$$
\begin{aligned}
X_n = \operatorname{Span}&\left\{\left[\frac{B(z)F(z)z^{-1}}{H(z,\theta_\circ^n)A^2(z)T_G(z)},0\right]^T,\ldots,\left[\frac{B(z)F(z)z^{-m_a}}{H(z,\theta_\circ^n)\,A^2(z)T_G(z)},0\right]^T\right\}\\
&\oplus \operatorname{Span}\left\{\left[\frac{F(z)z^{-1}}{H(z,\theta_\circ^n)A(z)T_G(z)},0\right]^T,\ldots,\left[\frac{F(z)z^{-m_b}}{H(z,\theta_\circ^n)\,A(z)T_G(z)},0\right]^T\right\}\\
&\oplus \operatorname{Span}\left\{\left[0,\frac{z^{-1}}{D(z)T_H(z)}\right]^T,\ldots,\left[0,\frac{z^{-m_d}}{D(z)T_H(z)}\right]^T,\left[0,\frac{z^{-1}}{C(z)T_H(z)}\right]^T,\ldots,\left[0,\frac{z^{-m_c}}{C(z)T_H(z)}\right]^T\right\}.
\end{aligned}
$$

## APPENDIX B

### A. Proof of Corollary 3.1

*Proof:* Since the conditions of Theorem 3.1 are satisfied, then the asymptotic in $N$ co-variance is given by (21). Furthermore, in the case that a Box-Jenkins model structure is employed, then under the assumptions (27)

$$
\begin{aligned}
\frac{d}{dc_k}G(z,\theta^n) = \frac{d}{dd_k}G(z,\theta^n) &= \frac{d}{da_k}H(z,\theta^n)\\
&= \frac{d}{db_k}H(z,\theta^n) = 0 \qquad \text{(B.15)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{da_k}G(z,\theta^n) &= -\frac{B(z)}{A^2(z)T_G(z)}\cdot z^{-k}\\
\frac{d}{db_\ell}G(z,\theta^n) &= \frac{z^{-\ell}}{A(z)T_G(z)} \qquad \text{(B.16)}\\
\frac{dH(z,\theta^n)}{dd_k} &= -\frac{C(z)z^{-k}}{D^2(z)T_H(z)}\\
\frac{dH(z,\theta^n)}{dc_\ell} &= \frac{z^{-\ell}}{D(z)T_H(z))}. \qquad \text{(B.17)}
\end{aligned}
$$

Therefore, since under the assumption of $\Phi_{ue}(\omega) = 0$

$$
S_\zeta(z) = \begin{bmatrix} F(z) & 0\\ 0 & \sigma \end{bmatrix} \qquad \text{(B.18)}
$$

then, according to (15) and (16), the equation at the top of the page holds. Furthermore, under the assumption (28) the space $X_n$ may be reformulated as

$$
\begin{aligned}
X_n = \operatorname{Span}\Big\{&f_1(z),\ldots,f_{m_a+m_b-m_{t_g}}(z),\\
&g_1(z),\ldots,g_{m_c+m_d-m_{t_h}}(z)\Big\} \qquad \text{(B.19)}
\end{aligned}
$$

$$
\begin{aligned}
f_k(z) &\triangleq \left[\tfrac{z^{-k}}{A_\dagger(z)},\quad 0\right]^T\\
g_k(z) &\triangleq [0,\quad \tfrac{z^{-k}}{C(z)D(z)T_H(z)}]^T. \qquad \text{(B.20)}
\end{aligned}
$$

Therefore, by [6, Lemma 3.4], the multivariable reproducing kernel for $X_n$ is of the form

$$
\varphi_n(\lambda,\mu) = \begin{bmatrix} \varphi_n^f(\lambda,\omega) & 0\\ 0 & \varphi_m^g(\lambda,\omega) \end{bmatrix} \qquad \text{(B.21)}
$$

where all the entries in the above matrix are scalar. In this case, [6, Lemmas 3.1 and 3.2] may be used to quantify $\varphi_n^f(\lambda,\omega)$, $\varphi_n^g(\lambda,\omega)$ Setting $\kappa(\omega) = \varphi_n^f(\omega,\omega)$, $\widetilde{\kappa}(\omega) = \varphi_n^g(\omega,\omega)$ completes the proof. □

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# Simultaneous Tracking and Stabilization of Mobile Robots: An Adaptive Approach

K. D. Do, Z. P. Jiang, and J. Pan

***Abstract*—This note presents a time-varying global adaptive controller at the torque level that simultaneously solves both tracking and stabilization for mobile robots with unknown kinematic and dynamic parameters. The controller synthesis is based on Lyapunov's direct method and backstepping technique. Simulations illustrate the effectiveness of the proposed controller.**

***Index Terms*—Global adaptive control, Lyapunov design, mobile robot, stabilization, tracking.**

## I. INTRODUCTION

The main difficulty of solving stabilization and tracking control of mobile robots is because the motion of the systems in question has more degrees of freedom than the number of inputs under nonholonomic constraints. Furthermore, the necessary condition of Brockett's

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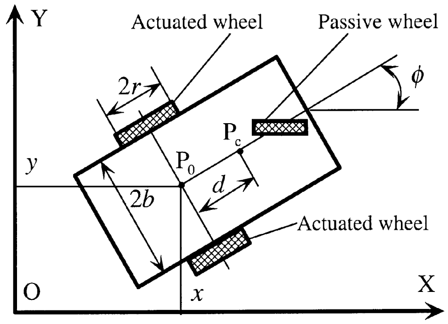


