1. (1 point) Honor code.

**Concept**: Linear Combinations

2. (2 points) Consider the vectors $[x, y]$, $[a, b]$ and $[c, d]$.
   (a) Express $[x, y]$ as a linear combination of $[a, b]$ and $[c, d]$.

   **Solution:** $[x, y]$=n$[a, b]$+m$[c, d]$, where n and m are real numbers.

   (b) Based on the expression that you have derived above, write down the condition under which $[x, y]$ cannot be expressed as a linear combination of $[a, b]$ and $[c, d]$. (Must: the condition should talk about some relation between the scalars $a, b, c, d, x$ and $y$)

   **Solution:**

   **Solution:** Let A = $[x, y]$ , U = $[a, b]$ and V = $[c, d]$.
   Now, $c_1$U + $c_2$V = A , where $c_1$ and $c_2$ belongs to R
   From above Derived equation
   $c_1\ [a, b] + c_2\ [c, d] = [x, y]$
   $$[c_1, c_2] \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [x, y]$$
   Applying Transpose on both sides we get :
   $$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$
   This is in the form of Ax=b
   Applying Gaussian elimination : (step: row2 = row2 - $\frac{b}{a}$ row1 ) we get
   $$\begin{bmatrix} a & c \\ 0 & \frac{ad-bc}{a} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ \frac{ay-bx}{a} \end{bmatrix}$$
   if $\frac{ad-bc}{a} = 0$ and $\frac{ay-bx}{a} \neq 0$ then the system has no solution which mean $[x, y]$ cannot be expressed as a linear combination of $[a, b]$ and $[c, d]$.
   Hence the conditions are : $ad = bc$ and $ay \neq bx$

   Solution of KARNIYANA HEMANTH BABU (cs21m027@smail.iitm.ac.in)

**Concept**: Elementary matrices

3. (1 point) Compute L and U for the matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get A = LU with 4 pivots

**Solution:**

**Solution:** $A = LU$

$EA = U$

$(E_{32}, E_{31}, E_{21})A = U$ E is Elementary matrixes

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

By changing R2=R2-R1

$$A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

By changing R3=R3-R1

$$A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix}$$

By changing R4=R4-R1

$$A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

By changing R4=R4-R3

$$A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By changing R3=R3-R2

$$A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-a \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$\Longrightarrow$

$$A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-a \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

all multipler $A_ij$ are equal to 1 so L lower triangular matrix with 1's on the diagonal and U is upper triangular matrix pivots are the non zero entries on the diagonal
$L = (E_{32}^{-1}, E_{31}^{-1}, E_{21}{}^{-}1)U$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-a \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

four conditions on a, b, c, d pivot condition
$a \neq 0$
$b \neq a$
$c \neq b$
$d \neq c$

Solution of Monica (monicad@tenet.res.in)

4. (1 point) Let $E_1, E_2, E_3, \ldots, E_n$ be $n$ lower triangular elementary matrices.

Let $(i_1, j_1), (i_2, j_2), \ldots (i_n, j_n)$ be the position of the non-zero off-diagonal element in each of these elementary matrices. Further, $if\ k \neq m\ then\ (i_k, j_k) \neq (i_m, j_m)$ (*i.e.*, no two elementary matrices in the sequence have a non-zero off-diagonal element in the same position). Prove that the product of these $n$ elementary matrices will have all diagonal entries as 1. (Proving this will help you understand why the diagonal elements of $L$ are always equal to 1.)

**Solution:**

**Solution:** Assumption: The matrices are unit lower triangular elementary matrices of arbitrarily chosen dimension n $\geq 1$.
Let us take any three consecutive matrices in the sequence, $E^i.E^{i+1}, E^{i+2}$.
Let $P = E^i * E^{i+1} * E^{i+2}$
From our assumption and the question, $E^i.E^{i+1}, E^{i+2}$ are unit lower triangular elementary matrices. If we can show that product of $Q = E^i.E^{i+1}$ is a unit lower triangular matrix, and subsequently, $Q * E^{i+2}$, is also a unit lower triangular matrix, then this can be recursively used to prove the statement asked in question. This is because every other multiplication will be between a unit lower triangular matrix with a unit lower triangular elementary matrix.

