# Appendix
## Description Logic Terminology

The purpose of this appendix is to summarise the syntax and semantics of the DL constructors and axioms used in this book. More information and explanations can be found in the relevant chapters. We will also comment on the naming schemes for DLs that are employed in the literature and in this book.

### A.1 Syntax and semantics of concept and role constructors

The *concept descriptions* of a DL are built from *concept names*, *role names* and *individual names* using the *concept* and *role constructors* available in the DL. Table A.1 lists the name, syntax and semantics of such constructors. In this table, $C, D$ stand for concepts (concept names or compound concepts), $r, s$ for roles (role names or compound roles) and $a$ for an individual name. The symbol $\#$ in the semantics of number restrictions maps a set to its cardinality. With $r^n$ we denote the $n$-fold composition of $r$ with itself, i.e., $r^1 = r$ and $r^{n+1} = r^n \circ r$. Note that, for historical reasons, role value maps are written $(r \sqsubseteq s)$, where $r$ and $s$ are role names or compositions of role names. Role value maps are concept descriptions – they denote the set of individuals whose role values satisfy the relevant inclusion – and should not be confused with role inclusion axioms.

*Predicate restrictions* need a bit more explanation. They presuppose that a fixed so-called *concrete domain* $\mathsf{D} = (\Delta^{\mathsf{D}}, \Phi^{\mathsf{D}})$ is given, where $\Delta^{\mathsf{D}}$ is a non-empty set and $\Phi^{\mathsf{D}}$ is a finite set of predicates. Each *predicate* in $\Phi^{\mathsf{D}}$ has a name $P$, an arity $k_P$ and an extension $P^{\mathsf{D}} \subseteq (\Delta^{\mathsf{D}})^{k_P}$. In the predicate restriction $\exists c_1, \ldots, c_k.P$, the symbol $P$ is the name of a predicate from $\Phi^{\mathsf{D}}$, which has arity $k$, and the symbols $c_1, \ldots, c_k$ stand for feature chains. A *feature chain* $c$ is a sequence of the form $g_1 \cdots g_n h$

| Name | Syntax | Semantics |
|---|---|---|
| Top | $\top$ | $\Delta^{\mathcal{I}}$ |
| Bottom | $\bot$ | $\emptyset$ |
| Conjunction | $C \sqcap D$ | $C^{\mathcal{I}} \cap D^{\mathcal{I}}$ |
| Disjunction | $C \sqcup D$ | $C^{\mathcal{I}} \cup D^{\mathcal{I}}$ |
| Negation | $\neg C$ | $\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ |
| Exist. restr. | $\exists r.C$ | $\{d \in \Delta^{\mathcal{I}} \mid \exists e \in \Delta^{\mathcal{I}}.(d,e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}}\}$ |
| Value restr. | $\forall r.C$ | $\{d \in \Delta^{\mathcal{I}} \mid \forall e \in \Delta^{\mathcal{I}}.(d,e) \in r^{\mathcal{I}} \rightarrow e \in C^{\mathcal{I}}\}$ |
| Self restr. | $\exists r.\mathsf{Self}$ | $\{d \in \Delta^{\mathcal{I}} \mid (d,d) \in r^{\mathcal{I}}\}$ |
| Unqualified number restr. | $(\leqslant n\, r)$<br>$(\geqslant n\, r)$ | $\{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}}\} \leq n\}$<br>$\{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}}\} \geq n\}$ |
| Qualified number restr. | $(\leqslant n\, r.C)$<br>$(\geqslant n\, r.C)$ | $\{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}}\} \leq n\}$<br>$\{d \in \Delta^{\mathcal{I}} \mid \#\{e \mid (d,e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}}\} \geq n\}$ |
| Nominal | $\{a\}$ | $\{a^{\mathcal{I}}\}$ |
| Role value map | $(r \sqsubseteq s)$ | $\{d \in \Delta^{\mathcal{I}} \mid \{e \mid (d,e) \in r^{\mathcal{I}}\} = \{e' \mid (d,e') \in s^{\mathcal{I}}\}\}$ |
| Predicate restr. | $\exists c_1, \ldots, c_k.P$ | $\{d \in \Delta^{\mathcal{I}} \mid (c_1^{\mathcal{I}}(d), \ldots, c_k^{\mathcal{I}}(d)) \in P^{\mathsf{D}}\}$ |
| Role composition | $r \circ s$ | $\{(d,f) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \exists e \in \Delta^{\mathcal{I}}.(d,e) \in r^{\mathcal{I}} \wedge (e,f) \in s^{\mathcal{I}}\}$ |
| Inverse role | $r^{-}$ | $\{(e,d) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (d,e) \in r^{\mathcal{I}}\}$ |
| Feature chain | $g_1 \cdots g_n h$ | $(g_1 \cdots g_n h)^{\mathcal{I}}(d) = h^{\mathcal{I}}(g_n^{\mathcal{I}}(\cdots(g_1^{\mathcal{I}}(d))\cdots))$ |

Table A.1. *Some Description Logic concept and role constructors.*

of $n \geq 0$ *abstract features* $g_i$ and one *concrete feature* $h$. Thus, from the syntactic point of view we need to assume that, in addition to concept, role and individual names, abstract and concrete feature names are also available.

