Lemma 2.10

Lemma 2.10. *Let $\mathcal{T}$ be an acyclic TBox, and $\mathcal{I}$ be an interpretation. Then there exists a model $\mathcal{J}$ of $\mathcal{T}$ that coincides with $\mathcal{I}$ on the interpretation of all role and concept names that are* not *defined in $\mathcal{T}$.*

By “not defined” it is meant “base symbols”

Note: the *Intro to DL* book authors do not define “base symbols”

Error on page 25: 2nd line ---

“not” - should be dropped..

Please download the errata file (list of known errors) from the book web site..

Formal Proof (1/2)

Let $\mathcal{T}$ be an acyclic TBox and $\mathcal{I}$ is an interpretation that interprets the primitive symbols only. There exists a *model* $\mathcal{J}$ of $\mathcal{T}$ that is an extension of $\mathcal{I}$.

--

$\mathcal{T} = \{ A_1 \equiv C_1,\ A_2 \equiv C_2,\ \ldots,\ A_k \equiv C_k \}$

$\mathcal{T}$ is *acyclic*: So, wlog, we can assume that the indices $\cdot_i$ of A's are such that, if A_i directly uses A_j, then $i > j$.

Define a sequence of interpretations $\mathcal{I}_i$ as modifications of $\mathcal{I}$ as follows:

For all *i*, $\mathcal{I}_i$ coincides with $\mathcal{I}$.

That is, $\Delta^{I_i} = \Delta^{I}$ and

$B^{I_i} = B^{I}$ for any primitive concept B in $\mathcal{T}$ and

$r^{I_i} = r^{I}$ for any role name r in $\mathcal{T}$

$\mathcal{I}_k$ would be the desired $\mathcal{J}$ ...

Formal Proof (2/2)

$\mathcal{T} = \{ A_1 \equiv C_1,\ A_2 \equiv C_2,\ \ldots,\ A_k \equiv C_k \}$

$\mathcal{T}$ is *acyclic*: So, wlog, we can assume that the indices $\cdot_i$ of A's are such that,
if A_i directly uses A_j, then $i > j$.

Define a *sequence* of interpretations $\mathcal{I}_i$ as modifications of $\mathcal{I}$ as follows:

For all *i*, $\mathcal{I}_i$ coincides with $\mathcal{I}$. That is,

$\Delta^{\mathcal{I}_i} = \Delta^{\mathcal{I}}$, $B^{\mathcal{I}_i} = B^{\mathcal{I}}$ for any primitive concept B in $\mathcal{T}$ and $r^{\mathcal{I}_i} = r^{\mathcal{I}}$ for any role r in $\mathcal{T}$

Now, for defined concepts, we set:

$A_1^{\mathcal{I}_1} = C_1^{\mathcal{I}}$, $A_j^{\mathcal{I}_1} = \Phi$ for all $j > 1$ //C_1 has only primitive names

$A_1^{\mathcal{I}_2} = A_1^{\mathcal{I}_1}$, $A_2^{\mathcal{I}_2} = C_2^{\mathcal{I}_1}$, $A_j^{\mathcal{I}_2} = \Phi$ for all $j > 2$ //C_2 has primitive names + A_1

….

$A_1^{I_k} = A_1^{I_{k-1}}$, $A_2^{I_k} = A_2^{I_{k-1}}$, … , $A_k^{I_k} = C_k^{I_{k-1}}$ //C_k - primitive + A_1, A_2, …, A_{k-1}

Expanding or Unfolding a TBox

Given a KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ where $\mathcal{T}$ - an acyclic TBox

Treat definitions as macros.

Carry out a (recursive) substitution in $\mathcal{A}$.

We get a new ABox $\mathcal{A}'$ and a KB $\mathcal{K}' = (\Phi, \mathcal{A}')$

One can show that $\mathcal{K}$ and $\mathcal{K}'$ have same models..

$\mathcal{K}'$ - called the expansion or unfolding of $\mathcal{K}$

Definition of Unfolding

Definition 2.11. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base, where $\mathcal{T}$ is acyclic and of the form $\mathcal{T} = \{A_i \equiv C_i \mid 1 \leq i \leq m\}$. Let $\mathcal{A}_0 = \mathcal{A}$ and let $\mathcal{A}_{j+1}$ be the result of carrying out the following replacement:

(i) find some $a : D \in \mathcal{A}_j$ in which some A_i occurs in D, for some $1 \leq i \leq m$;

(ii) replace all occurrences of A_i in D with C_i.

