Developments in Nonholonomic Control Problems

Ilya Kolmanovsky and N. Harris McClamroch

In this article, we provide a summary of recent developments in control of nonholonomic systems. The published literature has grown enormously during the last six years, and it is now possible to give a tutorial presentation of many of these developments. The objective of this article is to provide a unified and accessible presentation, placing the various models, problem formulations, approaches, and results into a proper context. It is hoped that this overview will provide a good introduction to the subject for nonspecialists in the field, while perhaps providing specialists with a better perspective of the field as a whole.

Introduction

Although nonholonomic systems have been studied in classical mechanics for more than 150 years, it is only recently that the study of control problems for such systems has been initiated. The published literature on control of nonholonomic systems has grown enormously during the last six years. The objective of this article is to provide a unified and accessible presentation, placing the various models, problem formulations, approaches, and results into a proper context. No attempt is made to make the presentation self-contained nor rigorous; rather, we emphasize concepts and ideas, and we do provide numerous references to the published literature. The presentation is organized as follows: introduction to nonholonomic control systems and where they arise in applications, classification of models of nonholonomic control systems, control problem formulations, motion planning results, stabilization results, and current and future research topics.

Nonholonomic systems most commonly arise in finite dimensional mechanical systems where constraints are imposed on the motion that are not integrable, i.e. the constraints cannot be written as time derivatives of some function of the generalized coordinates. Such constraints can usually be expressed in terms of nonintegrable linear velocity relationships. Nonholonomic control systems result from formulations of nonholonomic systems that include control inputs. This class of nonlinear control systems has been studied by many researchers, and the published literature is now extensive. The interest in such nonlinear control problems is motivated by the fact that such problems are not amenable to methods of linear control theory, and they are not transformable into linear control problems in any meaningful way. Hence, these are nonlinear control problems that require fundamentally nonlinear approaches. On the other hand, these nonlinear control problems are sufficiently special that good progress can be made.

There are numerous examples of nonholonomic systems, many of substantial engineering interest. The literature that deals with the formulation of the equations of motion and the dynamics of nonholonomic systems is vast; an excellent reference is the book by Neimark and Fufaev [112].The book by Murray, Li, and Sastry [102] provides a general introduction to nonholonomic control systems. Examples of nonholonomic control systems have been studied in the context of robot manipulation, mobile robots, wheeled vehicles, and space robotics. We now describe some of these applications. Specific examples of nonholonomic control systems include sledges or knife edge systems that slide on a plane [12, 14], simple wheels rolling without slipping on a plane [12, 14, 15], and spheres rolling without slipping on a plane [13,19,82] (Fig. 1). There is now an extensive literature on control of mobile robots and wheeled vehicles, including tractors with trailers (Fig. 2), that are described as nonholonomic control systems [8, 9, 20, 23, 25, 45, 74, 75, 76, 77, 102, 131, 132, 133, 134, 135, 145, 148, 158, 159, 160]. Related examples of nonholonomic control systems that arise in robot manipulation are described in [29, 82, 84, 85, 102, 107, 136].

In addition to the classical formulation, nonholonomic control systems can arise in other ways. If the motion of a mechanical system exhibits certain symmetry properties, it is well known that there exist conserved quantities. If these conserved quanti-

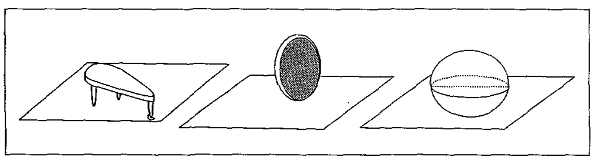

Fig. 1. Examples of classical nonholonomic systems: (a) knife edge, (b) rolling disk, (c) rolling sphere.

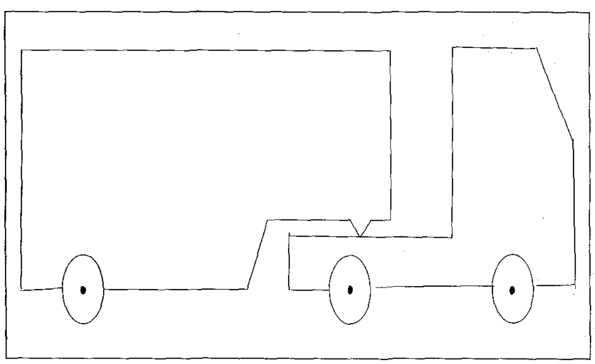

Fig. 2. A tractor with a single trailer.

The authors are with the Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2118. *Kolmanovsky's work was supported by a Francois-Xavier Bagnoud doctoral fellowship; McClamroch's by NSF Grant No. MSS-9114630.*

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ties are not integrable, then a class of nonholonomic systems is thereby obtained. Perhaps the most obvious case occurs when there is an angular momentum integral; that is, some function of the angular momentum is conserved. In such case, if this function of the angular momentum is not integrable, then a nonholonomic control system is obtained. Examples of such nonholonomic control systems include actuated multibody spacecraft [28, 47, 68, 95, 122, 150, 169] and underactuated symmetric rigid spacecraft [65, 66, 67, 149, 163, 167]. Related work on space robotics appears in [1, 10, 35, 44, 111, 164, 172]; other related multibody applications include [53, 83, 110].

Nonholonomic control systems also arise in mechanical systems that involve both symmetries and nonholonomic constraints. A formalism for such systems has recently been developed [16, 109]. Some control-theoretic results are available in [109, 55]. The papers [80, 117] deal with a wheeled Snakeboard, a simplified model of a commercially available skateboard, for which nonholonomic constraints arise as a result of no-slip condition for the wheels coupled with the conservation of the angular momentum.

Nonholonomic control systems can also arise as the result of the imposition of control design constraints; that is, it is desired to choose a controller so that certain motion constraints are imposed. If the imposed constraints are not integrable, then a nonholonomic system is obtained. This case has been identified for kinematically redundant underactuated robots in [6, 7]. There has been very little research on this source of nonholonomic control systems.

Nonholonomic systems are examples of underactuated mechanical systems; the general connection between nonholonomic control systems and underactuated systems is not completely understood, but some preliminary results have been presented in [116, 151]. Interesting connections between nonholonomic control systems and robotic locomotion have been presented in [55].

Nonholonomic control systems arise as models in these, and perhaps other, situations. In the literature, a variety of mathematical models has been studied. Although we do not know how to identify the most general class of nonholonomic control systems, we are able to identify several fundamental classes. However, the mathematical models for nonholonomic control systems that we subsequently present do not always arise directly from physical based models; it is often the case that state and control transformations are required to obtain these mathematical models. The models for nonholonomic control systems that we subsequently introduce include equations that represent nonholonomic constraints in a standard form, and they assume a sufficiently rich class of control inputs that the system is completely controllable. It is most convenient to make these assumptions in our presentation. However, there are also "nonholonomic control systems" of a more general form; brief comments are made in the last section about this class of systems.

Background

In this section we present models of nonholonomic control systems that have been widely studied in the literature, and we identify connections between these models. We refer to all of these as nonholonomic control systems, and we subsequently show the common features that they share. Motivated by the terminology of classical mechanics, we classify the models into kinematic models and dynamic models.

Kinematic Models

A general form of a nonholonomic control system, expressed in kinematic form, is given by the drift-free nonlinear control system

$$\dot{x} = g_1(x)u_1 + \ldots + g_m(x)u_m \,, \tag{1}$$

where $2 \le m < n$, $x = (x_1, \ldots, x_n)$ is the state vector, defined in an open subset of $\mathbb{R}^n$, u_i, $i = 1, \ldots, m$, are control variables, and g_i, $i = 1, \ldots, m$, are specified vector fields. The vector fields $g_1, \ldots, g_m$ can be thought of as a basis, at each x, for the null space of linear velocity constraints of the form $J^T(x)\dot{x} = 0$, where $J(x)$ is a full-rank $n \times (n - m)$ matrix. A technical but essential assumption (see [102] for details) is that the rank of the controllability Lie algebra generated by iterated Lie brackets of $g_1, \ldots, g_m$ is n, where the Lie bracket of two vector fields g_1 and g_2 is a new vector field $[g_1, g_2]$, defined by $[g_1, g_2](x) = \left(\frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2\right)(x)$. This assumption about the Lie algebra guarantees that there are no nontrivial functions which integrate the constraints represented by Equation (1). In this case, (1) is said to be completely nonholonomic, which is equivalent to complete controllability of (1). This model is referred to as a kinematic model since in applications from classical mechanics the controls are typically velocity variables and equation (1) is an expression of kinematic constraints on the motion.

The spanning properties of iterated Lie brackets are essential not only in identifying the class of nonholonomic control systems, but also in classifying nonholonomic control systems according to their degree of nonholonomy and their growth vector [165]. We do not digress to consider these notions here but refer the reader to [102] for details.