Fig. 1. Two-wheel driven mobile robot.

theorem [13] shows that any continuous time invariant feedback control law does not make the null solution of the wheeled mobile robots asymptotically stable in the sense of Lyapunov. Over the last decade, a lot of interest has been devoted to stabilization and tracking control of nonholonomic systems including wheeled mobile robots [1]–[10] to list a few, where tracking and stabilization were studied separately. Their objectives are mostly kinematic models. Recently, several authors focused on the dynamic model [10]–[12] using the backstepping technique [14]. From the aforementioned discussion, an open problem is to design a single controller that is able to solve both stabilization and tracking for mobile robots with unknown kinematic and dynamic parameters. This note contributes a positive answer to this problem. Our new result is facilitated by introducing a coordinate transformation to transform the tracking errors interpreted in a frame attached to the robot, to a triangular form, to which the backstepping technique can be applied.

## II. Problem Statement

We consider a mobile robot with two actuated wheels (Fig. 1) whose equations of motion are [12]

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{J}(\boldsymbol{\eta})\mathbf{v} \\ \mathbf{M}\dot{\mathbf{v}} + \mathbf{C}(\dot{\boldsymbol{\eta}})\mathbf{v} + \mathbf{D}\mathbf{v} &= \boldsymbol{\tau} \end{aligned} \tag{1}$$

where

$$\begin{aligned} &\boldsymbol{\eta} = [x\ \ y\ \ \phi]^T, \ \ \mathbf{v} = [v_1\ \ v_2]^T, \ \ \boldsymbol{\tau} = [\tau_1\ \ \tau_2]^T \\ &\mathbf{J}(\boldsymbol{\eta}) = 0.5r \begin{bmatrix} \cos(\phi) & \cos(\phi) \\ \sin(\phi) & \sin(\phi) \\ b^{-1} & -b^{-1} \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{11} \end{bmatrix} \\ &\mathbf{C}(\dot{\boldsymbol{\eta}}) = 0.5b^{-1}r^2 m_c d \begin{bmatrix} 0 & \dot{\phi} \\ -\dot{\phi} & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix} \\ &m_{11} = 0.25b^{-2}r^2(mb^2 + I) + I_w \\ &m_{12} = 0.25b^{-2}r^2(mb^2 - I) \\ &m = m_c + 2m_w \\ &I = m_c d^2 + 2m_w b^2 + I_c + 2I_m. \end{aligned}$$

In these expressions and Fig. 1, $b$ is half of the width of the mobile robot and $r$ is the radius of the wheel. $d$ is the distance between the center of mass, $P_c$, of the robot and the middle point, $P_0$, between the left and right wheels. $m_c$ and $m_w$ are the mass of the body and wheel with a motor. $I_c$, $I_w$ and $I_m$ are the moment of inertia of the body about the vertical axis through $P_c$, the wheel with a motor about the wheel axis, and the wheel with a motor about the wheel diameter, respectively. The positive terms $d_{ii}, i = 1, 2$ are the damping coefficients, ($x$, $y$, and $\phi$) are the position and orientation of the robot, $v_1$ and $v_2$ are the angular velocities of the wheels. $\tau_1$ and $\tau_2$ are the control torques applied to the wheels of the robot. We assume that the reference trajectory is generated by the virtual robot

$$\begin{aligned} \dot{x}_d &= \cos(\phi_d)u_{1d} \\ \dot{y}_d &= \sin(\phi_d)u_{1d} \\ \dot{\phi}_d &= u_{2d} \end{aligned} \tag{2}$$

where ($x_d$, $y_d$, and $\phi_d$) are the position and orientation of the virtual robot. $u_{1d}$ and $u_{2d}$ are the linear and angular velocities of the virtual robot, respectively.

*Control Objective:* Under Assumptions 1 and 2, design the control $\boldsymbol{\tau}$ to force the position and orientation, ($x$, $y$, and $\phi$) of the real robot (1) to globally asymptotically track ($x_d$, $y_d$, $\phi_d$) generated by (2).

*Assumption 1:* The reference signals $u_{1d}, \dot{u}_{1d}, \ddot{u}_{1d}, u_{2d}$ and $\dot{u}_{2d}$ are bounded. In addition, one of the following conditions holds:

$$\textbf{C1.} \quad \int_0^\infty (|u_{1d}(t)| + |u_{2d}(t)| + |\dot{u}_{1d}(t)|)\, dt \le \mu_1 \tag{3}$$

$$\textbf{C2.} \quad \int_0^\infty (|u_{1d}(t)| + |\dot{u}_{1d}(t)|)\, dt \le \mu_{21} \quad \text{and}$$

$$|u_{2d}(t)| \ge \mu_{22} \qquad \forall 0 \le t < \infty \tag{4}$$

$$\textbf{C3.} \quad |u_{1d}(t)| \ge \mu_3 \qquad \forall 0 \le t < \infty \tag{5}$$

where $\mu_1$ and $\mu_{21}$ are nonnegative constants, $\mu_{22}$ and $\mu_3$ are strictly positive constants.

*Assumption 2:* All of the robot parameters are constants but unknown, and lie in a compact set.

*Remark 1:* The problem of set-point regulation/stabilization, tracking a path approaching a set-point is included in C1. Tracking linear and circular paths belongs to C3. Condition C2 implies that the case, where the robot linear velocity is zero or approaches zero and its angular velocity is of sinusoidal type, is excluded. The reason is that our control approach is to introduce a sinusoid signal in the robot angular velocity virtual control to handle set-point stabilization/regulation. Therefore, this case is excluded to avoid two signals from canceling each other. If the reference velocity $u_{2d}$ is known completely in advance, the previous case can be included.

