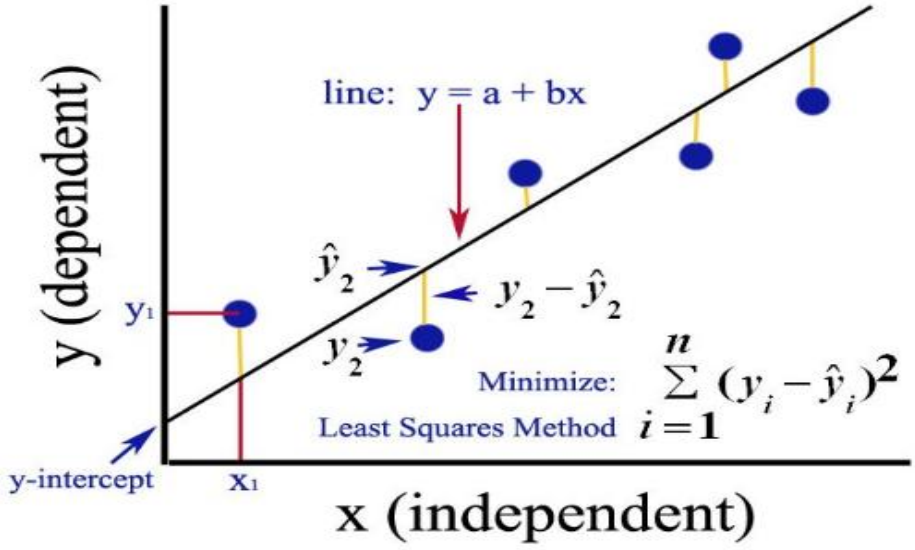
STATISTICS

Method of Least squares:

Suppose we are given n values of x_1, x_2, x_3,….., x_n of an independent variable x and the corresponding values y_1, y_2, y_3,….., y_n of a variable y depending on x. Then the pairs (x_1, y_1), (x_2, y_2),........, (x_n, y_n) give us n- points in the xy-plane. Generally it is not possible to find the actual curve y = f(x) that passes through these points. Hence we try to find a curve that serves as best approximation to the curve y = f(x). Such a curve is referred to as the curve of best fit. The process of determining a curve of best fit is called curve fitting. A method to find curve of best fit is called method of least squares.

The method of least squares tells that the curve should pass as closely as possible to meet all the points. Let y = f(x) be an approximate relation that fits into the data ($x_{i,}$ y_i), y_i are called observed values and $Y_i = f(x_i)$ are called the expected values. Then $E_{i\,=}$ y_i - Y_i are called the estimated error or residuals.

The method of least squares provides a relationship y = f(x) such that sum of the squares of the residues is least. Such a curve is known as least square curve.



Fitting of polynomial:

Approximating a data set using a polynomial equation is useful when conducting engineering calculations as it allows results to be quickly updated when inputs change without the need for manual lookup of the dataset. The most common method to generate a polynomial equation from a given data set is the least squares method. We will discuss the fitting of the following types of the curves.

Fitting of a straight line: $y = a + bx$

Let $y = a + bx$ be the equation of the straight line.

The error estimate is given by $E = y - (a + bx) = y - a - bx.$

By the principle of least squares we have to determine the constants a, b such that

$$E = \sum_{1}^{n} (y - a - bx)^2 \quad \text{is minimum.}$$

For E to be minimum the two necessary conditions are

$$\frac{\partial E}{\partial a} = 0, \qquad \frac{\partial E}{\partial b} = 0,$$

i.e, $\frac{\partial E}{\partial a} = 0 \Rightarrow 2\sum_{1}^{n}(y - a - bx)(-1) = 0,$

$$\Rightarrow 2\sum_{1}^{n}(y - a - bx) = 0,$$

$$\Rightarrow \sum y - \sum a - b\sum x = 0,$$

$$\Rightarrow \sum y = na + b\sum x,$$

$$\frac{\partial E}{\partial b} = 0 \Rightarrow 2\sum_{1}^{n}(y - a - bx)(-x) = 0,$$

$$\Rightarrow \sum xy = a\sum x + b\sum x^2 .$$

The normal equations for estimating the values of a and b are

$$\sum y = na + b\sum x,$$

$$\sum xy = a\sum x + b\sum x^2 .$$

Solving the above normal equations we estimate the values of a & b. With these values of a and b $y = a + bx$ is the line of best fit.

Fitting of a second degree equation (quadratic): $y = a + bx + cx^2$

Let $y = a + bx + cx^2$ be the equation of the curve.

The error estimate is given by $E = y - a - bx - cx^2.$

By the principle of least squares we have to determine the constants a, b and c such that

$$E = \sum_{1}^{n} (y - a - bx - cx^2)^2 \qquad \text{is minimum.}$$

For E to be minimum $\frac{\partial E}{\partial a} = 0, \frac{\partial E}{\partial b} = 0, \frac{\partial E}{\partial c} = 0,$

$$\frac{\partial E}{\partial a}=0\Rightarrow 2\sum_{1}^{n}(y-a-bx-cx^2)(-1)=0,$$

$$\Rightarrow \sum y-\sum a-b\sum x - c\sum x^2=0,$$

$$\Rightarrow \sum y = na + b\sum x + c\sum x^2,$$

$$\frac{\partial E}{\partial b}=0\Rightarrow 2\sum_{1}^{n}(y-a-bx-cx^2)(-x)=0,$$

$$\Rightarrow \sum xy = a\sum x + b\sum x^2 + c\sum x^3,$$

$$\frac{\partial E}{\partial c}=0\Rightarrow 2\sum_{1}^{n}(y-a-bx-cx^2)(-x^2)=0,$$

$$\Rightarrow \sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4.$$

The normal equations for estimating the values of a , b ,c are

$$\sum y = na + b\sum x + c\sum x^2,$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3,$$

$$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4.$$

Solving the above equations we estimate the values of a ,b & c. With these values of a , b & c, $y=a+bx+cx^2$ is the curve of best fit.

