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Linear controllers for exponential tracking of systems in chained-form

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SUMMARY

In this paper we address the tracking problem for a class of non-holonomic chained-form control systems. We present a simple solution for both the state feedback and the dynamic output feedback problem. The proposed controllers are linear and render the tracking error dynamics globally $\mathscr{K}$-exponentially stable. We also deal with both control problems under input saturation. Application of the results to the control of wheeled mobile robots is illustrated by means of simulations of a car pulling a single trailer. Copyright © 2000 John Wiley & Sons, Ltd.

1. INTRODUCTION

In recent years the control, and in particular the stabilization, of non-holonomic dynamic systems has received considerable attention. One of the reasons for this is that no smooth stabilizing static state-feedback control law exists for these systems, since Brockett's necessary condition for smooth stabilization is not met [3]. For an overview we refer to the survey paper [21] and references cited therein.

Although the stabilization problem for non-holonomic control systems is now well understood, the tracking control problem has received less attention. In fact, it is unclear how the stabilization techniques available can be extended directly to tracking problems for non-holonomic systems.

In References [7, 8, 17, 27, 28] tracking control schemes have been proposed based on the linearization of the corresponding error model. All these papers solve the local tracking problem for some classes of nonholonomic systems. To our knowledge, the first global tracking control law was proposed in Reference [36] for a two-wheel-driven mobile car. Other global results can be found in References [6, 12, 13, 15, 31].

In this paper we study the tracking problem for the class of non-holonomic systems in chained form [27]. It is well known that many mechanical systems with non-holonomic constraints

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can be locally, or globally, converted to the chained form under coordinate change and state feedback.

A disadvantage of most of the aforementioned tracking controllers is their lack of a clear interpretation. Complicated changes of co-ordinates and difficult Lyapunov analysis are needed to show that the proposed control laws yield asymptotic stability of the tracking error dynamics.

The purpose of this paper is to develop *simple* tracking controllers for the class of non-holonomic systems in chained form. Based on a result for (time-varying) cascaded systems [32] we divide the tracking error dynamics into a cascade of two linear sub-systems which we can stabilize independently of each other with simple (i.e., linear) controllers.

Using the same approach we also consider the tracking problem for chained form systems by means of dynamic output-feedback. To our knowledge, the only papers that addressed the dynamic output-feedback problem are References [1, 2] that concern the stabilization problem and References [12, 24] dealing with the tracking problem. A comparative separation in linear subsystems has been used in Reference [29] for solving the tracking problem for a chained-form system of order 3, and in Reference [35] for solving the stabilization of general chained-form systems.

Last, we partially deal with the tracking control problem under input constraints. The only results on saturated tracking control of non-holonomic systems that we are aware of, are Reference [12] which deals with this problem for a mobile robot with two degrees of freedom, and Reference [14] that deals with general chained form systems.

The organization of the paper is as follows: In Section 2 we present the class of systems and state the problem formulation. Based on the theory from Section 2, Section 3 deals with the design of simple tracking-controllers, for both the state-feedback case and for the output-feedback case. Also both control problems under input saturation are studied in this section. Section 4 illustrates the presented design methods with simulations of an articulated vehicle and comparisons with other recent design methods are made. Finally, Section 5 concludes the paper.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this section we introduce definitions and theorems used in the remainder of this paper and formulate the problem under consideration. We start with some basic stability concepts in Section 2.1, present a result for cascaded systems in Section 2.2 and recall some results in Section 2.3 from linear systems theory we use. We conclude this section with the problem formulation in Section 2.4.

2.1. Stability

To start with, we recall some basic concepts (see e.g. References [19, 42]).

Definition 2.1

A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

Definition 2.2

A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Consider the system

$$\dot{x} = f(t, x), \quad f(t, 0) = 0, \quad \forall t \geqslant 0 \tag{1}$$

with $x \in \mathbb{R}^n$ and $f(t, x)$ piecewise continuous in t and locally Lipschitz in x.

Definition 2.3

System (1) is uniformly stable if for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$, independent of t_0, such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geqslant t_0 \geqslant 0 \tag{2}$$

Definition 2.4

System (1) is globally uniformly asymptotically stable (GUAS) if it is uniformly stable and globally attractive, that is, there exists a class $\mathscr{KL}$ function $\beta(\cdot, \cdot)$ such that for every initial state $x(t_0)$:

$$\|x(t)\| \leqslant \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geqslant t_0 \geqslant 0 \tag{3}$$

Definition 2.5

System (1) is globally exponentially stable (GES) if there exist $k > 0$ and $\gamma > 0$ such that for any initial state $x(t_0)$:

$$\|x(t)\| \leqslant \|x(t_0)\| k \exp[-\gamma(t - t_0)] \tag{4}$$

A slightly weaker notion of exponential stability is the following:

Definition 2.6 (cf. [37])

We call System (1) *globally $\mathscr{K}$-exponentially stable* if there exist $\gamma > 0$ and a class $\mathscr{K}$ function $\kappa(\cdot)$ such that

$$\|x(t)\| \leqslant \kappa(\|x(t_0)\|) \exp[-\gamma(t - t_0)] \tag{5}$$

2.2. *Cascaded systems*

Consider the system

$$\begin{aligned} \dot{z}_1 &= f_1(t, z_1) + g(t, z_1, z_2) z_2 \\ \dot{z}_2 &= f_2(t, z_2) \end{aligned} \tag{6}$$

where $z_1 \in \mathbb{R}^n$, $z_2 \in \mathbb{R}^m$, $f_1(t, z_1)$ is continuously differentiable in (t, z_1) and $f_2(t, z_2)$, $g(t, z_1, z_2)$ are continuous in their arguments, and locally Lipschitz in z_2 and (z_1, z_2), respectively.

We can view system (6) as the system

$$\Sigma_1 : \dot{z}_1 = f_1(t, z_1) \tag{7}$$

that is perturbed by the state of the system

$$\Sigma_2 : \dot{z}_2 = f_2(t, z_2) \tag{8}$$

When Σ_2 is asymptotically stable, we have that z_2 tends to zero, which means that the z_1 dynamics in (6) asymptotically reduces to Σ_1. Therefore, we can hope that asymptotic stability of both Σ_1 and Σ_2 implies asymptotic stability of (6).