*Remark 2:* The problem of simultaneous stabilization and tracking not only is of theoretical interest but also possesses some advantages over the use of separate stabilization and tracking controllers such as only one controller and transient improvement because of no switching. Moreover, if the switching time is unknown, a separate stabilization and tracking control approach cannot be used.

## III. Control Design

As often done in tracking control of mobile robots, we interpret the tracking errors as

$$\begin{bmatrix} x_e \\ y_e \\ \phi_e \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_d \\ y - y_d \\ \phi - \phi_d \end{bmatrix}. \tag{6}$$

Indeed convergence of ($x_e$, $y_e$, $\phi_e$ to zero implies that of ($x - x_d$, $y - y_d$, $\phi - \phi_d$). Using (6), (2), and the kinematic part of (1), we have the kinematic tracking errors

$$\begin{aligned} \dot{x}_e &= ru_1 - u_{1d}\cos(\phi_e) + rb^{-1}y_e u_2 \\ \dot{y}_e &= u_{1d}\sin(\phi_e) - rb^{-1}x_e u_2 \\ \dot{\phi}_e &= rb^{-1}u_2 - u_{2d} \end{aligned} \tag{7}$$

where $u_1 = 0.5(v_1 + v_2)$, $u_2 = 0.5(v_1 - v_2)$. Now if $u_1$ and $u_2$ are considered as virtual controls, we can see directly from (7) that $x_e$

and $\phi_e$ can be stabilized by $u_1$ and $u_2$. Motivated by the car driving practice, we will use $\phi_e$ as a virtual control to stabilize $y_e$. Toward this end, we introduce the coordinate transformation

$$\begin{aligned} z_e &= \phi_e + \arcsin\left(\frac{k(t)y_e}{\varpi_1}\right) \\ k(t) &= \lambda_1 u_{1d} + \lambda_2 \cos(\lambda_3 t) \\ \varpi_1 &= \sqrt{1+x_e^2+y_e^2} \end{aligned} \tag{8}$$

where $\lambda_i$, $i = 1, 2, 3$ are positive constants such that $|k(t)| < 1, \forall t$. They will be specified later. It is seen that (8) is well defined. We now use (8) to write the tracking error dynamics as

$$\begin{aligned} \dot{x}_e &= ru_1 + f_x + rb^{-1}y_e u_2 \\ \dot{y}_e &= -ku_{1d}\varpi_1^{-1}y_e + f_y - rb^{-1}x_e u_2 \\ \dot{z}_e &= rb^{-1}u_2\left(1-k\varpi_1^{-1}x_e\right) + f_z + rg_z u_1 \\ \mathbf{M}\dot{\mathbf{v}} &+ \mathbf{C}(\dot{\boldsymbol{\eta}})\mathbf{v} + \mathbf{D}\mathbf{v} = \boldsymbol{\tau} \end{aligned} \tag{9}$$

where for simplicity of presentation, we have dropped the time argument of $k(t)$, and have defined

$$\begin{aligned} \varpi_2 &= \sqrt{1+x_e^2+(1-k^2)y_e^2} \\ f_x &= -u_{1d}\left(\cos(z_e)\varpi_1^{-1}\varpi_2 + \sin(z_e)k\varpi_1^{-1}y_e\right) \\ f_y &= -u_{1d}\left((\cos(z_e)-1)\varpi_1^{-1}\varpi_2 + \sin(z_e)k\varpi_1^{-1}y_e\right) \\ f_z &= -u_{2d} + \varpi_2^{-1}\left(\dot{k}y_e + kf_y - k\varpi_1^{-2}(x_e f_x + y_e f_y)\right) \\ g_z &= -k\varpi_1^{-2}\varpi_2^{-1}x_e y_e. \end{aligned} \tag{10}$$

The effort, we have made so far, is to have the term $-kv_d\varpi_1^{-1}y_e$ in the $y_e$-dynamics, and to put the tracking error dynamics in a triangular form of (9). We now design $\boldsymbol{\tau}$ to stabilize (9) in two steps.

*Step 1:* Define the virtual velocity tracking errors $\tilde{u}_1$ and $\tilde{u}_2$ as

$$\tilde{u}_1 = u_1 - u_{1c} \quad \tilde{u}_2 = u_2 - u_{2c} \tag{11}$$

where $u_{1c}$ and $u_{2c}$ are the virtual controls of $u_1$ and $u_2$. To design $u_{1c}$, we take the Lyapunov function

$$V_1 = r^{-1}(\varpi_1 - 1) + 0.5\gamma_{11}^{-1}\tilde{\theta}_{11}^2 \tag{12}$$

where $\gamma_{11}$ is a positive constant, $\tilde{\theta}_{11} = \theta_{11} - \hat{\theta}_{11}$ with $\hat{\theta}_{11}$ being an estimate of $\theta_{11} := r^{-1}$. Differentiating both sides of (12) along the solutions of (9), (11), and choosing $u_{1c}$ as

$$u_{1c} = -k_1 x_e - \hat{\theta}_{11}f_x \quad \dot{\hat{\theta}}_{11} = \gamma_{11}\varpi_1^{-1}x_e f_x \tag{13}$$

where $k_1$ is a positive constant, result in

$$\dot{V}_1 = -k_1\varpi_1^{-1}x_e^2 - ku_{1d}r^{-1}\varpi_1^{-2}y_e^2 + \varpi_1^{-1}x_e\tilde{u}_1 + r^{-1}\varpi_1^{-1}y_e f_y. \tag{14}$$

