TRACKING CONTROL OF NONLINEAR MECHANICAL SYSTEMS

PROEFSCHRIFT

ter verkrijging van
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door

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Dit proefschrift is goedgekeurd door de promotoren

prof. dr. ir. H. Kwakernaak,

prof. dr. H. Nijmeijer.

Ik droomde eens en zie ik liep
aan ’t strand bij lage tij.
Ik was daar niet alleen, want ook
de Heer liep aan mijn zij.
We liepen saam het leven door
en lieten in het zand,
een spoor van stappen, twee aan twee;
de Heer liep aan mijn hand.
Ik stopte en keek achter mij
en zag mijn levensloop,
in tijden van geluk en vreugd
van diepe smart en hoop.
Maar als ik goed het spoor bekeek,
zag ik langs heel de baan,
daar waar het juist het moeilijkst was,
maar één paar stappen staan . . .
Ik zei toen: “Heer, waarom dan toch?”
Juist toen ’k U nodig had,
juist toen ik zelf geen uitkomst zag
op ’t zwaarste deel van ’t pad . . .
De Heer keek toen vol liefd’ mij aan
en antwoordd’ op mijn vragen:
“Mijn lieve kind, toen ’t moeilijk was,
toen heb Ik jou gedragen . . . ”

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Preface

How to start a preface to a thesis that should give an overview of what I have been doing for almost four years? That is not an easy question to answer. Paradoxically, by writing down this question I also found an answer to it.

If one comes to think about it, this preface is full of paradoxes. It is funny to realize that this probably is the first page that people read (except from the title, and for Dutch readers also the poem, I presume), whereas it is one of the last pages I wrote. I could elaborate more on these paradoxes, but that would make this first page so philosophical and paradoxical that even this first page will not be read . . .

At a time like this, as a period of my life is about to finish, I can not avoid looking back at it, and looking forward too. While typing this sentence, my thoughts went out to several people. These include Henk, Guido, Anders, Kristin, Toño & Lena, my committee, colleagues, parents and not least of all Wieke (thinking about periods of my life . . .). They all are people who, amongst others, deserve credit for their contributions (in one way or the other) to this thesis.

First of all I would like to thank my supervisor and promotor Henk Nijmeijer. During the last five years we have come to know each other quite well. I am grateful for the opportunity of working with him, for the possibilities of visiting several colleagues, and for making me put things in the right perspective. I am really looking forward to our collaboration in Eindhoven in the near future.

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I would like to take the opportunity to thank all my (former) colleagues at the Faculty of Mathematical Sciences in Twente, and especially those of the Systems, Signals and Control group for having provided me with such a creative and friendly atmosphere to work in. Special thanks also go to the Systems Engineering group in Eindhoven (my current colleagues) for giving me the opportunity to finish my thesis quietly and showing me a glimpse of the challenges I will face in the near future.

A special word of thanks goes to Henk Ernst Blok, Paul Huijnen, Kristin Pettersen, and Phil Chimento who sacrificed themselves for going through (parts of) a draft version of this thesis. They all contributed to this thesis with their valuable comments.

I would finally like to thank all my friends, my beloved family, and all the people whose name I did not mention explicitly. A special word of thanks goes to my fiancée Wieke Fikse for all her support and help.

Erjen Lefeber

Contents

Chapter 1

Introduction

Nowadays, control systems are inevitable. They appear almost everywhere: in our homes (in e.g., radio, television, video, CD-player), in several types of vehicles (in e.g., automobiles, airplanes, spacecrafts, ships), in industry (e.g., robots, process control), in telecommunications, in biomedical engineering, and in numerous other places and situations.

Besides the growing usage of control systems, the requirements for a control system increase considerably, resulting in more, and more complex control systems. In order to be able to design more complex control systems for a larger variety of systems, a good understanding of control systems is crucial. In mathematical control theory the basic principles underlying the analysis and design of control systems are studied.

Nonlinear control is an important area in control, as virtually all physical systems are nonlinear in nature. In case a system does not deviate too much from the nominal set of operating conditions, often linear models can be used for describing the system and designing controllers. However, when the required operation range is large, a linear(ized) model usually is inadequate or inaccurate. Then nonlinear control comes into play. Nonlinear controllers are capable of handling the nonlinearities directly in large operating ranges. And even when the operation range is small enough, linearization does not always work, as controllable systems exist, like a car, whose linearization around any equilibrium point is uncontrollable. As a result nonlinear control theory has to be used for these systems.

In nonlinear control theory a large variety of approaches and mathematical tools for analysis exists. The main reason for this variety is that no tool or methodology is universally applicable in nonlinear systems analysis. As a result, systematic approaches and mathematical tools are only available for certain classes of nonlinear systems. This thesis is also concerned with the control of special classes of nonlinear systems.

The thesis consists of two parts. In the first part a new design approach, the cascaded approach, is presented. The main advantages of this new approach are that the expressions for the resulting control laws are not complex and that transforming the system is not necessary: all analysis can be done in the original co-ordinates. The cascaded design approach aims at arriving at a specific structure for the closed-loop system. It turns out that this may simplify the controller design, as part of the nonlinear dynamics can be forgotten. The tracking problem is first studied for mobile robots, then for the class of so called chained-form systems (including cars towing multiple trailers and a rigid spacecraft with two torque actuators) and finally for an under-actuated ship. The applicability of the method is illustrated by means of simulations. In case of the under-actuated ship, experiments on a scale model of an offshore supply vessel have been performed.

In the second part of this thesis three specific problems are considered. First, the regulation problem for a rigid robot manipulator under a constant disturbance is considered. It is shown that the common practice of using a PID-controller is guaranteed to work globally in case the integral action is turned on only after some time. Secondly the visual servoing problem for a rigid robot manipulator is considered. That is, a robot manipulator is considered operating in the plane, viewed on top with a camera. An image of the robot manipulator is displayed at a screen. The goal is to regulate the tip of the robot manipulator to a specified point at this screen using only position measurements. Extra difficulties are that both the camera position and orientation are assumed to be unknown, as well as certain intrinsic camera parameters (like scale factors, focal length and center offset). The problem is solved by using an adaptive controller. Thirdly the tracking control problem for nonlinear systems is considered in the presence of unknown parameters, i.e., the adaptive tracking control problem is considered. It turns out that finding a suitable problem formulation is a problem in itself, as not knowing certain parameters and specifying a reference trajectory are in conflict with each other. This conflict is illustrated by means of an example, for which an adaptive tracking control problem is not only formulated, but also solved.

1.1 Formulation of the tracking control problem

In this thesis the tracking control problem for nonlinear systems is considered. An accurate mathematical model is assumed to be given for the system under consideration, like a mobile robot, a car towing multiple trailers, a rigid spacecraft, a ship, or a rigid robot manipulator; which is of the form:

$$\dot{x} = f(t, x, u) \tag{1.1a}$$
$$y = h(t, x, u). \tag{1.1b}$$

Here $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ denotes the input by means of which the system can be controlled, and $y \in \mathbb{R}^k$ denotes the output of the system which represents the measurements.

Furthermore, a feasible reference state trajectory $x_r(t)$ is assumed to be given for the system to track. Feasible means that once being on the reference trajectory it is possible to stay on that trajectory. This means that also a reference input $u_r(t)$ is assumed to exist, which is such that

$$\dot{x}_r = f(t, x_r, u_r). \tag{1.2}$$

The problem of generating such a feasible reference trajectory for a system is a challenging problem, known as the motion planning problem. Although motion planning (including obstacle avoidance) is an interesting problem, this thesis is not concerned with it and a reference state $x_r(t)$ as well as a reference input $u_r(t)$ which satisfy (1.2) are assumed to be given.

Once a reference state trajectory $x_r(t)$ and a reference input $u_r(t)$ are given, also the resulting reference output $y_r(t)$ can be defined by means of

$$y_r = h(t, x_r, u_r).$$

An often studied problem is the problem of output tracking, that is the problem of finding a control law for the input u such that as t tends to infinity $y(t)$ converges to $y_r(t)$. This is not the problem this thesis deals with. For systems like a mobile robot or a ship, the measured output typically is the position. Tracking of the position might seem an interesting problem, but it is not all what is really of interest. In general, more is desired. When the only focus is on controlling the position, it might happen that the mobile robot or ship turns around and follows the reference trajectory backwards.

This is one of the reasons for insisting on state-tracking, that is, finding a control law for the input u which is such that as t tends to infinity $x(t)$ converges to $x_r(t)$. Two major state trajectory tracking problems can be distinguished, namely the state-feedback problem as well as the output-feedback problem. In case of the first problem the entire state can be used for feedback, whereas for the latter only the output can be used. To be more precise, the following two problems can be distinguished:

Problem 1.1.1 (State-feedback state-tracking problem). Consider the system (1.1). Assume that a feasible reference trajectory (x_r, u_r) is given (i.e., a trajectory satisfying (1.2)). Find an appropriate control law

$$u = u(t, x_r, u_r, x) \tag{1.3}$$

such that for the resulting closed-loop system (1.1, 1.3)

$$\lim_{t\to\infty} \|x(t) - x_r(t)\| = 0.$$

Problem 1.1.2 (Output-feedback state-tracking problem). Consider the system (1.1). Assume that a feasible reference trajectory (x_r, u_r) is given (i.e., a trajectory satisfying (1.2)). Find an appropriate dynamic control law

$$u = u(t, x_r, u_r, y, z) \tag{1.4a}$$
$$\dot{z} = g(t, x_r, u_r, y, z) \tag{1.4b}$$

such that for the resulting closed-loop system (1.1, 1.4)

$$\lim_{t\to\infty} \|x(t) - x_r(t)\| = 0.$$

Remark 1.1.3. Notice that the assumption of tracking a feasible trajectory is the same as requiring the zero tracking error to be an equilibrium. Therefore, the tracking problem can also (actually: better) be formulated as finding an appropriate control law that renders the zero tracking error equilibrium asymptotically stable.

In the problem formulations as presented, no constraint on the size of the input is given, whereas in practice the input that can be supplied to the system is limited, i.e., also the constraint

$$\|u(t)\| \leq u^{\max} \qquad \forall t \geq 0 \tag{1.5}$$

has to be met, where $u^{\max}$ is a given constant. In that case the state-tracking problem under input constraints can be formulated in a similar way, i.e., like the state-tracking control problem, with the additional constraint (1.5). Clearly, for obtaining a solvable problem it has to be assumed that the reference satisfies the input constraints, which results in the additional assumption that

$$u^{\max} > \sup_{t \geq 0} \|u_r(t)\| .$$

These are the control problems studied for several types of systems in this thesis.

1.2 Non-holonomic systems

Except for the rigid robot manipulator, all systems studied in this thesis have so-called *non-holonomic constraints*. What does this mean? To make this more clear, consider the simple model

$$\begin{aligned} \dot{x}_1 &= u x_2 \\ \dot{x}_2 &= -u x_1 \end{aligned} \tag{1.6}$$

where (x_1, x_2) is the state and u is the input.

Notice that model (1.6) contains a constraint on the velocities:

$$x_1 \dot{x}_1 + x_2 \dot{x}_2 = 0. \tag{1.7}$$

This constraint is a so-called holonomic constraint, since it can be integrated to obtain

$$\frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 = \text{constant}.$$

This teaches us that model (1.6) can be reduced. The change of co-ordinates

$$\begin{aligned} r &= x_1^2 + x_2^2 \\ \phi &= \arctan\left(\frac{x_1}{x_2}\right) \end{aligned}$$

leads to the "new model"

$$\dot{\phi} = u \qquad\qquad r = r(0).$$

As $x_1^2 + x_2^2$ is a conserved quantity, the model (1.6) which seems to be a second order model, turns out to be only a first order model.

Things become different when considering the model

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned} \tag{1.8}$$

where (x_1, x_2, x_3) is the state and (u_1, u_2) is the input. Model (1.8) also contains a constraint on the velocities, namely

$$x_1\dot{x}_2 - x_2\dot{x}_1 - \dot{x}_3 = 0. \tag{1.9}$$

However, contrary to (1.7) the constraint (1.9) can not be integrated, i.e., the constraint (1.9) can not be written as time derivative of some function of the state. The constraint (1.9) is called a non-holonomic constraint.

It turns out that it is really necessary to use three variables for modeling this system, which means that the constraint (1.9) is inherently part of the dynamics. As a result the system (1.8) fails to meet the conditions of Brockett (1983, Theorem 1) that are necessary conditions for the existence of a continuous static state-feedback law, i.e., a control law of the form $u = u(x)$, that asymptotically stabilizes the system (1.8).

Since in this thesis the tracking problem for non-holonomic systems is dealt with and stabilization is a specific case of tracking, in one way or the other this difficulty should be taken into account. As it turns out, conditions on the reference input u_r have to be imposed in order to circumvent this problem.

1.3 Outline of the thesis

This thesis consists of two parts, preceded by a chapter with preliminaries.

Chapter 2 provides an overview of notions and results that are used throughout the thesis. This chapter is included for making the thesis more or less self-contained. Section 2.4 is fundamental for Part I. The main contributions of Chapter 2 are Theorem 2.3.7, Theorem 2.3.8 and Lemma 2.4.5.

Part I

In the first part a cascaded design approach to the tracking problem for nonlinear systems is presented. This approach is illustrated by means of several examples: mobile robots in Chapter 4, general chained-form systems in Chapter 5 and an under-actuated ship in Chapter 6. The applicability of the method is illustrated by means of simulations. In case of the under-actuated ship also experiments have been performed. This first part is a composition of the papers

- J. Jakubiak, E. Lefeber, K. Tchón, and H. Nijmeijer, "Observer based tracking controllers for a mobile car," 2000, Submitted to the 39th Conference on Decision and Control, Sydney, Australia;

- Z.-P. Jiang, E. Lefeber, and H. Nijmeijer, "Stabilization and tracking of a nonholonomic mobile robot with saturating actuators," in *Proceedings of CONTROLO'98, Third Portuguese Conference on Automatic Control*, vol. 1, Coimbra, Portugal, 1998, pp. 315–320;

- Z.-P. Jiang, E. Lefeber, and H. Nijmeijer, "Saturated stabilization and tracking of a nonholonomic mobile robot," 1999, Submitted to *Systems and Control Letters*;
- E. Lefeber, K. Y. Pettersen, and H. Nijmeijer, "Tracking control of an under-actuated ship," 2000a, in preparation;
- E. Lefeber, A. Robertsson, and H. Nijmeijer, "Linear controllers for tracking chained-form systems," in *Stability and Stabilization of Nonlinear Systems*, D. Aeyels, F. Lamnabhi-Lagarrigue, and A. J. van der Schaft, Eds., no. 246 in Lecture Notes in Control and Information Sciences, pp. 183–199, London, United Kingdom: Springer-Verlag, 1999a;
- E. Lefeber, A. Robertsson, and H. Nijmeijer, "Output feedback tracking of nonholonomic systems in chained form," in *Proceedings of the 5th European Control Conference*, Karlsruhe, Germany, 1999b, paper 772;
- E. Lefeber, A. Robertsson, and H. Nijmeijer, "Linear controllers for exponential tracking of systems in chained form," *International Journal on Robust and Nonlinear Control*, vol. 10, no. 4, pp. 243–264, 2000b;
- E. Panteley, E. Lefeber, A. Loría, and H. Nijmeijer, "Exponential tracking control of a mobile car using a cascaded approach," in *Proceedings of the IFAC Workshop on Motion Control*, Grenoble, France, 1998, pp. 221–226,

and some additional unpublished material.

The main contribution of this part is the introduction of the cascaded design approach. New and simple time-varying state-feedback controllers are presented that achieve global and uniform tracking results for tracking mobile robots, chained-form systems and under-actuated ships. The state- and output-feedback control problems are considered, also under (partial) input saturation. No transformations are needed; all analysis is done in the original error co-ordinates.

Part II

In the second part solutions to three specific problems are presented. This part consists of three papers, respectively

- A. Loría, E. Lefeber, and H. Nijmeijer, "Global asymptotic stability of robot manipulators with linear PID and PI^{2}D control," 1999a, Submitted to *Stability and Control: Theory and Applications*;
- E. Lefeber, R. Kelly, R. Ortega, and H. Nijmeijer, "Adaptive and filtered visual servoing of planar robots," in *Proceedings of the Fourth IFAC Symposium on Nonlinear Control Systems Design (NOLCOS'98)*, vol. 2, Enschede, The Netherlands, 1998, pp. 563–568;
- E. Lefeber and H. Nijmeijer, "Adaptive tracking control of nonholonomic systems: an example," in *Proceedings of the 38th IEEE Conference on Decision and Control*, Phoenix, Arizona, USA, 1999, pp. 2094–2099.

The contributions of this part consist of

- *Global* asymptotic stability of linear PID controllers is shown by delaying the integral action;
- Classes of controllers are introduced that solve the visual servoing of planar robots under a fixed camera position for both the state- and output-feedback problem. These classes also contain saturated controllers. In case of unknown camera orientation a class of adaptive controllers is presented;
- Difficulties in formulating the adaptive state-tracking problem for nonlinear systems with unknown parameters are illustrated by means of an example. For this example a suitable problem formulation of the adaptive state-tracking problem is given and a solution is presented.

Chapter 11 contains the conclusions of this thesis and some recommendations for further research.

Appendix A contains the proofs of some theorems presented in this thesis.

Appendix B contains a backstepping control law for tracking an under-actuated ship. This expression which was too long to be incorporated in the text of Chapter 6.

Chapter 2

Preliminaries

In this chapter we recall a few notions and results that we use throughout this thesis. First, we consider some fundamental mathematical definitions. Next the concept of Lyapunov stability and some lemmas useful for showing stability are given. We review some basic notions for linear time-varying systems, introduce some crucial theorems, present results on (time-varying) cascaded systems, and briefly illustrate the method of backstepping.

2.1 Mathematical preliminaries

Definition 2.1.1. A **norm** $\|x\|$ of an n-dimensional vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is a real valued function with the properties

- $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$, with $\|x\| = 0$ if and only if $x = 0$;
- $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in \mathbb{R}^n$;
- $\|\alpha x\| = |\alpha| \, \|x\|$, for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Some commonly used norms are

$$\|x\|_p \triangleq (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}} \qquad\qquad 1 \leq p < \infty$$

and

$$\|x\|_\infty \triangleq \max_{i=1,\ldots,n} |x_i| \,.$$

Definition 2.1.2. We denote a sphere of radius r by B_r, i.e.,

$$B_r \triangleq \{x \in \mathbb{R}^n \,|\, \|x\| < r\} \,.$$

Definition 2.1.3. For functions of time $x : \mathbb{R}_+ \to \mathbb{R}^n$, we define the $\mathcal{L}_p$ **norm**

$$\|x\|_p \triangleq \left(\int_0^\infty \|x(\tau)\|^p \, \mathrm{d}\tau \right)^{\frac{1}{p}}$$

for $p \in [1, \infty)$ and say that $x \in \mathcal{L}_p$ when $\|x\|_p$ exists (i.e., when $\|x\|_p$ is finite). The $\mathcal{L}_\infty$ **norm** is defined as

$$\|x\|_\infty \triangleq \sup_{t \geq 0} \|x(t)\|$$

and we say that $x \in \mathcal{L}_\infty$ when $\|x\|_\infty$ exists.

Definition 2.1.4. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous at a point** x if given an $\epsilon > 0$ a constant $\delta > 0$ exists such that

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \epsilon \qquad x, y \in \mathbb{R}^n. \tag{2.1}$$

Definition 2.1.5. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous on a set** S if it is continuous at every point in S.

Definition 2.1.6. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **piecewise continuous on a set** S if it is continuous on S, except for a finite number of points.

Definition 2.1.7. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **uniformly continuous on a set** S if given an $\epsilon > 0$ a constant $\delta > 0$ exists (depending only on ϵ) such that (2.1) holds for all $x, y \in S$.

Notice that uniform continuity is defined on a set. Furthermore, for uniform continuity the same δ "works" for all points of the set. As a result, uniform continuity implies continuity, but not necessarily vice versa. Notice that the function $f(x) = e^x$ is continuous on $\mathbb{R}$, but *not* uniformly continuous on $\mathbb{R}$. However, the function $f(x) = e^x$ is uniformly continuous on any compact set $S \subset \mathbb{R}$.

Often uniform continuity of a function $f : \mathbb{R} \to \mathbb{R}$ can be verified by means of the following lemma.

Lemma 2.1.8. *Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$. If a constant $M \in \mathbb{R}$ exists such that*

$$\sup_{x \in \mathbb{R}} \left| \frac{\mathrm{d}f}{\mathrm{d}x}(x) \right| \leq M,$$

then f is uniformly continuous on $\mathbb{R}$.

Definition 2.1.9. We denote the class of n times continuously differentiable functions by $\mathcal{C}^n$.

Remark 2.1.10. From the fact that

$$\sin x = x - \frac{1}{6}x^3 + O(x^4) \qquad \text{and} \qquad \cos x = 1 - \frac{1}{2}x^2 + O(x^4)$$

we can conclude that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

As a result the functions

$$f_1(x) = \int_0^1 \cos(xs)\mathrm{d}s = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \tag{2.2}$$

and

$$f_2(x) = \int_0^1 \sin(xs)\mathrm{d}s = \begin{cases} \frac{1-\cos x}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \tag{2.3}$$

are continuous.

For simplicity of notation we use the expressions $\frac{\sin x}{x}$ and $\frac{1-\cos x}{x}$ throughout this thesis, whereas it would be more precise to use (2.2) and (2.3) respectively.

The same holds true for similar expressions that at first glance seem not to be defined for $x = 0$.

Definition 2.1.11. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to **class** $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is strictly increasing and $\alpha(0) = 0$.

Definition 2.1.12. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to **class** $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each fixed s the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and if for each fixed r the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Definition 2.1.13. A **saturation function with saturation level** ϵ is a $\mathcal{C}^1$ function $\sigma_\epsilon : \mathbb{R} \to [-\epsilon, \epsilon]$ that satisfies

$$x\sigma_\epsilon(x) > 0 \qquad \forall x \neq 0$$

and

$$\frac{\mathrm{d}\sigma_\epsilon}{\mathrm{d}x}(0) > 0.$$

2.2 Lyapunov stability

Consider a non-autonomous system described by

$$\dot{x} = f(t, x) \tag{2.4}$$

where $f : \mathbb{R}_+ \times D \to \mathbb{R}^n$ is piecewise continuous on $\mathbb{R}_+ \times D$ and locally Lipschitz in x on $\mathbb{R}_+ \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. We assume that the origin is an equilibrium point for (2.4) which is expressed by

$$f(t, 0) = 0, \qquad \forall t \geq 0.$$

For studying the stability of the origin we introduce the following notions (see e.g., (Khalil 1996)).

Definition 2.2.1. The equilibrium point $x = 0$ of (2.4) is said to be **(locally) stable (in the sense of Lyapunov)** if a positive constant $r > 0$ exists such that for all $(t_0, x(t_0)) \in \mathbb{R}_+ \times B_r$ a function $\alpha \in \mathcal{K}$ exists such that

$$\|x(t)\| \le \alpha(\|x(t_0)\|) \qquad \forall t \ge t_0 \ge 0, \forall x(t_0) \in B_r. \tag{2.5}$$

If the bound (2.5) holds for all $(t_0, x(t_0)) \in \mathbb{R}_+ \times \mathbb{R}^n$, then the origin is **globally stable**.

Definition 2.2.2. The equilibrium point $x = 0$ of (2.4) is said to be

- **(locally) asymptotically stable** if a constant $r > 0$ exists such that for all pairs $(t_0, x(t_0)) \in \mathbb{R}_+ \times B_r$ a function $\beta \in \mathcal{KL}$ exists such that

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) \qquad \forall t \ge t_0 \ge 0, \forall x(t_0) \in B_r; \tag{2.6}$$

- **semi-globally asymptotically stable** if for each constant $r > 0$ and for all pairs $(t_0, x(t_0)) \in \mathbb{R}_+ \times B_r$ a function $\beta \in \mathcal{KL}$ exists such that (2.6) holds;

- **globally asymptotically stable (GAS)** if a function $\beta \in \mathcal{KL}$ exists such that for all pairs $(t_0, x(t_0)) \in \mathbb{R}_+ \times \mathbb{R}^n$ (2.6) holds.

Definition 2.2.3. The equilibrium point $x = 0$ of (2.4) is said to be **(locally) exponentially stable** if it is (locally) asymptotically stable and (2.6) is satisfied with

$$\beta(r, s) = kre^{-\gamma s} \qquad k > 0, \gamma > 0.$$

In a similar way we can define the equilibrium point $x = 0$ of (2.4) to be **semi-globally exponentially stable** or **globally exponentially stable (GES)**.

For linear time-invariant systems $\dot{x} = Ax$ it is well-known that asymptotic stability is equivalent to GES and robustness with respect to perturbations is guaranteed, i.e., under a uniformly bounded additional perturbation $\delta(x, t)$ solutions of the system $\dot{x} = Ax + \delta$ remain bounded. Unfortunately this is in general not true for non-autonomous systems.

Example 2.2.4 (see (Panteley, Loría and Teel 1999)). Consider the system (2.4) with

$$f(t, x) = \begin{cases} -\frac{1}{1+t}\operatorname{sgn}(x) & \text{if } |x| \ge \frac{1}{1+t} \\ -x & \text{if } |x| \le \frac{1}{1+t} \end{cases}$$

where

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Then for each $r > 0$ and $t_0 \ge 0$ there exist constants $k > 0$ and $\gamma > 0$ such that for all $t \ge t_0$ and $|x(t_0)| \le r$

$$|x(t)| \le k\,|x(t_0)|\, e^{-\gamma(t - t_0)} \qquad \forall t \ge t_0 \ge 0. \tag{2.7}$$

However, always a bounded (arbitrarily small) additive perturbation $\delta(t, x)$ and a constant $t_0 \ge 0$ exist such that the trajectories of the perturbed system $\dot{x} = f(t, x) + \delta(t, x)$ are unbounded.

More details concerning the proof of the claims made in this example can be found in (Panteley et al. 1999). One of the reasons for this negative result is that in (2.7) the constants k and γ are allowed to depend on t_0, i.e., for each value of t_0 different constants k and γ may be chosen. Therefore, we introduce the notion of uniform stability.

Definition 2.2.5. The equilibrium point $x = 0$ of (2.4) is said to be **uniformly stable** if a positive constant $r > 0$ and an $\alpha \in \mathcal{K}$ exist, both independent of t_0, such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|) \qquad \forall t \geq t_0 \geq 0, \forall x(t_0) \in B_r. \tag{2.8}$$

If the bound (2.8) holds for all $(t_0, x(t_0)) \in \mathbb{R}_+ \times \mathbb{R}^n$, then the origin is **globally uniformly stable**.

Definition 2.2.6. The equilibrium point $x = 0$ of (2.4) is said to be

- **(locally) uniformly asymptotically stable** if a constant $r > 0$ and a function $\beta \in \mathcal{KL}$ exist, both independent of t_0, such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \qquad \forall t \geq t_0 \geq 0, \forall x(t_0) \in B_r; \tag{2.9}$$

- **semi-globally uniformly asymptotically stable** if for each constant $r > 0$ and for all $(t_0, x(t_0)) \in \mathbb{R}_+ \times B_r$ a function $\beta \in \mathcal{KL}$ exists such that (2.9) holds;
- **globally uniformly asymptotically stable (GUAS)** if a function $\beta \in \mathcal{KL}$ exists such that for all $(t_0, x(t_0)) \in \mathbb{R}_+ \times \mathbb{R}^n$ (2.9) holds.

Definition 2.2.7. The equilibrium point $x = 0$ of (2.4) is said to be **(locally) uniformly exponentially stable/semi-globally uniformly exponentially stable/globally uniformly exponentially stable (GUES)** if it is (locally) uniformly asymptotically stable/semi-globally uniformly asymptotically stable/globally uniformly asymptotically stable respectively and (2.9) is satisfied with

$$\beta(r, s) = kre^{-\gamma s} \qquad k > 0, \gamma > 0.$$

Having these definitions of uniform stability we are now able to formulate the following robustness result for uniformly asymptotically stable systems:

Lemma 2.2.8 ((Khalil 1996, Lemma 5.3)). *Let $x = 0$ be a uniformly asymptotically stable equilibrium point of the nominal system $\dot{x} = f(t, x)$ where $f : \mathbb{R}_+ \times B_r \to \mathbb{R}^n$ is continuously differentiable, and the Jacobian $\left[\frac{\partial f}{\partial x}\right]$ is bounded on B_r, uniformly in t. Then one can determine constants $\Delta > 0$ and $R > 0$ such that for all perturbations $\delta(t, x)$ that satisfy the uniform bound $\|\delta(t, x)\| \leq \delta < \Delta$ and all initial conditions $\|x(t_0)\| \leq R$, the solution $x(t)$ of the perturbed system $\dot{x} = f(t, x) + \delta(t, x)$ satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \qquad \forall t_0 \leq t \leq t_1$$

and

$$\|x(t)\| \leq \rho(\delta) \qquad \forall t \geq t_1$$

for some $\beta \in \mathcal{KL}$ and some finite time t_1, where $\rho(\delta)$ is a class $\mathcal{K}$ function of δ.

Furthermore, if $x = 0$ is a globally uniformly exponentially stable equilibrium point, we can allow for arbitrarily large δ by choosing $R > 0$ large enough.

This implies that uniform asymptotic stability gives rise to some robustness that is not guaranteed by asymptotic stability. This explains why in this thesis we do not aim for asymptotic stability, but for uniform asymptotic stability instead.

Notice that for autonomous systems $\dot{x} = f(x)$ we may drop the word "uniform" as the solution depends only on $t - t_0$.

From Lemma 2.2.8 it is also clear why exponential stability is a most favorable property. Unfortunately, global uniform exponential stability can not always be achieved, which could be an explanation for all the different notions of exponential stability that are available in literature. However, Example 2.2.4 clearly shows that exponential convergence in itself does not guarantee robustness; one needs uniformity. A notion that is equivalent to having both global uniform asymptotic stability and local uniform exponential stability (GUAS+LUES) is the following.

Definition 2.2.9 ((Sørdalen and Egeland 1995, Definition 2)). The equilibrium point $x = 0$ of (2.4) is said to be **globally $\mathcal{K}$-exponentially stable** if a function $\kappa \in \mathcal{K}$ and a constant $\gamma > 0$ exist such that for all $(t_0, x(t_0)) \in \mathbb{R}_+ \times \mathbb{R}^n$ we have

$$\|x(t)\| \leq \kappa(\|x(t_0)\|)e^{-\gamma(t-t_0)} \qquad \forall t \geq t_0 \geq 0.$$

A useful tool for showing asymptotic stability of a certain signal is:

Lemma 2.2.10 (see (Barbălat 1959)). *Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a uniformly continuous function. Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)\mathrm{d}\tau$ exists and is finite. Then*

$$\lim_{t\to\infty} \phi(t) = 0.$$

Corollary 2.2.11. *If $f \in \mathcal{L}_\infty$, $\dot{f} \in \mathcal{L}_\infty$, and $f \in \mathcal{L}_p$ for some $p \in [1, \infty)$, then*

$$\lim_{t\to\infty} f(t) = 0.$$

An extension of Barbălat's Lemma to functions ϕ that are not uniformly continuous (but can be written as the sum of a uniformly continuous function and a piecewise continuous function that decays to zero) was presented in (Micaelli and Samson 1993):

Lemma 2.2.12 ((Micaelli and Samson 1993, Lemma 1)). *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be any differentiable function. If $f(t)$ converges to zero as $t \to \infty$ and its derivative satisfies*

$$\dot{f}(t) = f_0(t) + \eta(t) \qquad t \geq 0$$

where f_0 is a uniformly continuous function and $\eta(t)$ tends to zero as $t \to \infty$, then $\dot{f}(t)$ and $f_0(t)$ tend to zero as $t \to \infty$.

2.3 Linear time-varying systems

Consider the linear time-varying (LTV) system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{aligned} \tag{2.10}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, and $A(t)$, $B(t)$, $C(t)$ are matrices of appropriate dimensions whose elements are piecewise continuous functions. Let $\Phi(t, t_0)$ denote the state-transition matrix for the system $\dot{x} = A(t)x$. We recall two definitions from linear control theory (see e.g., (Kailath 1980, Rugh 1996)).

Definition 2.3.1. The pair $(A(t), B(t))$ is **uniformly completely controllable (UCC)** if constants $\delta, \epsilon_1, \epsilon_2 > 0$ exist such that for all $t > 0$:

$$\epsilon_1 I_n \leq \int_t^{t+\delta} \Phi(t,\tau)B(\tau)B^T(\tau)\Phi^T(t,\tau)\mathrm{d}\tau \leq \epsilon_2 I_n.$$

Definition 2.3.2. The pair $(A(t), C(t))$ is **uniformly completely observable (UCO)** if constants $\delta, \epsilon_1, \epsilon_2 > 0$ exist such that for all $t > 0$:

$$\epsilon_1 I_n \leq \int_{t-\delta}^{t} \Phi^T(\tau, t-\delta)C^T(\tau)C(\tau)\Phi(\tau, t-\delta)\mathrm{d}\tau \leq \epsilon_2 I_n.$$

A very helpful theorem for showing UCC or UCO is

Theorem 2.3.3 ((Kern 1982, Theorem 2)). *Consider the linear time-varying system* (2.10)*. Suppose that $A(t)$ and $B(t)$ are bounded and that $A(t)$ is Lipschitz, i.e., constants K and L exist such that*

$$\begin{aligned} \|A(t)\| &\leq K && \text{for all } t \geq 0 \\ \|B(t)\| &\leq K && \text{for all } t \geq 0 \\ \|A(t) - A(t')\| &\leq L|t - t'| && \text{for all } t, t' \geq 0. \end{aligned}$$

Then the system (2.10) *is uniformly completely controllable if a constant $\delta_c > 0$ and an s with $t - \delta_c \leq s \leq t$ exist such that the matrix function $W(t - \delta_c, t)$ defined by*

$$W(t_0, t_1) = \int_{t_0}^{t_1} e^{A(s)(t_1-\tau)}B(\tau)B^T(\tau)e^{A^T(s)(t_1-\tau)}\mathrm{d}\tau$$

satisfies

$$0 < \alpha_1 I_n \leq W(t - \delta_c, t) \qquad \text{for all } t \geq 0$$

where $\alpha_1(\delta_c)$ is a constant.

Corollary 2.3.4. *Consider the system*

$$\begin{aligned} \dot{x} &= A(\phi(t))x + Bu \\ y &= Cx \end{aligned} \tag{2.11}$$

where $A(\phi)$ is continuous, $A(0) = 0$, $\phi : \mathbb{R} \to \mathbb{R}$ continuous. Assume that for all $s \neq 0$ the pair $(A(s), B)$ is controllable (respectively the pair $(A(s), C)$ is observable). If $\phi(t)$ is bounded, Lipschitz and constants $\delta_c > 0$ and $\epsilon > 0$ exist such that

$$\forall t \geq 0, \exists s : t - \delta_c \leq s \leq t \text{ such that } |\phi(s)| \geq \epsilon,$$

then the system (2.11) *is uniformly completely controllable (respectively observable).*

The condition imposed on $\phi(t)$ in Corollary 2.3.4 plays an important role, not only in this thesis, but also in identification and adaptive control systems. It is known as the "persistence of excitation condition".

Definition 2.3.5. A continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}$ is said to be **persistently exciting (PE)** if all of the following conditions hold:

- a constant $K > 0$ exists such that $|\phi(t)| \leq K$ for all $t \geq 0$,
- a constant $L > 0$ exists such that $|\phi(t) - \phi(t')| \leq L\,|t - t'|$ for all $t, t' \geq 0$, and
- constants $\delta_c > 0$ and $\epsilon > 0$ exist such that

$$\forall t \geq 0, \exists s : t - \delta_c \leq s \leq t \text{ such that } |\phi(s)| \geq \epsilon.$$

Remark 2.3.6. Notice that in the common definition of persistence of excitation usually the first two assumptions on $\phi(t)$ are made implicitly. The third condition is in general formulated for $\phi : \mathbb{R} \to \mathbb{R}^n$, assuming the existence of positive constants ϵ_1, ϵ_2, and δ such that for all $t \geq 0$

$$\epsilon_1 I_n \leq \int_t^{t+\delta} \phi(\tau)\phi^T(\tau)\mathrm{d}\tau \leq \epsilon_2 I_n.$$

Furthermore, notice that the third condition on $\phi(t)$ as in Definition 2.3.5 can be interpreted as follows: assume that we plot the graph of $|\phi(t)|$ and look at this plot through a window of width $\delta_c > 0$. Then, no matter where we put this window on the graph, always a time instant s exists where $|\phi(s)|$ is at least $\epsilon > 0$.

The following are some useful results.

Theorem 2.3.7. *The system*

$$\dot{x} = \begin{bmatrix} -k_1 & -k_2\phi(t) & -k_3 & -k_4\phi(t) & \dots \\ \phi(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \phi(t) & 0 \end{bmatrix} x \tag{2.12}$$

is globally uniformly exponentially stable (GUES) if $\phi(t)$ is persistently exciting and the k_i ($i = 1, \dots, n$) are such that the polynomial

$$\lambda^n + k_1\lambda^{n-1} + \dots + k_{n-1}\lambda + k_n$$

*is **Hurwitz** (i.e., all its roots have negative real parts).*

Proof. See Appendix A. ∎

Theorem 2.3.8. *The system*

$$
\dot{x} = \begin{bmatrix}
-k_1 & -k_2\phi(t) & -k_3 & -k_4\phi(t) & \dots & k_1 & k_2\phi(t) & k_3 & k_4\phi(t) & \dots \\
\phi(t) & 0 & \dots & \dots & 0 & 0 & \dots & \dots & \dots & 0 \\
0 & \ddots & \ddots & & \vdots & \vdots & & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\
0 & \dots & 0 & \phi(t) & 0 & 0 & \dots & \dots & \dots & 0 \\
0 & \dots & \dots & \dots & 0 & 0 & \dots & \dots & 0 & \vdots \\
\vdots & \ddots & & & \vdots & \phi(t) & \ddots & & \vdots & -l_4\phi(t) \\
\vdots & & \ddots & & \vdots & 0 & \ddots & \ddots & \vdots & -l_3 \\
\vdots & & & \ddots & \vdots & \vdots & \ddots & \ddots & 0 & -l_2\phi(t) \\
0 & \dots & \dots & \dots & 0 & 0 & \dots & 0 & \phi(t) & -l_1
\end{bmatrix} x \tag{2.13}
$$

is globally uniformly exponentially stable (GUES) if $\phi(t)$ is persistently exciting and the k_i, l_i ($i = 1, \dots, n$) are such that the polynomials

$$\lambda^n + k_1\lambda^{n-1} + \cdots + k_{n-1}\lambda + k_n \tag{2.14a}$$

and

$$\lambda^n + l_1\lambda^{n-1} + \cdots + l_{n-1}\lambda + l_n \tag{2.14b}$$

are Hurwitz.

Proof. See Appendix A. □

Theorem 2.3.9 ((Ioannou and Sun 1996, Theorem 3.4.6 v)). *The linear time-varying system* (2.10) *is globally uniformly exponentially stable (GUES) if and only if it is globally uniformly asymptotically stable (GUAS).*

Proposition 2.3.10. *Consider the system*

$$
\begin{aligned}
\dot{x}_1 &= -\sigma_\epsilon(x_1) + \phi(t)x_2 \\
\dot{x}_2 &= -\phi(t)x_1
\end{aligned} \tag{2.15}
$$

where σ_ϵ is a saturation function with saturation level ϵ as defined in Definition 2.1.13. If $\phi(t)$ is persistently exciting (PE), then the system (2.15) *is globally $\mathcal{K}$-exponentially stable.*

2.4 Cascaded systems

Consider a system $\dot{z} = f(t, z)$ that can be written as

$$\dot{z}_1 = f_1(t, z_1) + g(t, z_1, z_2)z_2 \tag{2.16a}$$
$$\dot{z}_2 = f_2(t, z_2) \tag{2.16b}$$

where $z_1 \in \mathbb{R}^n$, $z_2 \in \mathbb{R}^m$, $f_1(t, z_1)$ is continuously differentiable in (t, z_1) and $f_2(t, z_2)$, $g(t, z_1, z_2)$ are continuous in their arguments, and locally Lipschitz in z_2 and (z_1, z_2) respectively.

Notice that if $z_2 = 0$ (2.16a) reduces to

$$\dot{z}_1 = f_1(t, z_1).$$

Therefore, we can view (2.16a) as the system

$$\Sigma_1 : \qquad \dot{z}_1 = f_1(t, z_1) \tag{2.17}$$

that is perturbed by the output of the system

$$\Sigma_2 : \qquad \dot{z}_2 = f_2(t, z_2). \tag{2.18}$$

Assume that the systems Σ_1 and Σ_2 are asymptotically stable, i.e., for (2.17) we know $\lim_{t\to\infty} z_1(t) = 0$ and for (2.18) we have $\lim_{t\to\infty} z_2(t) = 0$. It is obvious that in that case also for (2.16b) $z_2(t)$ tends to zero. In that case the dynamics (2.16a) reduces to the dynamics (2.17). It seems plausible that therefore also (2.16a) and as a result the cascaded system (2.16) become asymptotically stable.

Unfortunately, this is not true in general as can be seen from the following example.

Example 2.4.1. Consider the system

$$\dot{z}_1 = -z_1 + z_1^2 z_2 \tag{2.19a}$$
$$\dot{z}_2 = -\gamma z_2 \qquad \gamma > 0 \tag{2.19b}$$

which can be seen as the system

$$\dot{z}_1 = -z_1 \tag{2.20a}$$

that is perturbed by the output of the system

$$\dot{z}_2 = -\gamma z_2 \qquad \gamma > 0. \tag{2.20b}$$

Both (2.20a) and (2.20b) are globally exponentially stable (GES). One would expect the system (2.19) to be asymptotically stable. However, solving the differential equations (2.19) yields

$$z_1(t) = \frac{2z_1(0)}{z_1(0)z_2(0)e^{-\gamma t} + [2 - z_1(0)z_2(0)]e^{\gamma t}} \tag{2.21a}$$
$$z_2(t) = z_2(0)e^{-\gamma t}. \tag{2.21b}$$

Notice that if $z_1(0)z_2(0) > 2$ the denominator of (2.21a) becomes zero at

$$t_{\text{esc}} = \frac{1}{2\gamma} \ln\left(\frac{z_1(0)z_2(0)}{z_1(0)z_2(0) - 2}\right),$$

so the solution of $z_1(t)$ goes to infinity in finite time. One could consider increasing the gain γ to make $z_2(t)$ converge to zero faster and have the dynamics (2.19a) converge to (2.20a) faster. Unfortunately, as a result the solution of $z_1(t)$ goes to infinity even quicker!

However, under certain conditions it is possible to conclude asymptotic stability of (2.16) when both Σ_1 and Σ_2 are asymptotically stable:

Lemma 2.4.2 ((Panteley and Loría 1999, Lemma 1)). *If the systems* (2.17) *and* (2.18) *are globally uniformly asymptotically stable (GUAS) and solutions of the cascaded system* (2.16) *are globally uniformly bounded, then the system* (2.16) *is globally uniformly asymptotically stable (GUAS).*

The question that remains is when solutions of (2.16) are globally uniformly bounded. To answer that question, we can use the following:

Theorem 2.4.3 ((Panteley and Loría 1999, Theorems 1, 2, 4)). *Consider the following assumptions*

A1. *The systems* (2.17) *and* (2.18) *are both globally uniformly asymptotically stable (GUAS) and we know explicitly a $\mathcal{C}^1$ Lyapunov function candidate $V(t,z_1)$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_4 \in \mathcal{K}$ and a positive* semi-*definite function $W(z_1)$ such that*

$$\alpha_1(\|z_1\|) \leq V(t,z_1) \leq \alpha_2(\|z_1\|) \tag{2.22a}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial z_1} f_1(t,z_1) \leq -W(z_1) \tag{2.22b}$$

$$\left\| \frac{\partial V}{\partial z_1} \right\| \leq \alpha_4(\|z_1\|). \tag{2.22c}$$

A2. *For each fixed z_2 a continuous function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ exists with $\lim_{s\to\infty} \lambda(s) = 0$ and such that*

$$\left\| \frac{\partial V}{\partial z_1} g(t,z_1,z_2) \right\| \leq \lambda(\|z_1\|) W(z_1)$$

with V and W as in Assumption A1.

A3. *Continuous functions $\theta_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and $\alpha_5 : \mathbb{R}_+ \to \mathbb{R}_+$ exist such that*

$$\|g(t,z_1,z_2)\| \leq \theta_1(\|z_2\|)\alpha_5(\|z_1\|) \tag{2.23}$$

and a continuous non-decreasing function $\alpha_6 : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $a \geq 0$ exist such that

$$\alpha_6(s) \geq \alpha_4\left(\alpha_1^{-1}(s)\right) \alpha_5\left(\alpha_1^{-1}(s)\right)$$

and

$$\int_a^\infty \frac{\mathrm{d}s}{\alpha_6(s)} = \infty \tag{2.24}$$

with α_1, α_4 as in Assumption A1.

A4. *For each $r > 0$, constants $\lambda > 0$ and $\eta > 0$ exist such that for all $t \geq 0$ and all $\|z_2\| < r$*

$$\left\| \frac{\partial V}{\partial z_1} g(t,z_1,z_2) \right\| \leq \lambda W(z_1) \qquad \forall \, \|z_1\| \geq \eta.$$

A5. *A function $\phi \in \mathcal{K}$ exists such that the solution $z_2(t)$ of* (2.18) *satisfies*

$$\int_{t_0}^{\infty} \|z_2(t)\| \, \mathrm{d}t \leq \phi(\|z_2(t_0)\|).$$

Then we can conclude

- *If Assumptions A1 and A2 hold, then the cascaded system* (2.16) *is globally uniformly asymptotically stable (GUAS).*
- *If Assumptions A1, A3 and A4 hold, then the cascaded system* (2.16) *is globally uniformly asymptotically stable (GUAS).*
- *If Assumptions A1, A3 and A5 hold, then the cascaded system* (2.16) *is globally uniformly asymptotically stable (GUAS).*

Corollary 2.4.4 (see (Panteley and Loría 1998)). *If Assumption A1 is satisfied with*

$$\begin{aligned} \alpha_1(\|z_1\|) &= c_1 \|z_1\|^2 \\ \alpha_4(\|z_1\|) &= c_4 \|z_1\|, \end{aligned}$$

continuous functions $k_1 : \mathbb{R}_+ \to \mathbb{R}$ and $k_2 : \mathbb{R}_+ \to \mathbb{R}$ exist such that

$$\|g(t, z_1, z_2)\| \leq k_1(\|z_2\|) + k_2(\|z_2\|) \|z_1\|, \tag{2.25}$$

and Assumption A5 is satisfied, then the cascaded system (2.16) *is globally uniformly asymptotically stable (GUAS).*

Proof. We have that (2.23) is satisfied with

$$\begin{aligned} \theta_1(\|z_2\|) &= \max(k_1(\|z_2\|), k_2(\|z_2\|)) \\ \alpha_5(\|z_1\|) &= 1 + \|z_1\|. \end{aligned}$$

Then we have

$$\alpha_4\left(\alpha_1^{-1}(s)\right) \alpha_5\left(\alpha_1^{-1}(s)\right) = \frac{c_4}{\sqrt{c_1}} \sqrt{s} \left(1 + \frac{1}{\sqrt{c_1}} \sqrt{s}\right),$$

so that we can take

$$\alpha_6(s) = \frac{c_4}{\sqrt{c_1}} \sqrt{s} + \frac{c_4}{c_1} s.$$

If we take $a > c_1$ in (2.24) we have that Assumption A3 is satisfied. □

Lemma 2.4.5 (see (Panteley, Lefeber, Loría and Nijmeijer 1998)). *Assume that both subsystems* (2.17) *and* (2.18) *are globally $\mathcal{K}$-exponentially stable, we know explicitly a $\mathcal{C}^1$ Lyapunov function candidate $V(t, z_1)$ that satisfies* (2.22) *with $\alpha_1(\|z_1\|) = c_1 \|z_1\|^2$, $\alpha_2 \in \mathcal{K}_\infty$, $\alpha_4(\|z_1\|) = c_4 \|z_1\|$ and positive semi-definite W and that $g(t, z_1, z_2)$ satisfies* (2.25)*. Then the cascaded system* (2.16) *is globally $\mathcal{K}$-exponentially stable.*

Corollary 2.4.6. *Assume that* (2.17) *is globally uniformly exponentially stable (GUES), that* (2.18) *is globally $\mathcal{K}$-exponentially stable and that $g(t, z_1, z_2)$ satisfies* (2.25)*. Then the cascaded system* (2.16) *is globally $\mathcal{K}$-exponentially stable.*

Proof. This follows immediately from Lemma 2.4.5, since the existence of a suitable Lyapunov function candidate is guaranteed from converse Lyapunov theory (see (Khalil 1996, Theorem 3.12)). □

2.5 Backstepping

A commonly used method of nonlinear controller design is backstepping. We illustrate this method by means of a simple example considering the special case of integrator backstepping. For a more detailed explanation the reader is referred to (Marino and Tomei 1995, Krstić, Kanellakopoulos and Kokotović 1995).

Example 2.5.1. Consider the second order system

$$\dot{x} = \cos x - x^3 + \xi \tag{2.26a}$$

$$\dot{\xi} = u \tag{2.26b}$$

where $[x, \xi]^T \in \mathbb{R}^2$ is the state and $u \in \mathbb{R}$ is the input. We want to design a state-feedback controller to render the equilibrium point $[x, \xi]^T = [0, -1]^T$ globally asymptotically stable (GAS).

If ξ were the input, then (2.26a) can easily be stabilized by means of $\xi = -c_1 x - \cos x$. A Lyapunov function would be $V(x) = \frac{1}{2}x^2$.

Unfortunately ξ is not the control but a state variable. Nevertheless, we could prescribe its *desired value*

$$\xi_{\text{des}} = -c_1 x - \cos x \triangleq \alpha(x).$$

Next, we define z to be the difference between ξ and its desired value:

$$z = \xi - \xi_{\text{des}} = \xi - \alpha(x) = \xi + c_1 x + \cos x.$$

We can now write the system (2.26) in the new co-ordinates (x, z):

$$\begin{aligned} \dot{x} &= -c_1 x - x^3 + z \\ \dot{z} &= u + (c_1 - \sin x)(-c_1 x - x^3 + z). \end{aligned} \tag{2.27}$$

To obtain a Lyapunov function candidate we simply augment the Lyapunov function with a quadratic term in z:

$$V_a(x, \xi) = V(x) + \frac{1}{2}z^2 = \frac{1}{2}x^2 + \frac{1}{2}(\xi + c_1 x + \cos x)^2.$$

The derivative of V_a along the solutions of (2.27) becomes

$$\dot{V}_a(x, z, u) = -c_1 x^2 - x^4 + z\left(x + u + (c_1 - \sin x)(-c_1 x - x^3 + z)\right).$$

The simplest way to arrive at a negative definite $\dot{V}_a$ is to choose

$$u = -c_2 z - x - (c_1 - \sin x)(-c_1 x - x^3 + z)$$

which in the original co-ordinates $[x, \xi]^T$ becomes

$$u = -(c_1 + c_2)\xi - (1 + c_1 c_2)x - (c_1 + c_2)\cos x + c_1 x^3 - x^3 \sin x + \xi \sin x + \sin x \cos x. \tag{2.28}$$

Usually ξ is called a *virtual control*, $\alpha(x)$ a *stabilizing function* and z the corresponding *error variable*.

From this example it is not difficult to see that the more general class of "triangular" nonlinear systems

$$\begin{aligned}
\dot{x} &= f(x) + g(x)\xi_1 \\
\dot{\xi}_1 &= f_1(x, \xi_1) + g_1(x, \xi_1)\xi_2 \\
\dot{\xi}_2 &= f_2(x, \xi_1, \xi_2) + g_2(x, \xi_1, \xi_2)\xi_3 \\
&\vdots \\
\dot{\xi}_{n-1} &= f_{n-1}(x, \xi_1, \ldots, \xi_{n-1}) + g_{n-1}(x, \xi_1, \ldots, \xi_{n-1})\xi_n \\
\dot{\xi}_n &= f_n(x, \xi_1, \ldots, \xi_n) + g_n(x, \xi_1, \ldots, \xi_n)u
\end{aligned}$$

can be stabilized in a similar way. First consider ξ_1 as a virtual input to stabilize the first subsystem, define the error variable z_1, consider ξ_2 as a virtual input to stabilize the $[x, z_1]^T$ subsystem, etc. Proceeding step by step along these lines one finally arrives at a control law for u.

One of the advantages of backstepping is that it provides a constructive systematic method to arrive at globally stabilizing control laws. Unfortunately, one usually obtains complex expressions (in the original co-ordinates) for the control law, as already can be seen from (2.28).

Part I

A cascaded approach to tracking

Chapter 3

Introduction to Part I

3.1 Cascaded design

In recent years recursive design methods for global stabilization of nonlinear systems have been developed. For applying these methods the nonlinear system has to have (or should be transformed into) a certain triangular form. Two major design techniques can be distinguished: backstepping for lower triangular systems (Koditschek 1987, Byrnes and Isidori 1989, Tsinias 1989, Marino and Tomei 1995, Krstić et al. 1995) and forwarding for upper triangular systems (Mazenc and Praly 1994, Janković, Sepulchre and Kokotović 1996).

One of the advantages of these methods is that they provide a systematic way of recursively designing feedback laws. Furthermore, associated Lyapunov functions for showing global stabilization are derived. However, a disadvantage is that the resulting control laws usually are complex expressions like in Example 2.5.1.

Our goal is to arrive at less complex expressions and to gain more insight in the control laws. This is why we follow a different approach. We use the results on cascaded systems (Ortega 1991, Mazenc and Praly 1996, Janković et al. 1996), or to be more precise the result for time-varying systems as initially presented by Panteley and Loría (1998) and further developed in (Panteley and Loría 1999).

Roughly speaking, we can summarize Theorem 2.4.3 by saying that under certain conditions the stability of the system

$$\dot{z}_1 = f_1(t, z_1) + g(t, z_1, z_2)z_2 \tag{3.1a}$$
$$\dot{z}_2 = f_2(t, z_2) \tag{3.1b}$$

can be concluded from the stability of the systems $\dot{z}_1 = f_1(t, z_1)$ and $\dot{z}_2 = f_2(t, z_2)$. This implies that in the analysis we can simply “forget” about the term $g(t, z_1, z_2)z_2$ since (under certain conditions) it does not play a crucial role.

