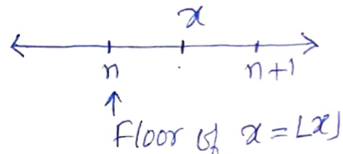
Some Special Functions:-

Floor and Ceiling function:-

Let 'x' be a real no. and floor functions assign 'x' the largest integer that is less than or equal to x. It is denoted as $\lfloor x \rfloor$. Symbolically, $\lfloor x \rfloor = n \iff n \le x < n+1$.
It is often also called the Greatest integer function.

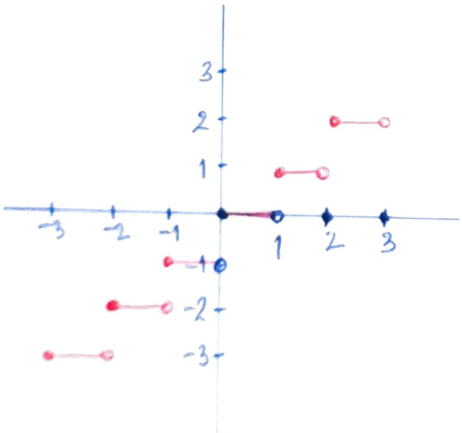
Eg:- $\lfloor -3.2 \rfloor = -4$ (not -3)



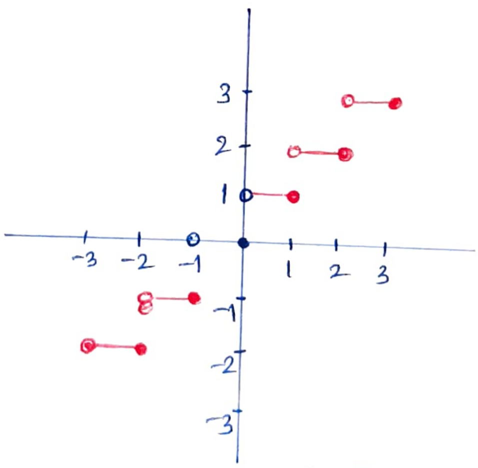
The ceiling function assign to x the smallest integer that is greater than or equal to x. It is denoted as $\lceil x \rceil$.
Symbolically, $\lceil x \rceil = n \iff n-1 < x \le n$.

The floor and ceiling functions are useful in data storage and data transmission.

Graphs:-



$y = \lfloor x \rfloor$



$y = \lceil x \rceil$

The open circles at the edges of each step are used to show that those points are not in the graphs

Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the value of x.

1) $8 \Rightarrow \lfloor 8 \rfloor = 8$ and $\lceil 8 \rceil = 8$
2) $6.01 \Rightarrow \lfloor 6.01 \rfloor = 6$ and $\lceil 6.01 \rceil = 7$
3) $-6.2 \Rightarrow \lfloor -6.2 \rfloor = -7$ and $\lceil -6.2 \rceil = -6$
4) $1/2 \Rightarrow \lfloor 1/2 \rfloor = 0$ and $\lceil 1/2 \rceil = 1$ 5) $\lfloor -1/2 \rfloor = -1$ and $\lceil -1/2 \rceil = 0$.

2) How many bytes are required to encode 'n' bits of data where 'n' equals (each byte is made up of 8 bits)
(i) 195 (ii) 1001 try.

$\Rightarrow$ (i) The no. of required bytes is the smallest integer that is greater than or equal to $195/8$. i.e, $\lceil 195/8 \rceil = \lceil 24.37 \rceil = 25$

Some useful properties:-

Given any real no. x;