To design the virtual control $u_{2c}$, we take the Lyapunov function

$$V_2 = V_1 + 0.5br^{-1}z_e^2 + 0.5\gamma_{12}^{-1}\tilde{\theta}_{12}^2 + 0.5\gamma_{13}^{-1}\tilde{\theta}_{13}^2 \tag{15}$$

where $\gamma_{12}$ and $\gamma_{13}$ are positive constants, $\tilde{\theta}_{1i} = \theta_{1i} - \hat{\theta}_{1i}$, $i = 2, 3$, with $\hat{\theta}_{1i}$ being an estimate of $\theta_{1i}$, $\theta_{12} = r^{-1}b$, $\theta_{13} = b$. Differentiating both sides of (15) along the solution of the third equation of (9), the second equation of (11), and choosing the virtual control $u_{2c}$ as

$$\begin{aligned} u_{2c} &= \frac{1}{1-k\varpi_1^{-1}x_e}\left(-k_2 z_e^2 - \hat{\theta}_{12}f_z - \hat{\theta}_{13}g_z u_{1c}\right) \\ \dot{\hat{\theta}}_{12} &= \gamma_{12}z_e f_z \\ \dot{\hat{\theta}}_{13} &= \gamma_{13}z_e g_z u_1 \end{aligned} \tag{16}$$

where $k_2$ is a positive constant to be specified later, give

$$\begin{aligned} \dot{V}_2 = &-k_1\varpi_1^{-1}x_e^2 - ku_{1d}\varpi_1^{-2}y_e^2 - k_2 z_e^2 + r^{-1}\varpi_1^{-1}y_e f_y \\ &+ \left(\varpi_1^{-1}x_e + \hat{\theta}_{13}z_e g_z\right)\tilde{u}_1 + \left(1-k\varpi_1^{-1}x_e\right)z_e\tilde{u}_2. \end{aligned} \tag{17}$$

*Remark 3:* From (8), (13), and (16), one can consider $u_{1c}$ and $u_{2c}$ as an interesting combination of time-varying stabilization and tracking controllers proposed in literature, if the robot velocities are used as the actual inputs. In fact, setting $\lambda_2 = 0$ results in $u_{1c}$ and $u_{2c}$ similar to a tracking control law proposed in [7], [8], [10], and [12]. On the other hand, setting $\lambda_1 = 0$ results in $u_{1c}$ and $u_{2c}$ similar to a stabilization control law presented in [4] and [6]. However, our approach is different from those existing ones in the sense that the robot orientation error $\phi_e$ is used as a virtual control to stabilize the $y_e$-dynamics.

*Step 2:* Defining $\tilde{v}_1 = v_1 - v_{1c}$, $\tilde{v}_2 = v_2 - v_{2c}$ with $v_{1c} = u_{1c} + u_{2c}$, $v_{2c} = u_{1c} - u_{2c}$, and using (11) result in $\tilde{u}_1 = 0.5(\tilde{v}_1 + \tilde{v}_2)$ and $\tilde{u}_2 = 0.5(\tilde{v}_1 - \tilde{v}_2)$. Therefore, we have $\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{v_c}$ with $\tilde{\mathbf{v}} = [\tilde{v}_1\ \tilde{v}_2]^T$, $\mathbf{v_c} = [v_{1c}\ v_{2c}]^T$. Differentiating both sides of $\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{v_c}$, multiplying by $\mathbf{M}$ and using the last equation of (9) result in

$$\mathbf{M}\dot{\tilde{\mathbf{v}}} = -\mathbf{D}\tilde{\mathbf{v}} + \boldsymbol{\Phi}\boldsymbol{\Theta}_2 + \boldsymbol{\tau} \tag{18}$$

where the regression matrix $\boldsymbol{\Phi}$ and the unknown parameter vector $\boldsymbol{\Theta}_2$ are defined as shown in (19) at the bottom of the page, with

$$\begin{aligned} \Omega_{i1} = &\frac{\partial v_{ic}}{\partial x_e}f_x + \frac{\partial v_{ic}}{\partial y_e}\left(-ku_{1d}\varpi_1^{-1} + f_y\right) \\ &+ \frac{\partial v_{ic}}{\partial z_e}f_z + \frac{\partial v_{ic}}{\partial k}\dot{k} + \frac{\partial v_{ic}}{\partial \dot{k}}\ddot{k} + \frac{\partial v_{ic}}{\partial u_{1d}}\dot{u}_{1d} \\ &+ \frac{\partial v_{ic}}{\partial u_{2d}}\dot{u}_{2d} + \sum_{j=1}^{3}\frac{\partial v_{ic}}{\partial \hat{\theta}_{1j}}\dot{\hat{\theta}}_{1j} \\ \Omega_{i2} = &\frac{\partial v_{ic}}{\partial x_e}u_1 + \frac{\partial v_{ic}}{\partial z_e}g_z u_1 \\ \Omega_{i3} = &\frac{\partial v_{ic}}{\partial x_e}y_e u_2 - \frac{\partial v_{ic}}{\partial y_e}x_e u_2 \\ &+ \frac{\partial v_{ic}}{\partial z_e}\left(1-k\varpi_1^{-1}x_e\right)u_2, \qquad i = 1, 2. \end{aligned} \tag{20}$$

