3
A Little Bit of Model Theory

The main purpose of this chapter is to show that sets of models of $\mathcal{ALC}$ concepts or knowledge bases satisfy several interesting properties, which can be used to prove expressivity and decidability results. To be more precise, we will introduce the notion of *bisimulation* between elements of $\mathcal{ALC}$ interpretations, and prove that $\mathcal{ALC}$ concepts cannot distinguish between bisimilar elements. On the one hand, we will use this to show restrictions of the *expressive power* of $\mathcal{ALC}$: number restrictions, inverse roles and nominals cannot be expressed within $\mathcal{ALC}$. On the other hand, we will employ bisimulation invariance of $\mathcal{ALC}$ to show that $\mathcal{ALC}$ has the *tree model property* and satisfies *closure under disjoint union* of models. We will also show that $\mathcal{ALC}$ has the *finite model property*, though not as a direct consequence of bisimulation invariance. These properties will turn out to be useful in subsequent chapters and of interest to people writing knowledge bases: for example, $\mathcal{ALC}$'s tree model property implies that it is too weak to describe the ring structure of many chemical molecules since any $\mathcal{ALC}$ knowledge base trying to describe such a structure will also have acyclic models. In the present chapter, we will only use the finite model property (or rather the stronger *bounded model property*) to show a basic, not complexity-optimal decidability result for reasoning in $\mathcal{ALC}$. For the sake of simplicity, we concentrate here on the terminological part of $\mathcal{ALC}$, i.e., we consider only concepts and TBoxes, but not ABoxes.

To obtain a better intuitive view of the definitions and results introduced below, one should recall that interpretations of $\mathcal{ALC}$ can be viewed as *graphs*, with edges labelled by roles and nodes labelled by sets of concept names. More precisely, in such a graph

- the nodes are the elements of the interpretation and they are labelled

with all the concept names to which this element belongs in the interpretation;
- an edge with label r between two nodes says that the corresponding two elements of the interpretation are related by the role r.

Examples for this representation of interpretations by graphs can be found in the previous chapter (see Figure 2.2) and in Figure 3.1.

3.1 Bisimulation

We define the notion of a bisimulation directly for interpretations, rather than for the graphs representing them.

Definition 3.1 (Bisimulation)**.** Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations. The relation $\rho \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a *bisimulation* between $\mathcal{I}_1$ and $\mathcal{I}_2$ if

(i) $d_1 \; \rho \; d_2$ implies

$$d_1 \in A^{\mathcal{I}_1} \quad \text{if and only if} \quad d_2 \in A^{\mathcal{I}_2}$$

for all $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$, and $A \in \mathbf{C}$;

(ii) $d_1 \; \rho \; d_2$ and $(d_1, d_1') \in r^{\mathcal{I}_1}$ implies the existence of $d_2' \in \Delta^{\mathcal{I}_2}$ such that

$$d_1' \; \rho \; d_2' \quad \text{and} \quad (d_2, d_2') \in r^{\mathcal{I}_2}$$

for all $d_1, d_1' \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$, and $r \in \mathbf{R}$;

(iii) $d_1 \; \rho \; d_2$ and $(d_2, d_2') \in r^{\mathcal{I}_2}$ implies the existence of $d_1' \in \Delta^{\mathcal{I}_1}$ such that

$$d_1' \; \rho \; d_2' \quad \text{and} \quad (d_1, d_1') \in r^{\mathcal{I}_1}$$

for all $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2, d_2' \in \Delta^{\mathcal{I}_2}$, and $r \in \mathbf{R}$.

Given $d_1 \in \Delta^{\mathcal{I}_1}$ and $d_2 \in \Delta^{\mathcal{I}_2}$, we define

$$(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2) \quad \text{if} \quad \text{there is a bisimulation } \rho \text{ between } \mathcal{I}_1 \text{ and } \mathcal{I}_2 \text{ such that } d_1 \; \rho \; d_2,$$

and say that $d_1 \in \mathcal{I}_1$ is *bisimilar* to $d_2 \in \mathcal{I}_2$.

Intuitively, d_1 and d_2 are bisimilar if they belong to the same concept names and, for each role r, have bisimilar r-successors. The reason for calling the relation ρ a *bi*simulation is that we require both property (ii) and (iii) in the definition. These two properties together are sometimes also called the back-and-forth property. Strictly speaking, the notion of

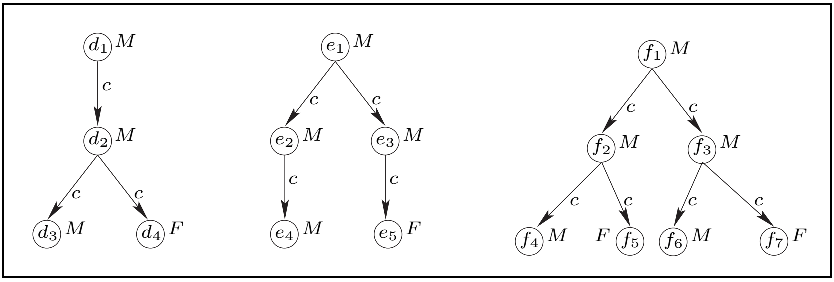

Fig. 3.1. Three interpretations $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ represented as graphs.

a bisimulation needs to be parametrised with respect to the employed set of concept names $\mathbf{C}$ and role names $\mathbf{R}$. In the following, we assume that these two sets are fixed, and thus do not mention them explicitly. It should also be noted that the interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ in Definition 3.1 are not required to be distinct. In addition, the empty relation is always a bisimulation, though not a very interesting one.

Given the three interpretations depicted in Figure 3.1 (where c is supposed to represent the role *child*, M the concept Male and F the concept Female), it is easy to see that $(d_1, \mathcal{I}_1)$ and $(f_1, \mathcal{I}_3)$ are bisimilar, whereas $(d_1, \mathcal{I}_1)$ and $(e_1, \mathcal{I}_2)$ are not.

The following theorem states that $\mathcal{ALC}$ cannot distinguish between bisimilar elements.

Theorem 3.2. *If $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$, then the following holds for all $\mathcal{ALC}$ concepts C:*

$$d_1 \in C^{\mathcal{I}_1} \quad \textit{if and only if} \quad d_2 \in C^{\mathcal{I}_2}.$$

Proof. Since $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$, there is a bisimulation ρ between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $d_1 \; \rho \; d_2$. We prove the theorem by induction on the structure of C. Since, up to equivalence, any $\mathcal{ALC}$ concept can be constructed using only the constructors conjunction, negation and existential restriction (see Lemma 2.16), we consider only these constructors in the induction step. The base case is the one where C is a concept name.

- Assume that $C = A \in \mathbf{C}$. Then

$$d_1 \in A^{\mathcal{I}_1} \quad \text{if and only if} \quad d_2 \in A^{\mathcal{I}_2}$$

is an immediate consequence of $d_1 \; \rho \; d_2$ (see part (i) of Definition 3.1).

