R. J. CAUDILL

Assistant Professor,
Transportation Program,
Princeton Univerisity,
Princeton, N. J.

W. L. GARRARD

Associate Professor,
Department of Aerospace
Engineering and Mechanics
and Center for Control Science,
University of Minnesota,
Minneapolis, Minn.

Vehicle-Follower Longitudinal Control for Automated Transit Vehicles[1]

This paper examines the effects of spacing policy and control system nonlinearities on the dynamic response of strings of automated transit vehicles operating under automatic velocity and spacing control. Both steady-state and transient responses are studied. Steady-state response is examined by a modification of the describing function technique and transient response is studied by Liapunov procedures. It is shown that a nonlinearity commonly encountered in automated transit vehicles, a limiter on acceleration and deceleration, can result in string instabilities even though a linearized analysis indicates that the string is stable. Although this paper is specifically focused on automated transit systems, some of the results obtained also appear to be applicable to strings of automobiles on freeways.

I Introduction

The objective of this paper is to examine basic relationships between spacing policy, control system design, and dynamic response for strings of automated transit vehicles operating at moderate speeds and short headways. Vehicles are assumed to operate under vehicle-follower longitudinal control. In vehicle-follower control, the control system maintains the vehicle velocity at the value specified for the section of guideway which the vehicle is traversing provided the preceding vehicle is distant. If the preceding vehicle is nearby, the control system maintains a specified spacing between the vehicles. If the spacing between vehicles decreases too rapidly, the control system applies emergency braking.

During normal operation, the control system must maintain proper velocity and spacing during (1) constant speed operation, (2) transitions between regions of high- and low-speed operation, (3) merges and diverges at stations and intersections, and (4) stops and starts within stations. Control must be accomplished without subjecting passengers to acceleration or jerk levels which may cause discomfort. In emergency situations, the longitudinal control system must stop or slow vehicles in order to avoid collisions and, in some cases, must control pushing of failed vehicles [1–3].[2]

The desired minimum spacing between vehicles is determined by the spacing policy chosen for system operation [4]. The following three basic spacing policies have been proposed:

1. *Constant Separation:* The command spacing is fixed and does not vary with velocity.
2. *Constant Time-Headway:* The command spacing varies linearly with velocity so that the time headway between vehicles is constant.
3. *Constant Safety-Factor:* The command spacing varies quadratically with velocity so that the separation is proportional to the safe stopping distance.

Each spacing policy has a number of implications in terms of overall system operations [5, 6]; however, in this paper only the relationship between spacing policy and vehicular dynamic response will be examined. Particular attention will be given to the stability of strings of vehicles. Although this paper is specifically focused on automated transit systems, the results obtained for constant-safety factor spacing policy are applicable to strings of automobiles traveling on freeways.

Two types of string stability have been defined. These are steady-state stability and transient stability. Steady-state stability has been widely discussed in the literature [7–9]. A string of vehicles is said to be steady-state stable if steady-state disturbances of either a random or periodic nature are attenuated as they propagate upstream along the vehicle string. Steady-state stability analysis involves examination of only the particular solutions of the differential equations describing the vehicle dynamics. If linear vehicle models are used, Fourier analysis techniques can be used to determine conditions which guarantee steady-state stability. However, as demonstrated in this paper, the presence of nonlinearities may result in string instabilities even though a linearized analysis indicates that the string is stable. To the authors' knowledge no other analytical results have been published on nonlinear string stability even though experimental studies have demonstrated that vehicle dynamics are highly nonlinear [10] and computer simulations have shown that system nonlinearities can cause unstable behavior [11, 12].

[1]This research was supported by the Urban Mass Transit Administration of the U. S. Department of Transportation Under Research and Training Grant No.MN-11-0002 and by a grant from the Graduate School of the University of Minnesota. The opinions expressed are those of the authors and not necessarily those of DOT.

[2]Numbers in brackets designate References at end of paper.

Contributed by the Automatic Control Division for publication in the JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENT, AND CONTROL. Manuscript received at ASME Headquarters September 26, 1977.

A vehicle string which is steady-state stable may exhibit unacceptable transient response when subjected to suddenly applied disturbances and commands. In order to determine transient stability, the complete solution of the differential equations describing the dynamics of the vehicle string must be examined. (Even if all of the characteristic roots for a vehicle string have negative real parts, the string may exhibit unacceptable transient and steady-state dynamic response.) If linear vehicle models are used, the response of the string can be calculated using any one of several techniques for solving linear differential equations. However, even in the linear case, analytical determination and evaluation of the response of a string of vehicles can be a very tedious and time consuming task.

If a constant-separation operating policy is used, a simple kinematic analysis shows that a velocity wave of increasing magnitude will be propagated upstream as the string passes through a speed reduction [11]. If constant-safety-factor or constant-time-headway spacing policies are used, a technique based on Liapunov stability theory can provide a preliminary assessment of transient stability. This technique has considerable computational advantages as it does not require explicit solution of the differential equations describing the vehicle motions. Since the Liapunov technique discussed in this paper is limited to linear systems, final assessment of the dynamic response of vehicle strings must be based on extensive computer simulations.