Unfortunately, this is not true in general. However, from the proof presented in Reference [32] it can be concluded that:

Theorem 2.7 (*based on* [32])
Cascaded system (6) is GUAS if the following three assumptions hold:

- assumption on Σ_1: the system $\dot{z}_1 = f_1(t, z_1)$ is GUAS and there exists a continuously differentiable function $V(t, z_1): \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ that satisfies

$$W_1(z_1) \leqslant V(t, z_1) \leqslant W_2(z_1), \quad \forall t \geqslant 0, \quad \forall z_1 \in \mathbb{R}^n, \tag{9}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial z_1} \cdot f_1(t, z_1) \leqslant 0, \quad \forall \|z_1\| \geqslant \eta \tag{10}$$

$$\left\| \frac{\partial V}{\partial z_1} \right\| \|z_1\| \leqslant c V(t, z_1), \quad \forall \|z_1\| \geqslant \eta \tag{11}$$

  where $W_1(z_1)$ and $W_2(z_1)$ are positive definite proper functions and $c > 0$ and $\eta > 0$ are constants,
- assumption on the interconnection: the function $g(t, z_1, z_2)$ satisfies for all $t \geqslant t_0$:

$$\|g(t, z_1, z_2)\| \leqslant \theta_1(\|z_2\|) + \theta_2(\|z_2\|)\|z_1\| \tag{12}$$

  where $\theta_1, \theta_2: \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions,
- assumption on Σ_2: the system $\dot{z}_2 = f_2(t, z_2)$ is GUAS and for all $t_0 \geqslant 0$:

$$\int_{t_0}^{\infty} \|z_2(t_0, t, z_2(t_0))\| dt \leqslant \kappa(\|z_2(t_0)\|) \tag{13}$$

  where the function $\kappa(\cdot)$ is a class $\mathcal{K}$ function.

Remark 2.8
Notice the assumption on Σ_1 is slightly weaker than the one presented in Reference [32]. However, the authors of Reference [32] showed the result also to hold under the assumptions mentioned above by (almost) exactly copying their proof.

Lemma 2.9 (*see* [31])
If in addition to the assumptions in Theorem 2.7 both $\dot{z}_1 = f_1(t, z_1)$ and $\dot{z}_2 = f_2(t, z_2)$ are globally $\mathcal{K}$-exponentially stable, then the cascaded system (6) is globally $\mathcal{K}$-exponentially stable.

2.3. *Linear time-varying systems*

Consider the linear time-varying system

$$\dot{z} = \underbrace{\begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ \psi(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \psi(t) & 0 \end{bmatrix}}_{A(t)} z + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}_{B(t)} u \tag{14a}$$

$$y = \underbrace{[0 \quad \cdots \quad \cdots \quad 0 \quad 1]}_{C(t)} z \tag{14b}$$

where $z \in \mathbb{R}^m$ and let $\Phi(t, t_0)$ denote the state-transition matrix for the system $\dot{z} = A(t)z$. We recall two definitions from linear control theory (cf. References [16, 34]).

Definition 2.10

The pair $(A(t), B(t))$ is *uniformly completely controllable* (UCC) if there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leqslant \int_t^{t+\delta} \Phi(t, \tau) B(\tau) B(\tau)^T \Phi^T(t, \tau)\, \mathrm{d}\tau \leqslant \varepsilon_2 I \tag{15}$$

Definition 2.11

The pair $(A(t), C(t))$ is *uniformly completely observable* (UCO) if there exist $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t > 0$:

$$\varepsilon_1 I \leqslant \int_{t-\delta}^{t} \Phi^T(\tau, t-\delta) C(\tau)^T C(\tau) \Phi(\tau, t-\delta)\, \mathrm{d}\tau \leqslant \varepsilon_2 I \tag{16}$$

From linear systems theory several methods are available to exponentially stabilize the linear time-varying system (14) via state or dynamic output-feedback, in case the pairs $(A(t), B(t))$ and $(A(t), C(t))$ are uniformly completely controllable and observable, respectively.

Assumption 2.12

We assume that $\psi(t): [0, \infty) \to \mathbb{R}$ is a bounded continuously differentiable Lipschitz function that does not converge to zero. More precise, we assume that

- there exists a constant M such that for all t: $|\psi(t)| \leqslant M$,
- $\psi(t)$ is a continuously differentiable function with respect to t,
- there exists a constant L such that for all $t_1, t_2 \in [0, \infty)$: $|\psi(t_1) - \psi(t_2)| \leqslant L|t_1 - t_2|$,
- there exist $\delta > 0$ and $\varepsilon > 0$ such that for all $t \geqslant 0$ there exists an $s \in [t, t+\delta]$ such that $|\psi(s)| \geqslant \varepsilon$.

Proposition 2.13

Assume $\psi(t)$ satisfies the conditions of Assumption 2.12. Then system (14) is uniformly completely controllable and uniformly completely observable.

Proof. This is a direct consequence of Theorem 2 in Reference [18]. □

Theorem 2.14

Consider system (14) in closed loop with the controller

$$u = -k_1 z_1 - k_2 \psi(t) z_2 - k_3 z_3 - k_4 \psi(t) z_4 - \cdots \tag{17}$$

where $k_i (i = 1, \ldots, m)$ are such that the polynomial

$$\lambda^m + k_1 \lambda^{m-1} + \cdots + k_{m-1}\lambda + k_m \tag{18}$$

is Hurwitz (i.e. has its roots in the left-half of the open complex plane). If $\psi(t)$ meets Assumption 2.12, then the closed-loop system (14, 17) is GES.

Proof. See the appendix. □

Remark 2.15

Notice we use a linear controller of the form $u = K(t)x$ with a special choice of the gain $K(t)$. Clearly, several other choices can be made. One possibility is to use the gain as known from 'standard linear control theory' [34] as we used in Reference [24], or a gain as proposed in Reference [5] (cf. Reference [25]), based on pole-placement [41, 40] or based on any robust design method for LTV systems.