- Assume that $C = D \sqcap E$. Then

$$\begin{array}{lll} d_1 \in (D \sqcap E)^{\mathcal{I}_1} & \text{if and only if} & d_1 \in D^{\mathcal{I}_1} \text{ and } d_1 \in E^{\mathcal{I}_1}, \\ & \text{if and only if} & d_2 \in D^{\mathcal{I}_2} \text{ and } d_2 \in E^{\mathcal{I}_2}, \\ & \text{if and only if} & d_2 \in (D \sqcap E)^{\mathcal{I}_2}, \end{array}$$

  where the first and third equivalences are due to the semantics of conjunction, and the second is due to the induction hypothesis applied to D and E.
- Negation ($\neg$) can be treated similarly.
- Assume that $C = \exists r.D$. Then

$$\begin{array}{lll} d_1 \in (\exists r.D)^{\mathcal{I}_1} & \text{if and only if} & \text{there is } d_1' \in \Delta^{\mathcal{I}_1} \text{ such that} \\ & & (d_1, d_1') \in r^{\mathcal{I}_1} \text{ and } d_1' \in D^{\mathcal{I}_1}, \\ & \text{if and only if} & \text{there is } d_2' \in \Delta^{\mathcal{I}_2} \text{ such that} \\ & & (d_2, d_2') \in r^{\mathcal{I}_2} \text{ and } d_2' \in D^{\mathcal{I}_2}, \\ & \text{if and only if} & d_2 \in (\exists r.D)^{\mathcal{I}_2}. \end{array}$$

  Here the first and third equivalences are due to the semantics of existential restrictions. The second equivalence is due to parts (ii) and (iii) of Definition 3.1 and the induction hypothesis.

This completes the proof of the theorem. □

Applied to our example, the theorem says that d_1 in $\mathcal{I}_1$ belongs to the same $\mathcal{ALC}$ concepts as f_1 in $\mathcal{I}_3$. For instance, both belong to the concept

$$\exists c.(M \sqcap \exists c.M \sqcap \exists c.F),$$

which contains those male individuals that have a son that has both a son and a daughter. In contrast, e_1 in $\mathcal{I}_2$ does not belong to this concept because e_1 does not have a son that has both a son and a daughter. It only has a son that has a son and another son that has a daughter.

3.2 Expressive power

In Section 2.5, we introduced extensions of $\mathcal{ALC}$ with the concept constructors *number restrictions* and *nominals*, and the role constructor *inverse roles*. How can we prove that these constructors really extend $\mathcal{ALC}$, i.e., that they *cannot be expressed* using just the constructors of $\mathcal{ALC}$? For this purpose, we need to show that, using any of these constructors (in addition to the constructors of $\mathcal{ALC}$), we can construct

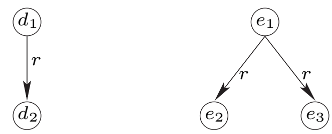

Fig. 3.2. Two interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ represented as graphs.

concepts that cannot be expressed by $\mathcal{ALC}$ concepts, i.e, there is no equivalent $\mathcal{ALC}$ concept. At first sight, this may sound like a formidable task. In fact, given such a concept C, we need to show that $C \not\equiv D$ holds for *all* $\mathcal{ALC}$ concepts D, and there are *infinitely many* such concepts D. This is where bisimulation comes into play: if we can show that C can distinguish between two bisimilar elements, then obviously it cannot be equivalent to an $\mathcal{ALC}$ concept by Theorem 3.2.

First, we consider the case of number restrictions. Remember that $\mathcal{ALCN}$ is the extension of $\mathcal{ALC}$ with unqualified number restrictions, i.e., concepts of the form $(\leqslant n\, r.\top)$ and $(\geqslant n\, r.\top)$, for $r \in \mathbf{R}$ and $n \geq 0$.

Proposition 3.3. *$\mathcal{ALCN}$ is more expressive than $\mathcal{ALC}$; that is, there is an $\mathcal{ALCN}$ concept C such that $C \not\equiv D$ holds for all $\mathcal{ALC}$ concepts D.*

Proof. We show that no $\mathcal{ALC}$ concept is equivalent to the $\mathcal{ALCN}$ concept $(\leqslant 1\, r.\top)$. Assume to the contrary that D is an $\mathcal{ALC}$ concept with $(\leqslant 1\, r.\top) \equiv D$. In order to lead this assumption to a contradiction, we consider the interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ depicted in Figure 3.2. Since

$$\rho = \{(d_1, e_1), (d_2, e_2), (d_2, e_3)\}$$

is a bisimulation, we have $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, e_1)$, and thus $d_1 \in D^{\mathcal{I}_1}$ if and only if $e_1 \in D^{\mathcal{I}_2}$. This contradicts our assumption $(\leqslant 1\, r.\top) \equiv D$ since $d_1 \in (\leqslant 1\, r.\top)^{\mathcal{I}_1}$, but $e_1 \notin (\leqslant 1\, r.\top)^{\mathcal{I}_2}$. $\square$

Recall that $\mathcal{ALCI}$ denotes the extension of $\mathcal{ALC}$ by inverse roles.

Proposition 3.4. *$\mathcal{ALCI}$ is more expressive than $\mathcal{ALC}$; that is, there is an $\mathcal{ALCI}$ concept C such that $C \not\equiv D$ holds for all $\mathcal{ALC}$ concepts D.*

Proof. We show that no $\mathcal{ALC}$ concept is equivalent to the $\mathcal{ALCI}$ concept $\exists r^-.\top$. Assume to the contrary that D is an $\mathcal{ALC}$ concept with $\exists r^-.\top \equiv D$. In order to lead this assumption to a contradiction, we consider the interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ depicted in Figure 3.3.

Since $\rho = \{(d_2, e_2)\}$ is a bisimulation, we have $(\mathcal{I}_1, d_2) \sim (\mathcal{I}_2, e_2)$, and

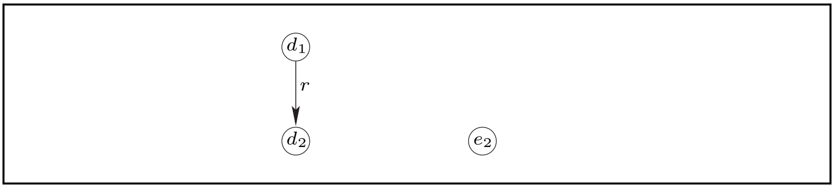

Fig. 3.3. Two more interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ represented as graphs.

thus $d_2 \in D^{\mathcal{I}_1}$ if and only if $e_2 \in D^{\mathcal{I}_2}$. This contradicts our assumption $\exists r^-.\top \equiv D$ since $d_2 \in (\exists r^-.\top)^{\mathcal{I}_1}$, but $e_2 \notin (\exists r^-.\top)^{\mathcal{I}_2}$. □

Recall that $\mathcal{ALCO}$ denotes the extension of $\mathcal{ALC}$ by nominals.