II Mathematical Models

Consider two adjacent members of a vehicle string as shown in Fig. 1. The positions of the vehicles are

$$X_{k+1} = V_N t + Z_{k+1} \tag{1a}$$

$$X_k = V_N t + \Delta_{k0} + Z_k + L \tag{1b}$$

where

V_N = nominal velocity of string (assumed constant)

Z_k, Z_{k+1} = deviations in the positions of the vehicles from their nominal values

Δ_k = spacing between vehicles (Δ_{k0} is the initial spacing between vehicles)

Δ_{ck} = commanded spacing

a_{ck+1} = commanded acceleration of k + 1st vehicle

L = length of vehicles

s = Laplace operator

The commanded acceleration is assumed to be a linear function of the difference in velocity between the vehicle and its immediate predecessor and the difference between the commanded and actual spacing, that is,

$$a_{ck+1}(s) = H(s)\,(\dot{X}_k(s) - \dot{X}_{k+1}(s)) + G(s)\,(\Delta_{ck}(s) - \Delta_k(s)) \tag{2}$$

where the functions $H(s)$ and $G(s)$ are determined by the control system design and except for constant separation, the command of spacing is a function vehicle velocity. For constant separation

$$\Delta_{ck} = \bar{\Delta}, \text{ a constant,} \tag{3}$$

for constant time-headway

$$\Delta_{ck} = \dot{X}_{k+1}\,h - L \tag{4}$$

where h is the nominal time headway, and for constant safety-factor

$$\Delta_{ck} = K\,D_{\text{stop}} \tag{5a}$$

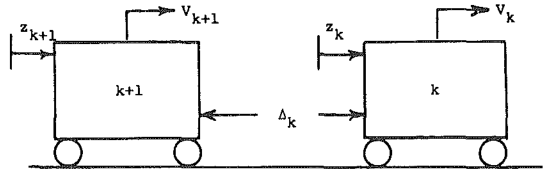

Fig. 1 Vehicle follower model

D_{stop} is the safe stopping distance and K is the safety factor.

In this paper a simplified expression for the stopping distance is assumed

$$D_{\text{stop}} = \dot{X}_{k+1}{}^2/2a_e \tag{5b}$$

where, a_e is the maximum assured emergency deceleration rate. This expression for minimum stopping distance can be modified to include jerk and time delays encountered in detecting a failure and initiating emergency braking [4]. However, the results obtained will be almost identical to those resulting from use of equation (5b).

Expanding Δ_{ck} in a Taylor series about the nominal velocity V_N and neglecting nonlinear terms, yields:

$$\Delta_{ck}(\dot{X}_{k+1}) \simeq \Delta_{ck}(V_N) + C\dot{Z}_{k+1} \tag{6a}$$

where

$$C = \left[\frac{\partial\,\Delta_{ck}}{\partial\,\dot{X}_{k+1}}\right] \text{evaluated at } \dot{X}_{k+1} = V_N$$

For constant separation,

$$C = 0; \tag{7}$$

for constant time-headway,

$$C = h; \tag{8}$$

and for constant safety-factor,

$$C = \frac{KV_N}{a_e} \tag{9}$$

Thus it can be seen that both constant-time-headway and constant-safety-factor spacing policies have the effect of increasing the velocity dependent forces acting on the vehicle For a second-order vehicle model this results in an increase in damping factor.

If the vehicles are initially at their proper positions and velocities, Z_k, Z_{k+1}, $\dot{Z}_k$, and $\dot{Z}_{k+1}$ are all initially zero and $\Delta_{k0} = \Delta_c(V_N)$. Then using equation (2) and (6), the command acceleration can be expressed as

$$a_{ck+1}(s) = H(s)s\,Z_k(s) - (H(s) + C\,G(s))s\,Z_{k+1} + G(s)\,(Z_k - Z_{k+1}) \tag{10}$$

Assuming that the acceleration of the k + 1st vehicle is

$$a_{k+1}(s) = s^2 Z_{k+1}(s) = F(s)\,a_{ck+1} \tag{11}$$

the transfer function between the k + 1st and kth vehicles is

$$\frac{Z_{k+1}}{Z_k} = T(s) = \frac{F(s)\,(sH(s) + G(s))}{s^2 + F(s)\,[s(H(s) + C\,G(s)) + G(s)]} \tag{12}$$

III Steady-State Stability

Using the linearized model given by equation (12), conditions for steady-state stability can be obtained in a straightforward manner. If $s_k(\omega)$ is the spectral density of Z_k then the expected value of $Z_{k+1}{}^2$ is

$$E\,[Z_{k+1}{}^2] = \int_{-\infty}^{\infty} T(-j\omega)\,T(j\omega)\,s_k(\omega)\,d\,\omega. \tag{13}$$

If $T(j\omega)\,T(-j\omega) < 1$ for all values of ω,

then

$$E\,[Z_{k+1}^2] < \int_{-\infty}^{\infty} s_k(\omega)d\,\omega = E\,[Z_k^2] \tag{14}$$

and steady-state disturbances will be attenuated along the vehicle string.

If $H(s)$ and $G(s)$ are constant, and $F(s) = 1$, then

$$T(s) = \frac{s\,H + G}{s^2 + s\,(H + CG) + G} \tag{15}$$

In order to insure $T(j\omega)\,T(-j\omega) < 1$ for all ω, the restriction on the controller gains are

$$\sqrt{G} > H,$$

and

$$\frac{H + C\,G}{2\sqrt{G}} \geq \frac{1}{\sqrt{2}} \tag{16}$$

If $C = 0$ the above conditions cannot be satisfied, consequently, it is impossible to obtain steady-state string stability for vehicles operating under a constant-separation spacing policy by using this particular control system. In reference [11] it is demonstrated by kinematic arguments that a constant-separation spacing policy also results in unstable transient response. For a constant-safety factor spacing policy, C is small if the nominal velocity is low. Thus even if stable dynamic response is obtained at line velocity, instabilities may result at low velocities characteristic of station operations. A method for modifying the constant-safety-factor spacing policy to guarantee steady-state stability for all velocities is discussed in Appendix A. It is impossible to operate at constant headway for velocities less than the length of the vehicle divided by the time headway [11].

The effects of various control laws on steady-state stability may be readily studied using linear vehicle models. However, longitudinal vehicle dynamics are invariably nonlinear and, as demonstrated below, this may result in steady-state instabilities even though a linearized stability analysis indicates stable behavior.

As in the linear case, the criterion for steady-state stability is that given a sinusoidal excitation of the lead vehicle, the amplitude of the response of the following vehicle is less than that of the lead vehicle for all values of frequency. A modification of the describing function technique can be used to analytically investigate steady-state stability. In this paper the effects of a saturation type nonlinearity limiting acceleration and deceleration are examined; however, the techniques used are equally applicable to other nonlinearities.