Example 3.1.1. Consider the second order system of Example 2.5.1 after the change of co-ordinates, i.e., consider the system

$$\dot{x} = -c_1 x - x^3 + z \tag{3.2a}$$

$$\dot{z} = u + (c_1 - \sin x)(-c_1 x - x^3 + z). \tag{3.2b}$$

The backstepping approach resulted into the control law

$$u = -c_2 z - x - (c_1 - \sin x)(-c_1 x - x^3 + z) \tag{3.3}$$

which results in the globally asymptotically stable (GAS) closed-loop system

$$\begin{aligned}\dot{x} &= -c_1 x - x^3 + z\\ \dot{z} &= -x - c_2 z.\end{aligned}$$

We could also have taken a slightly different approach before applying the final step in the backstepping design to arrive at (3.3). Notice that for $z = 0$ the system (3.2a) is globally asymptotically stable (GAS). This is not surprising, since z is precisely the difference between the virtual control and its desired value that would have stabilized the x-subsystem. As a result we can also view the system (3.2) as the (by means of the desired virtual control stabilized) system

$$\Sigma_1 : \qquad \dot{x} = -c_1 x - x^3$$

that is perturbed by the output z of the system

$$\Sigma_2 : \qquad \dot{z} = u + (c_1 - \sin x)(-c_1 x - x^3 + z). \tag{3.4}$$

As a result, if we are able to render (3.4) globally asymptotically stable we can claim global asymptotic stability of the overall system from the theory on cascaded systems.

It is clear that the control law

$$u = -c_2 z - (c_1 - \sin x)(-c_1 x - x^3 + z) \tag{3.5}$$

renders the system (3.4) globally exponentially stable (GES). With this control law we arrive at the overall closed-loop system

$$\begin{aligned}\dot{x} &= -c_1 x - x^3 \quad + \quad z\\ \dot{z} &= -c_2 z\end{aligned}$$

which according to Lemma 2.4.5 is globally asymptotically stable (GAS).

Notice that a slight difference exists between the control laws (3.3) and (3.5). It turns out that the "$g(t, z_1, z_2)z_2$-part" of the subsystem (3.2a) is left out (the term $+z$ in (3.2a) results in the extra term $+xz$ in the derivative of the Lyapunov function, which is accounted for by an additional $-x$ in the control law).

Although the difference is not remarkable, the main lesson that can be learned from Example 3.1.1 is not that we are able to leave out the term $-x$ in the control law, but that by recognizing a cascaded structure while designing a controller one might reduce the complexity of the controller.

3.2 An introductory example: tracking of a rotating rigid body

Example 3.1.1 gives rise to the question how to recognize a cascaded structure while designing a control law and how to guarantee that the closed-loop control system can be written in the form (3.1). One possible answer has been given there: follow a backstepping design and notice before applying the final step that one has a globally asymptotically stable (GAS) subsystem together with a corresponding Lyapunov function. Therefore, it might suffice to stabilize only the difference between the virtual control and its desired value, without taking into account the way this error enters the remaining dynamics. Clearly, also other directions can be taken. Instead of starting from a system Σ_1 and designing Σ_2, we can also start with designing Σ_2.

To make this more clear, we consider as an introductory example the tracking problem for a rotating rigid body, for instance a spacecraft. For reasons of simplicity, we consider not the entire model, but only the dynamics of the velocities. Then the dynamics for a rotating rigid body with two controls can be expressed as:

$$\begin{aligned}
\dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + u_1 \\
\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + u_2 \\
\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2
\end{aligned}$$

where ω_i ($i = 1, \ldots, 3$) are the angular velocities and $I_1 > I_2 > 0$ and $I_3 > 0$ are the principal moments of inertia.

Notice that we assume $I_1 \neq I_2$ in order to be able to control ω_3 by means of u_1 and u_2. Then the assumption $I_1 > I_2$ can be made without loss of generality.

Assume that a feasible reference trajectory (ω_r, u_r) is given, i.e., a trajectory satisfying

$$\begin{aligned}
\dot{\omega}_{1,r} &= \frac{I_2 - I_3}{I_1}\omega_{2,r}\omega_{3,r} + u_{1,r} \\
\dot{\omega}_{2,r} &= \frac{I_3 - I_1}{I_2}\omega_{3,r}\omega_{1,r} + u_{2,r} \\
\dot{\omega}_{3,r} &= \frac{I_1 - I_2}{I_3}\omega_{1,r}\omega_{2,r}.
\end{aligned}$$

When we define the tracking-error $\omega_e = \omega - \omega_r$ we obtain the tracking error dynamics

$$\dot{\omega}_{1,e} = \frac{I_2 - I_3}{I_1}(\omega_2\omega_3 - \omega_{2,r}\omega_{3,r}) + u_1 - u_{1,r} \tag{3.7a}$$

$$\dot{\omega}_{2,e} = \frac{I_3 - I_1}{I_2}(\omega_3\omega_1 - \omega_{3,r}\omega_{1,r}) + u_2 - u_{2,r} \tag{3.7b}$$

$$\dot{\omega}_{3,e} = \frac{I_1 - I_2}{I_3}(\omega_1\omega_2 - \omega_{1,r}\omega_{2,r}). \tag{3.7c}$$

We are interested in obtaining a closed-loop system of the form (3.1). That is what we focus on in the controller design. To start with, we look for a way to obtain in closed loop a subsystem Σ_2, i.e., a subsystem (3.1b). In that light it is good to remark that we can use one input for stabilization of a subsystem of the control system (3.7). If we for instance take

$$u_1 = u_{1,r} - \frac{I_2 - I_3}{I_1}(\omega_2\omega_3 - \omega_{2,r}\omega_{3,r}) - k_1\omega_{1,e} \qquad k_1 > 0, \tag{3.8}$$

then the subsystem (3.7a) is rendered globally uniformly exponentially stable (GUES). In the closed-loop system this stabilized subsystem can be considered as the system Σ_2, i.e., the system (3.1b). Now we still have one input left that should be chosen such that the overall closed-loop system is rendered asymptotically stable.

We aim for a closed-loop system of the form (3.1). Besides, for asymptotic stability of the system (3.1) it is necessary that the part

$$\dot{z}_1 = f_1(t, z_1) \tag{3.9}$$

is asymptotically stable. This should is something that be guaranteed by the controller design. From Theorem 2.4.3 we furthermore know that it might be sufficient too! As a result, we can conclude that it might suffice in the controller design for the remaining input to render the part (3.9) asymptotically stable and "forget" about the $g(t, z_1, z_2)z_2$ part.

So how to proceed? Notice that it is fairly easy to arrive from (3.1a) at (3.9). It is mainly a matter of substituting $z_2 \equiv 0$ in (3.1a). This is also the way to proceed in the controller design. In the first step we designed a control law for one of the two inputs in such a way that in closed loop a subsystem was stabilized. Before we proceed with the controller design we assume that the stabilization of this subsystem worked out.

For the example of the rotating body this boils down to substituting $\omega_{1,e} \equiv 0$ in the remaining dynamics, which results into the linear system

$$\begin{bmatrix} \dot{\omega}_{2,e} \\ \dot{\omega}_{3,e} \end{bmatrix} = \begin{bmatrix} 0 & \frac{I_3 - I_1}{I_2}\omega_{1,r} \\ \frac{I_1 - I_2}{I_3}\omega_{1,r} & 0 \end{bmatrix} \begin{bmatrix} \omega_{2,e} \\ \omega_{3,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u_2 - u_{2,r}]. \tag{3.10}$$

In general this assumption (i.e., the substitution $z_2 \equiv 0$) simplifies the remaining dynamics considerably, since part of it can be forgotten. What we are left with is the part of the closed-loop system that is described by (3.9) and the problem has reduced to finding a control law for the second input that is such that this remaining part becomes globally uniformly asymptotically stable (GUAS).

For the rotating body this can be guaranteed by a proper choice of the remaining input u_2. From Theorem 2.3.7 we know that the control law

$$u_2 = u_{2,r} - k_2\omega_{2,e} - k_3\omega_{1,r}\omega_{3,e} \tag{3.11}$$

with $k_2 > 0$ and $k_3 > \frac{I_3 - I_1}{I_2}$ makes that the closed-loop system (3.10, 3.11) is globally uniformly asymptotically stable (GUAS), provided that $\omega_{1,r}$ is persistently exciting.

In that case the closed-loop system (3.7, 3.8, 3.11) can be written as

$$\begin{bmatrix}\dot\omega_{2,e}\\ \dot\omega_{3,e}\end{bmatrix} = \underbrace{\begin{bmatrix}-k_2 & -\left(k_3 - \frac{I_3-I_1}{I_2}\right)\omega_{1,r}\\ \frac{I_1-I_2}{I_3}\omega_{1,r} & 0\end{bmatrix}\begin{bmatrix}\omega_{2,e}\\ \omega_{3,e}\end{bmatrix}}_{f_1(t,z_1)} + \underbrace{\begin{bmatrix}\frac{I_3-I_1}{I_2}(\omega_{3,e}+\omega_{3,r})\\ \frac{I_1-I_2}{I_3}(\omega_{2,e}+\omega_{2,r})\end{bmatrix}}_{g(t,z_1,z_2)}\omega_{1,e} \tag{3.12a}$$

$$\dot\omega_{1,e} = \underbrace{-k_1\omega_{1,e}}_{f_2(t,z_2)} \tag{3.12b}$$

which has a clear cascaded structure. That is, we can clearly recognize the systems $\dot z_1 = f_1(t,z_1)$ and $\dot z_2 = f_2(t,z_2)$, as well as the "connecting term" $g(t,z_1,z_2)$.

We now have found an overall closed-loop system with a cascaded structure, but does this enable us to conclude asymptotic stability of the overall closed-loop system? Fortunately the answer is: yes. Since the systems $\dot z_1 = f_1(t,z_1)$ and $\dot z_2 = f_2(t,z_2)$ both are globally uniformly exponentially stable (GUES), it follows from Corollary 2.4.6 that we can conclude global $\mathcal{K}$-exponential stability of the system (3.12) once we have that $g(t,z_1,z_2)$ satisfies (2.25). This follows immediately when we assume that both $\omega_{2,r}$ and $\omega_{3,r}$ are bounded.

We can summarize this result as follows.

Proposition 3.2.1. *Consider the tracking error dynamics* (3.7) *in closed loop with the control laws* (3.8, 3.11). *If ω_r is bounded and $\omega_{1,r}$ is persistently exciting (PE), then the resulting closed-loop system* (3.12) *is globally $\mathcal{K}$-exponentially stable.*

Remark 3.2.2. Instead of first using u_1 to render the subsystem (3.7a) GUES and then u_2 to stabilize the remaining dynamics, we can also first use u_2 to render the subsystem (3.7b) GUES and then use u_1 to stabilize the dynamics that remain then. This is similar to interchanging the indices $(\cdot)_1$ and $(\cdot)_2$ in both (3.8) and (3.11).

The example of tracking the kinematics of a rotating body learned us that another way of obtaining a closed-loop system of the form (3.1) for a system with two inputs is the following:

- use one input for stabilizing a subsystem of the dynamics. In the overall closed-loop system this is the system (3.1b);
- assume that the stabilization of z_2 has worked out (as guaranteed by the first step of this procedure), i.e., substitute $z_2 \equiv 0$ in the remaining system;
- use the other input to stabilize the simplified remaining system;
- apply Theorem 2.4.3 to conclude asymptotic stability of the overall closed-loop dynamics.

This is the approach that we follow in this part of the thesis, i.e., in the next three chapters. We study three different examples of systems with two inputs that can be stabilized using this procedure.

Chapter 4

Tracking of a mobile robot

4.1 Introduction

In this chapter we study the tracking problem for a wheeled mobile robot of the unicycle type, shown in Figure 4.1. It is assumed that the masses and inertias of the wheels are negligible

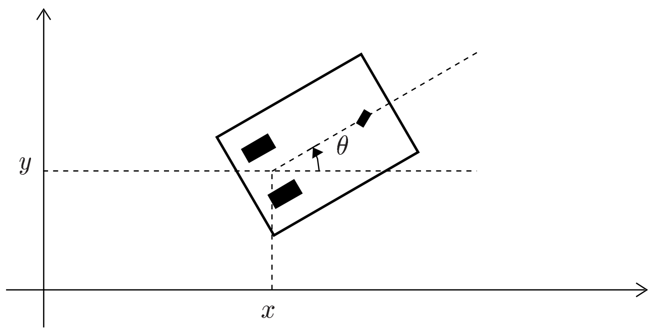

Figure 4.1: A two-wheel mobile robot.

and that both the forward velocity v and angular velocity ω can be controlled independently by motors. Let (x, y) denote the co-ordinates of the center of mass, and θ the angle between the heading direction and the x-axis. We assume that the wheels do not slide, which results in the following equations

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \end{aligned} \tag{4.1}$$

where v and ω are considered as inputs.

Notice that the no-slip condition imposes the non-holonomic constraint

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0.$$

As a result, the system (4.1) fails to meet Brockett's necessary condition for feedback stabilization (Brockett 1983). This implies that no smooth (or even continuous) time-invariant static state-feedback law $u = u(x)$ exists which makes a specified equilibrium of the closed-loop locally asymptotically stable. Consequently either discontinuous or time-varying (or both) controllers are needed for the stabilization problem, which explains the interest of many researchers in this simple model. For an overview we refer to the survey paper of Kolmanovsky and McClamroch (1995) and references cited therein.

Although the stabilization problem for wheeled mobile robots is now well understood, the tracking problem has received less attention. As a matter of fact, it is not clear that the current stabilization methodologies can be extended easily to tracking problems.

In (Kanayama, Kimura, Miyazaki and Noguchi 1990, Murray, Walsh and Sastry 1992, Micaelli and Samson 1993, Walsh, Tilbury, Sastry, Murray and Laumond 1994, Fierro and Lewis 1995) a linearization-based tracking control scheme was derived. The idea of input-output linearization was used by Oelen and van Amerongen (1994). Fliess, Levine, Martin and Rouchon (1995) dealt with the trajectory stabilization problem by means of a flatness approach. All these papers solve the local tracking problem.

The first global tracking control law that we are aware of was proposed by Samson and Ait-Abderrahim (1991). Another global tracking result was derived by Jiang and Nijmeijer (1997) using integrator backstepping.

Assume that feasible reference dynamics $(x_r, y_r, \theta_r, v_r, \omega_r)^T$ is given, i.e., dynamics that satisfies

$$\begin{aligned}
\dot{x}_r &= v_r\cos\theta_r \\
\dot{y}_r &= v_r\sin\theta_r \\
\dot{\theta}_r &= \omega_r.
\end{aligned}$$

For solving the tracking control problem the following global change of co-ordinates was proposed by Kanayama et al. (1990) (cf. Figure 4.2):

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \theta_r - \theta \end{bmatrix}. \tag{4.2}$$

This global change of co-ordinates from $[x_r - x, y_r - y]^T$ to $[x_e, y_e]^T$ makes that the error-variables become independent from the choice of the inertial co-ordinate frame; the errors are considered in a frame attached to the mobile robot. In these new co-ordinates the tracking error dynamics becomes:

$$\begin{aligned}
\dot{x}_e &= \omega y_e - v + v_r(t)\cos\theta_e \\
\dot{y}_e &= -\omega x_e + v_r(t)\sin\theta_e \\
\dot{\theta}_e &= \omega_r(t) - \omega.
\end{aligned} \tag{4.3}$$

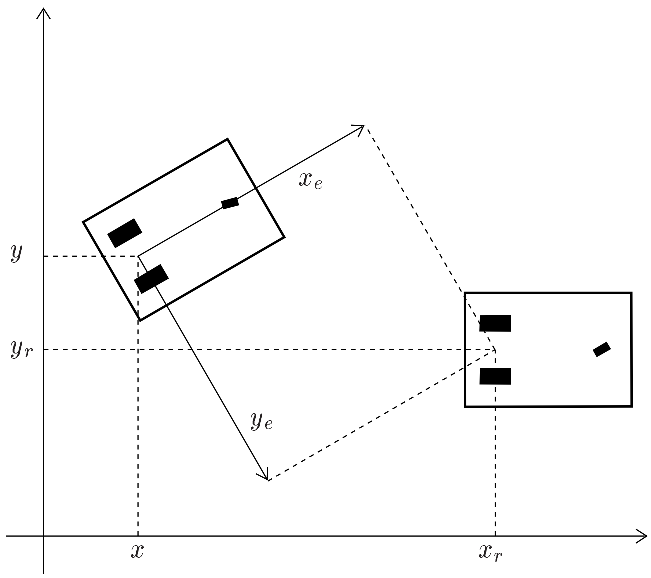

Figure 4.2: The new error co-ordinates.

The tracking control problem boils down to finding appropriate control laws for v and ω such that the tracking error $(x_e, y_e, \theta_e)^T$ converges to zero.

This is the problem studied in this chapter. Subsequently we study in Section 4.2 the state-feedback problem, in Section 4.3 the output-feedback problem, and in Section 4.4 the state-feedback problem under input saturation. The performance of the derived controllers is illustrated by means of simulations in Section 4.5. We conclude this chapter with some final remarks in Section 4.6.

4.2 State-feedback

In this section we study the state-feedback tracking control problem for a mobile robot. As mentioned in the introduction, we are aware of two global tracking results. First, we recover these two results that both achieve global asymptotic stability (GAS) of the tracking error dynamics. For reasons of robustness we would like to be able to conclude global *uniform* asymptotic stability (GUAS) (cf. Example 2.2.4). By means of a cascaded design approach we derive a controller that achieves global $\mathcal{K}$-exponential stability.

4.2.1 Previous results

We first summarize the available global tracking results.

Proposition 4.2.1 (Samson and Ait-Abderrahim (1991), Lyapunov based). *Consider the tracking error dynamics* (4.3) *in closed loop with the control law*

$$\omega = \omega_r(t) + \frac{k_3}{k_2}\theta_e + \frac{k_1}{k_2}v_r(t)y_e\frac{\sin\theta_e}{\theta_e} - \frac{k_6}{k_2}x_e \tag{4.4a}$$

$$\begin{aligned} v = {} & v_r(t) + k_3k_5x_e + (2k_3k_4 + v_r(t)\frac{\cos\theta_e - 1}{\theta_e} + k_6)\theta_e \\ & + (1-k_1)y_e(\omega_r(t) + \frac{k_3}{k_2}\theta_e + \frac{k_1}{k_2}v_r(t)y_e\frac{\sin\theta_e}{\theta_e} - \frac{k_6}{k_2}x_e) \end{aligned} \tag{4.4b}$$

where $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, $k_5 > k_4^2$, and k_4 and k_6 are arbitrary constants. If $\dot{v}_r(t)$ and $\dot{\omega}_r(t)$ are bounded and either $v_r(t)$ or $\omega_r(t)$ does not converge to zero, then the closed-loop system (4.3, 4.4) *is globally asymptotically stable (GAS).*

Proposition 4.2.2 (Jiang and Nijmeijer (1997), backstepping based). *Consider the tracking error dynamics* (4.3) *in closed loop with the control law*

$$\omega = \omega_r(t) + k_1k_4\theta_e + k_1v_r(t)y_e\frac{\sin\theta_e}{\theta_e} \tag{4.5a}$$

$$\begin{aligned} v = {} & v_r(t)\cos\theta_e + k_3x_e - k_2\dot{\omega}_r(t)y_e - k_1k_2\dot{v}_r(t)y_e^2\frac{\sin\theta_e}{\theta_e} + k_1^2k_2k_4^2y_e\theta_e \\ & - k_1k_2k_3k_4y_e\theta_e - 2k_1k_2v_r(t)^2y_e\frac{\sin^2\theta_e}{\theta_e} + 3k_1^2k_2k_4v_r(t)x_ey_e\sin\theta_e \\ & + 3k_1k_2v_r(t)\omega_r(t)x_ey_e\frac{\sin\theta_e}{\theta_e} - k_1k_2k_3v_r(t)y_e^2\frac{\sin\theta_e}{\theta_e} - k_2k_3\omega_r(t)y_e \\ & + k_1^2k_4v_r(t)y_e^2\cos\theta_e - k_2v_r(t)\omega_r(t)\sin\theta_e - k_1k_2k_4v_r(t)\theta_e\sin\theta_e \\ & + 2k_1k_2k_4\omega_r(t)x_e\theta_e + k_2\omega_r(t)^2x_e + k_1^2k_2k_4x_e\theta_e^2 + 2k_1^2k_2v_r(t)^2x_ey_e^2\frac{\sin^2\theta_e}{\theta_e^2} \\ & + k_1^2k_2v_r(t)^2y_e^3\frac{\theta_e\sin\theta_e\cos\theta_e - \sin^2\theta_e}{\theta_e^3} \end{aligned} \tag{4.5b}$$

where $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, $k_4 > 0$. If $v_r(t)$, $\dot{v}_r(t)$, $\omega_r(t)$ and $\dot{\omega}_r(t)$ are bounded and either $v_r(t)$ or $\omega_r(t)$ does not converge to zero, then the closed-loop system (4.3, 4.5) *is globally asymptotically stable (GAS).*

Remark 4.2.3. Jiang and Nijmeijer (1997) remarked that by means of Lyapunov theory the control law

$$\omega = \omega_r(t) + v_r(t)y_e\frac{\sin\theta_e}{\theta_e} + c_1\theta_e \qquad c_1 > 0 \tag{4.6a}$$

$$v = v_r(t)\cos\theta_e + c_2x_e \qquad c_2 > 0 \tag{4.6b}$$

can be shown to yield GAS of the closed-loop system (4.3, 4.6), provided that $v_r(t)$ and $\omega_r(t)$ are uniformly continuous and bounded, and either $v_r(t)$ or $\omega_r(t)$ does not converge to zero. This boils down to the controller (4.4) where we take $k_1 = 1$, $k_2 = 1$, $k_3 = c_1$, $k_4 = 0$, $k_5 = \frac{c_2}{c_1}$, $k_6 = 0$. However, the assumption on $v_r(t)$ and $\omega_r(t)$ is slightly weaker (uniform continuous and bounded, instead of differentiable with bounded derivative).

4.2.2 A cascaded design

In this section we derive a controller for the tracking control problem for a mobile robot. For that we use the cascaded systems based approach sketched in Section 3.2, to achieve globally uniformly asymptotically stable (GUAS) tracking error dynamics.

Recall that the tracking error dynamics for a mobile robot can be described by

$$\dot{x}_e = \omega y_e - v + v_r(t)\cos\theta_e \tag{4.7a}$$
$$\dot{y}_e = -\omega x_e + v_r(t)\sin\theta_e \tag{4.7b}$$
$$\dot{\theta}_e = \omega_r(t) - \omega \tag{4.7c}$$

where x_e and y_e are position errors, θ_e is the orientation error and $v_r(t)$ and $\omega_r(t)$ are the forward and angular velocity of the reference trajectory to be tracked. As inputs we have the forward velocity v and the angular velocity ω.

As pointed out in Section 3.2 we first use one input for stabilization of a subsystem. By means of the input ω the dynamics (4.7c) can easily be stabilized. The control law

$$\omega = \omega_r(t) + k_1\theta_e \qquad\qquad k_1 > 0 \tag{4.8}$$

results into the globally uniformly exponentially stable (GUES) subsystem

$$\dot{\theta}_e = -k_1\theta_e \qquad\qquad k_1 > 0. \tag{4.9}$$

We can think of (4.9) as the system Σ_2 (cf. Section 2.4).

The remaining dynamics is then given by

$$\begin{aligned} \dot{x}_e &= \omega_r(t)y_e + k_1\theta_e y_e - v + v_r(t)\cos\theta_e \\ \dot{y}_e &= -\omega_r(t)x_e - k_1\theta_e x_e + v_r(t)\sin\theta_e. \end{aligned} \tag{4.10}$$

We proceed by assuming that the stabilization of θ_e has been established. What we do is to "forget" about the $g(t, z_1, z_2)z_2$ part of the dynamics (3.1a) and focus on rendering (3.9) globally uniformly asymptotically stable (GUAS). The next step therefore is substituting $\theta_e(t) \equiv 0$ in the remaining dynamics (4.10), which results into

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} 0 & \omega_r(t) \\ -\omega_r(t) & 0 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [v_r(t) - v] \tag{4.11}$$

which simply is a linear time-varying system. From Theorem 2.3.7 we know that if $\omega_r(t)$ is persistently exciting (PE), then the control law

$$v = v_r(t) + k_2 x_e - k_3\omega_r(t)y_e \qquad\qquad k_2 > 0, k_3 > -1 \tag{4.12}$$

renders the resulting closed-loop system (4.11, 4.12) globally uniformly exponentially stable (GUES).

As a result we obtain the following.

Proposition 4.2.4. *Consider the tracking error dynamics* (4.7) *in closed loop with the control law*

$$\omega = \omega_r(t) + k_1\theta_e \qquad k_1 > 0 \tag{4.13a}$$

$$v = v_r(t) + k_2 x_e - k_3\omega_r(t) y_e \qquad k_2 > 0, k_3 > -1. \tag{4.13b}$$

If $v_r(t)$ is bounded and $\omega_r(t)$ is persistently exciting (PE) then the closed-loop system (4.7, 4.13) *is globally $\mathcal{K}$-exponentially stable.*

Proof. Due to the design we obtain a cascaded structure for the closed-loop system:

$$\begin{bmatrix}\dot{x}_e \\ \dot{y}_e\end{bmatrix} = \underbrace{\begin{bmatrix}-k_2 & (k_3+1)\omega_r(t) \\ -\omega_r(t) & 0\end{bmatrix}\begin{bmatrix}x_e \\ y_e\end{bmatrix}}_{f_1(t,z_1)} + \underbrace{\begin{bmatrix}k_1 y_e + v_r(t)\frac{\cos\theta_e - 1}{\theta_e} \\ -k_1 x_e + v_r(t)\frac{\sin\theta_e}{\theta_e}\end{bmatrix}}_{g(t,z_1,z_2)}\theta_e$$

$$\dot{\theta}_e = \underbrace{-k_1\theta_e}_{f_2(t,z_2)}.$$

Notice that the systems $\dot{z}_1 = f_1(t,z_1)$ and $\dot{z}_2 = f_2(t,z_2)$ are globally uniformly exponentially stable (GUES). Since $\frac{\cos\theta_e - 1}{\theta_e}$ and $\frac{\sin\theta_e}{\theta_e}$ are bounded, the boundedness of $v_r(t)$ guarantees that the assumption on $g(t,z_1,z_2)$ is met. Applying Corollary 2.4.6 completes the proof. □

Remark 4.2.5. This result was originally presented by Panteley et al. (1998), where $k_3 = 0$ was used.

When we compare the result of Proposition 4.2.4 with the results as presented in Propositions 4.2.1 and 4.2.2, a difference in complexity can be noticed in (4.4) and (4.5) versus (4.13). Furthermore, the controllers (4.4) and (4.5) were only shown to yield globally asymptotically stable (GAS) closed-loop tracking error dynamics, whereas for (4.13) we were able to show the more desirable property of global uniform asymptotic stability (GUAS)[1]. The price we pay is that (4.13) makes it impossible to track a reference for which $\omega_r(t)$ tends to zero but $v_r(t)$ does not, which is something that can be dealt with using (4.4) or (4.5).

Notice that due to the cascaded design approach we were able to reduce the problem of stabilizing the nonlinear tracking error dynamics (4.7) to the problem of stabilizing the linear systems

$$\dot{\theta}_e = \omega_r(t) - \omega \tag{4.14}$$

and

$$\begin{bmatrix}\dot{x}_e \\ \dot{y}_e\end{bmatrix} = \begin{bmatrix}0 & \omega_r(t) \\ -\omega_r(t) & 0\end{bmatrix}\begin{bmatrix}x_e \\ y_e\end{bmatrix} + \begin{bmatrix}1 \\ 0\end{bmatrix}[v_r(t) - v]. \tag{4.15}$$

So, a sense we reduced the nonlinear problem into two easy-to-solve linear problems.

[1]Most likely the control laws (4.4) and (4.5) also yield GUAS of the resulting closed-loop tracking error dynamics.

This observation can be very helpful in finding answers to questions that come from a more practical point of view. In practice we also have to deal with disturbances due to errors in the model or due to imperfect state measurements. Instead of solving the problem for a third order nonlinear system, it suffices to solve the problem for a first order and a second order linear system.

Assume for instance that a constant disturbance is perturbing the system (4.7). For design purposes we can simply assume that this constant disturbance is perturbing the systems (4.14) and (4.15). Therefore, the robust controller design for the nonlinear system (4.7) under constant disturbances simply reduces to the robust controller design for the linear system (4.14) and the linear time-varying system (4.15). Both can easily be solved by adding integral action. Similar reasoning can be used in case of more general additive disturbances. Then one can for instance use H_∞ control techniques for arriving at robust controllers for the two linear systems, instead of going through a nonlinear H_∞ design.

In case we have noisy measurements, it is common practice to filter the measurements and use the filtered state for feedback. Since the tracking error dynamics are nonlinear, this approach is in general not guaranteed to work. However, for linear systems this approach can be applied successfully. Therefore, we simply design filters such that the linear systems (4.14) and (4.15) are rendered asymptotically stable. Corollary 2.4.6 then guarantees $\mathcal{K}$-exponential stability of the nonlinear tracking error dynamics.

Remark 4.2.6. Not only can the cascaded design reduce the nonlinear controller design problem for the system (4.7) into two linear ones, it also provides an eye-opener to recognizing a simpler structure for backstepping. From the cascaded design we obtained for $k_3 = 0$ the Σ_1 subsystem

$$\begin{aligned} \dot{x}_e &= -k_2 x_e + \omega_r(t) y_e \\ \dot{y}_e &= -\omega_r(t) x_e \end{aligned} \tag{4.16}$$

which can be seen as the subsystem (4.7a, 4.7b) stabilized by means of the input $v = v_r(t)\cos\theta_e + k_2 x_e$ and the virtual control $\theta_e \equiv 0$. We can show global asymptotic stability (GAS) of the system (4.16) by means of the Lyapunov function candidate

$$V = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2$$

and some additional standard Lyapunov techniques. If we now ‘step back’ the virtual control θ_e to the true input ω, we obtain the control law

$$\begin{aligned} \omega &= \omega_r(t) + v_r(t) y_e \frac{\sin\theta_e}{\theta_e} + k_1\theta_e & \qquad k_1 > 0 \\ v &= v_r(t)\cos\theta_e + k_2 x_e & \qquad k_2 > 0 \end{aligned}$$

which is exactly the controller (4.6). Therefore, backstepping not necessarily has to lead to complex expressions for control laws as (4.5), but can also result in more simple expressions as (4.6). The only difficulty is to recognize the simpler structure for backstepping. This structure became clear from the cascaded controller design.

Remark 4.2.7. The requirement that ω_r has to be persistently exciting (PE)is a serious practical limitation, since it makes it impossible to follow straight lines, while this is the first thing one would like to do in practice. One way is to overcome this difficultie is sketched in the previous remark. An other idea is using the idea of uniform δ-persistence of excitation as introduced by Loría, Panteley and Teel (1999b). This weakened version of PE makes it not only possible to deal with tracking of straight lines, but also with stabilization. By using this concept, global uniform asymptotic stability (GUAS) can be shown.

4.3 Dynamic output-feedback

In this section we study the dynamic output-feedback tracking control problem for a mobile robot. That is, we study the problem of stabilizing the tracking error dynamics (4.7) where we are only allowed to use the measured output for designing the control laws for v and ω. With the cascaded design from the previous section in mind, the control laws derived in the previous section can easily be extended. In case we are able to measure only one of the state-components we end up with an unobservable system, which makes it impossible to reconstruct the state from the measurements. Therefore, we consider in the following sections the cases where we measure two of the state components.

4.3.1 Unmeasured x_e

First, we assume that we are unable to measure x_e, but that we can measure y_e and θ_e. In that case the available output is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_e \\ \theta_e \end{bmatrix}. \tag{4.17}$$

From a cascaded design point of view, we know that we only have to stabilize the systems

$$\dot{\theta}_e = \omega_r(t) - \omega \tag{4.18a}$$
$$y_2 = \theta_e \tag{4.18b}$$

and

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} 0 & \omega_r(t) \\ -\omega_r(t) & 0 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [v_r(t) - v] \tag{4.19a}$$
$$y_1 = y_e. \tag{4.19b}$$

It is clear that we can still use the control law (4.8) for stabilizing (4.18). The only problem is to stabilize (4.19).

However, from Theorem 2.3.8 we know that the dynamic output-feedback

$$v = v_r(t) + k_2\hat{x}_e - k_3\omega_r(t)\hat{y}_e \tag{4.20a}$$
$$\begin{bmatrix} \dot{\hat{x}}_e \\ \dot{\hat{y}}_e \end{bmatrix} = \begin{bmatrix} 0 & \omega_r(t) \\ -\omega_r(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_e \\ \hat{y}_e \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [v_r(t) - v] + \begin{bmatrix} -l_2\omega_r(t) \\ l_1 \end{bmatrix} [y_1 - \hat{y}_1] \tag{4.20b}$$
$$\hat{y}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_e \\ \hat{y}_e \end{bmatrix} \tag{4.20c}$$

with $k_2 > 0$, $k_3 > -1$, $l_1 > 0$, and $l_2 > -1$ renders the closed-loop system (4.18, 4.20) globally uniformly exponentially stable (GUES).

As a result we obtain:

Proposition 4.3.1. *Consider the tracking error dynamics* (4.7) *with output* (4.17) *in closed loop with the control laws* (4.8, 4.20). *Assume that $\omega_r(t)$ is persistently exciting (PE) and that $v_r(t)$ is bounded. Then the resulting closed-loop system is globally $\mathcal{K}$-exponentially stable.*

Proof. We can see the closed-loop system (4.7, 4.17, 4.8, 4.20) as a cascaded system, i.e., a system of the form (3.1) where

$$
\begin{aligned}
z_1 &= \begin{bmatrix} x_e & y_e & x_e - \hat{x}_e & y_e - \hat{y}_e \end{bmatrix}^T \\
z_2 &= \theta_e \\
f_1(t, z_1) &= \begin{bmatrix} -k_2 & (k_3+1)\omega_r(t) & k_2 & -k_3\omega_r(t) \\ -\omega_r(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & (l_2+1)\omega_r(t) \\ 0 & 0 & -\omega_r(t) & -l_1 \end{bmatrix} z_1 \\
f_2(t, z_2) &= -k_1 z_2 \\
g(t, z_1, z_2) &= \begin{bmatrix} k_1 y_e + v_r(t)\frac{\cos\theta_e - 1}{\theta_e} \\ -k_1 x_e + v_r(t)\frac{\sin\theta_e}{\theta_e} \\ k_1 y_e + v_r(t)\frac{\cos\theta_e - 1}{\theta_e} \\ -k_1 x_e + v_r(t)\frac{\sin\theta_e}{\theta_e} \end{bmatrix}.
\end{aligned}
$$

From Theorem 2.3.8 we know that the system $\dot{z}_1 = f_1(t, z_1)$ is globally uniformly exponentially stable (GUES). It is also clear that the system $\dot{z}_2 = f_2(t, z_2)$ is GUES. The boundedness of $v_r(t)$ guarantees that the assumption on $g(t, z_1, z_2)$ is met. Applying Corollary 2.4.6 completes the proof. □

Notice that (4.20b, 4.20c) is a full order observer for the system (4.19), i.e., even though we can measure y_e we also have generated an estimate for y_e. It is also possible to use a *reduced order* observer, i.e., to reconstruct only the unknown signal x_e.

In order to find a reduced observer for the system (4.19) we try to estimate some linear combination of the measured and the unknown signals. To be precise, we define a new variable z as

$$z = x_e - b(t)y_1$$

where $b(t)$ is a function still to be determined in order to guarantee asymptotic stability of the reduced order observer. Differentiating z with respect to time along the dynamics (4.19) yields

$$
\begin{aligned}
\dot{z} &= \omega_r(t)y_e + [v_r(t) - v] - \frac{\mathrm{d}b(t)}{\mathrm{d}t}y_e + b(t)\omega_r(t)x_e \\
&= b(t)\omega_r(t)(x_e - b(t)y_e) + b(t)^2\omega_r(t)y_e + \omega_r(t)y_e + [v_r(t) - v] - \frac{\mathrm{d}b(t)}{\mathrm{d}t}y_e \\
&= b(t)\omega_r(t)z + \left(b(t)^2\omega_r(t) + \omega_r(t) - \frac{\mathrm{d}b(t)}{\mathrm{d}t}\right)y_e + [v_r(t) - v].
\end{aligned}
$$

In case we define the reduced order observer dynamics as

$$\dot{\hat{z}} = b(t)\omega_r(t)\hat{z} + \left(b(t)^2\omega_r(t) + \omega_r(t) - \frac{\mathrm{d}b(t)}{\mathrm{d}t}\right) y_e + [v_r(t) - v]$$

we obtain for the observation-error $\tilde{z} = z - \hat{z}$

$$\dot{\tilde{z}} = b(t)\omega_r(t)\tilde{z}. \tag{4.21}$$

If we now take $b(t) = -l\omega_r(t)$ with l a positive constant and we furthermore assume that $\omega_r(t)$ is persistently exciting (PE), we are able to conclude global uniform exponential stability (GUES) of (4.21).

We can combine this reduced observer with the controller (4.12):

Proposition 4.3.2. *Consider the tracking error dynamics* (4.7) *with output* (4.17) *in closed-loop with the control law*

$$\omega = \omega_r(t) + k_1\theta_e \qquad k_1 > 0 \tag{4.22a}$$
$$v = v_r(t) + k_2\hat{x}_e - k_3\omega_r(t)y_e \qquad k_2 > 0, k_3 > -1 \tag{4.22b}$$

where $\hat{x}_e$ is generated by the reduced order observer

$$\hat{x}_e = \hat{z} - l\omega_r(t)y_e \qquad l > 0 \tag{4.23a}$$
$$\dot{\hat{z}} = -l\omega_r(t)^2\hat{z} + [l^2\omega_r(t)^3 + \omega_r(t) + l\dot{\omega}_r(t)]y_e + [v_r(t) - v]. \tag{4.23b}$$

If $v_r(t)$ is bounded and $\omega_r(t)$ is persistently exciting (PE), then the closed-loop system (4.7, 4.22, 4.23) *is globally $\mathcal{K}$-exponentially stable.*

Proof. We can view the closed-loop system (4.7, 4.22, 4.23) as a cascaded system, i.e., a system of the form (3.1) where

$$\begin{aligned}
z_1 &= \begin{bmatrix} x_e & y_e & x_e - \hat{x}_e \end{bmatrix}^T \\
z_2 &= \theta_e \\
f_1(t, z_1) &= \begin{bmatrix} -k_2 & (k_3+1)\omega_r(t) & k_2 \\ -\omega_r(t) & 0 & 0 \\ 0 & 0 & -l\omega_r(t)^2 \end{bmatrix} z_1 \\
f_2(t, z_2) &= -k_2 z_2 \\
g(t, z_1, z_2) &= \begin{bmatrix} k_1 y_e + v_r(t)\frac{\cos\theta_e - 1}{\theta_e} \\ -k_1 x_e + v_r(t)\frac{\sin\theta_e}{\theta_e} \\ k_1 y_e + v_r(t)\frac{\cos\theta_e - 1}{\theta_e} + l\omega_r(t)\left(-k_1 x_e + v_r(t)\frac{\sin\theta_e}{\theta_e}\right) \end{bmatrix}.
\end{aligned}$$

To be able to apply Corollary 2.4.6 we need to verify global uniform exponential stability (GUES) of the system $\dot{z}_1 = f_1(t, z_1)$, which can also be expressed as

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \underbrace{\begin{bmatrix} -k_2 & (k_3+1)\omega_r(t) \\ -\omega_r(t) & 0 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \end{bmatrix}}_{\bar{f}_1(t,\bar{z}_1)} + \underbrace{\begin{bmatrix} k_2 \\ 0 \end{bmatrix}}_{\bar{g}(t,\bar{z}_1,\bar{z}_2)} \bar{z}_2 \tag{4.24a}$$

$$\dot{\bar{z}}_2 = -l\omega_r(t)^2\bar{z}_2. \tag{4.24b}$$

Since $\omega_r(t)$ is persistently exciting (PE), we have the existence of constants $\delta, \varepsilon_1, \varepsilon_2 > 0$ such that for all $t \geq 0$:

$$\varepsilon_1 < \int_t^{t+\delta} \omega_r^2(\tau)\mathrm{d}\tau < \varepsilon_2.$$

Therefore, the subsystem (4.24b) is GUES. Furthermore, the term $\bar{g}(t, \bar{z}_1, \bar{z}_2)$ is bounded and according to Theorem 2.3.7 the system $\dot{\bar{z}}_1 = \bar{f}_1(t, \bar{z}_1)$ is GUES. From Corollary 2.4.6 we can conclude that the system $\dot{z}_1 = f_1(t, z_1)$ is globally uniformly asymptotically stable (GUAS). Since it is a linear time-varying system Theorem 2.3.9 enables us to conclude that $\dot{z}_1 = f_1(t, z_1)$ is GUES. Since also the system $\dot{z}_2 = f_2(t, z_2)$ is GUES and boundedness of both $v_r(t)$ and $\omega_r(t)$ (cf. Definition 2.3.5) guarantees that the condition on $g(t, z_1, z_2)$ is met, Corollary 2.4.6 yields the desired result. $\square$

4.3.2 Unmeasured y_e

In case we assume that we are unable to measure y_e, but we can measure x_e and θ_e, we have the outputs

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_e \\ \theta_e \end{bmatrix}. \tag{4.25}$$

Since we can repeat the reasoning of the previous section, we summarize this analysis in the following two propositions.

Proposition 4.3.3. *Consider the tracking error dynamics* (4.7) *with output* (4.25) *in closed loop with the control law*

$$\begin{aligned} \omega &= \omega_r(t) + k_1\theta_e & k_1 &> 0 \\ v &= v_r(t) + k_2\hat{x}_e - k_3\omega_r(t)\hat{y}_e & k_2 &> 0, k_3 > -1 \end{aligned}$$

where $\hat{x}_e$ and $\hat{y}_e$ are generated by the full order observer

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_e \\ \dot{\hat{y}}_e \end{bmatrix} &= \begin{bmatrix} 0 & \omega_r(t) \\ -\omega_r(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_e \\ \hat{y}_e \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [v_r(t) - v] + \begin{bmatrix} l_1 \\ l_2\omega_r(t) \end{bmatrix} [y_1 - \hat{y}_1] \\ \hat{y}_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_e \\ \hat{y}_e \end{bmatrix} \end{aligned}$$

where $l_1 > 0$, $l_2 > -1$. Assume that $\omega_r(t)$ is persistently exciting (PE) and that $v_r(t)$ is bounded. Then the resulting closed-loop system is globally $\mathcal{K}$-exponentially stable.

Proposition 4.3.4. *Consider the tracking error dynamics* (4.7) *with output* (4.25) *in closed-loop with the control law*

$$\omega = \omega_r(t) + k_1\theta_e \qquad k_1 > 0 \tag{4.26a}$$

$$v = v_r(t) + k_2 x_e - k_3\omega_r(t)\hat{y}_e \qquad k_2 > 0, k_3 > -1 \tag{4.26b}$$

where $\hat{y}_e$ is generated by the reduced order observer

$$\hat{y}_e = \hat{z} + l\omega_r(t)x_e \qquad l > 0 \tag{4.27a}$$

$$\dot{\hat{z}} = -l\omega_r(t)^2\hat{z} - [l^2\omega_r(t)^3 + \omega_r(t) + l\dot{\omega}_r(t)]x_e + l\omega_r(t)[v - v_r(t)]. \tag{4.27b}$$

If $v_r(t)$ is bounded and $\omega_r(t)$ is persistently exciting (PE), then the closed-loop system (4.7, 4.26, 4.27) *is globally $\mathcal{K}$-exponentially stable.*

4.4 Saturated control

All control laws mentioned in this chapter have one thing in common: the larger the errors, the larger the control. In practice, however, the input is constrained: the mobile car has a maximum forward and angular velocity. Therefore, it is interesting to take these input constraints into account when designing control laws. In this section we study the global tracking control problem for a mobile robot under input saturation, i.e., in the controller design we take into account the constraints

$$|v(t)| \leq v^{\max} \qquad \forall t \geq 0 \tag{4.28a}$$
$$|\omega(t)| \leq \omega^{\max} \qquad \forall t \geq 0. \tag{4.28b}$$

We would like to design controllers such that they never result into a forward and/or angular velocity exceeding the limits of the mobile car.

In order to be able to do so, we assume that once we are exactly on the desired trajectory we can stay on it. This means that the reference forward and angular velocity should not exceed the limits:

$$\sup_{t\geq 0} |v_r(t)| < v^{\max} \tag{4.29a}$$
$$\sup_{t\geq 0} |\omega_r(t)| < \omega^{\max}. \tag{4.29b}$$

Under these feasibility conditions we look for controllers for v and ω that always meet (4.28) and still guarantee global uniform asymptotic stability of the tracking error dynamics.

4.4.1 A Lyapunov design

Our first approach is a Lyapunov design similar to the one which results in the (unsaturated) control law (4.6). Inspired by Remark 4.2.6 and (Jiang and Praly 1992) we consider the Lyapunov function candidate

$$V = \frac{1}{2}\log(1 + x_e^2 + y_e^2) + \frac{1}{2\epsilon_1}\theta_e^2 \qquad \epsilon_1 > 0. \tag{4.30}$$

Differentiating V along the solutions of (4.7) yields

$$\begin{aligned}
\dot{V} &= \frac{x_e}{1 + x_e^2 + y_e^2}(-v + v_r(t)\cos\theta_e) + \frac{v_r(t)y_e}{1 + x_e^2 + y_e^2}\frac{\sin\theta_e}{\theta_e}\theta_e + \frac{1}{\epsilon_1}\theta_e(\omega_r(t) - \omega) \\
&= \frac{x_e}{1 + x_e^2 + y_e^2}(-v + v_r(t)\cos\theta_e) + \frac{1}{\epsilon_1}\theta_e\left(\omega_r(t) + \frac{\epsilon_1 v_r(t)y_e}{1 + x_e^2 + y_e^2}\frac{\sin\theta_e}{\theta_e} - \omega\right).
\end{aligned}$$

Choosing

$$\omega = \omega_r(t) + \frac{\epsilon_1 v_r(t) y_e}{1 + x_e^2 + y_e^2} \frac{\sin\theta_e}{\theta_e} + \sigma_{\epsilon_2}(\theta_e) \tag{4.31a}$$

$$v = v_r(t)\cos\theta_e + \sigma_{\epsilon_3}(x_e) \tag{4.31b}$$

with σ a saturation function as defined in Definition 2.1.13 and $\epsilon_2 > 0$, $\epsilon_3 > 0$, results in

$$\dot{V} = -\frac{x_e \sigma_{\epsilon_3}(x_e)}{1 + x_e^2 + y_e^2} - \frac{1}{\epsilon_1}\theta_e \sigma_{\epsilon_2}(\theta_e) \leq 0.$$

As a result, using (4.30) we can conclude that the trajectories $(x_e(t), y_e(t), \theta_e(t))$ are uniformly bounded. If we furthermore assume that $v_r(t)$ and $\omega_r(t)$ are uniformly continuous, we obtain that also $x_e(t)$, $y_e(t)$ and $\theta_e(t)$ are uniformly continuous. It follows by a direct application of Barbălat's Lemma (Lemma 2.2.10) that

$$\lim_{t\to\infty} \frac{x_e(t)\sigma_{\epsilon_3}(x_e(t))}{1 + x_e(t)^2 + y_e(t)^2} + \frac{1}{\epsilon_1}\theta_e(t)\sigma_{\epsilon_2}(\theta_e(t)) = 0$$

which, in turn, implies that

$$\lim_{t\to\infty} (|x_e(t)| + |\theta_e(t)|) = 0.$$

In order to show that $y_e(t)$ goes to zero as $t \to \infty$ we use Lemma 2.2.12 with

$$\begin{aligned} f(t) &= \theta_e \\ g(t) &= \frac{\epsilon_1 v_r(t) y_e}{1 + x_e^2 + y_e^2}\left(\frac{\sin\theta_e}{\theta_e} - 1\right) + \sigma_{\epsilon_2}(\theta_e) \end{aligned}$$

which enables us to conclude that

$$\lim_{t\to\infty} \frac{\epsilon_1}{1 + x_e(t)^2 + y_e(t)^2} v_r(t) y_e(t) = 0.$$

As a result also

$$\lim_{t\to\infty} v_r(t) y_e(t) = 0. \tag{4.32}$$

Next, we apply Lemma 2.2.12 with

$$\begin{aligned} f(t) &= x_e \\ g(t) &= \left(\frac{\epsilon_1 v_r(t) y_e}{1 + x_e^2 + y_e^2}\frac{\sin\theta_e}{\theta_e} + \sigma_{\epsilon_2}(\theta_e)\right) y_e - \sigma_{\epsilon_3}(x_e) \end{aligned}$$

in order to conclude that

$$\lim_{t\to\infty} \omega_r(t) y_e(t) = 0. \tag{4.33}$$

As a result from (4.32) and (4.33) we have that $y_e(t)$ tends to zero as t tends to infinity, provided that either $v_r(t)$ or $\omega_r(t)$ does not converge to zero.

We can summarize this result as follows.

Proposition 4.4.1 (see (Jiang, Lefeber and Nijmeijer 1999)). *Consider the tracking error dynamics* (4.7) *in closed loop with the control law* (4.31). *If $v_r(t)$ and $\omega_r(t)$ are uniformly continuous and bounded, and either $v_r(t)$ or $\omega_r(t)$ does not converge to zero, then the closed-loop system* (4.7, 4.31) *is GAS. Furthermore, given the constraints* (4.28) *and the feasibility condition* (4.29) *it is always possible to choose ϵ_1, ϵ_2, ϵ_3 such that the constraints* (4.28) *are satisfied.*

4.4.2 A cascaded design

As mentioned in Section 4.2 the cascaded approach learned us that for the tracking problem of a mobile robot, stabilization of the nonlinear tracking error dynamics (4.7) in a sense boils down to the separate stabilization of the linear systems (4.14) and (4.15).

The same holds true for the saturated controller design problem. Once we are able to find controllers for the systems (4.14) and (4.15) that meet the constraints (4.28), the same saturated controllers render the tracking error dynamics asymptotically stable too.

So also the nonlinear tracking problem under input constraints reduces to two separated linear problems. For linear systems globally asymptotically stabilizing saturated controllers and several anti-windup controllers are available in literature and can be used.

A saturated controller for the system (4.14) is given by

$$\omega = \omega_r(t) + \sigma_{\epsilon_1}(\theta_e) \tag{4.34}$$

which results in the globally $\mathcal{K}$-exponentially stable closed-loop dynamics

$$\dot{\theta}_e = -\sigma_{\epsilon_1}(\theta_e). \tag{4.35}$$

For stabilizing (4.15) we can use

$$v = v_r(t) + \sigma_{\epsilon_2}(x_e) \tag{4.36}$$

which results into

$$\begin{aligned} \dot{x}_e &= -\sigma_{\epsilon_2}(x_e) + \omega_r(t)y_e \\ \dot{y}_e &= -\omega_r(t)x_e. \end{aligned} \tag{4.37}$$

From Proposition 2.3.10 we know that if $\omega_r(t)$ is persistently exciting (PE), the system (4.37) is globally $\mathcal{K}$-exponentially stable. As a result we obtain the following.

Proposition 4.4.2. *Consider the tracking error dynamics* (4.7) *in closed loop with the control laws* (4.34, 4.36). *Assume that $\omega_r(t)$ is persistently exciting (PE) and that $v_r(t)$ is bounded. Then the resulting closed-loop system is globally $\mathcal{K}$-exponentially stable.*

Proof. We can write the closed-loop system (4.7, 4.34, 4.36) as a system in cascade:

$$\begin{aligned}\begin{bmatrix}\dot{x}_e\\ \dot{y}_e\end{bmatrix} &= \underbrace{\begin{bmatrix}-\sigma_{\epsilon_2}(x_e)+\omega_r(t)y_e\\ -\omega_r(t)x_e\end{bmatrix}}_{f_1(t,z_1)} + \underbrace{\begin{bmatrix}k_1 y_e\frac{\sigma_{\epsilon_1}(\theta_e)}{\theta_e}+v_r(t)\frac{\cos\theta_e-1}{\theta_e}\\ -k_1 x_e\frac{\sigma_{\epsilon_1}(\theta_e)}{\theta_e}+v_r(t)\frac{\sin\theta_e}{\theta_e}\end{bmatrix}}_{g(t,z_1,z_2)}\theta_e\\ \dot{\theta}_e &= \underbrace{-\sigma_{\epsilon_1}(\theta_e)}_{f_2(t,z_2)}.\end{aligned}$$

Almost all conditions of Lemma 2.4.5 are satisfied, since both (4.35) and (4.37) are globally $\mathcal{K}$-exponentially stable and $g(t,z_1,z_2)$ satisfies (2.25). We only need to find the Lyapunov function candidate of Assumption A1 for the system (4.37) satisfying the required properties. For that we can take

$$V = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2$$

whose time-derivative along solutions of (4.37) is

$$\dot{V} = -x_e\sigma_{\epsilon_2}(x_e) \le 0.$$

Applying Lemma 2.4.5 completes the proof. □

4.5 Simulations

This section is to illustrate that the cascaded approach as presented in this chapter can be easily extended. The simple non-holonomic example of a knife-edge moving on the plane was studied by Bloch, Reyhanoglu and McClamroch (1992). Let x_c, y_c denote the co-ordinates of the center of mass of the knife-edge on the plane and let ϕ denote the heading angle measured from the x-axis. Since the velocity of the center of mass is always perpendicular to the runner, there is a non-holonomic constraint of the form

$$\dot{x}_c\sin\phi - \dot{y}_c\cos\phi = 0.$$

The controls are the pushing force τ_1 in the direction of the heading angle and the steering torque τ_2 about the vertical axis through the center of mass. The d'Alembert's formulation of the knife-edge dynamics provide

$$\begin{aligned}\ddot{x}_c &= \frac{\lambda}{m}\sin\phi + \frac{\tau_1}{m}\cos\phi\\ \ddot{y}_c &= -\frac{\lambda}{m}\cos\phi + \frac{\tau_1}{m}\sin\phi\\ \ddot{\phi} &= \frac{\tau_2}{I_c}\\ \dot{x}_c\sin\phi &= \dot{y}_c\cos\phi\end{aligned}\tag{4.38}$$

where m is the mass of the knife-edge, I_c is the moment of inertia of the knife-edge, and λ is the scalar constraint multiplier. When we define the variables

$$\begin{aligned}v &= \dot{x}_c\cos\phi + \dot{y}_c\sin\phi\\ \omega &= \dot{\phi}\end{aligned}$$

the dynamics (4.38) can also be expressed as

$$\begin{aligned}
\dot{x}_c &= v\cos\phi \\
\dot{y}_c &= v\sin\phi \\
\dot{\phi} &= \omega \\
\dot{v} &= \frac{\tau_1}{m} \\
\dot{\omega} &= \frac{\tau_2}{I_c}
\end{aligned} \tag{4.39}$$

which is simply the model of a mobile robot and two additional integrators.

