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On the stabilization of nonholonomic systems

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Abstract

In the present paper we discuss the issue of local (global) stabilizability of non-holonomic systems. We show that if a nonholonomic system with less inputs as states, presents certain geometric features, mainly the existence of a controlled invariant distribution, and if it is non continuous, it is possible to design a locally (globally) stabilizing continuously differentiable (smooth) control law around a point of discontinuity. The main contribution of the paper rests on a sufficient condition for stabilizability of noncontinuous non-holonomic systems and on the use of a non smooth coordinate transformation to overcome the obstruction of stabilizability contained in Brockett's theorem.

1. Introduction

The feedback stabilization of nonlinear systems has occupied a central role in nonlinear systems literature for at least three decades. One of the most challenging topic in this field is the design of a local (global) stabilizing control law for nonholonomic systems with more degrees of freedom than controls. As pointed out in the famous paper of Brockett [1], such systems cannot be stabilized by continuously differentiable, time invariant control laws, even if they are strongly accessible. Therefore, a considerable effort has been done in order to find discontinuous or time varying feedback laws, see [2–8] for a complete survey on the topic.

In the present paper we assume an opposite point of view: we consider the problem of stabilization of nonholonomic discontinuous systems via smooth state feedback, and we bridge our approach, with the one adopted by other authors, using non smooth coordinate transformation in the state space.

We present a sufficient condition for local (global) stabilizability of non-holonomic systems having n states and $n-p$ inputs (with $p>0$). This is in contrast with Brockett theorem, yielding necessary and sufficient condition of stabilizability, thus we show that, if the hypotheses of our theorem are met then the system under consideration violates the regularity assumptions of Brockett: *i.e.* it is a discontinuous system.

As natural systems are seldom not continuous it could seem that our discussion has only a mathematical interest. However, every continuous system can be, via non smooth change of coordinate (the so called σ process, see [9] where the σ process is used to resolve vector field singularities), transformed into a discontinuous one.

Thus, C^1 nonholonomic systems that cannot be stabilized via continuously differentiable control (because of the obstruction of stabilizability contained in Brockett's theorem) can be transformed (using the σ process) into discontinuous ones. Such systems may be, if certain conditions are met, stabilized by smooth control (in the new coordinates). Note that in the initial coordinates the obtained control law is not continuous (as it must be).

The σ process has been used in geometric theory of differential equations to resolve singularities of vector fields around an equilibrium point [9]; it is mainly a special apparatus to observe fine details of dynamical systems near a fixed point. It consists of iterated non smooth coordinate transformation whose goal is to *increase the resolution*, like a microscope, in a given set.

Our approach (non smooth coordinate transformation - synthesis of a continuous control - inverse transformation) can be considered too artificial; however we believe that the design of continuously differentiable (smooth) control law is more natural than the design of discontinuous ones. For this reason, we think that the use of a non smooth coordinate transformation can result helpful in solving, easily and in a natural way, a certain number of *difficult* control problems.

The paper is organized as follows. In section II we recall the theorem of Brockett on stabilizability of non holonomic systems. In section III we present our main results, *i.e.* the geometric properties of a special class of non holonomic systems. Section IV deals with the use of non smooth coordinate transformation (the σ process); section V, VI and VII contain an example, possible extensions of our theory and concluding remarks.

2. The theorem of Brockett

The theorem of Brockett [1] provides necessary and sufficient conditions for feedback stabilizability of nonholonomic systems.

Theorem 1 *Let $\dot{q} = g(q)u$ be given, with $g(q_0)u_0 = 0$, $g(\cdot)$ continuously differentiable in a neighborhood of q_0 and $g(q)$ being a distribution of constant rank in a neighborhood of q_0.*

Then there exists a continuously differentiable control law which makes (q_0, u_0) asymptotically stable iff $dim(q) = dim(u)$.

The essence of Theorem 1 is that the only *interesting* nonholonomic systems are those for which the distribution $g(q)$ drops dimension precisely at q_0 or is not continuously differentiable at q_0. In such cases we cannot infer anything about the existence of a C^1 (smooth), time invariant, state feedback control law.

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3. Main results

This section contains our main results: we find sufficient conditions for stabilizability of a system having n states, $n-p$ inputs ($p>0$) and no drift term. Then, we show that if such conditions are fulfilled, the given system has a certain type of discontinuity.