Acceleration/deceleration limiting may be purposely introduced to insure passenger comfort or may be due to inherent limitations on the forces generated by the propulsion-braking system. The results discussed below are specific to the particular vehicle and control system models used; however, they indicate the importance of including significant nonlinearities in the analysis of string stability.

The block diagram representation for the $k + 1$st vehicle in the string is shown in Fig. 2. F_p represents the propulsive force, m is the mass of the vehicle, y is the input to the saturation nonlinearity. The actual acceleration, $\ddot{Z}_{k+1}$, is limited to $\pm A_{max}$, the maximum service rate.

A modification of the describing function technique can be used to determine the frequency response [13–14].

Following the procedure suggested by Hill [14], let

$$Z_k\,(t) = M_k \sin \omega\, t \tag{17}$$

Assuming $y(t)$ to be a sinusoid of amplitude A and frequency ω,

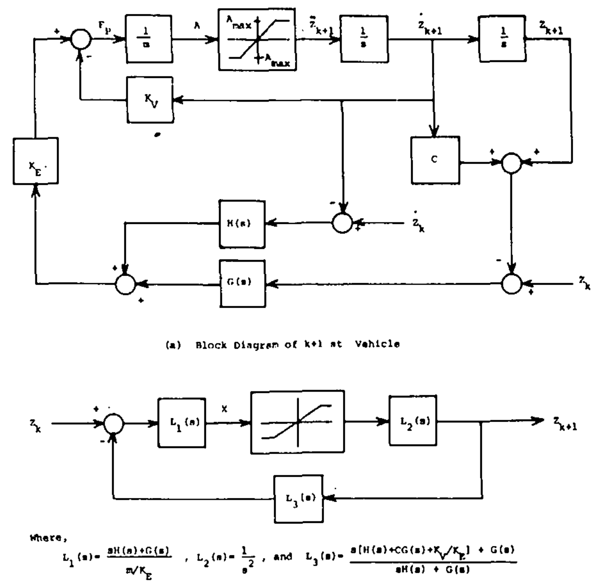

Fig. 2 Block diagrams of (k+1)st vehicle

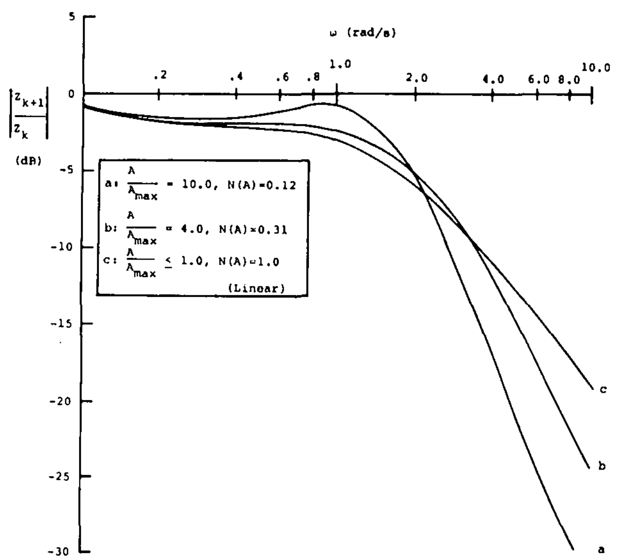

Fig. 3 Frequency response of the k+1st vehicle under constant-safety factor spacing for various levels of nonlinear operation

the transfer function from Z_k to Z_{k+1} becomes:

$$\frac{Z_{k+1}}{Z_k}\,(j\omega, A) = \frac{L_1\,(j\omega)\,N(A)\,L_2\,(j\omega)}{1 + L_1\,(j\omega)\,N(A)\,L_2\,(j\omega)\,L_3\,(j\omega)} \tag{18}$$

where $L_1(s)$, $L_2(s)$, and $L_3(s)$ are given in Fig. 2 and $N(A)$ is the describing function for the nonlinearity. The describing function for a saturation nonlinearity is well documented [13, 15]. The response of the system can be determined for various levels of nonlinear operation by choosing different values for A [16].

The response of a vehicle using a linear induction motor for propulsion and service braking will be examined. Based on previous work, a proportional-plus-integral control on the relative velocity error and proportional-integral-derivative control on the spacing error with a low-pass filter in the spacing loop is used [5, 6, 17]. That is,

$$H(s) = K_p + K_I/s$$

$$G(s) = (G_D\,s + G_p + G_I/s)\;\frac{1}{\tau s + 1} \tag{21}$$

Numerical values for the physical parameters specifying the controlled vehicle and propulsion system and various operating parameters used in this analysis are given in Table 1. These values are typical of many small automated transit systems under development.

Representing $H(j\omega)$ and $G(j\omega)$ as

$$H(j\omega) = \mathrm{Re}\, H + j\, ImH$$

$$G(j\omega) = \mathrm{Re}\, G + j\, ImG$$

and neglecting K_V/K_E, M_k can be expressed as

$$M_k = \frac{A\, N(A)}{\omega^2} \left\{ \frac{\left[\mathrm{Re}G - \omega ImH - \omega C ImG - \dfrac{m\omega^2}{N(A)K_E}\right]^2 + [\omega \mathrm{Re}H + \omega C \mathrm{Re}G + ImG]^2}{(\mathrm{Re}G - \omega ImH)^2 + (ImG + \omega \mathrm{Re}H)^2} \right\}^{1/2} \tag{19}$$

Table 1 Vehicle and control system data

Vehicle		
Mass (M)	2000	Kg
Length (L)	4	m
Aerodynamic drag coefficient (C_D)	.484	Kg/m[a]
Linear Induction Motor		
Resistance of primary (R_1)	.35	ohm
Resistance of secondary (R_2)	.20	ohm
Leakage reactance of primary (L_1)	.0033	henry
Leakage reactance of secondary (L_2)	.0004	henry
Magnetizing reactance (L_m)	.0033	henry
Capacitor (C)	.0021	farad
Synchronous frequency (fs)	60	hz
Pole pitch (P)	0.164	m
K_V	Negligible	
K_E	29.8	nt/volt
Velocity Controller - $H(s)$		
Proportional gain (K_P)	200	volt-s/m
Integral gain (K_I)	3200	volt/m
Spacing Controller - $G(s)$		
Proportional gain (G_P)	4000	volt/m
Integral gain (G_I)	800	volt/m-s
Derivative gain (G_D)	8000	volt-s/m
Filter time constant	5	sec
System Operating Parameters		
Nominal line velocity (V_N)	15	m/s
Service accel/deceleration (A_{max})	0.25	g
Service jerk	0.15	g/s
Emergency deceleration (a_e)	0.6	g
Nominal vehicle spacing (Δ_0)	19.133	m
Nominal time headway (h)	1.54	s
Nominal K-factor (K)	1.0	

[a] Assuming frontal area of 2.5 m² and a nondimensional drag coefficient of 0.3.