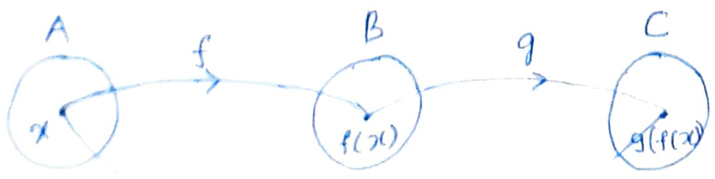
1) a) $\lfloor x \rfloor = n$ iff $n \le x < n+1$
b) $\lceil x \rceil = n$ iff $n-1 < x \le n$
c) $\lfloor x \rfloor = n$ iff $x-1 < n \le x$, d) $\lceil x \rceil = n$ iff $x \le n < x+1$

2) a) $\lfloor -x \rfloor = -\lceil x \rceil$ and b) $\lceil -x \rceil = -\lfloor x \rfloor$

3) a) $\lceil x+m \rceil = \lfloor x \rfloor + m$ (b) $\lfloor x+m \rfloor = \lceil x \rceil + m$ when $m \in I$

Composition of Functions:-

Let $f: A \rightarrow B$ and $g: B \rightarrow C$. The composition of f and g denoted by $g \circ f$ results in a new function from A to C is given by $(g \circ f)(x) = g(f(x))\ \forall x$ in A. [The range space of f becomes the domain space of g].



$g \circ f: A \rightarrow C$

Eg:- 1) Let $A = \{1, 2, 3\}$, $B = \{a, b\}$ and $C = \{r, s\}$ and $f: A \to B$ be defined by $f(1) = a$, $f(2) = a$, $f(3) = b$ and $g: B \to C$ be defined by $g(a) = s$, $g(b) = r$. Then $g\circ f: A \to C$ is defined by;

$$(g\circ f)(1) = g(f(1)) = g(a) = s$$

$$(g\circ f)(2) = g(f(2)) = g(a) = s, \quad (g\circ f)(3) = g(f(3)) = g(b) = r$$

2) If $f: R \to R$ and $g: R \to R$ are defined by the formulas $f(x) = x + 2 \ \forall x \in R$ and $g(x) = x^2 \ \forall x \in R$. Then

$$(g\circ f)(x) = g(f(x)) = g(x+2) = (x+2)^2 = x^2 + 4x + 4 \text{ and}$$

$$(f\circ g)(x) = f(g(x)) = f(x^2) = x^2 + 2$$

Note:- $g\circ f \neq f\circ g$ [Composition of functions is not commutative]

Theorem:- Let $f: A \to B$, $g: B \to C$ and $h: C \to D$. Then $h\circ(g\circ f) = (h\circ g)\circ f$ [composition of functions is associative].

Proof:- Since $f: A \to B$, $g: B \to C$ and $h: C \to D$ we write $g\circ f: A \to C$ and $h\circ g: B \to D$. Hence $h\circ(g\circ f): A \to D$.

Let $x \in A$, $y \in B$, $z \in C$ $\ni$ $f(x) = y$ and $g(y) = z$. Then

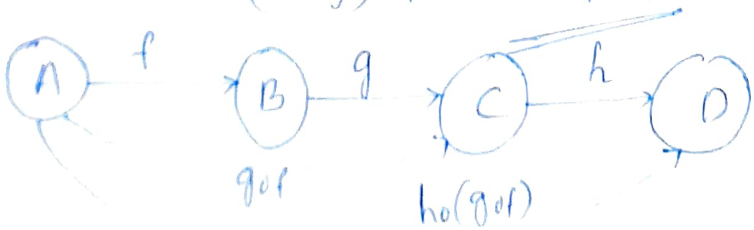
$$[(h\circ g)\circ f](x) = (h\circ g)[f(x)] = (h\circ g)(y)$$
$$= h[g(y)] = h(z) \quad \text{--- (1)}$$

also $[h\circ(g\circ f)](x) = [h\circ(g\circ f)(x)] = h[(g\circ f)(x)]$

$$= h[g(f(x))] = h[g(y)] = h(z) \text{ --- (2)}$$

from (1) & (2) we write;

$$(h\circ g)\circ f = h\circ(g\circ f)$$



Theorem :- Let $f : A \to B$ and $g : B \to C$ be functions.

a) If f and g are injections then $g \circ f : A \to C$ is an injection

b) If f and g are surjections then so is $g \circ f$

c) If f and g are bijections, then so is $g \circ f$.

