 0z = 1$

all 3 planes representing the equations meet at a point.

Hence, there is only 1 solution for this system and it is $x = 1, y = \frac{1}{6}, z = \frac{-1}{12}$

(c) (½ point) infinite solutions

Solution:

For ∞ solutions, we can add a equation which is linear combination of the above 2 equations. If we add the 2 equations we get,

$2x + 7y + 2z = 3$

Now the 3 planes representing the equations intersect at a line and hence, there are ∞ solutions to the resulting system of linear equations.