Part 1: $Q = E_i.E_{i+1}$ is a unit lower triangular matrix.
For any diagonal element at position (j,j) in matrix Q, it is given by multiplying row j of $E_i$ and column j of $E_{i+1}$.
$\therefore Q_{j,j} = \Sigma_{k=1}^{n} E^i_{j,k} * E^{i+1}_{k,j}$ [For all $k < j$, $E^{i+1}$ will be 0 and for all $k > j$, $E^i$ will be 0.]
$\therefore Q_{j,j} = E^i_{j,j} * E^{i+1}_{j,j}$
which gives, $Q_{j,j} = 1$
This shows its a unit diagonal matrix.

To show its a lower triangular matrix,
Any element in the upper triangle of Q, let it be $Q_{j,k}$ where $k > j$ is given by multiplying row j of $E^i$ with column k of $E^{i+1}$:
$Q_{j,k} = \Sigma_{l=1}^{n} E^i_{j,l} * E^{i+1}_{l,k}$ [For all $l > j$, $E^i$ will be 0, and for all $l < k$, $E^{i+1}$ will be 0. ]
$\therefore Q_{j,k} = 0$

Part 2: $P = Q * E^{i+2}$ is a unit lower triangular matrix
For any diagonal element at position (j,j) in matrix Q, it is given by multiplying row j of $E_i$ and column j of $E_{i+1}$.
$\therefore P_{j,j} = \Sigma_{k=1}^{n} Q_{j,k} * E_{k,j}^{i+2}$ [For all $k < j$, $E^{i+2}$ will be 0 and for all $k > j$, $Q$ will be 0.]
$\therefore P_{j,j} = Q_{j,j} * E_{j,j}^{i+2}$
which gives, $P_{j,j} = 1$
This shows its a unit diagonal matrix.

To show its a lower triangular matrix,
Any element in the upper triangle of P, let it be $P_{j,k}$ where $k > j$ is given by multiplying row j of $P$ with column k of $E^{i+2}$:
$P_{j,k} = \Sigma_{l=1}^{n} P_{j,l} * E_{l,k}^{i+1}$ [For all $l > j$, $Q$ will be 0, and for all $l < k$, $E^{i+2}$ will be 0. ]
$\therefore Q_{j,k} = 0$
Therefore the resultant matrix is a lower triangular matrix with unit diagonal.
Any further multiplication will be of the form shown in part 2, and will follow the same reasoning to produce lower triangular unit diagonal matrices and thus the fianl product will also be a matrix with unit diagonal.

**Concept**: Inverse

5. (½ point) Show that the matrix $B^\top AB$ is symmetric if $A$ is symmetric.

**Solution:** We know that $(XYZ)^\top = Z^\top Y^\top X^\top$
$(B^\top AB)^\top = B^\top A^\top B$
Now, if A is symmetric, then $A^\top = A$
$(B^\top AB)^\top = B^\top AB$

6. (2 points) Prove that a $n \times n$ matrix A is invertible if and only if Gaussian Elimination of A produces $n$ non-zero pivots.

**Solution:**
**Proof (the if part):**
To Prove : If a $n \times n$ matrix $A$ has n pivots, then $A$ is invertible.

Given, Matrix $A$ has n pivots.
Since $A$ has $n$ pivots, $A$ can be expressed as $n$ equations in Gaussian Elimination form and obtain a unique solution for $x_i$ in $Ax_i = e_i$ where $i = 1, 2, 3..., n$. The column vector $x_i$ can be taken as $A^{-1}$ when $e_i$ is the $i^{th}$ column of a $n \times n$ Identity

matrix. $\implies AA^{-1} = I$. This means that at least a right inverse of $A$ exists.

Further, considering the sequence of multiplications performed on $A$. This process of a sequence of multiplications of elementary matrices $E$s to get zeros below and above the pivots, permutation matrices $P$s to rearrange the rows of $A$ if needed and a diagonal matrix $D^{-1}$ to get all 1s after dividing the pivots, is called Gaussian Elimination.

$$CA = (D^{-1}..E...P...E)A = I$$

Since this exists, this means that a left inverse of $A$ exists $\implies$ we have reached $A^{-1}A = I$ with n pivots.