<u>Fitting of a curve of the form:</u> $y=ae^{bx}$

Let $y=ae^{bx}$ be the equation of the given curve.

Taking log on both sides we get, logy = loga + $\log e^{bx}$,

$\Rightarrow u = A + bx$, where $A = \log a$ & $u = \log y$.

This is linear in u and x.

Then the normal equations for estimating the values of A and b are

$$\sum u = nA + b\sum x,$$

$$\sum xu = A\sum x + b\sum x^2.$$

By solving these equations, we get the values of A and b .

But $A=\log a \Rightarrow a=\text{antilog}A$.

With these values of a and b, $y=ae^{bx}$ is the curve of best fit.

Fitting of a curve of the form: $y = ax^{b}$

Let $y = ax^{b}$.

Taking log on both sides we get

logy = loga + blogx,

$Y = A + bX$ where$Y = \log y, A = \log a, X = \log x$.

The normal equations are

$\sum Y = nA + b\sum X$,

$\sum XY = A\sum X + b\sum X^{2}$.

Solving the above equations we estimate the values of a & b. With these values of a and b, $y = ax^{b}$ is the curve of best fit.

Examples:

1. Fit a straight line to the following data.

x	1	6	11	16	20	26
y	13	16	17	23	24	31

Let y = a + b x be the straight line.
The normal equations for estimating the values of a and b are

$$\sum y = na + b\sum x,$$

$$\sum xy = a\sum x + b\sum x^{2}.$$

Given n = 6

x	**y**	**x^2**	**xy**
1	13	1	13
6	16	36	96
11	17	121	187
16	23	256	368
20	24	400	480
26	31	676	806
$\sum x = 80$	$\sum y = 124$	$\sum^{x2}$ =1490	$\sum xy = 1950$

Substituting the above values in the normal equations we get

6a + 80b = 124
80a + 1490 = 1950

Solving, we get a = 11.3227, b = 0.7008.

Therefore the equation of best fit is y = 11.3227 +0.7008x

2. Fit a straight line to the following data.

x	1	2	3	4	5	6
y	6	4	3	5	4	2

Soln:
Let y = a + b x be the straight line.
The normal equations for estimating the values of a and b are

$$\sum y = na + b\sum x, \quad \sum xy = a\sum x + b\sum x^2.$$

Here n = 6 and following the procedure as in example 1 we get

$$\sum x = 21, \ \sum y = 24, \ \sum xy = 75, \ \sum x^2 = 91.$$

Therefore, we get 24 = 6a + 21b, 75 = 21a + 91b.

Solving, we get a = 5.799, b = – 0.514.

Therefore the equation of best fit is y = 5.799 – 0.514x.

3. Fit a straight line of the form y = ax +b for the following data by the method of least squares.

x	5	10	15	20	25
y	16	19	23	26	30

Soln:

Let y = ax + b be the given straight line

The normal equations are $\sum y = a\sum x + nb, \ \sum xy = a\sum x^2 + b\sum x$.

Here n = 5 and following the procedure as in example 1 we get

$$\sum y = 114, \sum x = 75, \sum xy = 1885, \sum x^2 = 1375,$$

Substituting in the above equations we get a = 0.7, b = 12.3.

The best fit is y = 0.7 x + 12.3.

4. Fit an exponential curve of the type $y = a\,e^{b\,x}$ from the following data by the method of least squares.

x	1	2	4
y	5	10	30

Let $y = a\,e^{b\,x}$ ………….(1) be the required curve.
Taking log on both side of (1) and simplifying we get

Y = A + b x, where A = log a, Y = log y

The normal equations for estimating the values of a and b are

$$\sum Y = nA + b\sum x \ \text{ and } \ \sum xY = A\sum x + b\sum x^2$$

x	y	Y = log y	xY	x^2
1	5	0.6990	0.6990	1
2	10	1.0000	2.0000	4
4	30	1.4771	5.9085	16
$\sum = 7$		$\sum = 3.1761$	$\sum = 8.6095$	$\sum = 21$

Substituting the above values in the normal equations we get

$$3A + 7b = 3.1761$$
$$7A + 21b = 8.6095$$

Solving, we get A = 0.4604 but a = antilog (0.4604) = 2.8867, b = 0.2564.

Therefore the equation curve of best fit is $y = 2.8867\, e^{0.5624\, x}$.

5. Fit a curve of the form $y = a\, e^{b\,x}$ to the data by the method of least squares.

x	0	2	4
y	8.12	10	31.82

Soln:

Let $y = a\, e^{b\,x}$ ....(1) be the required curve.
Taking log on both side of (1) and simplifying we get

Y = A + b x, where A = log a, Y = log y

The normal equations for estimating the values of a and b are

$$\sum Y = nA + b\sum x \text{ and } \sum xY = A\sum x + b\sum x^2$$

Here n = 3 and following the procedure as in example 4 we get

$$\sum x = 6,\ \sum Y = 7.85,\ \sum xY = 18.44,\ \sum x^2 = 20.$$

Substituting the above values in the normal equations we get

$$3A + 6b = 7.85$$
$$6A + 20b = 18.44$$

By solving these equations, we get A = 1.932 but a = antilog (A) = 6.903, b = 0.3425.