Proposition 3.5. *$\mathcal{ALCO}$ is more expressive than $\mathcal{ALC}$; that is, there is an $\mathcal{ALCO}$ concept C such that $C \not\equiv D$ holds for all $\mathcal{ALC}$ concepts D.*

Proof. We show that no $\mathcal{ALC}$ concept is equivalent to the $\mathcal{ALCO}$ concept $\{a\}$. Using the same pattern as in the previous two proofs, it is enough to show that there are bisimilar elements that can be distinguished by this concept. For this, we consider the interpretation $\mathcal{I}_1$ with $\Delta^{\mathcal{I}_1} = \{d\}$, $a^{\mathcal{I}_1} = d$ and $A^{\mathcal{I}_1} = \emptyset = r^{\mathcal{I}_1}$ for all $A \in \mathbf{C}$ and $r \in \mathbf{R}$; and the interpretation $\mathcal{I}_2$ with $\Delta^{\mathcal{I}_2} = \{e_1, e_2\}$, $a^{\mathcal{I}_2} = e_1$ and $A^{\mathcal{I}_2} = \emptyset = r^{\mathcal{I}_2}$ for all $A \in \mathbf{C}$ and $r \in \mathbf{R}$.

Since $\rho = \{(d, e_2)\}$ is a bisimulation, we have $(\mathcal{I}_1, d) \sim (\mathcal{I}_2, e_2)$, but $d \in \{a\}^{\mathcal{I}_1}$ and $e_2 \notin \{a\}^{\mathcal{I}_2}$. □

In summary, we have now convinced ourselves that extending $\mathcal{ALC}$ with one of inverse roles, nominals or number restrictions indeed increases its ability to describe certain models. In the following sections, we will look more closely into statements that we cannot make in $\mathcal{ALC}$. For example, the results of the next section imply that $\mathcal{ALC}$ cannot enforce finiteness of a model, whereas the subsequent section shows that it cannot enforce infiniteness either. Finally, the tree model property proved in the last section of this chapter implies that $\mathcal{ALC}$ cannot enforce cyclic role relationships.

3.3 Closure under disjoint union

Given two interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ with disjoint domains, one can put them together into one interpretation $\mathcal{I}$ by taking as its domain the union of the two domains, and defining the extensions of concept and role names in $\mathcal{I}$ as the union of the respective extensions in $\mathcal{I}_1$ and $\mathcal{I}_2$. It can then be shown that the extension of a (possibly complex) concept C in $\mathcal{I}$ is also the union of the extensions of C in $\mathcal{I}_1$ and $\mathcal{I}_2$. Below, we define and prove this in the more general setting where the interpretation domains are not necessarily disjoint and where we may have more than two interpretation. Before we can then build the *disjoint* union of these interpretations, we must make them disjoint by an appropriate renaming of the domain elements.

Definition 3.6 (Disjoint union). Let $\mathfrak{N}$ be an index set and $(\mathcal{I}_\nu)_{\nu\in\mathfrak{N}}$ a family of interpretations $\mathcal{I}_\nu = (\Delta^{\mathcal{I}_\nu}, \cdot^{\mathcal{I}_\nu})$. Their *disjoint union* $\mathcal{J}$ is defined as follows:

$$
\begin{aligned}
\Delta^{\mathcal{J}} &= \{(d,\nu) \mid \nu \in \mathfrak{N} \text{ and } d \in \Delta^{\mathcal{I}_\nu}\};\\
A^{\mathcal{J}} &= \{(d,\nu) \mid \nu \in \mathfrak{N} \text{ and } d \in A^{\mathcal{I}_\nu}\} \text{ for all } A \in \mathbf{C};\\
r^{\mathcal{J}} &= \{((d,\nu),(e,\nu)) \mid \nu \in \mathfrak{N} \text{ and } (d,e) \in r^{\mathcal{I}_\nu}\} \text{ for all } r \in \mathbf{R}.
\end{aligned}
$$

In the following, we will sometimes denote such a disjoint union as $\biguplus_{\nu\in\mathfrak{N}} \mathcal{I}_\nu$. Note that the interpretations $\mathcal{I}_\nu$ are not required to be distinct from each other. In particular, if all members $\mathcal{I}_\nu$ of the family are the same interpretation $\mathcal{I}$ and $\mathfrak{N} = \{1,\ldots,n\}$, then we call $\biguplus_{\nu\in\mathfrak{N}} \mathcal{I}_\nu$ the n-fold disjoint union of $\mathcal{I}$ with itself. Similarly, if $\mathfrak{N} = \mathbb{N}$ and all elements of the family are equal to $\mathcal{I}$, then we call $\biguplus_{\nu\in\mathfrak{N}} \mathcal{I}_\nu$ the countably infinite disjoint union of $\mathcal{I}$ with itself.

As an example, consider the three interpretations $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ depicted in Figure 3.1. We can view the three graphs in this figure as a single graph, which then is the graph representation of the disjoint union of these three interpretations (modulo appropriate renaming of nodes).

Lemma 3.7. *Let $\mathcal{J} = \biguplus_{\nu\in\mathfrak{N}} \mathcal{I}_\nu$ be the disjoint union of the family $(\mathcal{I}_\nu)_{\nu\in\mathfrak{N}}$ of interpretations. Then we have*

$$
d \in C^{\mathcal{I}_\nu} \quad \textit{if and only if} \quad (d,\nu) \in C^{\mathcal{J}}
$$

for all $\nu \in \mathfrak{N}$, $d \in \Delta^{\mathcal{I}_\nu}$ and $\mathcal{ALC}$ concept descriptions C.

Proof. It is easy to see that, for all $\nu \in \mathfrak{N}$, the relation

$$
\rho = \{(d,(d,\nu)) \mid d \in \Delta^{\mathcal{I}_\nu}\}
$$

is a bisimulation between $\mathcal{I}_\nu$ and $\mathcal{J}$. Thus, the bi-implication in the statement of the lemma follows immediately from Theorem 3.2. □

As an easy consequence of this lemma, we obtain that the class of all models of a TBox is closed under disjoint union.

Theorem 3.8. *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and $(\mathcal{I}_\nu)_{\nu\in\mathfrak{N}}$ a family of models of $\mathcal{T}$. Then its disjoint union $\mathcal{J} = \biguplus_{\nu\in\mathfrak{N}} \mathcal{I}_\nu$ is also a model of $\mathcal{T}$.*

Proof. Assume that $\mathcal{J}$ is not a model of $\mathcal{T}$. Then there is a GCI $C \sqsubseteq D$ in $\mathcal{T}$ and an element $(d,\nu) \in \Delta^{\mathcal{J}}$ such that $(d,\nu) \in C^{\mathcal{J}}$, but $(d,\nu) \notin D^{\mathcal{J}}$. By Lemma 3.7, this implies $d \in C^{\mathcal{I}_\nu}$ and $d \notin D^{\mathcal{I}_\nu}$, which contradicts our assumption that $\mathcal{I}_\nu$ is a model of $\mathcal{T}$. □

As an example of an application of this theorem we show that extensions of satisfiable concepts can always be made infinite.