If no more replacements can be applied to $\mathcal{A}_k$, we call $\mathcal{A}_k$ the *result of unfolding* $\mathcal{T}$ *into* $\mathcal{A}$.

The unfolded version and the original are same

Lemma 2.12. *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base with $\mathcal{T}$ being acyclic. Then the result of unfolding $\mathcal{T}$ into $\mathcal{A}$ exists and, for $\mathcal{A}'$ the result of unfolding $\mathcal{T}$ into $\mathcal{A}$, we have that*

(i) *each model of $\mathcal{K}$ is a model of $\mathcal{A}'$, and*
(ii) *each model $\mathcal{I}$ of $\mathcal{A}'$ can be modified to one of $\mathcal{K}$ that coincides with $\mathcal{I}$ on the interpretation of roles and concepts that are not defined in $\mathcal{T}$.*

Proof is skipped: Read it from the book...

Expansion of (Family, { a:Grandmother })

$$
\begin{aligned}
\mathsf{Woman} &\equiv \mathsf{Person} \sqcap \mathsf{Female}\\
\mathsf{Man} &\equiv \mathsf{Person} \sqcap \neg(\mathsf{Person} \sqcap \mathsf{Female})\\
\mathsf{Mother} &\equiv (\mathsf{Person} \sqcap \mathsf{Female}) \sqcap \exists \mathsf{hasChild}.\mathsf{Person}\\
\mathsf{Father} &\equiv (\mathsf{Person} \sqcap \neg(\mathsf{Person} \sqcap \mathsf{Female})) \sqcap \exists \mathsf{hasChild}.\mathsf{Person}\\
\mathsf{Parent} &\equiv ((\mathsf{Person} \sqcap \neg(\mathsf{Person} \sqcap \mathsf{Female})) \sqcap \exists \mathsf{hasChild}.\mathsf{Person})\\
&\quad \sqcup ((\mathsf{Person} \sqcap \mathsf{Female}) \sqcap \exists \mathsf{hasChild}.\mathsf{Person})\\
\mathsf{Grandmother} &\equiv ((\mathsf{Person} \sqcap \mathsf{Female}) \sqcap \exists \mathsf{hasChild}.\mathsf{Person})\\
&\quad \sqcap \exists \mathsf{hasChild}.(((\mathsf{Person} \sqcap \neg(\mathsf{Person} \sqcap \mathsf{Female}))\\
&\qquad \sqcap \exists \mathsf{hasChild}.\mathsf{Person})\\
&\qquad \sqcup ((\mathsf{Person} \sqcap \mathsf{Female})\\
&\qquad \sqcap \exists \mathsf{hasChild}.\mathsf{Person}))
\end{aligned}
$$

Family

$\text{Woman} \equiv \text{Person} \sqcap \text{Female}$

$\text{Man} \equiv \text{Person} \sqcap \neg \text{Woman}$

$\text{Mother} \equiv \text{Woman} \sqcap \exists\, \text{hasChild}.\text{Person}$

$\text{Father} \equiv \text{Man} \sqcap \exists\, \text{hasChild}.\text{Person}$

$\text{Parent} \equiv \text{Father} \sqcup \text{Mother}$

$\text{Grandmother} \equiv \text{Mother} \sqcap \exists\, \text{hasChild}.\text{Parent}$

Unfolding a TBox - Exponential Bloating of ABox

$\mathcal{T}$ = { $A_1 \equiv \forall r.A_2 \sqcap \forall s.A_2$
$A_2 \equiv \forall r.A_3 \sqcap \forall s.A_3$
$A_3 \equiv \forall r.A_4 \sqcap \forall s.A_4$
….
$A_{n-1} \equiv \forall r.A_n \sqcap \forall s.A_n$ }

$\mathcal{A}$ = { a: A_1 }

$\mathcal{K} = (\mathcal{T}, \mathcal{A})$

Size: Linear in *n*

$\mathcal{K}' = (\Phi, \mathcal{A}')$

Size: Exponential in *n*

$\mathcal{A}$ = { a: A_1 }
= { a: $\forall r.A_2 \sqcap \forall s.A_2$ }
= { a: $\forall r.(\forall r.A_3 \sqcap \forall s.A_3) \sqcap \forall s.(\forall r.A_3 \sqcap \forall s.A_3)$ }
= …
= { a: expression with 2^n number of A_ns }

Basic Reasoning Problems and Services

So far,

- What are knowledge bases ?
- Components of DL knowledge bases
  - Terminological Box (TBox)
  - Assertion Box (ABox)
- Interpretations
- When are interpretations called ***models***?

