Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

N. Kw

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Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: Yes

Concept: System of linear equations

2. (2 points) This question has two parts as mentioned below:
   (a) Find a 2 x 3 system Ax = b whose complete solution is

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Solution:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, Now, $x_{particular} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $x_{nullspace} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Therefore, $Ax_{nullspace} = 0$,

$=> a_{11} + 2a_{12} + a_{13} = 0$

$=> a_{21} + 2a_{22} + a_{23} = 0$

Also, $Ax_{particular} = b = (\begin{bmatrix} b_1 & b_2 \end{bmatrix})^\top$,

If we fix A as $\begin{bmatrix} -1 & 1 & -1 \\ 1 & -2 & 3 \end{bmatrix}$, for $Ax_{nullspace} = 0$, $-1{+}2{-}1 = 0$ and $1{-}4{+}3 = 0$.

Now, $Ax_{particular} = (\begin{bmatrix} b_1 & b_2 \end{bmatrix})^\top$, $=> \begin{bmatrix} -1 & 1 & -1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$b = \begin{bmatrix} (-1+1-1) \\ (1-2+3) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Therefore system is,

$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -2 & 3 \end{bmatrix} x = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(b) Now find a 3 x 3 system which has these solutions exactly when $b_1 + b_2 + b_3 = 0$. (Note: $b = [b_1\ b_2\ b_3]^T$.)

Solution:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

Similar to part (a), if we fix A as, $\begin{bmatrix} -1 & 1 & -1 \\ 1 & -2 & 3 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

corresponding b is $\begin{bmatrix} -1 \\ 2 \\ b_3 \end{bmatrix}$, as $b_1 + b_2 + b_3 = 0 \Rightarrow b_3 = 1 - 2 = -1$.

Therefore $b = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$. So we need $a_{31} + 2a_{32} + a_{33} = 0$ for $x_{nullspace} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and

$a_{31} + a_{32} + a_{33} = b_3 = -1$ for $x_{particular} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Therefore, we can take $a_{33} = 0$, $a_{32} = 1$ and $a_{31} = -2$. Therefore system is,

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -2 & 3 \\ -2 & 1 & 0 \end{bmatrix} x = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

3. (2 points) Consider the matrices A and B below

(i) A = $\begin{bmatrix} 1 & 6 & -3 & 4 \\ 0 & 2 & 4 & 0 \\ 2 & 12 & -6 & 8 \end{bmatrix}$ (ii) B = $\begin{bmatrix} 3 & 1 & 2 \\ 12 & 4 & 8 \\ 6 & 2 & 4 \end{bmatrix}$

(a) Write down the row reduced echelon form of matrices A and B (also mention the steps involved).

Solution:

For A, $A = \begin{bmatrix} 1 & 6 & -3 & 4 \\ 0 & 2 & 4 & 0 \\ 2 & 12 & -6 & 8 \end{bmatrix}$

$R3 = R3 - 2R1 => \begin{bmatrix} 1 & 6 & -3 & 4 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $R1 = R1 - 3R2 => \begin{bmatrix} 1 & 0 & -15 & 4 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$R2 = 0.5R2 => \begin{bmatrix} 1 & 0 & -15 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ = Row reduced echelon form of A.

For B, $B = \begin{bmatrix} 3 & 1 & 2 \\ 12 & 4 & 8 \\ 6 & 2 & 4 \end{bmatrix}$, $R2 = R2 - 4R1 => \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 6 & 2 & 4 \end{bmatrix}$

$R3 = R3 - 2R1 => \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $R1 = (\frac{1}{3})R1 => \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = Row reduced echelon form of B.

(b) Find all solutions to $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$.

Solution:

For $Ax = 0$, Row reduced echelon form of $A = \begin{bmatrix} 1 & 0 & -15 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

If $x = (\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix})^\top$, x_3 and x_4 are free variables and x_1 and x_2 are pivot variables.

If we take $x_3 = 1$ and $x_4 = 0$, $x_2 + 2 + 0 = 0 => x_2 = -2$, also, $x_1 - 15 + 0 = 0 => x_1 = 15$.