Proof :- a) Let $a_1, a_2 \in A$. By definition of composition,

we have $(g \circ f)(a_1) = (g \circ f)(a_2) \Rightarrow g(f(a_1)) = g(f(a_2))$

(g is injective) $\Rightarrow f(a_1) = f(a_2)$

(f is injective) $\Rightarrow a_1 = a_2$

b) let $c \in C$. Then we can find an element $a \in A \ni'$ $(g \circ f)(a) = c$. Since g is onto C, there is an element $b \in B \ni' g(b) = c$. Then, since f is onto B, $\exists^{\circ}\ a \in A \ni'$ $f(a) = b$. Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$.

From this since f & g are both one-one & onto their composition will be bijective.

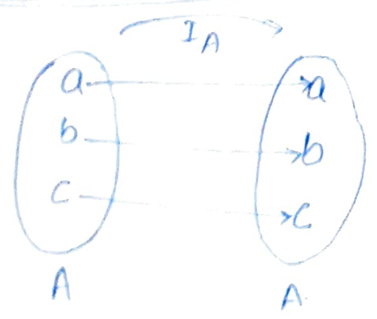
Identity Function :- The function $f : A \to A$ defined by $f(x) = x$ for every $x \in A$ is called the identity of A and is ~~det~~ denoted by I_A.

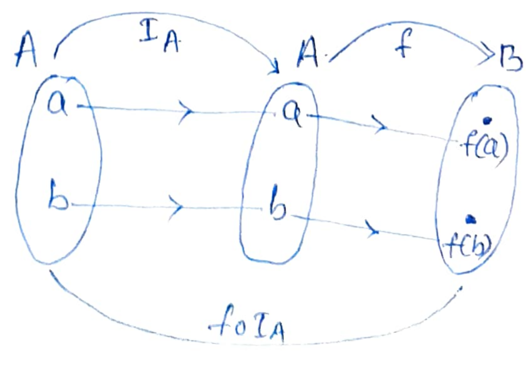
Result :- The composition of any function with the identity function is the function itself.

i.e., $(f \circ I_A)(x) = (I_B \circ f)(x) = f(x)$

In particular, if $f : A \to A$ then $f \circ I_A = I_A \circ f = f$

Inverse Function :-





Inverse Function :-

Let $f : A \to B$. A map $g : B \to A$ is called the inverse of f if $g \circ f = I_A$ and $f \circ g = I_B$.

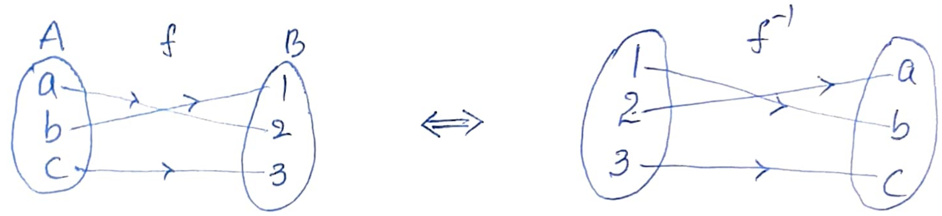
i.e., $g[f(x)] = x \; \forall \, x \in A$ and $f[g(y)] = y \; \forall \, y \in B$

Thus, if $f(x) = y$ then $g(y) = g[f(x)] = x$.

The Inverse g of f is denoted by f^{-1}. Thus $f(x) = y \Leftrightarrow x = f^{-1}(y)$.

Note :- A necessary and sufficient condition for $f : A \to B$ to have the inverse of $f^{-1} : B \to A$ is that f be Bijective.

Eg :-



Bijective $\longleftrightarrow$ Inverse

Problems :-

1) Show that the functions $f(x) = x^3$ and $g(x) = x^{1/3} \; \forall \, x \in R$ are inverses of one another.