Fig. 3 shows the magnitude-frequency response of the $(k + 1)$st vehicle for various values of A for constant-safety-factor operation at $V_N = 15$ m/s. As A increases, the effect of the nonlinearity becomes more pronounced and the ability of the system to attenuate disturbances is decreased. Figs. 4 and 5 clearly illustrate the dependence of steady-state string stability on spacing policy. For A equal to 10 A_{max}, Fig. 4 shows that the disturbance is amplified for frequencies between 1.0 and 3.2 rad/s for $C = 0.85$. This value of C corresponds to constant-safety-factor operation at $V_N = 5$ m/s. Even for the linear case (shown in Fig. 5), the string is no longer stable when C is less than 0.34 ($V_N = 2$ m/s).

Two important results regarding the steady-state behavior of a string may be concluded from the describing function analysis.

1. As A increases, the attenuation of the disturbance decreases and steady-state instability eventually results.
2. As C decreases, attenuation decreases. For the linear case, C must be greater than about 0.4s or the string is no longer stable. For constant safety-factor, this corresponds to $V_N = 2.35$ m/s and for constant time-headway to $h = 0.4$ s.

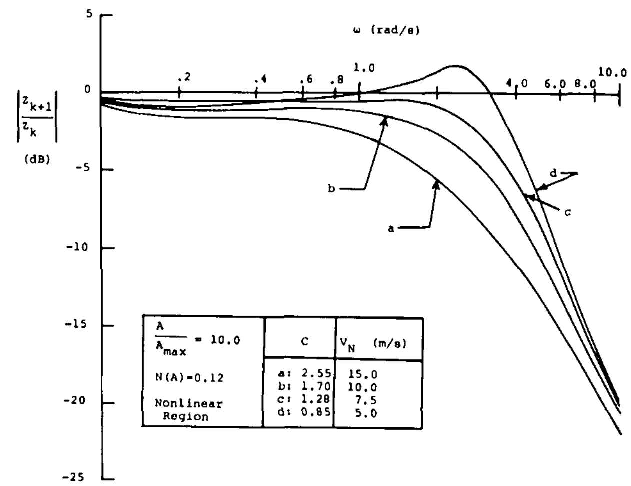

Fig. 4 Frequency response of k+1st vehicle for various velocities—nonlinear operation constant-safety factor spacing

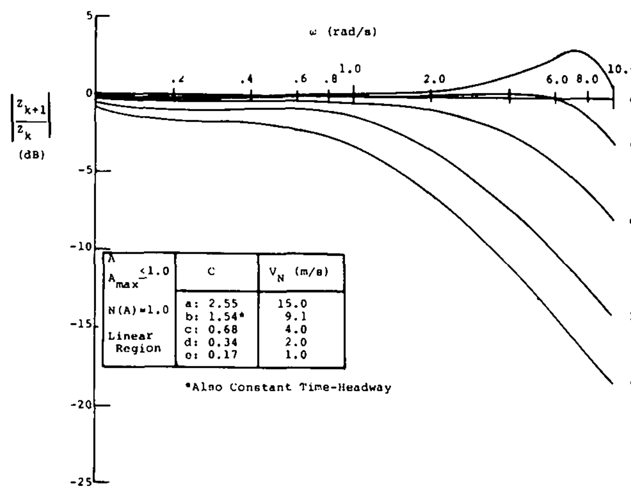

Fig. 5 Frequency response of k+1st vehicle for various velocities—linear operation, constant-safety factor spacing

Nonlinear systems often exhibit jump-resonance phenomena when the amplitude or frequency of the sinusoidal input varies. At the jump-resonance point, Z_{k+1}/Z_k is discontinuous and the same values of input amplitude M_k and frequency correspond to different values of A, the amplitude of the input to the nonlinearity. Fig. 6 shows a typical graph of A versus M_k for frequencies of 1 and 3 rad/s. The existence of discontinuous changes in A is clearly indicated. For example, at a frequency of 1 rad/s, A jumps from 9 m/s² to 540 m/s² as M_k approaches 4.45 m. These analytical results are verified by computer simulation as indicated. A similar jump occurs at 3 rad/s.

The presence of the jump-resonance phenomenon is further illustrated in Fig. 7. The intersection of the polar plot of the transfer function $L(j\omega) = L_1(j\omega)\, L_2(j\omega)\, L_3(j\omega)$ with circles with center coordinates