Theorem 2.16

Consider system (14) in closed loop with the controller

$$u = -k_1\hat{z}_1 - k_2\psi(t)\hat{z}_2 - k_3\hat{z}_3 - k_4\psi(t)\hat{z}_4 - \cdots \tag{19}$$

where $\hat{z}$ is generated from the observer

$$\dot{\hat{z}} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \psi(t) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \psi(t) & 0 \end{bmatrix} \hat{z} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} u + \begin{bmatrix} \vdots \\ l_4\psi(t) \\ l_3 \\ l_2\psi(t) \\ l_1 \end{bmatrix} (y - \hat{y}) \tag{20a}$$

$$\hat{y} = [0 \quad \cdots \quad \cdots \quad 0 \quad 1]\hat{z} \tag{20b}$$

and k_i, l_i $(i = 1, \ldots, m)$ are such that the polynomials

$$\begin{aligned} \lambda^m + k_1\lambda^{m-1} + \cdots + k_{m-1}\lambda + k_m \\ \lambda^m + l_1\lambda^{m-1} + \cdots + l_{m-1}\lambda + l_m \end{aligned} \tag{21}$$

are Hurwitz (i.e. have their roots in the left half of the open complex plane). If $\psi(t)$ meets Assumption 2.12, then the closed-loop system (14), (19) and (20) is GES.

Proof. See the appendix. □

2.4. Problem formulation

The class of chained-form non-holonomic systems we study in this paper is given by the following equations:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1 \end{aligned} \tag{22}$$

where $x = (x_1, \ldots, x_n)$ is the state, u_1 and u_2 are inputs.

Consider the problem of tracking a reference trajectory (x_r, u_r) generated by the chained-form system

$$\begin{aligned} \dot{x}_{1,r} &= u_{1,r} \\ \dot{x}_{2,r} &= u_{2,r} \end{aligned}$$

$$\begin{aligned}\dot{x}_{3,r} &= x_{2,r}u_{1,r}\\ &\vdots\\ \dot{x}_{n,r} &= x_{n-1,r}u_{1,r}\end{aligned} \tag{23}$$

where we assume $u_{1,r}(t)$ and $u_{2,r}(t)$ to be continuous functions of time. This reference trajectory can be generated by any of the motion planning techniques available from the literature.

When we define the tracking error $x_e = x - x_r$ we obtain as tracking error dynamics

$$\begin{aligned}\dot{x}_{1,e} &= u_1 - u_{1,r} &&= u_1 - u_{1,r}\\ \dot{x}_{2,e} &= u_2 - u_{2,r} &&= u_2 - u_{2,r}\\ \dot{x}_{3,e} &= x_2u_1 - x_{2,r}u_{1,r} &&= x_{2,e}u_{1,r} + x_2(u_1 - u_{1,r})\\ &\vdots &&\vdots\\ \dot{x}_{n,e} &= x_{n-1}u_1 - x_{n-1,r}u_{1,r} &&= x_{n-1,e}u_{1,r} + x_{n-1}(u_1 - u_{1,r})\end{aligned} \tag{24}$$

The state-feedback tracking control problem then can be formulated as

Problem 2.17 (*State-feedback tracking control problem*).

Find appropriate state feedback laws u_1 and u_2 of the form

$$u_1 = u_1(t, x, x_r, u_r) \quad \text{and} \quad u_2 = u_2(t, x, \mathrm{x}_r, u_r) \tag{25}$$

such that the closed-loop trajectories of (24,25) are globally uniformly asymptotically stable.

Consider system (22) with output

$$y = \begin{bmatrix} x_1 \\ x_n \end{bmatrix} \tag{26}$$

then it is easy to show (see e.g. Reference [1]) that system (22) with output (26) is locally observable at any $x \in \mathbb{R}^n$.

Now we can formulate the dynamic output-feedback tracking problem as

Problem 2.18 (Dynamic output-feedback tracking control problem)

Find appropriate control laws u_1 and u_2 of the form

$$u_1 = u_1(t, \hat{x}, y, x_r, u_r) \quad \text{and} \quad u_2 = u_2(t, \hat{x}, y, x_r, u_r) \tag{27}$$

where $\hat{x}$ is generated from an observer

$$\dot{\hat{x}} = f(t, \hat{x}, y, x_r, u_r) \tag{28}$$

such that the closed-loop trajectories of (24), (27), (28) are globally uniformly asymptotically stable.

3. CONTROLLER DESIGN

As mentioned in the introduction, our goal is to find simple controllers that globally stabilize the tracking error dynamics (24). The approach used in Reference [15] is based on the integrator backstepping idea [4, 20, 22, 39] which consists of searching a stabilizing function for a subsystem of (24), assuming the remaining variables to be controls. Then, new variables are defined, describing the difference between the desired dynamics and the true dynamics. Subsequently a stabilizing controller for this 'new system' is looked for.

This approach has the advantage that it can lead to globally stabilizing controllers for systems in chained form. A disadvantage, however, is that the controller is also expressed in these 'new coordinates'. When written in the 'original' chained-form coordinates, usually complex expressions are obtained. Especially since a change of coordinates is required to bring the dynamics (24) in a triangular form suitable for applying the integrator backstepping technique.

To arrive at simple controllers, our approach is different. We use the ideas of cascaded systems [11, 26, 30] but in particular the result for time-varying systems is presented [32]. With the result of Theorem 2.7 in mind, we try to look for a subsystem which, with a stabilizing control law, can be written in the form $\dot{z}_2 = f_2(t, z_2)$ and is asymptotically stable. In the remaining dynamics we can then replace the appearance of z_2 by 0, leading to the system $\dot{z}_1 = f_1(t, z_1)$. As a result we can write the system as (6). If both the subsystems $\dot{z}_1 = f_1(t, z_1)$ and $\dot{z}_2 = f_2(t, z_2)$ are asymptotically stable we might be able to conclude asymptotic stability of the overall system by means of Theorem 2.7.

One could remark that for arriving at the chained form, usually complex changes of coordinates and state feedback are needed. Therefore, a simple controller in chained-form co-ordinates is no guarantee for a simple controller in the co-ordinates of the original model. However, using the same idea simple controllers in the original co-ordinates can also be found, as was shown in Reference [31] for a two-wheel-driven mobile car.