Kolmanovsky and McClamroch (1996) studied the problem of making the knife-edge (4.38) follow the reference dynamics

$$\begin{aligned}
x_{c,r}(t) &= \sin t \\
y_{c,r}(t) &= -\cos t \\
\phi_r(t) &= t \\
\dot{x}_{c,r}(t) &= \cos t \\
\dot{y}_{c,r}(t) &= -\sin t.
\end{aligned}$$

This trajectory corresponds to the center of mass of the knife-edge moving along a circular path of unit radius with uniform angular rate.

Kolmanovsky and McClamroch (1996) solved this problem by defining the change of coordinates

$$\begin{aligned}
\theta &= -x_c\sin\phi + y_c\cos\phi \\
x_1 &= x_c\cos\phi + y_c\sin\phi \\
x_2 &= -\dot{x}_c\sin\phi + \dot{y}_c\cos\phi - \dot{\phi}(x_c\sin\phi - y_c\cos\phi) \\
x_3 &= \phi \\
x_4 &= \dot{\phi} \\
u_1 &= \frac{\tau_1}{m} + \frac{\tau_2}{I_c}(-x_c\sin\phi + y_c\cos\phi) - (x_c\cos\phi + y_c\sin\phi)\dot{\phi}^2 \\
u_2 &= \frac{\tau_2}{I_c}
\end{aligned}$$

and the tracking errors

$$\begin{aligned}
\tilde{\theta} &= \theta - \hat{\theta} \\
\tilde{x} &= x - \hat{x} \\
\tilde{u} &= u - \hat{u}.
\end{aligned}$$

The following hybrid controller was proposed

$$\tilde{u}_1(t) = -\tilde{x}_1 - \tilde{x}_2 + U_1(\alpha^k; t) \qquad kT \leq t < (k+1)T \tag{4.40a}$$

$$\tilde{u}_2(t) = -\tilde{x}_3 - \tilde{x}_4 + U_2(\alpha^k; t) \qquad kT \leq t < (k+1)T \tag{4.40b}$$

where $T = 2\pi$, and for a scalar parameter α

$$U_1(\alpha;t) = \frac{3}{2}\alpha\sin(2t) - \alpha\cos(2t) + \alpha\cos t$$
$$U_2(\alpha;t) = |\alpha|\sin t + \alpha.$$

Let $\theta^k = \theta(kT)$. Then the parameter α^k is updated via the following scheme:

$$\alpha^0 = \nu$$
$$\alpha^k = \begin{cases} \alpha^k = \alpha^{k-1} & \text{if } \alpha^{k-1}\theta^k > 0 \text{ or } \theta^k = 0 \\ \alpha^k = \gamma\left|\alpha^{k-1}\right|\operatorname{sgn}(\theta^k) & \text{if } \alpha^{k-1}\theta^k \leq 0 \text{ and } \theta^k \neq 0 \end{cases}$$

where the values $\gamma = \nu = 0.8$ were proposed.

Starting from the initial condition

$$\begin{bmatrix} x_c(0) & y_c(0) & \phi(0) & \dot{x}_c(0) & \dot{y}_c(0) & \dot{\phi}(0) \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 & 0.5 & 0.5 & 0.5 \end{bmatrix}^T \tag{4.41}$$

the resulting performance is depicted in Figure 4.3, where we assumed that $m = 1$ and $I_c = 1$. Notice that it takes almost 200 seconds for the knife-edge to converge to the reference trajectory.

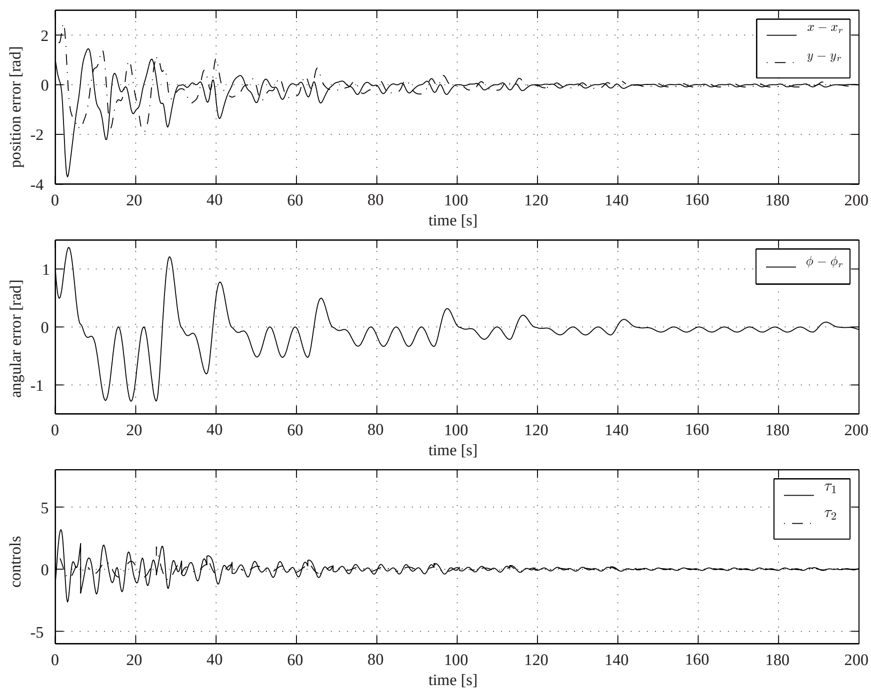

Figure 4.3: Tracking error and inputs for cascade controller (4.40).

Jiang and Nijmeijer (1999b) solved the same tracking problem by defining the global change of co-ordinates and preliminary feedback

$$\begin{aligned}
x_1 &= \phi \\
x_2 &= x_c \cos\phi + y_c \sin\phi \\
x_3 &= x_c \sin\phi - y_c \cos\phi \\
x_4 &= \dot{\phi} \\
x_5 &= \dot{x}_c \cos\phi + \dot{y}_c \sin\phi + \dot{\phi}(-x_c \sin\phi + y_c \cos\phi) \\
v_1 &= \frac{\tau_2}{I_c} \\
v_2 &= \frac{\tau_1}{m} + \frac{\tau_2}{I_c}(-x_c \sin\phi + y_c \cos\phi) - \dot{\phi}^2(x_c \cos\phi + y_c \sin\phi)
\end{aligned}$$

and correspondingly the tracking errors $x_e = x - x_d$. The following controller was proposed

$$v_1 = -2\bar{u}_1 - z_3 + z_2 x_5 - z_1 z_3 x_5^2 + z_1 v_2 - 2x_{4e} \tag{4.42a}$$
$$v_2 = -2\bar{u}_2 - 2z_2 - 2x_5 + z_3 x_5 \tag{4.42b}$$

where

$$\begin{aligned}
\bar{u}_1 &= x_4 - 1z_1 x_5 + 2z_3 \\
\bar{u}_2 &= x_5 + 2z_2 + z_1
\end{aligned}$$

and

$$\begin{aligned}
z_1 &= x_{3e} - x_{2e} z_{1e} \\
z_2 &= x_{2e} \\
z_3 &= x_{1e}
\end{aligned}$$

Starting from the same initial condition (4.41) the resulting performance is depicted in Figure 4.4.

We can also use the cascaded approach presented in this chapter for solving the tracking problem. Starting from the model (4.39) and using the change of co-ordinates (4.2) solving the tracking problem boils down to stabilizing the linear time-invariant ($\omega_r = 1$) systems:

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{v}_e \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_e \\ y_e \\ v_e \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} [\tau_{1,r} - \tau_1]$$

and

$$\begin{bmatrix} \dot{\theta}_e \\ \dot{\omega}_e \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_e \\ \omega_e \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I_c} \end{bmatrix} [\tau_{2,r} - \tau_2]$$

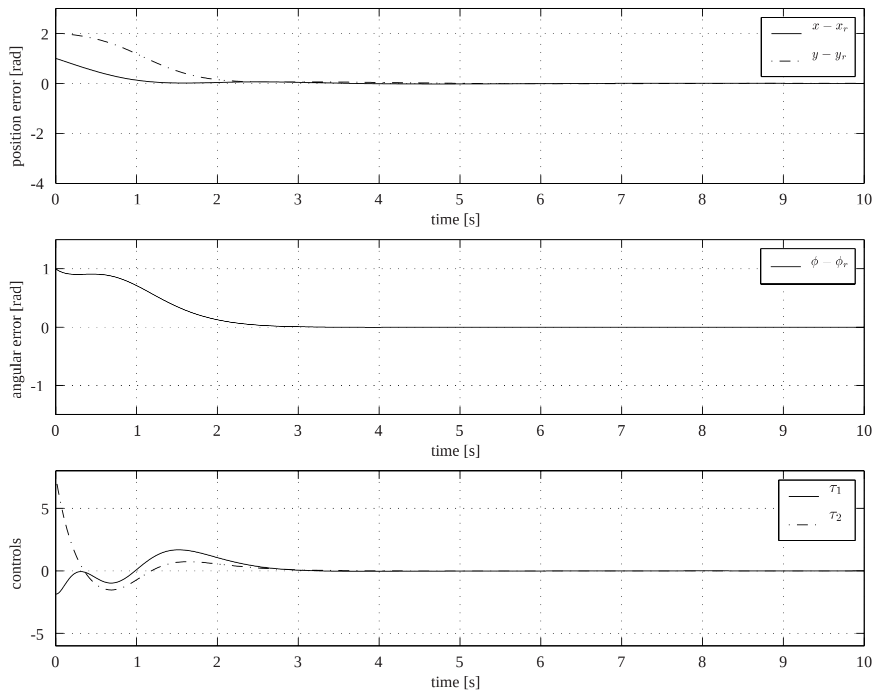

Figure 4.4: Tracking error and inputs for backstepping controller (4.42).

where we defined $v_e = v_r - v$ and $\omega_e = \omega_r - \omega$. Both systems can easily be stabilized by using linear controllers. For tuning these controllers we defined the costs

$$J_1 = \int_0^\infty x_e(t)^2 + y_e(t)^2 + v_e(t)^2 + 0.01(\tau_{1,r}(t) - \tau_1(t))^2 \mathrm{d}t$$
$$J_2 = \int_0^\infty \theta_e(t)^2 + 0.1(\tau_{2,r}(t) - \tau_2(t))^2 \mathrm{d}t$$

and used optimal control to minimize these costs. As a result we obtained as controllers for the tracking control problem

$$\tau_1 = \tau_{1,r} + 4.4705 x_e - 0.0012 y_e + 4.3521 v_e \tag{4.43a}$$
$$\tau_2 = \tau_{2,r} + 10.0000 \theta_e + 10.9545 \omega_e \tag{4.43b}$$

Starting also from the initial condition (4.41) the resulting performance is depicted in Figure 4.5. Notice that the resulting performance is comparable with the backstepping based

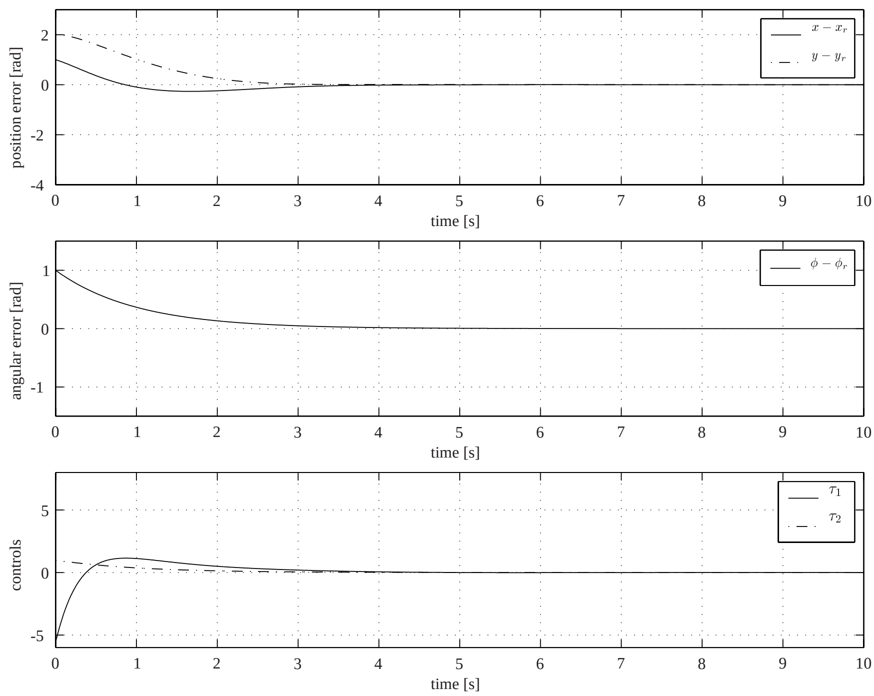

Figure 4.5: Tracking error and inputs for cascade controller (4.43).

controller of Jiang and Nijmeijer (1999b).

For comparison reasons we also considered the following quantities

$$J_r = \int_0^{200} [x_c(t) - x_{c,r}(t)]^2 + [y_c(t) - y_{c,r}(t)]^2 \mathrm{d}t$$

and

$$J_\tau = \int_0^{200} [\tau_1(t)]^2 + [\tau_2(t)]^2 \mathrm{d}t$$

where J_r can be thought of as an error measure and J_τ as a measure of the control effort. For the three different controllers we considered we obtained the following values:

	J_r	J_τ
hybrid controller (4.40)	103.558	63.789
backstepping controller (4.42)	3.752	8.796
cascade controller (4.43)	3.341	4.566

4.6 Concluding remarks

In this chapter we considered the tracking control problem for the kinematic model of a mobile robot. We solved this problem using state-feedback, output-feedback, and under input saturation. All results yield globally $\mathcal{K}$-exponentially stable closed-loop tracking error dynamics under a persistence of excitation condition on the reference angular velocity.

This persistence of excitation condition on the reference angular velocity makes that tracking of a straight line and stabilization is not possible with the "cascaded controllers" that have been derived. However, this problem can be overcome by weakening the persistence of excitation (PE) condition by assuming a so-called uniform δ-persistence of excitation (uδ-PE), as recently introduced by Loría et al. (1999b). In that case global uniform asymptotic stability (GUAS) can be shown.

We arrived at the results by means of the cascaded design approach as explained in Section 3.2. This approach revealed a nice structure in the tracking error dynamics, which makes that the nonlinear tracking problem can be reduced to two *linear* problems. This is the case for both the state- and output-feedback problem, as well as the control problem under input saturation and other interesting problems like incorporating robustness against uncertainties.

This simple structure is also maintained when we consider so called dynamic extensions of the model, when additional integrators are added. This was illustrated by means of simulations using the example of a knife-edge.

Chapter 5

Tracking of non-holonomic systems in chained form

5.1 Introduction

In this chapter we consider the tracking problem for a special class of non-holonomic systems, namely systems in chained form. Many mechanical systems with non-holonomic constraints can be locally or globally converted to the chained form under co-ordinate change and preliminary feedback, see (Murray and Sastry 1993).

Chained-form systems of order n with two inputs can be expressed as

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1
\end{aligned} \tag{5.1}$$

where $x = (x_1, \ldots, x_n)^T$ is the state, and u_1 and u_2 are two inputs.

Consider the kinematic model of a mobile robot that we studied in the previous chapter:

$$\begin{aligned}
\dot{x} &= v \cos\theta \\
\dot{y} &= v \sin\theta \\
\dot{\theta} &= \omega.
\end{aligned} \tag{5.2}$$

When we define the global change of co-ordinates

$$\begin{aligned}
x_1 &= \theta \\
x_2 &= x \cos\theta + y \sin\theta \\
x_3 &= x \sin\theta - y \cos\theta
\end{aligned}$$

and apply the preliminary feedback

$$\begin{aligned}\omega &= u_1 \\ v &= u_2 + \omega x_3\end{aligned}$$

where u_1 and u_2 are new inputs, then the system (5.2) is transformed to

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1\end{aligned}$$

which is a chained-form system of order 3 with two inputs.

It is well-known that many mechanical systems with non-holonomic constraints can be locally or globally converted to the chained form under co-ordinate change and preliminary feedback. Interesting examples of such mechanical systems include not only the mobile robot, but also cars towing several trailers, the knife edge, a vertical rolling wheel and a rigid spacecraft with two torque actuators (see e.g., (Murray and Sastry 1993, Kolmanovsky and McClamroch 1995) or Section 3.2).

As for the mobile robot, the system (5.1) fails to meet Brockett's necessary condition for smooth feedback stabilization, which implies that no smooth (or even continuous) time-invariant static state-feedback $u = u(x)$ exists that renders a specified equilibrium of the closed-loop locally asymptotically stable. For this reason the stabilization problem has received a lot of attention (see e.g., (Pomet 1992, Murray and Sastry 1993, Canudas de Wit, Berghuis and Nijmeijer 1994, Samson 1995, Sørdalen and Egeland 1995, Teel, Murray and Walsh 1995, Jiang 1996)).

However, the tracking problem for systems in chained form has received little attention. Most of the global tracking results we are aware of, are on the tracking control of a mobile robot, which is a chained-form system of order 3 (and dealt with in the previous chapter). Another global result for a chained-form system of order 3 is given by Escobar, Ortega and Reyhanoglu (1998), where they introduced a field-oriented control approach for the tracking of the non-holonomic integrator. We are not aware of any global results for general chained-form systems of order n. Jiang and Nijmeijer (1999b) derived semi-global tracking controllers for general chained-form systems by means of backstepping and they achieved global tracking results for some special cases.

In this chapter we solve the global tracking problem for general chained-form systems by means of a cascaded systems based approach. We first apply in Section 5.2 the idea explained in Section 3.2 to the tracking error dynamics. This results in a similar design separation principle as in the previous chapter for the mobile robot. That is, we reduce the problem of designing stabilizing controllers for the nonlinear tracking error dynamics to two linear controller design problems. With this knowledge we tackle in Section 5.3 the state-feedback tracking problem and in Section 5.4 the output-feedback tracking problem. In Section 5.5 we deal with both these tracking control problems under input saturation for a special class of reference trajectories. The effectiveness of the derived controllers is illustrated in Section 5.6 by means of simulations. Section 5.7 contains some concluding remarks for this chapter.

5.2 The search for a cascaded structure

For studying the tracking control problem for systems in chained form, assume we are given a reference trajectory $(x_r^T, u_r^T)^T$ satisfying

$$\begin{aligned}
\dot{x}_{1,r} &= u_{1,r}\\
\dot{x}_{2,r} &= u_{2,r}\\
\dot{x}_{3,r} &= x_{2,r}u_{1,r}\\
&\vdots\\
\dot{x}_{n,r} &= x_{n-1,r}u_{1,r}.
\end{aligned}$$

We define the tracking error $x_e = x - x_r$ and obtain as the tracking error dynamics

$$\begin{aligned}
\dot{x}_{1,e} &= u_1 - u_{1,r} && = u_1 - u_{1,r}\\
\dot{x}_{2,e} &= u_2 - u_{2,r} && = u_2 - u_{2,r}\\
\dot{x}_{3,e} &= x_2u_1 - x_{2,r}u_{1,r} && = x_{2,e}u_{1,r} + (x_{2,e} + x_{2,r})(u_1 - u_{1,r})\\
&\vdots && \vdots\\
\dot{x}_{n,e} &= x_{n-1}u_1 - x_{n-1,r}u_{1,r} && = x_{n-1,e}u_{1,r} + (x_{n-1,e} + x_{n-1,r})(u_1 - u_{1,r}).
\end{aligned} \tag{5.3}$$

The tracking control problem boils down to finding appropriate control laws for u_1 and u_2 such that the tracking error x_e converges to zero. For that we like to use the cascaded design approach as proposed in Section 3.2. We look for a control law for one of the two inputs which is such that a subsystem of (5.3) is asymptotically stabilized in closed loop. Preferably, this subsystem has to be such that the remaining dynamics reduces considerably in case we assume that the stabilization of this subsystem has been established.

Notice that either the $x_{1,e}$ dynamics or the $x_{2,e}$ dynamics can be easily rendered asymptotically stable by choosing an appropriate control law for u_1 or u_2, respectively. As the next step is to assume that the stabilization has been established, we could decide to use u_2 for stabilizing the $x_{2,e}$ dynamics, but this does not look too promising. On the other hand, if we decide to use u_1 for stabilizing the $x_{1,e}$ dynamics, the assumption that this stabilization has worked out simplifies almost all equations in (5.3). Therefore, we decide to first use u_1 for stabilizing the $x_{1,e}$ dynamics.

Next, we assume that the stabilization of $x_{1,e}$ has been established, that is, we substitute $x_{1,e}(t) \equiv 0$ in the remaining dynamics. Notice that as a result also $u_1(t) - u_{1,r}(t) \equiv 0$. After this substitution the remaining dynamics becomes

$$\begin{aligned}
\dot{x}_{2,e} &= u_2 - u_{2,r}(t)\\
\dot{x}_{3,e} &= x_{2,e}u_{1,r}(t)\\
&\vdots\\
\dot{x}_{n,e} &= x_{n-1,e}u_{1,r}(t)
\end{aligned}$$

which is the same as the linear time-varying system

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \vdots \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r}(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ \vdots \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} [u_2 - u_{2,r}(t)]. \tag{5.4}$$

All we need to do is to find a feedback controller for u_2 that stabilizes the system (5.4) and hope that all conditions for applying the cascaded theorem (Theorem 2.4.3) are met. Notice from Corollary 2.3.4 that if $u_{1,r}(t)$ is persistently exciting the system (5.4) is uniformly completely controllable.

To summarize: instead of solving the problem of finding stabilizing control laws for the nonlinear tracking error dynamics (5.3) we might as well look at the two separate problems of finding a stabilizing control law for the linear system

$$\dot{x}_{1,e} = u_1 - u_{1,r}(t) \tag{5.5}$$

and finding one for the linear time-varying system

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \vdots \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r}(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ \vdots \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} [u_2 - u_{2,r}(t)]. \tag{5.6}$$

When we do so, the stabilized system (5.6) plays the role of the system Σ_1 in Section 2.4 and (5.5) the role of the system Σ_2. Then we can use Theorem 2.4.3 to conclude asymptotic stability of the entire nonlinear tracking error dynamics. So instead of solving a nonlinear control problem, we have to solve two linear ones.

5.3 State-feedback

In this section we study the state-feedback tracking control problem for chained-form systems. As derived in the previous section from a cascaded systems point of views the problem of stabilizing (5.3) reduces to stabilizing the linear systems (5.5) and (5.6).

Clearly, the system (5.5) can easily be stabilized. A possible control law for u_1 is

$$u_1 = u_{1,r}(t) - k_1 x_{1,e} \qquad k_1 > 0 \tag{5.7}$$

since then the resulting closed-loop system

$$\dot{x}_{1,e} = -k_1 x_{1,e} \qquad k_1 > 0$$

is globally uniformly exponentially stable (GUES).

For stabilizing (5.6) we can use the result of Theorem 2.3.7, provided that $u_{1,r}(t)$ is persistently exciting (PE).

We can combine both results and solve the global state-feedback tracking control problem.

Proposition 5.3.1. *Consider the chained-form tracking error dynamics* (5.3)*. Assume that* $u_{1,r}(t)$ *is persistently exciting (PE) and that* $x_{2,r}, x_{3,r}, \ldots, x_{n-1,r}$ *are bounded.*

Then the control law

$$\begin{aligned} u_1 &= u_{1,r}(t) - k_1 x_{1,e} && \text{(5.8a)} \\ u_2 &= u_{2,r}(t) - k_2 x_{2,e} - k_3 u_{1,r}(t) x_{3,e} - k_4 x_{4,e} - k_5 u_{1,r}(t) x_{5,e} - \cdots && \text{(5.8b)} \end{aligned}$$

renders the closed-loop system (5.3, 5.8) *globally* $\mathcal{K}$*-exponentially stable, provided that* $k_1 > 0$ *and* k_i *(*$i = 2, \ldots, n$*) are such that the polynomial*

$$\lambda^{n-1} + k_2 \lambda^{n-2} + \cdots + k_{n-1}\lambda + k_n$$

is Hurwitz.

Proof. Due to the design, we can recognize a cascaded structure in the closed-loop system (5.3, 5.8):

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \vdots \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \underbrace{\begin{bmatrix} -k_2 & -k_3 u_{1,r}(t) & -k_4 & -k_5 u_{1,r}(t) & \ldots \\ u_{1,r}(t) & 0 & \ldots & \ldots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ \vdots \\ \vdots \\ x_{n,e} \end{bmatrix}}_{f_1(t,z_1)} + \underbrace{\begin{bmatrix} 0 \\ -k_1(x_{2,e}+x_{2,r}) \\ -k_1(x_{3,e}+x_{3,r}) \\ \vdots \\ -k_1(x_{n-1,e}+x_{n-1,r}) \end{bmatrix}}_{g(t,z_1,z_2)} x_{1,e}$$

$$\dot{x}_{1,e} = \underbrace{-k_1 x_{1,e}}_{f_2(t,z_2)}.$$

From Theorem 2.3.7 we know that the system $\dot{z}_1 = f_1(t, z_1)$ is globally uniformly exponentially stable (GUES). Also the system $\dot{z}_2 = f_2(t, z_2)$ is GUES. Since $x_{2,r}, \ldots, x_{n-1,r}$ are bounded we have that $g(t, z_1, z_2)$ satisfies (2.25). As a result, Corollary 2.4.6 completes the proof. □

Remark 5.3.2. Notice that the only property of the system $\dot{z}_1 = f_1(t, z_1)$ that we need in this proof, is the fact that it is globally uniformly exponentially stable (GUES). This is something that (according to Theorem 2.3.7) is guaranteed by the choice for the input u_2. However, under the assumption that $u_{1,r}(t)$ is persistently exciting (which yields uniform complete controllability according to Corollary 2.3.4), more control laws for u_1 are available in literature that also guarantee GUES. In case we replace u_2 with any of these, the proof still holds. Therefore, several other choices for u_2 can be made. For instance, one might consider

- a 'standard' linear control law (Rugh 1996) which involves using the state-transition matrix of the system (5.6) (see (Lefeber, Robertsson and Nijmeijer 1999a));

- a less complicated control law (which also needs the state-transition matrix of the system (5.6)) as presented by Chen (1997) (see (Lefeber, Robertsson and Nijmeijer 1999b));

- a pole-placement based control law, like for instance the one presented by Valášek and Olgac (1995) (which requires $u_{1,r} \in \mathcal{C}^{n-2}$ and the signals $\frac{\mathrm{d}u_{1,r}}{\mathrm{d}t}(t), \ldots, \frac{\mathrm{d}^{n-2}u_{1,r}}{\mathrm{d}t^{n-2}}(t)$ to be available),

or any other control law one prefers that guarantees GUES of the system (5.6).

Remark 5.3.3. As pointed out by Samson (1995), it is possible to normalize the system's equations in terms of the advancement velocity $|u_{1,r}(t)|$, in order to replace time by the distance gone by the reference vehicle. This "time normalization" makes the solutions "geometrically" unaffected by velocity changes, yielding convergence in terms of this distance, instead of time. In practice this has the advantage that the damping rate does not change with different values of $u_{1,r}(t)$.

Remark 5.3.4. The mobile robot we studied in the previous chapter is also a chained-form system. It would be interesting to compare the results of the previous chapter with the result derived here. It would be most reasonable to compare both results in the original error co-ordinates, i.e., in the variables x_e and y_e as defined by (4.2). In case we translate the result of this chapter using the original error co-ordinates, we obtain

$$\begin{aligned} \omega &= \omega_r + k_1\theta_e && k_1 > 0 \\ v &= v_r + k_2 x_e - (k_3 - 1)y_e + f(\theta_e, t) && k_2, k_3 > 0 \end{aligned}$$

where $f(\theta_e, t)$ is a quite complicated expression that satisfies $f(0, t) = 0$. In case we interpret $f(0, t)$ as a part of the connecting term "$g(t, z_1, z_2)z_2$" we can decide to forget about it. When we do so, we regain exactly the control law (4.13).

The conclusion that we are allowed to leave out the term $f(0, t)$, however, can only be drawn with the cascaded structure of the mobile robot in mind. Therefore, the work of the previous chapter can not be considered redundant. Furthermore, this makes clear that one should avoid changing co-ordinates in order to be able to apply a standard control design technique, since it can lead to unnecessary complicated controllers. One of the advantages of the cascaded control design approach is that all analysis can be done in the original co-ordinates.

5.4 Dynamic output-feedback

In this section we study the dynamic output-feedback tracking control problem for chained-form systems. That is, we study the problem of stabilizing the tracking error dynamics (5.3) where we are only allowed to use the measured output for designing the control laws for u_1 and u_2.

Notice that for the system (5.1) we are unable to reconstruct the variables x_1 and x_n, no matter what output we have. Furthermore, in case we are able to measure only x_1 and x_n, the system (5.1) is locally observable at any $x \in \mathbb{R}^n$ (see e.g., (Astolfi 1995)).

This is why we assume to have $[x_1, x_n]^T$ available as an output for the chained-form system (see (Astolfi 1995, Jiang and Nijmeijer 1999a)), since in a sense it represents the least amount of components of the state vector that is required for being able to control the system (5.1).

In light of Section 5.2 it is clear that the problem can be reduced to the problem of finding stabilizing dynamic output-feedback laws for the systems

$$\dot{x}_{1,e} = u_1 - u_{1,r}(t) \tag{5.9a}$$
$$y_1 = x_{1,e} \tag{5.9b}$$

and

$$\begin{bmatrix} \dot{x}_{2,e} \\ \dot{x}_{3,e} \\ \vdots \\ \vdots \\ \dot{x}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r}(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ x_{3,e} \\ \vdots \\ \vdots \\ x_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} [u_2 - u_{2,r}(t)] \tag{5.10a}$$

$$y_2 = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{2,e} \\ \vdots \\ x_{n,e} \end{bmatrix}. \tag{5.10b}$$

For stabilizing (5.9) we can use

$$u_1 = u_{1,r}(t) - k_1 x_{1,e} \qquad k_1 > 0 \tag{5.11}$$

since then the resulting closed-loop system becomes

$$\dot{x}_{1,e} = -k_1 x_{1,e} \qquad k_1 > 0$$

which is globally uniformly exponentially stable (GUES).

For stabilizing (5.10) by means of output-feedback we need both uniform complete controllability and uniform complete observability of the system (5.10). For this it suffices that $u_{1,r}(t)$ is persistently exciting. Uniform complete controllability follows from Corollary 2.3.4 as mentioned in the previous section. Uniform complete observability follows from duality. For stabilizing (5.10) we can use Theorem 2.3.8.

By combining both results we obtain a solution for the dynamic output-feedback tracking control problem.

Proposition 5.4.1. *Consider the chained-form tracking error dynamics* (5.3)*. Assume that $u_{1,r}(t)$ is persistently exciting (PE) and that $x_{2,r}, x_{3,r}, \dots, x_{n-1,r}$ are bounded.*

Then the control law

$$u_1 = u_{1,r}(t) - k_1 x_{1,e} \tag{5.12a}$$
$$u_2 = u_{2,r}(t) - k_2 \hat{x}_{2,e} - k_3 u_{1,r}(t) \hat{x}_{3,e} - k_4 \hat{x}_{4,e} - k_5 u_{1,r}(t) \hat{x}_{5,e} - \cdots \tag{5.12b}$$

where $\hat{x}$ is generated from the observer

$$\begin{bmatrix} \dot{\hat{x}}_{2,e} \\ \dot{\hat{x}}_{3,e} \\ \vdots \\ \vdots \\ \dot{\hat{x}}_{n,e} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & 0 \\ u_{1,r}(t) & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{2,e} \\ \hat{x}_{3,e} \\ \vdots \\ \vdots \\ \hat{x}_{n,e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} [u_2 - u_{2,r}(t)] + \begin{bmatrix} \vdots \\ l_5 u_{1,r}(t) \\ l_4 \\ l_3 u_{1,r}(t) \\ l_2 \end{bmatrix} [x_{n,e} - \hat{x}_{n,e}] \tag{5.13}$$

renders the closed-loop system (5.3, 5.12, 5.13) *globally $\mathcal{K}$-exponentially stable, provided that $k_1 > 0$ and k_i, l_i ($i = 2, \dots, n$) are such that the polynomials*

$$\lambda^{n-1} + k_2 \lambda^{n-2} + \dots + k_{n-1}\lambda + k_n$$

and

$$\lambda^{n-1} + l_2 \lambda^{n-2} + \dots + l_{n-1}\lambda + l_n$$

are Hurwitz.

Proof. We can see the closed-loop system (5.3, 5.12, 5.13) as a cascaded system, i.e., a system of the form (3.1) where

$$\begin{aligned} z_1 &= \begin{bmatrix} x_{2,e} & \dots & x_{n,e} & \tilde{x}_{2,e} & \dots & \tilde{x}_{n,e} \end{bmatrix}^T \\ z_2 &= x_{1,e} \\ f_1(t, z_1) &= \begin{bmatrix} -k_2 & -k_3 u_{1,r}(t) & -k_4 & -k_5 u_{1,r}(t) & \dots & k_2 & k_3 u_{1,r}(t) & k_4 & k_5 u_{1,r}(t) & \dots \\ u_{1,r}(t) & 0 & \dots & \dots & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\ 0 & \dots & 0 & u_{1,r}(t) & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & \dots & \dots & 0 & \vdots \\ \vdots & & & & \vdots & u_{1,r}(t) & \ddots & & \vdots & -l_5 u_{1,r}(t) \\ \vdots & & & & \vdots & 0 & \ddots & \ddots & \vdots & -l_4 \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots & 0 & -l_3 u_{1,r}(t) \\ 0 & \dots & \dots & \dots & 0 & 0 & \dots & 0 & u_{1,r}(t) & -l_2 \end{bmatrix} z_1 \\ f_2(t, z_2) &= -k_1 z_2 \\ g(t, z_1, z_2) &= \begin{bmatrix} 0 \\ -k_1(x_{2,e} + x_{2,r}) \\ \vdots \\ -k_1(x_{n-1,e} + x_{n-1,r}) \\ 0 \\ -k_1(x_{2,e} + x_{2,r}) \\ \vdots \\ -k_1(x_{n-1,e} + x_{n-1,r}) \end{bmatrix} \end{aligned}$$

and $\tilde{x}_{i,e} = x_{i,e} - \hat{x}_{i,e}$ ($i = 2, \dots, n$). From Theorem 2.3.8 we know that the system $\dot{z}_1 = f_1(t, z_1)$ is globally uniformly exponentially stable (GUES). Also the system $\dot{z}_2 = f_2(t, z_2)$ is GUES. Since $x_{2,r}, \dots, x_{n-1,r}$ are bounded $g(t, z_1, z_2)$ satisfies (2.25). Application of Corollary 2.4.6 completes the proof. ☐

5.5 Saturated control

In a similar way as in the previous sections, we can study the tracking problem for a system in chained form under input saturation. That is, we can study the problem of designing tracking controllers in such a way that we are always guaranteed to meet the constraints

$$|u_1(t)| \leq u_1^{\max} \tag{5.14a}$$
$$|u_2(t)| \leq u_2^{\max}. \tag{5.14b}$$

Obviously, we need to assume that once we are on the reference trajectory we can stay on the trajectory, that is, for the reference trajectory the condition (5.14) is met. Therefore, we assume that the reference trajectory that we would like to track satisfies

$$\sup_{t\geq 0} |u_{1,r}(t)| < u_1^{\max}$$
$$\sup_{t\geq 0} |u_{2,r}(t)| < u_2^{\max}.$$

Instead of the nonlinear tracking control problem under input constraints, we only have to address the two linear problems of stabilizing the subsystems (5.5) and (5.6) under the input constraints (5.14).

As can easily be seen, stabilizing (5.5) while meeting (5.14a) is not difficult. We can modify the control laws (5.7) and (5.11) into

$$u_1 = u_{1,r}(t) - \sigma_\epsilon(x_{1,e}) \tag{5.15}$$

where σ_ϵ is a saturation function as defined in Definition 2.1.13 and

$$\epsilon \leq u_1^{\max} - \sup_{t\geq 0} |u_{1,r}(t)|\,.$$

Then the resulting closed-loop $x_{1,e}$ dynamics becomes

$$\dot{x}_{1,e} = -\sigma_\epsilon(x_{1,e})$$

which is globally $\mathcal{K}$-exponentially stable.

Assume that we have a control law for u_2 for the system (5.6) which is such that for the resulting closed-loop system Assumption A1 of Theorem 2.4.3 is satisfied. Then Corollary 2.4.4 tells us that we have global uniform asymptotic stability of the tracking error dynamics (5.3) in closed loop with this control law for u_2 and (5.15). In addition, if u_2 guarantees local uniform exponential stability (LUES) we can conclude global $\mathcal{K}$-exponential stability of the closed-loop tracking error dynamics (cf. Lemma 2.4.5).

Therefore, the only problem that remains, is to find a control law for u_2 that in closed loop with (5.6) results in a globally uniformly asymptotically stable (GUAS) system (most preferable globally $\mathcal{K}$-exponentially stable), while meeting (5.14b). As far as we know, no result on stabilizing linear time-varying systems by means of saturated state or output-feedback is known (yet).

For linear time-invariant systems we have the results of Sussmann, Sontag and Yang (1994), which deals with the stabilization under input constraints, both by using state-feedback and output-feedback. For linear time-invariant systems we can also think of using anti-windup controllers, like for example the one proposed by Kapoor, Teel and Daoutidis (1998).

As a result, the tracking problem under input constraints (5.14) for chained-form systems (under both state- and output-feedback) can (yet) only be solved for a special class of reference trajectories, namely those reference trajectories for which $u_{1,r}(t)$ is a non-zero constant. However, as soon as we have a result on the stabilization of linear time-varying systems under input saturation, we also have a solution to the general tracking problem for chained-form systems.

5.6 Simulations

In this section we apply the proposed state-feedback design for the tracking control of a well-known benchmark problem: a towing car with a single trailer, see e.g., (Murray and Sastry 1993, Samson 1995, Jiang and Nijmeijer 1999b).

The state configuration of the articulated vehicle consists of the position of the car (x_c, y_c), the steering angle ϕ, and the orientations θ_0 respectively θ_1 of the car and the trailer with respect to the x-axis, see Figure 5.1 The rear wheels of the car and the trailer are aligned

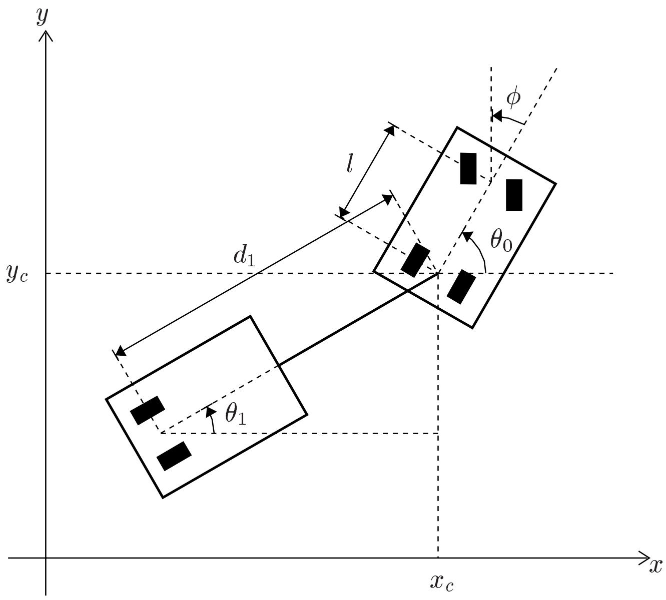

Figure 5.1: Kinematic model of a car with a single trailer, where the controls are the forward velocity v and the steering velocity ω of the tow car.

with the chassis and are not allowed to slip sideways. The two input signals are the driving velocity of the front wheels, v, and the steering velocity ω.

The kinematic equations of motion for the vehicle can be described by

$$\begin{aligned}
\dot{x}_c &= v\cos\theta_0 \\
\dot{y}_c &= v\sin\theta_0 \\
\dot{\phi} &= \omega \\
\dot{\theta}_0 &= \frac{v}{l}\tan\phi \\
\dot{\theta}_1 &= \frac{v}{d_1}\sin(\theta_0-\theta_1)
\end{aligned} \tag{5.16}$$

where l is the wheelbase of the tow car and d_1 is the distance from the wheels of the trailer to the rear wheels of the car.

Although the system (5.16) is not in chained form, it can locally be transformed into the chained form (5.1) via a change of co-ordinates and preliminary state-feedback (Murray and Sastry 1993):

$$\begin{aligned}
x_1 &= x_c \\
x_2 &= \frac{\sin\theta_1\sin^2(\theta_0-\theta_1)}{d_1^2\cos^2\theta_0\cos^3\theta_1} + \frac{\tan\phi}{ld_1\cos^3\theta_0\cos\theta_1} - \frac{\sin(\theta_0-\theta_1)}{d_1^2\cos\theta_0\cos^3\theta_1} \\
x_3 &= \frac{\sin(\theta_0-\theta_1)}{d_1\cos\theta_0\cos^2\theta_1} \\
x_4 &= \tan\theta_1 \\
x_5 &= y_c - d_1\log\left(\frac{1+\sin\theta_1}{\cos\theta_1}\right) \\
v &= \frac{u_1}{\cos\theta_0} \\
\omega &= \beta_1(\phi,\theta_0,\theta_1)u_1 + \beta_2(\phi,\theta_0,\theta_1)u_2
\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}
\beta_1 = &-\frac{3\sin\theta_0\sin^2\phi}{l\cos^2\theta_0} + \frac{4\sin\phi\cos\phi}{d_1\cos\theta_1} - \frac{3\sin\phi\cos\phi}{d_1\cos\theta_0} - \frac{7l\cos^2\phi}{d_1^2\cos\theta_1}\sin(\theta_0-\theta_1) \\
&- \frac{6l\cos^2\theta_0\cos^2\phi}{d_1^2\cos^3\theta_1}\sin(\theta_0-\theta_1) + \frac{12l\cos^2\theta_0\cos^2\phi}{d_1^2\cos\theta_1}\sin(\theta_0-\theta_1) \\
&+ \frac{6l}{d_1^2}\sin\theta_0\cos^2\phi - \frac{6l\cos^2\theta_0\sin\theta_0\cos^2\phi}{d_1^2\cos^2\theta_1} \\
\beta_2 = &\; ld_1\cos^3\theta_0\cos\theta_1\cos^2\phi.
\end{aligned}$$

This change of co-ordinates and preliminary feedback, results into the chained-form system

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2u_1 \\
\dot{x}_4 &= x_3u_1 \\
\dot{x}_5 &= x_4u_1.
\end{aligned}$$

(Jiang and Nijmeijer 1999b) studied for this system the problem of tracking the straight line

$$\begin{bmatrix} x_{c,r} & y_{c,r} & \phi_r & \theta_{0,r} & \theta_{1,r} & v_r & \omega_r \end{bmatrix} = \begin{bmatrix} t & 0 & 0 & 0 & 0 & 1 & 0. \end{bmatrix} \tag{5.18}$$

In chained-form co-ordinates the reference (5.18) can be expressed as

$$\begin{bmatrix} x_{1,r} & x_{2,r} & x_{3,r} & x_{4,r} & x_{5,r} & u_{1,r} & u_{2,r} \end{bmatrix} = \begin{bmatrix} t & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Starting from the initial condition

$$x_e(0) = \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}^T$$

it was shown by Jiang and Nijmeijer (1999b) that their backstepping based controller

$$\begin{aligned}
u_1 = u_{1,r} + \big(&-c_5 x_{1,e} - u_{2,r}x_{2,r}x_{1,e} - u_{2,r}x_{4,r}x_{1,e} + u_{2,r}x_{3,e} + u_{2,r}x_{5,e} \\
&- 2c_4 x_{2,r}x_{3,r}x_{1,e}^2 - 2c_4 x_{3,r}x_{4,r}x_{1,e}^2 - 3u_{1,r}x_{2,r}^2 x_{1,e}^2 - 4u_{1,r}x_{2,r}x_{4,r}x_{1,e}^2 \\
&- u_{1,r}x_{4,r}^2 x_{1,e}^2 - u_{2,r}x_{1,e}x_{2,e} + c_4 x_{2,r}x_{1,e}x_{2,e} + c_4 x_{4,r}x_{1,e}x_{2,e} \\
&+ 2c_4 x_{3,r}x_{1,e}x_{3,e} + 6u_{1,r}x_{2,r}x_{1,e}x_{3,e} + 4u_{1,r}x_{4,r}x_{1,e}x_{3,e} - u_{2,r}x_{1,e}x_{4,e} \\
&+ 2c_4 x_{2,r}x_{1,e}x_{4,e} + 2c_4 x_{4,r}x_{1,e}x_{4,e} + 2c_4 x_{3,r}x_{1,e}x_{5,e} + 4u_{1,r}x_{2,r}x_{1,e}x_{5,e} \\
&+ 2u_{1,r}x_{4,r}x_{1,e}x_{5,e} - c_4 x_{2,e}x_{3,e} - c_4 x_{2,e}x_{5,e} - 3u_{1,r}x_{3,e}^2 - 2c_4 x_{3,e}x_{4,e} \\
&- 4u_{1,r}x_{3,e}x_{5,e} - 2c_4 x_{4,e}x_{5,e} - u_{1,r}x_{5,e}^2 - 2c_4 x_{3,r}x_{1,e}^2 x_{2,e} \\
&- 6u_{1,r}x_{2,r}x_{1,e}^2 x_{2,e} - 4u_{1,r}x_{4,r}x_{1,e}^2 x_{2,e} - 2c_4 x_{2,r}x_{1,e}^2 x_{3,e} \\
&- 2c_4 x_{4,r}x_{1,e}^2 x_{3,e} - 2c_4 x_{3,r}{x_{1,e}}^2 x_{4,e} - 4u_{1,r}x_{2,r}x_{1,e}^2 x_{4,e} \\
&- 2u_{1,r}x_{4,r}x_{1,e}^2 x_{4,e} + c_4 x_{1,e}x_{2,e}^2 + 6u_{1,r}x_{1,e}x_{2,e}x_{3,e} + 3c_4 x_{1,e}x_{2,e}x_{4,e} \\
&+ 4u_{1,r}x_{1,e}x_{2,e}x_{5,e} + 2c_4 x_{1,e}x_{3,e}^2 + 4u_{1,r}x_{1,e}x_{3,e}x_{4,e} + 2c_4 x_{1,e}x_{3,e}x_{5,e} \\
&+ 2c_4 x_{1,e}x_{4,e}^2 + 2u_{1,r}x_{1,e}x_{4,e}x_{5,e} - 3u_{1,r}x_{1,e}^2 x_{2,e}^2 - 2c_4 x_{1,e}^2 x_{2,e}x_{3,e} \\
&- 4u_{1,r}x_{1,e}^2 x_{2,e}x_{4,e} - 2c_4 x_{1,e}^2 x_{3,e}x_{4,e} - u_{1,r}x_{1,e}^2 x_{4,e}^2\big) \;/\; \big(\lambda \\
&+ 6x_{2,r}x_{3,r}x_{1,e} + 2x_{3,r}x_{4,r}x_{1,e} - 2x_{2,r}x_{2,e} - x_{3,r}x_{3,e} - 5x_{2,r}x_{4,e} \\
&- 2x_{3,r}x_{5,e} + 6x_{3,r}x_{1,e}x_{2,e} + 6x_{2,r}x_{1,e}x_{3,e} + 2x_{4,r}x_{1,e}x_{3,e} \\
&+ 2x_{3,r}x_{1,e}x_{4,e} - 2x_{2,e}^2 - 5x_{2,e}x_{4,e} - x_{3,e}^2 - 2x_{3,e}x_{5,e} + 6x_{1,e}x_{2,e}x_{3,e} \\
&+ 2x_{1,e}x_{3,e}x_{4,e}\big)
\end{aligned} \tag{5.19a}$$

$$\begin{aligned}
u_2 = u_{2,r} &+ 2c_4 x_{3,r}x_{1,e} + 3u_{1,r}x_{2,r}x_{1,e} + u_{1,r}x_{4,r}x_{1,e} - c_4 x_{2,e} - 3u_{1,r}x_{3,e} \\
&- 2c_4 x_{4,e} - u_{1,r}x_{5,e} + 3u_{1,r}x_{1,e}x_{2,e} + 2c_4 x_{1,e}x_{3,e} + u_{1,r}x_{1,e}x_{4,e}
\end{aligned} \tag{5.19b}$$

where $\lambda = 5$, $c_4 = 2$, $c_5 = 2$ behaved as shown in Figure 5.2.

This behavior was compared to that of the control law (5.8). For tuning the gains we considered the two linear subsystems that result from the cascaded analysis. Both can be expressed as a standard linear time-invariant system of the form $\dot{x} = Ax + Bu$. We used optimal control to arrive at the control law $u = -Kx$ for which the costs

$$\int_0^\infty \|x(t)\|^2 + r\,\|u(t)\|^2 \,\mathrm{d}t$$

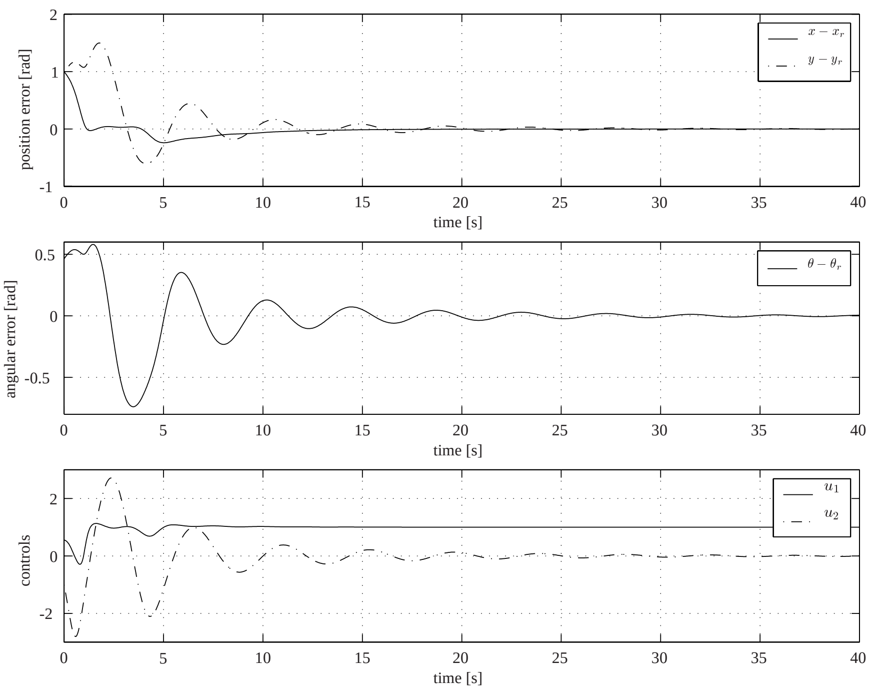

Figure 5.2: Tracking errors and inputs for backstepping controller (5.19).

are minimized, where we took the constant r such that similar control effort is needed as by Jiang and Nijmeijer (1999b). The control law we used was

$$u_1 = u_{1,r}(t) - 3.1623x_{1,e} \tag{5.20a}$$
$$u_2 = u_{2,r}(t) - 2.0770x_{2,e} - 2.1070x_{3,e} - 1.1969x_{4,e} - 0.3162x_{5,e} \tag{5.20b}$$

The resulting behavior is presented in Figure 5.3.

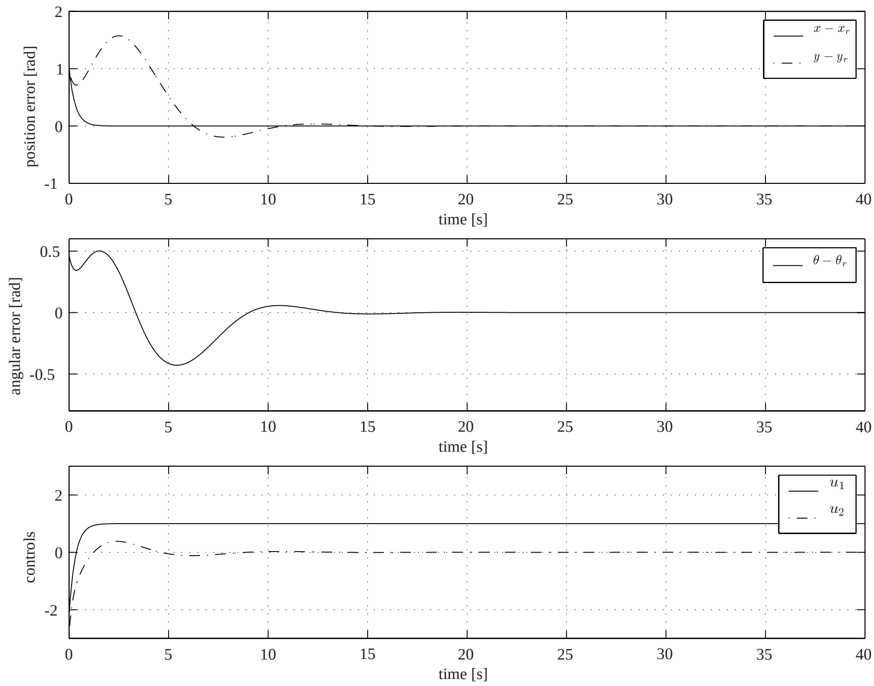

Figure 5.3: Tracking errors and inputs for cascade controller (5.20).

We see from this simulation that the cascaded control law (5.20) not only looks more appealing than (5.19) but also obtains better convergence and is much easier to tune. Notice that due to the local nature of the change of co-ordinates and preliminary feedback (5.17) the control laws (5.20) and (5.19) are only global respectively semi-global in the chained-form co-ordinates, but local in the original co-ordinates.

5.7 Concluding remarks

In this chapter we studied the tracking problem for a special class of non-holonomic systems, namely the class of chained-form systems. This class contains several interesting examples of mechanical systems with non-holonomic constraints.

Using a cascaded systems approach we solved both the state- and output-feedback tracking problems globally for this class of systems. We also solved the saturated tracking problem (both under state- and output-feedback) for only a special class of reference trajectories, namely those with constant $u_{1,r}(t)$. All results assume a persistence of excitation condition on $u_{1,r}(t)$.