$$\left[\frac{N(A)^{-1} + N^*(A)^{-1}}{2}, 0\right]$$

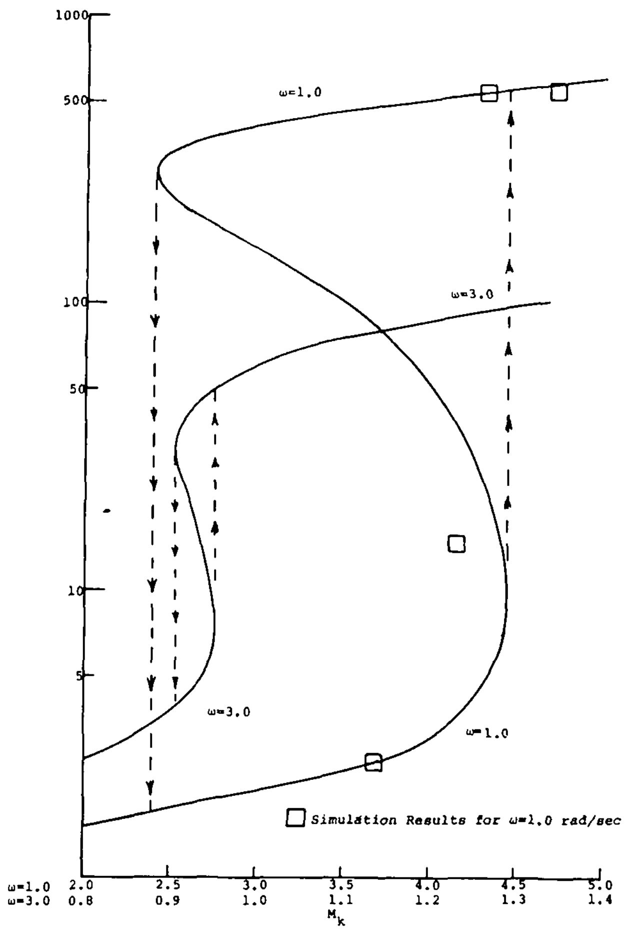

Fig. 6 *A as a function of the amplitude of the perturbation, M_k, for ω = 1.0 and 3.0 rad/s*

and radii

$$\left[\frac{N^*(A)^{-1} - N(A)^{-1}}{2}\right]$$

where

$$N^*(A) = N(A) + A\frac{d\,N(A)}{d\,A}$$

define amplitudes of A and frequencies at which jumps can occur [18, 19]. Once A and ω are known, M_k can be determined from equation (19). The presence of a jump can result in the system being driven well into the nonlinear region of operation resulting in a loss of steady-state stability.

For a jump to occur at ω less than 0.6 rad/s requires that M_k exceed 13 m. M_k must be constrained to be less than one-half the nominal separation distance of 19.13 m or collisions between vehicles can occur even if the perturbation is attenuated. If M_k exceeds this threshold value, the vehicles should undergo emergency braking. Thus it is physically impossible for jumps to occur at frequencies less than 0.6 rad/s.

Since jumps can result in unstable string behavior, they should be eliminated if possible. A simple way to avoid jumps is to change the acceleration limiter from a pure saturation to a piecewise linear element whose final slope, k_2, is small, but nonzero (see Fig. 8). As shown in Fig. 8, for k_2 = 0.05, the locus of $L(j\omega)$ never enters the jump region. Since k_2 is small, the overall system response will be only slightly altered but no jump will exist for any combination of M_k and ω.

IV Transient Stability

Transient stability is much more difficult to investigate analytically than steady-state stability [20]; however, a simple kinematic analysis shows constant-separation spacing to be unstable in speed transitions [11]. Although transient instability is inherent in constant-separation operation, its effects can be reduced by reducing the delay time in the response of the following vehicle to changes in the velocity of the lead vehicle and by limiting the length of vehicle strings.

Liapunov techniques can be used to provide a preliminary assessment of transient string stability for constant-time-headway and constant-safety-factor operation. Liapunov theory will be used to calculate bounds on the error in the state of each vehicle. These bounds, expressed in terms of the norm of the state error vector of each vehicle, define a region of stable string operation in the position-velocity-acceleration error space. The size of this region is determined by controller gains and spacing policy. The Liapunov method provides sufficient conditions only; consequently, the region of stable operation may be larger than predicted by the theory [21].

Two important cases are examined by use of the Liapunov technique. First, the lead vehicle of the string experiences a nonzero initial error. This is typical of merging or overtaking. Second, all vehicles in the string experience nonzero initial errors. This is typical of a sampled-data operation.

To illustrate the use of the Liapunov technique, consider a linear model of the dynamics of the k + 1st vehicle. Linearizing the spacing command and the propulsion dynamics, neglecting the acceleration limiter, and using proportional-integral control on the velocity error and proportional-integral-deviative control on the spacing error as described in the previous section, the equation of motion for the k + 1st vehicle is

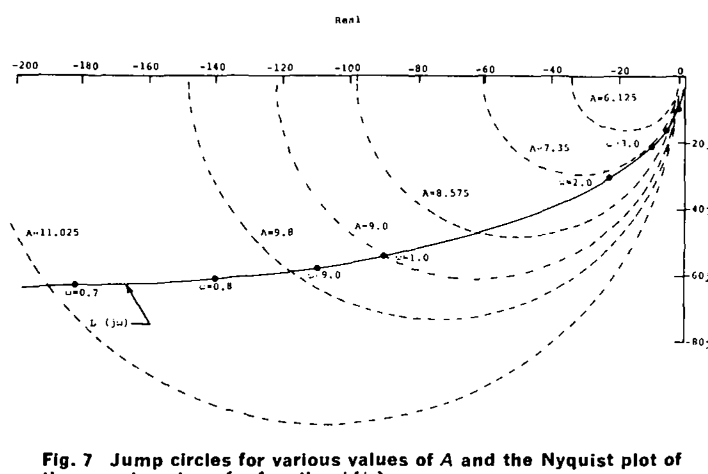

Fig. 7 **Jump circles for various values of** *A* **and the Nyquist plot of the open-loop transfer function** *L(jω)*