Consider the tracking error dynamics

$$\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} \\
\dot{x}_{2,e} &= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_{2,e}u_{1,r} + x_2(u_1 - u_{1,r}) \\
&\;\;\vdots \\
\dot{x}_{n,e} &= x_{n-1,e}u_{1,r} + x_{n-1}(u_1 - u_{1,r})
\end{aligned} \tag{29}$$

It is very easy to stabilize only the $x_{1,e}$ dynamics, for example by using

$$u_1 = u_{1,r} - k_1 x_{1,e}, \quad k_1 > 0 \tag{30}$$

Clearly, other choices can be made as well.

Once the $x_{1,e}$ dynamics are asymptotically stable, we have determined a subsystem of the form $\dot{z}_2 = f_2(t, z_2)$. In order to arrive at the $\dot{z}_1 = f_1(t, z_1)$ dynamics, we can assume we already have stabilized the $\dot{x}_{1,e}$ dynamics, e.g. we assume $x_{1,e}(t) \equiv 0$. As a result also $u_1(t) - u_{1,r}(t) \equiv 0$. Then the remaining dynamics become

$$\begin{aligned}
\dot{x}_{2,e} &= u_2 - u_{2,r} \\
\dot{x}_{3,e} &= x_{2,e}u_{1,r} \\
&\;\;\vdots \\
\dot{x}_{n,e} &= x_{n-1,e}u_{1,r}
\end{aligned} \tag{31}$$

which is equivalent to

$$\underbrace{\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \vdots \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix}}_{\dot{z}_1} = \underbrace{\begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r}(t) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r}(t) & 0 \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} x_{2,e} \\ x_{3,e} \\ \vdots \\ \vdots \\ x_{n,e} \end{bmatrix}}_{z_1} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}_{B} (u_2 - u_{2,r}) \tag{32}$$

Now we only have to make sure that system (32) in closed loop with a suitably chosen feedback controller for u_2 is asymptotically stable, and hope that Theorem 2.7 enables us to conclude asymptotic stability of the tracking error dynamics (29).

As a result, we have reduced the tracking control problem to the problem of finding a control law for u_1 that stabilizes the linear system

$$\dot{x}_{1,e} = u_1 - u_{1,r} \tag{33}$$

and finding a control law for u_2 that stabilizes the LTV system (32).

3.1. *State-feedback*

In order to solve the state-feedback tracking control problem (Problem 2.17) we stabilize systems (32) and (33). For stabilizing (32) we use the result of Theorem 2.14 and for stabilizing (33) we use (30). As a result we get

Theorem 3.1

Consider the tracking error dynamics (29). Assume that $u_{1,r}(t)$ satisfies Assumption 2.12 and that $x_{2,r}, \ldots, x_{n-1,r}$ are bounded.

Then the control law

$$\begin{aligned} u_1 &= u_{1,r} - k_1 x_{1,e} \\ u_2 &= u_{2,r} - k_2 x_{2,e} - k_3 u_{1,r}(t) x_{3,e} - k_4 x_{4,e} - k_5 u_{1,r}(t) x_{5,e} \ldots \end{aligned} \tag{34}$$

results in closed-loop dynamics that are globally $\mathcal{K}$-exponentially stable, provided $k_1 > 0$ and k_i $(i = 2, \ldots, n)$ are such that the polynomial

$$\lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_{n-1}\lambda + k_n \tag{35}$$

is Hurwitz (i.e. has its roots in the left-half of the open complex plane).

Proof. We can see the closed-loop system (29), (34) as a system of form (6) where

$$z_1 = [x_{2,e}, \ldots, x_{n,e}]^{\mathrm{T}} \tag{36}$$

$$z_2 = x_{1,e} \tag{37}$$

$$f_1(t, z_1) = (A(t) - BK(t))z_1 \tag{38}$$

$$f_2(t, z_2) = -k_1 z_2 \tag{39}$$

$$g(t, z_1, z_2) = -k_1[0, x_2, x_3, \ldots, x_{n-1}]^{\mathrm{T}} \tag{40}$$

with

$$A(t) = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r}(t) & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r}(t) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad K(t) = \begin{bmatrix} k_2 \\ k_3 u_{1,r}(t) \\ k_4 \\ k_5 u_{1,r}(t) \\ \vdots \end{bmatrix}^{\mathrm{T}} \tag{41}$$

To be able to apply Theorem 2.7 we need to verify the three assumptions:

- Assumption on Σ_1: Due to the assumption on $u_{1,r}(t)$ we have from Theorem 2.14 that $\dot{z}_1 = f_1(t, z_1)$ is GES (and therefore GUAS). From converse Lyapunov theory (see e.g. Reference [19]) the existence of a suitable V is guaranteed.
- Assumption on connecting term: Since $x_{2,r}, \ldots, x_{n-1,r}$ are bounded, we have

$$\|g(t, z_1, z_2)\| \leqslant k_1 \left(\left\| \begin{bmatrix} 0 \\ x_{2,r} \\ \vdots \\ x_{n-1,r} \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 \\ x_{2,e} \\ \vdots \\ x_{n-1,e} \end{bmatrix} \right\| \right) \tag{42}$$

$$\leqslant k_1 M + k_1 \|z_1\| \tag{43}$$

- Assumption on Σ_2: Follows from GES of $\dot{z}_2 = -k_1 z_2$.

Therefore, we conclude GUAS from Theorem 2.7. Since both Σ_1 and Σ_2 are GES, Lemma 2.9 gives the desired result. □

Remark 3.2

Since the control law (20) is a static state feedback we know from Brockett [3] that stabilization is not possible. This explains why we need to assume that $u_{1,r}(t)$ satisfies Assumption 2.12. In Reference [35] a stabilization result using a comparative separation in linear subsystems can be found.