Now,

- What are the basic reasoning problems considered in DL KBs?
- Are there any relationships between them?
- What are the services offered by DL ***reasoners***?

Reasoning Problems - Satisfiability

$\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is an $\mathcal{ALC}$ KB.

C, D are concept expressions/descriptions (possibly compound).

C is ***satisfiable*** wrt $\mathcal{T}$ if there exists a *model* $\mathcal{I}$ of $\mathcal{T}$ and $C^{\mathcal{I}}$ is non-empty.

That is, there is some d $\in \Delta^{\mathcal{I}}$ and d $\in C^{\mathcal{I}}$ for some model $\mathcal{I}$.

C is ***unsatisfiable*** wrt $\mathcal{T}$ if there exists no *model* $\mathcal{I}$ of $\mathcal{T}$ such that $C^{\mathcal{I}}$ is non-empty.

Most of our example concept descriptions - satisfiable

$C = A \sqcap \neg A$ is not satisfiable wrt $\mathcal{T} = \Phi$.

$C = \exists r.A \sqcap \forall r.\neg A$ is not satisfiable wrt any TBox.

One can write down infinitely many unsatisfiable concepts!

Some C -- *satisfiable* wrt a TBox and *not satisfiable* wrt some other TBox.

Reasoning Problems - Subsumption & Equivalence

$\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is an $\mathcal{ALC}$ KB.

C, D are concept expressions (possibly compound).

C is ***subsumed by*** D wrt $\mathcal{T}$, written $\mathcal{T} \models C \sqsubseteq D$, if for *every model* $\mathcal{I}$ of $\mathcal{T}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

C and D are ***equivalent*** wrt $\mathcal{T}$, written $\mathcal{T} \models C \equiv D$, if for *every model* $\mathcal{I}$ of $\mathcal{T}$, $C^{\mathcal{I}} = D^{\mathcal{I}}$.

$\mathcal{T} \models C \sqsubseteq D$ is also written as $C \sqsubseteq_{\mathcal{T}} D$

$\mathcal{T} \models C \equiv D$ is also written as $C \equiv_{\mathcal{T}} D$

$\Phi \models A \sqsubseteq A \sqcup B.$

$\Phi \models A \sqcap B \sqsubseteq A.$

$\Phi \models \exists r.A \sqcap \forall r.B \sqsubseteq \exists r.B$

There are infinitely many subsumption relations entailed by a TBox,
(even by an empty TBox!)

Example TBox $\mathcal{T}_{ex}$

$$
\begin{array}{rrcll}
\mathcal{T}_{ex} = \{ & \mathsf{Course} & \sqsubseteq & \neg\mathsf{Person}, & (\mathcal{T}_{ex}.1) \\
& \mathsf{UGC} & \sqsubseteq & \mathsf{Course}, & (\mathcal{T}_{ex}.2) \\
& \mathsf{PGC} & \sqsubseteq & \mathsf{Course}, & (\mathcal{T}_{ex}.3) \\
& \mathsf{Teacher} & \equiv & \mathsf{Person} \sqcap \exists \mathit{teaches}.\mathsf{Course}, & (\mathcal{T}_{ex}.4) \\
& \exists \mathit{teaches}.\top & \sqsubseteq & \mathsf{Person}, & (\mathcal{T}_{ex}.5) \\
& \mathsf{Student} & \equiv & \mathsf{Person} \sqcap \exists \mathit{attends}.\mathsf{Course}, & (\mathcal{T}_{ex}.6) \\
& \exists \mathit{attends}.\top & \sqsubseteq & \mathsf{Person} \} & (\mathcal{T}_{ex}.7)
\end{array}
$$

"Course $\sqcap$ $\exists$ teaches.Course" is not satisfiable wrt $\mathcal{T}_{ex}$.

- It is satisfiable wrt $\mathcal{T}_{ex} - \{ \mathcal{T}_{ex}.1 \}$

$\mathcal{T}_{ex} \models \exists$ teaches.Course $\sqsubseteq \neg$Course

$\mathcal{T}_{ex} \models$ PGC $\sqsubseteq \neg$Person

Note: Satisfiability, subsumption and equivalence definitions are wrt TBoxes alone.