Corollary 3.9. *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and C an $\mathcal{ALC}$ concept that is satisfiable with respect to $\mathcal{T}$. Then there is a model $\mathcal{J}$ of $\mathcal{T}$ in which the extension $C^{\mathcal{J}}$ of C is infinite.*

Proof. Since C is satisfiable with respect to $\mathcal{T}$, there is a model $\mathcal{I}$ of $\mathcal{T}$ and an element $d \in \Delta^{\mathcal{I}}$ such that $d \in C^{\mathcal{I}}$. Let $\mathcal{J} = \biguplus_{n\in\mathbb{N}} \mathcal{I}_n$ be the countably infinite disjoint union of $\mathcal{I}$ with itself. By Theorem 3.8, $\mathcal{J}$ is a model of $\mathcal{T}$, and by Lemma 3.7, $(d,n) \in C^{\mathcal{J}}$ for all $n \in \mathbb{N}$. □

In this section, we have restricted our attention to TBoxes. We can extend our observations to knowledge bases, but we need to be a little bit careful: in particular, since individual names can have only one extension in an interpretation, we would need to pick a single index $\nu \in \mathfrak{N}$ and set $a^{\mathcal{J}} = (a^{\mathcal{I}_\nu}, \nu)$ for all individual names occurring in this knowledge base. Then, being a model of a knowledge base is preserved when taking the disjoint union of such models.

3.4 Finite model property

As we saw in the previous chapter, in $\mathcal{ALC}$ we cannot force models to be finite. As we will see next, we cannot enforce them to be infinite either.

Definition 3.10. The interpretation $\mathcal{I}$ is a *model of a concept C with respect to a TBox $\mathcal{T}$* if $\mathcal{I}$ is a model of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$. We call this model *finite* if $\Delta^{\mathcal{I}}$ is finite.

In the following, we show that $\mathcal{ALC}$ has the *finite model property* (fmp), i.e., every $\mathcal{ALC}$ concept that is satisfiable with respect to an $\mathcal{ALC}$ TBox has a finite model. Interestingly, this can be used to show that satisfiability of $\mathcal{ALC}$ concepts with respect to $\mathcal{ALC}$ TBoxes is decidable since we can actually determine a concrete bound on the size of such a finite model.

Before we can prove that $\mathcal{ALC}$ has the fmp, we need to introduce some technical notions. Given an $\mathcal{ALC}$ concept C, we define its *size* $\mathsf{size}(C)$ and the set of its *subconcepts* $\mathsf{sub}(C)$ by induction on the structure of C:

- If $C = A \in N_C \cup \{\top, \bot\}$, then $\mathsf{size}(C) = 1$ and $\mathsf{sub}(C) = \{A\}$.
- If $C = C_1 \sqcap C_2$ or $C = C_1 \sqcup C_2$, then $\mathsf{size}(C) = 1 + \mathsf{size}(C_1) + \mathsf{size}(C_2)$ and $\mathsf{sub}(C) = \{C\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2)$.
- If $C = \neg D$ or $C = \exists r.D$ or $C = \forall r.D$, then $\mathsf{size}(C) = 1 + \mathsf{size}(D)$ and $\mathsf{sub}(C) = \{C\} \cup \mathsf{sub}(D)$.

The size just counts the number of occurrences of concept names (including $\top$ and $\bot$), role names and Boolean operators. For example,

$$\mathsf{size}(A \sqcap \exists r.(A \sqcup B)) = 1 + 1 + (1 + (1 + 1 + 1)) = 6.$$

For the same concept, the set of its subconcepts is

$$\mathsf{sub}(A \sqcap \exists r.(A \sqcup B)) = \{A \sqcap \exists r.(A \sqcup B), A, \exists r.(A \sqcup B), A \sqcup B, B\}.$$

We can extend these notions to $\mathcal{ALC}$ TBoxes as follows:

$$\mathsf{size}(\mathcal{T}) = \sum_{C \sqsubseteq D \in \mathcal{T}} \mathsf{size}(C) + \mathsf{size}(D) \text{ and } \mathsf{sub}(\mathcal{T}) = \bigcup_{C \sqsubseteq D \in \mathcal{T}} \mathsf{sub}(C) \cup \mathsf{sub}(D).$$

It is easy to see[1] that the number of subconcepts of a concept or TBox is bounded by the size of the concept or TBox:

Lemma 3.11. *Let C be an $\mathcal{ALC}$ concept and $\mathcal{T}$ be an $\mathcal{ALC}$ TBox. Then*

$$|\mathsf{sub}(C)| \leq \mathsf{size}(C) \text{ and } |\mathsf{sub}(\mathcal{T})| \leq \mathsf{size}(\mathcal{T}).$$

We call a set S of $\mathcal{ALC}$ concepts *closed* if $\bigcup\{\mathsf{sub}(C) \mid C \in S\} \subseteq S$. Obviously, if S is the set of subdescriptions of an $\mathcal{ALC}$ concept or TBox, then S is closed.

Definition 3.12 (S-type)**.** Let S be a set of $\mathcal{ALC}$ concepts and $\mathcal{I}$ an interpretation. The *S-type* of $d \in \Delta^\mathcal{I}$ is defined as

$$t_S(d) = \{C \in S \mid d \in C^\mathcal{I}\}.$$

Since an S-type is a subset of S, there are at most as many S-types as there are subsets:

Lemma 3.13. *Let S be a finite set of $\mathcal{ALC}$ concepts and $\mathcal{I}$ an interpretation. Then $|\{t_S(d) \mid d \in \Delta^\mathcal{I}\}| \leq 2^{|S|}$.*

[1] A formal proof can be done by induction on the structure of concepts.

The main idea underlying our proof that $\mathcal{ALC}$ has the fmp is that, in order to find a model of an $\mathcal{ALC}$ concept C with respect to an $\mathcal{ALC}$ TBox $\mathcal{T}$, it is sufficient to consider only interpretations in which every S-type is realised by at most one element, i.e., $d = d'$ if and only if $t_S(d) = t_S(d')$, where S is the set of subconcepts of C and $\mathcal{T}$. Starting with an arbitrary model of C with respect to $\mathcal{T}$, we can obtain a model satisfying this property by merging elements that have the same S-type into a single element using the filtration technique introduced below.

Definition 3.14 (S-filtration)**.** Let S be a finite set of $\mathcal{ALC}$ concepts and $\mathcal{I}$ an interpretation. We define the equivalence relation $\simeq_S$ on $\Delta^\mathcal{I}$ as follows:

$$d \simeq_S e \text{ if } t_S(d) = t_S(e).$$

The $\simeq_S$-*equivalence class* of $d \in \Delta^\mathcal{I}$ is denoted by $[d]_S$, i.e.,

$$[d]_S = \{e \in \Delta^\mathcal{I} \mid d \simeq_S e\}.$$

The *S-filtration of $\mathcal{I}$* is the following interpretation $\mathcal{J}$:

$$\begin{array}{rcl} \Delta^\mathcal{J} & = & \{[d]_S \mid d \in \Delta^\mathcal{I}\}; \\ A^\mathcal{J} & = & \{[d]_S \mid \text{there is } d' \in [d]_S \text{ with } d' \in A^\mathcal{I}\} \text{ for all } A \in \mathbf{C}; \\ r^\mathcal{J} & = & \{([d]_S, [e]_S) \mid \text{there are } d' \in [d]_S, e' \in [e]_S \text{ with } (d', e') \in r^\mathcal{I}\} \\ & & \text{for all } r \in \mathbf{R}. \end{array}$$

Lemma 3.15. *Let S be a finite,* closed *set of $\mathcal{ALC}$ concepts, $\mathcal{I}$ an interpretation and $\mathcal{J}$ the S-filtration of $\mathcal{I}$. Then we have*

$$d \in C^\mathcal{I} \textit{ if and only if } [d]_S \in C^\mathcal{J}$$

for all $d \in \Delta^\mathcal{I}$ and $C \in S$.

