Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

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Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: Yes

Concept: Projection

2. (2 points) Consider a matrix A and a vector $\mathbf{b}$ which does not lie in the column space of A. Let $\mathbf{p}$ be the projection of $\mathbf{b}$ on to the column space of A. If $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 7 \\ 4 \\ 2 \\ 5 \end{bmatrix}$, find $\mathbf{b}$.

Solution:

We are given A and p. Since A has 2 columns and both are independent and has 3 rows, column space of A is a plane in 3D space. Therefore, any vector which starts at origin and ends at a point on the line normal to the plane and passing through the point at the end of p will be projected onto the plane as p.

Therefore, $b = p + ke$ where e is a unit vector perpendicular to the column space plane and $k \in R$.

Therefore to find a normal to column space plane, it should be perpendicular to all column vectors of A, i.e. $A^\top e = 0$, if $e = (\begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix})^\top$, solutions to $A^\top e = 0$ is the null space of $A^\top$.

Getting row reduced echelon form of $A^\top$, $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 2 & 0 & 1 \end{bmatrix}$, $R2 = R2 - 3R1$

$=> \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -6 & -8 \end{bmatrix}$, $R2 = \frac{1}{2}R2 => \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & -4 \end{bmatrix}$

Hence, null space of $A^\top$ has basis vectors, $\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$

Hence any linear combination of these two vectors can be taken as the vector e.
$e = \begin{bmatrix} -2p - 3q \\ 3p + 4q \\ p \\ q \end{bmatrix}$ and so, $b = p + e = \begin{bmatrix} 7 - 2p - 3q \\ 4 + 3p + 4q \\ 2 + p \\ 5 + q \end{bmatrix}$ where $p, q \in R$.

Example, $\begin{bmatrix} 7 \\ 4 \\ 2 \\ 5 \end{bmatrix}$.

3. (2 points) Consider the following statement: Two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ cannot have the same projection $\mathbf{p}$ on the column space of A.

(a) Give one example where the above statement is True.

Solution:

If we consider, $A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = A^\top = A^{-1} = I$.

Hence, $P = A(A^\top A)^{-1}A^\top = I(II)^{-1}I = I(I)^{-1}I = I(I)I = I$.

For any vector b, its projection on column space of A is $p = Pb = Ib = b$.

Hence for any two vectors b_1 and b_2, their projections on column space of A will be b_1 and b_2 itself. Hence if they are two different vectors, their projections will also be different.

Hence the statement is True for this A.

(b) Give one example where the above statement is False.

Solution:

If we consider, $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A^\top = \begin{bmatrix} 1 & 0 \end{bmatrix}$,

$A^\top A = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$, Therefore, $(A^\top A)^{-1} = \begin{bmatrix} -1 \end{bmatrix}$.

Hence, $P = A(A^\top A)^{-1}A^\top = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$.

Therefore, for any vector b, its projection p is, $Pb = p$.

If we take $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, their projections are,

$p_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $p_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Clearly there are 2 vectors such that their projections are same on column space of A. Hence the statement is False for this A.

(c) Based on the above examples, state the generic condition under which the above statement will be True or False.

Solution:

The condition is False except when the matrix A is invertible. This is because if A is invertible, $A^\top$ is invertible and the projection matrix can we rewritten as, $p = Pb = A(A^\top A)^{-1}A^\top b = AA^{-1}(A^\top)^{-1}A^\top b = IIb = Ib = b$. $=> p = b$.

Hence for every vector b, its projection is itself and hence for any two different vectors b_1 and b_2, their projections can never be the same.

4. (2 points) (a) Find the projection matrix P_1 that projects onto the line through $\mathbf{a} = \begin{bmatrix}1\\2\end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to $\mathbf{a}$.