We like to emphasize that the fact that the kinematic model of a mobile car is contained in the class of chained-form systems does not make the results in the previous chapter redundant. On the contrary: a cascaded approach to the model of the mobile robot in the original co-ordinates leads to less complicated expressions for the controller. This is also what should be kept in mind when designing controllers for general systems that can be transformed into chained form: apply the approach in the original co-ordinates! Besides, the change of co-ordinates that brings the system in chained form is not global in general. Fortunately, very often x_1 and u_1 are natural co-ordinates of the system (as for the mobile robot). Then the cascaded approach used in this chapter can easily be applied to the system in original co-ordinates. That is, first stabilize the tracking error dynamics that corresponds to $x_{1,e}$ using the input that corresponds to u_1. Then assume that this stabilization has worked out (which boils down to substituting $x_{1,e} \equiv 0$ and $u_1 = u_{1,r}(t)$ in the remaining dynamics). Using the remaining input, this system can be stabilized, provided that $u_{1,r}(t)$ is persistently exciting. Next, Theorem 2.4.3 guarantees global uniform asymptotic stability of the cascaded system under some additional boundedness assumptions on the reference trajectory.

As mentioned in the previous chapter a persistence of excitation (PE) condition on the reference input $u_{1,r}(t)$ might not be required. Loría et al. (1999b) used the weaker notion of uδ-PE to study the stabilization problem for chained-form systems of order 3. It is worth investigating whether weakening the PE condition on $u_{1,r}$ to a uδPE condition can be done for general chained-form systems of order n too.

Furthermore, a uniform global stabilization result for linear time-varying systems under input saturation (and for the system (5.6) in particular) would extend the class of reference trajectories we can track under input saturation.

For the mobile robot the cascaded design was an eye-opener to recognize a simpler structure for backstepping. Since for the mobile robot this enabled us to weaken the assumption on the reference trajectory, it is worth trying the same for chained-form systems. That is, we can start from the nonlinear tracking error dynamics (5.3) with the input u_2 as in Proposition 5.3.1 and the virtual control $x_{1,e} = 0$ and "step back" the virtual control to obtain an expression for the input u_1.

Chapter 6

Tracking control of an under-actuated ship

6.1 Introduction

In this chapter we study the tracking problem for an under-actuated ship. For a conventional ship it is common to consider the motion in *surge* (forward), *sway* (sideways) and *yaw* (heading), see Figure 6.1. Often, we have surge and sway control forces and yaw control

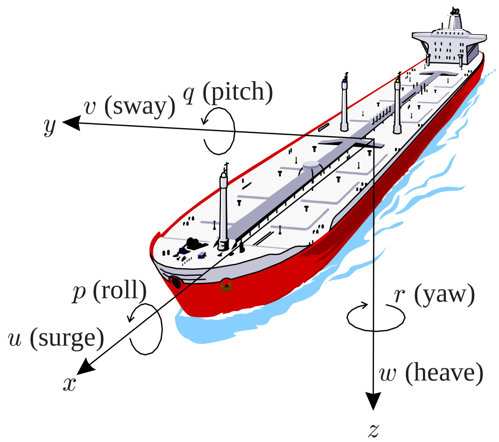

Figure 6.1: Definition of state variables in surge, sway, heave, roll, pitch and yaw for a marine vessel.

moment available for steering the ship. However, this assumption is not valid for all ships. It is very well possible that ships are either equipped with two independent aft thrusters or with one main aft thruster and a rudder, but are without any bow or side thrusters, like for instance many supply vessels. As a result, we have no sway control force. Therefore, we assume to have only surge control force and yaw control moment available. Since we need to control three degrees of freedom and have only two inputs available we are dealing with an under-actuated problem.

For modeling the ship we follow Fossen (1994). We neglect the dynamics associated with the motion in heave, roll and pitch and a slowly-varying bias term due to wind, currents, and wave drift forces. Furthermore we assume that the inertia, added mass and damping matrices are diagonal. In that case the ship dynamics can be described by (see e.g., (Fossen 1994)):

$$\dot{u} = \frac{m_{22}}{m_{11}}vr - \frac{d_{11}}{m_{11}}u + \frac{1}{m_{11}}u_1 \tag{6.1a}$$

$$\dot{v} = -\frac{m_{11}}{m_{22}}ur - \frac{d_{22}}{m_{22}}v \tag{6.1b}$$

$$\dot{r} = \frac{m_{11} - m_{22}}{m_{33}}uv - \frac{d_{33}}{m_{33}}r + \frac{1}{m_{33}}u_2 \tag{6.1c}$$

$$\dot{x} = u\cos\psi - v\sin\psi \tag{6.1d}$$

$$\dot{y} = u\sin\psi + v\cos\psi \tag{6.1e}$$

$$\dot{\psi} = r \tag{6.1f}$$

where u, v and r are the velocities in surge, sway and yaw respectively and x, y, ψ denote the position and orientation of the ship in the earth-fixed frame. The parameters $m_{ii} > 0$ are given by the ship inertia and added mass effects. The parameters $d_{ii} > 0$ are given by the hydrodynamic damping. The available controls are the surge force u_1, and the yaw moment u_2. The ship model (6.1) is neither static feedback linearizable, nor can it be transformed into chained form. It was shown by Pettersen and Egeland (1996) that no continuous or discontinuous static state-feedback law exists which makes the origin asymptotically stable.

The stabilization problem for an under-actuated ship has been studied in (Wichlund, Sørdalen and Egeland 1995, Pettersen and Egeland 1996, Reyhanoglu 1996, Pettersen and Nijmeijer 1998). Tracking control of ships has mainly been based on linear models, giving local results, and steering only two degrees of freedom. (Godhavn 1996) investigated output-tracking control based on a nonlinear model of the ship, and a controller providing global exponential stability (GES) of the desired trajectory was developed. As only the position variables are controlled, typically the ship may turn around and the desired position trajectory is followed backwards. That is why we focus on state-tracking instead of output-tracking.

The first complete state-tracking controller based on a nonlinear model was developed by Pettersen and Nijmeijer (1998) and yields global practical stability. Pettersen and Nijmeijer (2000) achieved semi-global asymptotic stability by means of backstepping, inspired by the results of Jiang and Nijmeijer (1999b). We are not aware of any global tracking results for the tracking control of an under-actuated ship in literature.

In this chapter we present a global solution to the tracking problem for an under-actuated ship. In Section 6.2 we derive the tracking error dynamics considered in (Pettersen and Nijmeijer 2000) and also more natural error dynamics. In Section 6.3 we solve the state-feedback tracking control problem. The controller derived in Section 6.3 is implemented for tracking control of a scale model of an offshore supply vessel. The experimental results are presented in Section 6.4. Some conclusions are drawn in Section 6.5.

6.2 The tracking error dynamics

Assume that a feasible reference trajectory $(u_r, v_r, r_r, x_r, y_r, \psi_r, u_{1,r}, u_{2,r})^T$ is given, i.e., a trajectory satisfying

$$\begin{aligned}
\dot{u}_r &= \frac{m_{22}}{m_{11}}v_r r_r - \frac{d_{11}}{m_{11}}u_r + \frac{1}{m_{11}}u_{1,r} \\
\dot{v}_r &= -\frac{m_{11}}{m_{22}}u_r r_r - \frac{d_{22}}{m_{22}}v_r \\
\dot{r}_r &= \frac{m_{11}-m_{22}}{m_{33}}u_r v_r - \frac{d_{33}}{m_{33}}r_r + \frac{1}{m_{33}}u_{2,r} \\
\dot{x}_r &= u_r \cos\psi_r - v_r \sin\psi_r \\
\dot{y}_r &= u_r \sin\psi_r + v_r \cos\psi_r \\
\dot{\psi}_r &= r_r.
\end{aligned} \tag{6.2}$$

Notice that a drawback exists in considering the error co-ordinates $x - x_r$ and $y - y_r$, since these position errors depend on the choice of the inertial frame. This problem is solved by defining the change of co-ordinates as proposed by Pettersen and Egeland (1996) which boils down to considering the dynamics in a frame with an earth-fixed origin having the x- and y-axis always oriented along the ship surge- and sway-axis:

$$\begin{aligned}
z_1 &= x\cos\psi + y\sin\psi \\
z_2 &= -x\sin\psi + y\cos\psi \\
z_3 &= \psi.
\end{aligned} \tag{6.3}$$

The reference variables $z_{1,r}$, $z_{2,r}$ and $z_{3,r}$ are defined correspondingly. Next, we define the tracking errors

$$u_e = u - u_r \tag{6.4a}$$
$$v_e = v - v_r \tag{6.4b}$$
$$r_e = r - r_r \tag{6.4c}$$
$$z_{1,e} = z_1 - z_{1,r} \tag{6.4d}$$
$$z_{2,e} = z_2 - z_{2,r} \tag{6.4e}$$
$$z_{3,e} = z_3 - z_{3,r}. \tag{6.4f}$$

In this way, we obtain the tracking error dynamics

$$\dot{u}_e = \frac{m_{22}}{m_{11}}(v_e r_e + v_e r_r(t) + v_r r_e) - \frac{d_{11}}{m_{11}}u_e + \frac{1}{m_{11}}(u_1 - u_{1,r}) \tag{6.5a}$$
$$\dot{v}_e = -\frac{m_{11}}{m_{22}}(u_e r_e + u_e r_r(t) + u_r r_e) - \frac{d_{22}}{m_{22}}v_e \tag{6.5b}$$
$$\dot{r}_e = \frac{m_{11}-m_{22}}{m_{33}}(u_e v_e + u_e v_r + u_r v_e) - \frac{d_{33}}{m_{33}}r_e + \frac{1}{m_{33}}(u_2 - u_{2,r}) \tag{6.5c}$$
$$\dot{z}_{1,e} = u_e + z_{2,e}r_e + z_{2,e}r_r(t) + z_{2,r}r_e \tag{6.5d}$$
$$\dot{z}_{2,e} = v_e - z_{1,e}r_e - z_{1,e}r_r(t) - z_{1,r}r_e \tag{6.5e}$$
$$\dot{z}_{3,e} = r_e. \tag{6.5f}$$

Like Pettersen and Nijmeijer (2000) we study the problem of stabilizing the tracking error dynamics (6.5).

As mentioned, the change of co-ordinates (6.3) boils down to considering the dynamics in a frame with an earth fixed origin having the x- and y-axis always oriented along the ship surge- and sway-axis. This is done for both the ship and the reference. Therefore, a physical interpretation of the error co-ordinates (6.4d, 6.4e, 6.4f) is less clear. More natural error co-ordinates would be to consider the position errors $x - x_r$ and $y - y_r$ in a frame attached to the body of the ship (as for the mobile robot). This leads to the error co-ordinates

$$\begin{bmatrix} x_e \\ y_e \\ \psi_e \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_r \\ y - y_r \\ \psi - \psi_r \end{bmatrix}. \tag{6.6}$$

When using the error co-ordinates (6.4a, 6.4b, 6.4c, 6.6) the tracking error dynamics becomes

$$\begin{aligned}
\dot{u}_e &= \frac{m_{22}}{m_{11}}(v_e r_e + v_e r_r(t) + v_r r_e) - \frac{d_{11}}{m_{11}} u_e + \frac{1}{m_{11}}(u_1 - u_{1,r}) \\
\dot{v}_e &= -\frac{m_{11}}{m_{22}}(u_e r_e + u_e r_r(t) + u_r r_e) - \frac{d_{22}}{m_{22}} v_e \\
\dot{r}_e &= \frac{m_{11} - m_{22}}{m_{33}}(u_e v_e + u_e v_r + u_r v_e) - \frac{d_{33}}{m_{33}} r_e + \frac{1}{m_{33}}(u_2 - u_{2,r}) \\
\dot{x}_e &= u - u_r \cos\psi_e - v_r \sin\psi_e + r_e y_e + r_r(t) y_e \\
\dot{y}_e &= v - v_r \cos\psi_e + u_r \sin\psi_e - r_e x_e - r_r(t) x_e \\
\dot{\psi}_e &= r_e.
\end{aligned} \tag{6.7}$$

Therefore, we could as well study the problem of stabilizing the tracking error dynamics (6.7). However, for comparison reasons we focus in Section 6.3 on stabilizing (6.5). Stabilizing (6.7) can be dealt with in a similar way.

6.3 State-feedback: a cascaded approach

We want to use the cascaded design approach for solving the state-feedback tracking problem. For that, we follow the ideas presented in Section 3.2 and look for a control law for one of the two inputs, which is such that in closed loop a subsystem is asymptotically stabilized.

By defining the preliminary feedback

$$u_2 = u_{2,r} - (m_{11} - m_{22})(uv - u_r v_r) + d_{33} r_e + m_{33}\nu \tag{6.8}$$

where ν is a new input, the subsystem (6.5c, 6.5f) reduces to the linear system

$$\begin{aligned} \dot{r}_e &= \nu \\ \dot{z}_{3,e} &= r_e \end{aligned} \tag{6.9}$$

which can easily be stabilized by choosing a suitable control law for ν, for example

$$\nu = -c_1 r_e - c_2 z_{3,e} \qquad\qquad c_1, c_2 > 0. \tag{6.10}$$

Next, we assume that the stabilization of (6.9) has been established, that is, we substitute $r_e \equiv 0$ and $z_{3,e} \equiv 0$ in (6.5a, 6.5b, 6.5d, 6.5e). This results in

$$\begin{aligned}
\dot{u}_e &= \frac{m_{22}}{m_{11}} v_e r_r(t) - \frac{d_{11}}{m_{11}} u_e + \frac{1}{m_{11}}(u_1 - u_{1,r}) \\
\dot{v}_e &= -\frac{m_{11}}{m_{22}} u_e r_r(t) - \frac{d_{22}}{m_{22}} v_e \\
\dot{z}_{1,e} &= u_e + z_{2,e} r_r(t) \\
\dot{z}_{2,e} &= v_e - z_{1,e} r_r(t)
\end{aligned}$$

which is just a linear time-varying system:

$$\begin{bmatrix} \dot{u}_e \\ \dot{v}_e \\ \dot{z}_{1,e} \\ \dot{z}_{2,e} \end{bmatrix} = \begin{bmatrix} -\frac{d_{11}}{m_{11}} & \frac{m_{22}}{m_{11}} r_r(t) & 0 & 0 \\ -\frac{m_{11}}{m_{22}} r_r(t) & -\frac{d_{22}}{m_{22}} & 0 & 0 \\ 1 & 0 & 0 & r_r(t) \\ 0 & 1 & -r_r(t) & 0 \end{bmatrix} \begin{bmatrix} u_e \\ v_e \\ z_{1,e} \\ z_{2,e} \end{bmatrix} + \begin{bmatrix} \frac{1}{m_{11}} \\ 0 \\ 0 \\ 0 \end{bmatrix} [u_1 - u_{1,r}]. \tag{6.11}$$

All that remains to be done, is to find a feedback controller for u_1 that stabilizes the system (6.11). It follows from Corollary 2.3.4 that the system (6.11) is uniformly completely controllable (UCC) if the reference yaw velocity $r_r(t)$ is persistently exciting. Notice that this condition is similar to that of the mobile robot studied in Chapter 4, where also the reference angular velocity had to be persistently exciting. As a result, if the reference yaw velocity $r_r(t)$ is persistently exciting, we can use any of the control laws available in literature for stabilizing linear time-varying systems, as mentioned in Remark 5.3.2.

In addition to these results we arrived at the following.

Proposition 6.3.1. *Consider the system* (6.11) *in closed loop with the control law*

$$u_1 = u_{1,r} - k_1 u_e + k_2 r_r(t) v_e - k_3 z_{1,e} + k_4 r_r(t) z_{2,e} \tag{6.12}$$

where k_i ($i = 1, \ldots, 4$) satisfy

$$k_1 > d_{22} - d_{11} \tag{6.13a}$$

$$k_2 = \frac{k_4(k_4 + k_1 + d_{11} - d_{22})}{\frac{m_{11}}{m_{22}}(d_{22}k_4 + m_{11}k_3)} \tag{6.13b}$$

$$0 < k_3 < (k_1 + d_{11} - d_{22}) \frac{d_{22}}{m_{11}} \tag{6.13c}$$

$$k_4 > 0. \tag{6.13d}$$

If $r_r(t)$ is persistently exciting (PE) then the closed-loop system (6.11, 6.12) *is globally uniformly exponentially stable (GUES).*

Proof. See Appendix A. □

Combining the controllers (6.8, 6.10) and (6.12) we are now able to formulate the cascaded systems based solution to the tracking control problem:

Proposition 6.3.2. *Consider the ship tracking error dynamics* (6.5) *in closed loop with the control law*

$$u_1 = u_{1,r} - k_1 u_e + k_2 r_r(t) v_e - k_3 z_{1,e} + k_4 r_r(t) z_{2,e} \tag{6.14a}$$

$$u_2 = u_{2,r} - (m_{11} - m_{22})(u_e v_e + v_r u_e + u_r v_e) - k_5 r_e - k_6 z_{3,e} \tag{6.14b}$$

where

$$\begin{aligned} k_1 &> d_{22} - d_{11} \\ k_2 &= \frac{k_4(k_4 + k_1 + d_{11} - d_{22})}{\frac{m_{11}}{m_{22}}(d_{22}k_4 + m_{11}k_3)} \\ 0 < k_3 &< (k_1 + d_{11} - d_{22})\frac{d_{22}}{m_{11}} \\ k_4 &> 0 \\ k_5 &> -d_{33} \\ k_6 &> 0. \end{aligned}$$

If u_r, v_r, $z_{1,r}$ and $z_{2,r}$ are bounded and $r_r(t)$ is persistently exciting (PE), then the closed-loop system (6.5, 6.14) *is globally $\mathcal{K}$-exponentially stable.*

Proof. Due to the design, the closed-loop system (6.5, 6.14) has a cascaded structure:

$$\begin{bmatrix} \dot u_e \\ \dot v_e \\ \dot z_{1,e} \\ \dot z_{2,e} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{k_1+d_{11}}{m_{11}} & \frac{k_2+m_{22}}{m_{11}} r_r(t) & -\frac{k_3}{m_{11}} & \frac{k_4}{m_{11}} r_r(t) \\ -\frac{m_{11}}{m_{22}} r_r(t) & -\frac{d_{22}}{m_{22}} & 0 & 0 \\ 1 & 0 & 0 & r_r(t) \\ 0 & 1 & -r_r(t) & 0 \end{bmatrix} \begin{bmatrix} u_e \\ v_e \\ z_{1,e} \\ z_{2,e} \end{bmatrix}}_{f_1(t,z_1)} + \underbrace{\begin{bmatrix} \frac{m_{22}}{m_{11}}(v_e+v_r) & 0 \\ -\frac{m_{11}}{m_{22}}(u_e+u_r) & 0 \\ z_{2,e}+z_{2,r} & 0 \\ -(z_{1,e}+z_{1,r}) & 0 \end{bmatrix}}_{g(t,z_1,z_2)} \begin{bmatrix} r_e \\ z_{3,e} \end{bmatrix}$$

$$\begin{bmatrix} \dot r_e \\ \dot z_{3,e} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{d_{33}+k_5}{m_{33}} & -\frac{k_6}{m_{33}} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_e \\ z_{3,e} \end{bmatrix}}_{f_2(t,z_2)}.$$

From Proposition 6.3.1 we know that the system $\dot z_1 = f_1(t, z_1)$ is globally uniformly exponentially stable (GUES) and from standard linear control that the system $\dot z_2 = f_2(t, z_2)$ is GUES. Furthermore, due to the fact that u_r, v_r, $z_{1,r}$, and $z_{2,r}$ are bounded, $g(t, z_1, z_2)$ satisfies (2.25). Applying Corollary 2.4.6 provides the desired result. □

Remark 6.3.3. As pointed out in Remark 5.3.2 the control law (6.14) is not the only control law that results in global $\mathcal{K}$-exponential stability. Any control law for u_1 that renders the system (6.11) globally uniformly exponentially stable works also.

Notice that the stabilization of the "more natural" tracking error dynamics (6.7) follows along the same lines. Using the control law (6.8, 6.10), i.e.,

$$u_2 = u_{2,r} - (m_{11} - m_{22})(uv - u_r v_r) - k_5 r_e - k_6 \psi_e \qquad k_5 > -d_{33}, k_6 > 0 \tag{6.15}$$

results in the globally uniformly exponentially stable closed-loop (sub)system

$$\begin{bmatrix} \dot r_e \\ \dot\psi_e \end{bmatrix} = \begin{bmatrix} -\frac{d_{33}+k_5}{m_{33}} & -\frac{k_6}{m_{33}} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_e \\ \psi_e \end{bmatrix}.$$

Substitution of $r_e \equiv 0$ and $\psi_e \equiv 0$ in (6.7) yields the linear time-varying system:

$$\begin{bmatrix} \dot{u}_e \\ \dot{v}_e \\ \dot{x}_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} -\frac{d_{11}}{m_{11}} & \frac{m_{22}}{m_{11}} r_r(t) & 0 & 0 \\ -\frac{m_{11}}{m_{22}} r_r(t) & -\frac{d_{22}}{m_{22}} & 0 & 0 \\ 1 & 0 & 0 & r_r(t) \\ 0 & 1 & -r_r(t) & 0 \end{bmatrix} \begin{bmatrix} u_e \\ v_e \\ x_e \\ y_e \end{bmatrix} + \begin{bmatrix} \frac{1}{m_{11}} \\ 0 \\ 0 \\ 0 \end{bmatrix} [u_1 - u_{1,r}]$$

which is identical to (6.11). For stabilizing this system we can use the control law

$$u_1 = u_{1,r} - k_1 u_e + k_2 r_r(t) v_e - k_3 x_e + k_4 r_r(t) y_e \tag{6.16}$$

where k_i ($i = 1, \ldots, 4$) satisfy (6.13).

Corollary 6.3.4. *Assume that $r_r(t)$ is persistently exciting and that u_r and v_r are bounded, then we have that the control law* (6.15, 6.16) *renders the closed-loop tracking error dynamics* (6.7) *globally $\mathcal{K}$-exponentially stable.*

6.4 Experimental results

To support our claims we performed some experiments at the Guidance, Navigation and Control Laboratory located at the Department of Engineering Cybernetics, NTNU, Trondheim, Norway. In the experiments we used Cybership I, which is a $1 : 70$ scale model of an offshore supply vessel. The model ship has a length of 1.19 m, and a mass of 17.6 kg and is equipped with four azimuth-controlled thrusters (i.e., thrusters where the direction of the propeller force can be controlled). The maximum force from one thruster is approximately 0.9 N. The vessel moves in a 10-by-6 meter pool with a depth of about 0.25 meters.

Three spheres are mounted on the model of the vessel that can be identified by infra red cameras (for the simulation of a global positioning system (GPS)). Three infra red cameras are mounted in such a way that (almost always) one or two cameras can see the boat. From each camera the positions of the spheres are transfered via a serial line to a dSPACE signal processor (DSP). From these positions the ship position and orientation can be calculated. A nonlinear passive observer of Fossen and Strand (1999) is used to estimate the unmeasured states. The estimates for position and velocities generated by this observer are used for feedback in the control law. No theoretical guarantee for a stable controller observer combination can be given (yet), as for nonlinear systems no separation principle exists. However, in the experiments it turned out to work satisfactory.

The control law and position estimates are implemented on a Pentium 166 MHz PC which is connected with the DSP via a dSPACE bus. By using Simulink® blocks, the software is compiled and then downloaded into the DSP. The DSP sends the thruster commands to the ship via a radio-transmitter. The sampling frequency used in the experiments was 50 Hz.

The reference trajectory to be tracked was similar to that in (Pettersen and Nijmeijer 2000), namely a circle with a radius of 1 meter using a constant surge velocity of 0.05 m/s. From

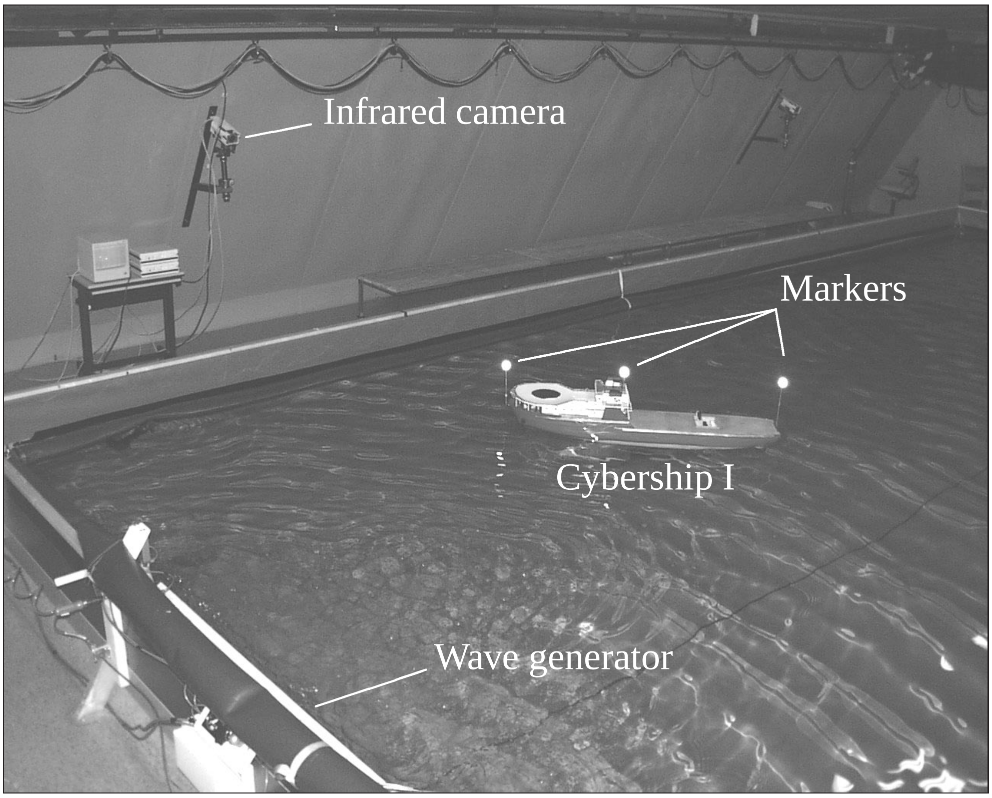

Figure 6.2: Guidance, Navigation and Control Laboratory.

the initial reference state

$$
\begin{aligned}
u_r(0) &= 0.05 \text{ m/s} \\
v_r(0) &= 0 \text{ m/s} \\
r_r(0) &= 0.05 \text{ rad/s} \\
x_r(0) &= 4.75 \text{ m} \\
y_r(0) &= 3.5 \text{ m} \\
\psi_r(0) &= \pi \text{ rad}
\end{aligned}
$$

and the requirement

$$
\begin{aligned}
u_r(t) &= 0.05 \text{ m/s} && \forall t \geq 0 \\
r_r(t) &= 0.05 \text{ rad/s} && \forall t \geq 0
\end{aligned}
$$

the reference trajectory $[u_r, v_r, r_r, x_r, y_r, \psi_r]^T$ can be generated, since it has to satisfy (6.2).

As did Pettersen and Nijmeijer (2000) we chose in the experiments not to cancel or compensate for the damping terms (i.e., assume $d_{11} = d_{33} = 0$), since these are restoring terms, and due to possible parameter uncertainties cancellations could result in destabilizing terms.

In the experiments we compare the control law (6.14) that we obtained by a cascaded design with the control of Pettersen and Nijmeijer (2000) that was derived by means of backstepping. This backstepping-based controller is given by

$$
\begin{aligned}
u_1 = &-\frac{m_{11}k_2^2 u_r z_{3,e}}{r_r} - \frac{m_{11}k_2^2 u_e z_{3,e}}{r_r} + \frac{m_{11}k_2 d v_e}{c r_r} + \frac{m_{11}k_2 d u_e z_{3,e}}{r_r} \\
&+ \frac{m_{11}k_2 d u_r z_{3,e}}{r_r} + m_{11}\gamma z_{2,e} z_{3,e} + m_{11}\gamma z_{3,e} z_{2,r} - m_{11}k_1 u_e - m_{11}\gamma z_{1,e} \\
&- m_{11}a_1 u_e - m_{22} v_r r_r - m_{22} v_r r_e - m_{22} v_e r_r - m_{22} v_e r_e - \frac{m_{11}a_1 k_2^2 z_{2,e}}{c r_r} \\
&+ m_{11}a_1 k_1 z_{3,e} z_{2,r} + m_{11}a_1 k_1 z_{2,e} z_{3,e} - \frac{m_{11}a_1 k_2^2 z_{3,e} z_{1,r}}{c r_r} \\
&+ m_{11}c^2 \alpha r_r u_e z_{3,e} + \frac{m_{11}a_1 k_2 d z_{3,e} z_{1,r}}{c r_r} + \frac{m_{11}a_1 k_2 d z_{1,e} z_{3,e}}{c r_r} \\
&+ m_{11}c\alpha r_r k_2 z_{2,e} + m_{11}c\alpha r_r k_2 z_{1,e} z_{3,e} - m_{11}a_1 k_1 z_{1,e} \\
&- m_{11}k_1 r_r z_{3,e} z_{1,r} + m_{11}c\alpha r_r v_e - m_{11}k_1 r_r z_{1,e} z_{3,e} + \frac{m_{11}a_1 k_2 d z_{2,e}}{c r_r} \\
&+ m_{11}c^2 \alpha r_r u_r z_{3,e} + m_{11}c\alpha r_r k_2 z_{3,e} z_{1,r} - \frac{m_{11}a_1 k_2^2 z_{1,e} z_{3,e}}{c r_r} \\
&- m_{11}k_1 r_r z_{2,e} - \frac{m_{11}k_2 d z_{1,e}}{c} + \frac{m_{11}k_2 d z_{2,e} z_{3,e}}{c} + \frac{m_{11}k_2 d z_{3,e} z_{2,r}}{c} \\
&- \frac{m_{11}k_2^2 z_{2,e} z_{3,e}}{c} - \frac{m_{11}k_2^2 z_{3,e} z_{2,r}}{c} - \frac{m_{11}k_2^2 v_e}{c r_r} + \frac{m_{11}k_2^2 z_{1,e}}{c}
\end{aligned}
\tag{6.17a}
$$

$$
u_2 = f_{k_1,k_2,a_1,a_2,a_3,\alpha,\lambda,\gamma}(u_e, v_e, r_e, z_{1,e}, z_{2,e}, z_{3,e}, t) \tag{6.17b}
$$

where $f_{k_1,k_2,a_1,a_2,a_3,\alpha,\lambda,\gamma}(u_e, v_e, r_e, z_{1,e}, z_{2,e}, z_{3,e}, t)$ is a complex expression of 2782 terms over a little less complex expression of 64 terms. For sake of not being incomplete, the control law for u_2 is contained in Appendix B. The controller gains are given by $k_1 = 0.45$, $k_2 = 0.25$, $a_1 = 0.5$, $a_2 = 2$, $a_3 = 15$, $\alpha = 0.75$, $\gamma = 0.005$, and $\lambda = 50$. These gains were found by trial and error, using a computer model of the ship.

For tuning the gains of (6.14) we prefer a more systematic approach. However, for comparison we first look for a set of control parameters for (6.14) more or less corresponding to the parameters of Pettersen and Nijmeijer (2000) for (6.17).

First we define the auxiliary signals

$$v_1 = u_1 - u_{1,r} \tag{6.18a}$$

and

$$v_2 = u_2 - u_{2,r} + (m_{11} - m_{22})(u_e v_e + v_r u_e + u_r v_e) \tag{6.18b}$$

and substituting for u_1 and u_2 the control laws (6.14) we obtain

$$v_1 = -k_1 u_e + k_2 r_r(t) v_e - k_3 z_{1,e} + k_4 r_r(t) z_{2,e}$$

and

$$v_2 = -k_5 r_r(t) - k_6 z_{3,e}$$

where for the experiment $r_r(t) = 0.05$. This is a linear controller. Therefore, the first approach for tuning the controller (6.14) is to consider a first order Taylor approximation of the auxiliary signals (6.18) where we take u_1 and u_2 as in (6.17). This results in

$$\begin{aligned}\bar{v}_1 \approx\ & -18.05u_e + 3.90v_e - 4.63z_{1,e} + 2.31z_{2,e} - 0.17r_e \\ & + (0.15 + 2.31z_{1,d} + 4.63z_{2,d})z_{3,e}\end{aligned} \tag{6.19a}$$

$$\begin{aligned}\bar{v}_2 \approx\ & -60.00r_e - 2.40z_{3,e} + 0.01u_e + 1.6v_e + 0.003z_{1,e} - 0.005z_{2,e} \\ & - (0.005z_{1,d} + 0.003z_{2,d})z_{3,e}.\end{aligned} \tag{6.19b}$$

In the cascaded analysis we first used $\bar{v}_2$ for stabilizing the $(r_e, z_{3,e})^T$ dynamics and while designing in the second stage $\bar{v}_1$ we assume that $r_e = z_{3,e} = 0$. Therefore, we leave out the r_e and $z_{3,e}$ terms of (6.19a) and neglect the small terms in (6.19b), as well as the term with v_e. This leads us to the control law

$$u_1 = u_{1,r} - 18.05u_e + 3.90v_e - 4.63z_{1,e} + 2.31z_{2,e} \tag{6.20a}$$

$$u_2 = u_{2,r} - (m_{11} - m_{22})(u_e v_e + v_r u_e + u_r v_e) - 60.00r_e - 2.40z_{3,e}. \tag{6.20b}$$

The second approach for tuning the controller (6.14) is to consider the two linear subsystems (6.9) and (6.11) that resulted from the cascaded analysis. Both can be expressed as a standard linear time-invariant system of the form $\dot{x} = Ax + Bu$. We use optimal control to arrive at the control law $u = -Kx$ for which the costs

$$\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) \mathrm{d}t$$

are minimized. For Q we choose a diagonal matrix with entries $q_{ii} = \frac{1}{\Delta x_i}$ ($i = 1, \ldots, 4$), where Δx_i is the maximum error we can tolerate in x_i. For R we take the inverse of maximum allowed input. This results in the choice

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \qquad R = 1.1$$

for the system (6.11) and

$$Q = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix} \qquad R = 1.1$$

for the system (6.9). In this way we obtain for the control law:

$$u_1 = u_{1,r} - 10.28u_e + 9.2v_e - 4.44z_{1,e} + 2.74z_{2,e} \tag{6.21a}$$
$$u_2 = u_{2,r} - (m_{11} - m_{22})(u_e v_e + v_r u_e + u_r v_e) - 9.02r_e - 6.74z_{3,e}. \tag{6.21b}$$

To summarize: we have three different control laws available, namely the backstepping-controller (6.17) as proposed in (Pettersen and Nijmeijer 2000), its cascaded-systems-based linearization (6.20) and the linear optimal control based cascaded controller (6.21).

We did experiments with all three controllers. The resulting performance of the cascaded controller (6.20), which was based on a linearization of the backstepping controller, is shown in Figure 6.3. In the first two graphs we compare the actual position of the ship with its desired position. The third graph contains the error in orientation. The fourth and fifth graph depict the controls applied to the ship. The bottom graph depicts the camera status. The reason for showing this is that the infrared cameras from time to time loose track of the ship. As long as the camera status equals zero we have position measurements from the camera-system, but as soon as the camera status is non-zero we no longer get correct position measurements. In Figure 6.3 we can see that for instance after about 120 seconds we had a temporary failure of the camera-system. This explains the sudden change in the orientation error ψ_e and in the control u_1.

The resulting performance of the backstepping controller (6.17) is presented in Figure 6.4.

The resulting performance of the cascaded controller (6.21), of which the gains were chosen by means of optimal control theory, is presented in Figure 6.5.

When we compare the backstepping controller (6.17) with the cascaded controller (6.21) the tracking of the reference positions $x_r(t)$ and $y_r(t)$ is comparable. However, the angular tracking error is considerably less for the cascaded controller.

From the fact that the presented controllers can be applied successfully in experiments, we might conclude that they possess some robustness with respect to modeling errors and with respect to disturbances due to currents and wave drift forces.

To illustrate this robustness even more, we performed one experiment using the “optimal gains” in which the author was wearing boots and walking through the pool, trying to create as much waves as possible and disturbing the ship as much as he could. The results are

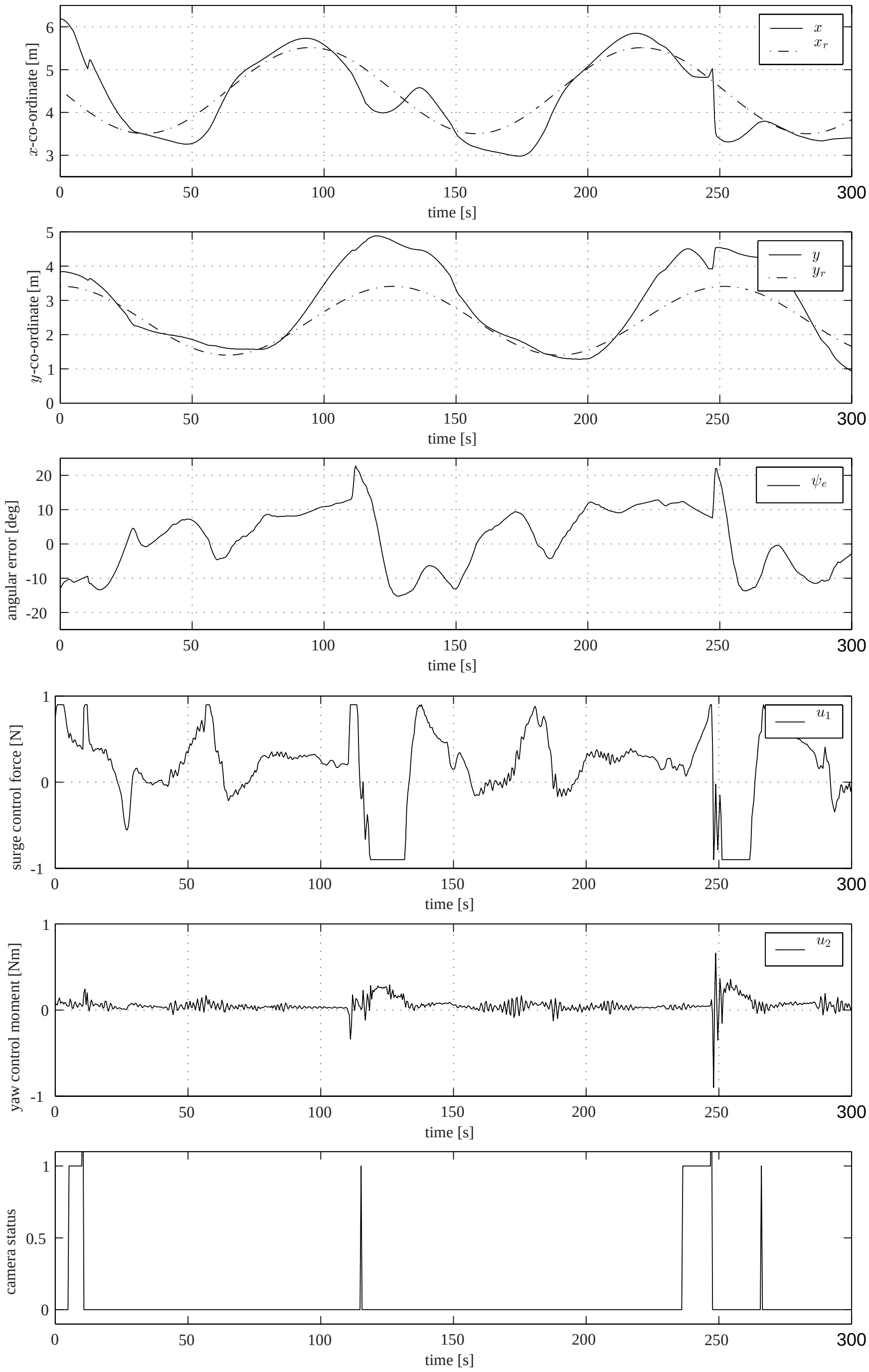

Figure 6.3: Cascade controller (6.20) with gains based on linearization of the backstepping controller.

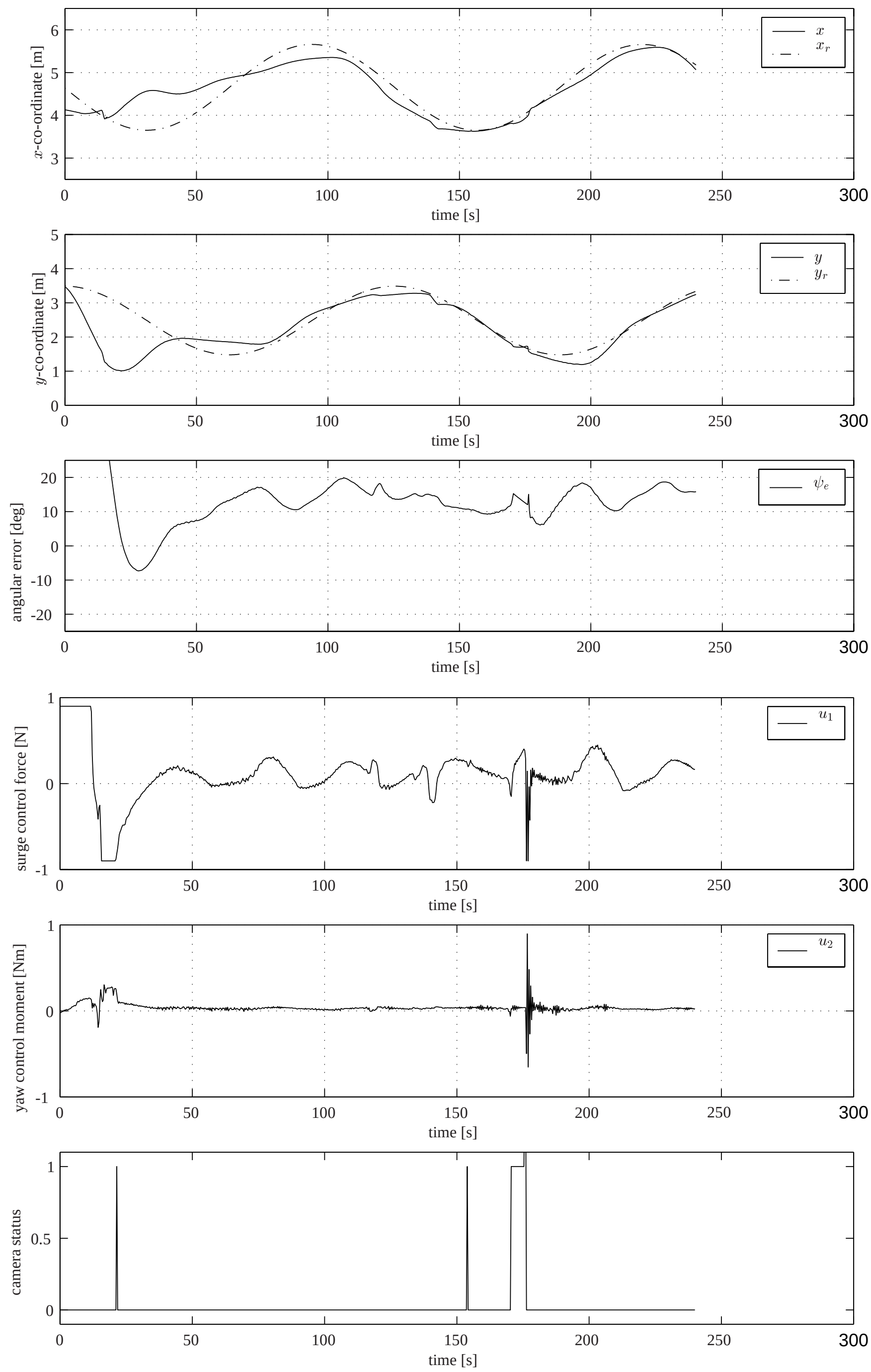

Figure 6.4: Backstepping controller (6.17).

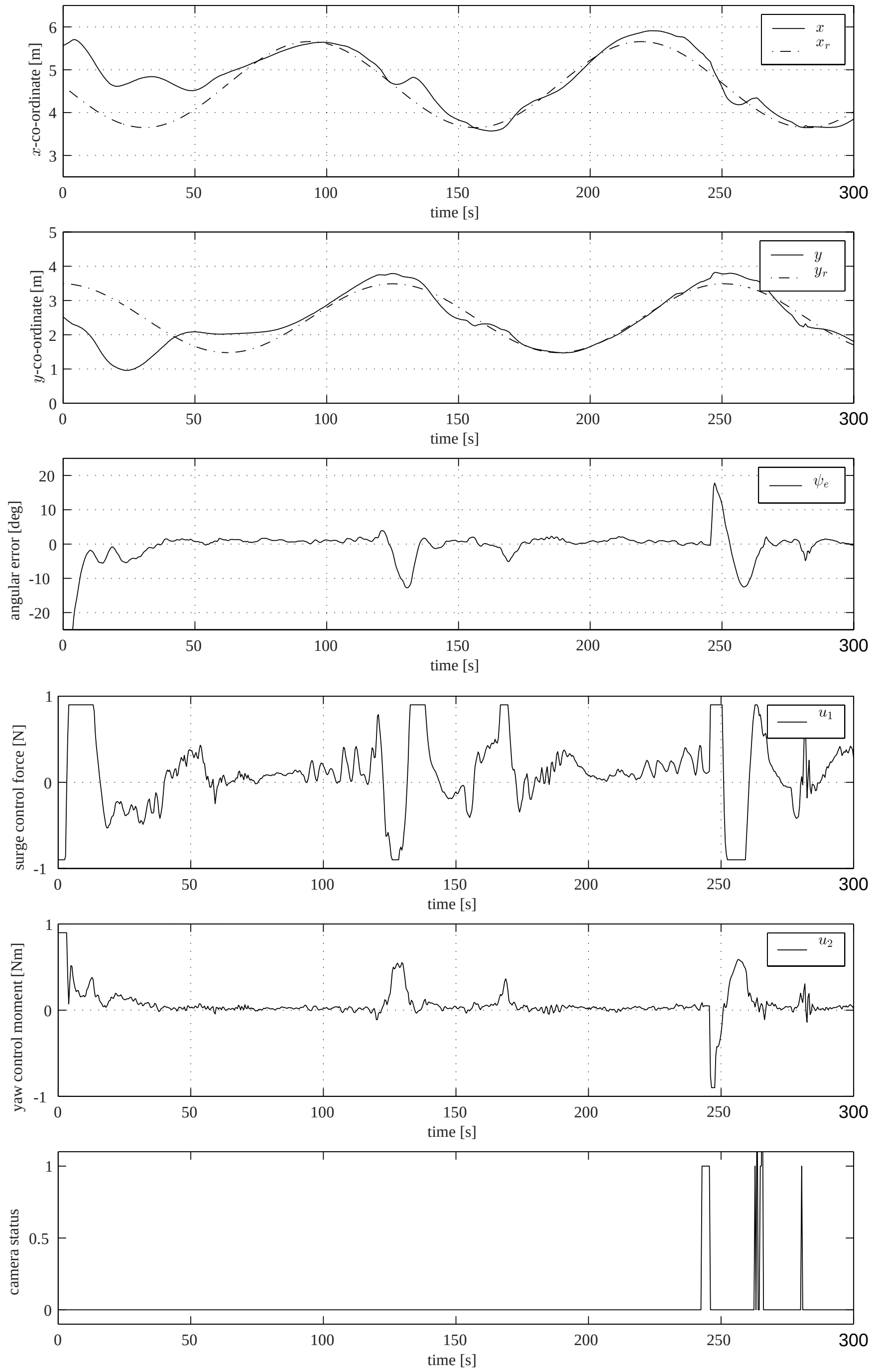

Figure 6.5: Cascade controller (6.21) with gains based on optimal control.

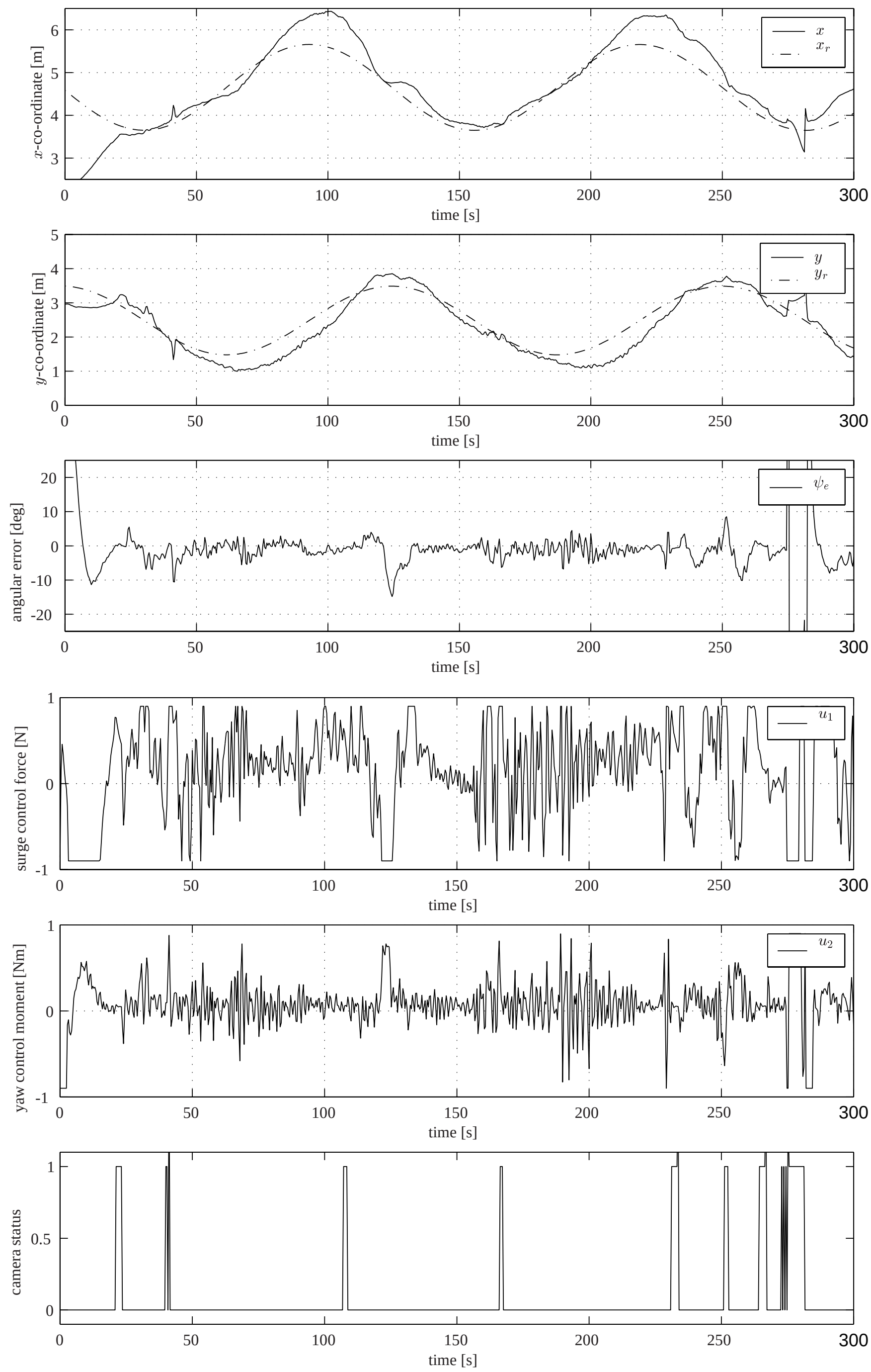

Figure 6.6: Cascade controller (6.21) under disturbance of author walking through the pool.

depicted in Figure 6.6. It can be noticed that due to the heavy waves, the camera system had much more difficulties in keeping track of the ship. Nevertheless a reasonable tracking performance was achieved.

6.5 Concluding remarks

In this chapter we studied the tracking problem for an under-actuated ship that has only surge control force and yaw control moment, which is a common situation for many supply vessels.

By means of a cascaded approach we developed a global tracking controller for this tracking problem. The resulting control law has a much simpler structure than the backstepping based controller that was available in literature and which guarantees semi-global tracking. The cascaded approach reduced the problem of stabilizing the nonlinear tracking error dynamics to two separated problems of stabilizing linear systems. This insight simplified the gain-tuning a lot, since optimal control could be used to arrive at suitable gains.

A disadvantage of both the backstepping and the cascade controller is the demand that the reference angular velocity does not tend to zero. As in the previous chapters, the cascaded approach leads us to a simpler structure for backstepping (cf. Remark 4.2.6). Starting from this we might be able to weaken this condition that the angular velocity of the reference should not tend to zero. Another possibility might be to consider the idea of uδ-PE.

The controllers presented in this chapter also proved to work reasonably well in experiments. This implies a certain robustness against modeling errors and disturbances due to currents and wave drift forces. In an attempt to get better robustness results, the cascaded approach might be helpful. Disturbances due to currents and wave drift forces are in general modeled by a constant force acting on the ship. The cascaded approach learned us that instead of looking at the nonlinear tracking error dynamics, we could as well consider two linear systems. Obtaining asymptotic stability of a linear system under a constant disturbance is a well-known problem that can be solved by using integral control. Therefore, adding integral control to the cascaded controller adds robustness against constant disturbances.

Part II

Robot manipulators and adaptive control

Chapter 7

Introduction to Part II

The second part of this thesis consists of three papers. All three deal with the tracking control of nonlinear mechanical systems, but each focuses on different aspects than those discussed so far in the first part of the thesis. Specifically, Chapter 8 and Chapter 9 deal with tracking of fully-actuated rigid robots, whereas Chapter 10 treats a new adaptive control problem for an under-actuated system. In what follows we briefly describe the subject of the three papers and finally give a short discussion on a number of relevant similarities in these papers.

7.1 Paper I: Global asymptotic stability of robot manipulators with linear PID and PI^{2}D control

This paper deals with the control of a special nonlinear mechanical system, namely a fully-actuated rigid robot manipulator. The tracking problem considered is to move the manipulator to a desired fixed point. This particular tracking problem also goes under the name set-point control or is called the regulation problem. Takegaki and Arimoto (1981) showed that a PD plus gravity compensation controller can globally asymptotically stabilize a rigid robot manipulator to any desired fixed point. One of the drawbacks of this approach is that the vector of gravitational forces is assumed to be known accurately. Whenever the gravitational vector is not known exactly and an estimate is used, the position error converges to a bounded steady state error. Common practice is to use PID control to overcome this problem. However, only local (or at best semi-global) asymptotic stability of this scheme has been proven so far (see e.g., (Kelly 1995)).