Before stating the main theorem we need the following preliminary result; it mainly points out some geometrical properties of a given class of nonlinear nonholonomic systems.

Lemma 1 *Consider the system*

$$\begin{array}{rcl} \dot{x}_1 & = & g_{11}(x_2)u_1 \\ \dot{x}_2 & = & g_{21}(x_1,x_2)u_1 + g_{22}(x_1,x_2)u_2 \end{array} \tag{1}$$

with $x=(x_1,x_2)\in\mathbb{R}^n$, $x_1\in\mathbb{R}^p$, $x_2\in\mathbb{R}^{n-p}$, $u_1\in\mathbb{R}^p$, $u_2\in\mathbb{R}^m$, $m+p<n$ and

$$g(x)=\begin{bmatrix} g_{11}(x_2) & 0 \\ g_{21}(x_1,x_2) & g_{21}(x_1,x_2)\end{bmatrix}$$

of constant rank in a neighborhood of the point of interest ($g_{ij}(x_1,x_2)$ are matrices of appropriate dimensions). Then:

1) the $n-p$ dimensional distribution

$$\Delta=\begin{bmatrix} 0 \\ I_{n-p}\end{bmatrix} \tag{2}$$

is locally controlled invariant (I_s denotes the identity matrix of dimension s);

2) the $n-p$ dimensional manifold $\mathcal{M}=\{x\in\mathbb{R}^n : x_1=0\}$ can be rendered invariant using any smooth control law $u=(u_1,u_2)$ such that

$$u_1(0,x_2)=0 \tag{3}$$

3) the projection of the motion $x(t)$ on the x_1 plane is locally (globally) asymptotically stable around $x_1=0$ if there exists a positive definite matrix X such that

$$x_1^T X g_{11}(x_2)u_1(x_1,x_2)<0$$

(with $x_1\neq 0$) and bounded for all x_1 in a neighborhood of $x_1=0$ (for all $x_1\in\mathbb{R}^p$) and for all x_2.

Remark 1 *System (1) is not really a special nonholonomic system. In fact, as discussed in [10], under mild hypotheses and with a proper choice of coordinates, it is always possible to write the kinematic equations of a nonholonomic system in the form of equations (1) with*

$$g_{11}(x_1)=I_p \qquad g_{21}=\begin{bmatrix} 0 \\ \star(x_1,x_2)\end{bmatrix} \qquad g_{22}=\begin{bmatrix} I_m \\ \star(x_1,x_2)\end{bmatrix}$$

where $\star(x_1,x_2)$ denotes a generic function of x_1 and x_2. This form is known as normal form [10], however in the sequel we do not treat nonholonomic systems in such a form, as many real systems are better described in their own coordinates yielding a system not in normal form.

Using Lemma 1 we can prove:

Theorem 2 *The continuously differentiable (smooth) control*

$$u=u(x_1,x_2)=\begin{bmatrix} u_1 \\ u_2\end{bmatrix}$$

with $u_2\in\mathbb{R}^{n-2p}$ and with u_1 satisfying equation (3), locally (globally) asymptotically stabilizes the system (1) around $(0,0)$ if

(A) $u(x_1,x_2)$ locally (globally) uniformly asymptotically stabilizes

$$\dot{x}_2=g_{21}(x_1,x_2)u_1+g_{22}(x_1,x_2)u_2 \tag{4}$$

for every $x_1(t)$ (this means that the $n-p$ dimensional system (4) parameterized by x_1 must be locally (globally) asymptotically stable for every possible evolution of the parameter).

(B) $x_1^T X g_1(x_2)u_1(x_1,x_2)<0$ (with $x_1\neq 0$ and $X>0$) and bounded for all x_1 in a neighborhood of $x_1=0$ (for all $x_1\in\mathbb{R}^p$) and for all x_2.

Remark 2 *Note that hypothesis (A) implies that the closed loop system (4) has a local (global) Lyapunov function independent of x_1. This condition is surely fulfilled if, for a certain system, there exists a control $u(x_1,x_2)$ such that the vector field $g_{21}(x_1,x_2)u_1(x_1,x_2)+g_{22}(x_1,x_2)u_2(x_1,x_2)$ is independent of x_1 and is locally (globally) asymptotically stable.*

Assume now that, for a given system, hypothesis (A) is fulfilled and in addition the $n-p$ dimensional distribution

$$\Delta_2=[g_{21},g_{22}]$$

is continuously differentiable (smooth) as a function of x_2 around $x_2=0$ (for all x_2).