Soln :- Since $(f \circ g)(x) = f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x = I_x$ and

$$(g \circ f)(x) = g(f(x)) = g(x^3) = (x^3)^{1/3} = x = I_x.$$

i.e., $f = g^{-1}$ (or) $g = f^{-1}$

2) If a mapping $f : A \to B$ is one-to-one and onto, then prove that inverse mapping $f^{-1} : B \to A$ is ~~also~~ Bijective.

Solution :- Here $f : A \to B$ is one-to-one and onto.

$a_1, a_2 \in A$ and $b_1, b_2 \in B$ so that

$f(a_1) = b_1$, $f(a_2) = b_2 \Rightarrow a_1 = f^{-1}(b_1)$, $a_2 = f^{-1}(b_2)$

As f is one to-one ; $f(a_1) = f(a_2) \Rightarrow \boxed{a_1 = a_2}$

(or) $b_1 = b_2 \Leftrightarrow f^{-1}(b_1) = f^{-1}(b_2)$

As f is onto, Every element of B is associated with a unique element of A ie, for any $a \in A$ is pre-image of some $b \in B$ where $f(a) = b \Rightarrow a = f^{-1}(b)$
ie. for $b \in B$, $\exists$ f^{-1} image $a \in A$. Hence f^{-1} is onto.

3) If $f : A \to B$ and $g : B \to C$ be one-to-one & onto functions, then prove that $g \circ f$ is also Bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Solution:-

Since f is 1-1, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for $x_1, x_2 \in R$

also since g is 1-1, $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$ for $y_1, y_2 \in R$

Now $g \circ f$ is 1-1 $\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow g[f(x_1)] = g[f(x_2)]$

$\Rightarrow f(x_1) = f(x_2)$ [g is 1-1]

$\Rightarrow x_1 = x_2$ [f is 1-1]

Since g is onto for $z \in C$ $\exists$ $y \in B \ni' g(y) = z$.

Also f being onto for $y \in B$ $\exists$ $x \in A$ $\ni' f(x) = y$.

Hence $z = g(y) = g[f(x)] = (g \circ f)(x) \Rightarrow g \circ f$ is onto.

ie., $g \circ f$ is 1-1 & onto (Bijective) & hence $(g \circ f)^{-1}$ exists

By the defn, $g \circ f : A \to C$. So $(g \circ f)^{-1} : C \to A$.

Also, $g^{-1} : C \to B$ and $f^{-1} : B \to A$

So we have $f^{-1} \circ g^{-1} : C \to A$

Therefore, the domain of $(g \circ f)^{-1}$ = the domain of $f^{-1} \circ g^{-1}$

$$
\begin{aligned}
(g \circ f)^{-1}(z) = x &\Leftrightarrow (g \circ f)(x) = z \\
&\Leftrightarrow g(f(x)) = z \\
&\Leftrightarrow g(y) = z \text{ where } y = f(x) \\
&\Leftrightarrow y = g^{-1}(z) \\
&\Leftrightarrow f^{-1}(y) = f^{-1}(g^{-1}(z)) = (f^{-1} \circ g^{-1})(z) \\
&\Leftrightarrow x = (f^{-1} \circ g^{-1})(z)
\end{aligned}
$$

Thus $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$. So $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

4) Let A and B be finite sets and $f: A \to B$. Then

i) If f is 1-1, then $|A| \leq |B|$

ii) If f onto, then $|B| \leq |A|$. (iii) If f is bijective, then $|A| = |B|$

5) If the function $f: R \to R$ defined by

$$f(x) = \begin{cases} 3x-4 & \text{if } x > 0 \\ -3x+2 & \text{if } x \leq 0 \end{cases}$$ determine