Based on the ideas presented by Lefeber (1996) and applied to the bounded tracking control of chaotic systems in (Lefeber and Nijmeijer 1996, Lefeber and Nijmeijer 1997a, Lefeber and Nijmeijer 1997b) we combine a PD controller with a PID controller. First we apply the global PD controller, which leads to a steady state error. Then we switch to the PID controller by activating the integral action, which results in asymptotic stability. Using this hybrid linear controller we are able to show *global* asymptotic stability.

7.2 Paper II: Adaptive and filtered visual servoing of planar robots

This paper also deals with the regulation problem for rigid robot manipulators, but now with time visual servoing (under a fixed camera configuration). In fixed-camera robotic systems, a vision system fixed in the world co-ordinate frame captures images of the robot and its environment. The objective is to move the robot in such a way that its end-effector reaches a desired target.

Among the existing approaches to solve this problem, it has been recognized that the "image-based" scheme possesses some degree of robustness against camera miscalibrations (Miyazaki and Masutani 1990, Lei and Ghosh 1993, Hager, Chang and Morse 1995). In this approach, the vision system provides an image position error, measured directly on the image plane as the visual distance between the target and end-effector positions. This error is used to drive the controller. Kelly and Marquez (1995) introduced a model of the vision system incorporating a perspective projection based on thin lens geometric optics and they derived a control law to solve the problem. Asymptotic stability of the resulting closed-loop system was shown under the assumption that the orientation of the camera is known. No intrinsic camera parameters were assumed to be known and neither the robot's inverse kinematics nor the inverse Jacobian was used. Kelly (1996) showed robustness of this controller in face of unknown radial lens distortions and uncertainty in the camera orientation.

In this paper we extend the controllers proposed in (Kelly and Marquez 1995, Kelly 1996) to a class of visual servoing controllers which also includes the controllers reported in (Miyazaki and Masutani 1990, Lei and Ghosh 1993, Coste-Manière, Couvignou and Khosla 1995, Kelly 1996, Kelly, Shirkey and Spong 1996). Our class of controllers also contains *saturated* controllers, which enables us to deal with constraints on the inputs. Furthermore, we extend in this paper the results of (Kelly and Marquez 1995, Kelly 1996) to the cases where velocity measurements are not available and the camera orientation parameter is unknown. The latter problem involves a nonlinear parameterized adaptive system for which special analysis and synthesis tools have to be developed, since this problem is almost unexplored in adaptive control.

7.3 Paper III: Adaptive tracking control of non-holonomic systems: an example

As in the previous paper, this third paper also studies a control problem where certain parameters are unknown. In the first part of this thesis we studied the tracking problem for nonlinear mechanical systems with non-holonomic constraints. We assumed all system parameters to be known exactly and no disturbances to be present. In case some system parameters are not known exactly, things change considerably. This paper deals with the adaptive state-tracking problem for the kinematic model of a four-wheel mobile robot with unknown length. The example illustrates that the formulation of the adaptive state-tracking problem is far from trivial. At first glance this might look surprising since adaptive tracking problems have been studied throughout in literature. However, for all these problems feasibility of the reference trajectory is not an issue, since it turns out to be a priori guaranteed. Most results in adaptive control deal either with the adaptive stabilization problem or the adaptive *output*-tracking problem. The adaptive stabilization problem usually is studied for systems without drift which can be stabilized to an arbitrary point, no matter what value the unknown parameters have. In case of the adaptive output tracking problem, one has as many inputs to the system as outputs. As a result an arbitrary signal can be specified for the output. This output can be tracked for any parameter.

As mentioned in Section 1.1, we insist on a state-tracking problem instead of an output-tracking problem. Usually it is undesirable in the tracking problem for a mobile robot or ship that the system turns around and follows the reference trajectory backwards. Therefore, output-tracking does not suffice and state-tracking is what we are looking for. We like to extend the results of the first part of this thesis to the case where certain system parameters are unknown. We illustrate in this paper that not knowing certain parameters, like in this example the length of the vehicle, and specifying a feasible reference trajectory is in conflict with each other. The question then arises how to formulate the adaptive state-tracking problem. One of the necessary conditions in formulating the adaptive state-tracking problem is that in case the parameters are known it reduces to the state-tracking problem (as formulated in Section 1.1).

By means of the kinematic vehicle model we illustrate in this paper the above mentioned conflict. We propose a natural formulation for the adaptive state-tracking problem and present a general methodology for solving this problem.

7.4 Discussion

This thesis is concerned with the tracking of nonlinear mechanical systems. In the first part we focussed on tracking of under-actuated systems and developed a new approach which can lead to simple controllers. However, all analysis is done on the known model, which is assumed to be accurate, whereas in practice all kinds of uncertainties play a role. That is why in this second part we shift attention to some of the uncertainties that are of interest when studying tracking of nonlinear mechanical systems.

To start with, in Chapter 8 and Chapter 9 we deal with uncertainties for fully-actuated systems. In Chapter 10 we return to the under-actuated problem studied in Part I, but this time in the presence of parametric uncertainties. The fully-actuated system studied in Chapter 8 and Chapter 9 is a rigid robot manipulator. In Chapter 8 we assume that the vector of gravitational forces is not known exactly. We show that the common practice of using PID control yields global asymptotic stability when the integral action is activated after some time. The question remains if using integral action from the beginning can also result in global asymptotic stability. The result we obtain is just a switch between a PD and a PID control. Switching controllers is closely related to the results of Teel and Kapoor (1997) and Prieur and Praly (1999).

An other approach of the problem would be to view the unknown constant gravitational vector as an unknown parameter. In that case the integrated error can be seen as an estimate for this unknown parameter and the additional integral action can be seen as an update law for this estimate.

In Chapter 9 we study a different adaptive control problem. Here we adapt for the unknown orientation of the camera that provides us with measurements. The main difficulty in solving this adaptive problem is given by the fact that the unknown parameter enters the dynamics in a nonlinear way. As a result none of the standard linear-in-the-parameters adaptive control techniques can be used. Our solution consists of switching between two controllers which results in a chattering behavior. Although not presented, also a smooth semi-global solution is available. In Chapter 4 we studied the output-feedback tracking problem where part of the position was unmeasured. However, if we consider the same problem where we assume that the position is measured but not the orientation, then we run into similar problems. In this case we also have to reconstruct an unknown angle that enters the dynamics nonlinearly. As illustrated by Jakubiak, Lefeber, Tchón and Nijmeijer (2000) the problems turn out to be the closely related. Actually, similar semi-global solutions can be derived. Results on adaptive control of nonlinearly parameterized systems (in a more general framework) can for instance be found in the work of Loh, Annaswamy and Skantze (1999) and Kojic and Annaswamy (1999).

Whereas in Chapter 8 and Chapter 9 we studied the adaptive regulation problem for fully-actuated systems, in Chapter 10 we study the adaptive *state-tracking* problem for under-actuated systems. In light of Chapter 8 and Chapter 9 the extension of the state-tracking problem as considered in Part I to an adaptive state-tracking problem seems to be a natural next step. As it turns out the first step to be made is arriving at a suitable problem formulation. A precise statement of the general adaptive state-tracking control problem is not so simple. An attempt is made in Chapter 10 by considering the example of a four-wheel mobile robot with unknown length. The reason for considering this example is not given from a practical point of view (since in practice the length of a mobile car can easily be measured), but mainly to illustrate the difficulties one runs into when considering the (also from a practical point of view interesting) adaptive state-tracking problem. For this specific example a problem formulation is presented which could form a basis to arrive at a general problem formulation of the adaptive tracking control problem as for instance presented by Lefeber and Nijmeijer (1998).

Chapter 8

Global asymptotic stability of robot manipulators with linear PID and PI^{2}D control

This chapter consists of the following paper:
A. Loría, E. Lefeber, and H. Nijmeijer, “Global asymptotic stability of robot manipulators with linear PID and PI^{2}D control,” 1999a, Submitted to *Stability and Control: Theory and Applications*.

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Global Asymptotic Stability of Robot Manipulators with Linear PID and PI^{2}D control

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Abstract

In this paper we address the problem of set-point control of robot manipulators with uncertain gravity knowledge by combining several previous contributions to PID control. The main contribution is a *linear* PID controller which ensures *global* asymptotic stability of the closed loop. The key feature of the controller, which allows to prove globality is that the integration is started after a short transient. In the case of unmeasurable velocities, a similar "delayed" PI^{2}D controller is shown to *globally* asymptotically stabilize the manipulator.

1 Introduction

1.1 Literature review

From the seminal paper [19] it is well known now that a PD plus gravity compensation controller can globally asymptotically stabilize a rigid-joints manipulator. However, this approach has two drawbacks (which have been already extensively studied): 1) the vector of gravitational forces is assumed to be known accurately and 2) velocities are needed to inject the necessary damping.

An *ad hoc* solution to the first problem is to compensate for the gravitational vector with the best estimate available. It is well known that in such case, the manipulator in closed loop with a simple PD controller will exhibit some robustness properties; more precisely, the position error will converge to a bounded steady state error.

This problem is not exclusive to robot control but it is often encountered in different industrial processes. A typical and efficient remedy is to use PID control, originally proposed by Nicholas Minorsky in 1922. In the western literature the first stability proof of a PID controller in closed loop with a rigid-joints manipulator is attributed to [3]. Unfortunately, due to some "mathematical technicalities" of the model only local (or at best semiglobal) asymptotic stability can be proven. (See for instance [11]).

Concerning the problem of unmeasurable velocities, we know at least the following *linear* dynamic position feedback controllers which appeared independently [10, 1, 5, 6], see also [16] where the concept of *EL controllers* was introduced and which generalizes the results of the previous references. As in the simple PD control case one can expect that if one compensates with a *constant* vector estimate of the gravitational forces instead of the true one, the manipulator error trajectories will converge to a bounded domain.

Furthermore, as in the case of measurable velocities, one can prove that the steady state error can be eliminated by adding an integrator. More precisely, the PI^{2}D controller originally introduced in [15], is based upon the PD structure and the approximate differentiation filter as proposed in [10], in combination with a double integrator: of the position error and of the filter output. Unfortunately, due to some technical difficulties, one can prove only semi-global asymptotic stability, see also [7].

For the case of measurable velocities one can design, with some smart modifications, *nonlinear* PID's which guarantee *global* asymptotic stability. As far as we know, the first nonlinear PID controller is due to [9][1] which was inspired upon the results of Tomei [21]. Tomei proposed a PD plus adaptive gravity *cancellation* and used a normalization (firstly introduced by Koditschek in [12]) to prove global asymptotic convergence. Using the same normalization idea, Kelly showed in [9] that global asymptotic convergence is still possible in the case when one *compensates* for the gravity forces evaluated at the *desired* position. The latter allows to reformulate the controller of [9] as a normalized PID.

Later, Arimoto [2] proposed to use a saturated proportional term. This idea helps in the same way as the normalization to cope with the third order terms which appear in the Lyapunov function derivative and impede claiming globality.

[1]Even though Kelly [9] presented his result as an "adaptive" controller, in section 4 it will become clear why we use the "PID" qualifier.

As far as we know, there exist no proof of global asymptotic stability of a *linear* PID controller in closed loop with a robot manipulator. In this paper we use some well known results to prove that the set-point regulation can be established by means of a linear *delayed* PID controller, that is a simple PD controller to which an integral action is added after some transient of time. In the sequel we will refer to this controller as "delayed PID", in short PI$_d$D.

Also, in case when no velocity measurements are available we show in a similar way that the integral action of a PI^{2}D controller can be delayed as to guarantee the *global* asymptotic stability of the closed loop. This linear controller will be referred to in the sequel as "delayed PI^{2}D" or in short PI^2_dD.

Our approach is inspired on the ideas of *composite control* developed in[2] [14]. The idea of the composite control approach is simple and practically appealing: to apply in a first phase, a *global* control law, which drives the closed loop trajectories inside some pre-specified bounded set. In the second phase, more precisely at time instant t_s when the trajectories are contained in the bounded domain, one switches to a *locally* stabilizing control law which drives the tracking error to zero. A successful usage of this approach hinges upon the ability of designing both controllers in a way such that the bounded set of the first phase, is contained within the domain of attraction designed for the closed loop in the second phase.

Finally it is also worth mentioning the related –but different– work [20] where the authors propose an algorithm to combine global with local controllers with the aim at improving both robustness and performance.

1.2 Model and problem formulation

The rigid-joints robot kinetic energy is given by $T(q,\dot q) = \frac{1}{2}\dot q^\top D(q)\dot q$, where $q \in \mathbb{R}^n$ represents the link positions, $D(q) = D^\top(q) > 0$ is the robot inertia matrix, and the potential energy generating gravity forces is denoted by $U_g(q)$. Applying the Euler-Lagrange equations we obtain the well known model

$$D(q)\ddot q + C(q,\dot q)\dot q + g(q) = u \tag{1}$$

where $g(q) := \frac{\partial U_g}{\partial q}(q)$, $C(q,\dot q)\dot q$ represents the Coriolis and centrifugal forces, and $u \in \mathbb{R}^n$ are the applied torques. It is also well known now (see for instance [17]) that the following properties hold.

P 1 For all $q \in \mathbb{R}^n$ the matrix $D(q)$ is positive definite and, with a suitable factorization (more precisely using the so-called Christoffel symbols of the first kind) the matrix $N(q,\dot q) = \dot D(q) - 2C(q,\dot q)$ is skew-symmetric. Moreover, there exist some positive constants d_m and d_M such that

$$d_m I < D(q) < d_M I. \tag{2}$$

P 2 There exists some positive constants k_g and k_v such that for all $q \in \mathbb{R}^n$

$$k_g \geq \sup_{q\in\mathbb{R}^n} \left\| \frac{\partial^2 U_g(q)}{\partial q^2} \right\|, \tag{3}$$

[2]It is worth mentioning that the name "composite control" has already been used by other authors to baptize different approaches than the one used in this paper, see for instance [18].

$$k_v \geq \sup_{q\in\mathbb{R}^n} \left\| \frac{\partial U_g(q)}{\partial q} \right\|. \tag{4}$$

P 3 The matrix $C(x,y)$ is bounded in x and linear in y, that is, for all $z \in \mathbb{R}^n$

$$C(x,y)z = C(x,z)y \tag{5}$$
$$\|C(x,y)\| \leq k_c\|y\|, \quad k_c > 0. \tag{6}$$

In this paper we are interested in the solution to the state and output (position) feedback set-point control problem of (1) assuming that the potential energy $U_g(q)$, is not exactly known. More precisely, consider the following problems.

Set-point control problem with uncertain gravity knowledge.

Assume that the gravitational energy function $U_g(q)$ is not exactly known but only its estimate $\hat{U}_g(q)$ is available. Moreover, assume that the estimate of the gravitational forces vector, $\hat{g}(q) \triangleq \frac{\partial \hat{U}}{\partial q}(q)$ satisfies

$$k_v \geq \sup_{q\in\mathbb{R}^n} \|\hat{g}(q)\|, \quad \forall\, q \in \mathbb{R}^n \tag{7}$$

where k_v is defined in (4). Under these conditions design continuous control laws

(state feedback) $\quad u = u(t, q, \dot{q})$
(output feedback) $\quad u = u(t, q, q_c),\ \dot{q}_c = f(q, q_c)$

such that, given any desired *constant* position the error $\tilde{q} \triangleq q - q_d$ be asymptotically convergent, that is, for *any* initial conditions

$$\lim_{t\to\infty} \tilde{q}(t) = 0. \tag{8}$$

In particular we are interested in PID-like control laws achieving this goal, more precisely we seek for controllers of the form

$$u = -k_p\tilde{q} - k_d\dot{q} + \nu(t), \quad k_p,\ k_d > 0 \tag{9}$$

$$\nu(t) = \begin{cases} \hat{g}(q_d), & \forall \quad 0 \leq t \leq t_s \\ -k_i \int_{t_s}^{t} \tilde{q}(s)ds + \hat{g}(q_d) & \forall \quad t \geq t_s,\ k_i > 0 \end{cases} \tag{10}$$

for the case when joint velocities are measured. In the sequel controllers like (9), (10) will be referred to as PI$_d$D. In the case when velocity measurements are not available we seek for a position feedback PI$_d^2$D controller, that is

$$u = -k_p\tilde{q} - k_d\vartheta + \nu(t) \tag{11}$$

$$\vartheta = \mathrm{diag}\left\{\frac{bp}{p+a}\right\} q, \quad a,\ b > 0,\ p = \frac{d}{dt} \tag{12}$$

$$\nu = \begin{cases} \hat{g}(q_d), & \forall \quad 0 \leq t \leq t_s \\ -k_i \int_{t_s}^{t} (\tilde{q}(s) - \vartheta(s))ds + \hat{g}(q_d) & \forall \quad t \geq t_s. \end{cases} \tag{13}$$

The paper is organized as follows. In section 2 we analyze in certain detail some results which appear fundamental to our main contributions. Section 3 contains our main results, that is, we show that the PI$_d$D and PI$_d^2$D controllers solve the *global* set-point control problem with uncertain gravity knowledge. In section 4 we discuss the advantages of our results over the mentioned nonlinear PID's. Section 5 presents a comparative simulation study of our contributions against previous results. We finish the paper with some concluding remarks in section 6.

Notation. In this paper we use $\|\cdot\|$ for the Euclidean norm of vectors and matrices. We denote by k_{p_m} and k_{p_M} the smallest and largest eigenvalues of matrix K_p and similarly for any matrix $\mathcal{M} \in \mathbb{R}^{n\times n}$.

2 Preliminary results

In order to put our contributions in perspective and to introduce some notation used in the sequel, we find it convenient to describe in more detail some of the above-mentioned PID and PD controllers. Even though some of these results are well known or can be easily derived they are fundamental to our main contributions.

2.1 First case: measurable velocities

Based on the results of [19] and [22] we present below a simple robustness result *vis-a-vis* the uncertainty of $g(q)$.

2.1 Proposition. *Consider the robot manipulator model (1) in closed loop with the PD control law*

$$u \;=\; -K_p\tilde{q} - K_d\dot{q} + \hat{g}(q_d). \tag{14}$$

Let $k_{p_m} > k_g$, then there exists a unique equilibrium point $(\dot{q}, q) = (0, q_s)$ for the closed loop system. The point $(\dot{q}, q) = (0, q_s)$ is globally asymptotically stable for (1), (14) and the steady state error $\tilde{q}_s \triangleq q_s - q_d$ satisfies

$$\|\tilde{q}_s\| \le \frac{2k_v}{k_{p_m}}. \tag{15}$$

□

Proof. The closed loop equation (1), (14) is given by

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - \hat{g}(q_d) + K_p\tilde{q} + K_d\dot{q} = 0. \tag{16}$$

System (16) is a Lagrangian system with potential energy

$$U_1(q) \triangleq U_g(q) - \hat{U}_g(q_d) - \tilde{q}^\top\hat{g}(q_d) + \frac{1}{2}\tilde{q}^\top K_p\tilde{q}.$$

It is well known that (16) has its equilibria at the minima of $U_1(q)$. To evaluate these equilibria we calculate the critical points of $U_1(q)$ say, all points $q = q_s$ satisfying

$$\frac{\partial U_1}{\partial q}(q_s) = 0 \;\Leftrightarrow\; K_p(q_s - q_d) + g(q_s) - \hat{g}(q_d) = 0, \tag{17}$$

moreover the equilibrium $q = q_s$ is global and unique if $k_{p_m} > k_g$ where k_g satisfies (3). The global asymptotic stability result is then established with help of the Lyapunov function candidate

$$V_1(q, \dot{q}) = \frac{1}{2}\dot{q}^\top D(q)\dot{q} + U_1(q) \tag{18}$$

which corresponds to the total energy of the closed loop system (16), hence it is positive definite and moreover has a global and unique minimum at $(\dot{q}, q) = (0, q_s)$ if $k_{p_m} > k_g$. The time derivative of V_1 along the trajectories of (16) is

$$\dot{V}_1(\dot{q}) = -\dot{q}^\top K_d \dot{q}.$$

Global asymptotic stability of the equilibrium $(\dot{q}, q) = (0, q_s)$ immediately follows using Krasovskii-LaSalle's invariance principle. Finally the bound for the steady state error defined in (15) is easily derived from (17) using the triangle inequality and the conditions on k_v given by (7), (4). ■

However, as it is well known the steady state error $\tilde{q}_s$ can be eliminated by the use of an integrator, this result was firstly proved in [3]. Reformulating (for further analysis) the original contribution of [3] we have the following

2.2 Proposition. *Consider the dynamic model (1) in closed loop with the PID control law*

$$u = -K_p\tilde{q} - K_d\dot{q} + \nu \tag{19}$$

$$\dot{\nu} = -K_i\tilde{q}, \qquad \nu(0) = \nu_0 \in I\!R^n. \tag{20}$$

where K_p, K_d, and K_i are diagonal positive definite matrices. If K_p is sufficiently large then the closed loop is locally *asymptotically stable at the origin $x \triangleq col[\tilde{q}, \dot{q}, \tilde{\nu}] = 0$.* □

Proof. Choose any positive definite diagonal matrix K'_p and let

$$K_p \triangleq K'_p + \frac{1}{\varepsilon}K_i \tag{21}$$

where $\varepsilon > 0$ is a (small) constant to be determined, clearly K_p is also positive definite and diagonal for any $\varepsilon > 0$. Then the error equation (1), (19), (20) can be written as

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - g(q_d) + K'_p\tilde{q} + K_d\dot{q} = -\frac{1}{\varepsilon}K_i\tilde{q} + \tilde{\nu} \tag{22}$$

$$\dot{\tilde{\nu}} = -K_i\tilde{q} \tag{23}$$

where we have defined $\tilde{\nu} \triangleq \nu - g(q_d)$ in order to compact the notation. A simple inspection shows that the unique equilibrium of the system (22), (23) is $\tilde{q} = 0$, $\tilde{\nu} = 0$ and $\dot{q} = 0$. Now we proceed to analyze the stability of the closed loop system, for this we use the Lyapunov function candidate

$$V_2(\tilde{q}, \dot{q}, \tilde{\nu}) = \frac{1}{2}\dot{q}^\top D\dot{q} + U_g - U_{g_d} - \tilde{q}^\top g_d + \frac{1}{2}\tilde{q}^\top K'_p\tilde{q} + \frac{\varepsilon}{2}(-\frac{1}{\varepsilon}K_i\tilde{q} + \tilde{\nu})^\top K_i^{-1}(-\frac{1}{\varepsilon}K_i\tilde{q} + \tilde{\nu}) + \varepsilon\tilde{q}^\top D\dot{q} \tag{24}$$

where we have dropped the arguments and defined $U_{g_d} \triangleq U_g(q_d)$, $g_d \triangleq g(q_d)$ to simplify the notation. It is worth mentioning at this point that, Lyapunov candidate functions with cross terms as V_2 have been widely used in the literature starting probably with [12] (see also [23, 11, 2, 15] and references therein).

We find it convenient to this point to split the kinetic, and part of the potential energy terms as

$$\begin{aligned}\tilde{q}^\top K_p' \tilde{q} &= (\lambda_1 + \lambda_2 + \lambda_3)\tilde{q}^\top K_p' \tilde{q} \\ \dot{q}^\top D(q)\dot{q} &= (\lambda_1 + \lambda_2 + \lambda_3)\dot{q}^\top D(q)\dot{q}\end{aligned}$$

with $1 > \lambda_i > 0$, $i = 1, 2, 3$. Then one can show that if

$$k'_{p_m} \geq \max\left\{\frac{k_g}{\lambda_1}, \frac{\varepsilon^2 d_M}{\lambda_1 \lambda_2}\right\}, \tag{25}$$

then the function $V_2(q, \dot{q}, \tilde{\nu})$ satisfies the lowerbound:

$$V_2(\tilde{q}, \dot{q}, \tilde{\nu}) \geq \frac{\lambda_3}{2}\tilde{q}^\top K_p' \tilde{q} + \frac{\lambda_2 + \lambda_3}{2}\dot{q}^\top D \dot{q} \tag{26}$$

hence it is positive definite and radially unbounded. The motivation for this partitioning of the energy terms will become more evident in the sequel. Next, using the well known bounds (6) and

$$\|g(q) - g(q_d)\| \leq k_g \|\tilde{q}\| \tag{27}$$

we obtain that the time derivative of $V_2(q, \dot{q}, \tilde{\nu})$ along the trajectories of (22), (23) is bounded by

$$\dot{V}_2(\tilde{q}, \dot{q}) \leq -\left(k_{d_m} - \frac{\varepsilon}{2}k_{d_M} - \varepsilon k_c \|\tilde{q}\| - \varepsilon d_M\right)\|\dot{q}\|^2 - \varepsilon\left(k'_{p_m} - k_g - \frac{1}{2}k_{d_M}\right)\|\tilde{q}\|^2 \tag{28}$$

which is negative semidefinite for instance if

$$k_{d_m} > \varepsilon(k_{d_M} + 2d_M) \tag{29}$$
$$k'_{p_m} > k_g + \frac{1}{2}k_{d_M} \tag{30}$$
$$\|\tilde{q}\| \leq \frac{k_{d_m}}{2\varepsilon k_c}. \tag{31}$$

Local asymptotic stability of the origin $x = 0$ follows using Krasovskii-LaSalle's invariance principle. Furthermore one can define a domain of attraction for the closed loop system (22), (23) as follows. Define the level set

$$B_\delta \stackrel{\triangle}{=} \left\{x \in \mathbb{R}^{3n} : V_2(x) \leq \delta\right\} \tag{32}$$

where δ is the largest positive constant such that $\dot{V}_2(x) \leq 0$ for all $x \in B_\delta$. Since V_2 is radially unbounded and positive definite, and $\dot{V}_2(x) \leq 0$ for all $x \in B_\delta$, this level set is positive invariant (i.e. if $x(0) \in B_\delta$ then $x(t) \in B_\delta$ for all $t \geq 0$) and qualifies as a domain of attraction for x. ■

2.2 Second case: *un*measurable velocities

In this section we briefly present some similar results to those contained in Propositions 2.1 and 2.2 for the case when only position feedback is available. First, based on the results of Kelly [11], consider the following

2.3 Proposition. *Consider the dynamic model (1) in closed loop with the PD control law*

$$u = -K_p\tilde{q} - K_d\vartheta + \hat{g}(q_d) \tag{33}$$
$$\dot{q}_c = -A(q_c + Bq) \tag{34}$$
$$\vartheta = q_c + Bq \tag{35}$$

where A, B, K_d and K_p are diagonal positive definite matrices. Then, if $k_{p_m} > k_g$, the equilibrium point $(\dot{q},\, \vartheta,\, q) = (0,\, 0,\, q_s)$ where q_s satisfies (15), of the closed loop system is globally asymptotically stable. □

Proof. The proof can be easily given along the lines of the proof of Proposition 2.1. First we write the error equation (1), (33)-(35) as

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - \hat{g}(q_d) + K_p\tilde{q} + K_d\vartheta = 0 \tag{36}$$
$$\dot{\vartheta} = -A\vartheta + Bq. \tag{37}$$

Then, consider the Lyapunov function candidate

$$V_3(\dot{q},\vartheta,q) = V_1(\dot{q},q) + \frac{1}{2}\vartheta^\top K_d B^{-1}\vartheta \tag{38}$$

which is positive definite and radially unbounded with a global and unique minimum at $\mathrm{col}[\dot{q},\, \vartheta,\, q] = \mathrm{col}[0,\, 0,\, q_s]$ if $k_{p_m} > k_g$. Its time derivative along the trajectories of (36), (37) is

$$\dot{V}_3(\vartheta) \leq -\vartheta^\top K_d B^{-1} A\vartheta,$$

then global asymptotic stability follows by invoking Krasovskii-LaSalle's invariance principle and using standard arguments. ■

The proposition above ensures the global asymptotic convergence of $\tilde{q} \to \tilde{q}_s$ as $t \to \infty$ where $\tilde{q}_s$ satisfies (15). However, as in the case of measurable velocities, one can eliminate the steady state error by using PID control. More precisely, in [15] we introduced the PI^{2}D controller which establishes semi-global asymptotic stability with uncertain gravity knowledge. For simplicity and for the purposes of this paper we formulate below a proposition which follows as a corollary of the main result contained in [15] (see also [7]). The result below guarantees *local* asymptotic stability.

2.4 Proposition. *Consider the robot model (1) in closed loop with the PI^{2}D control law*

$$\left\{\begin{array}{rcl} u &=& -K_p\tilde{q} - K_d\vartheta + \nu \\ \dot{\nu} &=& -K_i(\tilde{q} - \vartheta), \qquad \nu(0) = \nu_0 \in \mathbb{R}^n \\ \dot{q}_c &=& -A(q_c + Bq) \\ \vartheta &=& q_c + Bq. \end{array}\right. \tag{39}$$

Let K_p, K_i, K_d, A and B be positive definite diagonal matrices where B is such that $BD(q) = D(q)B > 0$. Under these conditions, we can always find a sufficiently large proportional gain K_p (or sufficiently small K_i) such that the equilibrium $\xi \stackrel{\triangle}{=} col[\tilde{q},\ \dot{q},\ \vartheta,\ \tilde{\nu}] = 0$ is locally *asymptotically stable.* □

Proof. Below we give an outline of the proof proposed in [15], which we will use in the sequel for our main results. First, the error equation (1), (39) can be written as

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) - g(q_d) + K_p'\tilde{q} + K_d\vartheta = \tilde{\nu} - \frac{1}{\varepsilon}K_i\tilde{q} \tag{40}$$

$$\dot{\tilde{\nu}} = -K_i(\tilde{q} - \vartheta) \tag{41}$$

$$\dot{\vartheta} = -A\vartheta + B\dot{q} \tag{42}$$

where K_p' is defined by (21).

From [15] we know that the Lyapunov function candidate

$$V_4(\tilde{q},\dot{q},\vartheta) = V_2(\tilde{q},\dot{q},\tilde{\nu}) + \frac{1}{2}\vartheta^\top K_d B^{-1}\vartheta - \varepsilon\vartheta^\top D(q)\dot{q}$$

is positive definite and radially unbounded with a global and unique minimum at the origin if ε is sufficiently small. For the sake of completeness we rewrite the conditions derived in [15] with a slight modification convenient for the purpose of this paper. Let us partition the term $\vartheta^\top K_d B^{-1}\vartheta = (\mu_1 + \mu_2)\vartheta^\top K_d B^{-1}\vartheta$ where $0 < \mu_1 + \mu_2 \leq 1$, $\mu_i > 0$ with $i = 1, 2$. With these definitions, one can prove that if (25) holds and

$$\varepsilon < \left(\frac{2k_{d_m}\lambda_2\mu_2}{d_M b_M}\right)^{1/2} \tag{43}$$

then $V_4(\tilde{q},\dot{q},\vartheta)$ satisfies the bound

$$V_4(\tilde{q},\dot{q},\vartheta) \geq \frac{\lambda_3}{2}\dot{q}^\top D\dot{q} + \frac{\lambda_3}{2}\tilde{q}^\top K_p'\tilde{q} + \frac{\mu_1}{2}\vartheta^\top K_d B^{-1}\vartheta. \tag{44}$$

Furthermore, it has also been shown in [15] that if the position error $\tilde{q}$ and the filter output ϑ satisfy

$$\|\vartheta\| + \|\tilde{q}\| \leq \frac{b_m d_m}{2k_c} \tag{45}$$

and if $\varepsilon > 0$ is sufficiently small to satisfy

$$\varepsilon < \min\left\{\frac{(k_{p_m}' - k_g)k_{d_m}a_m}{2b_M\left[k_{p_M}' + k_{d_M} + k_g\right]^2},\ \frac{k_{d_m}a_m d_m}{2[a_m d_M]^2},\ \frac{k_{d_m}a_m}{2b_M k_{d_M}}\right\} \tag{46}$$

then there exist strictly positive constants β_1, β_2, and β_3 such that the time derivative of V_4 along the closed loop trajectories (40), (41) is bounded by

$$\dot{V}_4(q,\dot{q},\vartheta,\tilde{\nu}) \leq -\beta_1\|\tilde{q}\|^2 - \beta_2\|\dot{q}\|^2 - \beta_3\|\vartheta\|^2. \tag{47}$$

Since V_4 is positive definite and $\dot{V}_4$ is locally negative semidefinite, *local* asymptotic stability of $\xi = 0$ can be proven by invoking Krasovskii-LaSalle's invariance principle and using standard arguments. Furthermore a domain of attraction for system (40), (42) with state

$\xi = \operatorname{col}[\tilde{q},\ \dot{q},\ \vartheta,\ \tilde{\nu}]$ can be defined similarly as in the proof of Proposition 2.2 as the level set

$$B_\rho \stackrel{\triangle}{=} \{\xi \in \mathbb{R}^{4n} : V_4(\xi) \le \rho\} \tag{48}$$

where ρ is the largest positive constant such that $\dot{V}_4(\xi) \le 0$ for all $\xi \in B_\rho$. The proof finishes using similar arguments as in the proof of Proposition 2.2. ■

We are ready now to present our main results: global asymptotic stability with PID and PI^{2}D control.

3 Main results

In this section we present our main results, which leans on the results derived in Propositions 2.1 - 2.4 and the composite control approach proposed in [13]. We show that one can achieve *global* asymptotic stability with PID and PI^{2}D control by simply *delaying* the integral action.

3.1 First case: measurable velocities

3.1 Proposition. *Consider the robot manipulator model (1) in closed loop with the PI$_d$D control law*

$$u = -K_p\tilde{q} - K_d\dot{q} + \nu(t) \tag{49}$$

$$\nu(t) = \begin{cases} \hat{g}(q_d), & \forall \quad 0 \le t \le t_s \\ \\ -K_i \int_{t_s}^{t} \tilde{q}(s)ds + \hat{g}(q_d) & \forall \quad t \ge t_s \end{cases} \tag{50}$$

where K_p, K_d, and K_i are diagonal positive definite matrices. There always exist a finite *time instant $t_s \ge 0$, a sufficiently large proportional gain K_p and/or a sufficiently small integral gain K_i, such that the closed loop system is* globally *asymptotically stable at the origin $x \stackrel{\triangle}{=} col[\dot{q}, \tilde{q}, \tilde{\nu}] = 0$.* □

Roughly speaking, in its first phase (that is $0 \le t \le t_s$), the delayed PID of Proposition 3.1 collapses to the robust controller of Proposition 2.1 which guarantees global *asymptotic* stability of a different equilibrium than desired but it also guarantees that the *steady* position error is confined to the closed ball of radius determined by (15). In its second phase (that is, for all $t \ge t_s$), the delayed PID collapses to the "conventional" PID controller of Proposition 2.2 with initial conditions $x_0 = x(t_s)$. From the proof of Proposition 2.2 we know that if the initial conditions x_0 are small enough then the trajectories $x(t)$ are asymptotically stable (hence $q(t) \to q_d$ as $t \to \infty$). Thus the main difficulty in the proof of Proposition 3.1 is to show that there exist suitable gains K_p and K_i such that the bounded set of convergence defined for the first phase is contained in the domain of attraction defined for the second phase. The latter implies that, before the integral action is incorporated (i.e. for all $t \le t_s$), the delayed PID drives the generalized positions and velocities into the domain of attraction B_δ in finite time. From this the existence of a finite start-integration time t_s to guarantee GAS, follows. We prove below that this is the case.

3.2 Remark. It is important to remark that even though the (delayed) PID controller proposed above is not smooth, it is continuous. This depends of course on the correct setting of the initial conditions of the integrator, that is $\nu(t_s) = \hat{g}(q_d)$.

Proof of Proposition 3.1. From Proposition 2.1 it follows that during the first phase of the delayed PID, $(\tilde{q},\, \dot{q},\, \tilde{\nu}) \to (\tilde{q}_s,\, 0,\, \hat{g}(q_d) - g(q_d))$ as $t \to \infty$. Furthermore, $\tilde{q}_s$ satisfies the upperbound (15). Define the set

$$\Gamma \triangleq \left\{ x \in \mathbb{R}^{3n} \;:\; \|\tilde{q}\| \le \frac{2k_v}{k_{p_m}},\;\; \dot{q} = 0,\;\; \|\tilde{\nu}\| \le 2k_v \right\},$$

from the discussion above, we must find a constant δ large enough so that $\Gamma \subset B_\delta$ where B_δ is defined in (32), henceforth a suitable time moment t_s to guarantee GAS of the closed loop. Notice that in order to give an explicit value to δ in terms of the control gains, $V_2(x)$ is needed, however the potential energy term $U_g(q)$ is not known explicitly. Therefore, let us define

$$V_{2M}(\tilde{q}, \dot{q}, \tilde{\nu}) = \frac{1}{2}\dot{q}^\top D \dot{q} + \frac{1}{2}(k_{p_M} + k_g)\,\|\tilde{q}\|^2 + \|\tilde{\nu}\|\,\|\tilde{q}\| + \frac{\varepsilon}{2k_{i_m}}\,\|\tilde{\nu}\|^2 + \varepsilon \tilde{q}^\top D \dot{q} \tag{51}$$

and the level set

$$B_\delta^M \triangleq \left\{ x \in \mathbb{R}^{3n} \;:\; V_{2M}(x) \le \delta \right\}.$$

It is not difficult to see that $V_{2M}(x) \ge V_2(x)$ hence $B_\delta^M \subset B_\delta$. Now we look for a δ such that $\Gamma \subset B_\delta^M \subset B_\delta$, it suffices that the four corners of the "plane" Γ be contained in B_δ^M hence, using (51) and (15) it is sufficient that

$$\delta > \frac{1}{2}(k_{p_M} + k_g)\left(\frac{2k_v}{k_{p_m}}\right)^2 + \frac{4k_v^2}{k_{p_m}} + \frac{2\varepsilon k_v^2}{k_{i_m}}. \tag{52}$$

In words, the lower-bound on δ given above, ensures that the delayed PID controller in its first phase will drive the trajectories into the domain of attraction B_δ in finite time. The second requirement on δ is that $\dot{V}_2$ be negative semi-definite for all $x \in B_\delta$, hence we proceed to calculate an upperbound for δ so that $\dot{V}_2(B_\delta) \le 0$.

From the proof of Proposition 2.2 (see (26)) we know that (25) implies that $V_2(x) \ge V_{2m}(x)$ where we defined

$$V_{2m}(x) \triangleq \frac{\lambda_3}{2} k'_{p_m}\,\|\tilde{q}\|^2,$$

Define the set $B_\delta^m \triangleq \left\{ x \in \mathbb{R}^{3n} \;:\; V_{2m}(x) \le \delta \right\}$. With these definitions we have that $B_\delta \subset B_\delta^m$ hence it suffices to prove that $\dot{V}_2(B_\delta^m) \le 0$. Notice that among the three sufficient conditions (29)-(31) to ensure $\dot{V}_2(x) \le 0$, the only one which affects the definition of the domain of attraction (hence of δ) is (31) thus, it should hold true that

$$\frac{2\delta}{\lambda_3 k'_{p_m}} < \frac{k_{d_m}^2}{4\varepsilon^2 k_c^2}. \tag{53}$$

In summary, recalling (52) it is sufficient that δ satisfies

$$\frac{1}{2}(k_{p_M} + k_g)\left(\frac{2k_v}{k_{p_m}}\right)^2 + \frac{4k_v^2}{k_{p_m}} + \frac{2\varepsilon k_v^2}{k_{i_m}} < \delta < \frac{\lambda_3 k'_{p_m} k_{d_m}^2}{8\varepsilon^2 k_c^2} \tag{54}$$

to ensure that the trajectories $x(t)$ converge to the domain of attraction B_δ in finite time. Finally, to ensure global asymptotic stability of the origin it suffices to choose the time t_s as the first time moment when the "initial conditions" $x(t_s) \in B_\delta$ that is, $t_s : V_2(x(t_s)) \le \delta$ however, since $V_2(x)$ is not accurately known consider the function

$$\bar{V}_{2M}(x) \triangleq \frac{1}{2}\dot{q}^\top D\dot{q} + \frac{1}{2}(k_{p_M} + k_g)\|\tilde{q}\|^2 + 2k_v\left(\|\tilde{q}\| + \frac{\varepsilon k_v}{k_{i_M}}\right) + \varepsilon\tilde{q}^\top D\dot{q}.$$

To this point, we recall that for all $t \le t_s$ the PI$_d$D controller is a robust PD with $\tilde{\nu} = \hat{g}(q_d) - g(q_d) =$ constant, hence $\|\tilde{\nu}\| \le 2k_v$. From this it follows that $\bar{V}_{2M}(x(t)) \ge V_{2M}(x(t))$ for all $t \le t_s$. Thus, the proof is completed by defining the start-integration time as

$$t_s \ : \ \bar{V}_{2M}(x(t_s)) \le \delta \tag{55}$$

and noticing that (54) holds for sufficiently small ε, hence due to (21) for sufficiently large k_{p_m} and/or sufficiently small $k_{i_M} < \varepsilon$.

■

Summarizing all conditions, we draw the following corollary from the proofs of Propositions 2.2 and 3.1. This gives an insight to the practitioner on how to choose the control gains and the switching time t_s to guarantee GAS of the origin.

3.3 Corollary. *Consider the dynamic model (1) in closed loop with the PI$_d$D control law (49), (50). Let K_p, K_d, and K_i be diagonal positive definite matrices, satisfying*

$$k_{d_m} \ > \ \varepsilon(k_{d_M} + 2d_M) \tag{56}$$

$$k'_{p_m} \ > \ \max\left\{\frac{k_g}{\lambda_1}, \frac{\varepsilon^2 d_M}{\lambda_2\lambda_1}, k_g + \frac{k_{d_M}}{2}\right\}, \tag{57}$$

and (54). Define the start-integration time t_s as in (55). Under these conditions, the closed loop system is globally *asymptotically stable at the origin.*

The first two parts of condition (57) ensure that $V_2(x) \ge V_{2m}(x)$, $V_2(x)$ is positive definite and radially unbounded. Then, condition (56) and the third part of condition (57) imply that $\dot{V}_2(x)$ is negative semi-definite. Thus all *sufficient* conditions derived in the previous proofs have been collected in the corollary above. In order to satisfy them one may proceed as follows:

1. Pick any ε and λ_i in the interval $(0,1)$ satisfying $\lambda_1+\lambda_2+\lambda_3 \le 1$ and any proportional and derivative gains K_p and K_d satisfying (56) and (57).
2. Pick any "small" integral gain K_i and check whether there is a δ satisfying (54). If not, then pick a smaller ε hence either larger k_{p_m} or smaller k_{i_M} according with (21).
3. Repeat steps 1 and 2 until all conditions are satisfied. Finally, define t_s as in (55).

Thus all conditions can be easily verified and the controller gains can be computed for *any* initial conditions $x(0)$.

3.2 Second case: *un*measurable velocities

3.4 Proposition. *Consider the robot model (1) in closed loop with the PI^2_dD control law*

$$\begin{aligned} u &= -K_p\tilde{q} - K_d\vartheta + \nu(t) &(58)\\ \dot{q}_c &= -A(q_c + Bq) &(59)\\ \vartheta &= q_c + Bq &(60)\\ \nu &= \begin{cases} \hat{g}(q_d), & \forall \quad 0 \le t \le t_s \\ -K_i \int_{t_s}^{t}(\tilde{q}(s) - \vartheta(s))ds + \hat{g}(q_d) & \forall \quad t \ge t_s. \end{cases} &(61) \end{aligned}$$

Let K_p, K_i, K_d, A and B be positive definite diagonal matrices where B is such that $BD(q) = D(q)B > 0$.Under these conditions, we can always find a finite time instant $t_s \ge 0$, sufficiently large gains K_p, B and/or a sufficiently small integral gain K_i such that the closed loop system is globally *asymptotically stable at the origin $\xi \triangleq$ col$[\dot{q}, \tilde{q}, \vartheta, \tilde{\nu}] = 0$.* □

Proof. The proof follows along the lines of the proof of Proposition 3.1, based on the results obtained in Propositions 2.3 and 2.4. We start by defining the set

$$\Gamma' \triangleq \left\{\xi \in \mathbb{R}^{4n} \ : \ \|\tilde{q}\| \le \frac{2k_v}{k_{p_m}}, \ \dot{q} = \vartheta = 0, \ \|\tilde{\nu}\| \le 2k_v\right\},$$

and denoting the level set

$$B_\rho^M \triangleq \left\{\xi \in \mathbb{R}^{4n} \ : \ V_{4M}(\xi) \le \rho\right\}.$$

where

$$V_{4M}(\xi) \triangleq V_{2M}(x) + \frac{1}{2}\vartheta^\top K_d B^{-1}\vartheta - \varepsilon\vartheta^\top D\dot{q}.$$

Notice from the proof of Proposition 2.4 that $V_{4M}(\xi) \ge V_4(\xi)$, hence $B_\rho^M \subset B_\rho$. Notice also that $V_{4M}(\Gamma') = V_{2M}(\Gamma)$ hence $\Gamma' \subset B_\rho^M$ if ρ satisfies a similar bound as (52). We only need to define an upperbound for ρ which ensures that $\dot{V}_4(B_\rho) \le 0$. Let

$$V_{4m}(\xi) \triangleq V_{2m}(x) + \frac{\mu_1 k_{d_m}}{2b_M}\|\vartheta\|^2 \tag{62}$$

and from (44) we have that $V_4(\xi) \ge V_{4m}(\xi)$ if condition (43) and (25) hold. Consider next the condition established by inequality (45), then analogously to (53) we have that

$$\max\left\{\left(\frac{2\rho}{\lambda_3 k'_{p_m}}\right), \left(\frac{2\rho b_M}{\mu_1 k_{d_m}}\right)\right\} < \frac{b_m^2 d_m^2}{16k_c^2}$$

and (46) imply that $\dot{V}_4(\xi) \le 0$ for all ξ such that $V_{4m}(\xi) \le \rho$, hence also for all $\xi \in B_\rho$. In summary, it is sufficient that ρ satisfies

$$\frac{1}{2}(k_{p_M} + k_g)\left(\frac{2k_v}{k_{p_m}}\right)^2 + \frac{4k_v^2}{k_{p_m}} + \frac{2\varepsilon k_v^2}{k_{i_m}} < \rho < \frac{b_m^2 d_m^2}{16k_c^2}\min\left\{\left(\frac{\lambda_3 k'_{p_m}}{2}\right), \left(\frac{\mu_1 k_{d_m}}{2b_M}\right)\right\}, \tag{63}$$

to ensure that the delayed PI^{2}D controller in its first phase drives the trajectories $\xi(t)$ into the domain of attraction defined for the second phase. Hence there exists a finite $t_s \ge 0$

ensuring GAS of the origin $\xi = 0$. As in the proof of Proposition 3.1, considering that for all $t \leq t_s$, the gravity compensation error $\|\tilde{\nu}\|$ is a *constant* bounded by $2k_v$, the instant t_s can be chosen as

$$t_s \,:\, \bar{V}_{4M}(x(t_s)) \leq \rho \tag{64}$$

where

$$\bar{V}_{4M}(x) \triangleq \frac{1}{2}\dot{q}^\top D\dot{q} + \frac{1}{2}(k_{p_M} + k_g)\,\|\tilde{q}\|^2 + 2k_v\left(\|\tilde{q}\| + \frac{\varepsilon k_v}{k_{i_m}}\right) + \varepsilon\tilde{q}^\top D\dot{q} + \frac{1}{2}\vartheta^\top K_d B^{-1}\vartheta - \varepsilon\vartheta^\top D\dot{q}.$$

The proof finishes noticing that (63) holds for sufficiently large b_m and sufficiently small ε, hence due to (21) for sufficiently large k_{p_m} and/or sufficiently small $k_{i_M} < \varepsilon$. ■

3.5 Remark. Notice from (64) that the switching time t_s does depend indeed on the *unmeasurable* velocities $\dot{q}(t_s)$. Hence, the precise *theoretical* result which is contained in Proposition 3.4 is that "there exists a start-integration time t_s such that the origin $\xi = 0$ is GAS". For practical purposes however, observe that the velocity measurements are *not* used in the controller equations (58) – (61). As it can be seen from the proof above, in practice the start-integration time t_s can be computed with knowledge of the best estimate available of the velocity measurement at a *precise* instant. For instance any t_s such that $\bar{V}_{4M}(x(t_s)) < \rho$ where we redefined

$$\bar{V}_{4M}(x(t_s)) \triangleq \frac{1}{2}\dot{\hat{q}}(t_s)^\top D\dot{\hat{q}}(t_s) + \frac{1}{2}(k_{p_M} + k_g)\,\|\tilde{q}(t_s)\|^2 + 2k_v\left(\|\tilde{q}(t_s)\| + \frac{\varepsilon k_v}{k_{i_m}}\right) + \varepsilon\tilde{q}(t_s)^\top D\dot{\hat{q}}(t_s). \tag{65}$$

and $\dot{\hat{q}}(t_s)$ is the best estimate available of $\dot{q}(t_s)$, that is, at the *precise* instant t_s. Such estimate can be computed for instance from the last two position measurements prior to the moment t_s.

We finally draw the following Corollary from Propositions 2.4 and 3.4.

3.6 Corollary. *Consider the dynamic model (1) in closed loop with the PI^2_dD control law (58)–(61). Let K_p, K_d, and K_i be diagonal positive definite matrices with K_p defined by (21), satisfying (43), (46), and (63). There exists time instant t_s (for instance given by (64)) such that the closed loop system is* globally *asymptotically stable at the origin $\xi = col[\tilde{q}, \dot{q}, \vartheta, \tilde{\nu}] = 0$.*

For practical applications however one may choose the start-integration time t_s according to (65). It is important to remark that the semiglobal stability results reported in [15] and [7] have the same practical drawback: the initial (*unmeasurable*) velocity must be known.

4 Discussion

As it is clear now from the proof of Proposition 2.2, what impedes claiming global asymptotic stability for a PID controller is the presence of the cubic term $\varepsilon k_c\,\|\tilde{q}\|\,\|\dot{q}\|^2$ in the Lyapunov function derivative $\dot{V}_2$. As mentioned in the introduction, this technical difficulty can be overcome by making some "smart" modifications to the PID control law, leading for instance, to the design of *nonlinear* PID controllers.

To the best of our knowledge, the first non-linear PID controllers appeared in the literature are [9] and [2]. In this section we discuss these controllers and show that seemingly these results cannot so easily be extended to the case of unmeasurable velocities.

4.1 The normalized PID of Kelly [9]

In order to cope with the cubic term $\varepsilon k_c \|\tilde{q}\| \|\dot{q}\|^2$ in (28), Kelly [9] has proposed the "adaptive" PD controller

$$u = -K_p'\tilde{q} - K_d\dot{\tilde{q}} + \Phi(q_d)\hat{\theta} \tag{66}$$

together with the update law

$$\dot{\tilde{\theta}} = \dot{\hat{\theta}} = -\frac{1}{\gamma}\Phi(q_d)^\top \left[\dot{q} + \frac{\varepsilon_0\tilde{q}}{1+\|\tilde{q}\|}\right] \tag{67}$$

where $\varepsilon_0 > 0$ is a small constant. Kelly [9] proved that this "adaptive" controller in closed loop with a rigid-joint robot results in a globally convergent system. However, since the regressor vector $\Phi(q_d)$ is *constant* the update law (67), together with the control input (66) can be implemented as a *nonlinear* PID controller by integrating out the velocities vector from (67):

$$\hat{\theta} = -\frac{1}{\gamma}\Phi(q_d)^\top \left[\tilde{q} + \int_0^t \frac{\varepsilon_0\tilde{q}}{1+\|\tilde{q}\|}d\tau\right] + \hat{\theta}(0). \tag{68}$$

Notice that the choice $K_p = K_p' + K_i$, with $K_i = \frac{1}{\gamma}\Phi(q_d)\Phi(q_d)^\top$, yields the controller implementation

$$\begin{aligned} u &= -K_p\tilde{q} - K_d\dot{q} + \nu & (69)\\ \dot{\nu} &= -\varepsilon_0 K_i \frac{\tilde{q}}{1+\|\tilde{q}\|}, \qquad \nu(0) = \nu_0 \in \mathbb{R}^n. & (70) \end{aligned}$$

Since controllers (66), (68) and (69), (70) are equivalent, following the steps of Kelly [9] one can prove *global* asymptotic stability of the closed loop system (1), (69)–(70). Evaluating the time derivative of the Lyapunov function candidate

$$V_k(\tilde{q}, \dot{q}, z_k) = \frac{1}{2}\dot{q}^\top D(q)\dot{q} + U_g(q) - U(q_d) - \tilde{q}^\top g(q_d) + \frac{1}{2}\tilde{q}^\top K_p'\tilde{q} + \frac{1}{2}z_k^\top K_i^{-1} z_k + \varepsilon_k \tilde{q}^\top D(q)\dot{q}. \tag{71}$$

where we have defined

$$\varepsilon_k \triangleq \frac{\varepsilon_0}{1+\|\tilde{q}\|} \tag{72}$$

$$z_k \triangleq -K_i\tilde{q} + \tilde{\nu}, \tag{73}$$

we obtain that

$$\dot{V}_k(\tilde{q}, \dot{q}, z_k) \leq -\left(k_{d_m} - \frac{\varepsilon_0 k_c}{1+\|\tilde{q}\|}\|\tilde{q}\| - \varepsilon_0 d_M - \frac{\varepsilon_0 k_{d_M}}{2}\right)\|\dot{q}\|^2 - \frac{\varepsilon_0}{1+\|\tilde{q}\|}(k_{p_m}' - k_g - \frac{\varepsilon_0 k_{d_M}}{2})\|\tilde{q}\|^2 \tag{74}$$

however, notice that the normalized term $\varepsilon_k k_c \|\tilde{q}\| \|\dot{q}\|^2 \leq k_c \|\dot{q}\|^2$, hence $\dot{V}_k(\tilde{q}, \dot{q}, z_k)$ is negative semi-definite for sufficiently small ε_0. Global asymptotic stability follows by invoking Krasovskii-LaSalle's invariance principle.

4.2 The saturated PID controller of Arimoto [2]

An alternative trick to achieve GAS is the scheme of Arimoto [2] who proposed the nonlinear PID:

$$u = -K_p' \,\mathrm{sat}(\tilde{q}) - \frac{1}{\varepsilon}K_i\tilde{q} - K_d\dot{q} + \nu \quad (75)$$

$$\dot{\nu} = -K_i\tilde{q}, \qquad \nu(0) = \nu_0 \in \mathbb{R}^n. \quad (76)$$

where the *saturation* function $\mathrm{sat} : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\mathrm{sat}(\psi)^\top\psi > 0$ for all $\psi \neq 0$, $\mathrm{sat}(0) = 0$ and it is bounded as $\|\mathrm{sat}(\psi)\| \leq 1$. Arimoto[3] [2] proved that if $k_{p_m} > k_g$, and K_i is sufficiently small, the closed loop is *globally* asymptotically stable.