If we take $x_3 = 0$ and $x_4 = 1$, $x_2 + 0 + 0 = 0 => x_2 = 0$, also, $x_1 + 0 + 4 = 0 => x_1 = -4$.

So, basis vectors for null-space of A are, $\begin{bmatrix} 15 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Solutions to $Ax = 0$ are p $\begin{bmatrix} 15 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ + q $\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, where $p, q \in R$.

For $Bx = 0$, Row reduced echelon form of $B = \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

If $x = (\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix})^\top$, x_2 and x_3 are free variables and x_1 is pivot variable.

If we take $x_2 = 1$ and $x_3 = 0$, $x_1 + \frac{1}{3} + 0 = 0 => x_1 = -\frac{1}{3}$.

If we take $x_2 = 0$ and $x_3 = 1$, $x_1 + 0 + \frac{2}{3} = 0 => x_1 = -\frac{2}{3}$.

So, basis vectors for null-space of B are, $\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$.

Solutions to $Bx = 0$ are p $\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$ + q $\begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$, where $p, q \in R$.

(c) Write down the basis for the four fundamental subspaces of A.

Solution:

To find basis of $C(A)$. Columns of A are, $a_1 = (\begin{bmatrix}1 & 0 & 2\end{bmatrix})^\top$, $a_2 = (\begin{bmatrix}6 & 2 & 12\end{bmatrix})^\top$, $(\begin{bmatrix}4 & 0 & 8\end{bmatrix})^\top = 4(a_1)$ and
$(\begin{bmatrix}-3 & 4 & -6\end{bmatrix})^\top = (\begin{bmatrix}12 & 4 & 24\end{bmatrix})^\top + (\begin{bmatrix}-15 & 0 & -30\end{bmatrix})^\top = 2(a_2) - 15(a_1)$.
Therefore $C(A)$ has basis vectors as independent columns of A,
$\begin{bmatrix}1\\0\\2\end{bmatrix}$ and $\begin{bmatrix}6\\2\\12\end{bmatrix}$.

To find basis of $N(A)$, As found on Q3 (b),
basis vectors of $N(A)$ are $\begin{bmatrix}15\\-2\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-4\\0\\0\\1\end{bmatrix}$.

To find basis of $C(A^\top)$, Rows of A are, $a_1 = \begin{bmatrix}1 & 6 & -3 & 4\end{bmatrix}$, $a_2 = \begin{bmatrix}0 & 2 & 4 & 0\end{bmatrix}$ and $\begin{bmatrix}2 & 12 & -6 & 8\end{bmatrix} = 2a_1$. Therefore $C(A^\top)$ has basis vectors as independent rows of A,
$\begin{bmatrix}1\\6\\-3\\4\end{bmatrix}$ and $\begin{bmatrix}0\\2\\4\\0\end{bmatrix}$.

To find basis of $N(A^\top)$,
If we apply the row operations performed on A to get R(A) in Q3 (b) to I,
$$\mathrm{I} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}, R3 = R3 - 2R1 => \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\-2 & 0 & 1 & 0\end{bmatrix},$$
$$R1 = R1 - 3R2 => \begin{bmatrix}1 & -3 & 0 & 0\\0 & 1 & 0 & 0\\-2 & 0 & 1 & 0\end{bmatrix}, R2 = 0.5R2 => \begin{bmatrix}1 & -3 & 0 & 0\\0 & \frac{1}{2} & 0 & 0\\-2 & 0 & 1 & 0\end{bmatrix} = \mathrm{E}$$
Since in row reduced echelon form of A, the last row was 0, therefore we get 0 for linear combination of rows of A with weights as the last row in E.
Therefore $N(A^\top)$ has basis vectors as last row of E, $\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}$.

(d) Write down the basis for the four fundamental subspaces of B.

Solution:

To find basis of $C(B)$. Columns of B are, $b_2 = (\begin{bmatrix}1 & 4 & 2\end{bmatrix})^\top$,
$(\begin{bmatrix}3 & 12 & 6\end{bmatrix})^\top = 3(b_2)$ and
$(\begin{bmatrix}2 & 8 & 4\end{bmatrix})^\top = 2(b_2)$.