Reasoning Problems - Consistency

$\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is an $\mathcal{ALC}$ KB.

$\mathcal{K}$ is ***consistent*** if there exists a model for $\mathcal{K}$.

Note: Consistency is defined wrt a KB, both TBox and ABox.

$\mathcal{K}_{ex} = (\mathcal{T}_{ex}, \mathcal{A}_{ex})$ is consistent. We have a model $\mathcal{I}'$ for $\mathcal{K}_{ex}$, seen earlier.

$(\mathcal{T}_{ex}, \mathcal{A}_2)$ where $\mathcal{A}_2$ = { ET: Course, (ET, Foo): teaches }

is not consistent.

It is possible that a concept C may be *unsatisfiable* wrt a *consistent* KB

({ X ≡ A ⊓ ¬A }, Φ) has infinite models but X is unsatisfiable in all of them!

Reasoning Problems - "instance of"

$\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is an $\mathcal{ALC}$ KB.

C, D are concept expressions (possibly compound).

b is an individual name.

b is an ***instance of*** C wrt to $\mathcal{K}$, written as $\mathcal{K} \vDash b{:}C$

if $b^{\mathcal{I}} \in C^{\mathcal{I}}$ for ***every*** model $\mathcal{I}$ of $\mathcal{K}$.

Note: *instance* notion is defined for individual *names*, not for elements of domain

Recall: if $a \in C^{\mathcal{I}}$ in an interpretation $\mathcal{I}$, a is in the ***extension*** of C under $\mathcal{I}$.

A KB can *enforce* that an individual name is **an instance of** some C.

Examples using $\mathcal{ALC}$ KB $\mathcal{K}_{ex} = (\mathcal{T}_{ex}, \mathcal{A}_{ex})$

$$
\begin{array}{rcll}
\mathcal{T}_{ex} = \{\mathsf{Course} & \sqsubseteq & \neg\mathsf{Person}, & (\mathcal{T}_{ex}.1)\\
\mathsf{UGC} & \sqsubseteq & \mathsf{Course}, & (\mathcal{T}_{ex}.2)\\
\mathsf{PGC} & \sqsubseteq & \mathsf{Course}, & (\mathcal{T}_{ex}.3)\\
\mathsf{Teacher} & \equiv & \mathsf{Person} \sqcap \exists \mathit{teaches}.\mathsf{Course}, & (\mathcal{T}_{ex}.4)\\
\exists \mathit{teaches}.\top & \sqsubseteq & \mathsf{Person}, & (\mathcal{T}_{ex}.5)\\
\mathsf{Student} & \equiv & \mathsf{Person} \sqcap \exists \mathit{attends}.\mathsf{Course}, & (\mathcal{T}_{ex}.6)\\
\exists \mathit{attends}.\top & \sqsubseteq & \mathsf{Person}\ \} & (\mathcal{T}_{ex}.7)
\end{array}
$$

$$
\begin{array}{rll}
\mathcal{A}_{ex} = \{\mathsf{Mary} : & \mathsf{Person}, & (\mathcal{A}_{ex}.1)\\
\mathsf{CS600} : & \mathsf{Course}, & (\mathcal{A}_{ex}.2)\\
\mathsf{Ph456} : & \mathsf{Course} \sqcap \mathsf{PGC}, & (\mathcal{A}_{ex}.3)\\
\mathsf{Hugo} : & \mathsf{Person}, & (\mathcal{A}_{ex}.4)\\
\mathsf{Betty} : & \mathsf{Person} \sqcap \mathsf{Teacher}, & (\mathcal{A}_{ex}.5)\\
(\mathsf{Mary}, \mathsf{CS600}) : & \mathit{teaches}, & (\mathcal{A}_{ex}.6)\\
(\mathsf{Hugo}, \mathsf{Ph456}) : & \mathit{teaches}, & (\mathcal{A}_{ex}.7)\\
(\mathsf{Betty}, \mathsf{Ph456}) : & \mathit{attends}, & (\mathcal{A}_{ex}.8)\\
(\mathsf{Mary}, \mathsf{Ph456}) : & \mathit{attends}\ \} & (\mathcal{A}_{ex}.9)
\end{array}
$$

$\mathcal{K}_{ex} \models$ Hugo: Teacher
$\mathcal{K}_{ex} \models$ Mary: Teacher
$\mathcal{K}_{ex} \models$ Betty: Teacher

Hugo, Mary and Betty are all **instances of** Teacher

Hugo is not an instance of Student wrt $\mathcal{K}_{ex}$.