The key idea used in [2] is to dominate the cubic terms in the Lyapunov function derivative by means of the saturated proportional feedback in (75). More precisely, it can be easily proven that the time derivative of the Lyapunov function

$$V_a(\tilde{q},\dot{q},z) = \frac{1}{2}\dot{q}^\top D(q)\dot{q} + U_g(q) - U(q_d) - \tilde{q}^\top g(q_d) + \frac{1}{2}\tilde{q}^\top K_p'\tilde{q} + \frac{\varepsilon}{2}z^\top K_i^{-1}z + \varepsilon\,\mathrm{sat}(\tilde{q})^\top D(q)\dot{q} \quad (77)$$

where $\varepsilon > 0$ is a small *constant* and z is defined by

$$z \triangleq \tilde{\nu} - \frac{1}{\varepsilon}K_i\tilde{q} \quad (78)$$

(78), along the trajectories of the closed loop system (1), (75)-(76) is bounded by

$$\dot{V}_a(\tilde{q},\dot{q},z) \leq -(k_{d_m} - \varepsilon k_c\|\mathrm{sat}(\tilde{q})\| - \varepsilon d_M - \frac{\varepsilon}{2}k_{d_M})\|\dot{q}\|^2 - \varepsilon(k'_{p_m} - k_g - \frac{1}{2}k_{d_M})\,\mathrm{sat}(\tilde{q})^2 \quad (79)$$

however, notice that the term $\varepsilon k_c\|\mathrm{sat}(\tilde{q})\|\,\|\dot{q}\|^2 \leq k_c\|\dot{q}\|^2$, hence $\dot{V}_a(\tilde{q},\dot{q},z)$ is negative semi-definite for sufficiently small ε and global asymptotic stability follows by observing that $\mathrm{sgn}(\mathrm{sat}(\tilde{q})) = \mathrm{sgn}(\tilde{q})$ and invoking Krasovskii-LaSalle's invariance principle.

Remarks.

1. Besides their complexity, a practical drawback of the nonlinear PID controllers of [2] and [9] with respect to the controller of Proposition 3.1 is that one may expect from the expressions of the Lyapunov derivatives $\dot{V}_a$ and $\dot{V}_k$, that they converge slower than our linear PI$_d$D controller. More precisely, notice that the saturation and normalization used in those approaches clearly attenuates the growth rate of $-\dot{V}_a$ and $-\dot{V}_k$ with respect to $\|\tilde{q}\|$.

2. In contrast to this, our approach guarantees global asymptotic stability with a simple *linear* PID as it is used in many practical applications. Roughly speaking the user can apply a simple robust PD controller as that of Proposition 2.1 and start the integration effect when the generalized velocities and the positions are small.

3. From a theoretical point of view, the trick of introducing cross terms in the Lyapunov function is not new [12, 23] however, the idea of using a saturated proportional feedback in (75) is due, as far as we know, to [2]. This trick in combination with the saturated cross term used in V_a were fundamental to prove GAS. The same observation is valid for the approach of Kelly where the normalization plays a crucial role.

[3] It is worth mentioning that in [2], Arimoto used a saturation function which is a particular case of sat considered here, however this point is not fundamental for the validity of the result.

4. Even though these "tricks" can be efficiently applied to bound the "position dependent" cubic terms $\varepsilon k_c \operatorname{sat}(\tilde{q})\dot{\tilde{q}}^2$, and $\varepsilon_0 \frac{k_c \tilde{q}}{1+\|\tilde{q}\|}\dot{\tilde{q}}^2$, when velocity measurements are supposed unavailable, it is not possible to apply the same approaches. To illustrate this idea, let – without loss of generality – $\operatorname{sat}(\vartheta) \stackrel{\triangle}{=} \tanh(\vartheta)$ then, if one uses the saturation *à la* Arimoto in the PI^{2}D scheme, seemingly the saturated cross term $\varepsilon \tanh(\vartheta)^\top D(q)\dot{\tilde{q}}$ should be used in the Lyapunov function candidate V_4, instead of $\varepsilon \vartheta^\top D(q)\dot{\tilde{q}}$. However, this yields the term $-\varepsilon b_m d_m \| \operatorname{sech}^2(\vartheta)\| \dot{\tilde{q}}^2$ in the Lyapunov function derivative, instead of $-\varepsilon b_m d_m \dot{\tilde{q}}^2$ and since $\| \operatorname{sech}^2(\vartheta)\|$ vanishes as $\vartheta \to \infty$ the Lyapunov function derivative will be *locally* negative semidefinite. Similar conclusions can be drawn if one tries to use the normalization used in [9]. For this reason there is not much hope to extend the approaches of [2] and [11] to the output feedback case. For its simplicity and the arguments exposed in Remark 3.5, our result of Corollary 3.6 seems more promising for practical applications.

5 Simulation results

To illustrate the working of the controllers derived in this paper, simulations have been carried out using MATLAB™. We compared the delayed PID controller derived in section 3.1 with the normalized PID of Kelly [9] and the saturated PID controller of Arimoto [2]. For our simulations we used the model presented in [4], where

$$
\begin{aligned}
D(q) &= \begin{bmatrix} 8.77 + 1.02\cos q_2 & 0.76 + 0.51\cos q_2 \\ 0.76 + 0.51\cos q_2 & 0.62 \end{bmatrix} \\
C(q,\dot{q}) &= 0.51 \sin q_2 \begin{bmatrix} -\dot{q}_2 & -(\dot{q}_1 + \dot{q}_2) \\ \dot{q}_1 & 0 \end{bmatrix} \\
g(q) &= 9.81 \begin{bmatrix} 7.6\sin q_1 + 0.63\sin(q_1+q_2) \\ 0.63\sin(q_1+q_2) \end{bmatrix}
\end{aligned}
$$

For this system we have $d_m = 0.45$, $d_M = 9.96$, $k_c = 1.53$, $k_v = 80.7$, $k_g = 81.2$. We assume to have no better estimate of the gravitational forces vector than $\hat{g}(q) = [0,0]^\top$.

We considered the problem of controlling the manipulator from the position $[2,0]^\top$ towards $[1,1]^\top$. For this we used $K_p = 240I$, $K_d = 75I$, $K_i = 150I$, where $K_p' = 120I$. From (21) it follows that $\epsilon = 1.25$. From (25) we see that $V_2 \geq 0$ and by choosing $\delta = 290$ we meet (52) and are guaranteed to enter the set $\bar{B}_\delta^M \stackrel{\triangle}{=} \{\xi \in \mathbb{R}^{4n} : \bar{V}_{2M}(x) \leq \delta\}$ and therefore the existence of t_s as defined in (55) is also guaranteed.

By choosing $\lambda_1 = 0.7$, $\lambda_2 = 0.2$ and $\lambda_3 = 0.1$ we see from (25) that

$$V_2(\tilde{q},\dot{q},\tilde{\nu}) \geq 6\|\tilde{q}\|^2$$

so that from $V_2(\tilde{q},\dot{q},\tilde{\nu}) \leq \delta$ we can conclude that $\|\tilde{q}\| < 7$, which results into

$$\dot{V}_2 < -2.2\|\dot{q}\|^2 - 1.6\|\tilde{q}\|^2.$$

Therefore, if we start integrating as soon as we enter $\bar{B}_\delta^M$ we have asymptotic stability of the second phase and global asymptotic stability of the PI$_d$D controller.

From section 4 we know that our selection of gains also guarantees global asymptotic stability of the normalized PID of Kelly [9] (we use $\epsilon_0 = 1$) and the saturated PID controller of Arimoto [2].

The resulting performance is depicted in Figure 1.

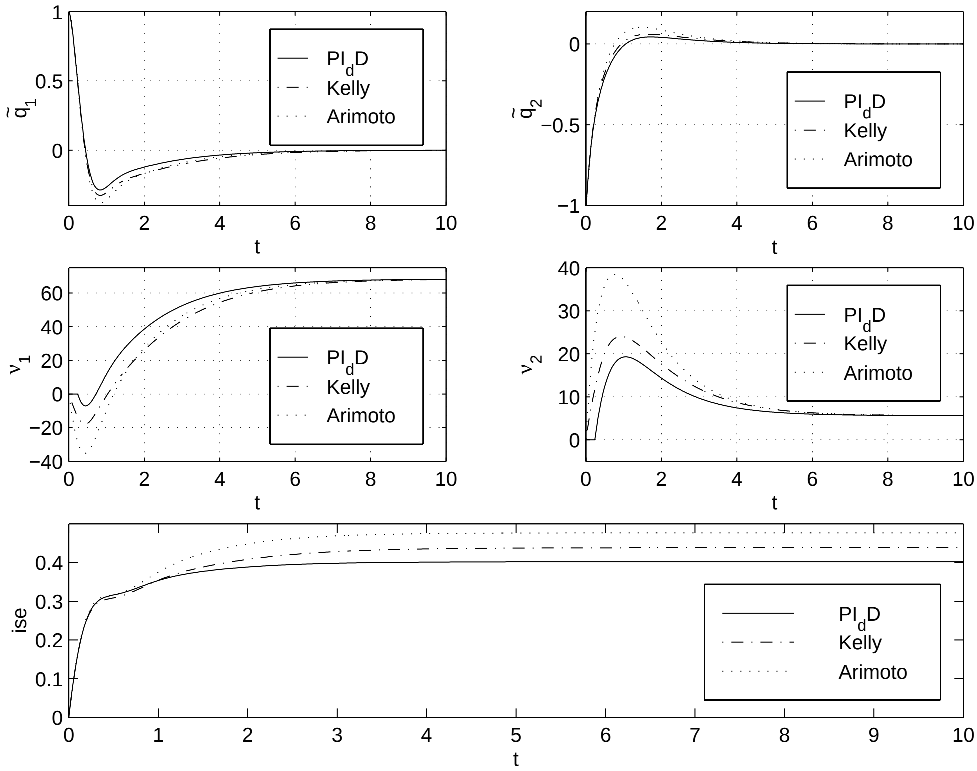

Figure 1: A comparitive study

We see that the partially saturated proportional term leads to a larger overshoot for Arimoto's controller [2], whereas the saturation in Kelly's controller [9] leads to a slower convergence of ν to $g(q_d)$. We can also see the delayed integration (starting at $t_s = 0.2533$) of the PI$_d$D controller.

To make not only a qualitative but also a quantitative comparison between the three controllers, we looked at the expression

$$ise(t) \triangleq \int_0^t \tilde{q}(s)^\top \tilde{q}(s) ds$$

Then we see that Arimoto's controller [2] all the time has the largest ise due to the partially saturated proportional term. We also see that during the first second, the ise of Kelly's controller [9] is a little bit lower than the ise of our delayed PID controller, however, due to the saturation in the integral part the final convergence of Kelly's controller is slower, resulting into a larger ise.

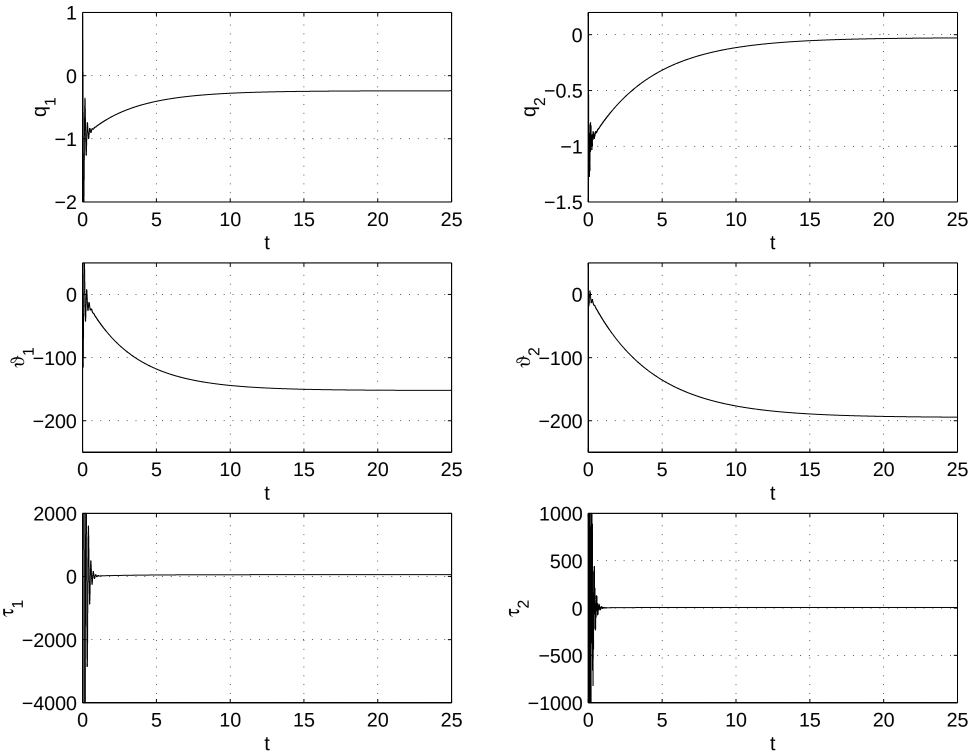

Figure 2: Delayed PI^{2}D controller (first 25 seconds)

In case of unmeasurable velocities we also consider the problem of controlling the manipulator from the position $[2, 0]^\top$ towards $[1, 1]^\top$. For this we used $K_p = 240I$, $K_d = 75I$, $K_i = 2I$, $A = 15I$, $B = 200I$. The smaller value for K_i in comparison with the state-feedback case is due to the more restrictive inequalities. By choosing $K_p' = 100I$ (which results into $\epsilon = 0.0143$) we have that $V_4 \geq 0$, and using $\lambda_1 = 0.82$, $\lambda_2 = 0.08$, $\lambda_3 = 0.10$, $\mu_1 = 0.95$, $\mu_2 = 0.05$, (62) becomes

$$V_{4m}(\tilde{q}, \dot{q}, \vartheta) \geq 5\|\tilde{q}\|^2 + 0.1781\|\vartheta\|^2 \tag{80}$$

By choosing $\rho = 275$ we meet the left hand side of (63) and are guaranteed to enter the set B_ρ^M and therefore the existence of t_s as defined in (64).

From $V_{4m} \leq \rho$ we conclude that

$$\|\tilde{q}\| + \|\vartheta\| \leq 40$$

so that

$$\dot{V}_4 \leq - \begin{bmatrix} \|\dot{q}\| \\ \|\tilde{q}\| \\ \|\vartheta\| \end{bmatrix}^\top \begin{bmatrix} 0.3974 & 0 & -1.0675 \\ 0 & 0.2679 & -0.6697 \\ -1.0675 & -0.6697 & 4.5535 \end{bmatrix} \begin{bmatrix} \|\dot{q}\| \\ \|\tilde{q}\| \\ \|\vartheta\| \end{bmatrix}$$

Therefore if we start integrating as soon as we are in B_ρ^M we have asymptotic stability of the second phase and global asymptotic stability of the delayed $\mathrm{PI}^2\mathrm{D}$ controller.

As already pointed out in Remark 3.5 the problem is to determine when we are in B_ρ^M, since we need velocity measurements for determining this. However, we are guaranteed that during the first phase we converge to a fixed point that is contained in B_ρ^M.

In Figure 2 we can see the behaviour of the signals during the first 25 seconds of simulation. It can be seen that at $t = 25$ we have almost converged a fixed point. Therefore, we decided to start integrating from $t = 25$ on. The resulting overall performance of the delayed $\mathrm{PI}^2\mathrm{D}$ controller is depicted in Figure 3. We can see that the integrating that started at $t_s = 25$ results into zero position error.

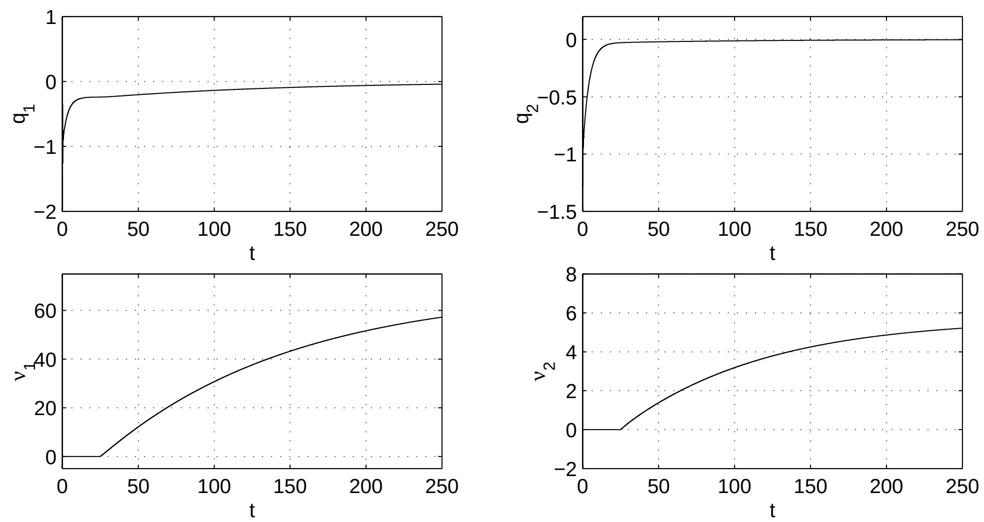

Figure 3: Delayed $\mathrm{PI}^2\mathrm{D}$ controller

6 Concluding remarks

We have addressed the practically important problem of *global* asymptotic stabilization of robot manipulators with PID and PI^{2}D control, i.e. the problem of set-point control with uncertain gravity knowledge and both by state and output feedback. Our main contribution is the proof that GAS is possible with *linear* PID and PI^{2}D controllers by simply delaying the integral action. We have called our new controllers PI$_d$D and PI$_d^2$D.

From a theoretical point of view we have shown for both cases state and position feedback, that there exists a "start-integration time" t_s such that GAS is guaranteed. From a practical point of view, we have given criteria on how to choose the instant t_s and the control gains. Unfortunately, in both cases the time t_s depend on the whole state however since the PI^{2}D does *not* use velocity feedback, this drawback can be overcome in practice by using an estimate of the generalized velocities at the *precise* instant t_s. Finally, we have shown in simulations the potential advantages of our schemes vis-a-vis existing *non*linear PID controllers.

From a theoretical point of view, the technique of switching controllers has recently become very popular (see e.g. [20, 14, 8] and references therein) in the nonlinear systems literature. Our results illustrate the impact that this theory has in practice, and an important issue of future research is how to combine local and global controllers avoiding the fact that the switching time t_s depend on unmeasurable state variables.

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Chapter 9

Adaptive and filtered visual servoing of planar robots

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ADAPTIVE AND FILTERED VISUAL SERVOING OF PLANAR ROBOTS

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Abstract: In this paper we address the visual servoing of planar robot manipulators with a fixed–camera configuration. The control goal is to place the robot end-effector over a desired static target by using a vision system equiped with a fixed camera to 'see' the robot end-effector and target. To achieve this goal we introduce a class of visual servo state feedback controllers and output (position) feedback controllers provided the camera orientation is known. For the case of unknown camera orientation a class of adaptive visual servo controllers is presented. All three classes contain controllers that meet input constraints.

Keywords: Visual servoing, robotics, stability

1. INTRODUCTION

External sensors such as visual systems enlarge the potential applications of actual robot manipulators evolving in unstructured environments. Although this fact has been recognized decades ago, it is until recent years that its effectiveness has reached the real world applications thanks to the technological improvement in cameras and dedicated hardware for image processing (Hashimoto, 1993; Hutchinson *et al.*, 1996).

This paper deals with a fixed camera configuration for visual servoing of robot manipulators. Most previous research has been started with the optics of the kinematic control where the robot velocity control (in joint or Cartesian space) is assumed to be computed in advance, and therefore the robot dynamics can be neglected (Allen *et al.*, 1993; Castaño and Hutchinson, 1994; Chaumette *et al.*, 1991; Espiau, 1993; Feddema *et al.*, 1991; Hager *et al.*, 1995; Nelson *et al.*, 1996; Mitsuda *et al.*, 1996). This approach is an example of a mechanical control system in which a kinematic model is used for control design, that is, the velocity of the system is assumed to be a direct input which can be manipulated. In physical systems, however, actuators exert forces or torques. This control philosophy is certainly effective for slow robot motion but its application is of a limited value when high speed motions are demanded.

We focus the visual servoing problem from an automatic control point of view by considering the full robot nonlinear dynamics with the applied torques as the control actions, and a rigorous stability analysis is given for an appropriate (adaptive) set point controller. Also, we are interested in simple control schemes avoiding the common procedures of camera calibration, inversion of the robot Jacobian and computation of the inverse kinematics. Previous efforts in

this subject have been reported in (Coste-Manière *et al.*, 1995; Kelly, 1996; Kelly *et al.*, 1996; Lei and Ghosh, 1993; Miyazaki and Masutani, 1990).
The main contributions of our work are extensions of the results in (Kelly, 1996) to the cases where velocity measurements are not available and the camera orientation parameter is unknown. The first problem is solved invoking the (by now) standard "dirty derivative" solution. However, the later problem involves a nonlinearly parametrized adaptive system, –a situation which is essentially unexplored in the field– hence special analysis and synthesis tools have to be developed for its solution. Furthermore, we provide a simple common framework to design standard proportional or saturated controllers.
The organisation of this paper is as follows. Section 2 contains the problem formulation, preliminaries and notation. In section 3 we introduce a class of visual servo controllers which includes the controllers reported in (Coste-Manière *et al.*, 1995; Kelly, 1996; Kelly *et al.*, 1996). In section 4 we derive a class of adaptive visual servo controllers in case the camera orientation is unknown. In section 5 we present a class of visual servo controllers in case we have no velocity measurements available. Section 6 contains our concluding remarks.

2. PROBLEM FORMULATION, PRELIMINARIES AND NOTATION

2.1 *Robot dynamics*

In the absence of friction or other disturbances, the dynamics of a serial 2–link rigid robot manipulator can be written as (see e.g. (Ortega and Spong, 1989; Spong and Vidyasagar, 1989)):

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \qquad (1)$$

where q is the 2×1 vector of joint displacements, τ is the 2×1 vector of applied joint torques, $M(q)$ is the 2×2 symmetric positive definite manipulator inertia matrix, $C(q,\dot{q})\dot{q}$ is the 2×1 vector of centripetal and Coriolis torques, and $g(q)$ is the 2×1 vector of gravitational torques. Two important properties of the robot dynamic model are the following:

Property 1. (see e.g. (Ortega and Spong, 1989; Spong and Vidyasagar, 1989)) The time derivative of the inertia matrix, and the centripetal and Coriolis matrix satisfy:

$$\dot{q}^T \left[\frac{1}{2}\dot{M}(q) - C(q,\dot{q}) \right] \dot{q} = 0; \;\; \forall \, q, \dot{q} \in I\!R^2. \qquad (2)$$

Property 2. (see e.g. (Craig, 1988)). The gravitational torque vector $g(q)$ is bounded for all $q \in I\!R^2$. This means there exist finite constants $k_i \geq 0$ such that

$$\max_{q \in I\!R^2} ||g_i(q)|| \leq k_i \quad i = 1,2$$

where $g_i(q)$ stands for the elements of $g(q)$.

For the purposes of this paper we consider a planar two degrees of freedom robot arm. For convenience we define a Cartesian reference frame anywhere in the robot base.

2.2 *Output equation*

We consider a fixed CDD camera whose optical axis is perpendicular to the plane where the robot tip evolves. The orientation of the camera with respect to the robot frame is denoted by θ.
The image acquired by the camera supplies a two–dimensional array of brightness values from a three–dimensional scene. This image may undergo various types of computer processing to enhance image properties and extract image features. In this paper we assume that the image features are the projection into the 2D image plane of 3D points in the scene space.
The output variable $y \in I\!R^2$ is defined as the position (in pixels) of the robot tip in the image. The mapping from the joint positions q to the output y involves a rigid body transformation, a perspective projection and a linear transformation (Feddema *et al.*, 1991; Hutchinson *et al.*, 1996). The corresponding output equation has the form (Kelly, 1996)

$$y = ae^{-J\theta}[k(q) - \vartheta_1] + \vartheta_2 \qquad (3)$$

where $a > 0$ and ϑ_1, ϑ_2 denote intrinsic camera parameters (scale factors, focal length, center offset), $k : I\!R^2 \to I\!R^2$ stands for the robot direct kinematics, and

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The direct kinematics yields $\dot{k} = \mathcal{J}(q)\dot{q}$, where $\mathcal{J}(q) \in I\!R^{2\times 2}$ is the analytic robot Jacobian. An important property of this Jacobian is the following (see e.g. (Spong and Vidyasagar, 1989)):

Property 3. The Jacobian is bounded for all $q \in I\!R^2$, i.e. there exists a finite constant J_M such that

$$||\mathcal{J}(q)|| \leq J_M \quad \forall q \in I\!R^2$$

2.3 *Problem formulation*

Consider the robotic system (1) together with the output equation (3), where the camera orientation θ is known, but the intrinsic camera parameters a, ϑ_1 and ϑ_2 are unknown. Suppose that together with the position y of the robot tip in the image also measurements of the joint positions q and velocities $\dot{q}$ are available. Let $y_d \in I\!R^2$ be a desired constant position for the robot tip in the image plane. This corresponds to the image of a point target which is assumed to be located strictly inside the robot workspace. Then the control problem can be stated as to design a control law for the actuator torques τ such that the robot tip reaches, in the image supplied on the screen, the target point placed anywhere in the robot workspace. In other words:

$$\lim_{t\to\infty} y(t) = y_d$$

Later in this paper the assumption that the camera orientation θ is known will be relaxed, as well as the assumption that measurements of the joint velocities $\dot{q}$ are available.
To be able to solve the problem formulated above we make the following assumptions:

Assumption 4. **(Problem solvability)** There exists a constant (unknown) vector $q_d \in I\!\!R^2$ such that

$$y_d = ae^{-J\theta}[k(q_d) - \vartheta_1] + \vartheta_2$$

Assumption 5. **(Nonsingularity at the desired configuration)** For the (unknown) vector $q_d \in I\!\!R^2$ it holds true that

$$\det\{\mathcal{J}(q_d)\} \neq 0.$$

Corollary 6. There exists a neighborhood around q_d for which $\det\{\mathcal{J}(q)\} \neq 0$ (by smoothness of the Jacobian).

It is worth noticing that in case y_d corresponds to the image of a point target located strictly inside the robot workspace, then Assumptions 4 and 5 are trivially satisfied. Also, under Assumptions 4 and 5 we have that $q = q_d \in I\!\!R^2$ is an isolated solution of

$$y_d = ae^{-J\theta}[k(q) - \vartheta_1] + \vartheta_2 \tag{4}$$

i.e. there exists a neighborhood around q_d for which $q = q_d$ is the only solution of (4).

2.4 *Notation*

Throughout we use the following notation.

Definition 7. Let $\mathcal{F}^n$ denote the class of continuous functions $f : I\!\!R^n \to I\!\!R^n$ for which there exists a positive definite $F : I\!\!R^n \to I\!\!R$ such that

$$f(x) = f(x_1,\ldots,x_n) = \begin{bmatrix} \dfrac{\partial F}{\partial x_1}(x_1,\ldots,x_n) \\ \vdots \\ \dfrac{\partial F}{\partial x_n}(x_1,\ldots,x_n) \end{bmatrix} \tag{5}$$

and for which $x^T f(x)$ is a positive definite function.

Definition 8. Let $\mathcal{B}^n$ denote the class of $f \in \mathcal{F}^n$ that are bounded, i.e the class of $f \in \mathcal{F}^n$ for which there exists a constant $f_M \in I\!\!R$ such that $||f(x)|| \leq f_M$ for all $x \in I\!\!R^n$.

An important property of $f \in \mathcal{F}^n$ is the following:

Property 9. Let $f \in \mathcal{F}^n$. Then $f(x) = 0$ if and only if $x = 0$.

In general it is not easy to verify whether a given $f : I\!\!R^n \to I\!\!R^n$ can be written as the gradient of a radially unbounded $F : I\!\!R^n \to I\!\!R$. However, a necessary condition for continuously differentiable f is that its Jacobian $\frac{\partial f}{\partial x}$ is symmetric.
It is easy to see that elements of $\mathcal{F}^n$ are the functions

$$f(x) = K_1[f_1(x_1),\ldots,f_n(x_n)]^T$$

and

$$f(x) = K_2 x$$

where $K_1 = K_1^T$ is a $n \times n$ diagonal positive definite matrix, $K_2 = K_2^T$ is a $n \times n$ (not necessarily diagonal) positive definite matrix, and f_i are continuous nondecreasing functions satisfying $f_i(0) = 0$ and $f_i'(0) > 0$ ($i = 1,\ldots,n$). By choosing $f_i(x) = \tanh(\lambda_i x)$, $f_i(x) = \mathrm{sat}(\lambda_i x)$ or $f_i(x) = \frac{x}{\lambda_i + |x|}$ ($\lambda_i > 0$) we obtain elements of $\mathcal{B}^n$, whereas $f(x) = K_2 x$ is an element of $\mathcal{F}^n$ but not of $\mathcal{B}^n$.
Throughout we denote for $f \in \mathcal{F}^n$ by $F(x)$ the associated function of which f is the gradient (cf. (5)).
Furthermore, we define

$$\tilde{q} = q - q_d \text{ and } \tilde{y} = y - y_d.$$

Since y is measurable and y_d is given,

$$\tilde{y} = ae^{-J\theta}(k(q) - k(q_d))$$

can be measured too. However, since q_d is unknown, $\tilde{q}$ is *not* available for measurement.
We conclude this section by noticing that since q_d is fixed, $\dot{\tilde{y}} = ae^{-J\theta}\mathcal{J}(q)\dot{q}$ and therefore

$$\dot{F}(\tilde{y}) = a\dot{q}^T\mathcal{J}(q)^T e^{J\theta} f(\tilde{y}).$$

3. A CLASS OF STABLE VISUAL SERVO CONTROLLERS

In this section we introduce a class of visual servo controllers which includes those reported in (Coste-Manière *et al.*, 1995; Kelly, 1996; Kelly *et al.*, 1996). Assuming that the camera orientation θ is known, and the full state $(q, \dot{q})$ is measured, these controllers ensure local regulation. This is formally stated in the next

Proposition 10. Consider the system (1) in closed-loop with the control law

$$\tau = g(q) - f_1(\dot{q}) - \mathcal{J}(q)^T e^{J\theta} f_2(\tilde{y}) \tag{6}$$

where $f_1, f_2 \in \mathcal{F}^2$. Under Assumptions 4–5 we have

$$\lim_{t\to\infty} \tilde{y}(t) = \lim_{t\to\infty} \dot{q}(t) = 0$$

provided the initial conditions $\dot{q}(0)$ and $\tilde{y}(0)$ are sufficiently small.

PROOF. Using the control law (6) results in the closed-loop dynamics

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + f_1(\dot{q}) + \mathcal{J}(q)^T e^{J\theta} f_2(\tilde{y}) = 0 \tag{7}$$

According to Assumptions 4–5 this equation has an isolated equilibrium at $[q^T \;\; \dot{q}^T]^T = [q_d^T \;\; 0^T]^T$.

Consider the Lyapunov function candidate

$$V(\tilde{q},\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{a}F_2(\tilde{y})$$

which is a (locally) positive definite function.
Along the closed-loop dynamics (7) its time-derivative becomes, using Property 1:

$$\begin{aligned}\dot{V}(\tilde{q},\dot{q}) &= -\dot{q}^T f_1(\dot{q}) - \dot{q}^T \mathcal{J}(q)^T e^{J\theta} f_2(\tilde{y}) + \\ &\quad + \dot{q}^T \mathcal{J}(q)^T e^{J\theta} f_2(\tilde{y}) \\ &= -\dot{q}^T f_1(\dot{q}) \leq 0\end{aligned}$$

which is negative semidefinite in the state $(\tilde{q},\dot{q})$. Using LaSalle's theorem and Corollary 6, for any initial condition in a small neighborhood of the equilibrium we have

$$\lim_{t\to\infty} \dot{q}(t) = \lim_{t\to\infty} f_2(\tilde{y}(t)) = 0$$

so we can conclude using Property 9:

$$\lim_{t\to\infty} \tilde{y}(t) = 0.$$

□

Consider the system (1) where we deal with the input constraints

$$|\tau_i(t)| \leq \tau_{i,max} \quad i = 1,2. \tag{8}$$

Then we can derive the following

Corollary 11. If $\tau_{i,max} > k_i$, where k_i has been defined in Property 2, then there exist $f_1, f_2 \in \mathcal{B}^2$ such that the controller (6) meets (8) and in closed-loop with the system (1) yields

$$\lim_{t\to\infty} \tilde{y}(t) = 0$$

provided the initial conditions are sufficiently small.

4. ADAPTIVE VISUAL SERVOING

In this section we consider the case in which, in contrast with the previous section, also the camera orientation θ is unknown. Still assuming that the full state $(q,\dot{q})$ is available for measurement we introduce a class of adaptive controllers that ensure local regulation:

Proposition 12. Consider the system (1) in closed-loop with the control law

$$\tau = \begin{cases} g(q) - f_1(\dot{q}) - \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y}) \\ \qquad \text{if } \dot{q}^T \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y}) \geq 0 \\ g(q) - f_1(\dot{q}) + \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y}) \\ \qquad \text{if } \dot{q}^T \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y}) < 0 \end{cases} \tag{9}$$

where $f_1, f_2 \in \mathcal{F}^2$. We update the parameter $\hat{\theta}$ as

$$\dot{\hat{\theta}} = \gamma \dot{q}^T \mathcal{J}(q)^T J e^{J\hat{\theta}} f_2(\tilde{y}) \tag{10}$$

where $\gamma > 0$ is a constant. Under Assumptions 4–5 we have, if we define $\tilde{\theta} = \hat{\theta} - \theta$:

$$\lim_{t\to\infty} \tilde{y}(t) = \lim_{t\to\infty} \dot{q}(t) = \lim_{t\to\infty} \dot{\tilde{\theta}}(t) = 0$$

provided the initial conditions $\dot{q}(0)$, $\tilde{y}(0)$ and $\tilde{\theta}(0)$ are sufficiently small.

PROOF. Using the control law (9) together with the parameter update law (10) results in the closed-loop dynamics

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + f_1(\dot{q}) &= \pm \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y}) \\ \dot{\tilde{\theta}} &= \gamma \dot{q}^T \mathcal{J}(q)^T J e^{J\hat{\theta}} f_2(\tilde{y}) \end{aligned} \tag{11}$$

where the '$\pm$' reads as a '$+$' if $\dot{q}^T \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y}) \geq 0$ and as a '$-$' if $\dot{q}^T \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y}) < 0$.
Consider the Lyapunov function candidate

$$V(\tilde{q},\dot{q},\tilde{\theta}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{a}F_2(\tilde{y}) + \frac{1}{\gamma}(1-\cos\tilde{\theta}) \tag{12}$$

which is a (locally) positive definite function.
Along the closed-loop dynamics (11) its time-derivative becomes, using Property 1:

$$\begin{aligned}\dot{V}(\tilde{q},\dot{q},\tilde{\theta}) &= -\dot{q}^T f_1(\dot{q}) - \left|\dot{q}^T \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y})\right| + \\ &\quad + \dot{q}^T \mathcal{J}(q)^T e^{J\theta} f_2(\tilde{y}) + \\ &\quad + \sin\tilde{\theta}\dot{q}^T \mathcal{J}(q)^T J e^{J\hat{\theta}} f_2(\tilde{y}) \\ &= -\dot{q}^T f_1(\dot{q}) - \left|\dot{q}^T \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y})\right| + \\ &\quad + \dot{q}^T \mathcal{J}(q)^T (e^{-J\tilde{\theta}} + \sin\tilde{\theta} J) e^{J\hat{\theta}} f_2(\tilde{y}) \\ &= -\dot{q}^T f_1(\dot{q}) - \left|\dot{q}^T \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y})\right| + \\ &\quad + \cos\tilde{\theta}\dot{q}^T \mathcal{J}(q)^T e^{J\hat{\theta}} f_2(\tilde{y}) \\ &\leq -\dot{q}^T f_1(\dot{q})\end{aligned}$$

which is negative semidefinite in the state $(\tilde{q},\dot{q},\tilde{\theta})$.
According to LaSalle's theorem, the closed-loop system tends to the largest invariant set of points $(\tilde{q},\dot{q},\tilde{\theta})$ for which $\dot{V} = 0$. From $0 = \dot{V} \leq -\dot{q}^T f_1(\dot{q}) \leq 0$ it follows that necessarily $\dot{q} = 0$. Then from the closed-loop dynamics (11) we know $\dot{\tilde{\theta}} = 0$ and using Corollary 6 also $f_2(\tilde{y}) = 0$. Therefore LaSalle's theorem gives us for any initial condition in a small neighborhood of the origin

$$\lim_{t\to\infty} \dot{\tilde{\theta}}(t) = \lim_{t\to\infty} \dot{q}(t) = \lim_{t\to\infty} \tilde{y}(t) = 0$$

□

Remark 13. The switching nature of the controller (9) leads to chattering, which is undesirable. Using a suitably smoothed control law might be a way to overcome the chattering.

As in the previous section we can derive the following

Corollary 14. If $\tau_{i,max} > k_i$, where k_i has been defined in Property 2, then there exist $f_1, f_2 \in \mathcal{B}^2$ such that the controller (9) meets (8) and in closed-loop with the system (1) yields

$$\lim_{t\to\infty} \tilde{y}(t) = \lim_{t\to\infty} \dot{q}(t) = \lim_{t\to\infty} \dot{\tilde{\theta}}(t) = 0$$

provided the initial conditions are sufficiently small.

Remark 15. For the system (1) it is well known (Ortega and Spong, 1989) that there exist a reparametrization of all unknown system parameters into a parameter vector $\Theta \in I\!R^p$ that enters linearly in the system dynamics (1). Therefore the following holds:

$$M(q,\Theta)\ddot{q} + C(q,\dot{q},\Theta)\dot{q} + g(q,\Theta) =$$
$$M_0(q)\ddot{q} + C_0(q,\dot{q})\dot{q} + g_0(q) + Y(q,\dot{q},\dot{q},\ddot{q})\Theta$$

We can cope with those unknown system parameters in the 'standard' way by adding $Y(q,\dot{q},\dot{q},\ddot{q})\hat{\Theta}$ to the control law (and replacing $g(q)$ with $g_0(q)$), where $\hat{\Theta}$ is updated according to

$$\dot{\hat{\Theta}} = -\Gamma Y^T(q,\dot{q},\dot{q},\ddot{q})\dot{q}$$

where $\Gamma = \Gamma^T > 0$ is a positive definite matrix. To prove asymptotic stability as in Proposition 12, we only add $\frac{1}{2}\tilde{\Theta}^T\Gamma^{-1}\tilde{\Theta}$ to the Lyapunov function (12), where we defined $\tilde{\Theta} = \hat{\Theta} - \Theta$.

5. FILTERED VISUAL SERVOING

In this section we consider the case in which, in contrast with section 3, no measurements of the joint velocities $\dot{q}$ are available. Assuming that the camera orientation θ is known and only measurements of the joint positions q are available we introduce a class of controllers and filters that ensure local regulation. This is formally stated in the next

Proposition 16. Consider the system (1) in closed-loop with the control law

$$\tau = g(q) - \mathcal{J}(q)^T e^{J\theta} f_1(z) - \mathcal{J}(q)^T e^{J\theta} f_2(\tilde{y}) \quad (13)$$

where $f_1, f_2 \in \mathcal{F}^2$, and z is generated from the filter

$$\begin{aligned} z &= \tilde{y} - w \\ \dot{w} &= \tilde{y} - w \end{aligned} \quad (14)$$

Under Assumptions 4–5 we have

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t) = \lim_{t\to\infty} \dot{q}(t) = \lim_{t\to\infty} \tilde{y}(t) = 0$$

provided the initial conditions $w(0)$, $\dot{q}(0)$, and $\tilde{y}(0)$ are sufficiently small.

PROOF. Using the control law (13) together with the filter (14) results in the closed-loop dynamics

$$\begin{aligned} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \mathcal{J}(q)^T e^{J\theta}[f_1(z) + f_2(\tilde{y})] &= 0 \\ \dot{z} = ae^{-J\theta}\mathcal{J}(q)\dot{q} - z \end{aligned} \quad (15)$$

Consider the Lyapunov function candidate

$$V(\tilde{q},\dot{q},z) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{a}F_2(\tilde{y}) + \frac{1}{a}F_1(z) \quad (16)$$

which is a (locally) positive definite function.
Along the closed-loop dynamics (15) its time-derivative becomes:

$$\begin{aligned} \dot{V}(\tilde{q},\dot{q},z) &= -\dot{q}^T\mathcal{J}(q)^T e^{J\theta} f_1(z) - \dot{q}^T\mathcal{J}(q)^T e^{J\theta} f_2(\tilde{y}) + \\ &\quad + \dot{q}^T\mathcal{J}(q)^T e^{J\theta} f_2(\tilde{y}) + \frac{1}{a}\dot{z}^T f_1(z) \\ &= -\dot{q}^T\mathcal{J}(q)^T e^{J\theta} f_1(z) + \dot{q}^T\mathcal{J}(q)^T e^{J\theta} f_1(z) - \\ &\quad -\frac{1}{a}z^T f_1(z) \\ &= -\frac{1}{a}z^T f_1(z) \leq 0 \end{aligned}$$

which is negative semidefinite in the state $(\tilde{q},\dot{q},z)$. Using LaSalle's theorem and Corollary 6, for any initial condition in a small neighborhood of the equilibrium we can conclude

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t) = \lim_{t\to\infty} \dot{q}(t) = \lim_{t\to\infty} \tilde{y}(t) = 0.$$

□

Remark 17. The filter (14) can, similar to the one presented in (Lefeber and Nijmeijer, 1997), be seen as a simple representative of a whole class of possible filters. For instance if $f_1 \in \mathcal{F}^2$ satisfies the property that also $\Lambda f \in \mathcal{F}^2$, where Λ is an arbitrary positive definite matrix, then it can easily be seen that instead of (14) also the filter

$$\begin{aligned} z &= \Lambda_1\tilde{y} - \Lambda_2 w \\ \dot{w} &= \Lambda_3(\Lambda_2\tilde{y} - \Lambda_2 w) \end{aligned} \quad (17)$$

can be used (replace in (16) $F_1(\tilde{y})$ with the $F(\tilde{y})$ associated with $\Lambda_1^{-1} f_1(\tilde{y})$ to obtain

$$\dot{V} = -\frac{1}{a}z^T\Lambda_3\Lambda_2\Lambda_1^{-1} f_1(z) = z^T\tilde{f}_1(z)$$

with $\tilde{f}_1(z) \in \mathcal{F}^2$). The filter (17) is similar to the ones presented in (Ailon and Ortega, 1993; Berghuis and Nijmeijer, 1993). Also the more general class of linear filters presented in (Arimoto *et al.*, 1994; Kelly and Santibañez, 1996) can similarly be seen as a special case of (14). Also a wide variety of nonlinear filters can be rewritten as (14).
In general one can say that the filter (14) is a representative of a whole class of controllers that takes its simple form due to a well chosen change of coordinates.
To obtain other possible filters, just apply a suitable change of coordinates in z and w (suitable in the sense that $\dot{V}$ remains negative definite). As far as the proof is concerned, one possibly has to replace $F_1(\tilde{y})$ in (16) with a different F, as we have seen in deriving (17), sometimes resulting in a different expression for $f_1(z)$ in (13).

As in the previous sections we can derive the following

Corollary 18. If $\tau_{i,max} > k_i$, where k_i has been defined in Property 2, then there exist $f_1, f_2 \in \mathcal{B}^2$ such that the controller (13) meets (8) and in closed-loop with the system (1) yields

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t) = \lim_{t\to\infty} \dot{q}(t) = \lim_{t\to\infty} \tilde{y}(t) = 0$$

provided the initial conditions are sufficiently small.

6. CONCLUDING REMARKS

In this paper we addressed the visual servoing of planar robot manipulators under a fixed camera configuration. In case the camera orientation is known, we introduced a class of visual servo controllers for both the state feedback and output feedback case (position measurements). In case of unknown camera orientation a class of adaptive controllers has been presented. The results include controllers that satisfy input constraints.

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Chapter 10

Adaptive tracking control of non-holonomic systems: an example

This chapter consists of the paper:
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Adaptive tracking control of nonholonomic systems: an example

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Abstract

We study an example of an adaptive (state) tracking control problem for a four-wheel mobile robot, as it is an illustrative example of the general adaptive state-feedback tracking control problem. It turns out that formulating the adaptive state-feedback tracking control problem is not straightforward, since specifying the reference state-trajectory can be in conflict with not knowing certain parameters. Our example illustrates this difficulty and we propose a problem formulation for the adaptive state-feedback tracking problem that meets the natural prerequisite that it reduces to the state-feedback tracking problem if the parameters are known. A general methodology for solving the problem is derived.

1 Introduction

In recent years a lot of interest has been devoted to (mainly) stabilization and tracking of nonholonomic dynamic systems, see e.g. [1, 2, 3, 4] and references therein. One of the reasons for the attention is the lack of a continuous static state feedback control since Brockett's necessary condition for smooth stabilization is not met, see [5]. The proposed solutions to this problem follow mainly two routes, namely discontinuous and/or time-varying control. For a good overview, see the survey paper [6].

Less studied is the adaptive control of nonholonomic systems. Results on adaptive stabilization can be found in [7, 8]. In [9, 10, 11, 12] the adaptive tracking problem is studied, but all papers are either concerned with adaptive *output* tracking, or the state trajectory to be tracked is feasable for any possible parameter. However, it is possible that specifying a reference-state trajectory and not knowing certain parameters are in conflict with each other. The question then arises how to formulate the adaptive tracking problem in such a way that it reduces to the state feedback tracking problem in case the parameters are known.

In this paper we consider a simple academic example that clearly illustrates the above mentioned conflict. We propose a formulation for the adaptive (state) tracking control problem and derive a general methodology for solving this problem.

The example we study is the kinematic model of a mobile car with rear wheel driving and front wheel steering:

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \frac{v}{L}\tan\phi \\ \dot{\phi} &= \omega \end{aligned} \tag{1}$$

The forward velocity of the rear wheel v and the angular velocity of the front wheel ω are considered as inputs, (x, y) is the center of the rear axis of the vehicle, θ is the orientation of the body of the car, ϕ is the angle between front wheel and car and $L > 0$ is a constant that denotes the length of the car (see also Figure 1), and is assumed to be unknown.

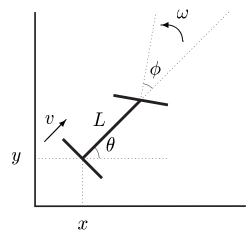

Figure 1: The mobile car

The organization of the paper is as follows. Section 2 contains the problem formulation of the tracking problem and illustrates the difficulties in arriving at the problem formulation for the adaptive tracking problem. Section 3 contains some definitions and preliminary results. Section 4 addresses the tracking problem and prepares for Section 5 in which the adaptive tracking problem is considered. Finally, Section 6 concludes the paper.

2 Problem formulation

2.1 Tracking control problem

Since we want the adaptive tracking control problem to reduce to the tracking problem for known L, we first have to formulate the tracking problem for the case L is known.

Consider the problem of tracking a feasible reference trajectory, i.e. a trajectory $([x_r, y_r, \theta_r, \phi_r]^T, [v_r, \omega_r]^T)$ satisfying

$$\begin{aligned} \dot{x}_r &= v_r \cos\theta_r \\ \dot{y}_r &= v_r \sin\theta_r \\ \dot{\theta}_r &= \frac{v_r}{L}\tan\phi_r \\ \dot{\phi}_r &= \omega_r \end{aligned} \tag{2}$$

This reference trajectory can be generated by any of the motion planning techniques available from literature. The tracking control problem then can be formulated as

Problem 2.1 (Tracking control problem) *Given a feasible reference trajectory $([x_r, y_r, \theta_r, \phi_r]^T, [v_r, \omega_r]^T)$, find appropriate control laws v and ω of the form*

$$v = v(t, x, y, \theta, \phi), \quad \omega = \omega(t, x, y, \theta, \phi) \tag{3}$$

such that for the resulting closed-loop system (1,3)

$$\lim_{t\to\infty} \big(|x(t) - x_r(t)| + |y(t) - y_r(t)| + \\ +|\theta(t) - \theta_r(t)| + |\phi(t) - \phi_r(t)|\big) = 0$$

Remark 2.2 *Notice that in general, the control laws (3) are not only a function of x, y, θ, and ϕ, but also of $v_r(t), \omega_r(t)$, $x_r(t)$, $y_r(t)$, $\theta_r(t)$, $\phi_r(t)$, and possibly their derivatives with respect to time. This explains the time-dependency in (3).*

Remark 2.3 *Notice that the tracking control problem we study here is not the same as an output tracking problem of the flat output $[x_r(t), y_r(t)]^T$. First of all, by specifying $x_r(t)$ and $y_r(t)$ the reference trajectory can not be uniquely specified (e.g. $v_r(t)$ can be either positive or negative). But more important is the fact that tracking of $x_r(t)$ and $y_r(t)$ does not guarantee tracking of the corresponding $\theta_r(t)$ and $\phi_r(t)$.*

2.2 Adaptive tracking control problem

In case the parameter L is unknown, however, we can not formulate the adaptive tracking problem in the same way. This is due to the fact that for unknown L we can not specify a feasible reference trajectory $([x_r, y_r, \theta_r, \phi_r]^T, [v_r, \omega_r]^T)$, satisfying (2). In specifying $v_r(t)$, $\phi_r(t)$ and $\theta_r(t)$ we have to make sure that

$$\dot{\theta}_r = \frac{v_r}{L}\tan\phi_r \tag{4}$$

in order to obtain a feasible reference trajectory. This is in conflict with the assumption that we do not know L, since once $v_r(t)$, $\phi_r(t)$ and $\theta_r(t)$ are specified it is possible to determine L from (4).

So the question is how to formulate the adaptive tracking problem for the nonholonomic system (1) in such a way that it reduces to the state-feedback tracking control problem for the case L is known? Appearently we can not both specify v_r, θ_r and ϕ_r as functions of time, and assume that L is unknown.

When generating a feasible reference trajectory satisfying (2), one usually generates some sufficiently smooth reference signals, e.g. $x_r(t)$ and $y_r(t)$, and then all other signals are derived from the equations (2). Notice that it is possible to specify $v_r(t)$, $x_r(t)$, $y_r(t)$, and $\theta_r(t)$ without assuming anything on L. These signals mainly cover the behaviour of the mobile car. However, as mentioned in Remark 2.3, tracking of the output x_r, y_r, θ_r is not what we are interested in, since it is possible to have $x(t)-x_r(t)$, $y(t)-y_r(t)$, and $\theta(t) - \theta_r(t)$ converge to zero as t goes to infinity, but $\phi(t)$ not converge to $\phi_r(t)$. Actually, $\phi(t)$ can even grow unbounded. That is why we insist on looking at the *state* tracking problem.

In case we know L it is possible, once $v_r(t)$, $x_r(t)$, $y_r(t)$ and $\theta_r(t)$ are given, to determine $\phi_r(t)$ uniquely. Notice that we can determine $\tan\phi_r(t)$, from which $\phi_r(t)$ is uniquely determined (since $\dot{\theta}_r$ has to exist and therefore $\phi_r \in]-\frac{\pi}{2}, \frac{\pi}{2}[$). Once $\phi_r(t)$ is known, also $\omega_r(t)$ can be uniquely determined using (2).

When L is unknown we still know that once $v_r(t)$, $x_r(t)$, $y_r(t)$ and $\theta_r(t)$ are given, $\phi_r(t)$ and $\omega_r(t)$ are uniquely determined. The only problem is that these signals are unknown, due to the fact that L is unknown. This is something we illustrate throughout by writing $\phi_r^L(t)$ and $\omega_r^L(t)$. Therefore, we can assume that a feasible reference trajectory $([x_r, y_r, \theta_r, \phi_r^L]^T, [v_r, \omega_r^L]^T)$, satisfying (2) is given and study the problem of finding a state-feedback law that assures tracking of this reference state.

Problem 2.4 (Adaptive tracking control problem) *Let a feasible reference trajectory $([x_r, y_r, \theta_r, \phi_r^L]^T, [v_r, \omega_r^L]^T)$ be given (i.e. $x_r(t)$, $y_r(t)$, $\theta_r(t)$ and $v_r(t)$ are known time-functions, but $\phi_r^L(t)$ and $\omega_r^L(t)$ are unknown, due to the fact that L is unknown). Find appropriate control laws v and ω of the form*

$$v = v(t, x, y, \theta, \phi), \quad \omega = \omega(t, x, y, \theta, \phi) \tag{5}$$

such that for the resulting closed-loop system

$$\lim_{t\to\infty} \big(|x(t) - x_r(t)| + |y(t) - y_r(t)| + \\ +|\theta(t) - \theta_r(t)| + |\phi(t) - \phi_r^L(t)|\big) = 0$$

Remark 2.5 *Notice that the time-dependency in (5) allows for using $v_r(t)$, $x_r(t)$, $y_r(t)$, $\theta_r(t)$ in the control laws (as well as their derivatives with respect to time), but in this case we can **not** use $\omega_r^L(t)$ or $\phi_r^L(t)$.*

Remark 2.6 *It is clear that once L is known this problem formulation reduces to that of the tracking problem for known L. Then also $\phi_r(t)$ and $\omega_r(t)$ can be used in the control laws again, since these signals are just functions (depending on L) of $v_r(t)$, $\theta_r(t)$ and their derivatives with respect to time.*

In order to be able to solve the (adaptive) tracking control problem, we need to make the following assumptions on the reference trajectory

Assumption 2.7 *First of all, the reference dynamics need to have a unique solution, which is why we need $\phi_r(t) \in]-M, M[$ with $M < \frac{\pi}{2}$. This is equivalent to assuming that $\frac{\dot{\theta}_r}{v_r}$ is bounded.*
Second, we assume that the reference is always moving in a forward direction with a bounded velocity, i.e. there exist constants v_r^{min} and v_r^{max} such that

$$0 < v_r^{min} \leq v_r(t) \leq v_r^{max}$$

Furthermore, we assume that the forward and angular acceleration, i.e. $\dot{v}_r$ and $\ddot{\theta}_r$, are bounded.

3 Preliminaries

In this section we introduce the definitions and theorems used in the remainder of this paper.