Therefore $C(B)$ has basis vectors as independent columns of B, $\begin{bmatrix}1\\4\\2\end{bmatrix}$.

To find basis of $N(B)$, As found on Q3 (b),
basis vectors of $N(B)$ are $\begin{bmatrix}-\frac{1}{3}\\1\\0\end{bmatrix}$, $\begin{bmatrix}-\frac{2}{3}\\0\\1\end{bmatrix}$.

To find basis of $C(B^\top)$, Rows of B are, $b_1 = \begin{bmatrix}3 & 1 & 2\end{bmatrix}$,
$b_2 = \begin{bmatrix}12 & 4 & 8\end{bmatrix} = 4(b_1)$,
$b_3 = \begin{bmatrix}6 & 2 & 4\end{bmatrix} = 2(b_1)$,

Therefore $C(B^\top)$ has basis vectors as independent rows of B, $\begin{bmatrix}3\\1\\2\end{bmatrix}$

To find basis of $N(B^\top)$,
If we apply the row operations performed on B to get R(B) in Q3 (b) to I,

$$\text{I} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, R2 = R2 - 4R1 => \begin{bmatrix}1 & 0 & 0\\-4 & 1 & 0\\0 & 0 & 1\end{bmatrix},$$

$$R3 = R3 - 2R1 => \begin{bmatrix}1 & 0 & 0\\-4 & 1 & 0\\-2 & 0 & 1\end{bmatrix}, R1 = \tfrac{1}{3}R1 => \begin{bmatrix}\frac{1}{3} & 0 & 0\\-4 & 1 & 0\\-2 & 0 & 1\end{bmatrix} = \text{E}$$

Since in row reduced echelon form of B, the last 2 rows were 0, therefore we get 0 for linear combination of rows of B with weights as the last 2 rows in E.

Therefore $N(B^\top)$ has basis vectors as last 2 rows of E, $\begin{bmatrix}-4\\1\\0\end{bmatrix}$, $\begin{bmatrix}-2\\0\\1\end{bmatrix}$.

Concept: Rank

4. (1 ½ points) Consider the matrices A and B as given below:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -6 & -4 & -2 \\ 3 & 2 & x \end{bmatrix} \text{ and B} = \begin{bmatrix} 7 & 2 & 7 \\ y & 2 & y \end{bmatrix}$$

Give the values for entries x and y such that the ranks of the matrices A and B are

(a) 1

Solution:

For A,
$R2 = \begin{bmatrix} -6 & -4 & -2 \end{bmatrix} = -2\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} = -2R1$. Therefore R2 is a dependent row.

For rank(A) = 1, we need R3 also to be a dependent row on R1. Therefore, $R3 = \begin{bmatrix} 3 & 2 & x \end{bmatrix} = pR1 = \begin{bmatrix} 3p & 2p & p \end{bmatrix}$. Hence, $p = 1$, therefore $x = 1$.

For B,

For rank(B) = 1, we need R2 to be a dependent row on R1. Therefore, $R2 = \begin{bmatrix} y & 2 & y \end{bmatrix} = pR1 = \begin{bmatrix} 7p & 2p & 7p \end{bmatrix}$. Hence, $p = 1$, therefore $y = 7$.

(b) 2

Solution:

For A,

$R2 = -2R1$ and hence R2 is a dependent row.

For rank(A) = 2, we need R3 to be independent of R1 and R2. Therefore, $R3 = \begin{bmatrix} 3 & 2 & x \end{bmatrix} \neq pR1 = \begin{bmatrix} 3p & 2p & p \end{bmatrix}$. Hence as $p = 1$, therefore $x \neq 1$ for rank 2. Therefore x can be any value other than 1. Example, x = 2.

For B,

For rank(B) = 2, we need R2 to be independent of R1. Therefore, $R2 = \begin{bmatrix} y & 2 & y \end{bmatrix} \neq pR1 = \begin{bmatrix} 7p & 2p & 7p \end{bmatrix}$. Hence as $p = 1$, therefore $y \neq 7$ for rank 2. Therefore y can be any value other than 7. Example, y = 1.