To design the actual control input vector $\boldsymbol{\tau}$, we take the Lyapunov function

$$V_3 = V_2 + 0.5\left(\tilde{\mathbf{v}}^{\mathrm{T}}\mathbf{M}\tilde{\mathbf{v}} + \tilde{\boldsymbol{\Theta}}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2^{-1}\tilde{\boldsymbol{\Theta}}_2\right) \tag{21}$$

where the adaptation gain matrix $\boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}_2^{\mathrm{T}}$ is positive definite, $\tilde{\boldsymbol{\Theta}}_2 = \boldsymbol{\Theta}_2 - \hat{\boldsymbol{\Theta}}_2$ with $\hat{\boldsymbol{\Theta}}_2$ being an estimate vector of $\boldsymbol{\Theta}_2$. Differentiating both

$$\begin{aligned} \boldsymbol{\Phi} &= \begin{bmatrix} -v_1u_2 & -v_{1c} & 0 & -\Omega_{11} & -\Omega_{21} & -\Omega_{12} & -\Omega_{22} & -\Omega_{13} & -\Omega_{23} \\ v_2u_2 & 0 & -v_{2c} & -\Omega_{21} & -\Omega_{11} & -\Omega_{22} & -\Omega_{12} & -\Omega_{23} & -\Omega_{13} \end{bmatrix} \\ \boldsymbol{\Theta}_2 &= \left[0.5b^{-2}r^3m_cd \quad d_{11} \quad d_{22} \quad m_{11} \quad m_{12} \quad m_{11}r \quad m_{12}r \quad b^{-1}m_{11}r \quad b^{-1}m_{12}r\right]^T \end{aligned} \tag{19}$$

sides of (21) along the solution of (18) and (17), and choosing the actual control $\boldsymbol{\tau}$ as

$$\begin{aligned}\boldsymbol{\tau} = & -\mathbf{K_2}\tilde{\mathbf{v}} - \boldsymbol{\Phi}\hat{\boldsymbol{\Theta}}_\mathbf{2} \\ & - 0.5\begin{bmatrix}\varpi_1^{-1}x_e + \hat{\theta}_{13}z_e g_z + \left(1 - k\varpi_1^{-1}x_e\right)z_e \\ \varpi_1^{-1}x_e + \hat{\theta}_{13}z_e g_z - \left(1 - k\varpi_1^{-1}x_e\right)z_e\end{bmatrix} \\ \dot{\hat{\boldsymbol{\Theta}}}_\mathbf{2} = & \boldsymbol{\Gamma}_\mathbf{2}\boldsymbol{\Phi}^\mathbf{T}\tilde{\mathbf{v}}\end{aligned} \tag{22}$$

where $\mathbf{K_2} = \mathbf{K_2^T} > 0$, result in

$$\begin{aligned}\dot{V}_3 = & -k_1\varpi_1^{-1}x_e^2 - ku_{1d}\varpi_1^{-2}y_e^2 - k_2 z_e^2 \\ & -\tilde{\mathbf{v}}^\mathbf{T}(\mathbf{D} + \mathbf{K_2})\tilde{\mathbf{v}} + r^{-1}\varpi_1^{-1}y_e f_y.\end{aligned} \tag{23}$$

It is noted from (10) that $|r^{-1}\varpi_1^{-1}y_e f_y| \leq \varepsilon\varpi_1^{-2}u_{1d}^2 y_e^2 + r^{-2}\varepsilon^{-1}z_e^2$ with $\varepsilon$ being a positive constant. Substituting $k(t)$ in (8) into (23) yields

$$\begin{aligned}\dot{V}_3 \leq & -k_1\varpi_1^{-1}x_e^2 - u_{1d}^2\varpi_1^{-2}(\lambda_1 - \varepsilon)y_e^2 - \lambda_2\cos(\lambda_3 t)u_{1d}\varpi_1^{-2}y_e^2 \\ & -(k_2 - r^{-2}\varepsilon^{-1})z_e^2 - \tilde{\mathbf{v}}^\mathbf{T}(\mathbf{D} + \mathbf{K_2})\tilde{\mathbf{v}}.\end{aligned} \tag{24}$$

We now present the main result of this note.

*Theorem 1:* Under Assumptions 1 and 2, there exist appropriate design constants $k_2, \lambda_i, i = 12, 3$, such that the adaptive control law (22) forces the mobile robot (1) to globally asymptotically track the virtual vehicle (2).

*Proof:* As required in (8), we first choose the design constants $\lambda_1$ and $\lambda_2$ such that $|k(t)| < 1$, i.e.,

$$\lambda_1 u_{1d}^{\max} + \lambda_2 < 1 \tag{25}$$

where $u_{1d}^{\max}$ denotes the maximum value of $|u_{1d}(t)|$. From (24), $\lambda_1$ and $k_2$ are chosen such that

$$\lambda_1 > \varepsilon \quad k_2 > r_{\min}^{-2}\varepsilon^{-1} \tag{26}$$

where $r_{\min}$ denotes the minimum value of $r$. We now consider each case of Assumption 1.