Therefore $y = 6.903\, e^{0.3425x}$ is the curve of best fit.

6. Fit a curve of the form $y = a\, b^{x}$ ....(1) to the data by the method of least squares.

x	2	4	6	8	10
y	1	3	6	12	24

Soln:

Let $y = a\,b^{x}$ …………(1) be the required curve.
Taking log on both side of (1) and simplifying we get

$Y = A + B\,x$, where $A = \log a$, $B = \log b$ and $Y = \log y$

The normal equations for estimating the values of a and b are

$\sum Y = nA + B\sum x$ and $\sum xY = A\sum x + B\sum x^2$

Here n = 5 and following the procedure as in example 4 we get

$\sum x = 30$, $\sum Y = 3.7147$, $\sum xY = 29.0130$, $\sum x^2 = 220$.

Substituting the above values in the normal equations we get

$$5A + 30B = 3.7147,\quad 30A + 220B = 29.0130$$

By solving these equations, we get A = -0.26566 but a = antilog (A) = 1.8436,

B = 0.1681 but b = antilog (B) = 1.4727.

Therefore $y = (1.8436)\,(1.4727)^{x}$ is the curve of best fit.

7. At constant temperature the pressure P and the volume V of a gas are connected by the relation $PV^{\gamma} = K$ (constant). Find the best fitting equation of this form to the following data and estimate V when P = 4.

P	0.5	1.0	1.5	2.0	2.5	3.0
V	1620	1000	750	620	520	460

Soln: Let $PV^{\gamma} = K$. …….. (1) be the given relation. Taking log on both side of (1) and simplifying we get

$$\sum \log V = 39.73, \sum \log P = 2.42,$$

$$\sum \log V \log P = 14.4786,\ \sum (\log V)^2 = 264.1689.$$

Here n = 3 and following the procedure as in example 4 we get

$\gamma = 1.42$ and K = 18144

Therefore $PV^{1.42} = 18144$ is the curve of best fit.

At P = 4, V = 375.9428 ≈ 376.

8. Fit a second degree parabola for the following data.

x	0	1	2	3	4
y	1	3	4	5	6

Soln: Let y = a + bx + cx^2 be the second degree polynomial and we have to determine a, b and c.

Normal equations for the second degree parabola are

$$\sum y = na + b\sum x + c\sum x^2,$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3,$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4.$$

x	**y**	**xy**	**x^2**	**x^2y**	**x^3**	**x^4**
0	1	0	0	0	0	0
1	3	3	1	3	1	1
2	4	8	4	16	8	16
3	5	15	9	45	27	81
4	6	24	16	96	64	256
$\sum x = 10$	$\sum y = 19$	$\sum xy = 50$	$\sum x^2 = 30$	$\sum x^2y = 160$	$\sum x^3 = 100$	$\sum x^4 = 354$

Substituting the above values in the normal equations and solving we get a = 1.114,

b = 1.7717, c = 0.1429.

Therefore the second degree of parabola of best fit is y = 1.114 + 1.7717 x – 0.1429x^2

9. Fit a curve of the form y = a + bx + cx^2 to the data by the method of least squares.

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Soln: Let y = a + bx + cx^2 be the second degree parabola and we have to determine a, b and c.

Normal equations for the second degree parabola are

$$\sum y = na + b\sum x + c\sum x^2,$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3,$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4.$$

Here n = 5 and following the procedure as in example 7 we get

$$\sum x = 10, \sum y = 12.9, \sum xy = 38.1, \sum x^2 = 30, \sum x^3 = 100, \sum x^4 = 354, \sum x^2 y = 131.3.$$

Substitute these values in normal equations

$$\sum y = na + b\sum x + c\sum x^2,$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3,$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \,.$$

Solving we get a = 0.7914, b = - 0.1128, c = 0.3357.

Then the curve of best fit is y = 0.7914 – 0.1128 x + 0.3357 x^2.

10. The following table gives the production (in thousand units) of a certain commodity in different years:

Year(x)	1968	1978	1988	1998	2008
Production(y)	8	10	12	10	16

Fit a straight line to the data and estimate the production in the year 2015.

Soln:

For convenience in computations, let us set X = x -1967 and Let y = a + b X be the straight line.

The normal equations for estimating the values of a and b are

$$\sum y = na + b\sum X, \quad \sum Xy = a\sum X + b\sum X^2.$$

Here n = 6 and following the procedure as in example 1 we get

$$\sum y_i = 56, \sum X_i = 105, \sum X_i y_i = 1336, \sum x_i^{\,2} = 3025,$$

Substitute these values in normal equations we get

56 = 5a + 10b,

1336 = 105a + 3205b.

Solving these equations, we get a = 7.84 and b = 0.16. Therefore the line of best fit is given by y = a +b X = 7.84 + 0.16X = 7.84 + 0.16 (x -1967) = 0.16x - 306.88.

For x = 2015, this gives y = 15.52.

Thus, for the year 2015, the estimated production is 15.52(thousand units).