(c) 3

Solution:

For A,

$R2 = -2R1$ and hence R2 is a dependent row.

Therefore even if R3 is independent, rank can only be 2. Therefore for no value of x, rank(A) can be 3.

For B,

Since B has only 2 rows, the maximum possible rank is 2. Therefore for no value of y, rank(B) can be 3.

Concept: Nullspace and column space

5. (½ point) State True or False and explain you answer: The nullspace of R is the same as the nullspace of U (where R is the row reduced echelon form of A and U is the matrix in LU decomposition of A).

Solution:

For A, let null space be N(A).

Therefore if $x \in N(A)$, $Ax = 0$. If we consider a invertible matrix M, now,

$M(Ax) = M(0) = 0 => (MA)x = 0$, i.e. N(A) is included in N(MA). I.e. for any $x \in N(A)$, it is also in $N(MA)$.

Let for some y, $(MA)y = 0$, as M is invertible, now, $M^{-1}(MA)y = M^{-1}0 => (M^{-1}M)Ay = 0 => Ay = 0$. I.e. for any $y \in N(MA)$, it is also in $N(A)$.

As all elements in N(A) are in N(MA) and vice versa, $N(A) = N(MA)$.

I.e. For any invertible matrix M, A and MA have same null space.

Row reduced echelon form of A, R is obtained by performing invertible operations on A (row operations, scaling diagonals, swapping rows). Therefore we can write, $R = E_R A$.

As E_R is invertible, $N(R) = N(E_R A) = N(A)$.

Similarly, for U, we can reach U by performing elementary operations on A which are invertible. $U = EA$

Therefore $N(U) = N(EA) = N(A)$.

Therefore N(R) = N(A) = N(U).

TRUE, null space of R is same as null space of U.

6. (1 point) Construct a matrix whose column space contains $[2,5,3]^\top$ and $[0,3,1]^\top$ and whose null space contains $[1,3,2]^\top$

Solution:

Since column space has the given 2 vectors, we can take those 2 vectors themselves as 2 columns of A. Also clearly the 2 vectors are independent as they arent multiples of each other.

Therefore, let $A = \begin{bmatrix} 2 & 0 & a \\ 5 & 3 & b \\ 3 & 1 & c \end{bmatrix}$

Since null space contains $y = (\begin{bmatrix} 1 & 3 & 2 \end{bmatrix})^\top$, $Ay = 0$.

2*1 + 0*3 + 2a = 0 => a = -1.

5*1 + 3*3 + 2b = 0 => b = -7.

3*1 + 1*3 + 2c = 0 => c = -3.

Therefore matrix is, $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 3 & -7 \\ 3 & 1 & -3 \end{bmatrix}$

7. (2 points) Consider the matrix $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 9 \end{bmatrix}$. The column space of this matrix is a 2 dimensional plane. What is the equation of this plane? (You need to write down the steps you took to arrive at the equation)

Solution:

Column space of A, C(A) is, $p\begin{bmatrix}3\\2\\1\end{bmatrix} + q\begin{bmatrix}0\\1\\9\end{bmatrix} = \begin{bmatrix}3p\\2p+q\\p+9q\end{bmatrix}$, where $p, q \in R$

Plane is the set of all 3D points generated by giving different values of p and q.

If we consider plane equation as, $ax + by + cz + d = 0$, where $x = 3p$, $y = 2p + q$, $z = p + 9q$.

$=> a(3p) + b(2p + q) + c(p + 9q) + d = 0 => (3a + 2b + c)p + (b + 9c)q = -d,$

Since all values of p and q must satisfy this equation,

For p=0, q=0, $0 + 0 = -d => d = 0$.

For p=0, q=1, $0 + b + 9c = 0 => b = -9c$.

For p=1, q=0, $3a+2b+c+0 = 0 => 3a = -c-(-18c) = -c+9c = 17c => a = \frac{17}{3}c$.

Therefore equation is, $ax + by + cz + d = 0 => \frac{17}{3}c - 9cy + cz + 0 = 0$

$=> 17x - 27y + 3z = 0$.