We know that a matrix cannot have two different inverses. This means, the right inverse equals the left inverse. Therefore, a $n \times n$ matrix $A$ with a full set of pivots ($n$ pivots in this case) will always have an inverse.

**Proof (the only if part):**
To Prove: If a $n \times n$ matrix $A$ is invertible, then $A$ has n pivots.

Given, Matrix $A$ is invertible.
Let us assume that $A$ does not have $n$ pivots, but rather, it has $n-1$ pivots. In this case, Gaussian Elimination will lead to a zero row.
The elimination steps for this are taken by an invertible matrix $X$. $\implies$ Gaussian Elimination leads to a row in $XA$ to be zero. With this, if $AA^{-1} = I$ was possible, then $XAA^{-1} = X$. The zero row of $XA$ times $A^{-1}$ gives a zero row of $X$ itself. Since $X$ is invertible, it cannot have any zero rows and this contradicts our initial assumption. Therefore, $A$ must have $n$ pivots if $A^{-1}$ exists.

**REFERRED FROM: Gilbert Strang - Introduction to Linear Algebra (2016, Wellesley-Cambridge Press).**

Solution from SANJANAA G V

7. (1 ½ points) If $A$ and $B$ are $n \times n$ and $n \times m$ matrices respectively and $a$ and $b$ are $n \times 1$ and $m \times 1$ vectors respectively, then what is the cost of:

   (a) Computing $AB$

**Solution:**
Given:- $A$ and $B$ are $n \times n$ and $n \times m$ matrices respectively .

Every element of AB can be computed by dot product of rows of A with columns of B . Hence , we need n multiplications and n-1 additions for each element of

AB.

$$\therefore \ Cost\ of\ computing\ each\ element\ of\ AB = n + n - 1$$
$$= 2n - 1$$

AB will result in a matrix containing $n \times m$ elements.

$$\therefore \ Cost\ of\ computing\ AB = n \times m \times (2n - 1)$$
$$= 2mn^2 - mn$$
$$= O\left(mn^2\right)$$

Solution of Hithesh M, hitheshm@tenet.res.in

(b) Computing $B^T a$

**Solution:**
Given:- B is a $n \times m$ matrix and a is $n \times 1$ vector .
$B^T$ will result in $m \times n$ matrix. Every element of $B^T a$ can be computed by dot product of rows of $B^T$ with a . Hence , we need n multiplications and n-1 additions for each element of $B^T a$ .

$$\therefore \ Cost\ of\ computing\ each\ element\ of\ B^T a = n + n - 1$$
$$= 2n - 1$$

$B^T a$ will result in a vector containing m elements.

$$\therefore \ Cost\ of\ computing\ B^T a = m \times (2n - 1)$$
$$= 2nm - m$$
$$= O\left(mn\right)$$

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(c) Computing $A^{-1}$

**Solution:** We will compute the cost of $A^{-1}$ using Gauss-Jordan method.

$\therefore$ We will perform some row operations on [A I] where I is a $n \times n$ identity matrix tacked onto A.
Now we have a matrix that has n rows and 2n columns.
To get a zero in (2,1) position of the matrix , we need to perform 'n' operations

on A since all elements in that row will change and only '1' operation on I since only one element will change in I.
$\therefore$ Cost for obtaining zero in (2,1) position in the matrix = n + 1
$\therefore$ Cost for obtaining '0' in all postions below pivot 1 = (n - 1) * (n + 1) , since there are (n-1) rows below pivot 1 .

To get a zero in (3,2) position of matrix , we need 'n + 1' operations since there are n - 1 elements to be changed in A and 2 elements will change in I .
$\therefore$ Cost for obtaining zero in (3,2) position in the matrix = n - 1 + 2 = n + 1 .
Cost for obtaining '0' in all postions below pivot 2 = (n - 2) * (n + 1) , since there are (n - 2) rows below pivot 2 .