It is not enforced by $\mathcal{K}_{ex}$.

Note the difference between being an “instance of” and being in the “extension of” a concept ...

Basic Reasoning Problems - All at one place..

Definition 2.14. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ALC}$ knowledge base, C, D possibly compound $\mathcal{ALC}$ concepts, and b an individual name. We say that

(i) C is *satisfiable* with respect to $\mathcal{T}$ if there exists a model $\mathcal{I}$ of $\mathcal{T}$ and some $d \in \Delta^{\mathcal{I}}$ with $d \in C^{\mathcal{I}}$;

(ii) C is *subsumed by* D with respect to $\mathcal{T}$, written $\mathcal{T} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$;

(iii) C and D are *equivalent* with respect to $\mathcal{T}$, written $\mathcal{T} \models C \equiv D$, if $C^{\mathcal{I}} = D^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$;

(iv) $\mathcal{K}$ is *consistent* if there exists a model of $\mathcal{K}$;

(v) b is an *instance of* C with respect to $\mathcal{K}$, written $\mathcal{K} \models b : C$, if $b^{\mathcal{I}} \in C^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{K}$.

Properties of Subsumption Relation (Lemma 2.15)

C, D, E - $\mathcal{ALC}$ concepts ; b - an individual name

$(\mathcal{T}, \mathcal{A})$, $(\mathcal{T}_1, \mathcal{A}_1)$ are $\mathcal{ALC}$ knowledge bases with $\mathcal{T} \subseteq \mathcal{T}_1$ and $\mathcal{A} \subseteq \mathcal{A}_1$

(i) $C \sqsubseteq_{\mathcal{T}} C$

(ii) If $C \sqsubseteq_{\mathcal{T}} D$ and $D \sqsubseteq_{\mathcal{T}} E$, then $C \sqsubseteq_{\mathcal{T}} E$

(iii) If $(\mathcal{T}, \mathcal{A}) \models b : C$ and $C \sqsubseteq_{\mathcal{T}} D$, then $(\mathcal{T}, \mathcal{A}) \models b : D$

(iv) If $(\mathcal{T}, \mathcal{A}) \models C \sqsubseteq D$ then $(\mathcal{T}_1, \mathcal{A}_1) \models C \sqsubseteq D$

(v) If $(\mathcal{T}, \mathcal{A}) \models C \equiv D$ then $(\mathcal{T}_1, \mathcal{A}_1) \models C \equiv D$

(vi) If $(\mathcal{T}, \mathcal{A}) \models b : E$ then $(\mathcal{T}_1, \mathcal{A}_1) \models b : E$

Lemma 2.5:
$\mathcal{T} \subseteq \mathcal{T}_1 \Rightarrow$ any model of $\mathcal{T}_1$ is also a model of $\mathcal{T}$

Minimality of Operators (Lemma 2.16)

C, D -- concepts r -- a role $\mathcal{T}_0 = \Phi$ (the empty TBox) $\mathcal{T}$ -- arbitrary TBox

(i) $\mathcal{T}_0 \models \top \equiv (\neg C \sqcup C)$

(ii) $\mathcal{T}_0 \models \bot \equiv (\neg C \sqcap C)$

(iii) $\mathcal{T}_0 \models C \sqcup D \equiv \neg(\neg C \sqcap \neg D)$

(iv) $\mathcal{T}_0 \models \forall r.C \equiv \neg(\exists r.\neg C)$

(v) $\mathcal{T} \models C \sqsubseteq D$ if and only if $\mathcal{T} \models \top \sqsubseteq (\neg C \sqcup D)$ $\leftarrow$

Consequences: We can do away with $\top$ $\bot$ $\sqcup$ $\forall$;   or alternatively, $\top$ $\bot$ $\sqcap$ $\exists$

These properties are satisfied by all TBoxes ($\because$ $\mathcal{T}_0 \sqsubseteq \mathcal{T}$ and Lemma 2.15)

Aka Tautologies ...