By Brockett theorem the distribution Δ_2 must be regular (*i.e.* must have constant dimension) around $x_2=0$ for every x_1; therefore the $(n-p)$ dimensional vector

$$g_{21}(0,x_2)u_1(0,x_2)$$

must be non zero. This implies that some entries of the matrix $g_{21}(x_1,x_2)$ are of the form

$$g_{21}^{ij}(x_1,x_2)=\frac{\tilde{g}_{21}^{ij}(x_1,x_2)}{\psi(x_1)}$$

with $\tilde{g}_{21}^{ij}(0,x_2)\neq 0$ and $\psi(x_1)$ homogeneous function of degree $\alpha>0$.

Thus the given system has a discontinuity at $x_1=0$ and the regularity hypotheses of Brockett theorem are not fulfilled.

4. The σ process

In the present section we discuss the use of non smooth coordinate changes to transform continuous systems into discontinuous ones.

For the study of fine details of every mathematical object near a given point, it is possible to device a special apparatus. We consider a choice of coordinate system in which, to a small displacement near a fixed point, there corresponds a great change in coordinates.

The polar coordinates system possesses such property; however the cartesian to polar transformation requires transcendental functions; therefore, when not necessary, we should avoid the use of polar coordinates, using another procedure: the so called σ process [9].

Mainly it consists of a non smooth rational transformation; but with abuse of notation we denote with the term σ process every non smooth coordinate transformations possessing the property of *increasing the resolution* around a given point.

Possible examples of σ processes are the already mentioned cartesian to polar transformation, and the following rational transformation in the plane

$$w = \frac{y^\alpha}{x^\beta} \qquad z = x$$

with $\alpha > 0$, $\beta > 0$ and defined for $x \neq 0$.
Straightforward calculations show that a continuous system is transformed by the σ process into a discontinuous one.

5. An example

In the present section we use the developed theory to design a discontinuous stabilizing control law for a real system: a mobile robot. Such problem has been already treated, with various technique, by several authors [2, 11, 4, 6]. According with our setup, we consider the kinematic equations of a non-holonomic mobile robot in polar coordinates (we apply a σ process):

$$\dot{q} = g(q)u$$

where $q = [\rho, \alpha, \phi]^T$, $u = [u_1, u_2]^T$ and

$$g(q) = \begin{bmatrix} -\cos\alpha & 0 \\ \frac{\sin\alpha}{\rho} & -1 \\ 0 & -1 \end{bmatrix} \tag{5}$$

See Fig. 1 for further details on the coordinate transformation.

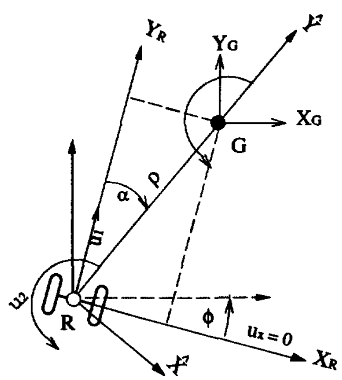

Figure 1: Robot kinematics

It is readily seen that the state feedback control $u = Kx$ with

$$K = \begin{bmatrix} k_\rho & 0 & 0 \\ 0 & k_\alpha & k_\phi \end{bmatrix} \tag{6}$$

and

$$\begin{array}{c} k_\rho > 0 \\ k_\phi < 0 \\ k_\alpha + k_\phi - k_\rho > 0 \end{array} \tag{7}$$

locally stabilizes system (5) and satisfies hypotheses (A) and (B) of Theorem 2. Actually it is even possible to show that for a certain choice of the state feedback matrix (6) the system is globally asymptotically stable [12].

In the original coordinates the control law is described by equations of the form

$$\begin{array}{rcl} u_1 & = & k_\rho\sqrt{x^2+y^2} \\ u_2 & = & k_\alpha atan(-\frac{x}{y}) + k_\phi\phi \end{array} \tag{8}$$

thus it is not defined at $x = y = 0$. To avoid such a problem we can just set

$$u_1(0,0,\phi) = 0 \qquad u_2(0,0,\phi) = k_\phi\phi$$

Figures 2 and 4 show the behavior of the robot with the proposed control law.