The semantics of concept and role descriptions is defined using the notion of an *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty set and the interpretation function $\cdot^{\mathcal{I}}$ maps concept names $A$ to sets $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, role names $r$ to binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and individual names $a$ to elements $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. In the presence of a concrete domain $\mathsf{D} = (\Delta^{\mathsf{D}}, \Phi^{\mathsf{D}})$, abstract features $g$ are interpreted as partial functions $g^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow \Delta^{\mathcal{I}}$ and concrete features $h$ as partial functions $h^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow \Delta^{\mathsf{D}}$. The interpretation function $\cdot^{\mathcal{I}}$ is inductively extended to compound concepts, roles and feature chains using the identities given in the semantics column of Table A.1. In the definition of the semantics of predicate restrictions, the condition that the tuple $(c_1^{\mathcal{I}}(d), \ldots, c_k^{\mathcal{I}}(d))$ belongs to $P^{\mathcal{D}}$ includes the requirement that all the elements of this tuple are well-defined, i.e., $d$ belongs to the domains of the partial functions $c_1^{\mathcal{I}}, \ldots, c_k^{\mathcal{I}}$. For the feature chain $c = g_1 \cdots g_n h$, the elements $d \in \Delta^{\mathcal{I}}$ belong to the domain of $c^{\mathcal{I}}$ if $d$ belongs to the domain of $g_1^{\mathcal{I}}$, $g_1^{\mathcal{I}}(d)$ belongs to the domain of $g_2^{\mathcal{I}}$ etc. and $g_n^{\mathcal{I}}(\cdots(g_1^{\mathcal{I}}(d))\cdots)$ belongs to the domain of $h^{\mathcal{I}}$.

## A.2 Syntax and semantics of knowledge bases

Knowledge bases consist of terminological axioms and assertions. Terminological axioms restrict the interpretation of concepts (concept axioms) and roles (role axioms), whereas assertions restrict the interpretation of individuals. In Table A.2, $C, D$ again stand for concepts (concept names or compound concepts) and $r, s$ for roles (role names or compound roles); in addition, $A$ stands for a concept name and $a, b$ stand for individual names. A *TBox* is a finite set of concept and role axioms, and an *ABox* is a finite set of assertions. A *knowledge base* $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ consists of a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$.

The semantics of axioms is defined using the notion of a *model*. An interpretation $\mathcal{I}$ *satisfies* an axiom if it satisfies the condition formulated in the semantics column of Table A.2. Recall that a binary relation $r^{\mathcal{I}}$ is *transitive* if it satisfies

$$(d, e) \in r^{\mathcal{I}} \wedge (e, f) \in r^{\mathcal{I}} \Rightarrow (d, f) \in r^{\mathcal{I}};$$

it is *functional* if it satisfies

$$(d, e) \in r^{\mathcal{I}} \wedge (d, f) \in r^{\mathcal{I}} \Rightarrow e = f;$$

it is *reflexive* if it satisfies

$$d \in \Delta^{\mathcal{I}} \Rightarrow (d, d) \in r^{\mathcal{I}};$$

it is *irreflexive* if it satisfies

$$d \in \Delta^{\mathcal{I}} \Rightarrow (d, d) \notin r^{\mathcal{I}};$$

| Name | Syntax | Semantics |
|---|---|---|
| General concept inclusion | $C \sqsubseteq D$ | $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ |
| Concept definition | $A \equiv C$ | $A^{\mathcal{I}} = C^{\mathcal{I}}$ |
| Role inclusion | $r \sqsubseteq s$ | $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$ |
| Role disjointness | $\mathsf{Disj}(r, s)$ | $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$ |
| Role transitivity | $\mathsf{Trans}(r)$ | $r^{\mathcal{I}}$ is transitive |
| Role functionality | $\mathsf{Func}(r)$ | $r^{\mathcal{I}}$ is functional |
| Role reflexivity | $\mathsf{Ref}(r)$ | $r^{\mathcal{I}}$ is reflexive |
| Role irreflexivity | $\mathsf{Irref}(r)$ | $r^{\mathcal{I}}$ is irreflexive |
| Role symmetry | $\mathsf{Sym}(r)$ | $r^{\mathcal{I}}$ is symmetrical |
| Role antisymmetry | $\mathsf{Asym}(r)$ | $r^{\mathcal{I}}$ is antisymmetrical |
| Concept assertion | $a : C$ | $a^{\mathcal{I}} \in C^{\mathcal{I}}$ |
| Role assertion | $(a, b) : r$ | $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ |

Table A.2. *Terminological and assertional axioms.*

it is *symmetrical* if it satisfies

$$(d, e) \in r^{\mathcal{I}} \Rightarrow (e, d) \in r^{\mathcal{I}};$$

and it is antisymmetrical if it satisfies

$$(d, e) \in r^{\mathcal{I}} \Rightarrow (e, d) \notin r^{\mathcal{I}}.$$

An interpretation that satisfies each axiom in a TBox $\mathcal{T}$ (ABox $\mathcal{A}$) is called a *model* of $\mathcal{T}$ ($\mathcal{A}$). It is a model of a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ if it is a model of both $\mathcal{T}$ and $\mathcal{A}$.