$\therefore$ Cost for reducing A to U ,

$$\begin{aligned} &= (n-1)*(n+1) + (n-2)*(n+1) + \cdots + 1*(n+1) \\ &= (n+1)(n-1+n-2+\cdots+1) \\ &= \frac{n^3-n}{2} \end{aligned} \tag{1}$$

Now we have to make the entries above the pivots to zero .
To get a zero in (n-1,n) position of the matrix , we need '1' operation on A and 'n' operations on I since all elements in $(n-1)^{th}$ row of I will change.
$\therefore$ Cost for obtaining '0' in (n-1,n) position = n + 1
$\therefore$ Cost for obtaining '0' in all postions above last pivot = (n - 1) * (n + 1) , since there are (n-1) rows above the last pivot.

$\therefore$ Cost for reducing U to D ,

$$\begin{aligned} &= (n-1)*(n+1) + (n-2)*(n+1) + \cdots + 1*(n+1) \\ &= (n+1)(n-1+n-2+\cdots+1) \\ &= \frac{n^3-n}{2} \end{aligned} \tag{2}$$

To reduce D to I , we need to divide each row by it's pivot.
$\therefore$ Cost for obtaining 1 in pivot position in one row of the matrix = n + 1 , since n elements in I will be divided and '1' element in A will change.
There are n rows in the matrix.

$\therefore$ Cost for reducing D to I ,

$$\begin{aligned} &= (n+1)*n \\ &= n^2+n \end{aligned} \tag{3}$$

$\therefore$ Total cost for computing $A^{-1}$,

$$= \frac{n^3 - n}{2} + \frac{n^3 - n}{2} + n^2 + n \qquad\qquad \mathit{From}\ (1)\ ,\ (2)\ ,\ (3)$$
$$= O(n^3)$$

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**Concept**: LU factorisation

8. (1 ½ points) (a) Under what conditions is the would A have a full set of pivots ?

A=$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

**Solution:** If we performed multiplication we get
A=$\begin{bmatrix} d_1 & -d_1 & 0 \\ -d_1 & d_1 + d_2 & -d_2 \\ 0 & -d_2 & d_2 + d_3 \end{bmatrix}$
After Gaussian elimination:
A=$\begin{bmatrix} d_1 & -d_1 & 0 \\ 0 & d_2 & -d_2 \\ 0 & 0 & d_3 \end{bmatrix}$
A would have full set of pivots if pivot elements are non-zero i.e. $d_1 \neq 0$ , $d_2 \neq 0$ and $d_3 \neq 0$

(b) Solve as two triangular systems, without multiplying LU to find A:
LUx=$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

**Solution:** Step 1) Lc = b

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

We get Equations :-

$$c_1 = 2 \tag{1}$$
$$c_1 + c_2 = 0 \tag{2}$$
$$c_1 + c_3 = 2 \tag{3}$$

Solving them We can easily get : $c_1 = 2, c_2 = -2, c_3 = 0$
Step 2) Ux = c

$$\begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

We get Equations :-

$$2x_1 + 4x_2 + 4x_3 = 2 \tag{4}$$
$$x_2 + 2x_3 = -2 \tag{5}$$
$$x_3 = 0 \tag{6}$$

Solving them We can easily get , x1 = 5, x2 = -2, x3 = 0

9. (2 points) Consider the following system of linear equations. Find the $LU$ factorisation of the matrix A corresponding to this system of linear equations. Show all the steps involved. (this is where you will see what happens when you have to do more than 1 permutations).

$$\begin{aligned} x + y &= -3 \\ w - x - y &= +2 \\ 3w - 3x - 3y - z &= -19 \\ -5x - 3y - 3z &= -2 \end{aligned}$$

**Solution:**

**Solution:** The matrix A corresponding to the given system of linear equations is :

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 3 & -3 & -3 & -1 \\ 0 & -5 & -3 & -3 \end{bmatrix}$$

For LU factorisation, we apply the following steps:

1. Swap Row1 and Row2 with $P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ to get:

$$P_1 A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & -3 & -3 & -1 \\ 0 & -5 & -3 & -3 \end{bmatrix}$$

2. $Row3 = Row3 - 3 \times Row1$ with $E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ to get:

$$E_{31} P_1 A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -5 & -3 & -3 \end{bmatrix}$$