3.2. *Dynamic output-feedback*

In order to solve the dynamic output-feedback tracking control problem (Problem 2.18) we stabilize the systems

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \vdots \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ u_{1,r}(t) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ \vdots \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} (u_2 - u_{2,r}) \tag{44}$$

$$y_1 = x_{n,e}$$

and

$$\begin{aligned} \dot{x}_{1,e} &= u_1 - u_{1,r} \\ y_2 &= x_{1,e} \end{aligned} \tag{45}$$

For stabilizing (44) we use the result of Theorem 2.16 and for stabilizing (33) we use again (30). As a result we obtain

Theorem 3.3

Consider the tracking error dynamics (29). Assume that $u_{1,r}(t)$ satisfies Assumption 2.12 and that $x_{2,r}, \ldots, x_{n-1,r}$ are bounded.

Then the control law

$$\begin{aligned} u_1 &= u_{1,r} - k_1 x_{1,e} \\ u_2 &= u_{2,r} - k_2 \hat{x}_{2,e} - k_3 u_{1,r}(t) \hat{x}_{3,e} - k_4 \hat{x}_{4,e} - k_5 u_{1,r}(t) \hat{x}_{5,e} \cdots \end{aligned} \tag{46}$$

where $[\hat{x}_{2,e}, \ldots, \hat{x}_{n,e}]^{\mathrm{T}}$ is generated by the observer

$$\begin{bmatrix} \dot{\hat{x}}_{2,e} \\ \dot{\hat{x}}_{3,e} \\ \vdots \\ \vdots \\ \dot{\hat{x}}_{n,e} \end{bmatrix} = \begin{bmatrix} -k_2 & -k_3 u_{1,r}(t) & \cdots & \cdots & \cdots \\ u_{1,r}(t) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{2,e} \\ \hat{x}_{3,e} \\ \vdots \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix} + \begin{bmatrix} \vdots \\ l_5 u_{1,r}(t) \\ l_4 \\ l_3 u_{1,r}(t) \\ l_2 \end{bmatrix} (x_{n,e} - \hat{x}_{n,e}) \tag{47}$$

results in closed-loop dynamics that are globally $\mathcal{K}$-exponentially stable, provided that $k_1 > 0$ and k_i, l_i ($i = 2, \ldots, n$) are such that the polynomials

$$\begin{aligned} &\lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_{n-1}\lambda + k_n \\ &\lambda^{n-1} + l_2 \lambda^{n-2} + \cdots + l_{n-1}\lambda + l_n \end{aligned} \tag{48}$$

are Hurwitz (i.e., have their roots in the left-half of the open complex plane).

Proof. We can see the closed-loop system (29) and (34) as a system of form (6) where

$$z_1 = [x_{2,e}, \ldots, x_{n,e}, \tilde{x}_{2,e}, \ldots, \tilde{x}_{n,e}]^{\mathrm{T}} \tag{49}$$

$$z_2 = x_{1,e} \tag{50}$$

$$f_1(t, z_1) = \begin{bmatrix} A(t) - BK(t) & -BK(t) \\ 0 & A(t) - L(t)C \end{bmatrix} z_1 \tag{51}$$

$$f_2(t, z_2) = -k_1 z_2 \tag{52}$$

$$g(t, z_1, z_2) = -k_1 [0, x_2, x_3, \ldots, x_{n-1}, \underbrace{0, \ldots, 0}_{(n-1)}]^{\mathrm{T}} \tag{53}$$

and $\tilde{x}_{i,e} = x_{i,e} - \hat{x}_{i,e} (i = 2, \ldots, n)$. To be able to apply Theorem 2.7 we need to verify the three assumptions:

- Assumption on Σ_1: Due to the assumption on $u_{1,r}(t)$ we have from Theorem 2.16 that $\dot{z}_1 = f_1(t, z_1)$ is GES (and therefore GUAS). From converse Lyapunov theory (see e.g. Reference [19]) the existence of a suitable V is guaranteed.

- Assumption on connecting term: Since $x_{2,r}, \ldots, x_{n-1,r}$ are bounded, we have again

$$\|g(t, z_1, z_2)\| \leqslant k_1 M + k_1 \|z_1\| \tag{54}$$

- Assumption on Σ_2: Follows from GES of $\dot{z}_2 = -k_1 z_2$.

Therefore, we conclude GUAS from Theorem 2.7. Since both Σ_1 and Σ_2 are GES, Lemma 2.9 gives the desired result. □

3.3. Saturated control

As in Reference [14] we can consider Problems 2.17 and 2.18 under the additional design constraint that

$$|u_1(t)| \leqslant u_{1,\max} \quad \forall t \geqslant 0 \tag{55}$$

where $u_{1,\max}$ is a constant such that $\sup_t |u_{1,r}(t)| < u_{1,\max}$.

It is obvious that if we replace the expression $u_1 = u_{1,r} - k_1 x_{1,e}$ with

$$u_1 = u_{1,r} - \sigma(x_{1,e}) \tag{56}$$

where $\sigma(\cdot)$ is any differentiable function that satisfies

- $x\sigma(x) > 0$ for all $x \neq 0$,
- $\sup_s |\sigma(s)| \leqslant u_{1,\max} - \sup_t |u_{1,r}(t)|$,
- $\mathrm{d}\sigma/\mathrm{d}x\,(0) > 0$.

the results of Theorems 3.1 and 3.3 still hold.

More interesting is the case where we not only assume the design constraint (55) on u_1, but also a design constraint on u_2:

$$|u_2(t)| \leqslant u_{2,\max} \quad \forall t \geqslant 0 \tag{57}$$

where $u_{2,\max}$ is a constant such that $\sup_t |u_{2,r}(t)| < u_{2,\max}$. To our knowledge, no saturated controller for stabilizing the general LTV system (14) has been derived in the literature yet. However, for the case that $u_{1,r}$ is constant for all t, system (14) reduces to a time-invariant linear system. In that case the results of Reference [38] can be used to solve the problem for both the state and dynamic output-feedback problem.

4. SIMULATIONS: CAR WITH TRAILER

In this section we apply the proposed state- and output-feedback designs for the tracking control of a well-known benchmark problem; a towing car with a single trailer, see e.g. References [15, 27, 35].

The state configuration of the articulated vehicle consists of the position of the car, (x_c, y_c), the steering angle ϕ, and the angles, (θ_0, θ_1), of the car and the trailer with respect to the x-axis, see Figure 1. The rear wheels of the car and the trailer are aligned with the chassis and are not allowed to slip sideways. The two input signals are the driving velocity of the front wheels, v, and the steering velocity, ω.