Solution:

For P_1,

for line passing through $a = \begin{bmatrix}1\\2\end{bmatrix}$, we consider $A = a$ and hence,

$P_1 = A(A^\top A)^{-1}A^\top = \begin{bmatrix}1\\2\end{bmatrix} (\begin{bmatrix}1 & 2\end{bmatrix} \begin{bmatrix}1\\2\end{bmatrix})^{-1} \begin{bmatrix}1 & 2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} (\begin{bmatrix}5\end{bmatrix})^{-1} \begin{bmatrix}1 & 2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} (\begin{bmatrix}\frac{1}{5}\end{bmatrix}) \begin{bmatrix}1 & 2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}1 & 2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$ Hence, $P_1 = \frac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$.

For P_2,

for line perpendicular to $a = \begin{bmatrix}1\\2\end{bmatrix}$, if a_p is a vector along that perpendicular line,

$a_p = \begin{bmatrix}2\\-1\end{bmatrix}$ as $a^\top . a_p = 1 * 2 - 2 * 1 = 0$.

we consider $A = a_p$ and hence,

$P_2 = A(A^\top A)^{-1}A^\top = \begin{bmatrix}2\\-1\end{bmatrix} (\begin{bmatrix}2 & -1\end{bmatrix} \begin{bmatrix}2\\-1\end{bmatrix})^{-1} \begin{bmatrix}2 & -1\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix} (\begin{bmatrix}5\end{bmatrix})^{-1} \begin{bmatrix}2 & -1\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix} (\begin{bmatrix}\frac{1}{5}\end{bmatrix}) \begin{bmatrix}2 & -1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}2\\-1\end{bmatrix} \begin{bmatrix}2 & -1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$ Hence, $P_2 = \frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$.

(b) Compute $P_1 + P_2$ and $P_1 P_2$ and explain the result.

Solution:

$P_1 = \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ and $P_2 = \frac{1}{5}\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix}$

$P_1 + P_2 = \frac{1}{5}\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix} = I$

Projection of b along a line and perpendicular to the same line produces 2 perpendicular vectors whose sum gives back b. If the projections are p_{p1} and p_{p2}, we can write, $b = p_{p1} + p_{p2}$.

$=> b = (P_1 b) + (P_2 b) => b = (P_1 + P_2)b.$

Hence we get $P_1 + P_2$ as I.

$P_1 P_2 = \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\frac{1}{5}\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

Here, when we do $P_1 P_2$, we are simply projecting the columns of P_2 onto the subspace corresponding to P_1, i.e. along the line through vector a.

But as P_2 corresponds to projecting onto line perpendicular to vector a, columns of P_2 will be along the perpendicular line and hence will be perpendicular to the vector a.

Hence projection of every column of P_2 onto line through vector a will 0 as they are perpendicular.

Hence we get a 0 matrix.

Concept: Dot product of vectors

5. (1 point) For all the vectors u, v $\in$ R^n, u^{T}v $\leq ||u||_2||v||_2$.
   Prove the statement if true, or give counterexample if false.

Solution:

$u^\top v$ is same as dot product, $u.v$.

Also, $u.v = ||u||_2||v||_2 Cos\theta$ where θ is the angle between the two vectors u and v.

Since $Cos\theta$ always lies in the range $[-1, 1]$,

$||u||_2||v||_2 Cos\theta$ lies in range $[-||u||_2||v||_2, ||u||_2||v||_2]$.

Therefore, $u^\top v$ lies in range $[-||u||_2||v||_2, ||u||_2||v||_2]$.

Therefore, $u^\top v \leq ||u||_2||v||_2$ for all vectors $u, v \in R$.

Hence Proved.

Concept: Vector norms

6. (1 point) The L_p-norm of a vector $\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ is defined as:

$$||\mathbf{x}||_p = (|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

(a) Prove that $||\mathbf{x}||_\infty = max_{1 \leq i \leq n} |x_i|$

Solution:

$||\mathbf{x}||_\infty = \lim_{p \to +\infty} ||\mathbf{x}||_p = \lim_{p \to \infty} (|x_1|^p + ... + |x_n|^p)^{\frac{1}{p}}$
Suppose $max_{1 \leq i \leq n} |x_i| = |x_j|$, Is we take out that term in the summation,
$= \lim_{p \to \infty} (|x_j|^p (\frac{|x_1|^p}{|x_j|^p} + ... + 1 + ... + \frac{|x_n|^p}{|x_j|^p}))^{\frac{1}{p}}$.
$= \lim_{p \to \infty} (|x_j|^p ((\frac{|x_1|}{|x_j|})^p + ... + 1 + ... + (\frac{|x_n|}{|x_j|})^p))^{\frac{1}{p}}$.
Since x_j is the largest element, $\frac{x_i}{x_j} < 1$ for $1 \leq i \leq n$ and $i \neq j$.
Therefore, as $p \to \infty$, $(\frac{|x_1|}{|x_j|})^p \to 0 => \lim_{p \to \infty} (|x_j|^p (0 + ... + 1 + ... + 0)^{\frac{1}{p}}$.
$= \lim_{p \to \infty} (|x_j|^p (1)^{\frac{1}{p}} = \lim_{p \to \infty} (|x_j|^p)^{\frac{1}{p}} = \lim_{p \to \infty} x_j = x_j = max_{1 \leq i \leq n} |x_i|$.
Hence proved that, $||\mathbf{x}||_\infty = max_{1 \leq i \leq n} |x_i|$.

(b) True or False (explain with reason): $||\mathbf{x}||_0$ is a norm.

Solution:

FALSE
Since the $||x||_0$ norm simply counts the number of non-zero entries in x, suppose we scale x by a factor k $k \in R - \{0\}$.
Since scaling by a non-zero value cannot make a non-zero value zero or vice versa, $||kx||_0 = ||x||_0$ as both will have same number of non-zero entries.
This violates the property of norms which says that $||kx|| = k||x||$.
Hence $||x||_0$ is not a norm.

Concept: Orthogonal/Orthornormal vectors and matrices

7. (1 point) Consider the following questions:

(a) Construct a 2×2 Orthonormal matrix, such that none of its entries are real.

Solution:

If we consider the matrix, $A = \begin{bmatrix} \sqrt{\frac{3}{2}} + i & \sqrt{\frac{3}{2}} - i \\ -\sqrt{\frac{3}{2}} + i & \sqrt{\frac{3}{2}} + i \end{bmatrix}$

Dot product of columns is, $\begin{bmatrix} \sqrt{\frac{3}{2}} + i \\ -\sqrt{\frac{3}{2}} + i \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\frac{3}{2}} - i \\ \sqrt{\frac{3}{2}} + i \end{bmatrix}$

$= (\sqrt{\frac{3}{2}}+i)(\sqrt{\frac{3}{2}}-i)+(-\sqrt{\frac{3}{2}}+i)(\sqrt{\frac{3}{2}}+i) = (\sqrt{\frac{3}{2}}+i)(\sqrt{\frac{3}{2}}-i)-(\sqrt{\frac{3}{2}}+i)(\sqrt{\frac{3}{2}}-i) = 0$

Hence the columns are orthogonal.

The norms of the columns are,

$||A_1||_2 = \sqrt{(\sqrt{\frac{3}{2}} + i)^2 + (-\sqrt{\frac{3}{2}} + i)^2} = \sqrt{\frac{3}{2} - 1 + 2\frac{3}{2}i + \frac{3}{2} - 1 - 2\frac{3}{2}i}$

$= \sqrt{3 - 1 - 1} = 1$

$||A_2||_2 = \sqrt{(\sqrt{\frac{3}{2}} - i)^2 + (\sqrt{\frac{3}{2}} + i)^2} = \sqrt{\frac{3}{2} - 1 - 2\frac{3}{2}i + \frac{3}{2} - 1 + 2\frac{3}{2}i}$

$= \sqrt{3 - 1 - 1} = 1$

Since the norms of the columns are 1 and thier dot product is 0, the columns of A are orthonormal.

Hence A is a orthogonal matrix.

(b) Now, construct a 4×4 Orthogonal matrix, such that all its entries are +1, -1, +2 or -2.

Solution:

Since allowed values are 1, -1, 2, -2, the minimum possible $||A_i||_2$ (norm of ith column) is when the column has all elements 1 or -1. Minimum norm is $\sqrt{(\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2} = 2$.

Hence norm of a column with elements from 1, -1, 2, -2 cannot be 1.