(a) $f(0), f(2/3), f(-2)$ (b) $f^{-1}(0), f^{-1}(2), f^{-1}(-7)$

Solution:- a) $f(0) = -3(0) + 2 = 2$

$f(2/3) = 3(2/3) - 4 = -2, \quad f(-2) = -3(-2) + 2 = 8$

b) $f^{-1}(0) = \{x \in R \mid f(x) = 0\}$

$= \{x \in R \mid x > 0 \ \& \ \underset{=0}{3x-4}\} \cup \{x \in R \mid x \leq 0 \ \& \ -3x+2 = 0\}$

$= \{x \in R \mid x > 0 \ \& \ x = 4/3\} \cup \{x \in R \mid x \leq 0 \ \& \ x = 2/3\}$

$= \{4/3\} \cup \phi = \{4/3\}$

$f^{-1}(2) = \{x \in R \mid f(x) = 2\}$

$= \{x \in R \mid x > 0 \ \& \ 3x-4 = 2\} \cup \{x \in R \mid x \leq 0 \ \& \ -3x + \cancel{2} = \cancel{2}\}$

$= \{x \in R \mid x > 0 \ \& \ x = 2\} \cup \{x \in R \mid x \leq 0 \ \& \ x = 0\}$

$= \{2\} \cup \{0\} = \{0, 2\}$

$f^{-1}(-7) = \{x \in R \mid f(x) = -7\}$

$= \{x \in R \mid x > 0 \ \& \ 3x-4 = -7\} \cup \{x \in R \mid x \leq 0 \ \& \ -3x+2 = -7\}$

$= \{x \in R \mid x > 0 \ \& \ x = -1\} \cup \{x \in R \mid x \leq 0 \ \& \ x = 3\}$

$= \phi \cup \phi$

$= \phi$

6) Let the function $f: R \to R$ be defined by

$$f(x) = \begin{cases} 3x-12 & \text{for } x > 3 \\ 2x^2+3 & \text{for } -2 < x \leq 3 \\ 3x^2-7 & \text{for } x \leq -2 \end{cases}$$

Find $f^{-1}(5)$

Unit-1 - Questions

1) Which of the following sets are nonempty?

a) $\{x \mid x \in N,\ 2x+7=3\}$ b) $\{x \in Z \mid 3x+5=9\}$

c) $\{x \mid x \in Q,\ x^2+4=6\}$ d) $\{x \mid x \in C,\ x^2+3x+3=0\}$

2) If $A=[0,3]$, $B=[2,7)$ with $U=R$ determine each of the following:

a) $A \cap B$ b) $\bar{A}$ c) $A \Delta B$ d) $A-B$

3) Determine the sets A, B where $A-B=\{1,3,7,11\}$ and $B-A=\{2,6,8\}$ and $A \cap B=\{4,9\}$

4) Using the laws of set theory, simplify each of the following:

a) $A \cap (B-A)$ b) $(A-B) \cup (A \cap B)$ c) $\bar{A} \cup \bar{B} \cup (A \cap B \cap \bar{C})$

5) Use Venn diagrams to check the validity of the below:

a) $A-(B \cup C) = (A-B) \cap (A-C)$

6) If $A=\{1,2,3,4\}$, $B=\{2,5\}$ and $C=\{3,4,7\}$ determine

a) $A \cup (B \times C)$ b) $(A \cup B) \times C$ c) $(A \times C) \cup (B \times C)$

7) Let A, B be sets with $|B|=3$. If there are 4096 relations from A to B, what is $|A|$?

8) Given $A=\{1,2,3,4,5\}$. Give an example of a relation R on 'A' that is; a) Reflexive and symmetric but not transitive.

b) reflexive and transitive but not symmetric.

9) If $A=\{w,x,y,z\}$, determine the number of relations on A that are (a) Reflexive (b) Symmetric (c) Reflexive & Symmetric

10) Given a set A with $|A|=n$ and a relation R on A, let M denote the relation matrix for R. Then show that

a) R is reflexive iff $I_n \leq M$ b) R is symmetric iff $M = M^{tr}$

c) R is transitive iff $M \cdot M = M^2 \leq M$.