Exercise:

1. An experiment gave the following data:

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

It is known that x and y are connected by the relation y = $a_0 + a_1 x$. Find the best values of a and b using least square method.

Ans. $a_0 = 1$, $a_1 = 0.5420$ and y = 1 + 0.5420 x

2. The number y of bacteria per unit volume present in a culture after x hours is given by the following table :

x	0	1	2	3	4	5	6
y	32	47	65	92	132	190	275

Fit a curve of the form $y = a\, b^{x}$ to the data. Estimate the value of y when x = 7.

Ans. a = 32.14, b= 1.4270 and $y = 32.14\,(1.4270)^{x}$, $y_7 = 387$.

3. The following table gives the production (in thousands units) of a certain commodity in different years:

Year (x)	1941	1951	1961	1971	1981	1991	2001
Production (y)	3.9	5.3	7.3	9.6	12.9	17.1	23.2

Fit a curve of the form $y = a\, b^{x}$ to this data and estimate the production in the year 2006.

Ans. a = 9.5735, b = 1.3433 and $y = 9.5735\,(1.3433)^{x}$, y_{2006} = 27.5 x1000 quintals

4. The latent heat of vaporization of steam r is given in the following table at different temperatures t: For this range of temperature fit a relation of the form r = a + b t using the method of least squares.

t	40	50	60	70	80	90	100	110
r	1069.1	1063.6	1058.2	1052.7	1049.3	1041.8	1036.3	1030.8

Ans. a = 1090.26, b = -0.534 and r = 1090.26 – 0.534 t.

5. The following table gives the results of the measurements of train resistances; V is the velocity in mile per hour and R is the resistance in pound per ton.

V	20	30	40	50	60	70
R	54	90	138	206	292	396

If R is related to V by the relation $R = a + bV + cV^2$. Find a, b and c by the method of least squares and estimate R when V = 45 miles / hour.

Ans. a = 41.77, b = -1.096 and c = 0.08786 $R = 41.77 + (-1.096)\,V + 0.08786\,V^2$,

R = 170 Pound when V = 45 miles / hour.

Correlation and Regression:

The word correlation is used in everyday life to denote some form of association. In statistical terms we use correlation to denote association between two quantitative variables. We also assume that the association is linear, that one variable increases or decreases a fixed amount for a unit increase or decrease in the other. The other technique that is often used in these circumstances is regression, which involves estimating the best straight line to summarize the association.

Correlation:

Correlation means simply a relation between two or more variables.

Two variables are said to be correlated if the change in one variable results in a corresponding change in the other.

Ex: 1. x: supply y: price

2. x: demand y: Price.

Positive correlation:

If **an** increase or decrease in one variable corresponds to an increase or decrease in the other then the correlation is said to be positive correlation or direct correlation.

Ex: 1. Demand and price of commodity. 2. Income and expenditure.

Negative correlation:

If an increase or decrease in one variable corresponds to an decrease or increase in the other then the correlation is said to be negative correlation or inversely correlated.

Ex: 1.Supply and Price of a commodity.

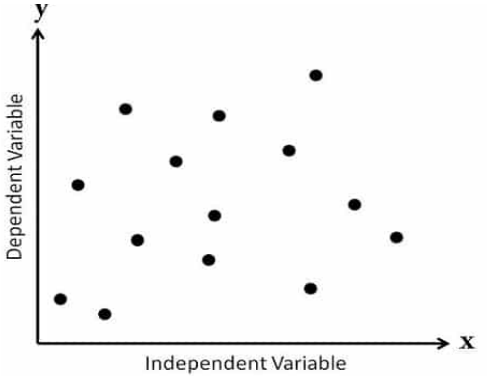
2. Correlation between Volume and pressure of a perfect gas.

No correlation

If there exist no relationship between two variables then they are said to be non correlated.

Scatter diagram

To obtain a measure of relationship between two variables x and y we plot their corresponding values in the xy - plane. The resulting diagram showing the collection of the dots is called the dot diagram or scatter diagram.



Correlation Coefficient (Karl Pearson correlation coefficient)

The degree of association is measured by a correlation coefficient, denoted by r. It is sometimes called Karl Pearson's correlation coefficient and is a measure of linear association. If a curved line is needed to express the relationship, other and more complicated measures of the correlation must be used.

Let $x_1, x_2, x_3, \ldots\ldots, x_n$ be n values of x and $y_1, y_2, y_3, \ldots\ldots y_n$ be the corresponding n values of y, then the coefficient of correlation between x and y is

$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}$, where $\sigma_x{}^2$ - variance of the x series, $\sigma_y{}^2$ - variance of the y series,

$\bar{x} = \frac{\Sigma x}{n}$ → Mean of the x series $\bar{y} = \frac{\Sigma y}{n}$ → mean of the y series.

For computation purpose we can use the formula

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{\{n\Sigma x^2 - (\Sigma x)^2\}\{n\Sigma y^2 - (\Sigma y)^2\}}}.$$

Limits for correlation coefficient

The coefficient of correlation numerically does not exceed unity ($-1 \le r \le 1$).