Hence a 4x4 matrix orthogonal matrix cannot be constructed using these values since it is not possible to have a column with norm of 1 which is necessary condition for all columns in a orthogonal matrix.

8. (1 point) Consider the vectors $\mathbf{a} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(a) What multiple of $\mathbf{a}$ is closest to $\mathbf{b}$?

Solution:

The closest multiple of a to b is the projection of b on a.

Projection of b on a is, $p = \frac{aa^\top}{a^\top a} b = \frac{1}{\begin{bmatrix} 4 & 6 & 2 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \end{bmatrix}} \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \end{bmatrix} \begin{bmatrix} 4 & 6 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$= \frac{1}{81} \begin{bmatrix} 16 & 24 & 8 & 20 \\ 24 & 36 & 12 & 30 \\ 8 & 12 & 4 & 10 \\ 20 & 30 & 10 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{81} \begin{bmatrix} 16+20 \\ 24+30 \\ 8+10 \\ 20+25 \end{bmatrix} = \frac{1}{81} \begin{bmatrix} 36 \\ 54 \\ 18 \\ 45 \end{bmatrix} = \frac{9}{81} \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \end{bmatrix} = \frac{1}{9} a$

Therefore the multiple of a closest to b is $\frac{1}{9} a$.

(b) Find orthonormal vectors $\mathbf{q_1}$ and $\mathbf{q_2}$ that lie in the plane formed by $\mathbf{a}$ and $\mathbf{b}$?

Solution:

$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. $||q_1||_2 = \frac{1}{2}(1+1) = 1$ and since q_1 is a multiple of b, it lies on the plane formed by a and b. Since any vector on the plane formed by a and b is a linear combination of a and b, $q_2 = \begin{bmatrix} 4x+y \\ 6x \\ 2x \\ 5x+y \end{bmatrix}$. Since q_1 and q_2 are perpendicular (orthogonal), $q_1.q_2 = 0$.

Therefore, $\frac{1}{\sqrt{2}}(4x+y+5x+y) = 0 => 9x+2y = 0 => y = \frac{-9x}{2}$

If we take x as 2, y is -9. Therefore $\begin{bmatrix} 8-9 \\ 12 \\ 4 \\ 10-9 \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \\ 4 \\ 1 \end{bmatrix}$

Since q_2 is orthonormal, it must have $||q_2||_2 = 1$, $q_2 = \frac{1}{\sqrt{1+144+16+1}} \begin{bmatrix} -1 \\ 12 \\ 4 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{162}} \begin{bmatrix} -1 \\ 12 \\ 4 \\ 1 \end{bmatrix} = \frac{1}{9\sqrt{2}} \begin{bmatrix} -1 \\ 12 \\ 4 \\ 1 \end{bmatrix}$

$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $q_2 = \frac{1}{9\sqrt{2}} \begin{bmatrix} -1 \\ 12 \\ 4 \\ 1 \end{bmatrix}$

9. (1 point) True or False : If A is Unitary matrix then A^2 must be an Unitary matrix. Prove the statement if True, or give counterexample if false.

Solution:

TRUE

Given that A is a unitary matrix, i.e. $A\overline{A}^\top = \overline{A}^\top A = I$.

$A = \overline{A}^\top$

Also we know that for any two matrices A and B, $\overline{AB}^\top = (\overline{A}\overline{B})^\top$ as taking conjugate after multiplying two complex values / matrices is same as taking conjugate and then multiplying.

$= \overline{B}^\top \overline{A}^\top$.

Therefore, $A^2 = AA$ and hence, $(A^2)\overline{(A^2)}^\top = AA\overline{A}^\top\overline{A}^\top = A(A\overline{A}^\top)\overline{A}^\top$.

Since A is unitary, $A\overline{A}^\top = I$, $=> A(I)\overline{A}^\top = A\overline{A}^\top = I$.

Therefore, $(A^2)\overline{(A^2)}^\top = I$ and hence A^2 is a unitary matrix.