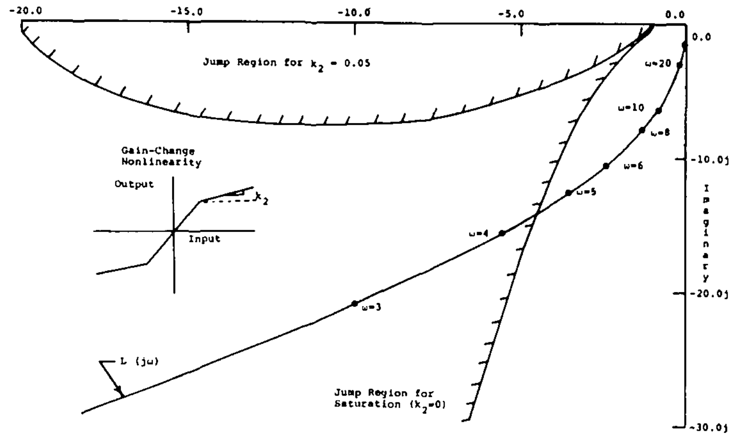

Fig. 8 Jump regions for the pure saturation and piecewise linear elements and Nyquist plot of $L(j\omega)$

$$\dddot{Z}_{k+1} + a_3 \ddot{Z}_{k+1} + a_2 \dot{Z}_{k+1} + a_1 Z_{k+1} = b_3 \ddot{Z}_k + b_2 \dot{Z}_k + b_1 Z_k$$

where

$$a_3 = \frac{K_p + C\,G_p + G_D + \dfrac{K_V}{K_E}}{m/K_E + C\,G_D}$$

$$a_2 = \frac{K_I + G_P + C\,G_I}{m/K_E + C\,G_D}$$

$$a_1 = \frac{G_I}{m/K_E + C\,G_D}$$

$$b_3 = \frac{K_P + G_D}{m/K_E + C\,G_D} \tag{21}$$

$$b_2 = \frac{K_I + G_P}{m/K_E + C\,G_D}$$

$$b_1 = \frac{G_I}{m/K_E + C\,G_D}$$

Defining the state error vector x_k for the k + 1st vehicle as

$$x_{k+1}{}^T = [Z_{k+1}\ \dot{Z}_{k+1}\ \ddot{Z}_{k+1}] \tag{22}$$

The set of equations describing the error vector for each vehicle in a string of N vehicles can be written in matrix form as:

$$\dot{x}_1 = A\,x_1 \tag{23}$$

$$\dot{x}_{k+1} = A\,x_{k+1} + B\,x_k \qquad \text{for } k = 1, 2, 3, \ldots, N-1$$

where the matrices A and B are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \tag{24}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_1 & b_2 & b_3 \end{bmatrix} \tag{25}$$

The norm of the vector x_k is chosen to be

$$||x_k(t)||_P = [x_k{}^T P\,x_k]^{1/2} \tag{26}$$

where P is a positive definite matrix. The Liapunov function is selected as the sum of the norms of the individual state error vectors

$$V(x(t)) = \sum_{j=1}^{N} ||x_j(t)||_P{}^2$$

P can be chosen such that the time derivative of V is negative semidefinite [22]; consequently,

$$V(x(t)) \leq V(x(0)) \text{ for all } t > 0 \tag{28}$$

$V(x(0))$ bounds the sum of the norms and can be used as a conservative estimate of the bound for the norm of each state error vector. The elements of P can be selected to define a region in the error space such that if the initial errors in the state of each vehicle are located within this region; the subsequent motion of each vehicle will be such that no collisions will occur. In order to avoid collisions, the position error for any vehicle in the string should never exceed one-half of the nominal spacing. The elements of P are functions of the vehicle parameters and control system gains and are given in Appendix B.

Consider the case where the first J vehicles in the string have nonzero initial state errors, $x_j{}^0$. The norm of these errors can be written as:

$$||x_j{}^0||_P = M_{Pj} \qquad \text{for } j = 1, 2, 3, \ldots J, \qquad J \leq N \tag{29}$$

The system bound becomes

$$V(x(0)) = \sum_{j=1}^{J} ||X_j{}^0||_p{}^2 \tag{30}$$

Letting $M_P \geq M_{Pj}$ for all j = 1, 2, 3, ... J, the system bound becomes,

$$V(x(0)) = J\,M_P{}^2 \tag{31}$$

and the norm for each vehicle is bounded by

$$||x_k(t)||_p \leq \sqrt{J}\,M_P \tag{32}$$

Since, no position error should exceed one-half the nominal spacing, Δ_0. M_P can be defined as

$$M_P = \begin{bmatrix} {}^1/_2\Delta_0 \\ 0 \\ 0 \end{bmatrix}^T P \begin{bmatrix} {}^1/_2\Delta_0 \\ 0 \\ 0 \end{bmatrix} = {}^1/_4\,p_{11}\,\Delta_o{}^2 \tag{33}$$

M_P is also defined from the maximum initial error state of vehicles 1, 2, 3, ... J. If the allowable initial position, velocity, and acceleration errors for each vehicle are denoted as ΔX, ΔV, and ΔA, then M_P becomes

$$M_P = \begin{bmatrix} \Delta X \\ \Delta V \\ \Delta A \end{bmatrix}^T P \begin{bmatrix} \Delta X \\ \Delta V \\ \Delta A \end{bmatrix} \tag{34}$$

Equating the two expressions for M_P yields

$$p_{11}\,\Delta X^2 + 2(p_{12}\,\Delta V + p_{13}\,\Delta A)\,\Delta X + p_{22}{}^2\,\Delta V + 2p_{23}\,\Delta V\,\Delta A + p_{33}\Delta A^2 \leq {}^1/_4 J^{-1/2}\,p_{11}\Delta_0{}^2 \tag{34}$$

where p_{ij} is the element in the ith row and jth column of matrix P. As long as the values of ΔX, ΔV, and ΔA satisfy the above equation, there is no possibility of a collision between vehicles.

Let $\Delta \bar{X}$, $\Delta \bar{V}$, $\Delta \bar{A}$ correspond to the maximum position, velocity, and acceleration errors permissible when the other two components of the initial allowable error vector equation are zero. That is $\Delta \bar{X} = {}^1/_2\Delta_0$ when $\Delta V = \Delta A = 0$. Then from equation (34) the following can be obtained:

$$\Delta \bar{X} = {}^1/_2\, J^{-1/4}\Delta_0 \tag{35}$$

$$\Delta \bar{V} = {}^1/_2\, J^{-1/4}\left(\frac{p_{11}}{p_{22}}\right)^{1/2}\Delta_0 \tag{36}$$

$$\Delta \bar{A} = {}^1/_2\, J^{-1/4}\left(\frac{p_{11}}{p_{33}}\right)^{1/2}\Delta_0 \tag{37}$$

$\Delta \bar{X}$ is dependent on J and Δ_0 but not on P; consequently, $\Delta \bar{X}$ can be increased by decreasing J or increasing Δ_0 but not by changing controller gains. On the other hand ,$\Delta \bar{V}$ and $\Delta \bar{A}$ are dependent on P. If p_{11} is increased with respect to p_{22} and p_{33}, the region of initial errors in which no collisions occur will be enlarged. Since the elements of P are functions of the control system gains, the no-collision region can be increased by proper selection of these gains.