Proof:

We have $r = \frac{\frac{1}{n}\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{1}{n}\Sigma(x_i-\bar{x})^2}\sqrt{\frac{1}{n}\Sigma(y_i-\bar{y})^2}}$, i=1,2,………n,

$$r = \frac{\frac{1}{n}\Sigma a_i \Sigma b_i}{\sqrt{\frac{1}{n}\Sigma a_i{}^2}\sqrt{\frac{1}{n}\Sigma b_i{}^2}}, \quad r^2 = \frac{(\Sigma a_i \, \Sigma b_i)^2}{\Sigma a_i{}^2 \;\; \Sigma b_i{}^2}. \qquad (1)$$

By Schwartz inequality, which states that if a_i, b_i i=1, 2… n are real quantities then

$(\Sigma a_i \, \Sigma b_i)^2 \le \Sigma a_i{}^2 \;\; \Sigma b_i{}^2$ and the sign of equality holding if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots\ldots\ldots\ldots = \frac{a_n}{b_n}.$$

Using this equation (1) becomes $r^2 \le 1$,

$$\Rightarrow |r| \le 1,$$

$$\Rightarrow -1 \le |r| \le 1.$$

Hence correlation coefficient cannot exceed unity numerically.

Note:

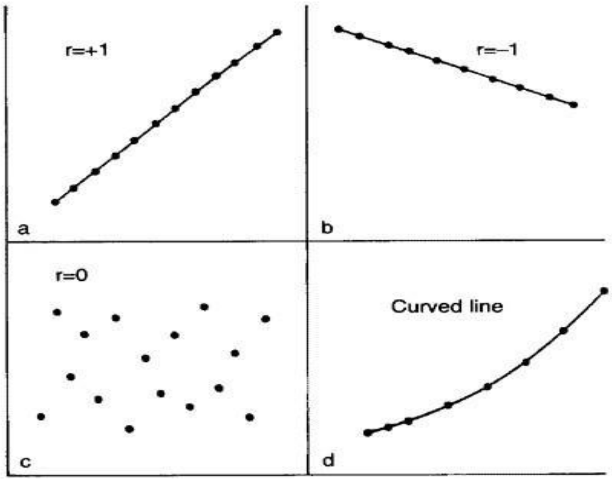


Figure 1.1 Correlation illustrated.

1. If r =-1 there is a perfect negative correlation.
2. If r =1 there is a perfect positive correlation.
3. If r =0 then the variables are non-correlated.
4. When r = 0, $\theta = \frac{\pi}{2}$. i.e, when the variables are independent the two lines of regression are perpendicular to each other.
5. When $r = \pm 1, \theta = 0$ or π. i,e the lines of regression coincide.

Examples:

1. If r is the correlation coefficient between x and y and z= ax+by. Show that

$$r = \frac{\sigma_z^2 - (a^2\sigma_x^2 + b^2\sigma_y^2)}{2ab\sigma_x\sigma_y}.$$

Soln:

Let z = ax + by $\Rightarrow \frac{1}{n}\sum z = \frac{a}{n}\sum x + \frac{b}{n}\sum y \Rightarrow \bar{z} = a\bar{x} + b\bar{y}$,

$$\frac{1}{n}\sum(z-\bar{z})^2 = a^2\frac{1}{n}\sum(x-\bar{x})^2 + b^2\frac{1}{n}\sum(y-\bar{y})^2 + 2ab\frac{1}{n}\sum(x-\bar{x})(y-\bar{y}),$$

$$\Rightarrow \quad \sigma_z^2 = a^2\sigma_x^2 + b^2\sigma_y^2 + 2abr\sigma_x\sigma_y,$$

$$\Rightarrow r = \frac{\sigma_z^2 - (a^2\sigma_x^2 + b^2\sigma_y^2)}{2ab\sigma_x\sigma_y}.$$

2. While calculating the correlation coefficient between x and y from 25 pairs of observations a person obtained the following values. $\sum x_i = 125, \sum x_i^2 = 650,$

$\sum y_i = 100, \sum y_i^2 = 460, \sum x_i y_i = 508$. It was later discovered that he had copied down the pairs (8,12) and (6,8) as (6,12) and (8,6) respectively. Obtain the correct value of the correlation coefficient.

Soln:

Correct $\sum x_i = 125, \sum x_i^2 = 650, \sum y_i = 102, \sum y_i^2 = 488, \sum x_i y_i = 532$,

n = 25,

$$r = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{\left\{n\sum x^2 - (\sum x)^2\right\}\left\{n\sum y^2 - (\sum y)^2\right\}}} = 0.51912.$$

3. The following Table gives the age (in years) of 10 married couples. Calculate the coefficient of correlation between these ages.

Age of Husband(x)	23	27	28	29	30	31	33	35	36	39
Age of wife(y)	18	22	23	24	25	26	28	29	30	32

Soln:

Here n=10

We find $\bar{x} = \frac{1}{n}\sum x_i = \frac{311}{10} = 31.1 \quad \bar{y} = \frac{1}{n}\sum y_i = \frac{257}{10} = 25.7.$

x_i	$X_i = x_i - \bar{x}$	X_i^2	$Y_i = y_i - \bar{y}$	Y_i^2	X_iY_i
23	-8.1	65.61	-7.7	59.29	62.37
27	-4.1	16.81	-3.7	13.69	15.17
28	-3.1	9.61	-2.7	7.29	8.37
29	-2.1	4.41	-1.7	2.89	3.57
30	-1.1	1.21	-0.7	0.49	0.77
31	-0.1	0.01	0.3	0.09	-0.03
33	1.9	3.61	2.3	5.29	4.37
35	3.9	15.21	3.3	10.89	12.87
36	4.9	24.01	4.3	18.49	21.07
39	7.9	62.41	6.3	39.69	49.77
		$\sum X_i^2 = 202.9$	$\sum Y_i^2 = 158.10$		$\sum X_iY_i = 178.$

$$r = \frac{\sum X_i Y_i}{\sqrt{\sum X_i^2 \sum Y_i^2}} = 0.9955 \approx 1.$$

i.e, the ages of husbands and wives are almost perfectly correlated.