Proof of part (v) of Lemma 2.16

(v) $\mathcal{T} \models C \sqsubseteq D$ if and only if $\mathcal{T} \models \top \sqsubseteq (\neg C \sqcup D)$

(Only if)

Assume that $\mathcal{T} \models C \sqsubseteq D$.

For any model $\mathcal{I}$ of $\mathcal{T}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

Let $x \in \Delta^{\mathcal{I}}$. Either $x \in C^{\mathcal{I}}$ or $x \notin C^{\mathcal{I}}$.

$x \in C^{\mathcal{I}} \Rightarrow x \in D^{\mathcal{I}} \Rightarrow x \in (\neg C \sqcup D)^{\mathcal{I}}$

$x \notin C^{\mathcal{I}} \Rightarrow x \in (\neg C)^{\mathcal{I}} \Rightarrow x \in (\neg C \sqcup D)^{\mathcal{I}}$

Hence, $\top \sqsubseteq (\neg C \sqcup D)$

Thus $\mathcal{T} \models \top \sqsubseteq (\neg C \sqcup D)$

(If)

Assume that $\mathcal{T} \models \top \sqsubseteq (\neg C \sqcup D)$

For any model $\mathcal{I}$ of $\mathcal{T}$, $\Delta^{I} \subseteq (\neg C)^{\mathcal{I}} \cup D^{\mathcal{I}}$.

Suppose $x \in C^{\mathcal{I}}$. Then $x \notin (\neg C)^{\mathcal{I}}$

Since $x \in \Delta^{I}$, $x \in (\neg C)^{\mathcal{I}} \cup D^{\mathcal{I}}$

So, $x \in D^{\mathcal{I}}$

Hence $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for any $\mathcal{I}$.

Thus, $\mathcal{T} \models C \sqsubseteq D$.

Relationships Among Reasoning Problems (Theorem 2.17)

$\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is an $\mathcal{ALC}$ KB. C, D are concept expressions (possibly compound).

And b is an individual name.

(i) $C \equiv_{\mathcal{T}} D$ iff $C \sqsubseteq_{\mathcal{T}} D$ and $D \sqsubseteq_{\mathcal{T}} C$

(ii) $C \sqsubseteq_{\mathcal{T}} D$ iff $C \sqcap \neg D$ is not satisfiable wrt $\mathcal{T}$ $\leftarrow$

(iii) C is satisfiable wrt $\mathcal{T}$ iff $C \not\sqsubseteq_{\mathcal{T}} \bot$

(iv) C is satisfiable wrt $\mathcal{T}$ iff $(\mathcal{T}, \{ b{:}C \})$ is consistent $\leftarrow$

(v) $(\mathcal{T}, \mathcal{A}) \models b{:}C$ iff $(\mathcal{T}, \mathcal{A} \cup \{ b{:}\neg C \})$ is not consistent $\leftarrow$

(vi) if $\mathcal{T}$ is acyclic, and $\mathcal{A}'$ is the result of unfolding $\mathcal{T}$ into $\mathcal{A}$, then

$\mathcal{K}$ is consistent iff $(\Phi, \mathcal{A}')$ is consistent.

Consequences of Thm 2.17

(i) $C \equiv_{\mathcal{T}} D$ iff $C \sqsubseteq_{\mathcal{T}} D$ and $D \sqsubseteq_{\mathcal{T}} C$

(ii) $C \sqsubseteq_{\mathcal{T}} D$ iff $C \sqcap \neg D$ is not satisfiable wrt $\mathcal{T}$

(iv) C is satisfiable wrt $\mathcal{T}$ iff $(\mathcal{T}, \{ b{:}C \})$ is consistent

1) Equivalence problem reduces to subsumption problem
2) Subsumption problem reduces to (un)satisfiabilty problem
3) Satisfiability problem reduces to KB consistency problem

(v) $(\mathcal{T}, \mathcal{A}) \models b{:}C$ iff $(\mathcal{T}, \mathcal{A} \cup \{ b{:}\neg C \})$ is not consistent

4) Instance-checking reduces to KB consistency problem.

So, all the reasoning problems are reducible to KB consistency problem !!

Proof of Thm 2.17(ii)

(ii) $C \sqsubseteq_{\mathcal{T}} D$ iff $C \sqcap \neg D$ is not satisfiable wrt $\mathcal{T}$

(Only If)

Suppose $C \sqsubseteq_{\mathcal{T}} D$.