In many applications, nonholonomic control systems have a special form, or can be transformed into a special form, that should be recognized and exploited. A special class of nonholonomic control systems is given by

$$\begin{aligned} \dot{z} &= \sum_{i=1}^{m} \tilde{g}_i(z, y)\dot{y}_i \,, \\ \dot{y}_i &= u_i, \; i = 1, \ldots, m, \end{aligned} \tag{2}$$

where $m \ge 2$, and $y = (y_1, \ldots, y_m)$ is referred to as the base vector, $z = (z_1, \ldots, z_{n-m})$ is referred to as the fiber vector, u_i, $i = 1, \ldots, m$, are the controls, and $\tilde{g}_i(z, y)$, $i = 1, \ldots$, m, are specified vector fields. We assume that Equations (2) are completely nonholonomic in the sense of Equation (1).

Several special classes of systems described by Equation (2), where the vector fields have a special form, have been widely studied in the literature. Equations (2) are said to be in Chaplygin (kinematic) form if the vector fields $\tilde{g}_1, \ldots, \tilde{g}_m$ depend only on the base vector y but not on the fiber vector z. For the case of two controls, $m = 2$, several special classes of nonholonomic control systems have been studied; for example, systems in chained form are given by

$$
\begin{aligned}
\dot{z}_1 &= y_1 \dot{y}_2, \\
\dot{z}_2 &= z_1 \dot{y}_2, \\
\dot{z}_3 &= z_2 \dot{y}_2, \\
&\vdots \\
\dot{z}_{n-m} &= z_{n-m-1} \dot{y}_2, \\
\dot{y}_1 &= u_1, \\
\dot{y}_2 &= u_2,
\end{aligned} \tag{3}
$$

and systems in power form, a special class of Chaplygin form, are given by

$$
\begin{aligned}
&\dot{z}_1 = y_1 \dot{y}_2, \\
&\dot{z}_2 = \frac{1}{2}(y_1)^2 \dot{y}_2, \\
&\vdots \\
&\dot{z}_{n-m} = \frac{1}{(n-m)!}(y_1)^{n-m} \dot{y}_2, \\
&\dot{y}_1 = u_1, \\
&\dot{y}_2 = u_2.
\end{aligned} \tag{4}
$$

Both the chained form and the power form satisfy the completely nonholonomic assumption mentioned previously. Extensions of the chained form and the power form are also available for the case of more than two control inputs [20, 168]. The chained form and the power form are equivalent via a state transformation [103, 168]. Sufficient conditions on the vector fields in (1) that guarantee that (1) can be transformed into the chained form via state and control transformations have been developed by Murray and Sastry in [105] for $m = 2$, and by Bushnell et al. in [20] for $m > 2$. If these sufficient conditions hold, the desired state and control transformations can be determined by solving a system of partial differential equations [20, 105]. Under weaker conditions, Tilbury et al. [158] developed a transformation of (1) into a perturbed chained form, where the perturbation terms are higher order. Using the development in [20], Walsh and Bushnell [168] obtained sufficient conditions for conversion to power form in the case $m \geq 2$. References [108, 160, 161, 162] use exterior differential system techniques to develop necessary and sufficient conditions and procedures for conversion into the chained or power form.

A more complete presentation of kinematic models of nonholonomic control systems can be found in the recent book by Murray, Li, and Sastry [102]. It can be shown that chained or power forms can be used to model a variety of kinematic constraints in mechanical systems, including knife edge systems [12], rolling wheels [12], front-wheel drive automobiles [105], tractors with multiple trailers [148], underactuated symmetric rigid spacecraft [67], and space multibody spacecraft [61]. The transformations required to obtain the chained or power form in these specific problems are not trivial, and we refer to the cited literature for the details. The fact that these models arise in such applications has provided substantial motivation for the extensive study of these classes of nonholonomic systems.

Dynamic Models

Although models that include kinematic relationships may be suitable for certain control objectives, models that include dynamic effects (generalized forces) are required for other purposes. Dynamic models of nonholonomic systems can be obtained by a natural extension of the kinematic models as

$$\dot{x} = g_1(x)v_1 + \ldots + g_m(x)v_m, \tag{5a}$$

$$v_i^{r_i} = u_i, \quad i = 1, \ldots, m, \tag{5b}$$

where $2 \leq m < n$, $x = (x_1, \ldots, x_n)$ is an n-vector, and $v = (v_1, \ldots, v_m)$ is an m-vector. The superscripts $r_1, \ldots, r_m$ on v_i denote the order of time differentiation. As previously, we assume that Equation (5a) is completely nonholonomic [12, 21]. Note that v is the output of a linear system consisting of chains of integrators. This model is referred to as a dynamic model since in applications from classical mechanics, where $r_i = 1$, $i = 1, \ldots, m$, the controls are typically generalized force variables and the governing equations include both the constraints on the motion (5a) and the dynamic equations of motion (5b).

To make this connection clearer, consider a generalized mechanical system with nonholonomic constraints, expressed in the d'Alembert-Lagrange form,

$$M(x)\ddot{x} + f(x, \dot{x}) = J(x)\lambda + B(x)\tau, \tag{6a}$$

$$J^T(x)\dot{x} = 0, \tag{6b}$$

where $x = (x_1, \ldots, x_n)$ is an n-vector of generalized coordinates, $M(x)$ is an $n \times n$ positive definite symmetric inertia matrix, $J(x)$ is a full rank $n \times (n\text{-}m)$ matrix, $2 \leq m < n$, λ is an $(n\text{-}m)$-vector of Lagrange multipliers, $B(x)\tau$ is a vector of generalized forces applied to the system, $B(x)$ is an $n \times p$ matrix, τ is a p-vector control, and the superscript T denotes the transpose. We assume that the constraints (6b) are completely nonholonomic in the sense described previously. Equation (6b) constrains the velocity $\dot{x}$, at each x, to the null space of $J^T(x)$. Let the vector fields $g_1, \ldots, g_m$ form the basis for the null space of $J^T(x)$ at each x, and let $g(x) = (g_1(x), \ldots, g_m(x))$. Then, $J^T(x)g(x) = 0$ for each x and,

$$\dot{x} = g(x)v = g_1(x)v_1 + \ldots + g_m(x)v_m, \tag{7}$$

for an appropriately defined m-vector $v = (v_1, \ldots, v_m)$. Depending on the choice of the basis for the nullspace of $J^T(x)$, the v_i, $i = 1, \ldots, m$, may or may not have a physical interpretation as velocities. By differentiating Equation (7),

$$\ddot{x} = g(x)\dot{v} + \frac{\partial g(x)}{\partial(x)}\dot{x}.$$

Substituting this expression into Equation (6a) and multiplying by $g^T(x)$, we obtain

$$g^T(x)M(x)g(x)\dot{v} + F(x, \dot{x}) = g^T(x)B(x)\tau, \tag{8}$$

for an appropriately defined vector function $F(x, \dot{x})$. Assume now that $g^T(x)\,B(x)$ is onto. This assumption requires that the independent degrees of freedom of the nonholonomic system are completely actuated [12,21]. Then (8) can be feedback-linearized to the form

$$\dot{v} = u, \tag{9}$$

where $u = (u_1, ..., u_m)$ is an m-vector control. Equations (7), (9) have the form of Equation (5) with $r_1 = r_2 = ... = r_m = 1$. Consequently, under a reasonable set of assumptions, d'Alembert's formulation of classical mechanics with control inputs and nonholonomic constraints can be transformed into the form of Equation (5), which is precisely the dynamic extension of the kinematic model (1). This reduction procedure to the form of Equation (5) was proposed by Campion et al. in [21].

Several special classes of dynamic nonholonomic systems, where the vector fields have a special form, have been studied in the literature. For example, in the above reduction procedure it is often possible to select the basis vector fields for the nullspace of $J^T(x)$ so that

$$g(x) = \begin{bmatrix} \tilde{g}(x) \\ I_m \end{bmatrix},$$

where $\tilde{g}(x)$ is an $(n\text{-}m)$ x m matrix and I_m is an m x m identity matrix. Partition the vector x as $x = (z,y)$, where $z = (z_1, ..., z_{n-m})$ is an $(n\text{-}m)$-vector and $y = (y_1, ..., y_m)$ is an m-vector. Then,

$$\dot{z} = \sum_{i=1}^{m} \tilde{g}_i(z, y)\dot{y}_i ,$$
$$\ddot{y}_i = u_i , \quad i = 1, \dots , m, \tag{10}$$

which is the dynamic extension of the kinematic model (2). The reduction procedure to the form of Equation (10) was proposed by Bloch et al. in [12]. As in the kinematics case, the vector $y = (y_1, ..., y_m)$ is referred to as the base vector and the vector $z = (z_1, ..., z_{n-m})$ is referred to as the fiber vector.