10. (1 point) If Q is an orthogonal matrix ,show that for any two vectors x and y of the proper dimension :

$$||Qx - Qy|| = ||x - y||$$

Solution:

Since Q is a matrix, $||Qx - Qy|| = ||Q(x - y)||$.

Since Q is a orthogonal matrix, $Q^{-1} = Q^\top$.

Now, if we take, $x - y = z$, $||Qz||^2 = (Qz)^\top Qz = z^\top Q^\top Qz = z^\top Q^{-1}Qz = z^\top (I)z$ $= z^\top z = ||z||^2$.

Therefore, $||Qz||^2 = ||z||^2$ and since $||Qz||$ and $||z||$ are always positive, $||Qz|| = ||z||$.

Hence, $||Qx - Qy|| = ||x - y||$.

Concept: Determinants

11. (2 points) Let A be a n × n matrix such that $A[i][j] = \begin{cases} 1 & i - j = 1 \text{ OR } i = j \\ -1 & j - i = 1 \\ 0 & otherwise \end{cases}$

Prove $|A_n| = |A_{n-1}| + |A_{n-2}|$.

Solution:

Here, $i = j$ represents all the diagonal elements of A.

$i - j = 1 => i = j + 1$ represents elements $A[2][1], A[3][2], ...A[j+1][j]$. These are the elements just below the diagonal elements of A.

$j - i = 1 => i = j - 1$ represents elements $A[1][2], A[2][3], ...A[j-1][j]$. These are the elements just above the diagonal elements of A.

Therefore, A is of the form, $A = \begin{bmatrix} 1 & -1 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & ... & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & 1 \end{bmatrix}$

Using $det(A)$ formula, $det(A_n) = (-1)^{1+1}(1)(det(M_{11})) + (-1)^{2+1}(-1)(det(M_{12}))$, (other terms are not taken as they are multiplied with 0) where M_{ij} is the matrix formed by removing ith row and jth column from A_n.

$det(A_n) = det(M_{11}) - (-1)det(M_{12}) = det(M_{11}) + det(M_{12})$

$M_{11} = \begin{bmatrix} 1 & -1 & 0 & ... & 0 & 0 & 0 \\ 1 & 1 & -1 & ... & 0 & 0 & 0 \\ 0 & 1 & 1 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & -1 & 0 \\ 0 & 0 & 0 & ... & 1 & 1 & -1 \\ 0 & 0 & 0 & ... & 0 & 1 & 1 \end{bmatrix}$ $M_{12} = \begin{bmatrix} 1 & -1 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & -1 & ... & 0 & 0 & 0 \\ 0 & 1 & 1 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 & -1 & 0 \\ 0 & 0 & 0 & ... & 1 & 1 & -1 \\ 0 & 0 & 0 & ... & 0 & 1 & 1 \end{bmatrix}$

We can see clearly that M_{11} is a $(n-1)x(n-1)$ matrix with central diagonal elements 1, elements above diagonal -1, elements below diagonal 1 and rest of the elements 0. Hence, $M_{11} = A_{n-1}$.

Also, $det(M_{12}) = det(m_{11}) - (-1)det(m_{12})$ (other terms are not taken as they are multiplied with 0), where m_{ij} is the $(n-2)x(n-2)$ matrix we get after removing ith row and jth column from M_{12}. Here, if we remove 1st row and 2nd column from M_{12}, the resulting matrix has all zeros in its first column. Hence, $det(m_{12}) = 0$.

Hence, $det(M_{12}) = det(m_{11}) = det(A_{n-2})$ as after removing 1st row and 1st column from M_{12}, we get back the same form as A with 1s in central diagonal, -1s above and 1s below it. Hence $m_{11} = A_{n-2}$ (as m_{11} is a $(n-2)x(n-2)$ matrix).

Therefore, $det(A_n) = det(A_{n-1}) + det(A_{n-2})$. Hence Proved.

12. (1 point) What is the least number of zeros in a $n \times n$ matrix that will guarantee $det(A) = 0$. Construct such matrix for n = 4.
    On the other hand, what is the maximum numbers of zeros in a $n \times n$ matrix that will guarantee $det(A) \neq 0$. Construct such matrix for n = 4.