Remark 3 *We stress that the synthesis of the control law is extremely simple in the transformed coordinate system, and also the local stability issue can be easily checked. Thus, the non smooth coordinate transformation allows us to design and evaluate noncontinuous control laws in a straightforward way.*

6. Further extensions

In the present section we propose a second example, which can be used as starting point to extend our considerations to systems with drift.
Again the example deals with a real system, *i.e.* we consider the dynamical model of a non-holonomic mobile robot in polar coordinates (remember that, in cartesian coordinates, it is not stabilizable by continuously differentiable feedback [2]):

$$\begin{array}{rcl} \dot{\eta}_1 & = & u_1 \\ \dot{\eta}_2 & = & u_2 \\ \dot{\rho} & = & -\eta_1\cos\alpha \\ \dot{\alpha} & = & \frac{\eta_1}{\rho}\sin\alpha - \eta_2 \\ \dot{\phi} & = & -\eta_2 \end{array} \tag{9}$$

This system does not meet the regularity assumption of Brockett theorem [1] on the stabilization of nonlinear systems and in fact it can be locally stabilized. Using the result in [13, Theorem 19.2] it is possible to show that the state feedback control

$$u_1 = -k_{z_1}(\eta_1 - k_\rho\rho) - k_\rho(\eta_1 - k_\rho\rho)\cos\alpha$$

$$u_2 = k_\alpha\frac{\eta_1 - k_\rho\rho}{\rho}\sin\alpha - (k_\alpha + k_\phi + k_{z_2})(\eta_2 - k_\alpha\alpha - k_\phi\phi)$$

with $k_{z_i} > 0$ (for $i = 1,2$) and $[k_\rho, k_\phi, k_\alpha]$ fulfilling conditions (7) locally asymptotically stabilizes the system (9). As in the previous example, the use of a non smooth change of coordinates yields an easy solution of a *difficult* control problem.
Figure 3 shows the floor trajectories of the mobile robot with the kinematic and dynamic based control laws. The gains k_ρ, k_α and k_ϕ are equal in both control laws, while $k_{z_i} = 3k_\rho$ (for $i = 1,2$). Figure 5 shows the behavior of the mobile robot with the above control law.

7. Conclusions and Acknowledgments

In the present paper we have discussed the existence of a stabilizing controller for nonholonomic systems with n states and $n-p$ inputs ($p > 0$). In particular we have shown that if the system has a discontinuity point, it is

possible to render such point locally (globally) asymptotically stable by mean of a continuously differentiable, state feedback control law. We point out that this stabilizability property is valid only for the singular point, while for any other points in a neighborhood of it the theorem of Brockett prevents the existence of a continuous stabilizing controller.
The proposed approach can result helpful in solving a certain number of control problem, *i.e.* when theoretic results prevent the existence of smooth control laws. In particular it has been used in [14] to design discontinuous stabilizing control laws for chained systems and other classes of nonholonomic systems.
The author wish to express his gratitude to Dr. E. Badreddin for having proposed him the first example, which acts as starting point to develop his results, and for is continuous encouragement.

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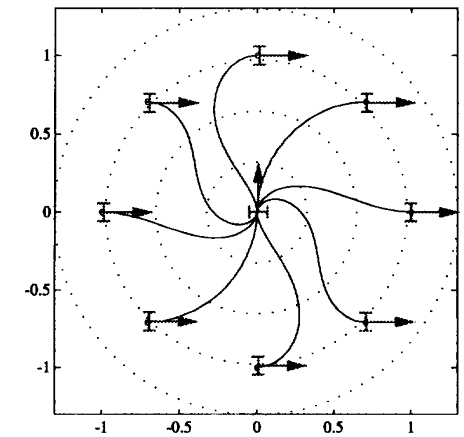

Figure 2: Resulting paths when the robot is initially on the unit circle in the xy-plane. Observe the symmetry of the motions

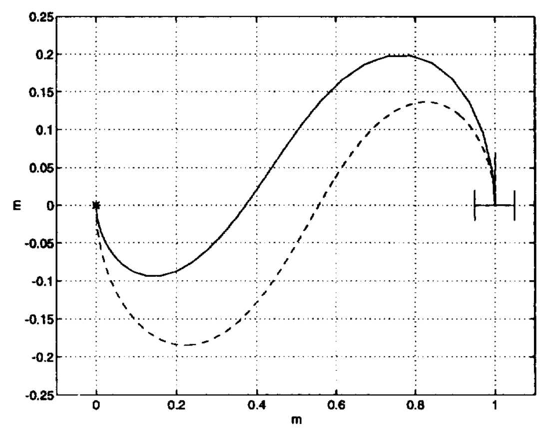

Figure 3: Parallel parking maneuver. Robot trajectory based on the kinematic model (cont.) and on the dynamic one (dashed).