Equations (10) are said to be in Chaplygin (dynamic) form if $\tilde{g}_1, ..., \tilde{g}_m$ depend only on the base vector y but not on the fiber vector x. Nonholonomic control systems in extended power form, which is the dynamic extension of the kinematic power form, have been studied by Kolmanovsky et al. in [57, 58] and by M'Closkey and Murray [93]. Similarly, it is possible to introduce nonholonomic control systems in extended chained form. Following the developments for the kinematic models, extended chained or power forms can be used to model a variety of mechanical systems, where both dynamic relationships and kinematic constraints are included.

Equation (5) with $r_i > 1$ can be used to model nonholonomic control systems with augmented actuator dynamics [57, 125]. For example, the control inputs in Equation (5) can be viewed as force/torque variables if $r_i = 1$. If these force/torque variables are generated as the outputs of servo motors, with inputs which are motor voltages, then the complete system, including servo motor dynamics, can be written in the form of Equation (5) with $r_i > 1$. The definitions of Chaplygin systems and nonholonomic systems in power or chained form can be generalized to this case.

Generalizations of both kinematic nonholonomic control systems and dynamic nonholonomic control systems are possible in a number of different ways. For example, in Equation (5) the vector fields $g_1, ..., g_m$ might also depend on v. Equation (5a) might include an additional, perhaps state-dependent and time-dependent, drift term. In addition, Equation (5b), assumed to be defined in terms of chains of integrators, can be generalized to a more general linear, or even nonlinear, system. These types of generalizations can be easily made, but only a few concrete control theoretic results are yet available [60, 116, 125]. Consequently, our subsequent development is restricted to nonholonomic control systems introduced above.

Example

A simple example of a nonholonomic control system is provided by a wheeled mobile robot of unicycle type [25], shown in Fig. 3. The two rear wheels of the robot are controlled independently by motors, and a front castor wheel prevents the robot from tipping over as it moves on a plane. By commanding the same velocity to both wheels the robot moves along the straight line. By commanding velocities with the same magnitude but opposite direction the robot can turn. It is assumed that the masses and inertias of the wheels are negligible and that the center of mass of the robot is located in the middle of the axis connecting the rear wheels. Let x_c, y_c denote the coordinates of the center of mass on the plane, θ denote the heading angle measured from the x-axis, v denote the magnitude of the translational velocity of the center of mass, and w denote the angular velocity of the robot. We assume that the wheels do not slide, i.e., that the velocity of the center of mass of the robot is orthogonal to the axis connecting the rear wheels. This assumption imposes a nonholonomic constraint on the motion of the robot of the form $\dot{x}_c \sin\theta - \dot{y}_c \cos\theta = 0$. The kinematics of the robot are modeled by the following equations:

$$\dot{x}_c = v\cos\theta,$$
$$\dot{y}_c = v\sin\theta,$$
$$\dot{\theta} = w. \tag{11}$$

This model of the robot is the same as that of Reeds-Shepp's car [121]. The control inputs are the forward velocity v and the

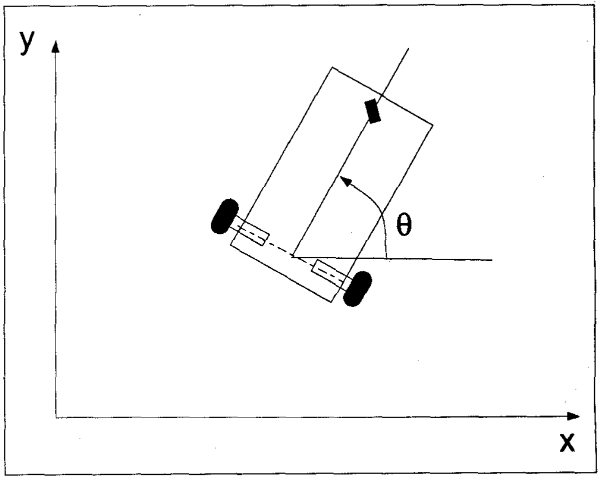

Fig. 3. A wheeled mobile robot.

steering velocity w. It is possible to transform the equations of motion of the robot to the power form. Consider a state and control transformation defined by

$x_1 = x_c \cos\theta + y_c \sin\theta$,

$x_2 = \theta$,

$x_3 = x_c \sin\theta - y_c \cos\theta$,

$u_1 = v - wx_3$,

$u_2 = w$.

It is not hard to verify that the transformed equations are in the (kinematic) power form

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_1 \dot{x}_2. \end{aligned} \tag{12}$$

In Equation (12) the variable x_3 is the fiber variable, and (x_1, x_2) is the base variable vector. It turns out [105] that any kinematic completely nonholonomic system with three states and two control inputs can be converted to the power form (12).

It is also possible to obtain a dynamic model, where the control inputs are the pushing force τ_1 in the direction of the heading angle, and the steering torque τ_2 about the vertical axis through the center of mass:

$$\begin{aligned} \ddot{x}_c &= \frac{\lambda}{m}\sin\theta + \frac{\tau_1}{m}\cos\theta, \\ \ddot{y}_c &= -\frac{\lambda}{m}\cos\theta + \frac{\tau_1}{m}\sin\theta, \\ \ddot{\theta} &= \frac{\tau_2}{I_c}, \\ \dot{x}_c \sin\theta &- \dot{y}_c \cos\theta = 0, \end{aligned}$$

where m is the mass of the robot, I_c is the moment of inertia of the robot, and λ is the scalar constraint multiplier. By transforming these equations as described in the previous subsection, the extended power form representation of the robot dynamics can be obtained as

$$\begin{aligned} \dot{x}_3 &= x_1 \dot{x}_2, \\ \ddot{x}_1 &= u_2, \\ \ddot{x}_2 &= u_2. \end{aligned}$$

The details of this transformation for the case of the knife-edge, whose model is the same as the mobile robot, can be found in [12].

Equilibrium Properties of Nonholonomic Control Systems

We mention several fundamental properties of all nonholonomic control systems, whether kinematic or dynamic. Our assumptions imply that nonholonomic control systems, as introduced here, are necessarily completely controllable. Assuming there is no control ($u = 0$), there is a manifold of equilibria containing the origin, and each equilbrium is non-hyperbolic, i.e., linearization of the dynamics at an equilibrium always has zero eigenvalues. In other words, no nonholonomic control system has an isolated equilibrium and hence no equilibrium can be locally asymptotically stable. For example, for the unforced kinematic nonholonomic system (1) all points in the configuration space are equilibria, and linearization about any of them is uncontrollable, of the form $\dot{x} = 0$.

It is easy to show that nonholonomic control systems of the form considered are invariant with respect to diffeomorphic state transformations and with respect to time-invariant smooth (continuously differentiable) static state feedback. Specifically, if a diffeomorphic state transformation is made and smooth static state feedback which vanishes at the origin is applied, the resulting closed loop system remains completely nonholonomic. In particular, the closed loop system has a manifold of equilibria. This closure property of nonholonomic control systems, under state and (smooth) feedback transformations, is an important characteristic of this class of control systems.

Control Problems for Nonholonomic Systems

A variety of theoretical and applied control problems have been studied for various classes of nonholonomic control systems. The relative difficulty of the control problem depends not only on the nonholonomic nature of the system but also on the control objective. For some control objectives, classical nonlinear control approaches (e.g., feedback linearization and dynamic inversion, as developed in [48]) are effective. Examples of such control objectives include stabilization to a suitably defined manifold that contains the equilibria manifold [12, 21, 54, 88, 116], stabilization to certain trajectories [166], dynamic path following [138], and output tracking [40, 129]. Consequently, there are classes of control problems for nonholonomic systems for which standard nonlinear control methods can be applied.

However, many of the most common control objectives, e.g., motion planning and stabilization to an equilibrium state, cannot be solved using the standard nonlinear control methods, and new approaches have been developed. Substantial research has been devoted to motion planning, i.e., the study of (open loop) controls that transfer the system from a specified initial state to a specified final state. Conditions have been developed that guarantee when motion planning problems have solutions, and a variety of construction procedures for determining such controls have been proposed. In addition, feedback control of nonholonomic systems has been studied where the objective has been to accomplish specified closed-loop performance objectives, including the classical control objectives of stabilization, asymptotic tracking, disturbance rejection, robustness improvement, etc.

In the following two sections of this article, we summarize a few fundamental control theoretic results and we describe recent developments in motion planning and in stabilization. These issues have received the most attention in the literature, and it is now possible to provide a general outline of the major themes of this research.

Motion Planning for Nonholonomic Control Systems

Motion planning problems are concerned with obtaining *open loop* controls which steer a nonholonomic control system from an initial state to a final state over a given finite time interval. To understand why nonholonomic motion planning may be difficult, it is convenient to compare it with motion planning for holonomic mechanical systems. For a holonomic system, a set of independent generalized coordinates can be found, and thus an arbitrary motion in the space of independent generalized coordinates is feasible.