Proof. By induction on the structure of C, where we again restrict our attention to concept names, negation, conjunction and existential restriction (see Lemma 2.16):

- Assume that $C = A \in \mathbf{C}$.
  - If $d \in A^\mathcal{I}$, then $[d]_S \in A^\mathcal{J}$ by the definition of $\mathcal{J}$ since $d \in [d]_S$.
  - If $[d]_S \in A^\mathcal{J}$, then there is $d' \in [d]_S$ with $d' \in A^\mathcal{I}$. Since $d \simeq_S d'$ and $A \in S$, $d' \in A^\mathcal{I}$ implies $d \in A^\mathcal{I}$.

- Assume that $C = D \sqcap E$. Then the following holds:

$$\begin{aligned} d \in (D \sqcap E)^{\mathcal{I}} \quad &\text{if and only if} \quad d \in D^{\mathcal{I}} \text{ and } d \in E^{\mathcal{I}} \\ &\text{if and only if} \quad [d]_S \in D^{\mathcal{J}} \text{ and } [d]_S \in E^{\mathcal{J}} \\ &\text{if and only if} \quad [d]_S \in (D \sqcap E)^{\mathcal{J}}. \end{aligned}$$

  The first and last bi-implications hold because of the semantics of conjunction. The second holds by induction: since S is closed, we have $D, E \in S$, and thus the induction hypothesis applies to D and E.
- Negation $C = \neg D$ can be treated similarly to conjunction.
- Assume that $C = \exists r.D$. Since S is closed, we have $D \in S$, and thus the induction hypothesis applies to D.
  - If $d \in (\exists r.D)^{\mathcal{I}}$, then there is $e \in \Delta^{\mathcal{I}}$ such that $(d, e) \in r^{\mathcal{I}}$ and $e \in D^{\mathcal{I}}$. We have $([d]_S, [e]_S) \in r^{\mathcal{J}}$ since $d \in [d]_S$ and $e \in [e]_S$. In addition, induction (applied to $D \in S$) yields $[e]_S \in D^{\mathcal{J}}$. This shows $[d]_S \in (\exists r.D)^{\mathcal{J}}$.
  - If $[d]_S \in (\exists r.D)^{\mathcal{J}}$, then there is $[e]_S \in \Delta^{\mathcal{J}}$ such that $([d]_S, [e]_S) \in r^{\mathcal{J}}$ and $[e]_S \in D^{\mathcal{J}}$. Induction (applied to $D \in S$) yields $e \in D^{\mathcal{I}}$. In addition, there are $d' \in [d]_S$ and $e' \in [e]_S$ such that $(d', e') \in r^{\mathcal{I}}$. Since $e \simeq_S e'$ and $D \in S$, we know that $e \in D^{\mathcal{I}}$ implies $e' \in D^{\mathcal{I}}$. Consequently, we have $d' \in (\exists r.D)^{\mathcal{I}}$. But then $d \simeq_S d'$ and $\exists r.D \in S$ yield $d \in (\exists r.D)^{\mathcal{I}}$. □

One may be tempted to show the lemma in a simpler way using bisimulation invariance of $\mathcal{ALC}$ and the relation

$$\rho = \{(d, [d]_S) \mid d \in \Delta^{\mathcal{I}}\}$$

between elements of the domain of $\mathcal{I}$ and elements of the domain of $\mathcal{J}$. Unfortunately, this relation is in general not a bisimulation. First of all, (i) of Definition 3.1 is obviously only guaranteed to hold if S contains all concept names in $\mathbf{C}$. But even if this is assumed, (iii) of Definition 3.1 need not hold. In fact, assume that $S = \{\top, A, \exists r.\top\}$ where $\mathbf{C} = \{A\}$ and $\mathbf{R} = \{r\}$, and consider the interpretation $\mathcal{I}$ consisting of the elements d_1, d_2, d'_1, d'_2 depicted on the left-hand side of Figure 3.4. Then $\simeq_S$ has three equivalence classes, $[d_1]_S = [d_2]_S, [d'_1]_S$ and $[d'_2]_S$, and the S-filtration $\mathcal{J}$ of $\mathcal{I}$ is the interpretation depicted on the right-hand side of Figure 3.4. It is easy to see that the relation ρ defined above is not a bisimulation in this example. In fact, we have $(d_1, [d_1]_S) \in \rho$, but $[d_1]_S$ has an r-successor in $\mathcal{J}$ that does not belong to the extension of A, whereas d_1 does not have such an r-successor in $\mathcal{I}$.

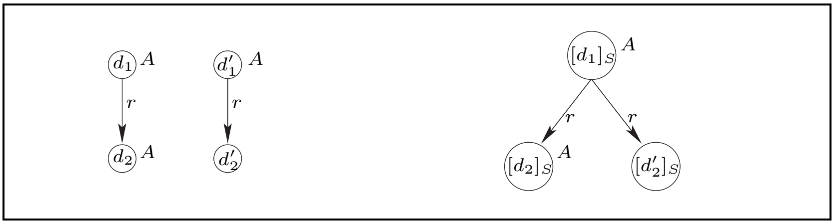

Fig. 3.4. An interpretation $\mathcal{I}$ and its S-filtration $\mathcal{J}$ for $S = \{\top, A, \exists r.\top\}$.

As a consequence of Lemma 3.15, we can show that $\mathcal{ALC}$ satisfies a property that is even stronger than the finite model property: the *bounded model property*. For the bounded model property, it is not sufficient to know that there is a finite model. One also needs to have an explicit bound on the cardinality of this model in terms of the size of the TBox and concept.

Theorem 3.16 (Bounded model property)**.** *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox, C an $\mathcal{ALC}$ concept and $n = \mathsf{size}(\mathcal{T}) + \mathsf{size}(C)$. If C has a model with respect to $\mathcal{T}$, then it has one of cardinality at most 2^n.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{T}$ with $C^\mathcal{I} \neq \emptyset$, and $S = \mathsf{sub}(\mathcal{T}) \cup \mathsf{sub}(C)$. Then we have $|S| \leq n$, and thus the domain of the S-filtration $\mathcal{J}$ of $\mathcal{I}$ satisfies $|\Delta^\mathcal{J}| \leq 2^n$ by Lemma 3.13. Thus, it remains to show that $\mathcal{J}$ is a model of C with respect to $\mathcal{T}$.

Let $d \in \Delta^\mathcal{I}$ be such that $d \in C^\mathcal{I}$. Since $C \in S$, we know that $d \in C^\mathcal{I}$ implies $[d]_S \in C^\mathcal{J}$ by Lemma 3.15, and thus $C^\mathcal{J} \neq \emptyset$. In addition, it is easy to see that $\mathcal{J}$ is a model of $\mathcal{T}$. In fact, let $D \sqsubseteq E$ be a GCI in $\mathcal{T}$, and $[e]_S \in D^\mathcal{J}$. We must show $[e]_S \in E^\mathcal{J}$. Since $D \in S$, Lemma 3.15 yields $e \in D^\mathcal{I}$, and thus $e \in E^\mathcal{I}$ since $\mathcal{I}$ is a model of $\mathcal{T}$. But then $E \in S$ implies $[e]_S \in E^\mathcal{J}$, again by Lemma 3.15. $\square$

Obviously, the finite model property of $\mathcal{ALC}$ is an immediate consequence of the bounded model property.