Definition 3.1 *We call $w(t) = [w_1(t), \ldots, w_n(t)]^T$* persistently exciting *if there exist constants $\delta, \epsilon_1, \epsilon_2 > 0$ such that for all $t > 0$:*

$$\epsilon_1 I \leq \int_t^{t+\delta} w(\tau)w(\tau)^T d\tau \leq \epsilon_2 I$$

Lemma 3.2 (cf. e.g. [13, 14]) *Consider the system*

$$\begin{bmatrix} \dot{e} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} A_m & b_m w^T(t) \\ -\gamma w(t) c_m^T & 0 \end{bmatrix} \begin{bmatrix} e \\ \phi \end{bmatrix} \qquad (6)$$

where $e \in \mathbb{R}^n$, $\phi \in \mathbb{R}^m$, $w \in \mathbb{R}^k$, $\gamma > 0$. Assume that $M(s) = c_m^T(sI - A_m)^{-1}b_m$ is a strictly positive real transfer function, then $\phi(t)$ is bounded and

$$\lim_{t\to\infty} e(t) = 0.$$

If in addition $w(t)$ and $\dot{w}(t)$ are bounded for all $t \geq t_0$, and $w(t)$ is persistently exciting then the system (6) is globally exponentially stable.

Lemma 3.3 ([15]) *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be any differentiable function. If $f(t)$ converges to zero as $t \to \infty$ and its derivative satisfies*

$$\dot{f}(t) = f_0(t) + \eta(t) \qquad t \geq 0$$

where f_0 is a uniformly continuous function and $\eta(t)$ tends to zero as $t \to \infty$, then $\dot{f}(t)$ and $f_0(t)$ tend to zero as $t \to \infty$.

Using standard techniques it is easy to show that

Lemma 3.4 *Assume that origin of the system*

$$\dot{x} = f(t, x) \qquad f(t, 0) = 0 \quad \forall t$$

where $x \in \mathbb{R}^n$ is globally exponentially stable. Then the disturbed system

$$\dot{x} = f(t, x) + \Delta(t)$$

where $\Delta(t)$ is a bounded vanishing disturbance, i.e.

$$\sup_t \|\Delta(t)\| \leq M \quad \textit{and} \quad \lim_{t\to\infty} \Delta(t) = 0$$

is globally asymptotically stable.

Remark 3.5 *Throughout this paper we use the expressions $\frac{x\cos x - \sin(x)}{x^2}$, $\frac{x - \sin(x)}{x^2}$, $\frac{\cos(x)-1}{x}$, $\frac{1 - x\sin x - \cos(x)}{x^2}$, $\frac{\cos(x)-1}{x^2}$, and $\frac{\sin(x)}{x}$. These functions are discontinuous in $x = 0$, but if we define their values for $x = 0$ as respectively 0, 0, 0, $-\frac{1}{2}$, $-\frac{1}{2}$, and 1 it is easy to verify that all functions are continuous and bounded.*

4 A tracking controller

First we consider the tracking problem for the case L is known. To overcome the problem that the errors $x - x_r$ and $y - y_r$ depend on how we choose the inertial reference frame, we define errors in a body reference frame, i.e. in a coordinate-frame attached to the car (cf. [16]):

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \theta_r - \theta \end{bmatrix} \qquad (7)$$

In order to be able to control the orientation θ of our mobile car by means of the input ω, we prefer to have $v(t) \neq 0$ for all $t \geq 0$. Since $v_r(t) \geq v_r^{\min} > 0$ we know that if $\sigma(\cdot)$ is a function that fulfills

$$\sigma(x) > -v_r^{\min} \qquad \forall x \in \mathbb{R}$$

the control law

$$v = v_r + \sigma(x_e) \qquad (8)$$

automatically guarantees $v(t) > 0$ for all $t \geq 0$. Furthermore, we assume that $\sigma(x)$ is continuously differentiable and satisfies

$$x\sigma(x) > 0, \quad \forall x \neq 0$$

Examples of possible choices for $\sigma(x)$ are

$$\begin{aligned} \sigma(x) &= v_r^{\min} \cdot \tanh(x) \\ \sigma(x) &= v_r^{\min} \cdot \frac{x}{1 + |x|} \end{aligned}$$

With the control law (8) the dynamics in the new coordinates (7) and ϕ become

$$\begin{aligned} \dot{x}_e &= y_e \tfrac{v_r + \sigma(x_e)}{L} \tan\phi + v_r(\cos\theta_e - 1) - \sigma(x_e) \\ \dot{y}_e &= -x_e \tfrac{v_r + \sigma(x_e)}{L} \tan\phi + v_r \sin\theta_e \\ \dot{\theta}_e &= \tfrac{v_r}{L} \tan\phi_r - \tfrac{v_r + \sigma(x_e)}{L} \tan\phi \\ \dot{\phi} &= \omega \end{aligned} \qquad (9)$$

Differentiating the function $V_1 = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2$ along the solutions of (9) yields

$$\dot{V}_1 = -x_e\sigma(x_e) + v_r\left(\frac{\cos\theta_e - 1}{\theta_e}x_e + \frac{\sin\theta_e}{\theta_e}y_e\right)\theta_e$$

When we consider ϕ as a *virtual control* we could design an intermediate control law for ϕ that achieves ($k_1, k_2 > 0$):

$$\dot{\theta}_e = -k_1\theta_e - k_2 v_r\left(\frac{\cos\theta_e - 1}{\theta_e}x_e + \frac{\sin\theta_e}{\theta_e}y_e\right)$$

Using the Lyapunov function candidate $V_2 = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2k_2}\theta_e^2$ and similar reasoning as in [2], we can then claim that x_e, y_e and θ_e converge to zero, provided that Assumption 2.7 is satisfied.

It would be the 'standard procedure' to define the *error variable*

$$\begin{aligned}\bar{z} \;&=\; \frac{v_r}{L}\tan\phi_r - \frac{v_r + \sigma(x_e)}{L}\tan\phi + k_1\theta_e + \\ &\quad + k_2 v_r\left(\frac{\cos\theta_e - 1}{\theta_e}x_e + \frac{\sin\theta_e}{\theta_e}y_e\right)\end{aligned}$$

However, for simplicity of analysis we prefer to consider the error variable $z = L\bar{z}$, i.e. we define ($c_1, c_2 > 0$):

$$z = v_r\tan\phi_r - v\tan\phi + c_1\theta_e + c_2 v_r\left(\tfrac{\cos\theta_e - 1}{\theta_e}x_e + \tfrac{\sin\theta_e}{\theta_e}y_e\right) \tag{10}$$

With this definition the error-dynamics (9) now become

$$\begin{aligned}
\dot{x}_e &= y_e\frac{v}{L}\tan\phi + v_r(\cos\theta_e - 1) - \sigma(x_e) && \text{(11a)}\\
\dot{y}_e &= -x_e\frac{v}{L}\tan\phi + v_r\sin\theta_e && \text{(11b)}\\
\dot{\theta}_e &= -\frac{c_1}{L}\theta_e - \frac{c_2}{L}v_r\left(\frac{\cos\theta_e - 1}{\theta_e}x_e + \frac{\sin\theta_e}{\theta_e}y_e\right) + \\
&\quad + \frac{1}{L}z && \text{(11c)}\\
\dot{z} &= \frac{v}{\cos^2\phi}\left(\frac{1}{L}\alpha(t)\sin\phi\cos\phi - \omega\right) + \beta(t) && \text{(11d)}
\end{aligned}$$

where

$$\begin{aligned}
\alpha(t) &= y_e\tan\phi + c_1 - c_2 v_r\left(\tfrac{1-\cos\theta_e}{\theta_e^2}x_e + \tfrac{\theta_e - \sin\theta_e}{\theta_e^2}y_e\right)\\
\beta(t) &= \dot{v}_r\tan\phi_r + \frac{v_r\omega_r}{\cos^2\phi_r} + (v_r\cos\theta_e - v)\tan\phi + \\
&\quad + c_2(\dot{v}_r x_e - v v_r - v_r^2)\tfrac{\cos\theta_e - 1}{\theta_e} + c_2\dot{v}_r y_e\tfrac{\sin\theta_e}{\theta_e} + \\
&\quad + c_1\tfrac{v_r}{L}\tan\phi_r + c_2 v_r\left(\tfrac{1-\theta_e\sin\theta_e - \cos\theta_e}{\theta_e^2}x_e + \right.\\
&\quad \left. + \tfrac{\theta_e\cos\theta_e - \sin\theta_e}{\theta_e^2}y_e\right)\tfrac{v_r}{L}\tan\phi_r
\end{aligned}$$

When we choose the input ω

$$\omega = \frac{1}{L}\alpha(t)\sin\phi\cos\phi + \frac{\cos^2\phi}{v}\left(\beta(t) + c_3 z\right) \tag{12}$$

we obtain

$$\dot{z} = -c_3 z$$

Consider the Lyapunov function candidate

$$V_3 = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{L}{2c_2}\theta_e^2 + \frac{1}{2c_1c_2c_3}\bar{z}^2 \tag{13}$$

Differentiating (13) along solutions of (11,12) yields

$$\begin{aligned}\dot{V}_3 &= -x_e\sigma(x_e) - \tfrac{c_1}{c_2}\theta_e^2 + \tfrac{1}{c_2}\theta_e z - \tfrac{1}{c_1c_2}z^2\\ &\leq -x_e\sigma(x_e) - \tfrac{c_1}{2c_2}\theta_e^2 - \tfrac{1}{2c_1c_2}z^2 \leq 0\end{aligned} \tag{14}$$

We establish the following result

Proposition 4.1 *Assume that Assumption 2.7 is satisfied. Then all trajectories of (11,12) are globally uniformly bounded. Furthermore, all closed-loop solutions converge to zero, i.e.*

$$\lim_{t\to\infty}\left(|x_e(t)| + |y_e(t)| + |\theta_e(t)| + |z(t)|\right) = 0$$

Proof: Since V is positive-definite and radially unbounded, we conclude from (14) that x_e, y_e, θ_e and z are uniformly bounded. From (10) and Assumption 2.7 it follows that also v, $\tan\phi$ and as a result also ω and $\phi - \phi_r$ are uniformly bounded. Also the derivatives of all these signals are bounded. With Barbalat's Lemma it follows that x_e, θ_e and z converge to zero as t goes to infinity. Using Lemma 3.3 with $f = \theta_e$, $f_0 = -k_2 v_r y_e$ and $\eta = -k_1\theta_e - k_2 v_r\left(\frac{\cos\theta_e - 1}{\theta_e}x_e + (\frac{\sin\theta_e}{\theta_e} - 1)y_e\right) + z$ gives also that y_e tends to zero as t goes to infinity. ∎

Corollary 4.2 *Consider the system (1) in closed loop with the control laws (8,12) where the reference trajectory satisfies (2) and Assumption 2.7. For the resulting closed-loop system we have*

$$\begin{aligned}\lim_{t\to\infty}\; &\left(|x(t) - x_r(t)| + |y(t) - y_r(t)| + \right.\\ &\left. + |\theta(t) - \theta_r(t)| + |\phi(t) - \phi_r(t)|\right) = 0\end{aligned}$$

Proof: Using (7) it follows from x_e and y_e tending to zero that also $x - x_r$ and $y - y_r$ converge to zero. It only remains to show that $\phi(t) - \phi_r(t)$ tends to zero as t tends to infinity. This comes down to showing that $\tan\phi(t) - \tan\phi_r(t)$ tends to zero as t tends to infinity, which is a direct result from the fact that z tends to zero (and x_e, y_e and θ_e). ∎

5 An adaptive tracking controller

From now on we assume that the parameter L is unknown. As mentioned in section 2 we have the difficulty that not only L is unknown, but also the reference signals $\phi_r^L(t)$ and $\omega_r^L(t)$ (that appear in the expression $\beta(t)$) can not be used in the control law.

Fortunately, we are not only allowed to use x_e, y_e, θ_e and ϕ, but also $\dot{\theta}_r$ and $\ddot{\theta}_r$. Notice that in (10) we can replace the occurence of ϕ_r^L by means of the signal $\dot{\theta}_r$:

$$z = L\dot{\theta}_r - v\tan\phi + c_1\theta_e + c_2 v_r\left(\tfrac{\cos\theta_e - 1}{\theta_e}x_e + \tfrac{\sin\theta_e}{\theta_e}y_e\right) \tag{15}$$

However, using the variable z as defined in (10) or (15) makes it hard to design a controller using conventional adaptive techniques because z includes the unknown parameter L. Therefore, we define

$$\hat{z} = \hat{L}\dot{\theta}_r - v\tan\phi + c_1\theta_e + c_2 v_r\left(\tfrac{\cos\theta_e - 1}{\theta_e}x_e + \tfrac{\sin\theta_e}{\theta_e}y_e\right) \tag{16}$$

which can be seen as an estimate for z. Using $\hat{z}$ the tracking error dynamics (9) can be expressed as

$$\begin{aligned}
\dot{x}_e &= y_e\frac{v}{L}\tan\phi + v_r(\cos\theta_e - 1) - \sigma(x_e) && \text{(17a)}\\
\dot{y}_e &= -x_e\frac{v}{L}\tan\phi + v_r\sin\theta_e && \text{(17b)}\\
\dot{\theta}_e &= -\frac{c_1}{L}\theta_e - \frac{c_2}{L}v_r\left(\frac{\cos\theta_e - 1}{\theta_e}x_e + \frac{\sin\theta_e}{\theta_e}y_e\right) + \\
&\quad + \frac{1}{L}\hat{z} - \tilde{L}\frac{1}{L}\dot{\theta}_r && \text{(17c)}\\
\dot{\hat{z}} &= \frac{v}{\cos^2\phi}(\varrho\alpha(t)\sin\phi\cos\phi - \omega) + \hat{\beta}(t) && \text{(17d)}
\end{aligned}$$

where we introduced the parameter $\varrho = \frac{1}{L}$. Furthermore, we defined $\tilde{L} = \hat{L} - L$ and

$$\begin{aligned}
\alpha(t) &= y_e\tan\phi + c_1 - c_2 v_r\left(\tfrac{1-\cos\theta_e}{\theta_e^2}x_e + \tfrac{\theta_e - \sin\theta_e}{\theta_e^2}y_e\right)\\
\hat{\beta}(t) &= \dot{\hat{L}}\dot{\theta}_r + \hat{L}\ddot{\theta}_r + (v_r\cos\theta_e - v)\tan\phi + \\
&\quad + c_2(\dot{v}_r x_e - v v_r - v_r^2)\tfrac{\cos\theta_e - 1}{\theta_e} + c_2\dot{v}_r y_e\tfrac{\sin\theta_e}{\theta_e} + \\
&\quad + c_1\dot{\theta}_r + c_2 v_r\left(\tfrac{1-\theta_e\sin\theta_e - \cos\theta_e}{\theta_e^2}x_e + \right.\\
&\quad \left. \tfrac{+\theta_e\cos\theta_e - \sin\theta_e}{\theta_e^2}y_e\right)\dot{\theta}_r
\end{aligned}$$

When we choose the input

$$\omega = \hat{\varrho}\alpha(t)\sin\phi\cos\phi + \frac{\cos^2\phi}{v}\left(\hat{\beta}(t) + k_3\hat{z}\right) \tag{18}$$

we obtain ($\tilde{\varrho} = \hat{\varrho} - \varrho$):

$$\dot{\hat{z}} = -k_3\hat{z} - \tilde{\varrho}\alpha(t)v\tan\phi$$

Consider the Lyapunov function candidate ($\gamma_1, \gamma_2 > 0$)

$$V_4 = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{L\theta_e^2}{2c_2} + \frac{\hat{z}^2}{2c_1c_2c_3} + \frac{\tilde{L}^2}{2c_2\gamma_1} + \frac{\tilde{\varrho}^2}{2c_1c_2c_3\gamma_2} \tag{19}$$

Differentiating (19) along solutions of (17,18) yields

$$\begin{aligned}
\dot{V}_4 &\le -x_e\sigma(x_e) - \frac{c_1}{2c_2}\theta_e^2 - \frac{1}{2c_1c_2}\hat{z}^2 + \\
&\quad + \frac{1}{c_2\gamma_1}\left(\dot{\hat{L}} - \gamma_1\theta_e\dot{\theta}_r\right)\tilde{L} + \\
&\quad + \frac{1}{c_1c_2c_3\gamma_2}\left(\dot{\hat{\varrho}} - \gamma_2\hat{z}\alpha(t)v\tan\phi\right)\tilde{\varrho}
\end{aligned}$$

So, if we define the parameter-update-laws

$$\begin{aligned}
\dot{\hat{L}} &= \gamma_1\theta_e\dot{\theta}_r && \text{(20a)}\\
\dot{\hat{\varrho}} &= \gamma_2\hat{z}\alpha(t)v\tan\phi && \text{(20b)}
\end{aligned}$$

we get

$$\dot{V}_4 \le -x_e\sigma(x_e) - \frac{c_1}{2c_2}\theta_e^2 - \frac{1}{2c_1c_2}\hat{z}^2 \le 0$$

and can establish the following result

Proposition 5.1 *Assume that Assumption 2.7 is satisfied. Then all trajectories of (17,18,20) are globally uniformly bounded. Furthermore,*

$$\lim_{t\to\infty}|x_e(t)| + |\theta_e(t)| + |\hat{z}(t)| + |\phi(t) - \phi_r(t)| = 0$$

If in addition $\dot{\theta}_r(t)$ is persistently exciting, we also have that

$$\lim_{t\to\infty}\left(|y_e(t)| + \left|\tilde{L}(t)\right| + |\tilde{\varrho}(t)|\right) = 0$$

Proof: Similar to the proof of Proposition 4.1 we can show uniform boundedness of all signals and their derivatives with respect to time. From Barbalat's Lemma it follows that x_e, θ_e, and $\hat{z}$ converge to zero as t goes to infinity. From (16) we conclude that $\hat{L} - v\tan\phi + c_2v_ry_e$ converges to zero too. Using Lemma 3.3 we can conclude that also $c_2v_ry_e + \tilde{L}\dot{\theta}_r$ converges to zero. Combining these two results, we obtain that $L\dot{\theta}_r - v\tan\phi$ and therefore $v_r[\tan\phi_r^L - \tan\phi]$ converges to zero. As a result

$$\lim_{t\to\infty}\left|\phi(t) - \phi_r^L(t)\right| = 0$$

Assume that in addition $\dot{\theta}_r(t)$ is persistently exciting. Notice that the (x_e, y_e)-dynamics (17a,17b) can also be seen as a LTV subsystem with an additional disturbance that is bounded and goes to zero as t goes to infinity:

$$\underbrace{\begin{bmatrix}\dot{x}_e\\ \dot{y}_e\end{bmatrix} = \begin{bmatrix}-k & \dot{\theta}_r(t)\\ -\dot{\theta}_r(t) & 0\end{bmatrix}\begin{bmatrix}x_e\\ y_e\end{bmatrix}}_{\text{LTV subsystem}} + \underbrace{\begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}}_{\text{disturbance}} \tag{21}$$

From Lemma 3.2 we know that the LTV subsystem of (21) is globally exponentially stable and therefore, from Lemma 3.4 that also y_e tends to zero as t tends to infinity. Also, the $(\theta_e, \tilde{L})$ dynamics can be seen as a cascade of a LTV subsystem with an additional disturbance that is bounded and goes to zero as t goes to infinity:

$$\begin{bmatrix}\dot{\theta}_e\\ \dot{\tilde{L}}\end{bmatrix} = \begin{bmatrix}-\frac{c_1}{L} & -\frac{1}{L}\dot{\theta}_r(t)\\ \gamma_1\dot{\theta}_r(t) & 0\end{bmatrix}\begin{bmatrix}\theta_e\\ \tilde{L}\end{bmatrix} + \begin{bmatrix}f_3(t)\\ f_4(t)\end{bmatrix}$$

In the same way we can conclude that also $\tilde{L}$ tends to zero as t tends to infinity.

Since we have shown that y_e tends to zero, also the $(\hat{z}, \tilde{\varrho})$ dynamics can be seen as a cascade of a LTV subsystem with an additional disturbance that is bounded and goes to zero as t goes to infinity:

$$\begin{bmatrix} \dot{\hat{z}} \\ \dot{\tilde{\varrho}} \end{bmatrix} = \begin{bmatrix} -k_3 & -c_1 L\dot{\theta}_r(t) \\ \gamma_2 L\dot{\theta}_r(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{z} \\ \tilde{\varrho} \end{bmatrix} + \begin{bmatrix} f_5(t) \\ f_6(t) \end{bmatrix}$$

Therefore, also $\tilde{\varrho}$ tends to zero as t tends to infinity, which concludes the proof. ∎

Corollary 5.2 *Consider the system (1) in closed loop with the control laws (8,18) where the parameter estimates $\hat{L}$ and $\hat{\varrho}$ are updated accoding to (20) and assume that the reference trajectory satisfies (2), Assumption 2.7, and that $\dot{\theta}_r$ is persistently exciting. For the resulting closed-loop system we have*

$$\lim_{t\to\infty} \big(|x(t) - x_r(t)| + |y(t) - y_r(t)| + \\ + |\theta(t) - \theta_r(t)| + \left|\phi(t) - \phi_r^L(t)\right|\big) = 0$$

and convergence of the parameter-estimates to their true value, i.e.

$$\lim_{t\to\infty} \left(\left|\hat{L}(t) - L\right| + \left|\hat{\varrho}(t) - \frac{1}{L}\right| \right) = 0$$

6 Concluding remarks

In this paper we addressed the problem of adaptive state tracking control for a four wheel mobile robot with unknown length. This simple example clearly illustrates that for the general state tracking problem specifying the state trajectory to be tracked and not knowing certain parameters can be in conflict with each other. We propose a formulation for the adaptive tracking problem that is such that it reduces to the tracking problem in case the parameters are known. Not only did we formulate the problem, also a solution was derived.

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Chapter 11

Conclusions

From a practical point of view, the problem of making a system follow a certain trajectory is interesting. One can think of robots that have to perform a desired motion, mobile cars in the harbor driving a prescribed trajectory, a spacecraft moving along a predetermined path, autopilots for airplanes, and numerous other examples. One not necessarily has to think of robots, also a factory following a predetermined production schedule can be thought of as a tracking problem. Since the tracking control problem for arbitrary systems is too complex to be solved in general, all we can do is restrict ourselves to classes of systems with a specific structure. This thesis is concerned with certain nonlinear mechanical systems described by continuous-time models.

11.1 Discussion

In Part I of this thesis we studied the tracking control problem for several non-holonomic systems with two inputs. We introduced the cascaded design approach as a new method for controller design. The key idea is to use one input for stabilizing a subsystem of the tracking error dynamics. Assuming that this stabilization has been achieved, part of the remaining dynamics can be ignored. Next, the second input can be used for stabilizing this simplified tracking error dynamics. One of the advantages of this method is that it is not necessary to transform the system into a certain form by means of a change of co-ordinates. As a result, the controllers derived by means of this method are less complex expressions.

The cascaded design approach was applied successfully for several classes of mechanical systems, including a rotating rigid body, mobile robots, chained-form systems, and an under-actuated ship. Under a persistence of excitation condition on one of the two reference inputs we arrived at *globally* $\mathcal{K}$-exponentially stable tracking error dynamics. This is a type of *uniform* stability which guarantees a certain robustness to disturbances.

Backstepping is a widely used controller design method for nonlinear systems with a triangular structure. In case we compare the cascaded design to a backstepping design, we can say that backstepping assumes a triangular structure of the system and uses this structure to arrive at a triangular closed-loop system in a systematic way. However, what is most important in our eyes is not the triangular structure of the open-loop system, but the triangular structure of the closed-loop system. Using backstepping, this triangular structure for the closed-loop system readily follows starting from a triangular open-loop system. However, the open-loop system not necessarily has to be in triangular form to obtain triangular closed-loop dynamics. This is where the cascaded design approach comes into play. A triangular structure for the open-loop system not necessarily has to be assumed beforehand. All we focus on, is a triangular structure for the closed-loop system. This is exactly what the cascaded approach does. As a result, transformation of the open-loop system for obtaining a triangular structure suitable for backstepping is not necessary.

In all the examples studied in the first part of this thesis we were able to perform the analysis in the original error co-ordinates, leading to a clear structure of the closed-loop dynamics and to much simpler expressions for the control laws (in the original co-ordinates) than achieved so far by means of backstepping. For the systems under consideration, the nonlinear tracking control problem essentially reduced to two linear stabilization problems. As a result, the gain tuning turned out to be not very difficult, since we could rely on linear techniques. Also the extension to other problems is straightforward. If one, for instance, is interested in an H_∞ design, one simply has to solve two linear problems instead of solving partial differential equations that come from the nonlinear H_∞ control problem. Notice that all analysis can be done in the original error co-ordinates, without using state-feedback or input transformations.

In all the examples studied in the first part of this thesis, the connecting term $g(t, z_1, z_2)$ could be upper bounded by a linear function of z_1. At first glance this might seem to be a restriction to the general applicability of the cascaded approach. This is not the case. First of all, the usage of the linearity assumption on $g(t, z_1, z_2)$ is to guarantee boundedness of the solutions of the cascaded system. Whenever one is able to show boundedness of solutions, no assumption on $g(t, z_1, z_2)$ is needed. In case the linearity assumption on $g(t, z_1, z_2)$ is not satisfied, one way to proceed might be to use the original cascaded theorem (Theorem 2.4.3) instead of Corollary 2.4.6. Another way to overcome the problem would be using different co-ordinates, as is done in backstepping. Using as new co-ordinates the difference between the actual value and the desired value of the virtual control assures a linear connecting term in a backstepping design. For a cascaded design the problem can be overcome, similarly.

To summarize, the main contributions in Part I are:

- we introduce the *cascaded design approach*, which does not require the system to be transformed into a specific structure, leading to simpler controllers than found so far;
- we present controllers for solving both the state- and *output*-feedback tracking problem for mobile robots. These controllers yield global *uniform* asymptotic stability and also deal with *input saturations*;
- we present controllers that solve both the state- and *output*-feedback tracking problem for chained-form systems with two inputs. These controllers yield *global uniform* asymptotic stability and partially deal with *input saturations*;
- we present controllers that solve the tracking problem for under-actuated ships by means of state-feedback. These controllers yield *global uniform* asymptotic stability.

In the second part of this thesis we studied three specific problems. First, we addressed the practically important problem of global set-point stabilization of robot manipulators with PID control. We showed that for both the state- and output-feedback problem a “start-integration time” exists such that global asymptotic stability is guaranteed. We also presented criteria on how to choose the start-integration time and the control gains. Finally, we showed in simulations the potential advantages of our linear scheme in comparison to existing nonlinear controllers.

Secondly, we addressed the visual servoing problem of planar robot manipulators under a fixed camera configuration. That is, we considered a robot manipulator operating in the plane, viewed from above with a camera, and of which an image is displayed at a screen. We were able to regulate the tip of the robot manipulator to a specified point at this screen using only position measurements. As an extra difficulty we took into account the possibility that both the camera position and orientation are unknown, as well as certain intrinsic camera parameters (like scale factors, focal length and center offset). In case the camera orientation is known, we introduced a class of visual servoing controllers for both the state- and output-feedback case. In case of unknown camera orientation, adaptive controllers were presented.

Thirdly we addressed the problem of adaptive state tracking control for nonlinear systems. It turned out that formulating the adaptive state-feedback tracking control problem is not straightforward, since specifying the reference state-trajectory can be in conflict with not knowing certain parameters. We showed this difficulty by means of an illustrative example of a four-wheel mobile car with unknown length. We formulated the adaptive state-feedback tracking control problem in such a way that it reduced to the tracking problem in case the parameters are known. Furthermore, we presented a general methodology for solving the problem by means of Lyapunov techniques.

To summarize, the main contributions of Part II are:

- we show *global* asymptotic stability of linear PID controllers when delaying the integral action;
- we introduce *classes of controllers* that solve the visual servoing of planar robots under a fixed camera position for both the state- and *output*-feedback problem. These classes also contain *saturated* controllers. In case of unknown camera orientation a class of adaptive controllers is presented;
- we illustrate difficulties in formulating the adaptive state-tracking problem for nonlinear systems with unknown parameters by means of an example. For this example a suitable *problem formulation of the adaptive state-tracking problem* is given and a *solution* is presented.

11.2 Further research

One of the key properties of the backstepping design methodology is its recursive nature. Once a system is written in a triangular form that is suitable for backstepping, a control law can be built step by step. In a sense, the cascaded design approach can be seen as a one-step vectorial backstepping design where part of the dynamics can be forgotten. The fact that we do not have a Lyapunov function for showing stability of the overall closed-loop system can be seen as a shortcoming of the cascaded design approach (or at least of Theorem 2.4.3), since it makes a recursive application of Theorem 2.4.3 difficult. It would be interesting to find out whether Theorem 2.4.3 could be extended by showing the existence of a Lyapunov function candidate satisfying Assumption A1 along trajectories of the overall closed-loop system. In that way the cascaded design approach can be applied recursively (as in the proof of Proposition 4.3.2).

For the systems considered in the first part of this thesis, the cascaded design approach reduced the nonlinear tracking control problem to two linear stabilization problems. This led us to a clear structure and gave us a simple strategy for tuning the gains. As a result, the from a practical point of view more interesting methods like using filtered measurements, adding integral action or using H_∞-control can be applied without any difficulties. It would be interesting to perform this exercise and study the resulting performance and compare this to existing results.

Another interesting question concerns the actual controller design, namely the gain-tuning. The method we used in this thesis is simply applying optimal control to the resulting linear systems. Notice however, that we have more freedom in choosing the control laws. For the mobile robot for instance, we could add an (almost) arbitrary term $g(t, x_e, y_e, \theta_e)\theta_e$ to v. As long as the function g can be bounded by a linear function of $[x_e, y_e]^T$ the proof still follows the same lines. In a similar way changes to the control laws for the chained-form system or the under-actuated ship can be constructed. This extra freedom enables us to improve performance and it is worth investigating how this freedom can be used in designing a well performing control law.

As mentioned in the beginning of this section, one could view the cascaded design approach as one-step vectorial backstepping. However, the cascaded design approach is more than that, since the design methodology as proposed also turns out to be an eye-opener to recognizing structures that had not been noticed before. However, once a cascaded design leads us to a simple structure, instead of applying the cascaded theorem for concluding asymptotic stability, one could also apply backstepping and redesign the control law for the input that was used for first stabilizing a subsystem. As mentioned in Remark 4.2.6 this can be done for the mobile robot, and leads to a weakening of the persistence of excitation condition on one of the two reference inputs. It is worth investigating whether this idea leads to similar results for the chained-form system and the under-actuated ship.

Another idea to weaken the persistence of excitation condition on one of the two reference inputs could be using the concept of uniform δ-persistence of excitation (uδ-PE) as introduced in (Loría et al. 1999b). In that paper the stabilization problem for a chained-form system of order 3 was solved using this concept. Since the mobile car can be transformed to a chained-form system of order 3 the stabilization problem for a mobile car can be solved in a similar way. It is worth investigating whether the concept of uδ-PE can be successful in weakening the PE condition for general chained-form systems and the under-actuated ship.

For all systems studied in the first part of this thesis it turned out that we could conclude global $\mathcal{K}$-exponential stability, provided that one of the reference inputs was persistently exciting. As it happened to be, the reference input that has to be persistently exciting, also was the first input that we used for stabilizing part of the tracking error dynamics. It is interesting to determine whether this is purely a coincidence or that it can be explained. This observation might also be a third way to overcome the “problem of the PE-condition”. The main reason why this PE-condition on the reference input is annoying, is that for the mobile robot and the under-actuated ship it makes it impossible to follow straight lines. A PE-condition on the other reference input would be less annoying, since that would imply only that the reference has to “keep on moving”. As for the mobile robot this can indeed be done (i.e., first use v to stabize the x_e-dynamics, then use ω for stablizing the simplified remaining dynamics, then conclude GUAS of the overall system under a PE-condition on v_r), it would be worth investigating if this idea can be applied to the tracking problem for the under-actuated ship as well.

In Chapter 10 it was shown, by means of an illustrative example, that the formulation of the general adaptive (state)-tracking problem is a problem in itself. As mentioned in the beginning of this chapter, the problem of following a prescribed trajectory is important from a practical point of view. Unfortunately, in practice we always have to deal with the fact that certain parameters are not known exactly. Therefore, the adaptive tracking control problem is even more interesting to study. However, before we are able to do so, we need to have a correct problem formulation. Obviously, arriving at a proper problem formulation is an interesting problem in itself that deserves attention. And once we have a proper problem formulation, solving the problem is even more interesting.

Appendix A

Proofs

We present the proofs of Theorem 2.3.7, Theorem 2.3.8 and Proposition 6.3.1. However, first we consider the stability of the differential equation

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}y(t) + a_1\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}y(t) + \cdots + a_{m-1}\frac{\mathrm{d}}{\mathrm{d}t}y(t) + a_m y(t) = 0 \tag{A.1}$$

For this system we can define the Hurwitz-determinants

$$\Delta_i = \begin{vmatrix} a_1 & a_3 & a_5 & \dots & a_{2i-1} \\ 1 & a_2 & a_4 & \dots & a_{2i-2} \\ 0 & a_1 & a_3 & \dots & a_{2i-3} \\ 0 & 1 & a_2 & \dots & a_{2i-4} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_i \end{vmatrix} \qquad (i = 1, \dots, m)$$

where, if an element a_j appears in Δ_i with $j > i$, it is assumed to be zero. It was shown by Hurwitz (1895) that the system (A.1) is asymptotically stable if and only if the determinants Δ_i are all positive.

A proof of this result by means of the second method of Lyapunov is less known (see also (Morin and Samson 1997)). If we define

$$b_1 = \Delta_1, \qquad b_2 = \frac{\Delta_2}{\Delta_1}, \qquad b_3 = \frac{\Delta_3}{\Delta_1\Delta_2}, \qquad b_i = \frac{\Delta_{i-3}\Delta_i}{\Delta_{i-2}\Delta_{i-1}} \qquad (i = 4, \dots, m)$$

as was shown by Parks (1962), the system (A.1) can also be represented as

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2 & 0 & \dots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} w. \tag{A.2}$$

Differentiating the Lyapunov function candidate

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \cdots + b_1 b_2 \cdots b_{m-1} w_{m-1}^2 + b_1 b_2 \cdots b_m w_m^2$$

(which is positive definite if and only if the determinants Δ_i are all positive) along solutions of (A.2) results in

$$\dot{V} = -b_1^2 w_1^2.$$

Asymptotic stability then can be shown by invoking LaSalle's theorem (LaSalle 1960).

Inspired by the result of Parks (1962) we look for a state-transformation $z = Sw$, that transforms the system (A.2) into

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_m \\ 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} z \tag{A.3}$$

To start with, we define

$$z_m = w_m.$$

Since $\dot{w}_m = w_{m-1}$, and we would like $\dot{z}_m = z_{m-1}$, we define

$$z_{m-1} = w_{m-1}.$$

Since $\dot{w}_{m-1} = w_{m-2} - b_m w_m$, and we would like $\dot{z}_{m-1} = z_{m-2}$, we define

$$z_{m-2} = w_{m-2} - b_m w_m.$$

Proceeding similarly, we define all z_k and obtain an expression that looks like

$$z_k = w_k + s_{k,k+2} w_{k+2} + s_{k,k+4} w_{k+4} + \dots. \tag{A.4}$$

By this construction of the state-transformation, we are guaranteed to meet the $m-1$ final equations of (A.3). The only thing that remains to be verified is whether the equation for $\dot{z}_1$ holds. From the structure displayed in (A.4) we know that the matrix S is nonsingular. Therefore, we can write

$$\dot{z}_1 = -\alpha_1 z_1 - \alpha_2 z_2 - \cdots - \alpha_n z_n, \qquad \alpha_i \in \mathbb{R} \qquad (i = 1, \dots, m).$$

The characteristic polynomial of the transformed system then becomes

$$\lambda^m + \alpha_1 \lambda^{m-1} + \cdots + \alpha_{m-1} \lambda + \alpha_m.$$

Since a state-transformation does not change the characteristic polynomial and we know from Parks (1962) that the characteristic polynomial of (A.2) equals

$$\lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1} \lambda + a_m,$$

clearly $\alpha_i = a_i$ ($i = 1, \ldots, m$).

Before we can prove Theorem 2.3.7 and Theorem 2.3.8 we need to remark one thing about this transformation. When we define $T = S^{-1}$, we know that

$$\begin{aligned} w_1 &= z_1 + t_{1,3}z_3 + t_{1,5}z_5 + \ldots \\ w_2 &= z_2 + t_{2,4}z_4 + t_{2,6}z_6 + \ldots. \end{aligned}$$

But also $\dot{w}_1 = -a_1 w_1 - b_2 w_2$ (notice that $b_1 = a_1$). Therefore,

$$\begin{aligned} \dot{w}_1 &= \dot{z}_1 + t_{1,3}\dot{z}_3 + t_{1,5}\dot{z}_5 + \ldots \\ &= (-a_1 z_1 - a_2 z_2 - \cdots - a_n z_n) + t_{1,3}z_2 + t_{1,5}z_4 + \ldots \\ &= [-a_1 z_1 - a_3 z_3 - \ldots] + [(t_{1,3} - a_2)z_2 + (t_{1,5} - a_4)z_4 + \ldots]. \end{aligned}$$

So obviously

$$w_1 = z_1 + \frac{a_3}{a_1}z_3 + \frac{a_5}{a_1}z_5 + \ldots. \tag{A.5}$$

Knowing this state-transformation and (A.5) we can start proving Theorems 2.3.7 and 2.3.8.

A.1 Proof of Theorem 2.3.7

Proof. We need to show global uniform exponential stability (GUES) of the system (2.12), which is described by

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2\phi(t) & -a_3 & -a_4\phi(t) & \ldots \\ \phi(t) & 0 & \ldots & \ldots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \phi(t) & 0 \end{bmatrix} z. \tag{A.6}$$

We can also write the system (A.6) as

$$\dot{z} = \phi(t) \begin{bmatrix} -a_1 & -a_2 & \ldots & \ldots & -a_m \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix} z + (\phi(t) - 1) \begin{bmatrix} a_1 z_1 + a_3 z_3 + \ldots \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}.$$

When we apply the change of co-ordinates $z = Sw$ as defined before, we obtain

$$\dot{w} = \phi(t) \begin{bmatrix} -b_1 & -b_2 & 0 & \ldots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix} w + (\phi(t) - 1) \begin{bmatrix} 1 & * & \ldots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 w_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

which (using $a_1 = b_1$) can we rewritten as

$$\dot{w} = \begin{bmatrix} -b_1 & -b_2\phi(t) & 0 & \dots & 0 \\ \phi(t) & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m\phi(t) \\ 0 & \dots & 0 & \phi(t) & 0 \end{bmatrix} w. \tag{A.7}$$

Consider the Lyapunov function candidate

$$V = b_1 w_1^2 + b_1 b_2 w_2^2 + \dots + b_1 b_2 \cdots b_{m-1} w_{m-1}^2 + b_1 b_2 \cdots b_m w_m^2 \tag{A.8}$$

which is positive definite if and only if

$$\lambda^m + a_1 \lambda^{m-1} + \dots + a_{m-1}\lambda + a_m$$

is a Hurwitz-polynomial. Differentiating (A.8) along solutions of (A.7) results in

$$\dot{V} = -b_1^2 w_1^2$$

which is negative semi-definite.

It is well-known (Khalil 1996) that the origin of the system (A.7) is globally uniformly exponentially stable (GUES) if the pair

$$\left(\begin{bmatrix} -b_1 & -b_2\phi(t) & 0 & \dots & 0 \\ \phi(t) & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -b_m\phi(t) \\ 0 & \dots & 0 & \phi(t) & 0 \end{bmatrix}, \begin{bmatrix} b_1, 0, \dots, 0 \end{bmatrix} \right) \tag{A.9}$$

is uniformly completely observable (UCO).

If $\phi(t)$ is persistently exciting, it follows immediately from Corollary 2.3.4 that the pair (A.9) is UCO, which completes the proof. □

A.2 Proof of Theorem 2.3.8

Proof. The system (2.13) can be written as

$$\dot{z}_1 = \underbrace{\begin{bmatrix} -k_2 & -k_3\phi & -k_4 & -k_5\phi & \dots \\ \phi & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \phi & 0 \end{bmatrix} z_1}_{f_1(t,z_1)} + \underbrace{\begin{bmatrix} -k_2 & -k_3\phi & -k_4 & -k_5\phi & \dots \\ 0 & \dots & \dots & \dots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} z_2}_{g(t,z_1,z_2)}$$

$$\dot{z}_2 = \underbrace{\begin{bmatrix} 0 & \dots & \dots & 0 & \vdots \\ \phi & \ddots & & \vdots & -l_5\phi \\ 0 & \ddots & \ddots & \vdots & -l_4 \\ \vdots & & \ddots & 0 & -l_3\phi \\ 0 & \dots & 0 & \phi & -l_2 \end{bmatrix} z_2}_{f_2(t,z_2)}$$

where

$$z_1 = \begin{bmatrix} x_{2,e} & x_{3,e} & \dots & x_{n,e} \end{bmatrix}^T$$

and

$$z_2 = \begin{bmatrix} x_{2,e} - \hat{x}_{2,e} & \tilde{x}_{3,e} - \hat{x}_{3,e} & \dots & \tilde{x}_{n,e} - \hat{x}_{n,e} \end{bmatrix}^T.$$

Since $\phi(t)$ is persistently exciting (PE) and k_i, l_i are such that the polynomials (2.14) are Hurwitz, we know from Theorem 2.3.7 that the systems $\dot{z}_1 = f_1(t, z_1)$ and $\dot{z}_2 = f_1(t, z_2)$ are globally uniformly exponentially stable (GUES).

Then the result follows immediately from Corollary 2.4.6, (since $g(t, z_1, z_2)$ satisfies (2.25)) and the fact that a LTV system which is globally uniformly asymptotically stable (GUAS) is also GUES (Theorem 2.3.9). □

A.3 Proof of Proposition 6.3.1

Before we prove Proposition 6.3.1 we first prove the following lemma:

Lemma A.3.1. *Let the following conditions be given:*

$$k_1 > d_{22} - d_{11} \tag{A.10a}$$

$$k_2 = \frac{k_4(k_4 + k_1 + d_{11} - d_{22})}{\frac{m_{11}}{m_{22}}(d_{22}k_4 + m_{11}k_3)} \tag{A.10b}$$

$$0 < k_3 < (k_1 + d_{11} - d_{22})\frac{d_{22}}{m_{11}} \tag{A.10c}$$

$$k_4 > 0. \tag{A.10d}$$

Define λ and μ ($\lambda < \mu$) by means of

$$\lambda + \mu = \frac{k_1 + d_{11}}{m_{11}} \tag{A.11a}$$

$$\lambda\mu = \frac{k_3}{m_{11}}, \tag{A.11b}$$

which is similar to saying that λ and μ are the roots of the polynomial

$$p(x) = m_{11}x^2 - (k_1 + d_{11})x + k_3. \tag{A.12}$$

Then λ and μ are well-defined, and furthermore

$$0 < \mu - \lambda \tag{A.13a}$$

$$0 < d_{22} - m_{11}\lambda \tag{A.13b}$$

$$0 < m_{11}\mu - d_{22} \tag{A.13c}$$

$$0 < m_{11}^2\lambda\mu + d_{22}k_4 \tag{A.13d}$$

$$0 < k_4 + m_{11}\lambda \tag{A.13e}$$

$$0 < k_4 + m_{11}\mu. \tag{A.13f}$$

Proof. First, we remark that from (A.10a) and the fact that $d_{22} > 0$, $m_{11} > 0$ we have:

$$0 < \frac{d_{22}}{m_{11}} < \frac{k_1 + d_{11}}{m_{11}}.$$

Consider the polynomial (A.12). Then obviously

$$p(0) = p\left(\frac{k_1 + d_{11}}{m_{11}}\right) = k_3 > 0$$

and

$$p\left(\frac{d_{22}}{m_{11}}\right) = m_{11}\left(\frac{d_{22}}{m_{11}}\right)^2 - (k_1 + d_{11})\frac{d_{22}}{m_{11}} + k_3 = (d_{22} - k_1 - d_{11})\frac{d_{22}}{m_{11}} + k_3 < 0.$$

Therefore, from the intermediate value theorem we know that a constant λ exists, $0 < \lambda < \frac{d_{22}}{m_{11}}$, such that $p(\lambda) = 0$ and also a μ, $\frac{d_{22}}{m_{11}} < \mu < \frac{k_1+d_{11}}{m_{11}}$, such that $p(\mu) = 0$. As a result we obtain that λ and μ are well-defined by means of (A.11). From (A.10) and

$$0 < \lambda < \frac{d_{22}}{m_{11}} < \mu$$

we can conclude the inequalities (A.13). $\square$

Proof of Proposition 6.3.1. The closed-loop system (6.11, 6.12) is given by

$$\begin{bmatrix} \dot{u}_e \\ \dot{v}_e \\ \dot{z}_{1,e} \\ \dot{z}_{2,e} \end{bmatrix} = \begin{bmatrix} -\frac{k_1+d_{11}}{m_{11}} & \frac{k_2+m_{22}}{m_{11}} r_r(t) & -\frac{k_3}{m_{11}} & \frac{k_4}{m_{11}} r_r(t) \\ -\frac{m_{11}}{m_{22}} r_r(t) & -\frac{d_{22}}{m_{22}} & 0 & 0 \\ 1 & 0 & 0 & r_r(t) \\ 0 & 1 & -r_r(t) & 0 \end{bmatrix} \begin{bmatrix} u_e \\ v_e \\ z_{1,e} \\ z_{2,e} \end{bmatrix}. \tag{A.14}$$

If we define λ and μ as in (A.11) and use (A.10b), the closed-loop dynamics (A.14) can be expressed as

$$\begin{bmatrix} \dot{u}_e \\ \dot{v}_e \\ \dot{z}_{1,e} \\ \dot{z}_{2,e} \end{bmatrix} = \begin{bmatrix} -(\lambda+\mu) & \frac{m_{22}(k_4+m_{11}\lambda)(k_4+m_{11}\mu)}{m_{11}(m_{11}^2\lambda\mu+d_{22}k_4)} r_r(t) & -\lambda\mu & \frac{k_4}{m_{11}} r_r(t) \\ -\frac{m_{11}}{m_{22}} r_r(t) & -\frac{d_{22}}{m_{22}} & 0 & 0 \\ 1 & 0 & 0 & r_r(t) \\ 0 & 1 & -r_r(t) & 0 \end{bmatrix} \begin{bmatrix} u_e \\ v_e \\ z_{1,e} \\ z_{2,e} \end{bmatrix}.$$

Using the change of co-ordinates

$$\begin{bmatrix} u_e \\ v_e \\ z_{1,e} \\ z_{2,e} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\mu}{\mu-\lambda} & -\frac{\lambda}{\mu-\lambda} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{\mu-\lambda} & \frac{1}{\mu-\lambda} \\ \frac{m_{22}}{d_{22}} & -\frac{m_{22}}{d_{22}} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

which, due to (A.13a), is well-defined, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{m_{11}\mu-d_{22}}{m_{22}(\mu-\lambda)} r_r & -\frac{d_{22}-m_{11}\lambda}{m_{22}(\mu-\lambda)} r_r \\ 0 & -\frac{d_{22}}{m_{22}} & -\frac{m_{11}\mu}{m_{22}(\mu-\lambda)} r_r & \frac{m_{11}\lambda}{m_{22}(\mu-\lambda)} r_r \\ \frac{m_{22}(k_4+m_{11}\lambda)}{m_{11}d_{22}} r_r & \frac{m_{22}(k_4+m_{11}\lambda)(d_{22}-m_{11}\lambda)\mu}{d_{22}(m_{11}^2\lambda\mu+d_{22}k_4)} r_r & -\mu & 0 \\ \frac{m_{22}(k_4+m_{11}\mu)}{m_{11}d_{22}} r_r & -\frac{m_{22}(k_4+m_{11}\mu)(m_{11}\mu-d_{22})\lambda}{d_{22}(m_{11}^2\lambda\mu+d_{22}k_4)} r_r & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{A.15}$$

Differentiating the positive definite (cf. (A.13)) Lyapunov-function candidate

$$V = \frac{m_{22}^2}{2m_{11}d_{22}} x_1^2 + \frac{m_{22}^2(d_{22}-m_{11}\lambda)(m_{11}\mu-d_{22})}{2m_{11}d_{22}(m_{11}^2\lambda\mu+d_{22}k_4)} x_2^2 + \frac{(m_{11}\mu-d_{22})}{2(k_4+m_{11}\lambda)(\mu-\lambda)} x_3^2 + \\ + \frac{(d_{22}-m_{11}\lambda)}{2(k_4+m_{11}\mu)(\mu-\lambda)} x_4^2$$

along solutions of (A.15) yields

$$\dot{V} = -\frac{(m_{11}\mu - d_{22})(d_{22} - m_{11}\lambda)m_{22}}{m_{11}(m_{11}^2\lambda\mu + d_{22}k_4)}x_2^2 - \frac{\mu(m_{11}\mu - d_{22})}{(\mu-\lambda)(k_4+\lambda)}x_3^2 - \\ - \frac{\lambda(d_{22} - m_{11}\lambda)}{(\mu-\lambda)(k_4 + m_{11}\mu)}x_4^2$$

which is negative semi-definite (cf. (A.13)).