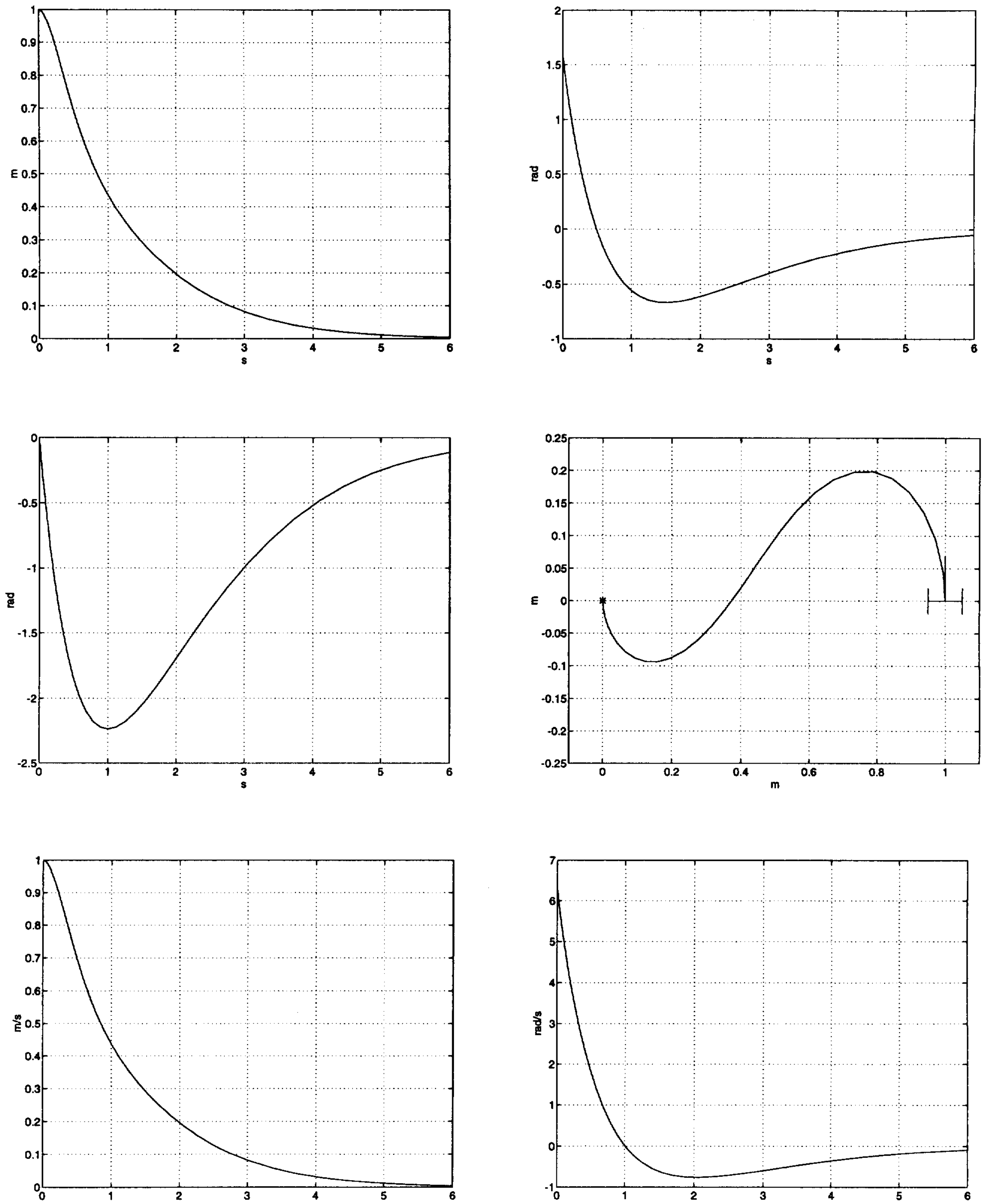

Figure 4: Parallel parking maneuver. Note the exponential convergence. $\rho(t)$ (up-left), $\alpha(t)$ (up-right), $\phi(t)$ (center-left), floor trajectory (center-right), $u_1(t)$ (down-left), $u_2(t)$ (down-right).