In contrast, for a nonholonomic system, a set of independent generalized coordinates does not exist. Consequently, not every motion is feasible, but only those motions which satisfy the instantaneous nonholonomic constraints. Nevertheless, the completely nonholonomic assumption guarantees that feasible motions do exist which steer an arbitrary initial state to an arbitrary final state.

Although motion planning for nonholonomic control systems must be performed without constraint violation, efficient techniques for motion planing have been developed. Optimal control problems and problems of motion planning with collision-avoidance have also been addressed. These developments are reviewed below.

Methodologies for Nonholonomic Motion Planning

A variety of motion planning techniques are described in the book [81], which is a collection of research articles on nonholonomic motion planning. Besides [81], an excellent introduction to motion planning for nonholonomic robots is contained in the book by Murray, Li, and Sastry [102]. The book by Latombe [74] also contains a nice chapter on nonholonomic motion planning. The motion planning methodologies can be loosely classified into differential-geometric and differential-algebraic techniques, geometric phase (holonomy) methods, and control parameterization approaches. These approaches may, at first glance, appear different, but, in fact, there are many connections, and they all lead to essentially equivalent developments.

Differential-Geometric and Differential-Algebraic Techniques. Many available motion planning tools are based on Lie-algebraic techniques whose in-depth treatment falls beyond the scope of the present article. A simple kinematic example with $m = 2$ can be used to motivate the importance of Lie brackets for motion planning. Suppose g_1 and g_2 are two smooth vector fields associated with the nonholonomic control system defined by (1). The motion of the system along the vector field g_1 can be generated by setting $u_1 = 1$, $u_2 = 0$, and along g_2 by setting $u_1 = 0$, $u_2 = 1$. Consider the motion of the system, starting at the origin, first along the vector field g_1 for Δt seconds, then along the vector field g_2 for Δt seconds, then along the vector field $-g_1$ for Δt seconds, and, finally, along the vector field $-g_2$ for Δt seconds. For small Δt, the final state resulting from this motion can be well approximated by $[g_1, g_2](0)\ (\Delta t)^2$, where $[g_1, g_2](0) = \left(\frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2\right)(0)$ is the Lie bracket of g_1 and g_2 evaluated at the origin [102]. Thus motions in the potentially new direction of $[g_1, g_2](0)$ can be generated by switching between the motions along g_1 and g_2, i.e., between the motions which satisfy the instantaneous nonholonomic constraints. By using more complex switchings it is possible to generate net motions in the directions provided by the iterated Lie brackets of g_1 and g_2. The assumption that the system is completely nonholonomic, i.e., that $g_1, ..., g_m$ and their iterated Lie brackets span $\mathbb{R}^n$, is the basis for the guarantee that any initial state can be steered to any final state. Reyhanoglu et al. [125] extended this result to dynamic models of nonholonomic control systems: if the iterated Lie brackets of $g_1, ..., g_m$ span $\mathbb{R}^n$, then system (5) can be steered from any initial state to any final state over a specified time interval. For the special case $r_1 = r_2 = ... = r_m = 1$ this result is known from the paper by Sussmann [155].

To make the ideas more concrete, consider the mobile robot example whose kinematic model is given by Equation (11). In this example, $g_1 = (\cos\theta, \sin\theta, 0)^T$, $g_2 = (0, 0, 1)^T$. The motion along g_1 corresponds to forward translation of the robot, and the motion along g_2 corresponds to counterclockwise rotation of the robot about its mass center. Consider the motion of the robot, which includes first moving along g_1 for Δt seconds (forward translation), then along g_2 for Δt seconds (counterclockwise rotation), then along $-g_1$ (backward translation), and along $-g_2$ for Δt seconds (clockwise rotation). It is not hard to verify that for small Δt the net motion of the robot is essentially translation sideways with respect to its original configuration. And, in fact, the Lie bracket, $[g_1, g_2] = (-\sin\theta, \cos\theta, 0)^T$, predicts precisely this motion. Although the instantaneous sideways motion is impossible because of the imposed no-slip condition, it can be generated by switching between the motions which satisfy the instantaneous nonholonomic constraint.

The idea of employing piecewise constant inputs to generate motions in the directions of iterated Lie brackets has been exploited by Lafferriere [72] and Lafferriere and Sussmann [73]. They proposed a general motion planning algorithm for kinematic models of nonholonomic systems. The algorithm is based on expressing the flow resulting from piecewise constant inputs as a formal exponential product expansion involving iterated Lie brackets. If the initial and final states are sufficiently close, the algorithm produces a path which moves the system closer to the desired state by at least a half. By repeated application of the algorithm it is possible to move the system into an arbitrary neighborhood of the desired state. For nilpotent systems, i.e., systems for which all iterated Lie brackets of sufficiently high order are zero, the algorithm provides exact steering. Examples of nilpotent systems include systems in chained and in power form. Actually, the algorithm can be based on other types of switching inputs, not necessarily piecewise constant inputs; see [73] for details. In a related paper, Jacob [49] proposed an algorithm for exact steering of nilpotent systems using piecewise constant or polynomial inputs. His algorithm is similar to Lafferriere and Sussmann's but with some modifications in the construction procedure, resulting in simpler paths. Reyhanoglu [123] developed a similar motion planning algorithm for Chaplygin systems. As opposed to Lafferriere and Sussmann's work, Reyhanoglu uses more elementary tools, Stokes theorem and Taylor series expansion, and his version of the algorithm results in simpler computations.

Another set of tools, based on averaging theory, has been developed by Gurvits and Li [43], Leonard and Krishnaprasad [78], Liu [86], and Sussmann and Liu [154] for kinematic models of nonholonomic control systems. The basic idea is to use high-frequency, high-amplitude periodic control inputs to generate motions in the directions of iterated Lie brackets. The averaged system, obtained in the limit of these high-frequency inputs, is steered exactly, while the original system is steered approximately, within a specified tolerance. Tilbury et al. [158] examine a variety of implementation issues pertinent to the asymptotic sinusoidal steering algorithm of Sussmann and Liu [154] in the context of steering kinematic car-like systems with trailers. Specifically, it is shown that preliminary state and control transformations may facilitate convergence to the averaged trajectory. Although high-frequency control inputs may be undesirable from an implementation point of view, the high frequency can be avoided by selecting the time interval, over which the system is steered approximately, to be large.

A dual approach to motion planning, which relies on exterior differential forms instead of Lie brackets, has been developed by Murray [106], Sluis et al. [139], and Tilbury et al. [159, 160, 162]. The basic idea is to represent nonholonomic constraints as a Pfaffian system of exterior differential forms. The differential form formulation is dual to the vector field formulation but it provides certain advantages in terms of computations, constructing state and control transformations and understanding the geometry of the problems.

The concept of a flat nonlinear system [38, 39, 113] is useful in solving certain nonholonomic motion planning problems. Consider the nonholonomic control system (1). If there exists an output, with dimension equal to the dimension of the control input, which is a function of the state, of the control input, and of the derivatives of the control input, such that the state and the control input can be expressed as functions of the output and the derivatives of the output, then (1) is called differentially flat and the output is called the flat output. For a differentially flat system motion planning reduces to prescribing a smooth output function satisfying boundary conditions imposed by the initial and final state specification. The desired control input and the trajectory can be obtained by differentiating the prescribed output function and no integration is required. Rouchon et al. [127, 128] showed that many mobile robot systems, e.g., an automobile with multiple trailers, are flat. They exploited this notion of flatness in solving motion planning problems for the front-wheel drive automobile with multiple trailers. The flat output is provided by the Cartesian coordinates of the last trailer. For system (12) the flat output is given by $y = (x_2, x_3)$; then $x_1 = \dot{y}_2 / \dot{y}_1$, $x_2 = y_1$, $x_3 = y_2$, $u_1 = (\ddot{y}_2\dot{y}_1 - \dot{y}_2\ddot{y}_1)/(\dot{y}_1)^2$, $u_2 = \dot{y}_1$, and the motion planning problem reduces to prescribing output functions $y_1(t)$, $y_2(t)$ satisfying boundary conditions imposed by the initial and final state specification and $\dot{y}_1(t) \neq 0$. In [87] it is shown that any kinematic nonholonomic system of the form (1) with $n = 5$ and $m = 2$ is flat. An example of a system which is not flat is provided by the ball rolling on the plane without slipping [13, 19, 82].