8. (1 point) True or false? (If true give logical, valid reasoning or give a counterexample if false)
a. If the row space equals the column space then $A^T = A$

Solution:

FALSE

As for any skew-symmetric matrix, $A^\top = -A$.

Example, $A = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$, $A^\top = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = -A$

For the example, $C(A) = p\begin{bmatrix}0\\1\end{bmatrix} + q\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}-q\\p\end{bmatrix}$ for $p, q \in R$.

Row space of A $= C(A^\top) = a\begin{bmatrix}0\\-1\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\-b\end{bmatrix}$ for $a, b \in R$.

As a, b, p, q can be any value in R, for every vector in C(A), we can put $a = -q$ and $b = -p$ to get the same vector in $C(A^\top)$. Similarly vice versa every vector in $C(A^\top)$, we can put $q = -a$ and $p = -b$ and get the same vector in C(A).

Therefore $C(A)$ is same as $C(A^\top)$ and hence is same as row space of A.

Hence for this matrix, row space and column space are same but $A^\top \neq A$.

b. If $A^T = -A$ then the row space of A equals the column space.

Solution:

TRUE

Given that $A^\top = -A$.

Since row space of A = column space of $A^\top$, we need to prove that, $C(A) = C(A^\top)$.

$C(A^\top) = C(-A)$

Let A have n columns and its columns be denoted by $a_{c1}, a_{c2}, ...a_{cn}$.

$C(A) = w_{c1}a_{c1} + ...w_{cn}a_{cn}$ for $w_{c1}, w_{c2}, ...w_{cn} \in R$.

$C(A^\top) = C(-A) = w_{r1}(-a_{c1}) + ...w_{rn}(-a_{cn})$ for $w_{r1}, w_{r2}, ...w_{rn} \in R$. $=> C(-A) = (-w_{r1})a_{c1} + ...(-w_{rn})a_{cn}$.

Since all belong to R, for any $x \in R$, there exists a $y \in R$ such that y = -x.

Therefore for any vector in $C(A)$, we can put $w_{r1} = -w_{c1}, ...w_{rn} = -w_{cn}$ to get the same vector in $C(-A)$.

Similarly, vice versa, for any vector in $C(-A)$, we can put $w_{c1} = -w_{r1}, ...w_{cn} = -w_{rn}$ to get the same vector in $C(A)$.

Combining these two statements, we get that $C(A)$ is same as $C(-A)$ and hence same as $C(A^\top)$ and further same as the row space of A.

Therefore given that $A^\top = -A$, the row space and column space of A are same.

9. (1 point) What are the dimensions of the four subspaces for **A, B,** and **C,** if I is the 3×3 identity matrix and 0 is the 3×2 zero matrix?

A=$\begin{bmatrix} I & 0 \end{bmatrix}$ and B =$\begin{bmatrix} I & I \\ 0^\top & 0^\top \end{bmatrix}$ and C=$\begin{bmatrix} 0 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For A,

$dim(\mathcal{C}(A)) = dim(\mathcal{C}(A^\top)) =$ rank of A = number of non zero pivot rows = 3

$dim(N(A)) =$ number of columns - rank = 5-3 = 2

$dim(N(A^\top)) =$ number of rows - rank = 3-3 = 0

For B,

$dim(\mathcal{C}(B)) = dim(\mathcal{C}(B^\top)) =$ rank of B = number of non zero pivot rows = 3

$dim(N(B)) =$ number of columns - rank = 6-3 = 3

$dim(N(B^\top)) =$ number of rows - rank = 5-3 = 2

For C,

$dim(\mathcal{C}(C)) = dim(\mathcal{C}(C^\top)) =$ rank of C = number of non zero pivot rows = 0

$dim(N(C)) =$ number of columns - rank = 2-0 = 2

$dim(N(C^\top)) =$ number of rows - rank = 3-0 = 3

10. (2 points) Solve the following questions.