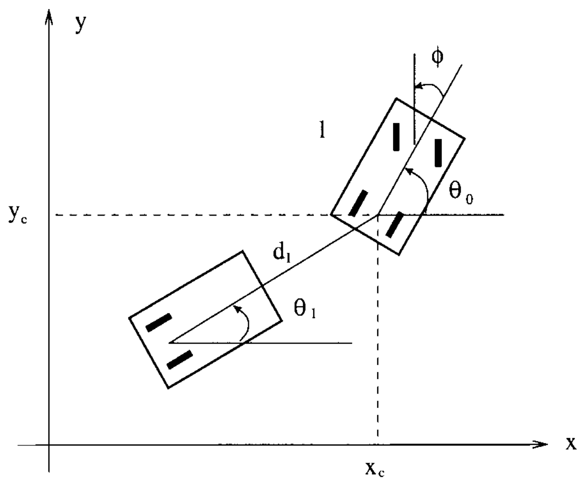

Figure 1. Car with a trailer, see Reference [27].

The kinematic equations of motion for the vehicle can be described by (cf. Reference [27]),

$$
\begin{aligned}
\dot{x}_c &= v\cos\theta_0 \\
\dot{y}_c &= v\sin\theta_0 \\
\dot{\phi} &= \omega \\
\dot{\theta}_0 &= \frac{1}{l}\tan\phi \\
\dot{\theta}_1 &= \frac{1}{d_1} v\sin(\theta_0 - \theta_1)
\end{aligned}
\tag{58}
$$

Via a (local) change of co-ordinates the system can be transformed to the following system in chained form:

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= u_1 x_2 \\
\dot{x}_4 &= u_1 x_3 \\
\dot{x}_5 &= u_1 x_4
\end{aligned}
\tag{59}
$$

We refer to Reference [15] for explicit expressions of the transformation.

For the simulations, we have considered tracking of a reference model (23) moving along a straight line,

$$u_{1,r} = 1, \quad u_{2,r} = 0$$

with the initial conditions

$$x_{ir}(0) = 0.0, \quad i = 1, \ldots, 5$$

$$x_1(0) = 1.0, \quad x_2(0) = x_3(0) = x_4(0) = x_5(0) = 0.5 \tag{60}$$

The state-feedback (SF) and the output feedback controller (OF) used in the simulations are

$$u_{1,\mathrm{SF}} = u_{1,r} - k_1 x_{1,e} \tag{61}$$

$$u_{2,\mathrm{SF}} = u_{2,r} - k_2 x_{2,e} - k_3 u_{1,r} x_{3,e} - k_4 x_{4,e} - k_5 u_{1,r} x_{5,e} \tag{62}$$

$$u_{1,\mathrm{OF}} = u_{1,r} - k_1 x_{1,e} \tag{63}$$

$$u_{2,\mathrm{OF}} = u_{2,r} - k_2 \hat{x}_{2,e} - k_3 u_{1,r} \hat{x}_{3,e} - k_4 \hat{x}_{4,e} - k_5 u_{1,r} \hat{x}_{5,e} \tag{64}$$

where the 'controller polynomial' (48) has all the roots in $\lambda = -2$ and the 'observer polynomial' (48) has its roots in $\lambda = -3$.

In Figure 2 the behaviour of the closed-loop system for the state-feedback controller (SF) and the output-feedback controller (OF) are compared to a recently presented state-feedback controller, *J&N(106-7)*, based on a backstepping design [15].

$$u_{2,JN} = -k_4 z_4 - 2k_4 z_2 - u_{1r}(3z_3 + z_1) \tag{65}$$

$$u_{1,JN} = u_{1r} - k_5 z_5 - [k_4 z_4 + 2k_4 z_2 + u_{1r}(3z_3 + z_1)]\left[z_1 + z_3 - \frac{5}{2} z_2 z_5 - z_4 z_5 + \frac{(2z_1 + z_3) z_5^2}{6}\right] \tag{66}$$

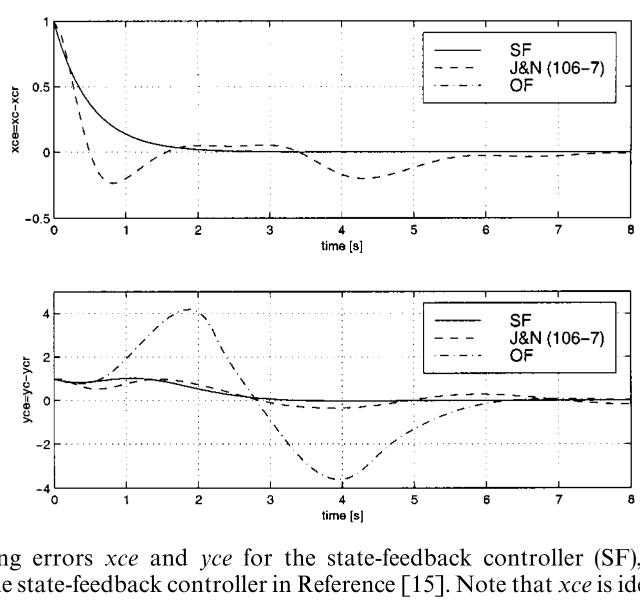

Figure 2. The tracking errors *xce* and *yce* for the state-feedback controller (SF), the output-feedback controller (OF) and the state-feedback controller in Reference [15]. Note that *xce* is identical for SF and OF.

where

$$
\begin{aligned}
z_1 &= x_5 - x_4 x_{1,e} + \tfrac{1}{2} x_3 x_{1,e}^2 - \tfrac{1}{6} x_2 x_{1,e}^3 \\
z_2 &= x_4 - x_3 x_{1,e} + \tfrac{1}{2} x_2 x_{1,e}^2 \\
z_3 &= x_3 - x_2 x_{1,e} \\
z_4 &= x_2 \\
z_5 &= x_{1,e}
\end{aligned}
\tag{67}
$$