Solution:

For $det(A) = 0$, it is enough to have a dependent row in A which can be written as linear combinations of other rows in A. It is not necessary that the row should have zeroes.

Hence minimum number of zeroes in a $n \times n$ matrix for $det(A) = 0$ is 0.

For $n = 4$, $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Here, if we do gaussian elimination, the first 3 rows become 0. Due to zero rows, $det(A) = 0$.

For $det(A) \neq 0$, A should have all rows as independent rows. Hence to get maximum number of zeroes in A and for it to still have all rows independent, we can make every row have all elements 0 except for 1 element such that the position of the non-zero element in each row is unique.

If we take the value of the non-zero element as 1, we simply get the identity matrix.

Since for a $n \times n$ identity matrix has all elements 0 except along the diagonal, it has number of zeros = (total number of elements in A) - (number of non-zero elements in A). Number of zeroes = (n*n) - (n) = n(n-1).

This is the maximum zeroes possible as to get even 1 more zero, we need to make one of the non-zero elements as 0. This causes its corresponding row to become a zero row which makes $det(A)$ to 0.

Hence maximum number of zeroes in a $n \times n$ matrix for $det(A) \neq 0$ is $n(n-1)$.

For $n = 4$, $A = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Here, since it is a identity matrix, it is invertible and hence $det(A) \neq 0$ and it has 12 zeros.

13. (1 point) This question is about properties 9 and 10 of determinants.

(a) Prove that $det(AB) = det(A)det(B)$

Solution:

Since any matrix can either be invertible or be non-invertible, there are 4 possibilities for the matrices A and B, and if we prove that the equality holds for all the cases then it is proved for any A and B.

(i) B is not invertible and A can be invertible or not-invertible.

I.e. $det(B) = 0$. Since B is not invertible, B must have a non-trivial (non-zero) vector x such that $Bx = 0$.

Hence if we compute the multiplication of the same non-trivial vector with AB, $(AB)x = A(Bx) = A0 = 0$. Hence there is a non-trivial solution to (AB)x = 0 and therefore AB must also be non-invertible. Hence, $det(AB) = 0$. Hence, $LHS = det(AB) = 0 = det(A) * 0 = det(A) * det(B) = RHS$. Hence proved for this case when B is not invertible.

(ii) B is invertible and A is not invertible

I.e. $det(B) \neq 0$ and $det(A) = 0$.

Since A is not invertible, there exists a non-trivial vector x such that $Ax = 0$.

Also since B is invertible, B^{-1} exists. For a vector y, such that $B^{-1}y = 0$, y must be a trivial vector (zero vector) as B^{-1} is invertible. Hence, for the non-trivial (non-zero) vector x, $B^{-1}x$ is also non-trivial as for $B^{-1}x$ to be trivial/zero vector, x must also be trivial which is not the case.

If we take the non-trivial vector $B^{-1}x$, Hence in $(AB)(B^{-1}x) = A(BB^{-1})x = Ax = 0$. I.e. AB has a non-trivial solution for $(AB)y = 0$ and hence AB is not invertible and therefore, $det(AB) = 0$. In the formula, $LHS = det(AB) = 0$, $RHS = det(A)det(B) = 0 * det(B) = 0$

LHS = RHS and hence the equality holds.

(iii) B is invertible and A is invertible.

I.e. $det(B) \neq 0$ and $det(A) \neq 0$. Therefore, we can replace A and B by a product of elementary matrices, $A = E_{a1}...E_{am}$ and $B = E_{b1}...E_{bn}$.

From properties of determinants, we know that $det(E_1M) = det(E_1)det(M)$ if E_1 is a elementary matrix. Hence, $det(A) = det(E_{a1}...E_{am}) = det(E_{a1})...det(E_{am})$ and $det(B) = det(E_{b1}...E_{bn}) = det(E_{b1})...det(E_{bn})$.

$RHS = det(A)det(B) = det(E_{a1})...det(E_{am})det(E_{b1})...det(E_{bn})$, using the above mentioned property, $= det(E_{a1}...E_{am}E_{b1}...E_{bn}) = det(AB) = LHS$. Hence proved for this case.