Corollary 3.17 (Finite model property)**.** *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and C an $\mathcal{ALC}$ concept. If C has a model with respect to $\mathcal{T}$, then it has one of finite cardinality.*

Another interesting consequence of the bounded model property of $\mathcal{ALC}$ is that the *satisfiability problem* for $\mathcal{ALC}$ concepts with respect to $\mathcal{ALC}$ TBoxes is decidable.

Corollary 3.18 (Decidability)**.** *Satisfiability of $\mathcal{ALC}$ concepts with respect to $\mathcal{ALC}$ TBoxes is decidable.*

Proof. Let $n = \mathsf{size}(\mathcal{T}) + \mathsf{size}(C)$. If C is satisfiable with respect to $\mathcal{T}$, then it has a model of cardinality at most 2^n. Up to isomorphism (i.e., up to renaming of the domain elements), there are only finitely many interpretations satisfying this size bound. Thus, we can enumerate all of these interpretations, and then check (using the inductive definition of the semantics of concepts) whether one of them is a model of C with respect to $\mathcal{T}$. □

Not all description logics have the fmp. For example, if we add number restrictions and inverse roles to $\mathcal{ALC}$, then the fmp is lost.

Theorem 3.19 (No finite model property)**.** *$\mathcal{ALCIN}$ does not have the finite model property.*

Proof. Let $C = \neg A \sqcap \exists r.A$ and $\mathcal{T} = \{A \sqsubseteq \exists r.A, \top \sqsubseteq (\leqslant 1\, r^-)\}$. We claim that C does not have a finite model with respect to $\mathcal{T}$.

Assume to the contrary that $\mathcal{I}$ is such a finite model, and let $d_0 \in \Delta^\mathcal{I}$ be such that $d_0 \in C^\mathcal{I}$. Then $d_0 \in (\exists r.A)^\mathcal{I}$, and thus there is $d_1 \in \Delta^\mathcal{I}$ such that $(d_0, d_1) \in r^\mathcal{I}$ and $d_1 \in A^\mathcal{I}$. Because of the first GCI in $\mathcal{T}$, there is $d_2 \in \Delta^\mathcal{I}$ such that $(d_1, d_2) \in r^\mathcal{I}$ and $d_2 \in A^\mathcal{I}$. We can continue this argument to obtain a sequence $d_0, d_1, d_2, d_3, \ldots$ of individuals in $\Delta^\mathcal{I}$ such that

- $d_0 \notin A^\mathcal{I}$,
- $(d_{i-1}, d_i) \in r^\mathcal{I}$ and $d_i \in A^\mathcal{I}$ for all $i \geq 1$.

Since $\Delta^\mathcal{I}$ is finite, there are two indices $0 \leq i < j$ such that $d_i = d_j$. We may assume without loss of generality that i is chosen minimally, i.e., for all $k < i$ there is no $\ell > k$ such that $d_k = d_\ell$.

Since $j > 0$, we have $d_i = d_j \in A^\mathcal{I}$, and thus $i = 0$ is not possible. However, $i > 0$ and $j > 0$ imply that d_{i-1} and d_{j-1} are r-predecessors of $d_i = d_j$, i.e., $(d_i, d_{i-1}) \in (r^-)^\mathcal{I}$ and $(d_i, d_{j-1}) \in (r^-)^\mathcal{I}$. Consequently, the second GCI in $\mathcal{T}$ enforces $d_{i-1} = d_{j-1}$, which contradicts our minimal choice of i. □

So, we have seen that $\mathcal{ALC}$ cannot enforce infinity of models, but $\mathcal{ALCIN}$ can. In fact, it is known that both $\mathcal{ALCI}$ and $\mathcal{ALCN}$ still enjoy the fmp (and so does $\mathcal{ALCQ}$). Thus, it is indeed the *combination* of number restrictions and inverse roles that destroys the fmp.

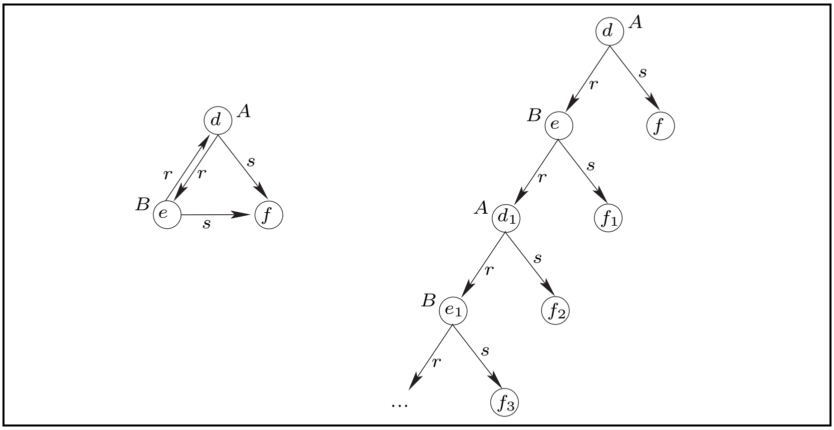

Fig. 3.5. Unravelling of a model $\mathcal{I}$ into a tree model $\mathcal{J}$.

3.5 Tree model property

Another interesting model-theoretic property of $\mathcal{ALC}$ is that every satisfiable concept has a tree model. For the purpose of this section, a *tree* is a directed graph $\mathcal{G} = (V, E)$ such that

- V contains a unique *root*, i.e., a node $v_r \in V$ such that there is no $v \in V$ with $(v, v_r) \in E$;
- every node $v \in V \setminus \{v_r\}$ has a unique predecessor, i.e., there is a unique node $v' \in V$ such that $(v', v) \in E$.

Basically, a tree model is a model whose graph representation is a tree.

Definition 3.20 (Tree model)**.** Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and C an $\mathcal{ALC}$ concept description. The interpretation $\mathcal{I}$ is a *tree model* of C with respect to $\mathcal{T}$ if $\mathcal{I}$ is a model of C with respect to $\mathcal{T}$, and the graph

$$\mathcal{G}_\mathcal{I} = \left(\Delta^\mathcal{I}, \bigcup_{r \in \mathbf{R}} r^\mathcal{I}\right)$$

is a tree whose root belongs to $C^\mathcal{I}$.

In order to show that every concept that is satisfiable with respect to $\mathcal{T}$ has a tree model with respect to $\mathcal{T}$, we use the well-known *unravelling* technique. Before introducing unravelling formally, we illustrate it by an example. The graph on the left-hand side of Figure 3.5 describes an interpretation $\mathcal{I}$. It is easy to check that $\mathcal{I}$ is a model of the concept A

with respect to the TBox

$$\mathcal{T} = \{A \sqsubseteq \exists r.B, \quad B \sqsubseteq \exists r.A, \quad A \sqcup B \sqsubseteq \exists s.\top\}.$$

The graph on the right-hand side of Figure 3.5 describes (a finite part of) the corresponding unravelled model $\mathcal{J}$, where d was used as the start node for the unravelling. Basically, one considers all paths starting with d in the original model but, rather than re-entering a node, one makes a copy of it. Like $\mathcal{I}$, the corresponding unravelled interpretation $\mathcal{J}$ is a model of $\mathcal{T}$ and it satisfies $d \in A^{\mathcal{J}}$.