For the case of constant $u_{1,r}$ we can apply the ideas from [38] for bounded control also on u_2. Figure 3 and 4 show the tracking error in the y-direction using bounded control of u_2 for the state-feedback and the output-feedback case. The saturated state-feedback controller [38] has the structure

$$u_{1,\mathrm{sat}} = u_{1,r} - \sigma_1(x_{1,e}) \tag{68}$$

$$u_{2,\mathrm{sat}} = u_{2,r} - \sum_{i=1}^{4} \varepsilon^i \sigma_2(y_i) \tag{69}$$

where

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \varepsilon & 0 & 0 \\ 1 & \varepsilon^2 + \varepsilon & \varepsilon^3 & 0 \\ 1 & \varepsilon^3 + \varepsilon^2 + \varepsilon & \varepsilon^5 + \varepsilon^4 + \varepsilon^3 & \varepsilon^6 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ x_{4,e} \\ x_{5,e} \end{bmatrix} \tag{70}
$$

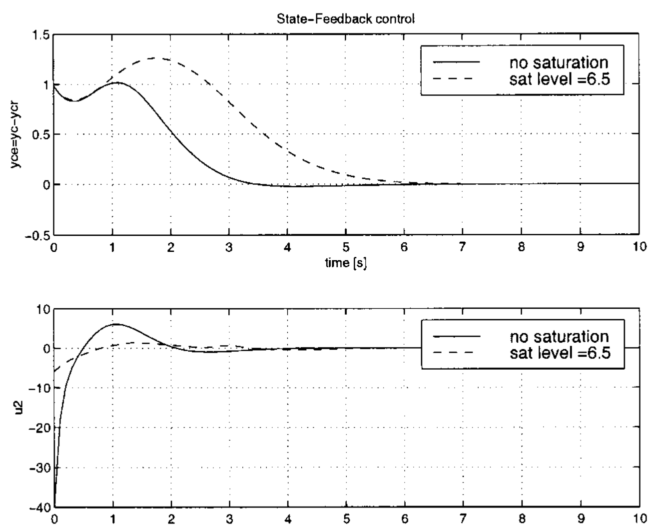

Figure 3. State feedback control with and without saturated u_2.

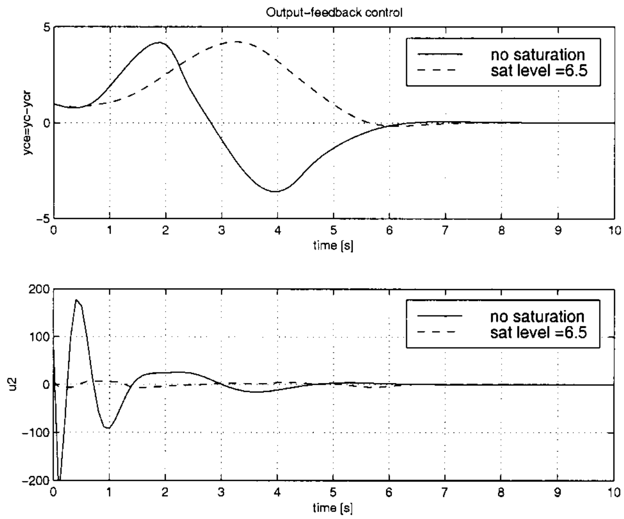

Figure 4. Output feedback control with and without saturated u_2.

and the saturated output-feedback controller uses the state estimations from observer (47) in a certainty equivalence sense.

5. CONCLUDING REMARKS

In this paper we addressed the problem of designing simple global tracking controllers for non-holonomic systems in chained form under both state and dynamic output feedback.

We divided the (nonlinear) tracking control problem into two simpler and 'independent' linear control problems. We showed by means of cascaded system theory that the two linear controllers that solve the two linear control problems also solve the tracking problem.

The state and dynamic output feedback tracking problem under input saturation were globally solved in case we have input saturation only on u_1. In case of input saturation on u_1 and u_2 both problems were solved for constant $u_{1,r}$.

We illustrated our results by means of a simulation of a car with a trailer.

Challenging questions that remain open are the tracking problem under input saturation on u_1 and u_2 for arbitrary $u_{1,r}$ and the study for robustness of the proposed schemes.

APPENDIX A. PROOFS OF THEOREMS 2.14 AND 2.16

To start with, we consider the stability of the differential equation

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m} y(t) + a_1 \frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}} y(t) + \cdots + a_{m-1} \frac{\mathrm{d}}{\mathrm{d}t} y(t) + a_m y(t) = 0 \tag{A1}$$

For this system we can define the Hurwitz determinants

$$\Delta_i = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2i-1} \\ 1 & a_2 & a_4 & \cdots & a_{2i-2} \\ 0 & a_1 & a_3 & \cdots & a_{2i-3} \\ 0 & 1 & a_2 & \cdots & a_{2i-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_i \end{vmatrix} \quad (i = 1, \ldots, m) \tag{A2}$$

where if an element a_j appears in Δ_i with $j > i$ it is assumed to be zero. It is well known [9] that system (A1) is asymptotically stable, if and only if the determinants Δ_i are all positive. Less known is a proof of this result by means of the second method of Lyapunov. If we define

$$b_1 = \Delta_1, \quad b_2 = \frac{\Delta_2}{\Delta_1}, \quad b_3 = \frac{\Delta_3}{\Delta_1 \Delta_2}, \quad b_i = \frac{\Delta_{i-3}\Delta_i}{\Delta_{i-2}\Delta_{i-1}} \quad (i = 4, \ldots, m) \tag{A3}$$

it was shown in Reference [33] that system (A1) can also be represented as

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} w \tag{A4}$$

Differentiating the Lyapunov function candidate

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{m-1} w_{m-1}^2 + b_1 b_2 \cdots b_m w_m^2 \tag{A5}$$

(which is positive definite if and only if the determinants Δ_i are all positive) along solutions of (A4) results in

$$\dot{V} = -b_1^2 w_1^2 \tag{A6}$$

Asymptotic stability then can be shown by invoking LaSalle's theorem [23].