3. $Row4 = Row4 + 5 \times Row2$ with $E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{bmatrix}$ to get:

$$E_{42} E_{31} P_1 A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -3 \end{bmatrix}$$

4. Swap Row3 and Row4 with $P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ to get:

$$P_2 E_{42} E_{31} P_1 A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ We get $P_2E_{42}E_{31}P_1A = U$
$\Rightarrow A = (P_2E_{42}E_{31}P_1)^{-1}U$
$\Rightarrow A = (P_1)^{-1}(E_{31})^{-1}(E_{42})^{-1}(P_2)^{-1}U$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} U$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & -5 & 1 & 0 \end{bmatrix} U$$

But L should be lower triangular, so pre-multiply both sides by $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

to get:

$$PA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & -5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow PA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\Rightarrow PA = LU$, where:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Solution of SALTANAT FIRDOUS ALLAQBAND

10. (1 point) For a square matrix A, prove that $LDU$ factorisation is unique.

**Solution:**
Given :- A is a square matrix.
Assume A is an invertible matrix that has two LDU factorisations.

$$\therefore A = L_1D_1U_1 \tag{1}$$
$$A = L_2D_2U_2 \tag{2}$$

To prove :-

$$L_1 = L_2$$
$$D_1 = D_2$$
$$U_1 = U_2$$

From (1) and (2) ,

$$L_1 D_1 U_1 = L_2 D_2 U_2$$

Since A is invertible , inverses for $L_1, D_1, U_1, L_2, D_2, U_2$ exists.

$$\begin{aligned} L_2^{-1} L_1 D_1 U_1 &= D_2 U_2 && Pre-multiplying\ by\ L_2^{-1}\ on\ both\ sides \\ L_2^{-1} L_1 D_1 &= D_2 U_2 U_1^{-1} && Post-multiplying\ by\ U_1^{-1}\ on\ both\ sides \\ L_2^{-1} L_1 &= D_2 U_2 U_1^{-1} D_1^{-1} \quad (3) && Post-multiplying\ by\ D_1^{-1}\ on\ both\ sides \end{aligned}$$

In LDU factorisation , L and U have 1's along their diagonal .
Inverse of an upper triangular matrix is upper triangular .
Inverse of a lower triangular matrix is lower triangular.

$\therefore$ From (3) we can say that ,
$L_2^{-1} L_1$ is a lower triangular matrix.
$D_2 U_2 U_1^{-1} D_1^{-1}$ is an upper traingular matrix. (multiplication of an upper triangular matrix with a diagonal matrix results in an upper triangular matrix).

Equation (3) will hold true only when both the sides of the equation is a diagonal matrix.(Diagonal matrices are both upper and lower triangular matrices)

$$\therefore L_2^{-1} L_1 = D_2 U_2 U_1^{-1} D_1^{-1} = D \qquad (4)$$

where D is a diagonal matrix.

$$D_2 U_2 U_1^{-1} D_1^{-1} = D \qquad (5) \qquad From\ (4)$$

The above equation will hold true only when $U_2 U_1^{-1}$ is a digonal matrix.
$\therefore U_2 U_1^{-1}$ is a diagonal matrix.

We know that $U_2\ and\ U_1$ have 1's in their diagonal.
$\implies U_2 U_1^{-1}$ is a digonal matrix with 1's in their diagonal.

A diagonal matrix with 1's in their diagonal is an Identity matrix (I).

$$\begin{aligned} &\therefore U_2 U_1^{-1} = I \\ &\implies U_2 = U_1 \qquad (6) \end{aligned}$$

$\therefore$ From (5) and (6) ,

$$D_2 I D_1^{-1} = D$$
$$\therefore D_2 D_1^{-1} = D \qquad (7)$$

$$L_2^{-1} L_1 = D \qquad (8) \qquad From\ (4)$$

$L_2^{-1}L_1$ will be a diagonal matrix with 1's in their diagonal since $L_2$ and $L_1$ have 1's in their diagonal.
A diagonal matrix with 1's in their diagonal is an Identity matrix (I).

$$L_2^{-1} L_1 = I \qquad (9)$$
$$\implies L_2 = L_1 \qquad (10)$$

From (7) , (8) , (9)

$$D_2 D_1^{-1} = I$$
$$\implies D_2 = D_1 \qquad (11)$$

$\therefore$ From (6) , (10) , (11) , we have shown that

$$L_1 = L_2$$
$$D_1 = D_2$$
$$U_1 = U_2$$

$\implies$ LDU factorisation is unique when A is invertible.