*Cases C1 and C2:* From (24), we have

$$\begin{aligned}\dot{V}_3 \leq & -k_1\varpi_1^{-2}x_e^2 - u_{1d}^2\varpi_1^{-2}(\lambda_1 - \varepsilon)y_e^2 + |\lambda_2 u_{1d}| \\ & -(k_2 - r^{-2}\varepsilon^{-1})z_e^2 - \tilde{\mathbf{v}}^\mathbf{T}(\mathbf{D} + \mathbf{K_2})\tilde{\mathbf{v}}.\end{aligned} \tag{27}$$

It is noted that in these cases, either (3) or (4) hold. By integrating both sides of (27), it is seen that $V_3(t) \leq \pi_3(\bullet)$ with $\pi_3(\bullet)$ being a class-$K$ function of $\|\mathbf{X}_e(t_0)\|$ with $X_e := [x_e, y_e, z_e, \tilde{\mathbf{v}}, \tilde{\theta}_{1i}, \tilde{\boldsymbol{\Theta}}_\mathbf{2}]^T$. An application of Barbalat's lemma in [15] to (27) shows that $\lim_{t\to\infty}(x_e(t), z_e(t), \tilde{\mathbf{v}}(t)) = 0$. Also from $\tilde{u}_1 = 0.5(\tilde{v}_1 + \tilde{v}_2)$, $\tilde{u}_2 = 0.5(\tilde{v}_1 - \tilde{v}_2)$, we have $\lim_{t\to\infty}(\tilde{u}_1(t), \tilde{u}_2(t)) = 0$. Since $V_3(t) \leq \pi_3(\bullet)$, we have that $y_e(t)$, $\tilde{\theta}_{1i}$, $i = 1, 2, 3$, and $\tilde{\boldsymbol{\Theta}}_\mathbf{2}$ are bounded. Hence, $\hat{\theta}_{1i}, i = 1, 2, 3$, and $\hat{\boldsymbol{\Theta}}_\mathbf{2}$ are bounded as well.

To prove that $\lim_{t\to\infty} y_e(t) = 0$, we substitute $u_1 = u_{1c} + \tilde{u}_1$ and $u_2 = u_{2c} + \tilde{u}_2$ with $u_{1c}$ and $u_{2c}$ being given in (13) and (16) to the first equation of (9) to have

$$\begin{aligned}\dot{x}_e = & -krx_e + \Omega(\bullet) \\ \Omega(\bullet) := & r\tilde{\theta}_{11}f_x + r\tilde{u}_1 + \frac{ry_e}{b\left(1 - k\varpi_1^{-1}x_e\right)} \\ & \times\left(-k_2 z_e^2 - \hat{\theta}_{12}f_z - \hat{\theta}_{13}g_z(-k_1 x_e - \hat{\theta}_{11}f_x)\right) \\ & + \frac{r}{b}y_e\tilde{u}_2.\end{aligned} \tag{28}$$

Applying [10, Lemma 2] to (28) yields $\lim_{t\to\infty}\Omega(\bullet) = 0$. From (10), $\lim_{t\to\infty} u_{1d}(t) = 0$ and $\lim_{t\to\infty}(x_e(t), z_e(t), \tilde{u}_1(t), \tilde{u}_2(t)) = 0$, it is readily shown that $\lim_{t\to\infty}\Omega(\bullet) = 0$ is equivalent to

$$\lim_{t\to\infty}\left(y_e(t)\left[-u_{2d} + \frac{(\lambda_1\dot{u}_{1d} - \lambda_2\lambda_3\sin(\lambda_3 t))\,y_e(t)}{\sqrt{1 + \left(1 - (\lambda_2\cos(\lambda_3 t))^2\right)y_e^2(t)}}\right]\right) = 0. \tag{29}$$

To show that $\lim_{t\to\infty} y_e(t) = 0$ based on (29), we investigate C1 and C2 separately.

For case C1, since $\lim_{t\to\infty}(u_{2d}(t), \dot{u}_{1d}(t)) = 0$, we can equivalently write (29) as

$$\lim_{t\to\infty}\left(\lambda_2\lambda_3 y_e^2\sin(\lambda_3 t)\right) = 0. \tag{30}$$

From (27), we have $(d/dt)(V_3 - \int_0^t |\lambda_2 u_{1d}(\tau)|d\tau) \leq 0$, which implies that $V_3 - \int_0^t |\lambda_2 u_{1d}(\tau)|d\tau$ is nonincreasing. Since $V_3$ is bounded from below by zero, $V_3$ tends to a finite nonnegative constant depending on $\|\mathbf{X_e}(t_0)\|$. This implies that the limit of $|y_e(t)|$ exists and is finite, say $l_{y_e}$. If $l_{y_e}$ were not zero, there would have existed a sequence of increasing time instants $\{t_i\}_{i=1}^\infty$ with $t_i \to \infty$, such that both of the limits of $\sin(\lambda_3 t_i)$ and $|y_e(t_i)|$ are not zero, which is impossible because of (30) for any $\lambda_2 > 0$ and $\lambda_3 > 0$. Hence, $l_{y_e}$ must be zero, i.e., $\lim_{t\to\infty} y_e(t) = 0$ in this case.