Motion Planning Using Geometric Phases. For the class of nonholonomic Chaplygin systems, a variety of techniques based on the use of the geometric phase, or holonomy [89], is available, see the papers by Bloch et al. [12], Gurvits and Li [43], Krishnaprasad et al. [68, 69], Li and Montgomery [83], Mukherjee and Anderson [101], and Reyhanoglu et al. [122, 124, 126]. Referring to nonholonomic control systems of kinematic Chaplygin type (2), suppose the base vector y undergoes a cyclic motion $y(t)$, $0 \leq t \leq 1$, satisfying $y(0) = y(1)$. The resulting change in the fiber vector, $z(1) - z(0)$, can be written as a line integral along the path of the base vector:

$$z(1) - z(0) = \oint_\gamma g(y)dy \tag{13}$$

where $\gamma = \{y(t) : 0 \leq t \leq 1\}$ is the base vector path. The value of this line integral is independent of any specific parameterization of the path and depends only on the geometry of the path; hence, this value is referred to as the geometric phase. Thus for Chaplygin systems the motion planning problem reduces to finding an appropriate base space path which produces the desired geometric phase. By considering a parameterized finite dimensional family of base space paths, the problem can be reduced to a root-finding problem for the value of the parameters.

To solve this root-finding problem, it is essential to be able to evaluate the geometric phase produced by a given base space path. Using Stokes theorem and Taylor series expansion, it is possible to expand the line integral (13) into a series which involves iterated Lie brackets of $g_1, ..., g_m$ [123]. For a nilpotent system this series terminates after a finite number of terms, thereby providing an explicit expression for the geometric phase. If the system is not nilpotent, numerical evaluation of (13) may be feasible. The use of Stokes theorem to reduce the line integral in (13) to a surface integral has been effective in a number of motion planning problems, e.g., [68, 83, 101, 122].

Gurvits and Li [43] showed that for a completely nonholonomic Chaplygin system any value of the geometric phase can be obtained by tracing paths in the base space which consist of appropriately constructed rectangular subpaths or other types of "elementary" subpaths. Rectangular subpaths are particularly convenient and have been used in a number of other studies [12, 82, 122, 124, 126].

Motion Planning Using Parameterization of the Input. A more elementary method for motion planning is also available. This method is based on parameterization of the input within a given finite dimensional family of functions. Consider the kinematic model of a nonholonomic control system of the form (1). The objective is to steer the system from a given initial state $x^i \in \mathbb{R}^n$ to a specified final state $x^f \in \mathbb{R}^n$ over a time interval $[0, T]$. Let $\{U(\alpha; \cdot) : \alpha \in \mathbb{R}^q\}$ be a parameter-dependent family of control inputs $U(\alpha; \cdot) : [0,1] \to \mathbb{R}^m$, where $\alpha \in \mathbb{R}^q$ is a parameter. Let $\hat{x}(\alpha; t)$, $0 \leq t \leq 1$ denote the solution to (1) with $\hat{x}(\alpha; 0) = 0$ and $u(t) = U(\alpha; t)$, $0 \leq t \leq 1$. Let $G : \mathbb{R}^q \to \mathbb{R}^n$ be defined by $G(\alpha) = \hat{x}(\alpha; 1)$. If the control family $\{U(\alpha; \cdot) : \alpha \in \mathbb{R}^q\}$ is sufficiently rich, G is onto $\mathbb{R}^n$. In this case the control input $\hat{u}(x; t)$, $0 \leq t \leq 1$, which steers the system from the origin to $x \in \mathbb{R}^n$, can be defined by setting $\hat{u}(x; t) = U(\alpha; t)$, $0 \leq t \leq 1$, where α is a solution to $G(\alpha) = x$. Since system (1) is drift-free, by time-rescaling it can be verified that the control

$$u(t) = \begin{cases} -\frac{2}{T}\hat{u}\left(x^i; 1 - 2t/T\right), & 0 \leq t \leq .5T, \\ \frac{2}{T}\hat{u}\left(x^f; 2t/T - 1\right), & .5T \leq t \leq T \end{cases}$$

steers the system (1) from x^i to x^f over the time-interval $[0, T]$.

This simple idea appears in the work of many researchers. For example, Bushnell et al. [20], Murray [103], Murray and Sastry [105], show how to steer systems in power or chained form using a family of sinusoids at integrally related frequencies. Lewis et al. [80] showed that sinusoids at integrally related frequencies can be used to steer the Snakeboard. The use of other control families, e.g., piecewise constant inputs or polynomial inputs, has also been investigated by Jacob [49], Tilbury [159], and Tilbury et al. [160]. The multirate digital control approaches developed by Chelouah et al. [27], Monaco and Norman-Cyrot

[97], Sordalen and Egeland [144], Tilbury and Chelouah [157], can be also viewed as a way of steering a system via parameterization of the input within a family of piecewise constant inputs. The basic idea of the multirate digital control approach is to generate the input by a zero order hold and steer the resulting discrete time system; typically, different sampling rates are used for different input channels. Depending on the nature of control strategies and their interpretation, the multirate digital control approaches may be viewed as feedback strategies [144]. Divelbiss and Wen [33] propose to use gradient descent algorithms for determining a solution to $G(\alpha) = x$. The approach can be used to steer both kinematic and dynamic models of nonholonomic systems.

To illustrate this simple approach with an example, consider steering the mobile robot whose kinematics is modeled by Equation (12). Let $\alpha \in \mathbb{R}^3$ and consider the family of control inputs $U(\alpha; t) = (U_1(\alpha; t), U_2(\alpha; t))$ defined as

$$U_1(\alpha; t) = \begin{cases} \alpha_1, & 0 \le t \le .5, \\ \alpha_3 \sin 4\pi t, & .5 \le t \le 1, \end{cases}$$

$$U_2(\alpha; t) = \begin{cases} \alpha_2, & 0 \le t \le .5, \\ |\alpha_3| \cos 4\pi t, & .5 \le t \le 1, \end{cases}$$

By integrating the equations of motion with $x_1(0) = x_2(0) = x_3(0) = 0$ over $[0, 1]$, we obtain

$$G(\alpha) = \begin{bmatrix} \dfrac{\alpha_1}{2} \\ \dfrac{\alpha_2}{2} \\ \dfrac{\alpha_1 \alpha_2}{2} - \alpha_3 |\alpha_3| / (8\pi) \end{bmatrix}.$$

Clearly, G is onto $\mathbb{R}^3$ and the system can be steered to any configuration as described above.

The input parameterization approach provides a convenient basis for introducing neural networks or other learning schemes. They can be employed to identify the map G on-line, and, after the training phase is completed, they can be used to approximate the solution to $G(\alpha) = x$. A neural network approach to nonholonomic motion planning has been studied in the context of free-flying multibody systems in space by Gorinevsky et al. [44], Sadegh [130], and in the context of mobile robots by Gorinevsky et al. [45].

Optimal Motion Planning

Although the methodologies of the previous section provide a solution to the motion planning problem, there often exist many solutions. A specific solution can be selected using optimization.

Brockett [18] and Brockett and Dai [19] demonstrated the optimality of, respectively, sinusoidal and elliptic control functions for certain minimum norm nonholonomic optimal control problems. The optimality of elliptic functions has been also addressed by Krishnaprasad and Yang in [68]. Reeds and Shepp [121] obtained a complete characterization of the shortest paths connecting any two given configurations for the mobile robot (11). They showed that the shortest path is one of 48 extremal paths that can be explicitly computed. Each of the extremal paths has no more than five segments and requires no more than two direction reversals. Montgomery [99] studied the so called isoholonomic optimal control problem. For Chaplygin systems, this isoholonomic problem is to determine a closed base space path which produces the desired geometric phase and has the minimal "length." The "length" of the path is essentially the energy required to trace the path. Montgomery showed that solving this isoholonomic problem is equivalent to solving certain Hamiltonian differential equations with appropriate boundary conditions. As an application he considered the optimal control of deformable bodies with zero angular momentum. Further use of optimal control for free-floating deformable bodies has been investigated by Krishnaprasad and Yang [68] and by Li and Montgomery in [83].

Conditions for optimality in various nonholonomic optimal control problems are discussed by Bloch and Crouch [13], Sastry and Montgomery [137], and Montgomery [100]. In particular, Sastry and Montgomery [137] and Montgomery [100] study the optimal control problem of minimizing the L_2-norm of the control subject to given initial and final states and subject to Equation (1). Using the maximum principle, they show that the optimal control is such that the quantity $\sum_{i=1}^{m} |u_i(t)|^2$ remains constant. For the same problem, Montgomery [100] considers in detail the case of abnormal (singular) extremals. He demonstrates that abnormal extremals may provide an optimal solution and thus cannot be neglected in analysis. He also considers a time-optimal control problem for nonholonomic control system (1). Walsh et al. [171] study related minimal norm control problems for kinematic systems evolving on Lie groups. Numerical strategies for constructing optimal trajectories in various nonholonomic control problems have been proposed by Agrawal and Xu [1], Fernandes et al. [36, 37], Hussein and Kane [47], and Murthy and Keerthi [110].