Various system parameters were perturbed from their nominal values to determine their effect on the stable region defined by equations (35)–(37). The spacing policy, the propulsion system gain K_E, the vehicle mass, and the controller gains were varied. For each set of parameters examined, a new Liapunov function must be derived. Table 2 shows the results of perturbing these parameters for $J = 1$ (the lead vehicle experiences an initial error). The results are summarized as follows:

1. Spacing policy, vehicle mass, and propulsion system gain have little effect on the size of the stable region.
2. The integral gain in the spacing controller, G_I, is the only controller gain which significantly influences the size of the stable region. Increasing G_I from 800 to 1200 volt/m-s increases the values of $\Delta \bar{V}$ and $\Delta \bar{A}$ by 50 percent.

The Liapunov technique described above is limited to linear vehicle models. Although in this paper the technique was used to determine the region of initial conditions for which no collision can occur, it can be generalized to determine the set of initial conditions for which no subsequent motion of any vehicle in the string will leave a prespecified region in the error space.

V Conclusion

A definite relationship exists between spacing policy and string stability. If a constant-separation spacing policy is used, a simple kinematic analysis shows that regardless of control system design, a velocity wave will be amplified as it propagates upstream along the vehicle string. The level of amplification can be reduced but not eliminated by control system design. If a constant-separation policy is used, the number of vehicles in any string must be limited. Constant-safety-factor operation can result in string instabilities at low velocities; however, it is possible to eliminate this problem by a slight modification of the constant-safety-factor policy. There is no fundamental reason why a constant-time-headway spacing policy will result in string instabilities provided a sufficient nominal headway is specified. For the vehicle system model considered in this paper minimum headways of four-tenths of a second were necessary to insure string stability. For both constant-time-headway and modified-constant-safety-factor spacing policies, string stability is primarily a function of longitudinal control system design.

Nonlinearities play an important role in determining the dynamic response of vehicle strings. Vehicular propulsion and braking systems typically exhibit nonlinear behavior, particularly during rapid acceleration and deceleration. In addition, nonlinearities may be intentionally added to the control system to limit acceleration and jerk. Nonlinearities can result in unstable behavior of strings of vehicles even though a linearized analysis may indicate that the string is stable. A modification of the describing function technique can be readily adapted to determine steady-state stability of vehicle strings; however, a vehicle string which is steady-state stable may exhibit undesirable transient response when subjected to step commands and/or sudden disturbances such as wind gusts. Even if linearized vehicle models are used, analytical determination of the transient behavior of vehicle strings can be very tedious if the response is determined explicitly by solution of the differential equations describing the behavior of the vehicle string. Liapunov stability theory can be used to provide a preliminary assessment of the behavior of strings of vehicles if linearized vehicle models are used. If nonlinear models are used, transient response must be determined by computer simulation.

Table 2 Effect of various parameters on the region in which no collision is guaranteed

	Spacing policy: Constant time-headway	Spacing policy: Constant safety-factor	Vehicle mass** (kg): 1500	Vehicle mass** (kg): 2000	K^*_E (nt/volt): 10	K^*_E (nt/volt): 29.9	G^*_I (volt/m-s): 800	G^*_I (volt/m-s): 1200
$\Delta \bar{X}$ (m)	9.5665	9.5665	9.5665	9.5665	9.5665	9.5665	9.5665	9.5665
$\Delta \bar{V}$ (m/s)	1.0620	1.0644	1.0614	1.0644	1.0637	1.0644	1.0644	1.5800
$\Delta \bar{A}$ (m/s^2)	0.9420	0.9440	0.9435	0.9440	0.9408	0.9440	0.9440	1.4200

APPENDIX A

Modification to the Constant-Safety-Factor Policy

Let

$$\Delta_c = K\,\frac{V^2}{2a_c} + f(V)$$

where the added term $f(V)$ must satisfy the following conditions:

1. The magnitude of $f(V)$ be small compared to $K\,\dfrac{V^2}{2a_c}$ at $V = V_N$
2. $f(V) = 0$ at $V = 0$
3. $\dfrac{\partial f(V)}{\partial V}$ at low velocity must be sufficiently large to make the system stable

A suitable choice for $f(V)$ is

$$f(V) = \bar{K}\,[1 - \exp\,(-\,V/b)]$$

where $\bar{K}$ and b are parameters to be chosen so that $f(V)$ satisfies the above requirements.

Let

$$C = \frac{\partial}{\partial V}[\Delta_c] = \frac{KV}{a_c} + \frac{\bar{K}}{b}\exp(-V/b)$$

then $\bar{K}/b$ must be chosen so that

$$C = \frac{K V_N}{a_e} + \frac{\bar{K}}{b}\exp(-V_N/b) > 0.4$$

The minimum value of C over the velocity range can be found by setting

$$\frac{\partial}{\partial V}[C] = 0$$

The velocity corresponding to minimal damping is

$$V_{\min} = b \ln \frac{\bar{K}a_e}{Kb^2}$$

b and C are chosen so that at $V = V_{\min}$, C is such that the system is stable (for the example in the paper, $C \geq 0.4$)

Using this value of C,

$$\ln \frac{\bar{K}a_e}{Kb^2} \geq \frac{0.4a_e}{bK} - 1$$

For $a_e = 0.6$ g and $K = 1$ appropriate values for $\bar{K}$ and b are 0.75m and 1.5 m/s, respectively. These values make $C \geq 0.4$ for all nominal operating velocities. Command spacing for the modified-constant-safety-factor policy is

$$\Delta_c(V) = \frac{KV^2}{2a_e} + 0.75\,[1 - \exp(-V/1.5)]$$

at $V = 15$ m/s the spacing is increased by 0.75 m, the headway is increased by 0.05 s and the capacity is decreased by 74 vehicles/hr out of a nominal capacity of 2337 vehicles/hr, a decrease of 3 percent.