More formally, let $\mathcal{I}$ be an interpretation and $d \in \Delta^{\mathcal{I}}$. A *$d$-path in $\mathcal{I}$* is a finite sequence $d_0, d_1, \ldots, d_{n-1}$ of $n \geq 1$ elements of $\Delta^{\mathcal{I}}$ such that

- $d_0 = d$,
- for all $i, 1 \leq i < n$, there is a role $r_i \in \mathbf{R}$ such that $(d_{i-1}, d_i) \in r_i^{\mathcal{I}}$.

Given a d-path $p = d_0, d_1, \ldots, d_{n-1}$, we define its *length* to be n and its *end node* to be $\mathsf{end}(p) = d_{n-1}$.

In the unravelled model, such paths constitute the elements of the domain. In our example, the node with label d_1 corresponds to the path d, e, d, the one with label f_1 to d, e, f, the one with label e_1 to d, e, d, e etc.

Definition 3.21 (Unravelling)**.** Let $\mathcal{I}$ be an interpretation and $d \in \Delta^{\mathcal{I}}$. The *unravelling of $\mathcal{I}$ at d* is the following interpretation $\mathcal{J}$:

$$\begin{aligned}
\Delta^{\mathcal{J}} &= \{p \mid \ p \text{ is a } d\text{-path in } \mathcal{I}\},\\
A^{\mathcal{J}} &= \{p \in \Delta^{\mathcal{J}} \mid \mathsf{end}(p) \in A^{\mathcal{I}}\} \text{ for all } A \in \mathbf{C},\\
r^{\mathcal{J}} &= \{(p, p') \in \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}} \mid p' = (p, \mathsf{end}(p')) \text{ and } (\mathsf{end}(p), \mathsf{end}(p')) \in r^{\mathcal{I}}\}\\
&\qquad \text{for all } r \in \mathbf{R}.
\end{aligned}$$

In our example, $d_1 = d, e, d \in A^{\mathcal{J}}$ because $\mathsf{end}(d_1) = d \in A^{\mathcal{I}}$, and $((d, e, d),\ (d, e, d, e)) \in r^{\mathcal{J}}$ because $(d, e) \in r^{\mathcal{I}}$.

Next, we will see that the relation that connects a d-path with its end node is a bisimulation.

Lemma 3.22. *The relation*

$$\rho = \{(p, \mathsf{end}(p)) \mid p \in \Delta^{\mathcal{J}}\}$$

is a bisimulation between $\mathcal{J}$ and $\mathcal{I}$.

Proof. By definition of the extensions of concept names in the interpretation $\mathcal{J}$, we have $p \in A^{\mathcal{J}}$ if and only if $\mathsf{end}(p) \in A^{\mathcal{I}}$, and thus Condition (i) of Definition 3.1 is satisfied.

To show that Condition (ii) of Definition 3.1 is also satisfied, we assume that $(p, p') \in r^{\mathcal{J}}$ and $(p, e) \in \rho$. Since $\mathsf{end}(p)$ is the only element of $\Delta^{\mathcal{I}}$ that is ρ-related to p, we have $e = \mathsf{end}(p)$. Thus, we must show that there is an $f \in \Delta^{\mathcal{I}}$ such that $(p', f) \in \rho$ and $(\mathsf{end}(p), f) \in r^{\mathcal{I}}$. We define $f = \mathsf{end}(p')$. Because $(p', \mathsf{end}(p')) \in \rho$, it is thus enough to show $(\mathsf{end}(p), \mathsf{end}(p')) \in r^{\mathcal{I}}$. This is, however, an immediate consequence of the definition of the extensions of roles in $\mathcal{J}$.

To show that Condition (iii) of Definition 3.1 is satisfied, we assume that $(e, f) \in r^{\mathcal{I}}$ and $(p, e) \in \rho$ (i.e., $\mathsf{end}(p) = e$). We must find a path p' such that $(p', f) \in \rho$ and $(p, p') \in r^{\mathcal{J}}$. We define $p' = p, f$. This is indeed a d-path since p is a d-path with $\mathsf{end}(p) = e$ and $(e, f) \in r^{\mathcal{I}}$. In addition, $\mathsf{end}(p') = f$, which shows $(p', f) \in \rho$. Finally, we clearly have $p' = p, \mathsf{end}(p')$ and $(\mathsf{end}(p), \mathsf{end}(p')) \in r^{\mathcal{I}}$ since $\mathsf{end}(p) = e$ and $\mathsf{end}(p') = f$. This yields $(p, p') \in r^{\mathcal{J}}$. □

The following proposition is an immediate consequence of this lemma and Theorem 3.2.

Proposition 3.23. *For all $\mathcal{ALC}$ concepts C and all $p \in \Delta^{\mathcal{J}}$, we have*

$$p \in C^{\mathcal{J}} \quad \textit{if and only if} \quad \mathsf{end}(p) \in C^{\mathcal{I}}.$$

We are now ready to show the tree model property of $\mathcal{ALC}$.

Theorem 3.24 (Tree model property)**.** *$\mathcal{ALC}$ has the* tree model property*, i.e., if $\mathcal{T}$ is an $\mathcal{ALC}$ TBox and C an $\mathcal{ALC}$ concept such that C is satisfiable with respect to $\mathcal{T}$, then C has a tree model with respect to $\mathcal{T}$.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{T}$ and $d \in \Delta^{\mathcal{I}}$ be such that $d \in C^{\mathcal{I}}$. We show that the unravelling $\mathcal{J}$ of $\mathcal{I}$ at d is a tree model of C with respect to $\mathcal{T}$.

(i) To prove that $\mathcal{J}$ is a model of $\mathcal{T}$, consider a GCI $D \sqsubseteq E$ in $\mathcal{T}$, and assume that $p \in \Delta^{\mathcal{J}}$ satisfies $p \in D^{\mathcal{J}}$. We must show $p \in E^{\mathcal{J}}$. By Proposition 3.23, we have $\mathsf{end}(p) \in D^{\mathcal{I}}$, which yields $\mathsf{end}(p) \in E^{\mathcal{I}}$ since $\mathcal{I}$ is model of $\mathcal{T}$. But then Proposition 3.23 applied in the other direction yields $p \in E^{\mathcal{J}}$.

(ii) We show that the graph

$$\mathcal{G}_{\mathcal{J}} = \Big(\Delta^{\mathcal{J}}, \bigcup_{r \in N_R} r^{\mathcal{J}}\Big)$$

is a tree with root d, where d is viewed as a d-path of length 1. First, note that d is the only d-path of length 1. By definition of the extensions of roles in $\mathcal{J}$ and the definition of d-paths, all and only d-paths of length > 1 have a predecessor with respect to some role. Consequently, d is the unique node without predecessor, i.e., the root. Assume that p is a d-path of length > 1. Then there is a unique d-path p' such that $p = p', \mathsf{end}(p)$. Thus, p' is the unique d-path with $(p', p) \in E$, which completes our proof that $\mathcal{G}_{\mathcal{J}}$ is a tree with root d.