(a) If A is an m×n matrix, find dim($\mathcal{R}(A)$) + dim($\mathcal{C}(A)$) + dim($\mathcal{N}(A)$) + dim($\mathcal{N}(A^\top)$). (in terms of n & m)

Solution:

$dim(R(A))$ depends on the number of independent rows in A and that is determined by number of non-zero pivots in A which is same as rank of A.

Therefore, if r is rank of A, then $dim(R(A)) = r$.

Also by same logic, $dim(C(A)$ is number of independent columns in A which is also determined by non-zero pivots and is equal to rank of A. $dim(C(A)) = r$.

Since we know, $dim(N(A)) = n - r$, $dim(N(A^\top)) = m - r$.

Therefore,

$dim(R(A)) + dim(C(A)) + dim(N(A)) + dim(N(A^\top)) = r + r + n - r + m - r = n + m$

(b) Let A and B be two $n \times n$ matrices such that $AB = 0$. Show that the row space of A is contained in the left null space of B.

Solution:

Let A have rows $a_1, ...a_n$.
As AB = 0,
$a_1 B = 0, a_2 B = 0, ...a_n B = 0$.
If we consider $w_1, w_2, ...w_n \in R$, then $(w_1 a_1 + w_2 a_2 + ...w_n a_n)B = w_1(a_1 B) + w_2(a_2 B) + ...w_n(a_n B) = w_1(0) + ...w_n(0) = 0$. Therefore any linear combination of rows of A are included in the left null space of B.

Since row space of A is just all the linear combinations of rows of A and all of those are included in the left null space of B, we can say that row space of A is contained in the left null space of B.

11. (1 point) True or false? If A is a $n \times n$ square matrix then $\mathcal{N}(A) = \mathcal{N}(AA^T)$ (If true give logical, valid reasoning or give a counterexample if false)

Solution:

FALSE

If we take $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $A^\top = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ $AA^\top = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$.

Now, $N(A)$ contains the vector $x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ as

$Ax = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2*1+1*2 \\ -2*0+1*0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0vector.$

Therefore x is in N(A).

However, $AA^\top x = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2*5+1*0 \\ -2*0+1*0 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \end{bmatrix} \neq 0vector.$

Therefore x is not in $N(AA^\top)$

Hence there is a vector in N(A) but not in $N(AA^\top)$. Therefore, $N(A) \neq N(AA^\top)$ for given A.

12. (2 points) Without explicitly computing the product of given two matrices, find bases

for each of its four sub-spaces.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And also explain the four sub-spaces along with the method you followed to compute them.

Solution:

Let us denote the given 2 matrices as L and U. Since L has all diagonal elements as 1 and is a lower triangle matrix, it is invertible.

Basis vectors of C(U) are the pivot columns of U. Therefore U has basis vectors for C(U) as $(\begin{bmatrix}1 & 0 & 0\end{bmatrix})^\top$ and $(\begin{bmatrix}3 & 1 & 0\end{bmatrix})^\top$.

Since multiplying L to U is same as taking linear combinations of columns of L with weights as values in each column of U, to get basis vectors of C(A) = C(LU), we need to apply the same linear combination to the columns of L taking weights as values in the basis vectors of U.

Applying these, we get basis vectors of A as, $(\begin{bmatrix}1 & 1 & 0\end{bmatrix})^\top$ and $3(\begin{bmatrix}1 & 1 & 0\end{bmatrix})^\top + \begin{bmatrix}0 & 1 & 1\end{bmatrix})^\top = (\begin{bmatrix}3 & 4 & 1\end{bmatrix})^\top$. Therefore C(A) has basis vectors $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}3\\4\\1\end{bmatrix}$.

Basis vectors of Row space of U are the pivot rows of U. Therefore U has basis vectors for $C(U^\top)$ as $\begin{bmatrix}0 & 1 & 2 & 3 & 4\end{bmatrix}$ and $\begin{bmatrix}0 & 0 & 0 & 1 & 2\end{bmatrix}$. Since A = LU, rows of A are linear combinations of rows of U.

Row 1 of A is row 1 of U and $a_1 = u_1$.