For case **C2**, since $\lim_{t\to\infty}\dot{u}_{1d}(t) = 0$, we can equivalently write (29) as

$$\lim_{t\to\infty}\left(y_e(t)\left[-u_{2d} + \frac{\lambda_2\lambda_3\sin(\lambda_3 t)y_e(t)}{\sqrt{1 + \left(1 - (\lambda_2\cos(\lambda_3 t))^2\right)y_e^2(t)}}\right]\right) = 0. \tag{31}$$

Since $|y_e(t)/\sqrt{1 + (1 - (\lambda_2\cos(\lambda_3 t))^2)y_e^2(t)}| \leq 1/\sqrt{1 - \lambda_2^2}$ and $|u_{2d}(t)| \geq \mu_{22} > 0$ in this case, if we choose any $\lambda_2 > 0$ and $\lambda_3 > 0$ such that

$$\frac{\lambda_2\lambda_3}{\sqrt{1 - \lambda_2^2}} < \mu_{22} \tag{32}$$

then, (31) implies that $\lim_{t\to\infty} y_e(t) = 0$. It is noted again that in this case if we do not assume that $u_{2d}$ is not of sinusoid signal, the square bracket in (31) might be zero at certain time instants $t_i$ with $y_e(t_i) \neq 0$ as noted in Remark 1. Consequently, one cannot prove $\lim_{t\to\infty} y_e(t) = 0$ based on (31).

*Case C3:* In this case, we rewrite (24) as

$$\begin{aligned}\dot{V}_3 \leq & -k_1\varpi_1^{-1}x_e^2 - \left(u_{1d}^2(\lambda_1 - \varepsilon) - \lambda_2|u_{1d}|\right)\varpi_1^{-2}y_e^2 \\ & -(k_2 - r^{-2}\varepsilon^{-1})z_e^2 - \tilde{\mathbf{v}}^\mathbf{T}(\mathbf{D} + \mathbf{K_2})\tilde{\mathbf{v}}.\end{aligned} \tag{33}$$

Therefore, if we choose the design constants $\lambda_1$ and $\lambda_2$ such that

$$u_{1d}^2(\lambda_1 - \varepsilon) - \lambda_2|u_{1d}| \geq \mu_4 \tag{34}$$

where $\mu_4$ is a strictly positive constant, (33) is equivalent to

$$\begin{aligned}\dot{V}_3 \leq & -k_1\varpi_1^{-1}x_e^2 - \mu_4\varpi_1^{-2}y_e^2 \\ & -(k_2 - r^{-2}\varepsilon^{-1})z_e^2 - \tilde{\mathbf{v}}^\mathbf{T}(\mathbf{D} + \mathbf{K_2})\tilde{\mathbf{v}}.\end{aligned} \tag{35}$$

Integrating both sides of (35) and using Barbalat's lemma shows that $\lim_{t\to\infty}(x_e(t), y_e(t), z_e(t), \tilde{\mathbf{v}}(t)) = 0$. From (5), one can write (34) as

$$\lambda_2 \leq \frac{\left((\lambda_1 - \varepsilon)\mu_3^2 - \mu_4\right)}{u_{1d}^{\max}}. \tag{36}$$

In summary, the design constants $\lambda_1$, $\lambda_2$, $\lambda_3$ and $k_2$ are chosen such that (25), (26), (32) and (36) hold.

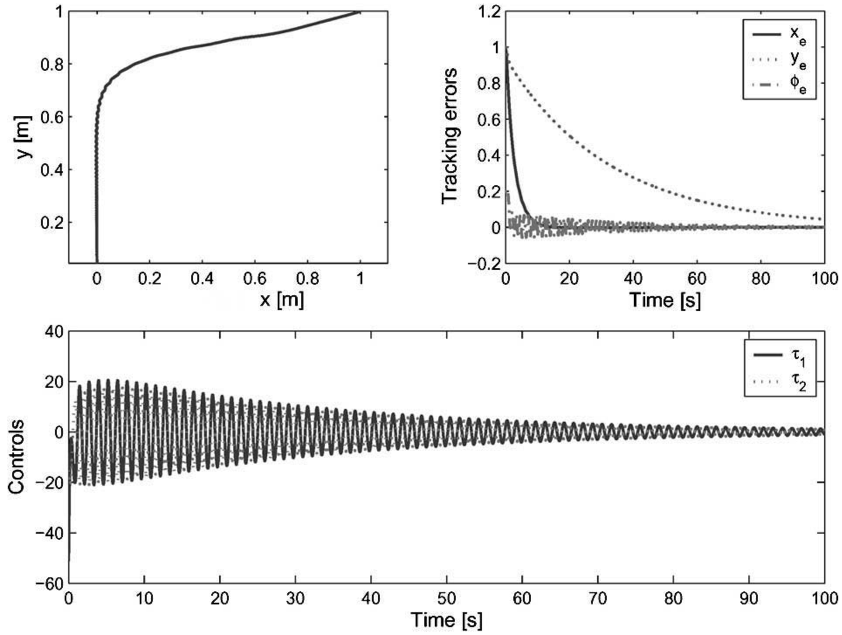


Fig. 2. Simulation results subject to C1. Top left: robot position in (x,y) plane. Top right: tracking errors. Bottom: controls.