Regression :

Correlation describes the strength of an association between two variables, and is completely symmetrical, the correlation between A and B is the same as the correlation between B and A. However, if the two variables are related it means that when one changes by a certain amount the other changes on an average by a certain amount. The relationship can be represented by a simple equation called the regression equation. In this context "regression" (the term is a historical anomaly) simply means that the average value of y is a "function" of x, that is, it changes with x.

Regression analysis is a mathematical measure of the average relationship between two or more variables in terms of the original units of data.

Line of regression:

Line of regression is the line which gives the best estimate to the value of one variable for any specific value of the other variable. So the line of regression is the line of best fit.

Regression line of y on x:

Let regression line of y on x be $y = a + bx$.

The normal equations by the method of least squares is

$$\sum y = na + b\sum x\,,$$

$$\sum xy = a\sum x + b\sum x^2\,,$$

$$\frac{1}{n}\sum y = a + \frac{b}{n}\sum x\,.$$

$\bar{y} = a + b\bar{x}$ is the regression line passing through ($(\bar{x}, \bar{y})$

$$b = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{\Sigma(XY)}{\Sigma X^2} = \frac{\Sigma(XY)}{n\sigma_x^{\;2}} = r\frac{\sigma_y}{\sigma_x}\,,$$

$y - \bar{y} = r\dfrac{\sigma_y}{\sigma_x}(x - \bar{x}) \Rightarrow Y = b_{yx}X$ is the regression line of y on x.

Note:

1. Regression coefficient of y on x

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{n\Sigma xy - \Sigma x \Sigma y}{n\Sigma x^2 - (\Sigma x)^2} = r\frac{\sigma_y}{\sigma_x}.$$

2. Regression coefficient of x on y

$$b_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2} = \frac{n\sum xy - \sum x\sum y}{n\sum y^2 - (\sum y)^2} = r\frac{\sigma_x}{\sigma_y}.$$

Examples:

1. If two regression equations of the variables x and y are x = 19.13 - .87y, y = 11.6 – 0.5x, find

   (a) mean of x

   (b) mean of y

   (c)The correlation coefficient between x and y.

Soln:

Since $\bar{x}$ and $\bar{y}$ lie on two regression lines,

$$\bar{x} = 19.13 - 0.87\bar{y},\ \ \bar{y} = 11.64 - 0.5\bar{x},$$
$$\text{Solving we get } \bar{x} = 15.79,\ \bar{y} = 3.74.$$

$$b_{yx} = -0.5, b_{xy} = -0.87, r = \sqrt{-0.5 \times -0.87} = -0.66.$$

2. In the following table data is showing the test scores made by sales man on an intelligent test and their weekly sales.

Test scores(x)	1	2	3	4	5	6	7	8	9	10
sales(y)	2.5	6	4.5	5	4.5	2	5.5	3	4.5	3

Calculate the regression line of sales on test scores and estimate the most possible weekly volume if a sales man scores 70.

Soln:

$\bar{x} = 60, \bar{y} = 4.05,$ Regression line of y on x is $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}),$

y = 0.06x + 0.45.

When x = 70, y = 4.65.

3. In a partially destroyed laboratory, record of an analysis of correlation data, the following results only are legible.

Variance of x = 9, Regression equations 8x -10y + 66 = 0, 40x - 18y = 214

what are (i) the mean values of x and y

(ii) the correlation coefficient between x and y

(iii) the standard deviation of y.

Soln:

(i) Since both the lines of regression pass through the point $(\overline{x}, \overline{y})$

$8\,\overline{x} - 10\,\overline{y} + 66 = 0,$

$40\overline{x} - 18\overline{y} - 214 = 0.$

Solving these equations we get $\overline{x} = 13$, $\overline{y} = 17$

(ii) $\sigma_x{}^2 = 9$
$\sigma_x = 3$

Let 8x - 10y + 66 = 0 and 40x - 18y = 214 be the lines of regression of y on x and x on y respectively

$$b_{yx} = \frac{4}{5}, b_{xy} = \frac{18}{40} = \frac{9}{20}, \text{Hence } r^2 = b_{yx}\, b_{xy} = \frac{9}{25}, \; r = \pm\frac{3}{5} = \pm 0.6.$$

Since both the regression coefficients positive we take r = 0.6.

Standard deviation of y = 4.

4. The following table gives the stopping distance y in meters of a motor bike Moving at a speed of x Kms/hour when the breaks are applied

x	16	24	32	40	48	56
y	0.39	0.75	1.23	1.91	2.77	3.81

Find the correlation coefficient between the speed and the stopping distance, and the equations of regression lines. Hence estimate the maximum speed at which the motor bike could be driven if the stopping distance is not to exceed 5 meters.

Soln:

$\overline{x} = 36, \;\; \overline{y} = 1.81, \;\; , \sigma_x = 13.663, \sigma_y = 1.1831,$

$b_{yx} = 0.0851, b_{xy} = 11.352,$

$$r = r = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{\{n\sum x^2 - (\sum x)^2\}\{n\sum y^2 - (\sum y)^2\}}} \quad = 0.983.$$

The equation of the line of regression of y on x is y = 0.0851x - 1.2536 (i)

and the equation of the line of regression of x on y is x =11.352y + 15.453. (ii)

For y = 5, equation (ii) gives x = 72.213.