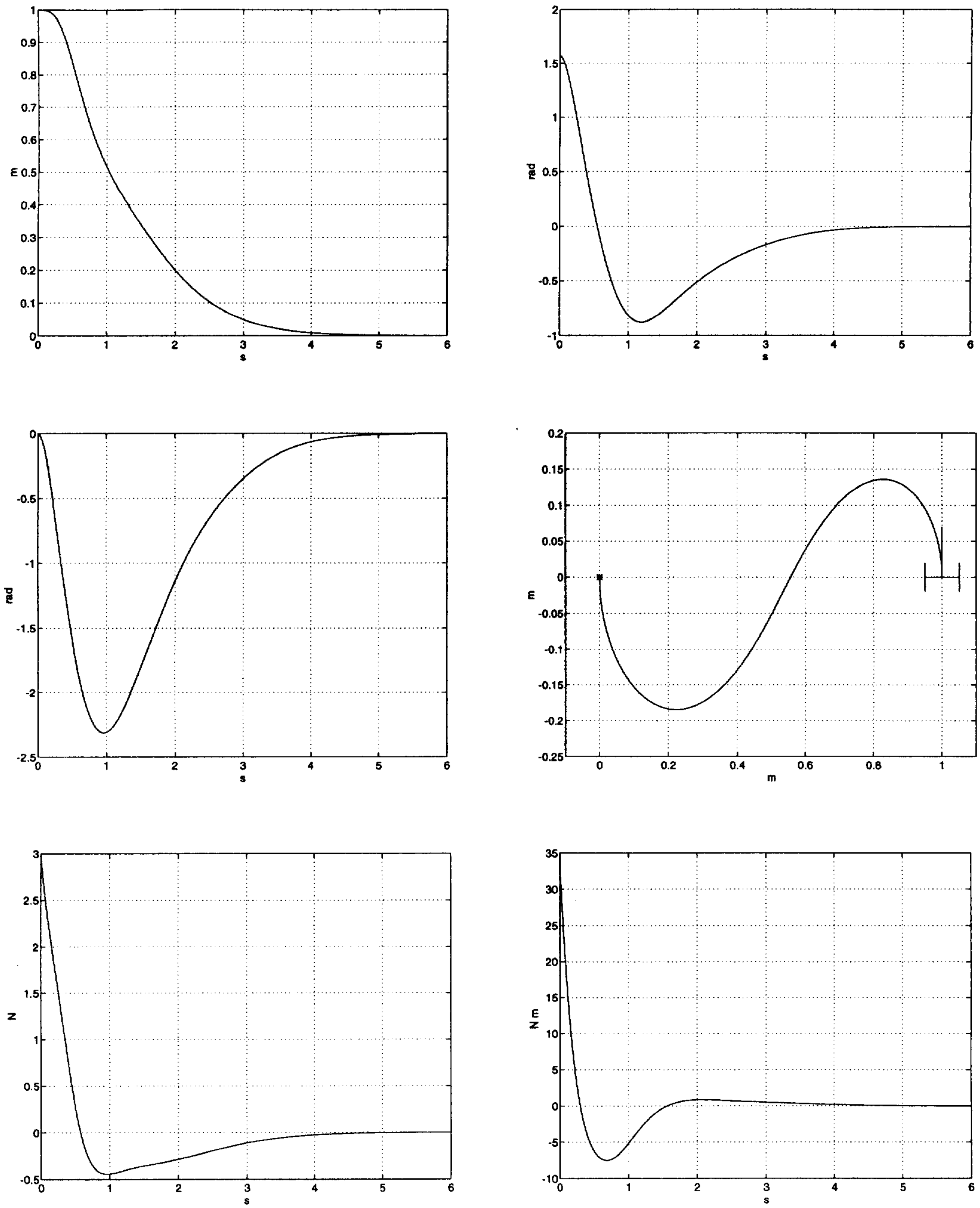

Figure 5: Parallel parking maneuver when torque inputs are applied. Observe the exponential convergence. $\rho(t)$ (up-left), $\alpha(t)$ (up-right), $\phi(t)$ (center-left), floor trajectory (center-right), $u_1(t)$ (down-left), $u_2(t)$ (down-right).