Motion Planning with Obstacle-Avoidance

Another area of active research is nonholonomic motion planning with obstacle-avoidance. This problem has been addressed by Barraquand and Latombe [8], Divelbiss and Wen [34], Gurvits and Li [43], Jacobs et al. [50], Laumond [75], Laumond et al. [77], Mirtich and Canny [96], and Sahai et al. [131]. The problem of planning a collision-free path for a nonholonomic control system is more difficult than for a holonomic control system. This is because an arbitrary collision-free path in the configuration space does not necessarily correspond to a feasible trajectory for the nonholonomic system (Fig. 4).

One general approach is to first construct a path connecting initial and final states which avoids obstacles but is not necessarily feasible, i.e., it does not necessarily satisfy the instantaneous nonholonomic constraints. Then, high-frequency high amplitude periodic inputs can be used to generate an approximating feasible path [43,154]. Although this method produces a collision-free path, this path may not be particularly nice.

For simple nonholonomic systems of mobile robot type it is possible to plan better paths using skeletons [96]. The skeleton is a collection of fixed (typically nonfeasible) paths which stay maximally clear from the obstacles. The system is forced to loosely follow the skeleton from an initial state to a final state

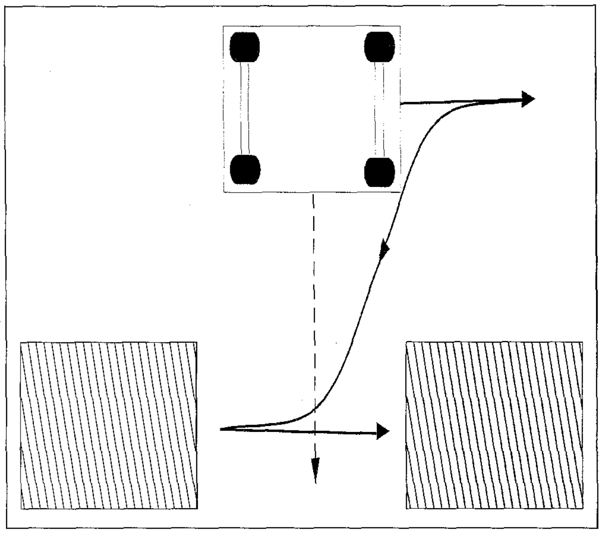

Fig. 4. Parallel parking of a front-wheel drive automobile: non-feasible motion (dashed) and feasible motion (solid).

while avoiding obstacles. Since the skeleton stays maximally clear from the obstacles, the resulting path tends to be of low complexity.

A different approach, also based on approximating a nonfeasible path by feasible path segments, is developed in [50] for a car-like mobile robot. This method has been extended in [77] to a car-like mobile robot towing a trailer. Barraquand and Latombe [8] use the potential field method to produce collision-free paths for mobile robots, while attempting to minimize the number of maneuvers required.

Feedback Stabilization of Nonholonomic Systems

Stabilization problems are concerned with obtaining *feedback laws* which guarantee that an equilibrium of the closed-loop system is asymptotically stable. For a linear time-invariant system, if all unstable eigenvalues are controllable, then the origin can be asymptotically stabilized by a linear time-invariant static state feedback. For nonholonomic systems the situation is more complex. The linearization of a nonholonomic system about any equilibrium is not asymptotically stabilizable. Consequently, linear stabilization tools cannot be used even locally. Moreover, there is a fundamental obstruction to existence of smooth (or even continuous) time-invariant feedback laws.

To gain insight to why this is the case, consider the problem of stabilizing system (12) to the origin. Suppose a time-invariant static state feedback is provided by smooth functions $u_1(x_1, x_2, x_3)$, $u_2(x_1, x_2, x_3)$ with $u_1(0,0,0) = 0$, $u_2(0,0,0) = 0$. The equilibria of the closed-loop system are solutions of $u_1(x_1, x_2, x_3) = 0$ and $u_2(x_1, x_2, x_3) = 0$. Consequently, there exists (locally) a one-dimensional manifold of equilibria which passes through the origin. Thus the origin cannot be locally asymptotically stabilized by any smooth static time-invariant state-feedback. A generalization of this informal observation is Brockett's necessary condition for feedback stabilization [17]. Its implication for nonholonomic control systems is that there exists no smooth (or even continuous) time-invariant static state-feedback which makes a specified equilibrium of the closed loop locally asymptotically stable [12, 21]. Moreover, there exists no dynamic continuous time-invariant feedback controller which renders the closed loop locally asymptotically stable [119]. Consequently, a nonholonomic control system cannot be asymptotically stabilized to an equilibrium using feedback linearization or any other control design approach that uses smooth time-invariant feedback. These conclusions are applicable to both kinematic and dynamic models of nonholonomic control systems. Although there exists no closed loop stabilizing smooth time-invariant dynamic feedback controller, there still may exist a smooth time-invariant dynamic controller which stabilizes the plant states only [46]. Further information on necessary conditions for feedback stabilization and their implications can be found in the books by Sontag [140] and Nikitin [114].

Despite the limitations imposed by Brockett's necessary condition, the completely nonholonomic assumption guarantees that there do exist feedback strategies which do (locally) asymptotically stabilize an equilibrium. A number of approaches have been proposed for stabilization of nonholonomic control systems to an equilibrium. The approaches can be classified as discontinuous time-invariant stabilization, time-varying stabilization, and hybrid stabilization. These developments are reviewed below. Several feedback laws will be illustrated numerically for the mobile robot whose kinematics is modeled by Equation (12).

Discontinuous Time-Invariant Stabilization

Discontinuous time-invariant feedback controllers for stabilization of nonholonomic systems to the origin can be classified into two types: piecewise continuous controllers and sliding mode controllers. These two types of discontinuous controllers are now described in more detail.

In [153] Sussmann proved existence of stabilizing piecewise continuous static state-feedback control for a class of nonlinear controllable systems. This class includes kinematic and dynamic models of nonholonomic control systems satisfying real analyticity assumptions. Lafferierre and Sontag [71] presented a formula for a piecewise continuous feedback law, obtained from a piecewise smooth control Lyapunov function. The resulting feedback is globally stabilizing; it has discontinuities on a separating surface of codimension one. While there are no general methods for constructing control Lyapunov functions satisfying assumptions of [71], specific examples of piecewise continuous stabilization have been reported by Lafferriere and Sontag in [71], by Canudas de Wit and Sordalen in [22] and by Khennouf and Canudas de Wit [56, 26]. In these examples exponential convergence of the states to the equilibrium has been demonstrated. In [149] Sordalen et al. proposed a piecewise continuous feedback law for local stabilization of the attitude of an underactuated rigid spacecraft with only two angular velocity controls. The feedback law results in exponential convergence rates of the states to the equilibrium.

A different approach to constructing piecewise continuous controllers has been developed by Aicardi et al. [2], Astolfi [3, 4], and Badreddin and Mansour [5]. There a nonsmooth state transformation is used to overcome the obstruction to stabilizability due to Brockett's theorem. A smooth time-invariant feedback is used to stabilize the transformed system. In the original coordinates the resulting feedback law is discontinuous. In [2, 3] this approach has been used for stabilization of kinematic and dynamic models of simple mobile robots. In these examples the nonsmooth state transformation is provided by changing Cartesian coordinates to polar coordinates. The potential of this approach for more complicated nonholonomic control problems remains to be investigated.

Time-invariant feedback laws can be developed using the sliding mode approach proposed by Bloch and Drakunov [11] and by Guldner and Utkin [41]. These discontinuous feedback laws force the trajectory to eventually slide along a manifold of codimension one towards the equilibrium. Consider, for example, the problem of stabilizing system (12) to the origin. Define the feedback law according to [11]

$$u_1 = -x_1 + 2x_2 \operatorname{sign}\left(x_3 - \frac{x_1 x_2}{2}\right), \tag{14a}$$

$$u_2 = -x_2 + 2x_1 \operatorname{sign}\left(x_3 - \frac{x_1 x_2}{2}\right), \tag{14b}$$

where $\operatorname{sign}(\cdot)$ denotes the signum function. Let $V(x_1, x_2) = \frac{1}{2}\left(\frac{x_1^2}{4} + x_2^2\right)$. Then the derivative of V along the closed-loop trajectories of (12) satisfies $\dot{V} = \frac{1}{4}x_1 u_1 + x_2 u_2 = -2V$. Thus $V(t) = V(0)e^{-2t} \to 0$ as $t \to \infty$ and $x_1 \to 0$, $x_2 \to 0$ as $t \to \infty$. Let $\theta = x_3 - \frac{x_1 x_2}{2}$. Then, $\dot{\theta} = -2V \operatorname{sign}(\theta)$. Clearly, $|\theta(t)|$ is nonincreasing and, in fact, can reach zero in finite time provided that

$$V(x_1(0), x_2(0)) > |\theta(0)| . \tag{15}$$

Once $\theta(t)$ reaches the origin, it must stay at the origin; hence, the trajectory will slide along the surface $x_3 = \frac{x_1 x_2}{2}$ toward the origin. If the initial conditions do not satisfy Inequality (15), a preliminary control can be used to force the trajectory into the region where Inequality (15) holds and then the feedback law (14) should be switched on. The sliding mode construction is also available for certain classes of higher-dimensional kinematic nonholonomic control systems and dynamic models of nonholonomic control systems; see [11, 41] for details. For a general class of nonholonomic systems, construction of sliding mode controllers is not available and remains a subject of future research.