Accordingly, for the stopping distance not to exceed 5 meters, the speed must not exceed 72 Kms/hour.

Exercise:

1. If the coefficient of correlation between the variables x and y is 0.5 and the acute angle between their lines of regression is $\tan^{-1}\left(\frac{3}{5}\right)$.Find the ratio of the standard deviation of x and y.

   Ans. $\frac{\sigma_x}{\sigma_y} = \frac{1}{2}$ or .$\frac{\sigma_x}{\sigma_y} = \frac{2}{1}$.

2. The following table shows the ages x and the systolic pressures of 12 persons.

Age (x)	56	42	72	36	63	47	55	49	38	42	68	60
Blood Pressure (**y**)	147	125	160	118	149	128	150	145	115	140	152	155

Calculate the coefficient of correlation between x and y. Estimate the blood pressure of a person whose age is 45 years.

Ans. r = 0.8961, y = 80.78 + 1.138 x , when x = 45, y = 132.

4. The height (inches) and weight (pounds) of baseball players are given below:
(76, 212), (76, 224), (72, 180), (74, 210), (75, 215), (71, 200), (77, 235), (78, 235), (77, 194), (76, 185).

   (i) Estimate the coefficient of correlation between weight and height of baseball players.

   (ii) Find the regression line between weight and height. Use the regression equation to find the weight of a baseball player that is 68 inches tall.

   Ans. r = 0.5529, y = 4.737 x – 147.227, x = 0.064 y + 61.712, when x = 68, y = 97.37.

5. The equations of regression lines of two variables x and y are 4 x – 5y + 33 = 0 and 20x - 9y = 107. Find the correlation coefficient and the means of x and y.

   Ans. r = 0.6, Mean of x = 13 and Mean of y = 17.

6. If the tangent of the angle between the lines of regression of y on x and x on y is 0.6 and the standard deviation of y is twice the standard deviation of x. find the coefficient of correlation between x and y.

   Ans. r = 0.5.

CONDITIONAL PROBABILITY:

The probability of an event B occurring when it is known that some event A has already occurred is called a **conditional probability** and is denoted by P(B|A). The symbol P(B|A) is usually read as “the probability that B occurs given that A occurs” or simply “the probability of B, given A.”

i.e., The conditional probability of B, given A, denoted by P(B|A), is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) > 0.$$

Two events A and B are **independent** if and only if P(B | A) = P(B) or P(A | B) = P(A), assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**. The condition P(B | A) = P(B) implies that P(A | B) = P(A), and conversely.

Multiplication theorem for Conditional Probability:

Suppose A and B are events in a sample space S with P(A) > 0. By definition of conditional probability, multiplying the conditional probability formula by P(A), we obtain the following important **multiplicative rule** (or **product rule**), which enables us to calculate the probability that two events will both occur. $P(A \cap B) = P(A)\,P(B|A)$, provided $P(A) > 0$.

Thus, the probability that both *A* and *B* occur is equal to the probability that *A* occurs multiplied by the conditional probability that *B* occurs, given that *A* occurs.

Since the events A ∩ B and B ∩ A are equivalent, thus we can also write

$P(A \cap B) = P(B \cap A) = P(B)\,P(A \mid B)$.

Theorem on Total Probability: If the events $B_1, B_2, \dots B_k$ constitute a partition of the sample space *S* such that $P(Bi) \neq 0$ for $i = 1, 2, \dots, k$, then for any event *A* of *S*,

$$P(A) = \sum_{i=i}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$

BAYE’S THEOREM (RULE):

If $B_1, B_2, \dots, B_n$ are mutually disjoint events with $P(B_i) \neq 0$ $(i = 1,2,\dots,n)$ then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} B_i$ such that $P(A) > 0$, then

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}.$$

Problems:

1. In a school 25% of the students failed in first language, 15% of the students failed in second language and 10% of the students failed in both. If a student is selected at random find the probability that

   (i) He failed in first language if he had failed in the second language.

   (ii) He failed in second language if he had failed in the first language.

   (iii) He failed in either of the two languages.

Solution: Let A be set of students failing in the first language and B be the set of students failing in the second language. We have by data

$P(A) = 25/100 = 1/4$, $P(B) = 15/100 = 3/20$, $P(A \cap B) = \frac{10}{100} = 1/10$.

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{10}}{\frac{3}{20}} = 2/3$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{1}{10}}{\frac{1}{4}} = 2/5$$

$P(AUB) = P(A) + P(B) - P(A\cap B) = 1/4 + 3/20 - 1/10 = 3/10$

2. Three machines A, B, C produces 50%, 30% and 20% of the items in factory. The percentage of defective outputs are respectively 3%, 4% and 5%. If an item is selected at random. What is the probability that it is defective? What is the probability that it is from A?

Solution: Let D denotes the defective item.

Given $P(A) = 0.5$ and $P(D \mid A) = 0.03$, $P(B) = 0.3$ and $P(D \mid B) = 0.04$,

$P(C) = 0.2$ and $P(D \mid C) = 0.05$.