## A.3 Naming schemes for description logics

A particular DL is determined by the constructors and axioms available in the DL. In order to distinguish between different DLs, certain naming schemes have been introduced in the DL community. These schemes start with (the name for) a basic DL, and then add letters or symbols to indicate additional concept constructors, role constructors and kinds of role axiom.

| Name | Syntax | Sym | $\mathcal{AL}$ | $\mathcal{EL}$ | $\mathcal{S}$ |
|---|---|---|---|---|---|
| Top | $\top$ | | ✓ | ✓ | ✓ |
| Bottom | $\bot$ | | ✓ | | ✓ |
| Conjunction | $C \sqcap D$ | | ✓ | ✓ | ✓ |
| Atomic negation | $\neg A$ | | ✓ | | ✓ |
| Value restr. | $\forall r.C$ | | ✓ | | ✓ |
| Disjunction | $C \sqcup D$ | $\mathcal{U}$ | | | ✓ |
| Negation | $\neg C$ | $\mathcal{C}$ | | | ✓ |
| Exist. restr. | $\exists r.C$ | $\mathcal{E}$ | | ✓ | ✓ |
| Unqualified number restr. | $(\leqslant n\, r)$ $(\geqslant n\, r)$ | $\mathcal{N}$ | | | |
| Qualified number restr. | $(\leqslant n\, r.C)$ $(\geqslant n\, r.C)$ | $\mathcal{Q}$ | | | |
| Nominal | $\{a\}$ | $\mathcal{O}$ | | | |
| Inverse role | $r^-$ | $\mathcal{I}$ | | | |
| Role inclusion | $r \sqsubseteq s$ | $\mathcal{H}$ | | | |
| Complex role inclusion | $r_1 \circ \ldots \circ r_n \sqsubseteq s$ | $\mathcal{R}$ | | | |
| Functionality | $\mathsf{Func}(r)$ | $\mathcal{F}$ | | | |
| Transitivity | $\mathsf{Trans}(r)$ | $_{R^+}$ | | | ✓ |

Table A.3. *The $\mathcal{AL}$, $\mathcal{EL}$, and $\mathcal{S}$ naming schemes.*

Three common such schemes are illustrated in Table A.3, where the columns $\mathcal{AL}$ , $\mathcal{EL}$ and $\mathcal{S}$ show the features of the corresponding basic DL, and the column Sym shows the symbols used to indicate additional features. As above, $C, D$ stand for concepts (concept names or compound concepts), $r, s$ stand for roles (role names or compound roles), $A$ stands for a concept name and $a, b$ stand for individual names.

The most common scheme starts with the basic DL $\mathcal{AL}$; for example, $\mathcal{ALC}$ is the DL obtained from $\mathcal{AL}$ by adding (full) negation. Note that we consider DLs modulo expressivity of constructors. Since negation can be used to define disjunction from conjunction and existential restriction from value restriction, $\mathcal{ALC}$ is the same DL as $\mathcal{ALCEU}$. Similarly, the fact that every $\mathcal{ALC}$ concept can be transformed into an equivalent one in negation normal form shows that $\mathcal{ALC}$ is actually the same DL as $\mathcal{ALEU}$.

The second naming scheme illustrated in Table A.3 starts with the basic DL $\mathcal{EL}$; for example, $\mathcal{ELI}$ stands for $\mathcal{EL}$ extended with inverse roles, and $\mathcal{ELIRO}$ for $\mathcal{ELI}$ extended with complex role inclusions and nominals.

The $\mathcal{S}$ naming scheme was introduced to avoid very long names for DLs. Its basic DL $\mathcal{S}$ is $\mathcal{ALC}$ extended with transitive roles. The DL $\mathcal{SHIQ}$ , for example, extends this basic DL with role inclusion axioms,[1] inverse roles and qualified number restrictions, while $\mathcal{SROIQ}$ also includes a role box (RBox) and nominals. Note that in this context $\mathcal{R}$ signifies an RBox, which can include not only complex role inclusion axioms but also disjointness, transitivity, reflexivity, irreflexivity, symmetry and antisymmetry axioms (see Table A.2), as well as the self restriction concept constructor (see Table A.1).

Unfortunately, things are not quite so simple since the unrestricted combination of the constructors indicated by the name $\mathcal{SHIQ}$ would lead to a DL with undecidable inference problems. For this reason, the qualified number restrictions in $\mathcal{SHIQ}$ are restricted to *simple roles*, i.e., roles that do not have transitive subroles (see [HST00] for details). Similarly, the use of complex role inclusions in DLs like $\mathcal{SROIQ}$ must be restricted to so-called regular collections of role inclusion axioms [HKS06].

[1] Role inclusion axioms are named with an $\mathcal{H}$ as they can be used to define a role hierarchy.