Row 2 of A is sum of first two rows of U, $a_2 = u_1 + u_2 = \begin{bmatrix}0 & 1 & 2 & 4 & 6\end{bmatrix}$ and hence it is independent from Row 1 of A.

Row 3 of A is sum of last two rows of U, $a_3 = u_2 + u_3 = u_2 + 0 = u_2 = a_2 - a_1$. It is a dependent row. Hence basis vectors for Row space of A are $\begin{bmatrix}0 & 1 & 2 & 3 & 4\end{bmatrix}$ and $\begin{bmatrix}0 & 1 & 2 & 4 & 6\end{bmatrix}$.

Row reduced echelon form of U = Row reduced form of A since A = LU and L is invertible lower triangle matrix. Therefore, finding rref(U), R1 = R1 - 3R2, $\begin{bmatrix}0 & 1 & 2 & 0 & -2\\0 & 0 & 0 & 1 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$. Therefore N(A) has basis vectors, $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\-2\\1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\2\\0\\-2\\1\end{bmatrix}$.

Since to get row reduced echelon form of A, we did R1 = R1 - 3R2, if we apply the operation to I, We get E as $\begin{bmatrix}1 & -3 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$. Since basis vectors of $N(A^\top)$ are the rows of E which produce zero rows in row reduced echelon form of A, Here, row 3 of rref(A) is a zero row. Basis vector of $N(A^\top)$ is, $\begin{bmatrix}0\\0\\1\end{bmatrix}$.

Concept: Free variables

13. (2 ½ points) True or False (with reason if true or example to show it is false).

(a) An matrix m×n can have zero pivots.

Solution:

TRUE

If all of the rows in the matrix consists of only zeroes, that matrix will have a zero pivots. Example, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Therefore it is possible for a matrix mxn to have zero pivots.

(b) A real-symmetric matrix m×m has no free variables.

Solution:

FALSE

If we consider matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A = A^{\top}$ and all values are real and hence it is a real-symmetric matrix.

However from A, we can see that since its 2nd column is all 0s, it has 1 free variable.

Therefore there are real-symmetric matrices with free variables.

(c) If A & B be are two m×n matrices with non-zero pivots, then a matrix C = A + B can have zero pivots

Solution:

TRUE

Suppose A and B are such that in one common row, A has elements $a_{r1}, ...a_{rn}$ and B has elements $b_{r1} = -a_{r1}, ...b_{rn} = -a_{rn}$, then $A + B$ will have all zeroes in that particular row which results in a zero pivot for $A + B$.

Example, $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$

Here, Gaussian Elimination of A is (R2 = R2 - R1) = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Here, Gaussian Elimination of B is (R2 = R2 + R1) = $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Therefore A and B both have all non-zero pivots.

However, $C = A + B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

Here, since there is a zero row, $A + B$ has zero pivot.

Therefore it is possible that two matrices A and B have all non-zero pivots but their sum A+B has zero pivots.

(d) A free variable in a matrix always implies that there is either a zero-row or zero-column in the matrix.

Solution:

FALSE

If we consider matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$,

Row reduced echelon form of A, $R2 = R2 - 2R1 => \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = R(A)$.

Here since there is a zero pivot in R(A), A has 1 free variable. However the matrix does not have a zero row or column.

Therefore a matrix can have free variables without a zero row or column.

(e) For any matrix A, does A^T and A^{-1} have the same number of pivots.

Solution:

FALSE

If we consider matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,

Since A has a zero row, A is not invertible.

Therefore, A^{-1} does not exist and hence here, $A^{\top}$ and A^{-1} does not have same number of pivots.

Concept: Reduced Echelon Form

14. (½ point) Suppose R is $m \times n$ matrix of rank r, with pivot columns first:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(a) Find a right-inverse B with $RB = I$ if $r = m$.

Solution:

Since r=m, the matrix R becomes, $R = \begin{bmatrix} I & F \end{bmatrix}$.

Therefore to get RB = I, we can take $B = \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix}$ where I is a rxr identity matrix and **0** is a (n-r)xr matrix with all elements as 0. Here, $RB = I.I + F.0 = I$.

$B = \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix}$.