(iii) It remains to show that the root d of this tree belongs to the extension of C in $\mathcal{J}$. However, this follows immediately by Proposition 3.23 since $d = \mathsf{end}(d)$ and $d \in C^{\mathcal{I}}$.

This completes the proof of the theorem. □

Note that, in case the model we start with has a cycle, the tree constructed in the proof is an infinite tree, i.e., it has infinitely many nodes. Although $\mathcal{ALC}$ has the finite model property and the tree model property, it does not have the finite tree model property. In fact, it is easy to see that the concept A does not have a finite tree model with respect to the TBox $\{A \sqsubseteq \exists r.A\}$.

It should also be noted that, in our definition of a tree model, we do not consider edge labels. Thus, $(u, v) \in r^{\mathcal{I}}$ yields the same edge (u, v) in E as $(u, v) \in s^{\mathcal{I}}$. Consequently, in a tree model as introduced in Definition 3.20, there can be several roles connecting two nodes u and v with $(u, v) \in E$. Alternatively, we could have required that, for every element d of $\Delta^{\mathcal{I}}$ excepting the root, there is exactly one role r and element $d' \in \Delta^{\mathcal{I}}$ such that $(d', d) \in r^{\mathcal{I}}$. One can show that $\mathcal{ALC}$ also satisfies the tree model property for this stronger notion of tree model, but the definition of unravelling gets a bit more complicated since role names need to be remembered in the paths.

We remark that many extensions of $\mathcal{ALC}$, such as $\mathcal{ALCIQ}$, also enjoy the tree model property. However, in the presence of inverse roles, a more liberal definition of trees is needed that also allows edges to be oriented towards the root. An example of a description logic that does not enjoy the tree model property is $\mathcal{ALCO}$: the concept $\{o\} \sqcap \exists r.\{o\}$ can clearly only have a non-empty extension in an interpretation that has a reflexive r-edge.

Finally, let us point out that the tree model property can also be used to show decidability of satisfiability of concepts with respect to TBoxes in $\mathcal{ALC}$, using the so-called *automata-based approach*. The automata used in this approach are automata working on infinite trees. In general, there are various types of such automata such as *Büchi*, *Rabin* and *parity automata*, but for $\mathcal{ALC}$ the simpler *looping automata* (which have a trivial acceptance condition) are sufficient. An important property of all these automata is that their emptiness problem (i.e., the question whether a given automaton accepts at least one tree) is decidable, for looping automata even in linear time. In principle, the automata approach for $\mathcal{ALC}$ works as follows:

- Devise a translation from each pair $C, \mathcal{T}$, where C is an $\mathcal{ALC}$ concept description and $\mathcal{T}$ is an $\mathcal{ALC}$ TBox, into a looping tree automaton $\mathcal{A}_{C,\mathcal{T}}$ such that $\mathcal{A}_{C,\mathcal{T}}$ accepts exactly the tree models of C with respect to $\mathcal{T}$.
- Apply the emptiness test for looping tree automata to $\mathcal{A}_{C,\mathcal{T}}$ to test whether C has a (tree) model with respect to $\mathcal{T}$: if $\mathcal{A}_{C,\mathcal{T}}$ accepts some trees, then these are (tree) models of C with respect to $\mathcal{T}$; if $\mathcal{A}_{C,\mathcal{T}}$ accepts no trees, then C has no tree models with respect to $\mathcal{T}$, and thus no models.

We do not go into more detail here, but just want to point out that the states of these automata are types (as introduced in Section 3.4) and that the emptiness test for them boils down to the type elimination procedure described in Section 5.1.2.

3.6 Historical context and literature review

In Section 2.6.2, the close relationship between description and modal logics was described. The model-theoretic notions and properties considered in this chapter in the context of description logics have originally been introduced and proved for modal logics (see, e.g., [BdRV01, Bv07, GO07]). For the modal logic $\mathbf{K_{(m)}}$, it was also shown that bisimulations satisfy additional interesting properties that are stronger than Theorem 3.2. Expressed for the syntactic variant $\mathcal{ALC}$ of $\mathbf{K_{(m)}}$, two famous ones are the following:

(a) A formula of first-order logic with one free variable is equivalent to the translation of an $\mathcal{ALC}$ concept if and only if it is invariant under bisimulation.

(b) If $\mathcal{I}_1$ and $\mathcal{I}_2$ are interpretations of finite outdegree (that is, in which every element has only finitely many role successors) and $d_1 \in \Delta^{\mathcal{I}_1}$ and $d_2 \in \Delta^{\mathcal{I}_2}$, then d_1 and d_2 belong to the same $\mathcal{ALC}$ concepts if and only if $(d_1, \mathcal{I}_1) \sim (d_2, \mathcal{I}_2)$. Note that every finite interpretation satisfies this property. For interpretations in which elements can have an unrestricted number of successors, this bi-implication need not hold.

Basically, Property (a) says that the notion of bisimulation that we have introduced in Definition 3.1 is exactly the right one for $\mathcal{ALC}$. An analogue of Property (a) also exists for TBoxes instead of for concepts, saying that a sentence of first-order logic is equivalent to the translation of an $\mathcal{ALC}$ TBox if and only if it is invariant under bisimulation and disjoint unions [LPW11].

It is important to note that many notions and constructions in this section are tailored specifically to the description logic $\mathcal{ALC}$. For example, proving non-expressibility results for logics other than $\mathcal{ALC}$ requires other versions of bisimulations. For $\mathcal{ALCI}$, one needs to admit also inverse roles in Conditions (ii) and (iii) of bisimulations. For $\mathcal{ALCQ}$, these conditions need to consider more than one successor at the time and involve some counting. An overview is given in [LPW11]; other relevant references are [KdR97, Kd99] in the description logic literature and [dR00, GO07] in that of modal logic. An interesting case is provided by description logics that are weaker than $\mathcal{ALC}$, such as $\mathcal{EL}$, which admits only the constructors $\top$, $\sqcap$ and $\exists r.C$, and which we study in Chapters 6 and 8. For this DL, bisimulations need to be replaced by simulations, which intuitively are "half a bisimulation" as they only go "forth" from $\mathcal{I}_1$ to $\mathcal{I}_2$, but not "back" from $\mathcal{I}_2$ to $\mathcal{I}_1$. Concepts formulated in a logic without disjunction such as $\mathcal{EL}$ also have another important model-theoretic property, namely that they are preserved under forming direct products, an operation well known from classical model theory [Hod93]. In fact, in analogy to Property (a), a formula of first-order logic with one free variable is equivalent to the translation of an $\mathcal{EL}$ concept if and only if it is preserved under simulations and direct products [LPW11].

Properties like the tree model property and the finite model property can also be shown as a consequence of the completeness of tableau algorithms (see Chapter 4). For example, the original tableau algorithm for satisfiability of $\mathcal{ALC}$ concepts (without TBoxes) [SS91] in principle constructs a finite tree model whenever the input concept is satisfiable.