APPENDIX B

Required Form of the Liapunov Function for Transit String Stability Analysis

The matrix P must have the form

$$P = \begin{bmatrix} \delta_{11} & \delta_{12} & b_1 \\ \delta_{12} & \delta_{22} & b_2 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$\delta_{11} = a_2b_1 - a_1b_2$$

$$\delta_{12} = {}^1/_2(-\alpha \pm \sqrt{\alpha^2 - 4\beta})$$

$$\alpha = a_1b_3 + \frac{a_1}{b_1}b_2{}^2 - a_2b_2 - a_3b_1$$

$$\beta = a_2a_3b_1b_2 - a_1a_2b_2b_3 + \frac{a_1{}^2}{b_1}b_2{}^2b_3 - a_1a_3b_2{}^2$$

α_{22} is chosen to make P positive definite and the time-derivative of $V(x(t))$ negative semi-definite. The coefficients a_1, a_2, a_3, b_1, b_2, and b_3 are given by equation (21). Derivation of the form of P given above and the conditions on δ_{22} are too long to be given in this paper but can be found in reference [5].

References

1 Hadjн, L. P., Gardiner, K. W., Tamura, H., and Pressman, G. L., "Design and Control Considerations for Automated Ground Transportation Systems," *Proceedings IEEE*, Vol. 56, 1968, pp. 493–513.

2 Brown, S. J., "Characteristics of a Linear Regulator Control Law for Vehicles in an Automatic Transit System," 1971 AIAA Guidance and Control Conference, Stony Brook, New York, Aug. 1971.

3 Brown, S. J., "Adaptive Merging Under Car-Follower Control," *Personal Rapid Transit III*, Ed. D. A. Gary, W. L. Garrard, A. L. Kornhauser, University of Minnesota, Department of Audio-Visual Extension, 1976.

4 MacKinnon, D., "High Capacity Personal Rapid Transit Systems Development," *IEEE Trans. Vehicular Technology*, Vol. VT-24, No. 1, Feb., 1975, pp. 8–14.

5 Caudill, R. J., "Vehicle-Follower Longitudinal Control for Small Automated Transit Vehicles," PhD thesis, University of Minnesota, Dec., 1976. Also, Published by Urban Mass Transit Administration U. S. DOT as Rept. No. UMTA-MN-11-0002-77-1).

6 Garrard, W. L., et al., "Longitudinal Control and Crashworthiness for Small Automated Transit Vehicles," NTIS PB-243-353, Springfield, VA., Jan. 1976.

7 Pipes, L. A., "A Proposed Dynamic Analogy of Traffic," *Journal of Applied Physics*, Vol. 24, 1953.

8 Chandler, R. E., Herman, R., and Montroll, E. W., "Traffic Dynamics: Studies in Car Following," *Operations Research*, Vol. 6, 1958.

9 Herman, R., Montroll, E. W., Potts, R. B., and Rothery, R. W., "Traffic Dynamics: Analysis of Stability in Car Following," *Operations Research*, Vol. 7, 1959.

10 Bender, J. G., and Fenton, R. E., "On Vehicle Longitudinal Dynamics," *Traffic Flow and Transportation*, Gordon Newell, Ed., Elsevier Press, 1973.

11 Garrard, William L., and Caudill, Reggie J., "Dynamic Behavior of Strings of Automated Transit Vehicles," SAE Paper 770288, Mar. 1977.

12 Chiu, H. Y., Stupp, G. B., and Brown, S. J., "Vehicle-Follower Controls for Short Headway AGT Systems-Functional Analysis and Conceptual Designs," Johns Hopkins University Applied Physics Lab. Report No. #CP 051/TPR 035, Dec. 1976.

13 Gelb, A., and Vander Velde, W. E., *Multiple Input Describing Functions and Nonlinear Systems Design*, McGraw-Hill, 1968.

14 Hill, J. C., "Closed-Loop Response of Nonlinear Systems by a Functional Transformation Approach," *IRE Trans. Automatic Control*, Vol. AC-7, No. 4, 1962.

15 Truxal, J. G., *Automatic Feedback Control System Synthesis*, McGraw-Hill, New York, 1955.

16 McAllister, A. S., "A Graphical Method for Finding the Frequency Response of Nonlinear Closed-Loop Systems," *Trans. IEEE Part II, Application in Industry*, Vol. 80, 1961.

17 Caudill, R. J., and Garrard, W. L., "Vehicle-Follower Longitudinal Control for Automated Transit Vehicles," IFAC/IFIP/IFORS Conference on Control in Transportation Systems, Columbus, Ohio, Aug. 1976.

18 Hatanaka, H., "The Frequency Responses and Jump-Resonance Phenomena of Nonlinear Feedback Control Systems," *Journal of Basic Engineering*, TRANS. ASME, Series D, June 1963, pp. 236–212.

19 West, J. C., Douce, J. L., and Livesley, R. K., "The Dual-Input Describing Function and Its Use in the Analysis of Non-Linear Feedback Systems," *Proc. IEE*, Vol. 103, B, 1956, pp. 463–474.

20 Voss, P. J., "Stability of a String of Vehicles Employing Vehicle Follower Control," APL/JHU MCS -6-153, Johns Hopkins Applied Physics Lab, Laurel, Md., 1972.

21 LaSalle, J. P., and Lefschetz, S., *Stability by Liapunov's Direct Method with Applications*, Academic Press, 1961.

22 Erickson, P. A., "Dynamics and Stability of Sequential Systems," PhD thesis, University of Texas at Austin, Dec. 1975.