Inspired by the result of Reference [33] we look for a state-transformation $z = Sw$, that transforms the system (74) into

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_m \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} z \tag{A7}$$

To start with, we define

$$z_m = w_m \tag{A8}$$

Since $\dot{w}_m = w_{m-1}$ and we would like $\dot{z}_m = z_{m-1}$ we define

$$z_{m-1} = w_{m-1} \tag{A9}$$

Since $\dot{w}_{m-1} = w_{m-2} - b_m w_m$ and we would like $\dot{z}_{m-1} = z_{m-2}$ we define

$$z_{m-2} = w_{m-2} - b_m w_m \tag{A10}$$

Proceeding similarly we define all z_k and obtain an expression that looks like

$$z_k = w_k + s_{k,k+2} \cdot w_{k+2} + s_{k,k+4} \cdot w_{k+4} + \cdots \tag{A11}$$

By this construction of the state-transformation, we are guaranteed to meet the $m-1$ final equations of (A7). The only thing that remains to be verified is if the equation for $\dot{z}_1$ holds. From the structure displayed in (A11) we know the matrix S is non-singular, so therefore we can write

$$\dot{z}_1 = -\alpha_1 z_1 - \alpha_2 z_2 - \cdots - \alpha_n z_n, \quad \alpha_i \in \mathbb{R}, \quad (i = 1, \ldots, m). \tag{A12}$$

The characteristic polynomial of the transformed system then becomes

$$\lambda^m + \alpha_1 \lambda^{m-1} + \cdots + \alpha_{m-1} \lambda + \alpha_m \tag{A13}$$

Since a state-transformation does not change the characteristic polynomial and we know from Reference [33] that the characteristic polynomial of (A4) equals

$$\lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1} \lambda + a_m \tag{A14}$$

clearly $\alpha_i = a_i$ $(i = 1, \ldots, m)$.

Before we can prove Theorems 2.14 and 2.16 we need to remark one thing about this transformation. When we define $T = S^{-1}$, we know that

$$w_1 = z_1 + t_{1,3} z_3 + t_{1,5} z_5 + \cdots \tag{A15}$$

$$w_2 = z_2 + t_{2,4} z_4 + t_{2,6} z_6 + \cdots \tag{A16}$$

But also $\dot{w}_1 = -a_1 w_1 - b_2 w_2$ (notice that $b_1 = a_1$). Therefore,

$$\dot{w}_1 = \dot{z}_1 + t_{1,3} \dot{z}_3 + t_{1,5} \dot{z}_5 + \cdots \tag{A17}$$

$$= (-a_1 z_1 - a_2 z_2 - \cdots - a_n z_n) + t_{1,3} z_2 + t_{1,5} z_4 + \cdots \tag{A18}$$

$$= [-a_1 z_1 - a_3 z_3 - \cdots] + [(t_{1,3} - a_2) z_2 + (t_{1,5} - a_4) z_4 + \cdots] \tag{A19}$$

So obviously

$$w_1 = z_1 + \frac{a_3}{a_1} z_3 + \frac{a_5}{a_1} z_5 + \cdots \tag{A20}$$

Knowing this state-transformation and (A20) we can start proving Theorems 2.14 and 2.16.

Proof of Theorem 2.14. The closed-loop system (14) ad (17) is given by

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 u_{1,r}(t) & -a_3 & -a_4 u_{1,r}(t) & \cdots \\ u_{1,r}(t) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1,r}(t) & 0 \end{bmatrix} z \tag{A21}$$

This can we rewritten as

$$\dot{z} = u_{1,r}(t) \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_m \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} z + (u_{1,r}(t) - 1) \begin{bmatrix} a_1 z_1 + a_3 z_3 + \cdots \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tag{A22}$$

When we apply the change of co-ordinates $z = Sw$ as defined before, we obtain

$$\dot{w} = u_{1,r}(t) \begin{bmatrix} -b_1 & -b_2 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} w + (u_{1,r}(t) - 1) \begin{bmatrix} 1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 w_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tag{A23}$$

which (using $a_1 = b_1$) can we rewritten as

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 u_{1,r}(t) & 0 & \cdots & 0 \\ u_{1,r}(t) & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m u_{1,r}(t) \\ 0 & \cdots & 0 & u_{1,r}(t) & 0 \end{bmatrix} w \tag{A24}$$

Consider the Lyapunov function candidate

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{m-1} w_{m-1}^2 + b_1 b_2 \cdots b_m w_m^2 \tag{A25}$$

which is positive definite if and only if

$$\lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1} \lambda + a_m \tag{A26}$$

is a Hurwitz-polynomial. Differentiating (A25) along solutions of (A24) results in

$$\dot{V} = -b_1^2 w_1^2 \tag{A27}$$

It is well known [19] that the origin of system (A24) is GES if the pair

$$\left(\begin{bmatrix} -b_1 & -b_2 u_{1,r}(t) & 0 & \dots & 0 \\ u_{1,r}(t) & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m u_{1,r}(t) \\ 0 & \cdots & 0 & u_{1,r}(t) & 0 \end{bmatrix}, [b_1, 0, \dots, 0] \right) \tag{A28}$$

is uniformly completely observable (UCO).

If $u_{1,r}(t)$ satisfies Assumption 2.12 it follows immediately from Theorem 2 in Reference [18] that the pair (A28) is UCO, which completes the proof. □

Proof of Theorem 2.16. We can write the closed-loop system (14, 19, 20) as

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{z}} \end{bmatrix} = \begin{bmatrix} A(t) - BK(t) & -BK(t) \\ 0 & A(t) - L(t)C \end{bmatrix} \begin{bmatrix} z \\ \tilde{z} \end{bmatrix} \tag{A29}$$

where $\tilde{z} = z - \hat{z}$.

Since $u_{1,r}(t)$ satisfies Assumption 2.12 and k_i, l_i are such that the polynomials (21) are Hurwitz, we know from Theorem 2.14 that the systems $\dot{z} = [A(t) - BK(t)]z$ and $\dot{\tilde{z}} = [A(t) - L(t)\, C]\tilde{z}$ are GES.

Then the result follows immediately from Theorem 2.7, (since $K(t)$ is bounded), and the fact that a LTV system which is GUAS also is GES [10, 19]. □

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