The disadvantage of the sliding mode controllers is that they may cause chattering. Guldner et al. [42] have proposed to use smoothing between gradients on either side of the sliding surface to prevent chattering. Piecewise continuous controllers usually avoid chattering as the trajectory does not "get stuck" at the discontinuities. It should be also pointed out that controlling kinematic nonholonomic systems with discontinuous (velocity) controls may be difficult to implement. Formulations involving dynamic nonholonomic control systems seem preferable if discontinuous controllers are used.

Time-Varying Stabilization

Various methods for designing time-varying controllers have been proposed in the literature. The use of time-varying feedbacks originated in the mobile robot work by Samson [132, 133, 134]. Coron [30] showed that kinematic nonholonomic control systems can be asymptotically stabilized to an equilibrium point by smooth time-periodic static state feedback. The existence proof in [30], however, does not provide feedback laws. Explicit feedback construction procedures are available. Murray et al. [103], Teel et al. [156], and Walsh and Bushnell [168] used the method of averaging and saturation type functions to construct smooth time-periodic feedback laws for systems in power and chained form. These feedback laws achieve global asymptotic stabilization. In [168] numerical simulations illustrate the resulting feedback laws for a fire truck example, a nonholonomic system with three inputs and five states. Samson and Ait-Abderrahim [134], and Walsh et al. [166] provide a different asymptotic stabilization scheme based on construction of a nominal motion which asymptotically approaches the equilibrium. A linear controller is constructed which stabilizes the variational system about the nominal motion. This approach can be used to construct time-varying stabilizing controllers for nonholonomic systems but it does require a priori selection of a nominal motion. Another general constructive approach has been proposed by Pomet [119] and Coron and Pomet [32]. This approach, widely known as Pomet's method, is based on Lyapunov's direct method and is, to some extent, similar to the well-known technique of Jurdjevic and Quinn [52]. Pomet's method generates smooth time-periodic feedback laws and also provides closed loop Lyapunov functions.

As an illustration, consider the following smooth feedback law for system (12) provided by Pomet's method:

$$u_1(x, t) = -x_1 + x_3(\sin t - \cos t) , \tag{16a}$$

$$u_2(x, t) = -x_2 = x_1 x_3 - x_1(x_1 + x_3 \cos t) \cos t . \tag{16b}$$

This feedback law is obtained from the closed-loop Lyapunov function given by

$$V(x, t) = \frac{1}{2}(x_1 + x_3 \cos t)^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2$$

By applying the Krasovskii-LaSalle invariance principle for periodic systems it can be verified that the origin is the globally asymptotically stable equilibrium of the closed loop. Fig. 5 shows a typical planar path of the center of mass of the mobile robot (11) under this feedback law.

Unfortunately, the rates of convergence provided by smooth time-periodic feedback laws are necessarily nonexponential [103]. For system (12) smooth time-periodic controllers can provide time rates of convergence of at most $1/\sqrt{t}$ [132]. Furthermore, in experimental work [91], M'Closkey and Murray have demonstrated that smooth time-periodic feedback laws do not steer mobile robots to a small neighborhood of the desired configuration in a reasonable amount of time. Thus feedback laws which provide faster convergence rates are desirable. These feedback laws must necessarily be nonsmooth (i.e., nondifferentiable). Further information on connections between the rates of convergence and smoothness of feedback laws can be found in references [43, 94].

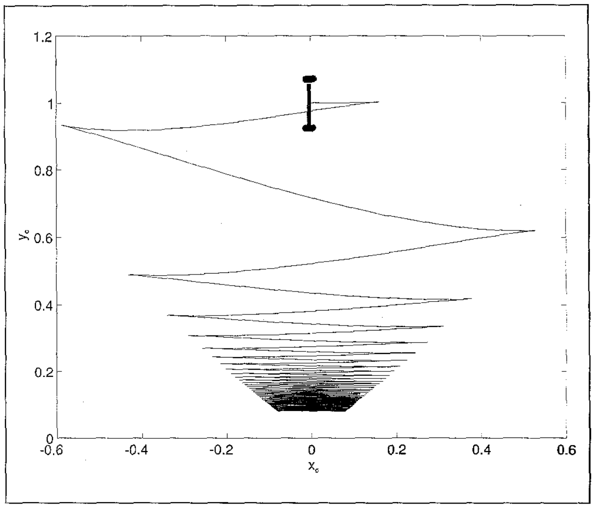

Fig. 5. Trajectory of the center of mass under smooth time-periodic feedback.

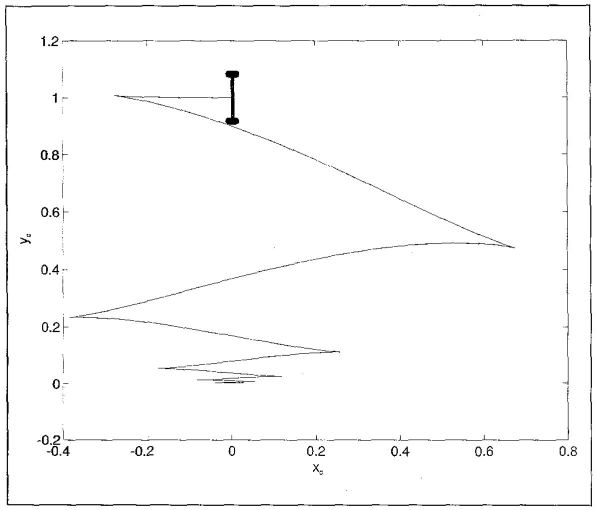

Fig. 6. Trajectory of the center of mass under nonsmooth time-periodic feedback.

The issue of existence of time-periodic feedback laws that provide exponential convergence rates for kinematic models of nonholonomic systems has been resolved in a paper by Coron [31]. There the existence of continuous time-periodic static state feedback laws that are smooth everywhere except at the origin and result in exponential convergence rates is established. As in the smooth case, the existence proof does not explicitly provide feedback laws. A construction procedure which provides nonsmooth feedback laws with exponential convergence rates has been proposed by M'Closkey and Murray in [90]. The resulting feedback laws are continuous, smooth everywhere except at the origin, and homogeneous with respect to a nonstandard dilation [90, 92]. The construction procedure can be viewed as an extension of Pomet's algorithm to the case of nonsmooth homogeneous feedback laws. For systems in power form, explicit expressions for the feedback laws can be obtained. For example, for system (12) a nonsmooth time-periodic feedback law which results in exponential convergence rates is of the form:

$$u_1(x,t) = -x_1 + \frac{x_3}{\rho(x)}\cos t, \quad x \neq 0\,, \tag{17a}$$

$$u_2(x,t) = -x_2 - \frac{x_3^2}{\rho^3(x)}\sin t\,, \quad x \neq 0\,, \tag{17b}$$

$$u_1(0,t) = u_2(0,t) = 0\,, \tag{17c}$$

where $\rho(x) = \left(x_1^4 + x_2^4 + x_3^2\right)^{(1/4)}$. The closed-loop system is globally exponentially stable with respect to the homogeneous norm $\rho(x)$, i.e., there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $\rho(x(t)) \leq \lambda_1\rho(x(0)) \exp(-\lambda_2 t)$. This notion of exponential stability with respect to a homogeneous norm is different from the standard notion of exponential stability, but it does imply exponential rates of convergence. Fig. 6 shows a typical planar path of the center of mass of the mobile robot (11) under this feedback law.

Time-periodic feedback laws for stabilization of dynamic models of nonholonomic control systems can be derived from kinematic controllers using the integrator backstepping or "error tracking" approaches. For details the reader is referred to references [57, 93, 168].

Besides mobile robots [91, 132, 133, 134, 135, 168], time-varying stabilization has been used for knife-edge models with augmented actuator dynamics [57], underactuated rigid spacecraft controlled by only two rotors [98, 167], and free-floating multibody spacecraft [61].

Hybrid Feedback Laws

The stabilization techniques described so far provide continuous-time feedback laws. Other methods, which generate hybrid controllers, are also available. Typically, hybrid controllers combine continuous-time features with either discrete-event features or discrete-time features. The operation of hybrid controllers is based on switchings at discrete-time instants between various low-level continuous-time controllers. The time-instants at which switches occur may be either specified a priori or be determined in the process of controller operation.