Hence proved.
Solution of Hithesh M, hitheshm@tenet.res.in

11. (1 ½ points) Consider the matrix $A$ which factorises as:

$$\begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Without computing $A$ or $A^{-1}$ argue that

(a) A is invertible (I am looking for an argument which relies on a fact about elementary matrices)

**Solution:** Let us take L = $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$ and U = $\begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

L can be written as product of two elementary matrices, $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$.

Similarly U can also be written as product of two elementary matrices , $E_3 = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

$\implies$ A = $E_1E_2E_3E_4$, where each elementary matrix is invertible, and thus A is invertible.

(b) A is symmetric (convince me that $A_{ij} = A_{ji}$ without computing $A$)

**Solution:**

**Solution:** Here we can observe that $U = L^T$ and vice-versa. $\therefore A = LU = LL^T$, and we know $LL^T$ is symmetric as $(LL^T)^T = LL^T$. Hence, $A$ is symmetric.

(c) A is tridiagonal (again, without computing $A$ convince me that all elements except along the 3 diagonals will be 0.)

**Solution:** For the above A=LU, L matrix does the following operation on the U matrix.
First row of U remains unchanged (L's first row is [1 0 0])
Second row of U : R2 $->$ 7R1+R2
Third row of U : R3 $->$ 5R2+R3
So, first row of A will be [1 7 0].
$\implies$ $A_{31} = 0$
$\implies$ $A_{13}$=0 , since A is symmteric as proved in (b).
So, the only two elements other than the three diagonals(main diagonal, digonal just below it and the diagonal just above it) element are 0.
$\implies$ A is tridiagonal.

**Concept**: Lines and planes

12. (1 ½ points) Consider the following system of linear equations

$$a_1x_1 + b_1y_1 + c_1z_1 = 1$$
$$a_2x_2 + b_2y_2 + c_2z_2 = 2$$
$$a_3x_3 + b_3y_3 + c_3z_3 = 3$$

Each equation represents a plane, so find out the values for the coefficients such that the following conditions are satisfied:

1. All planes intersect at a line
2. All planes intersect at a point
3. Every pair of planes intersects at a different line.

**Solution:** The matrix form of above system of equations is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

**1. Line**
Equations intersect in a line implies that we have $\infty$ solutions.

So, there must be two independent rows and one dependent row

One simple example will be considering $R_3 \longleftarrow R_1 + R_2$
So, one good example is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

**2. Point**
Equations intersect at a point means the solution is unique.
Hence all the rows must be independent

One simple example is

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 4 \\ 3 & 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

**3. Pair wise intersection**
Each intersect at a different line $\implies$ 0 Solutions

Let, $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 1 & x & y & 3 \end{array}\right]$

Applying $R_2 \longleftarrow R_2 - 2R_1$ and $R_3 \longleftarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & x-2 & y-3 & 2 \end{array}\right]$$

Applying $R_3 \longleftarrow R_3 + 3R_2$ (Say we are assuming some row operation)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & x-5 & y-9 & 2 \end{array}\right]$$

Now to let x = 5 and y = 9

So, our original matrix becomes $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 9 \end{bmatrix}$

which has 0 solutions for the above system.

13. (1 ½ points) Starting with a first plane $u - v - w = -1$, find the equation for

(a) the parallel plane through the origin.

**Solution:** $u - v - w = 0$

(b) a second plane that also contains the points (-1,-1,1) and (-7,-5,-1).

**Solution:** Many solutions possible. One such solution is $u - 4v + 5w = 8$

(c) a third plane that meets the first and second in the point (2, 1, 2).

**Solution:** Many solutions possible. One such solution is $u + v + w = 5$

**Concept**: Transpose

14. (2 points) Consider the transpose operation.