$\Leftrightarrow$ for every model of $\mathcal{I}$ of $\mathcal{T}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$

Consider an arbitrary $x \in \Delta^{\mathcal{I}}$

Case1: $x \in C^{\mathcal{I}} \Rightarrow x \in D^{\mathcal{I}} \Rightarrow x \notin (\neg D)^{\mathcal{I}}$

$\Rightarrow x \notin C^{\mathcal{I}} \cap (\neg D)^{\mathcal{I}} \Rightarrow x \notin (C \sqcap \neg D)^{\mathcal{I}}$

Case2: $x \notin C^{\mathcal{I}}$

$\Rightarrow x \notin C^{\mathcal{I}} \cap (\neg D)^{\mathcal{I}} \Rightarrow x \notin (C \sqcap \neg D)^{\mathcal{I}}$

Hence, $C \sqcap \neg D$ is not satisfiable

(If)

Suppose $C \sqcap \neg D$ is not satisfiable.

For every model of $\mathcal{I}$ of $\mathcal{T}$,

$C^{\mathcal{I}} \cap (\neg D)^{\mathcal{I}} = \Phi$

Consider $x \in C^{\mathcal{I}}$.

x either is in $D^{\mathcal{I}}$ or in $(\neg D)^{\mathcal{I}}$

If $x \in (\neg D)^{\mathcal{I}}$ then $C^{\mathcal{I}} \cap (\neg D)^{\mathcal{I}} \neq \Phi$.

So, $x \in D^{\mathcal{I}}$

That is, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$

Hence $C \sqsubseteq_{\mathcal{T}} D$.

Proof of Thm 2.17(iv)

(iv) C is satisfiable wrt $\mathcal{T}$ iff ($\mathcal{T}$, { b:C }) is consistent

(only if)

C is satisfiable wrt $\mathcal{T}$

There is a model $\mathcal{I}$ of $\mathcal{T}$ st $C^{\mathcal{I}} \neq \Phi$

Take some $x \in C^{\mathcal{I}}$

and extend $\mathcal{I}$ by setting $b^{\mathcal{I}} = x$.

Extended $\mathcal{I}$ continues to be a model of $\mathcal{T}$

And also, becomes a model of {b:C}.

Hence, ($\mathcal{T}$, { b:C }) is consistent.

(if)

($\mathcal{T}$, { b:C }) is consistent

There is a model $\mathcal{I}$ of $\mathcal{T}$ st $b^{\mathcal{I}} \in C^{\mathcal{I}}$

Thus, C is satisfiable wrt $\mathcal{T}$

Proof of Thm 2.17(v)

(v) $(\mathcal{T}, \mathcal{A}) \vDash b{:}C$ iff $(\mathcal{T}, \mathcal{A} \cup \{ b{:}\neg C \})$ is not consistent

(Only If)

$(\mathcal{T}, \mathcal{A}) \vDash b{:}C$

$\Rightarrow$ for every model $\mathcal{I}$ of $(\mathcal{T}, \mathcal{A})$, $b^{\mathcal{I}} \in C^{\mathcal{I}}$

$\Rightarrow$ for every model $\mathcal{I}$ of $(\mathcal{T}, \mathcal{A})$, $b^{\mathcal{I}} \notin (\neg C)^{\mathcal{I}}$

$\Rightarrow$ for no model $\mathcal{I}$ of $(\mathcal{T}, \mathcal{A})$, $b^{\mathcal{I}} \in (\neg C)^{\mathcal{I}}$

$(\mathcal{T}, \mathcal{A} \cup \{ b{:}\neg C \})$ is not consistent.

(If)

$(\mathcal{T}, \mathcal{A} \cup \{ b{:}\neg C \})$ is not consistent

Let $\mathcal{I}$ be *any* model of $(\mathcal{T}, \mathcal{A})$

In $\mathcal{I}$, it must be that $b^{\mathcal{I}} \notin (\neg C)^{\mathcal{I}}$

Otherwise, it contradicts our assumption.

So, $b^{\mathcal{I}} \in C^{\mathcal{I}}$.

Hence, $(\mathcal{T}, \mathcal{A}) \vDash b{:}C$

If $(\mathcal{T}, \mathcal{A})$ has no models, $(\mathcal{T}, \mathcal{A}) \vDash b{:}C$

is vacuously true.