Controllers which combine continuous time features with discrete event features have been proposed by Bloch et al. [12] and by Kolmanovsky et al. [58, 59] for the class of nonholonomic Chaplygin systems. These controllers consist of a discrete event supervisor and low-level time-invariant feedback controllers. The supervisor configures the low-level feedback controllers and accomplishes switchings between them in a way that provides stabilization of the system. Each of the low-level feedback controllers forces the base variables to trace a specific straight line segment of the base space path, which is selected by the supervisor to produce the desired geometric phase change. The feedback law provides finite time (dead-beat) responses. This approach has also been used by Krishnan et al. [65, 66, 67] for the attitude stabilization of rigid underactuated spacecraft with only two control torques.

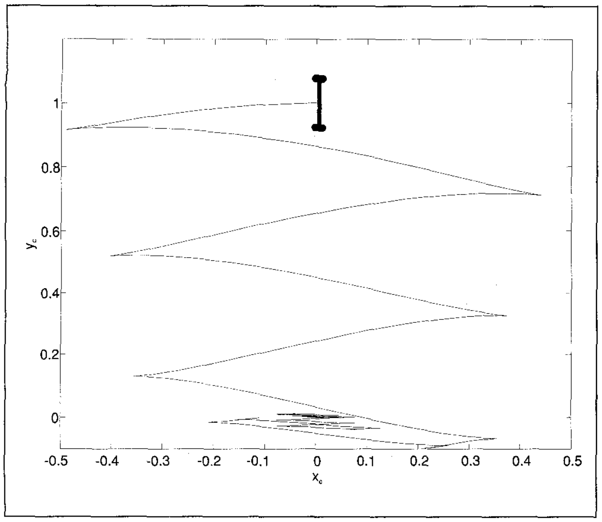

Fig. 7. Trajectory of the center of mass under hybrid feedback.

Hybrid controllers which combine continuous time and discrete time features have been developed by a number of authors. For example, Sordalen et al. [145, 146] developed hybrid controllers for stabilization of kinematic nonholonomic systems in chained form. These controllers result in exponential convergence rates of the states to the origin. Hybrid controllers of a different type for systems in chained form have been proposed by Canudas de Wit et al. [24]. These controllers provide practical stabilization, i.e., stabilization to a small neighborhood of the origin. The construction procedure in [24] is also applicable to dynamic extensions of the chained form. The approach proposed by Sontag [141, 143] is applicable to larger classes of kinematic nonholonomic control systems, but is less explicit. Sontag makes use of a family of periodic inputs that are universal nonsingular controls [141] and result in periodic trajectories. Linearization about each of these trajectories is controllable. Consequently, a perturbation of a periodic input can be constructed to bring the state closer to the origin at the end of each cycle. A good introduction to some of hybrid stabilization techniques in the mobile robot context is contained in the article [25].

A different approach, which makes use of a family of periodic inputs, has been proposed by Kolmanovsky and McClamroch [60]. This approach is applicable to a large class of kinematic and dynamic models of nonholonomic Chaplygin systems, including systems in power form. We illustrate this approach with an example. Consider the problem of stabilizing system (12) to the origin. A hybrid feedback law can be defined as [60]:

$$u_1(t) = -x_1 + \alpha^k \sin t\,,\ 2\pi k \le t \le 2\pi(k+1) \qquad (18a)$$

$$u_2(t) = -x_2 + |\alpha^k| \cos t\,,\ 2\pi k \le t \le 2\pi(k+1) \qquad (18b)$$

where $\{\alpha^k : k = 0, 1, 2, ...\}$ is a sequence of scalar parameters. The control family (18) can be viewed as a one-parameter family of low-level continuous-time controllers. The controllers are indexed by the value of the parameter α^k. To render the origin globally attractive, the sequence $\{\alpha^k\}$ should be selected according to the following feedback algorithm [60]: Let $0 < \gamma < 1$.

1. For k = 0: If $x_1(0) = x_2(0) = x_3(0) = 0$, set $\alpha^0 = 0$; else select any $\alpha^0 \neq 0$.
2. For k > 0: If $x_3(2\pi k)\alpha^{k-1} \ge 0$, set $\alpha^k = \alpha^{k-1}$. If $x_3(2\pi k)\alpha^{k-1} < 0$, set $\alpha^k = \gamma|\alpha^{k-1}|\mathrm{sign}(x_3(2\pi k))$.

Fig. 7 shows a typical planar path of the center of mass of the mobile robot (11) under this feedback law.

Current and Future Research Topics

As we have indicated, significant progress has been made in the study of nonholonomic control systems. Research continues on models of nonholonomic control systems, and on control design for motion planning and stabilization. There is now in place a strong theoretical base for research applications, particular those associated with wheeled vehicles and spacecraft (with a conserved angular momentum function); research in these, and other, applications is proceeding. The knowledge base about nonholonomic control systems is slowly maturing, as we have demonstrated, but the research momentum is expected to continue for some time. However, in our opinion research progress has been spotty, and there remain important topics that require attention; in the subsequent paragraphs we provide a brief overview of some of these topics.

The formulation of models of nonholonomic control systems is a central feature of this research. The models that we have introduced can be generalized in a number of ways. We now identify certain of these generalizations and we indicate where they arise in applications. Noncatastatic nonholonomic control systems, expressed in dynamic form, are a generalization of Equation(5) by the addition of a state-dependent and time-dependent drift term to (5a). Such models can arise from multibody systems where the angular momentum is conserved but is not zero [63, 150]. Another important generalization is the class of underactuated nonholonomic control systems. Recall that to transform d'Alembert's formulation of equations of motion (6) into the form of Equation (5), we assumed a sufficient set of inputs were available so that (8) could be feedback-linearized to obtain (9); if this condition is not satisfied, a new class of underactuated nonholonomic systems is obtained [116]. A related infinite-dimensional generalization, motivated by a free-floating beam with distributed flexibility, has been studied in [61]. An important theoretical research topic, with practical implications, is establishing equivalence relations between different models of nonholonomic control systems, under state and feedback transformations; the methods of exterior calculus and differential forms are important tools as indicated in [106, 139, 159]. There is a close connection between the coordinate dependent models of nonholonomic control systems that we have described here and the abstract formulation of control problems for rigid body motions defined on Lie groups; this latter approach has been developed in [69, 70, 78, 79, 170, 171].

There are many important research problems for nonholonomic control systems that have been little studied. Here we identify the problem of control of nonholonomic systems when there are model uncertainties, as arise from parameter variations or from neglected dynamics. It is not necessary to motivate the importance of this problem, but it is curious that there is little published literature that deals directly with these questions for nonholonomic control systems; a few preliminary results are available in [26]. The difficulties are primarily technical. General methods for study of robustness for this class of nonlinear systems are not available. Consequently, methods for design of robust controllers for nonholonomic systems are unknown. We are aware of a single paper by Jiang and Pomet [51] on adaptive control of nonholonomic systems in chained form. A very important situation requiring attention arises from model perturbations that destroy the "nonholonomic assumption." For example, the no-slip condition for bodies moving in contact or the zero-angular-momentum condition for multibody spacecraft may only hold approximately. There is much to be gained by studying the effects of these perturbations and by developing control designs that perform well in the presence of uncertainties.

In contrast with the extensive theoretical developments, there have been relatively few experimental results reported that make use of the developed theory. Experimental work on nonholonomic control systems has been described in [91, 115, 117, 169]. It is important that more experimental based research be undertaken to demonstrate the value and limitations of the theory of nonholonomic control systems that now exist.

In summary, we believe that much progress has been made in the development of a theoretical foundation for nonholonomic control systems. However, there remain many unsolved, and even unexplored, problem areas that should provide a source of challenge for the future. We also believe that there is much to be gained by maintaining a close connection between the theory and the applications in this future research.

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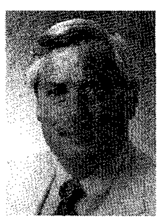
Ilya Kolmanovsky has received M.S. and Ph.D. degrees in aerospace engineering from the University of Michigan, Ann Arbor, MI. His research interests are in the area of dynamic systems and nonlinear control theory with applications to control of mechanical systems.



N. Harris McClamroch received a Ph.D. degree in engineering mechanics from The University of Texas at Austin. Since 1967 he has been at The University of Michigan, Ann Arbor, MI, where he is currently a professor and chair of the Department of Aerospace Engineering. His main research interests are in the stability and control of nonlinear feedback systems, including applications to a variety of mechanical systems problems. He has also published in the areas of sensitivity and robustness, optimal control, digital feedback, estimation, and stochastic control. He has worked on applications problems arising in flexible space structures, robotics and automated manufacturing, and space multibody systems. McClamroch is the author of numerous technical papers and one book. He is a Fellow of the IEEE and a past editor of the *IEEE Transactions on Automatic Control.*