11) Let $A=\{1,2,3\}$, $B=\{w,x,y,z\}$ and $C=\{4,5,6\}$. Define the relations $R_1 \subseteq A\times B$, $R_2 \subseteq B\times C$ & $R_3 \subseteq B\times C$ where

$R_1=\{(1,w),(3,w),(2,x),(1,y)\}$

$R_2=\{(w,5),(x,6),(y,4),(y,6)\}$ and $R_3=\{(w,4),(w,5),(y,5)\}$

Determine (a) $R_1\circ(R_2\cup R_3)$ (b) $(R_1\circ R_2)\cup(R_1\circ R_3)$

(c) $R_1\circ(R_2\cap R_3)$ (d) $(R_1\circ R_2)\cap(R_1\circ R_3)$

12) Given $A=\{1,2,3,4\}$, let $R=\{(1,1),(1,2),(2,3),(3,3),(3,4),(4,4)\}$ be a relation on A. Find two relations S and T where $S\neq T$ but $R\circ S=R\circ T=\{(1,1),(1,2),(1,4)\}$.

13) Draw the digraph $G_1=(V_1,E_1)$ where $V_1=\{a,b,c,d,e,f\}$ and $E_1=\{(a,b),(a,d),(b,c),(b,e),(d,b),(d,e),(e,c),(e,f),(f,d)\}$

14) For $A=\{v,w,x,y,z\}$, each of the following is the (0,1) matrix for a relation R on A. Here the rows & columns are indexed in the order v,w,x,y,z. Determine the relation $R\subseteq A\times A$ in each case, and draw the directed Graph G associated with R.

a) $M(R)=\begin{bmatrix}0&1&1&0&0\\1&0&1&1&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$ b) $M(R)=\begin{bmatrix}0&1&1&1&0\\1&0&1&0&0\\1&1&0&0&1\\1&0&0&0&1\\0&0&1&1&0\end{bmatrix}$

15) For $A=\{1,2,3,4\}$, let $R=\{(1,1),(1,2),(2,3),(3,3),(3,4)\}$ be a relation on A. Draw the directed graph G on A that is associated with R, R^2 and R^3

16) Draw the Hasse diagram for the POSET $(P(A), \subseteq)$, where $A = \{1, 2, 3, 4\}$.

17) Let $A = \{1, 2, 3, 6, 9, 18\}$, and define R on A by xRy if x/y. Draw the Hasse diagram for the poset (A, R).

18) Let $f, g: R \to R$ where $g(x) = 1 - x + x^2$ and $f(x) = ax + b$ if $(g \circ f)(x) = 9x^2 - 9x + 3$, determine a, b.

19) For the following functions $f: R \to R$ determine whether f is invertible, and if so, determine f^{-1}.

a) $f = \{(x, y) \mid 2x + 3y = 7\}$ b) $f = \{(x, y) \mid y = x^4 + x\}$

20) Let $f: R \to R$ be defined by

$$f(x) = \begin{cases} x + 7, & \text{if } x \le 0 \\ -2x + 5, & \text{if } 0 < x < 3 \\ x - 1, & \text{if } 3 \le x \end{cases}$$

determine;

ⓐ $f^{-1}(-10)$, ⓑ $f^{-1}(4)$ ⓒ $f^{-1}(6)$, $f^{-1}(8)$.

21) Let f & g be functions from the positive integers to the positive integers defined by $f(n) = n^2$, $g(n) = 2^n$. Find $f \circ f$, $g \circ g$, $f \circ g$, $g \circ f$

22) Let $f, g, h: Z \to Z$ be defined by $f(x) = x - 1$, $g(x) = 3x$

$$h(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd} \end{cases}$$

determine; ⓐ $f \circ g$ ⓑ $g \circ f$ ⓒ $g \circ h$ ⓓ $f \circ (g \circ h)$ ⓔ $(f \circ g) \circ h$ ⓕ f^2