Now $P(D) = P(A)\, P(D \mid A) + P(B)\, P(D \mid B) + P(C)\, P(D \mid C)$

$= 0.037$

By Baye's theorem,

Probability that the defective item is from A = $P(A \mid D) = P(A)P(D \mid A)/P(D)$

$= (0.5)\,(0.03)/0.037$

$= 0.4054$

3. In a college where boys and girls are equal proportion, it was found that 10 out of 100 boy and 25 out of 100 girls were using the same brand of a two wheeler. If a student using that was selected at random what is the probability of being a boy?

Solution:

$P(Boy) = P(B) = 1/2 = P(Girl) = P(G)$

Let E be the event of choosing a student using that brand of vehicle.

Therefore, $P(E \mid B) = 10/100 = 0.1$ and $P(E \mid G) = 25/100 = 0.25$

Now, $P(E) = P(B)\, P(E \mid B) + P(G)\, P(E \mid G) = 0.175$.

We have to find P(B│E) and by Baye's theorem

$P(B|E) = \frac{P(B)\,P(E|B)}{P(E)} = \frac{(0.5)(0.1)}{0.175} = 2/5$ = 0.2857.

4. In a recent survey in a statistics class, it was determined that only 60% of the students attend class on thursday. From past data it was noted that 98% of those who went to class on thursday pass the course, while only 20% of those who did not go to class on thursday passed the course.

a) What percentage of students is expected to pass the course?

b) Given that a student passes the course, what is the probability that he/she attended classes on thursday.

Solution:

A_1: the students attend class on thursday

A_2: the students do not attend class on thursday

B_1: the students pass the course

B_2: the students do not pass the course

a) $P(A_1) = 0.6, P(A_2) = 1 - P(A_1) = 0.4, P(B_1|A_1) = 0.98, P(B_1|A_2) = 0.2$

$$P(B_1) = P(B_1 \cap A_1) + P(B_1 \cap A_2)$$
$$= P(A_1) * P(B_1|A_1) + P(A_2) * P(B_1|A_2)$$
$$= 0.6 * 0.98 + 0.4 * 0.2 = 0.668$$

Therefore, percentage of students who pass the course = 66%

b) By Bayes' theorem,

$$P(A_1|B_1) = \frac{P(A_1 \cap B_1)}{P(B_1)} = \frac{P(A_1) * P(B_1|A_1)}{P(A_1) * P(B_1|A_1) + P(A_2) * P(B_1|A_2)}$$
$$= \frac{0.6 * 0.98}{0.6 * 0.98 + 0.4 * 0.2}$$

= 0.854

5. In an electronics laboratory, there are identically looking capacitors of three makes A_1, A_2 and A_3 in the ratio 2:3:4. It is known that 1% of A_1, 1.5% of A_2 and 2% of A_3 are defective. What percentage of capacitors in the laboratory is defective? If a capacitor picked at defective is found to be defective, what is the probability it is of make A_3?

Solution: Let D be the event that the item is defective.

Here we have to find $P(D)$and $P(A_3|D)$.

Here $P(A_1) = \frac{2}{9}, P(A_2) = \frac{1}{3}$ and $P(A_3) = \frac{4}{9}$.

The conditional probabilities are $P(D|A_1) = 0.01, P(D|A_2) = 0.015$ and $P(D|A_3) = 0.02$

$$P(D) = P(A_1) * P(D|A_1) + P(A_2) * P(D|A_2) + P(A_3) * P(D|A_3)$$
$$= \frac{2}{9} * 0.01 + \frac{1}{3} * 0.015 + \frac{4}{9} * 0.02$$

$$= 0.0167$$

$$\text{and } P(A_3|D) = \frac{P(A_3) * P(D|A_3)}{P(D)}$$

$$= \frac{\frac{4}{9} * 0.02}{0.0167} = 0.533\,.$$

Exercise:

1. There are three bags; first containing 1 white, 2 red, 3 green balls; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag. Ans: 6/11.
2. A factory uses three machines X, Y, Z to produce certain items.
   i. Machine X produces 50% of the items of which 3% are defective.
   ii. Machine Y produces 30% of the items of which 4% are defective.
   iii. Machine Z produces 20% of the items of which 5% are defective.

   Suppose a defective item is found among the output. Find the probability that it came from each of the machines. Ans: 40.5%, 32.5% and 27%.
3. In a certain college 25% of boys and 10% of girls are studying Mathematics. The girls constitute 60% of the student body.
   i. What is the probability that Mathematics is being studied? Ans: 0.16
   ii. If a student is selected at random and is found to be studying Mathematics, find the probability that the student is a (i) girl (ii) boy. Ans: (i) 0.375 (ii) 0.625.
4. A large industrial firm uses three local motels to provide overnight accommodations for its clients. From past experience it is known that 20% of the clients are assigned rooms at the Ramada Inn, 50% at the Sheraton, and 30% at the Lakeview Motor Lodge. If the plumbing is faulty in 5% of the rooms at the Ramada Inn, in 4% of the rooms at the Sheraton, and in 8% of the rooms at the Lakeview Motor Lodge, what is the probability that
   i. a client will be assigned a room with faulty plumbing? Ans: 0.054
   ii. a person with a room having faulty plumbing was assigned accommodations at the Lakeview Motor Lodge? Ans: 4/9.

In answering a question on a multiple choice test a student either knows the answer or he guesses. Let p be the probability that he knows the answer and 1 – p the probability that he guesses. Assume that a student who guesses at the answer will be correct with probability 1/5, where 5 is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he answered it correctly. Ans: 5p/4p + 1.