Since the equality holds for all the cases, it is proved.

(b) Prove that $det(A^\top) = det(A)$

Solution:

We can prove using mathematical induction,

If we consider nxn matrix A,

For the base case, n=1, since it is a single value in the matrix, $A = A^\top$ and hence. $det(A_1) = det(A_1^\top)$.

Induction Step, we assume that for n=k-1, $det(A_{k-1}) = det(A_{k-1}^\top)$.

Now, for n=k, Using cofactor expansion of determinant,

$det(A_k) = a_{11}det(M_{11}) - a_{12}det(M_{12}) + ... + (-1)^{k+1}a_{1k}det(M_{1k})$, where M_{ij} is the (k-1)x(k-1) matrix we get after removing ith row and jth column from A_k.

Similarly, $det(A_k^\top) = a_{11}det(M_{11}) - a_{21}det(M_{21}) + ... + (-1)^{k+1}a_{k1}det(M_{k1}) = a_{11}det(M_{11}^\top) + ... + (-1)^{k+1}a_{k1}det(M_{1k}^\top)$

Since $M_{11}...M_{1k}$ are (k-1)x(k-1) matrices, $det(M_{11}) = det(M_{11}^\top)$ ... $det(M_{1k}) = det(M_{1k}^\top)$. Hence, $det(A_k^\top) = a_{11}det(M_{11}) + ...(-1)^{k+1}a_{1k}det(M_{1k}) = det(A_k)$.

Hence it is shown that for n=k, $det(A_k) = det(A_k^\top)$.

Hence by induction, $det(A) = det(A^\top)$.

14. (1 point) Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

(a) Find the area of the triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}$

Solution:

Here by inspection, $v = 2u$. Hence $u + v = 3u$. Therefore, all 3 vectors lie on the same line (along u).

Hence area of triangle is 0.

(b) Suppose you rotate these vectors along the origin such that the heads of vectors **u** and **v** trace two concentric circles, then find the area of figure trapped between circles

Solution:

By rotating these vectors along origin, the heads of the vectors trace a circle of radius = magnitude of the vector.

Hence r_u = radius of trace by rotating the u vector = $||u||_2 = \sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13}$

r_v = radius of trace by rotating the v vector = $||v||_2 = \sqrt{6^2+4^2} = \sqrt{36+16} = \sqrt{52}$.

Hence area of figure trapper between the two circles = area of circle by v - area of circle by u = $\pi r_v^2 - \pi r_u^2 = \pi((\sqrt{52})^2 - (\sqrt{13})^2) = \pi(52-13) = 39\pi$

$= 39\pi \approxeq 122.522\ units^2$.

15. (2 points) The determinant of the following matrix can be computed as a sum of 120 (5!) terms.

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

State true or false with an appropriate explanation: All the 120 terms in the determinant will be 0.

Solution:

TRUE

Following the cofactor expansion formula for determinants, $det(A)$ will be a linear combination of terms of the form, $a_{1j_1}a_{2j_2}...a_{5j_5}$ where $j_1, ...j_5$ is a permutation of

$\{1, 2, 3, 4, 5\}$.

Therefore each term can be obtained by selecting one unique column for each row and multiplying the intersection elements of those rows and columns. To get a non-zero term, we should select the 5 elements in the term such that not even one of them is a 0.

If $x \neq 0$, in row 1 and 2 we can pick any column since all the elements are x.

But, the last 3 rows have only 2 non-zero elements each in them in the same last 2 columns. Hence we can avoid picking a 0 in row 4 and 5 by picking the 4th and 5th column elements respectively (elements = x).

But for the 3rd row, we cannot choose column 1, 2, 3 as their elements are 0. We also cannot pick column 4 or 5 as they have already been picked by rows 4 and 5.

Hence it is not possible to pick any permutation of $\{1, 2, 3, 4, 5\}$ such that every element in $\{a_{1j_1}, a_{2j_2}..., a_{5j_5}\}$ is non zero. Hence their product will always be 0. Hence every term in the cofactor expansion of determinant will be 0.

Hence Proved.