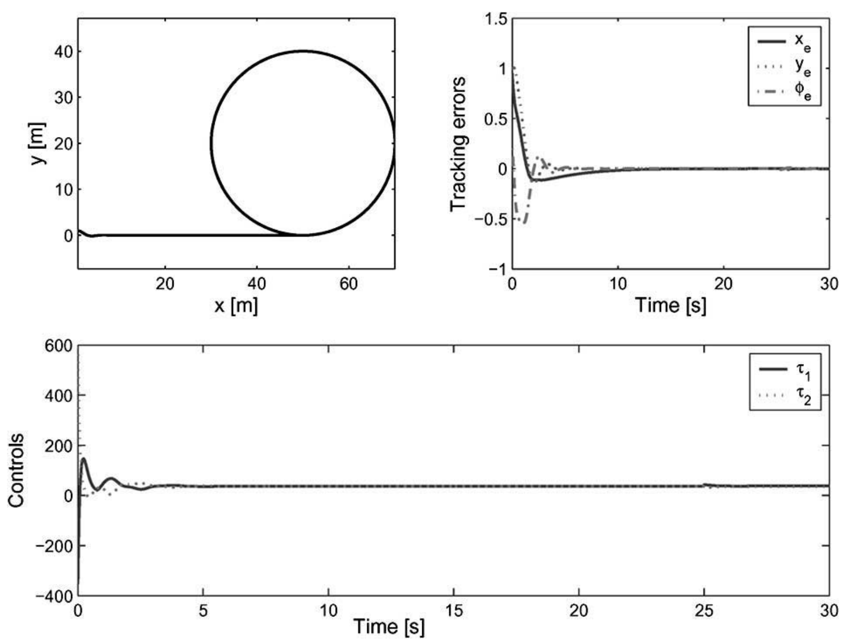


Fig. 3. Simulation results subject to C3. Top eft: robot position in (x,y) plane. Top right: tracking errors. Bottom: controls.

## IV. Simulations

Due to space limitation, we only simulate cases C1 and C3. The physical parameters are taken from [12]: $b = 0.75, d = 0.3, r = 0.15$, $m_c = 30$, $m_w = 1$, $I_c = 15.625$, $I_w = 0.005$, $I_m = 0.0025$, and $d_{11} = d_{22} = 5$. The reference velocities are chosen as: For case C1: $u_{1d} = u_{2d} = 0$; for case C3: $u_{1d} = 2$, $u_{2d} = 0$ for the first 20 s and $u_{1d} = 2$, $u_{2d} = 0.1$ for the rest. The initial conditions are picked as: $(\boldsymbol{\eta}^{\mathbf{T}}, \mathbf{v}^{\mathbf{T}}) = ((1, 1, 0.2), (0, 0))$, $(x_d, y_d, \phi_d) = (0, 0, 0)$. We take all of initial values of the parameter estimates to be 75% of their true values. Based on Theorem 1, control and adaptation gains are chosen as $k_1 = 2$, $k_2 = 5$, $\mathbf{K_2} = \mathrm{diag}(2, 2)$, $\lambda_1 = 0.4$, $\lambda_2 = 0.05$, $\lambda_3 = 4$, $\gamma_{1i} = 2$, $i = 1, 2, 3$, $\boldsymbol{\Gamma_2} = \mathrm{diag}(2)$. It is verified that the above choice satisfies requirements in Theorem 1. Results are plotted in Figs. 2 and 3. Slow convergence of the errors in the case C1 is a well-know effect when using smooth time-varying controllers [4], [7]. Note that we only plot the tracking errors and control inputs in the case C3 for the first 30 s.

## V. Conclusion

A time-varying global adaptive controller has been presented to solve both tracking and stabilization for mobile robots simultaneously at the torque level. The key to success of our proposed control design is the coordinate transformation (8). Current work is underway to extend our proposed methodology to a class of mechanical systems; see [16] for earlier results on adaptive control of a class of nonholonomic systems.

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# Multivariable Adaptive Control Using High-Frequency Gain Matrix Factorization

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***Abstract*—In this note, we extend the application of a less restrictive condition about the high-frequency gain matrix to design stable direct model reference adaptive control for a class of multivariable plants with relative degree greater than one. The new approach is based on a control parametrization derived from a factorization of the high-frequency gain matrix $K_p$ in the form of a product of three matrices, one of them being diagonal. Three possible factorizations are presented. Only the signs of the diagonal factor or, equivalently, the signs of the leading principal minors of $K_p$, are assumed known.**

***Index Terms*—Direct adaptive control, high-frequency gain, matrix factorization, model reference adaptive control (MRAC), multiple-input–multiple-output (MIMO) systems.**

## I. INTRODUCTION

The problem of adaptive control of linear multiple-input–multiple-output (MIMO) systems has been focused by many authors for almost two decades for both continuous and discrete-time systems. For continuous-time systems, the case considered in this note, the main approaches are discussed in [8], [10], and [12] (see also references therein). One difficulty about the existing methods is that, when using direct adaptive control, one has to assume stringent prior knowledge or constraints on the so called *high-frequency gain matrix* (HFGM) [1]. In the case of indirect adaptation algorithms, one has to avoid the singularity of certain estimated matrices. In [3], this is done with the help of hysteresis and the knowledge of some upper bound of the norm of $K_p$. The more general adaptive scheme of [15] requires very little prior knowledge of the plant, however, the corresponding algorithm is based on a large number of estimators to be run in parallel with some switching logic, this being admittedly not practical.

Considering direct adaptive schemes, in [8] it is assumed that a matrix $S_p$ is known such that $K_pS_p = (K_pS_p)^T > 0$. However, this symmetry condition is too restrictive. Recently, in [11] it was shown that the weaker assumption $K_pS_p + (K_pS_p)^T > 0$ is sufficient for global stabilization and global tracking. A noncertainty equivalent adaptive scheme stemming from the application of the immersion and invariance (I & I) approach was employed. The result, however, was restricted to plants with relative degree $n^* = 1$ and does not seem easily extendable to the more general case considered here.

The essential feature of the recent parametrizations stemming from some $K_p$ factorization is that they focus on the elements whose signs (or lower bounds) need to be known. While for the parametrization of [15] a lower bound is assumed to be known for each diagonal entry of an upper triangular factor $U$ of $K_p$, in [2], [4], [5], and [16], the common

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