(a) Show that it is a linear transformation.

**Solution:** Define $T : M_{m\times n} \to M_{n\times m}$ by $T(A) = A^t = B$ where $b_{ij} = a_{ji}$
Let $A, B$ be $m \times n$ matrices where $A = (a_{ij)}$ and $B = (b_{ij})$

Consider $T(A + B) = (A + B)^t$:
Then $(i, j)$ element of $(A + B)^t = (j, i)$ element of $(A + B)$
$= (j, i)$ element of $A+$ $(j, i)$ element of $B$
$= (i, j)$ element of $A^t+$ $(i, j)$ element of $B^t$

$(i, j)$ element of $A^t + B^t = (i, j)$ element of $A^t$ $+(i, j)$ element of $B^t$
Since this is true for all $(i, j)$ we have:
$(A + B)^t = A^t + B^t = T(A) + T(B)$ (15)

Now consider $T(\lambda A) = (\lambda A)^t$
$(i, j)$ element of $(\lambda A)^t = (\lambda((i, j)$ entry of $A^t)) = \lambda((i, j)$ entry of $A^t)$
Since this is true for all $(i, j)$ we have $T(\lambda A) = \lambda A^t = \lambda T(A)$ (16)

From 15 and 16 we have, $T$ is a linear transformation

(b) Find the matrix corresponding to this linear transformation.

**Solution:** $T : M_{m\times n} \to M_{n\times m}$ by $T(A) = A^T = B$ where $b_{ij} = a_{ji}$
Let $e_{ij}$ be the $m \times n$ matrix with a value of 1 at entry $(i, j)$ and 0 elsewhere. This is a basis for the space $M_{m\times n}$. Let us fix the order of our basis as $\{e_{11}, e_{12}, \dots e_{1n}, e_{21}, \dots e_{2n}, \dots, e_{mn}\}$. Then define transpose operation on that space as follows:

$$T(e_{ij}) = f_{ji} \ \forall \ i \in \{1 \ .... \ m\} \text{ and } j \in \{1 \ ... \ n\}$$

where $f_{ji}$ is an $n \times m$ matrix with a value of 1 at entry $(j, i)$ and 0 elsewhere. Now to represent the transpose transformation as matrix we observe that there exists a isomorphism between $M_{m\times n}$ and $\mathbb{R}^{mn}$ as well as between $M_{n\times m}$ and $\mathbb{R}^{mn}$

This is done as follows:
**Rule 1:** We take a matrix in $M_{m\times n}$ or $M_{n\times m}$ and stack the rown in order to produce a vector in $\mathbb{R}^{mn}$

**Rule 2:** To move from $\mathbb{R}^{mn}$ to $M_{m\times n}$ we take first $n$ entries and make them as first row and so on $m$ times.

**Rule 3:** To move from $\mathbb{R}^{mn}$ to $M_{n\times m}$ we take first $m$ entries and make them as first row and so on $n$ times.

Now we calculate $f_{ji}$ as $T(e_{ij})$ $\forall$ $i \in \{1 \text{ .... } m\}$ and $j \in \{1 \text{ ... } n\}$

Now we represent all these $f_{ij}$ in $\mathbb{R}^{mn}$ using above Rule 1 and put all these vectors in a matrix $A$ which is our transformation matrix for transpose operation.

$$A = \begin{bmatrix} f_{11}^{mn} & f_{21}^{mn} & \dots & f_{n1}^{mn} & f_{12}^{mn} & f_{22}^{mn} & \dots & f_{nm}^{mn} \end{bmatrix}$$ , where $f_{ij}^{mn}$ is the representation of $f_{ij}$ in $\mathbb{R}^{mn}$.

**Note:** The transpose operation using $A$ is done as follows:
We take a matrix $X$ in $M_{m\times n}$, morph it into a vector $\mathbf{v}$ in $\mathbb{R}^{mn}$ using Rule 1, then apply $A$ on this vector to get another vector $\mathbf{u}$ in $R^{mn}$, then morph it into $M_{n\times m}$ using Rule 3 and that is our $X^T$.