It is well-known (Khalil 1996) that the origin of the system (A.15) is globally uniformly exponentially stable (GUES) if the pair $(A(t), C)$ is uniformly completely observable (UCO), where

$$A(t) = \begin{bmatrix} 0 & 0 & -\frac{m_{11}\mu - d_{22}}{m_{22}(\mu-\lambda)}r_r & -\frac{d_{22}-m_{11}\lambda}{m_{22}(\mu-\lambda)}r_r \\ 0 & -\frac{d_{22}}{m_{22}} & -\frac{m_{11}\mu}{m_{22}(\mu-\lambda)}r_r & \frac{m_{11}\lambda}{m_{22}(\mu-\lambda)}r_r \\ \frac{m_{22}(k_4+m_{11}\lambda)}{m_{11}d_{22}}r_r & \frac{m_{22}(k_4+m_{11}\lambda)(d_{22}-m_{11}\lambda)\mu}{d_{22}(m_{11}^2\lambda\mu+d_{22}k_4)}r_r & -\mu & 0 \\ \frac{m_{22}(k_4+m_{11}\mu)}{m_{11}d_{22}}r_r & -\frac{m_{22}(k_4+m_{11}\mu)(m_{11}\mu-d_{22})\lambda}{d_{22}(m_{11}^2\lambda\mu+d_{22}k_4)}r_r & 0 & -\lambda \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & \sqrt{\frac{(m_{11}\mu-d_{22})(d_{22}-m_{11}\lambda)m_{22}}{m_{11}(m_{11}^2\lambda\mu+d_{22}k_4)}} & \sqrt{\frac{\mu(m_{11}\mu-d_{22})}{(\mu-\lambda)(k_4+\lambda)}} & \sqrt{\frac{\lambda(d_{22}-m_{11}\lambda)}{(\mu-\lambda)(k_4+m_{11}\mu)}} \end{bmatrix}.$$

If $r_r(t)$ is persistently exciting, it follows from Corollary 2.3.4 that the pair $(A(t), C)$ is UCO, which completes the proof. □

Appendix B

Backstepping controller for an under-actuated ship

The expression for $f_{k_1,k_2,a_1,a_2,a_3,\alpha,\lambda,\gamma}\left(u_e, v_e, r_e, z_{1,e}, z_{2,e}, z_{3,e}, t\right)$ of (6.17) is:

$$
\begin{aligned}
u_2 = -(&-m_{33}a_3cd^2u_rk_2^2z_{1,e}z_{3,e} + m_{33}a_3k_2^4u_rz_{2,e} + m_{33}a_3a_2z_{3,e}c^2r_r^2 \\
&+ m_{33}a_3\gamma c^2r_r^2z_{3,e}z_{2,r}v_r + m_{33}a_3\alpha c^4r_r^2u_rz_{3,e}^2\gamma z_{2,e} \\
&+ m_{33}a_3\gamma c^2r_r^2z_{3,e}z_{1,r}u_e - m_{33}a_3\gamma c^2r_r^2z_{1,e}v_r \\
&- m_{22}u_rv_r\gamma c^2r_r^2z_{1,e}^2 - m_{33}a_3\gamma c^2r_r^2z_{1,e}v_e \\
&+ m_{33}a_3\gamma c^2r_r^2z_{3,e}z_{2,r}v_e + m_{33}a_3\alpha c^3r_r^2v_e\gamma z_{2,e}z_{3,e} \\
&- m_{22}u_rv_r\gamma c^2r_r^2z_{2,e}^2 + m_{33}a_3\alpha c^3r_r^2k_2z_{2,e}^2\gamma z_{3,e} \\
&- m_{33}a_3\alpha c^4r_r^2u_rz_{3,e}\gamma z_{1,e} + m_{33}a_3\alpha c^3r_r^2k_2z_{1,e}z_{3,e}^2\gamma z_{2,r} \\
&+ m_{33}a_3\alpha c^3r_r^2k_2z_{2,e}\gamma z_{3,e}z_{2,r} + m_{33}a_3\alpha c^3r_r^2v_e\gamma z_{3,e}z_{2,r} \\
&- m_{33}a_3\alpha c^4r_r^2u_ez_{3,e}\gamma z_{1,e} - m_{33}a_3\alpha c^3r_r^2v_e\gamma z_{1,e} \\
&+ m_{33}a_3\alpha c^4r_r^2u_ez_{3,e}^2\gamma z_{2,e} + m_{33}a_3\alpha c^4r_r^2u_ez_{3,e}^2\gamma z_{2,r} \\
&+ m_{33}a_3\alpha c^3r_r^2k_2z_{1,e}z_{3,e}^2\gamma z_{2,e} - m_{22}u_rv_r\gamma c^2r_r^2z_{2,e}z_{2,r} \\
&+ m_{33}a_3\alpha c^3r_r^2k_2z_{3,e}^2z_{1,r}\gamma z_{2,e} - m_{33}a_3\alpha c^3r_r^2k_2z_{2,e}\gamma z_{1,e} \\
&- m_{33}a_3\gamma c^3r_r^2z_{1,e}z_{3,e}u_r + m_{33}a_3r_e\gamma c^2r_r^2z_{2,e}z_{2,r} \\
&+ m_{33}a_3r_e\gamma c^2r_r^2z_{1,e}z_{1,r} - m_{33}a_3\alpha c^3r_r^2k_2z_{1,e}^2z_{3,e}\gamma \\
&+ m_{33}a_3\alpha c^3r_r^2k_2z_{3,e}^2z_{1,r}\gamma z_{2,r} + m_{33}a_3\alpha c^4r_r^2u_rz_{3,e}^2\gamma z_{2,r} \\
&- m_{33}a_3\gamma c^3r_r^2z_{2,e}u_r + m_{33}a_3\gamma c^2r_r^2z_{1,e}z_{3,e}u_r \\
&- m_{33}a_3\gamma c^3r_r^2z_{1,e}z_{3,e}u_e + m_{33}a_3r_e\gamma c^2r_r^2z_{2,e}^2 \\
&+ m_{33}a_3r_e\gamma c^2r_r^2z_{1,e}^2 + m_{11}u_ev_r\gamma c^2r_r^2z_{1,e}^2 \\
&+ m_{33}a_3\gamma c^2r_r^2z_{3,e}z_{1,r}u_r + m_{33}a_3\gamma c^2r_r^2z_{2,e}u_e + m_{33}a_3\gamma c^2r_r^2z_{2,e}u_r \\
&- m_{33}a_3\gamma c^3r_r^2z_{2,e}u_e + m_{33}a_3\gamma c^2r_r^2z_{1,e}z_{3,e}u_e \\
&+ m_{11}u_rv_e\gamma c^2r_r^2z_{2,e}z_{2,r} + m_{11}u_rv_e\gamma c^2r_r^2z_{1,e}z_{1,r} \\
&+ m_{11}u_ev_r\gamma c^2r_r^2z_{2,e}z_{2,r} + m_{11}u_rv_r\gamma c^2r_r^2z_{1,e}z_{1,r} \\
&+ m_{11}u_rv_e\gamma c^2r_r^2z_{2,e}^2 + m_{11}u_rv_e\gamma c^2r_r^2z_{1,e}^2 \\
&+ m_{11}u_ev_r\gamma c^2r_r^2z_{1,e}z_{1,r} + m_{11}u_ev_r\gamma c^2r_r^2z_{2,e}^2 \\
&- m_{22}u_ev_e\gamma c^2r_r^2z_{1,e}^2 + m_{11}u_rv_r\gamma c^2r_r^2z_{2,e}^2 \\
&- m_{22}u_ev_e\gamma c^2r_r^2z_{2,e}z_{2,r} - m_{22}u_ev_e\gamma c^2r_r^2z_{1,e}z_{1,r} \\
&- m_{22}u_ev_r\gamma c^2r_r^2z_{1,e}z_{1,r} - m_{22}u_ev_e\gamma c^2r_r^2z_{2,e}^2 \\
&+ m_{11}u_rv_r\gamma c^2r_r^2z_{1,e}^2 + m_{11}u_rv_r\gamma c^2r_r^2z_{2,e}z_{2,r} \\
&- m_{22}u_ev_r\gamma c^2r_r^2z_{1,e}^2 - m_{22}u_ev_r\gamma c^2r_r^2z_{2,e}z_{2,r} \\
&- m_{22}u_ev_r\gamma c^2r_r^2z_{2,e}^2 - m_{22}u_rv_e\gamma c^2r_r^2z_{2,e}^2 - m_{22}u_rv_e\gamma c^2r_r^2z_{1,e}^2 \\
&- m_{22}u_rv_e\gamma c^2r_r^2z_{1,e}z_{1,r} - m_{22}u_rv_r\gamma c^2r_r^2z_{1,e}z_{1,r} \\
&- m_{33}a_3\alpha c^3r_r^2k_2z_{3,e}z_{1,r}\gamma z_{1,e} + m_{11}u_ev_e\gamma c^2r_r^2z_{1,e}z_{1,r} \\
&+ m_{11}u_ev_e\gamma c^2r_r^2z_{1,e}^2 - m_{22}u_rv_e\gamma c^2r_r^2z_{2,e}z_{2,r} \\
&- m_{33}a_3\gamma c^3r_r^2z_{3,e}z_{1,r}u_e - m_{33}a_3\gamma c^3r_r^2z_{3,e}z_{1,r}u_r \\
&+ m_{11}u_ev_e\gamma c^2r_r^2z_{2,e}z_{2,r} + m_{33}a_3\gamma c^2r_r^2z_{2,e}z_{3,e}v_e \\
&+ m_{33}a_3\gamma c^2r_r^2z_{2,e}z_{3,e}v_r + m_{11}u_ev_e\gamma c^2r_r^2z_{2,e}^2 \\
&- m_{11}u_e^2v_rk_2z_{2,e}dcr_r - m_{11}u_ev_rk_2^2k_1cr_rz_{2,e}^2z_{3,e} \\
&- m_{11}u_ev_rk_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,r} + m_{11}u_ev_rk_1cr_rk_2^2z_{1,e}^2z_{3,e} \\
&+ m_{11}u_ev_r\alpha k_2^2c^2r_r^2z_{1,e}z_{3,e}z_{2,r} + m_{11}u_rv_ek_2^2z_{2,e}c^2\alpha r_r^2z_{3,e}z_{1,r} \\
&- m_{11}u_rv_ek_2z_{2,e}dcr_ru_e + m_{11}u_rv_ek_2^2z_{2,e}r_rcu_e \\
&+ m_{11}u_rv_ek_2^2d^2z_{3,e}z_{1,r}z_{2,r} - 2m_{11}u_rv_edk_2^3z_{1,e}z_{3,e}z_{2,e} \\
&- 2m_{11}u_rv_edk_2^3z_{1,e}z_{3,e}z_{2,r} - 2m_{11}u_rv_edk_2^3z_{3,e}z_{1,r}z_{2,e} \\
&- 2m_{11}u_rv_edk_2^3z_{3,e}z_{1,r}z_{2,r} + m_{11}u_rv_ek_2^2r_rcu_ez_{2,r} \\
&+ m_{11}u_rv_ek_2^2k_1cr_rz_{1,e}z_{2,r} - 2m_{11}u_rv_ek_2^2k_1cr_rz_{2,e}z_{3,e}z_{2,r} \\
&+ m_{11}u_rv_ek_2dk_1cr_rz_{3,e}z_{2,r}^2 + m_{11}u_rv_ek_2dk_1cr_rz_{2,e}^2z_{3,e} \\
&- m_{11}u_rv_ek_2dcr_ru_ez_{2,r} - m_{11}u_rv_ek_2dk_1cr_rz_{1,e}z_{2,r} \\
&+ 2m_{11}u_rv_ek_2dk_1cr_rz_{2,e}z_{3,e}z_{2,r} + m_{11}u_rv_ek_2^2d^2z_{2,e}^2 \\
&- 2m_{11}u_ev_edk_2^3z_{2,e}z_{2,r} + m_{11}u_ev_ek_2^4z_{1,e}z_{3,e}z_{2,e} \\
&+ m_{11}u_ev_ek_2^4z_{1,e}z_{3,e}z_{2,r} + m_{11}u_ev_ek_2^4z_{3,e}z_{1,r}z_{2,e} \\
&+ m_{11}u_ev_ek_2^4z_{3,e}z_{1,r}z_{2,r} - m_{11}u_ev_ek_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,r} \\
&- m_{11}u_ev_ek_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,e} - m_{11}u_ev_ek_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,r} \\
&- 2m_{11}u_ev_ek_1cr_rk_2dz_{2,e}z_{1,e} - m_{11}u_ev_ek_1cr_rk_2dz_{2,e}z_{1,r} \\
&- m_{11}u_ev_ek_1cr_rk_2dz_{1,e}^2z_{3,e} - 2m_{11}u_ev_ek_1cr_rk_2dz_{1,e}z_{3,e}z_{1,r} \\
&- m_{11}u_ev_ek_1cr_rk_2dz_{3,e}z_{1,r}^2 + 2m_{11}u_ev_ek_1cr_rk_2^2z_{2,e}z_{1,e} \\
&+ m_{11}u_ev_ek_1cr_rk_2^2z_{2,e}z_{1,r} + m_{11}u_ev_ek_1cr_rk_2^2z_{1,e}^2z_{3,e} \\
&+ m_{11}u_e^2v_ek_1c^2r_r^2z_{1,e} - m_{11}u_ev_ek_2dk_1cr_rz_{1,e}z_{2,r} \\
&+ 2m_{11}u_ev_ek_2dk_1cr_rz_{2,e}z_{3,e}z_{2,r} + m_{11}u_ev_ek_2^2d^2z_{2,e}^2 \\
&- 2m_{11}u_ev_edk_2^3z_{2,e}^2 + m_{11}u_ev_ek_2^4z_{2,e}z_{2,r} \\
&- m_{11}u_ev_ek_2^2k_1cr_rz_{2,e}^2z_{3,e} - m_{11}u_ev_ek_2^2k_1cr_rz_{3,e}z_{2,r}^2 \\
&+ m_{11}u_ev_ek_2^2d^2z_{1,e}z_{3,e}z_{2,e} + m_{11}u_ev_ek_2^2d^2z_{1,e}z_{3,e}z_{2,r} \\
&+ m_{11}u_ev_ek_2^2d^2z_{2,e}z_{2,r} - 2m_{11}u_ev_edk_2^3z_{1,e}z_{3,e}z_{2,r} \\
&- 2m_{11}u_ev_edk_2^3z_{3,e}z_{1,r}z_{2,e} - 2m_{11}u_ev_edk_2^3z_{3,e}z_{1,r}z_{2,r} \\
&+ m_{11}u_e^2v_ek_2^2r_rcz_{2,r} + m_{11}u_ev_ek_2^2k_1cr_rz_{1,e}z_{2,r} \\
&- 2m_{11}u_ev_ek_2^2k_1cr_rz_{2,e}z_{3,e}z_{2,r} + m_{11}u_ev_ek_2dk_1cr_rz_{3,e}z_{2,r}^2 \\
&+ m_{11}u_ev_ek_2dk_1cr_rz_{2,e}^2z_{3,e} - 2m_{11}u_ev_edk_2^3z_{1,e}z_{3,e}z_{2,e} \\
&- m_{11}u_e^2v_ek_2dcr_rz_{2,r} + m_{11}u_ev_ek_2^2d^2z_{3,e}z_{1,r}z_{2,e} \\
&+ 2m_{11}u_ev_ek_1cr_rk_2^2z_{1,e}z_{3,e}z_{1,r} + m_{11}u_ev_ek_1cr_rk_2^2z_{3,e}z_{1,r}^2 \\
&+ m_{11}u_e^2v_ek_1c^2r_r^2z_{1,r} + m_{11}u_ev_ek_1^2c^2r_r^2z_{1,e}z_{1,r} \\
&+ m_{11}u_ev_e^2\alpha k_2c^2r_r^2z_{2,r} + m_{11}u_e^2v_e\alpha k_2c^3r_r^2z_{3,e}z_{2,r} \\
&+ 2m_{11}u_ev_e\alpha k_2c^3r_r^2u_rz_{3,e}z_{2,r} + m_{11}u_ev_e\alpha k_2^2c^2r_r^2z_{2,e}z_{2,r} \\
&+ m_{11}u_ev_e\alpha k_2^2c^2r_r^2z_{1,e}z_{3,e}z_{2,r} + m_{11}u_ev_e\alpha k_2^2c^2r_r^2z_{3,e}z_{1,r}z_{2,r} \\
&+ m_{11}u_ev_ek_1^2c^2r_r^2z_{1,e}^2 - m_{11}u_ev_ek_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,e} \\
&+ m_{11}u_ev_e\lambda c^2r_r^2 + m_{11}u_e^2v_rc^3z_{3,e}\alpha r_r^2k_2z_{2,e} \\
&+ 2m_{11}u_ev_rc^3u_rz_{3,e}\alpha r_r^2k_2z_{2,e} + m_{11}u_ev_rv_ec^2\alpha r_r^2k_2z_{2,e} \\
&+ m_{11}u_ev_rk_2^2z_{2,e}^2c^2\alpha r_r^2 + m_{11}u_ev_rk_2^2z_{2,e}c^2\alpha r_r^2z_{1,e}z_{3,e} \\
&+ m_{11}u_ev_rk_2^2z_{2,e}c^2\alpha r_r^2z_{3,e}z_{1,r} - m_{22}u_r^2v_rc^3z_{3,e}\alpha r_r^2k_2z_{2,e} \\
&- m_{22}u_rv_rv_ec^2\alpha r_r^2k_2z_{2,e} - m_{22}u_rv_rk_2^2z_{2,e}^2c^2\alpha r_r^2 \\
&- m_{22}u_rv_rk_2^2z_{2,e}c^2\alpha r_r^2z_{1,e}z_{3,e} - m_{22}u_rv_rk_2^2z_{2,e}c^2\alpha r_r^2z_{3,e}z_{1,r} \\
&+ m_{22}u_rv_rk_2z_{2,e}dcr_ru_e - m_{22}u_rv_rk_2^2z_{2,e}r_rcu_e \\
&- m_{22}u_rv_rk_2^2d^2z_{3,e}z_{1,r}z_{2,r} + 2m_{22}u_rv_rdk_2^3z_{1,e}z_{3,e}z_{2,e} \\
&+ 2m_{22}u_rv_rdk_2^3z_{1,e}z_{3,e}z_{2,r} + 2m_{22}u_rv_rdk_2^3z_{3,e}z_{1,r}z_{2,e} \\
&+ 2m_{22}u_rv_rdk_2^3z_{3,e}z_{1,r}z_{2,r} - m_{22}u_rv_rk_2^2r_rcu_ez_{2,r} \\
&+ m_{22}u_rv_ek_1cr_rk_2dz_{1,e}^2z_{3,e} + 2m_{22}u_rv_ek_1cr_rk_2dz_{1,e}z_{3,e}z_{1,r} \\
&+ m_{22}u_rv_ek_1cr_rk_2dz_{3,e}z_{1,r}^2 - 2m_{22}u_rv_ek_1cr_rk_2^2z_{2,e}z_{1,e} \\
&- m_{22}u_rv_ek_1cr_rk_2^2z_{2,e}z_{1,r} - m_{22}u_rv_ek_1cr_rk_2^2z_{1,e}^2z_{3,e} \\
&- 2m_{22}u_rv_ek_1cr_rk_2^2z_{1,e}z_{3,e}z_{1,r} - m_{22}u_rv_ek_1cr_rk_2^2z_{3,e}z_{1,r}^2
\end{aligned}
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\begin{aligned}
&- m_{22}u_rv_ek_1c^2r_r^2u_ez_{1,e} - m_{22}u_rv_ek_1c^2r_r^2u_ez_{1,r}\\
&- m_{22}u_rv_ek_1^2c^2r_r^2z_{1,e}z_{1,r} - m_{22}u_rv_e^2\alpha k_2c^2r_r^2z_{2,r}\\
&- m_{22}u_r^2v_e\alpha k_2c^3r_r^2z_{3,e}z_{2,r} - m_{22}u_rv_e\alpha k_2^2c^2r_r^2z_{2,e}z_{2,r}\\
&- m_{22}u_rv_e\alpha k_2^2c^2r_r^2z_{1,e}z_{3,e}z_{2,r} - m_{22}u_rv_e\alpha k_2^2c^2r_r^2z_{3,e}z_{1,r}z_{2,r}\\
&- m_{22}u_rv_ek_1^2c^2r_r^2z_{1,e}^2 + m_{22}u_rv_ek_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,e}\\
&- m_{22}u_rv_ek_2^4z_{2,e}^2 - m_{22}u_rv_ek_2^2d^2z_{1,e}z_{3,e}z_{2,r}\\
&- m_{22}u_rv_ek_2^2d^2z_{3,e}z_{1,r}z_{2,e} - m_{22}u_rv_ek_2^2d^2z_{2,e}z_{2,r}\\
&+ 2m_{22}u_rv_edk_2^3z_{2,e}z_{2,r} - m_{22}u_rv_ek_2^4z_{1,e}z_{3,e}z_{2,e}\\
&- m_{22}u_rv_ek_2^4z_{1,e}z_{3,e}z_{2,r} - m_{22}u_rv_ek_2^4z_{3,e}z_{1,r}z_{2,e}\\
&- m_{22}u_rv_ek_2^4z_{3,e}z_{1,r}z_{2,r} + m_{22}u_rv_ek_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,r}\\
&+ m_{22}u_rv_ek_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,e} + m_{22}u_rv_ek_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,r}\\
&+ 2m_{22}u_rv_ek_1cr_rk_2dz_{2,e}z_{1,e} - m_{22}u_rv_ek_2^2d^2z_{1,e}z_{3,e}z_{2,e}\\
&+ m_{22}u_rv_ek_1cr_rk_2dz_{2,e}z_{1,r} - m_{22}u_rv_rk_2^2k_1cr_rz_{1,e}z_{2,r}\\
&+ 2m_{22}u_rv_rk_1cr_rk_2dz_{2,e}z_{1,e} - m_{22}u_rv_r\alpha k_2c^2r_r^2v_ez_{2,r}\\
&- m_{33}a_3\alpha c^3r_r^2v_ek_1u_e - m_{33}a_3\alpha c^3r_r^3v_ek_1z_{2,e}\\
&+ m_{33}a_3\alpha c^4r_r^2u_ez_{3,e}^2a_1k_1z_{2,r} + m_{33}a_3\alpha c^3r_ru_ez_{3,e}a_1k_2dz_{2,e}\\
&- m_{33}a_3\alpha c^4r_r^2u_ez_{3,e}a_1k_1z_{1,e} + m_{33}a_3\alpha c^4r_ru_r^2z_{3,e}^2k_2d\\
&+ m_{33}a_3\alpha c^3r_r^2u_rz_{3,e}^2k_2dz_{2,e} + m_{33}a_3\alpha c^3r_r^2u_rz_{3,e}^2k_2dz_{2,r}\\
&- m_{33}a_3\alpha c^3r_r^2u_rz_{3,e}^2k_2^2z_{2,e} - m_{33}a_3\alpha c^3r_r^2u_rz_{3,e}^2k_2^2z_{2,r}\\
&- m_{33}a_3\alpha c^4r_r^3u_rz_{3,e}^2k_1z_{1,r} + m_{33}a_3k_2^4u_ez_{2,e}\\
&+ m_{33}a_3\alpha c^4r_r^2u_ez_{3,e}^2a_1k_1z_{2,e} + 2m_{33}a_3\alpha^2c^5r_r^3u_rz_{3,e}k_2z_{2,e}\\
&- m_{33}a_3\alpha c^2r_rv_ea_1k_2^2z_{1,e}z_{3,e} + m_{33}a_3\alpha c^3r_r^2k_2z_{3,e}z_{1,r}du_r\\
&- m_{33}a_3\alpha c^3r_r^2k_2z_{3,e}z_{1,r}k_1u_e - m_{33}a_3\alpha c^3r_r^3v_ek_1z_{1,e}z_{3,e}\\
&+ 2m_{33}a_3\alpha c^3r_rv_ek_2du_rz_{3,e} + m_{33}a_3\alpha c^2r_r^2v_ek_2dz_{2,e}z_{3,e}\\
&- 2m_{33}a_3\alpha c^3r_rv_ek_2^2u_ez_{3,e} - m_{22}u_r^2v_ec^3z_{3,e}\alpha r_r^2k_2z_{2,e}\\
&- m_{22}u_rv_e^2c^2\alpha r_r^2k_2z_{2,e} - m_{22}u_rv_ek_2^2z_{2,e}^2c^2\alpha r_r^2\\
&- m_{22}u_rv_ek_2^2z_{2,e}c^2\alpha r_r^2z_{1,e}z_{3,e} - m_{22}u_rv_ek_2^2z_{2,e}c^2\alpha r_r^2z_{3,e}z_{1,r}\\
&+ m_{22}u_rv_ek_2z_{2,e}dcr_ru_e - m_{22}u_rv_ek_2^2z_{2,e}r_rcu_e\\
&- m_{22}u_rv_ek_2^2d^2z_{3,e}z_{1,r}z_{2,r} + 2m_{22}u_rv_edk_2^3z_{1,e}z_{3,e}z_{2,e}\\
&+ 2m_{22}u_rv_edk_2^3z_{1,e}z_{3,e}z_{2,r} + 2m_{22}u_rv_edk_2^3z_{3,e}z_{1,r}z_{2,e}\\
&+ 2m_{22}u_rv_edk_2^3z_{3,e}z_{1,r}z_{2,r} - m_{22}u_rv_ek_2^2r_rcu_ez_{2,r}\\
&- m_{22}u_rv_ek_2^2k_1cr_rz_{1,e}z_{2,r} + 2m_{22}u_rv_ek_2^2k_1cr_rz_{2,e}z_{3,e}z_{2,r}\\
&- m_{22}u_rv_ek_2dk_1cr_rz_{3,e}z_{2,r}^2 - m_{22}u_rv_ek_2dk_1cr_rz_{2,e}^2z_{3,e}\\
&+ m_{22}u_rv_ek_2dcr_ru_ez_{2,r} + m_{22}u_rv_ek_2dk_1cr_rz_{1,e}z_{2,r}\\
&- 2m_{22}u_rv_ek_2dk_1cr_rz_{2,e}z_{3,e}z_{2,r} - m_{22}u_rv_ek_2^2d^2z_{2,e}^2\\
&+ 2m_{22}u_rv_edk_2^3z_{2,e}^2 - m_{22}u_rv_ek_2^4z_{2,e}z_{2,r}\\
&+ m_{22}u_rv_ek_2^2k_1cr_rz_{2,e}^2z_{3,e} + m_{22}u_rv_ek_2^2k_1cr_rz_{3,e}z_{2,r}^2\\
&- m_{22}u_rv_e\lambda c^2r_r^2 + m_{22}u_ev_rk_1cr_rk_2dz_{3,e}z_{1,r}^2\\
&- 2m_{22}u_ev_rk_1cr_rk_2^2z_{2,e}z_{1,e} - m_{22}u_ev_rk_1cr_rk_2^2z_{2,e}z_{1,r}\\
&- m_{22}u_ev_rk_1cr_rk_2^2z_{1,e}^2z_{3,e} - 2m_{22}u_ev_rk_1cr_rk_2^2z_{1,e}z_{3,e}z_{1,r}\\
&- m_{22}u_ev_rk_1cr_rk_2^2z_{3,e}z_{1,r}^2 - m_{22}u_e^2v_rk_1c^2r_r^2z_{1,e}\\
&- m_{22}u_e^2v_rk_1c^2r_r^2z_{1,r} - m_{22}u_ev_rk_1^2c^2r_r^2z_{1,e}z_{1,r}\\
&- m_{22}u_ev_r\alpha k_2c^2r_r^2v_ez_{2,r} - m_{22}u_e^2v_r\alpha k_2c^3r_r^2z_{3,e}z_{2,r}\\
&- 2m_{22}u_ev_r\alpha k_2c^3r_r^2u_rz_{3,e}z_{2,r} - m_{22}u_ev_r\alpha k_2^2c^2r_r^2z_{2,e}z_{2,r}\\
&- m_{22}u_ev_r\alpha k_2^2c^2r_r^2z_{1,e}z_{3,e}z_{2,r} - m_{22}u_ev_r\alpha k_2^2c^2r_r^2z_{3,e}z_{1,r}z_{2,r}\\
&- m_{22}u_ev_rk_1^2c^2r_r^2z_{1,e}^2 + m_{22}u_ev_rk_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,e}\\
&- m_{22}u_ev_r\lambda c^2r_r^2 + 2m_{22}u_ev_rdk_2^3z_{3,e}z_{1,r}z_{2,r}\\
&- m_{22}u_e^2v_rk_2^2r_rcz_{2,r} - m_{22}u_ev_rk_2^2k_1cr_rz_{1,e}z_{2,r}\\
&+ 2m_{22}u_ev_rk_2^2k_1cr_rz_{2,e}z_{3,e}z_{2,r} - m_{22}u_ev_rk_2dk_1cr_rz_{3,e}z_{2,r}^2\\
&- m_{22}u_ev_rk_2dk_1cr_rz_{2,e}^2z_{3,e} + m_{22}u_e^2v_rk_2dcr_rz_{2,r}\\
&+ m_{22}u_ev_rk_2dk_1cr_rz_{1,e}z_{2,r} - m_{22}u_ev_e\alpha k_2^2c^2r_r^2z_{2,e}z_{2,r}\\
&- m_{22}u_ev_e\alpha k_2^2c^2r_r^2z_{1,e}z_{3,e}z_{2,r} - m_{22}u_ev_e\alpha k_2^2c^2r_r^2z_{3,e}z_{1,r}z_{2,r}\\
&- m_{22}u_ev_ek_1^2c^2r_r^2z_{1,e}^2 + m_{22}u_ev_ek_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,e}\\
&- m_{22}u_e^2v_rc^3z_{3,e}\alpha r_r^2k_2z_{2,e} - 2m_{22}u_ev_rc^3u_rz_{3,e}\alpha r_r^2k_2z_{2,e}\\
&- m_{22}u_ev_rv_ec^2\alpha r_r^2k_2z_{2,e} - m_{22}u_ev_rk_2^2z_{2,e}^2c^2\alpha r_r^2\\
&- m_{22}u_ev_rk_2^2z_{2,e}c^2\alpha r_r^2z_{1,e}z_{3,e} - m_{22}u_ev_rk_2^2z_{2,e}c^2\alpha r_r^2z_{3,e}z_{1,r}\\
&+ m_{22}u_e^2v_rk_2z_{2,e}dcr_r - m_{22}u_e^2v_rk_2^2z_{2,e}r_rc\\
&- m_{22}u_ev_rk_2^2d^2z_{3,e}z_{1,r}z_{2,r} + 2m_{22}u_ev_rdk_2^3z_{1,e}z_{3,e}z_{2,e}\\
&+ 2m_{22}u_ev_rdk_2^3z_{1,e}z_{3,e}z_{2,r} + 2m_{22}u_ev_ek_1cr_rk_2dz_{1,e}z_{3,e}z_{1,r}\\
&+ m_{22}u_ev_ek_1cr_rk_2dz_{3,e}z_{1,r}^2 - 2m_{22}u_ev_ek_1cr_rk_2^2z_{2,e}z_{1,e}\\
&- m_{22}u_ev_ek_1cr_rk_2^2z_{2,e}z_{1,r} - m_{22}u_ev_ek_1cr_rk_2^2z_{1,e}^2z_{3,e}\\
&- 2m_{22}u_ev_ek_1cr_rk_2^2z_{1,e}z_{3,e}z_{1,r} - m_{22}u_ev_ek_1cr_rk_2^2z_{3,e}z_{1,r}^2
\end{aligned}
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\begin{aligned}
&- m_{22}u_e^2v_ek_1c^2r_r^2z_{1,e} - m_{22}u_e^2v_ek_1c^2r_r^2z_{1,r}\\
&- m_{22}u_ev_ek_1^2c^2r_r^2z_{1,e}z_{1,r} - m_{22}u_ev_e^2\alpha k_2c^2r_r^2z_{2,r}\\
&- m_{22}u_e^2v_e\alpha k_2c^3r_r^2z_{3,e}z_{2,r} - m_{22}u_ev_ek_2^4z_{2,e}^2\\
&+ m_{22}u_ev_ek_1cr_rk_2dz_{1,e}^2z_{3,e} - 2m_{22}u_ev_e\alpha k_2c^3r_r^2u_rz_{3,e}z_{2,r}\\
&+ 2m_{22}u_ev_rdk_2^3z_{3,e}z_{1,r}z_{2,e} - 2m_{22}u_ev_rk_2dk_1cr_rz_{2,e}z_{3,e}z_{2,r}\\
&- m_{22}u_ev_rk_2^2d^2z_{2,e}^2 + 2m_{22}u_ev_rdk_2^3z_{2,e}^2 - m_{22}u_ev_rk_2^4z_{2,e}z_{2,r}\\
&+ m_{22}u_ev_rk_2^2k_1cr_rz_{2,e}^2z_{3,e} + m_{22}u_ev_rk_2^2k_1cr_rz_{3,e}z_{2,r}^2\\
&- m_{22}u_ev_rk_2^2d^2z_{1,e}z_{3,e}z_{2,e} - m_{22}u_ev_rk_2^2d^2z_{1,e}z_{3,e}z_{2,r}\\
&- m_{22}u_ev_rk_2^2d^2z_{3,e}z_{1,r}z_{2,e} - m_{22}u_ev_rk_2^2d^2z_{2,e}z_{2,r}\\
&+ 2m_{22}u_ev_rdk_2^3z_{2,e}z_{2,r} - m_{22}u_ev_rk_2^4z_{1,e}z_{3,e}z_{2,e}\\
&- m_{22}u_ev_rk_2^4z_{1,e}z_{3,e}z_{2,r} - m_{22}u_ev_rk_2^4z_{3,e}z_{1,r}z_{2,e}\\
&- m_{22}u_ev_rk_2^4z_{3,e}z_{1,r}z_{2,r} + m_{22}u_ev_rk_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,r}\\
&+ m_{22}u_ev_rk_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,e} + m_{22}u_ev_rk_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,r}\\
&+ 2m_{22}u_ev_rk_1cr_rk_2dz_{2,e}z_{1,e} + m_{22}u_ev_rk_1cr_rk_2dz_{2,e}z_{1,r}\\
&+ m_{22}u_ev_rk_1cr_rk_2dz_{1,e}^2z_{3,e} + 2m_{22}u_ev_rk_1cr_rk_2dz_{1,e}z_{3,e}z_{1,r}\\
&+ m_{22}u_e^2v_ek_2z_{2,e}dcr_r - m_{22}u_ev_ek_2^2d^2z_{3,e}z_{1,r}z_{2,r}\\
&+ 2m_{22}u_ev_edk_2^3z_{1,e}z_{3,e}z_{2,e} + 2m_{22}u_ev_edk_2^3z_{1,e}z_{3,e}z_{2,r}\\
&+ 2m_{22}u_ev_edk_2^3z_{3,e}z_{1,r}z_{2,e} + 2m_{22}u_ev_edk_2^3z_{3,e}z_{1,r}z_{2,r}\\
&- m_{22}u_e^2v_ek_2^2r_rcz_{2,r} - m_{22}u_ev_ek_2^2k_1cr_rz_{1,e}z_{2,r}\\
&+ m_{11}u_rv_r\lambda c^2r_r^2 - m_{22}u_e^2v_ec^3z_{3,e}\alpha r_r^2k_2z_{2,e}\\
&- 2m_{22}u_ev_ec^3u_rz_{3,e}\alpha r_r^2k_2z_{2,e} - m_{22}u_ev_e^2c^2\alpha r_r^2k_2z_{2,e}\\
&- m_{22}u_ev_ek_2^2z_{2,e}^2c^2\alpha r_r^2 - m_{22}u_ev_ek_2^2z_{2,e}c^2\alpha r_r^2z_{1,e}z_{3,e}\\
&- m_{22}u_e^2v_ek_2^2z_{2,e}r_rc + m_{11}u_r^2v_r\alpha k_2c^3r_r^2z_{3,e}z_{2,r}\\
&+ m_{11}u_rv_r\alpha k_2^2c^2r_r^2z_{2,e}z_{2,r} + m_{11}u_rv_r\alpha k_2^2c^2r_r^2z_{1,e}z_{3,e}z_{2,r}\\
&+ m_{11}u_rv_r\alpha k_2^2c^2r_r^2z_{3,e}z_{1,r}z_{2,r} + m_{11}u_rv_rk_1^2c^2r_r^2z_{1,e}^2\\
&- m_{11}u_rv_rk_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,e} + m_{11}u_rv_rk_2^4z_{2,e}^2\\
&+ m_{11}u_rv_rk_2^2d^2z_{1,e}z_{3,e}z_{2,r} + m_{11}u_rv_rk_2^2d^2z_{3,e}z_{1,r}z_{2,e}\\
&+ m_{11}u_rv_rk_2^2d^2z_{2,e}z_{2,r} - 2m_{11}u_rv_rdk_2^3z_{2,e}z_{2,r}\\
&+ m_{11}u_rv_rk_2^4z_{1,e}z_{3,e}z_{2,e} + m_{11}u_rv_rk_2^4z_{1,e}z_{3,e}z_{2,r}\\
&+ m_{11}u_rv_rk_2^4z_{3,e}z_{1,r}z_{2,e} + m_{11}u_rv_rk_2^4z_{3,e}z_{1,r}z_{2,r}\\
&- m_{11}u_rv_rk_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,r} - m_{11}u_rv_rk_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,e}\\
&- m_{11}u_rv_rk_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,r} - 2m_{11}u_rv_rk_1cr_rk_2dz_{2,e}z_{1,e}\\
&- m_{11}u_rv_rk_1cr_rk_2dz_{2,e}z_{1,r} - m_{11}u_rv_rk_1cr_rk_2dz_{1,e}^2z_{3,e}\\
&- 2m_{11}u_rv_rk_1cr_rk_2dz_{1,e}z_{3,e}z_{1,r} - m_{11}u_rv_rk_1cr_rk_2dz_{3,e}z_{1,r}^2\\
&+ 2m_{11}u_rv_rk_1cr_rk_2^2z_{2,e}z_{1,e} + m_{11}u_rv_rk_1cr_rk_2^2z_{2,e}z_{1,r}\\
&+ m_{11}u_rv_rk_1cr_rk_2^2z_{1,e}^2z_{3,e} + 2m_{11}u_rv_rk_1cr_rk_2^2z_{1,e}z_{3,e}z_{1,r}\\
&+ m_{11}u_rv_rk_1cr_rk_2^2z_{3,e}z_{1,r}^2 + m_{11}u_rv_rk_1c^2r_r^2u_ez_{1,e}\\
&+ m_{11}u_rv_rk_1c^2r_r^2u_ez_{1,r} + m_{11}u_rv_rk_1^2c^2r_r^2z_{1,e}z_{1,r}\\
&+ m_{11}u_rv_r\alpha k_2c^2r_r^2v_ez_{2,r} + m_{11}u_rv_rk_2^2d^2z_{1,e}z_{3,e}z_{2,e}\\
&- m_{22}u_ev_ek_2^2z_{2,e}c^2\alpha r_r^2z_{3,e}z_{1,r} + 2m_{22}u_ev_ek_2^2k_1cr_rz_{2,e}z_{3,e}z_{2,r}\\
&- m_{22}u_ev_ek_2dk_1cr_rz_{3,e}z_{2,r}^2 - m_{22}u_ev_ek_2dk_1cr_rz_{2,e}^2z_{3,e}\\
&+ m_{22}u_e^2v_ek_2dcr_rz_{2,r} + m_{22}u_ev_ek_2dk_1cr_rz_{1,e}z_{2,r}\\
&- 2m_{22}u_ev_ek_2dk_1cr_rz_{2,e}z_{3,e}z_{2,r} - m_{22}u_ev_ek_2^2d^2z_{2,e}^2\\
&+ 2m_{22}u_ev_edk_2^3z_{2,e}^2 - m_{22}u_ev_ek_2^4z_{2,e}z_{2,r}\\
&+ m_{22}u_ev_ek_2^2k_1cr_rz_{2,e}^2z_{3,e} + m_{22}u_ev_ek_2^2k_1cr_rz_{3,e}z_{2,r}^2\\
&- m_{22}u_ev_ek_2^2d^2z_{1,e}z_{3,e}z_{2,e} - m_{22}u_ev_ek_2^2d^2z_{1,e}z_{3,e}z_{2,r}\\
&- m_{22}u_ev_ek_2^2d^2z_{3,e}z_{1,r}z_{2,e} - m_{22}u_ev_ek_2^2d^2z_{2,e}z_{2,r}\\
&+ 2m_{22}u_ev_edk_2^3z_{2,e}z_{2,r} - m_{22}u_ev_ek_2^4z_{1,e}z_{3,e}z_{2,e}\\
&- m_{22}u_ev_ek_2^4z_{1,e}z_{3,e}z_{2,r} - m_{22}u_ev_ek_2^4z_{3,e}z_{1,r}z_{2,e}\\
&- m_{22}u_ev_ek_2^4z_{3,e}z_{1,r}z_{2,r} + m_{22}u_ev_ek_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,r}\\
&+ m_{22}u_ev_ek_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,e} + m_{22}u_ev_ek_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,r}\\
&+ 2m_{22}u_ev_ek_1cr_rk_2dz_{2,e}z_{1,e} + m_{22}u_ev_ek_1cr_rk_2dz_{2,e}z_{1,r}\\
&- m_{22}u_ev_e\lambda c^2r_r^2 + m_{11}u_rv_e\alpha k_2^2c^2r_r^2z_{1,e}z_{3,e}z_{2,r}\\
&+ m_{11}u_rv_e\alpha k_2^2c^2r_r^2z_{3,e}z_{1,r}z_{2,r} + m_{11}u_rv_ek_1^2c^2r_r^2z_{1,e}^2\\
&- m_{11}u_rv_ek_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,e} + m_{11}u_rv_e\lambda c^2r_r^2\\
&+ m_{11}u_r^2v_rc^3z_{3,e}\alpha r_r^2k_2z_{2,e} + m_{11}u_rv_rv_ec^2\alpha r_r^2k_2z_{2,e}\\
&+ m_{11}u_rv_rk_2^2z_{2,e}^2c^2\alpha r_r^2 + m_{11}u_rv_rk_2^2z_{2,e}c^2\alpha r_r^2z_{1,e}z_{3,e}\\
&+ m_{11}u_rv_rk_2^2z_{2,e}c^2\alpha r_r^2z_{3,e}z_{1,r} - m_{11}u_rv_rk_2z_{2,e}dcr_ru_e\\
&- 2m_{11}u_rv_ek_1cr_rk_2dz_{1,e}z_{3,e}z_{1,r} - m_{11}u_rv_ek_1cr_rk_2dz_{3,e}z_{1,r}^2\\
&+ 2m_{11}u_rv_ek_1cr_rk_2^2z_{2,e}z_{1,e} + m_{11}u_rv_ek_1cr_rk_2^2z_{2,e}z_{1,r}\\
&+ m_{11}u_rv_ek_1cr_rk_2^2z_{1,e}^2z_{3,e} + 2m_{11}u_rv_ek_1cr_rk_2^2z_{1,e}z_{3,e}z_{1,r}\\
&+ m_{11}u_rv_ek_1cr_rk_2^2z_{3,e}z_{1,r}^2 + m_{11}u_rv_ek_1c^2r_r^2u_ez_{1,e}
\end{aligned}
$$

$$+ m_{11}u_rv_ek_1c^2r_r^2u_ez_{1,r} + m_{11}u_rv_ek_1^2c^2r_r^2z_{1,e}z_{1,r}$$
$$+ m_{11}u_rv_e^2\alpha k_2c^2r_r^2z_{2,r} + m_{11}u_r^2v_e\alpha k_2c^3r_r^2z_{3,e}z_{2,r}$$
$$+ m_{11}u_rv_ek_2^4z_{2,e}^2 + m_{11}u_rv_ek_2^4z_{2,e}z_{2,r} - m_{11}u_rv_ek_2^2k_1cr_rz_{2,e}^2z_{3,e}$$
$$- m_{11}u_rv_ek_2^2k_1cr_rz_{3,e}z_{2,r}^2 + m_{11}u_rv_ek_2^2d^2z_{1,e}z_{3,e}z_{2,e}$$
$$+ m_{11}u_rv_ek_2^2d^2z_{1,e}z_{3,e}z_{2,r} + m_{11}u_rv_ek_2^2d^2z_{3,e}z_{1,r}z_{2,e}$$
$$+ m_{11}u_rv_ek_2^2d^2z_{2,e}z_{2,r} - 2m_{11}u_rv_edk_2^3z_{2,e}z_{2,r}$$
$$+ m_{11}u_rv_ek_2^4z_{1,e}z_{3,e}z_{2,e} + m_{11}u_rv_ek_2^4z_{1,e}z_{3,e}z_{2,r}$$
$$+ m_{11}u_rv_ek_2^4z_{3,e}z_{1,r}z_{2,e} + m_{11}u_rv_ek_2^4z_{3,e}z_{1,r}z_{2,r}$$
$$- m_{11}u_rv_ek_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,r} - m_{11}u_rv_ek_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,e}$$
$$- m_{11}u_rv_ek_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,r} - 2m_{11}u_rv_ek_1cr_rk_2dz_{2,e}z_{1,e}$$
$$- m_{11}u_rv_ek_1cr_rk_2dz_{2,e}z_{1,r} - 2m_{11}u_rv_edk_2^3z_{2,e}^2$$
$$- m_{11}u_rv_ek_1cr_rk_2dz_{1,e}^2z_{3,e} + m_{11}u_rv_e\alpha k_2^2c^2r_r^2z_{2,e}z_{2,r}$$
$$+ m_{11}u_rv_rk_2^2z_{2,e}r_rcu_e + m_{11}u_rv_rk_2^2d^2z_{3,e}z_{1,r}z_{2,r}$$
$$- 2m_{11}u_rv_rdk_2^3z_{1,e}z_{3,e}z_{2,e} - 2m_{11}u_rv_rdk_2^3z_{1,e}z_{3,e}z_{2,r}$$
$$- 2m_{11}u_rv_rdk_2^3z_{3,e}z_{1,r}z_{2,e} - 2m_{11}u_rv_rdk_2^3z_{3,e}z_{1,r}z_{2,r}$$
$$+ m_{11}u_rv_rk_2^2r_rcu_ez_{2,r} + m_{11}u_rv_rk_2^2k_1cr_rz_{1,e}z_{2,r}$$
$$- 2m_{11}u_rv_rk_2^2k_1cr_rz_{2,e}z_{3,e}z_{2,r} + m_{11}u_rv_rk_2dk_1cr_rz_{3,e}z_{2,r}^2$$
$$+ m_{11}u_rv_rk_2dk_1cr_rz_{2,e}^2z_{3,e} - m_{11}u_rv_rk_2dcr_ru_ez_{2,r}$$
$$- m_{11}u_rv_rk_2dk_1cr_rz_{1,e}z_{2,r} + 2m_{11}u_rv_rk_2dk_1cr_rz_{2,e}z_{3,e}z_{2,r}$$
$$+ m_{11}u_rv_rk_2^2d^2z_{2,e}^2 - 2m_{11}u_rv_rdk_2^3z_{2,e}^2 + m_{11}u_rv_rk_2^4z_{2,e}z_{2,r}$$
$$- m_{11}u_rv_rk_2^2k_1cr_rz_{2,e}^2z_{3,e} - m_{11}u_rv_rk_2^2k_1cr_rz_{3,e}z_{2,r}^2$$
$$+ m_{11}u_ev_r\alpha k_2^2c^2r_r^2z_{3,e}z_{1,r}z_{2,r} + m_{11}u_ev_rk_1^2c^2r_r^2z_{1,e}^2$$
$$- m_{11}u_ev_rk_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,e} + m_{11}u_ev_r\lambda c^2r_r^2$$
$$+ m_{11}u_r^2v_ec^3z_{3,e}\alpha r_r^2k_2z_{2,e} + m_{11}u_rv_e^2c^2\alpha r_r^2k_2z_{2,e}$$
$$+ m_{11}u_rv_ek_2^2z_{2,e}^2c^2\alpha r_r^2 + m_{11}u_rv_ek_2^2z_{2,e}c^2\alpha r_r^2z_{1,e}z_{3,e}$$
$$+ 2m_{11}u_ev_rk_1cr_rk_2^2z_{1,e}z_{3,e}z_{1,r} + m_{11}u_ev_rk_1cr_rk_2^2z_{3,e}z_{1,r}^2$$
$$+ m_{11}u_e^2v_rk_1c^2r_r^2z_{1,e} + m_{11}u_e^2v_rk_1c^2r_r^2z_{1,r}$$
$$+ m_{11}u_ev_rk_1^2c^2r_r^2z_{1,e}z_{1,r} + m_{11}u_ev_r\alpha k_2c^2r_r^2v_ez_{2,r}$$
$$+ m_{11}u_e^2v_r\alpha k_2c^3r_r^2z_{3,e}z_{2,r} + 2m_{11}u_ev_r\alpha k_2c^3r_r^2u_rz_{3,e}z_{2,r}$$
$$+ m_{11}u_ev_r\alpha k_2^2c^2r_r^2z_{2,e}z_{2,r} - m_{22}u_ev_rk_2^4z_{2,e}^2$$
$$- 2m_{11}u_ev_rk_1cr_rk_2dz_{2,e}z_{1,e} - m_{11}u_ev_rk_1cr_rk_2dz_{2,e}z_{1,r}$$
$$- m_{11}u_ev_rk_1cr_rk_2dz_{1,e}^2z_{3,e} - 2m_{11}u_ev_rk_1cr_rk_2dz_{1,e}z_{3,e}z_{1,r}$$
$$- m_{11}u_ev_rk_1cr_rk_2dz_{3,e}z_{1,r}^2 + 2m_{11}u_ev_rk_1cr_rk_2^2z_{2,e}z_{1,e}$$
$$+ m_{11}u_ev_rk_1cr_rk_2^2z_{2,e}z_{1,r} + m_{11}u_ev_rk_2^4z_{2,e}^2$$
$$- m_{11}u_ev_rk_2^2k_1cr_rz_{3,e}z_{2,r}^2 + m_{11}u_ev_rk_2^2d^2z_{1,e}z_{3,e}z_{2,e}$$
$$+ m_{11}u_ev_rk_2^2d^2z_{1,e}z_{3,e}z_{2,r} + m_{11}u_ev_rk_2^2d^2z_{3,e}z_{1,r}z_{2,e}$$
$$+ m_{11}u_ev_rk_2^2d^2z_{2,e}z_{2,r} - 2m_{11}u_ev_rdk_2^3z_{2,e}z_{2,r}$$
$$+ m_{11}u_ev_rk_2^4z_{1,e}z_{3,e}z_{2,e} + m_{11}u_ev_rk_2^4z_{1,e}z_{3,e}z_{2,r}$$
$$+ m_{11}u_ev_rk_2^4z_{3,e}z_{1,r}z_{2,e} + m_{11}u_ev_rk_2^4z_{3,e}z_{1,r}z_{2,r}$$
$$- m_{11}u_ev_rk_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,r} - m_{11}u_ev_rk_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,e}$$
$$+ m_{11}u_e^2v_rk_2^2z_{2,e}r_rc + m_{11}u_ev_rk_2^2d^2z_{3,e}z_{1,r}z_{2,r}$$
$$- 2m_{11}u_ev_rdk_2^3z_{1,e}z_{3,e}z_{2,e} - 2m_{11}u_ev_rdk_2^3z_{1,e}z_{3,e}z_{2,r}$$
$$- 2m_{11}u_ev_rdk_2^3z_{3,e}z_{1,r}z_{2,e} - 2m_{11}u_ev_rdk_2^3z_{3,e}z_{1,r}z_{2,r}$$
$$+ m_{11}u_e^2v_rk_2^2r_rcz_{2,r} + m_{11}u_ev_rk_2^2k_1cr_rz_{1,e}z_{2,r}$$
$$- 2m_{11}u_ev_rk_2^2k_1cr_rz_{2,e}z_{3,e}z_{2,r} + m_{11}u_ev_rk_2dk_1cr_rz_{3,e}z_{2,r}^2$$
$$+ m_{11}u_ev_rk_2dk_1cr_rz_{2,e}^2z_{3,e} - m_{11}u_e^2v_rk_2dcr_rz_{2,r}$$
$$- m_{11}u_ev_rk_2dk_1cr_rz_{1,e}z_{2,r} + 2m_{11}u_ev_rk_2dk_1cr_rz_{2,e}z_{3,e}z_{2,r}$$
$$+ m_{11}u_ev_rk_2^2d^2z_{2,e}^2 - 2m_{11}u_ev_rdk_2^3z_{2,e}^2 + m_{11}u_ev_rk_2^4z_{2,e}z_{2,r}$$
$$- m_{33}a_3r_ek_2z_{2,e}dcr_ru_e + m_{33}a_3r_ek_2^2z_{2,e}r_rcu_e$$
$$+ m_{33}a_3r_ek_2^2d^2z_{3,e}z_{1,r}z_{2,r} - 2m_{33}a_3r_edk_2^3z_{1,e}z_{3,e}z_{2,e}$$
$$- 2m_{33}a_3r_edk_2^3z_{1,e}z_{3,e}z_{2,r} - 2m_{33}a_3r_edk_2^3z_{3,e}z_{1,r}z_{2,e}$$
$$- 2m_{33}a_3r_edk_2^3z_{3,e}z_{1,r}z_{2,r} + m_{33}a_3r_ek_2^2r_rcu_ez_{2,r}$$
$$+ m_{33}a_3r_ek_2^2k_1cr_rz_{1,e}z_{2,r} - 2m_{33}a_3r_ek_2^2k_1cr_rz_{2,e}z_{3,e}z_{2,r}$$
$$+ m_{33}a_3r_ek_2^2d^2z_{2,e}z_{2,r} + m_{33}a_3r_ek_2^2d^2z_{2,e}^2 - 2m_{33}a_3r_edk_2^3z_{2,e}^2$$
$$+ m_{33}a_3r_ek_2^4z_{2,e}z_{2,r} + m_{33}a_3r_e\lambda c^2r_r^2$$
$$+ m_{33}a_3r_ec^3u_ez_{3,e}\alpha r_r^2k_2z_{2,e} + m_{33}a_3r_ec^3u_rz_{3,e}\alpha r_r^2k_2z_{2,e}$$
$$+ m_{33}a_3r_ev_ec^2\alpha r_r^2k_2z_{2,e} + m_{33}a_3r_ek_2^2z_{2,e}^2c^2\alpha r_r^2$$
$$+ m_{33}a_3r_ek_2^2z_{2,e}c^2\alpha r_r^2z_{1,e}z_{3,e} + m_{33}a_3r_ek_2^4z_{1,e}z_{3,e}z_{2,r}$$
$$+ m_{33}a_3r_ek_2^4z_{3,e}z_{1,r}z_{2,e} + m_{33}a_3r_ek_2^4z_{3,e}z_{1,r}z_{2,r}$$
$$- m_{33}a_3r_ek_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,r} - m_{33}a_3r_ek_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,e}$$
$$- m_{33}a_3r_ek_1^2c^2r_r^2z_{3,e}z_{2,r}z_{1,r} - m_{33}a_3r_ek_1cr_rk_2dz_{2,e}z_{1,r}$$
$$+ 2m_{33}a_3r_ek_1cr_rk_2^2z_{2,e}z_{1,e} + m_{33}a_3r_ek_2dk_1cr_rz_{3,e}z_{2,r}^2$$
$$+ m_{33}a_3r_ek_2dk_1cr_rz_{2,e}^2z_{3,e} - m_{33}a_3r_ek_2dcr_ru_ez_{2,r}$$
$$- m_{33}a_3r_ek_2dk_1cr_rz_{1,e}z_{2,r} + 2m_{33}a_3r_ek_2dk_1cr_rz_{2,e}z_{3,e}z_{2,r}$$
$$- m_{33}a_3r_ek_2^2k_1cr_rz_{2,e}^2z_{3,e} - m_{33}a_3r_ek_2^2k_1cr_rz_{3,e}z_{2,r}^2$$
$$+ m_{33}a_3r_ek_2^2d^2z_{1,e}z_{3,e}z_{2,r} + m_{33}a_3r_ek_2^2d^2z_{3,e}z_{1,r}z_{2,e}$$
$$- 2m_{33}a_3r_edk_2^3z_{2,e}z_{2,r} - m_{33}a_3r_ek_1^2c^2r_r^2z_{2,e}z_{3,e}z_{1,e}$$
$$+ m_{33}a_3r_ek_2^4z_{1,e}z_{3,e}z_{2,e} - m_{33}a_3r_ek_1cr_rk_2dz_{1,e}^2z_{3,e}$$
$$+ m_{33}a_3r_ek_2^2z_{2,e}c^2\alpha r_r^2z_{3,e}z_{1,r} + m_{33}a_3\alpha c^3r_r^2k_2z_{3,e}^2z_{1,r}a_1k_1z_{2,r}$$
$$- m_{33}a_3\alpha c^3r_r^2k_2z_{3,e}z_{1,r}a_1k_1z_{1,e} + m_{33}a_3\alpha c^2r_r^2k_2^2z_{3,e}z_{1,r}u_e$$
$$+ m_{33}a_3\alpha c^2r_r^2k_2^2z_{3,e}z_{1,r}u_r - m_{33}a_3\alpha c^3r_r^2k_2z_{3,e}z_{1,r}a_1u_e$$
$$+ m_{33}a_3\alpha c^3r_r^2k_2z_{3,e}z_{1,r}du_e - m_{33}a_3\alpha c^3r_r^2k_2^2z_{3,e}z_{1,r}u_e$$
$$- m_{33}a_3\alpha c^3r_r^2k_2^2z_{3,e}z_{1,r}u_r + m_{33}a_3\alpha c^2r_rk_2dv_e^2$$
$$- m_{33}a_3\alpha c^2r_rk_2^3z_{2,e}^2a_1 - m_{33}a_3\alpha c^2r_rk_2^3z_{2,e}v_e$$
$$- m_{33}a_3\alpha c^3r_r^2k_2z_{1,e}z_{3,e}k_1u_e + m_{33}a_3\alpha c^3r_rk_2^2z_{3,e}^2z_{1,r}du_r$$
$$+ m_{33}a_3\alpha c^2r_r^2k_2^2z_{3,e}^2z_{1,r}dz_{2,e} + m_{33}a_3\alpha c^2r_r^2k_2^2z_{3,e}^2z_{1,r}dz_{2,r}$$
$$+ m_{33}a_3\alpha c^3r_rk_2^2z_{3,e}^2z_{1,r}du_e + m_{33}a_3\alpha c^2r_rk_2^2z_{3,e}z_{1,r}dv_e$$
$$+ m_{33}a_3\alpha c^2r_rk_2^2z_{1,e}^2z_{3,e}^2a_1d + 2m_{33}a_3\alpha c^2r_rk_2^2z_{1,e}z_{3,e}^2a_1dz_{1,r}$$
$$+ m_{33}a_3\alpha c^3r_r^2k_2z_{1,e}z_{3,e}^2a_1k_1z_{2,e}$$
$$+ m_{33}a_3\alpha c^3r_r^2k_2z_{1,e}z_{3,e}^2a_1k_1z_{2,r} - m_{33}a_3\alpha c^3r_r^2k_2z_{1,e}^2z_{3,e}a_1k_1$$
$$- 2m_{33}a_3\alpha c^2r_rk_2^3z_{1,e}z_{3,e}^2a_1z_{1,r} + m_{33}a_3\alpha c^2r_r^2k_2^2z_{1,e}z_{3,e}u_e$$
$$+ m_{33}a_3\alpha c^2r_r^2k_2^2z_{1,e}z_{3,e}u_r + m_{33}a_3\alpha^2c^4r_r^3k_2^2z_{3,e}^2z_{1,r}^2$$
$$+ m_{33}a_3\alpha c^2r_rk_2^2z_{3,e}^2z_{1,r}^2a_1d + m_{33}a_3\alpha^2c^4r_r^3k_2^2z_{2,e}^2$$
$$- m_{33}a_3\alpha c^2r_r^2k_2^3z_{1,e}z_{3,e}^2z_{2,r} - m_{33}a_3\alpha c^2r_r^2k_2^2z_{1,e}^2z_{3,e}d$$
$$- 2m_{33}a_3\alpha c^3r_r^3k_2z_{1,e}z_{3,e}^2k_1z_{1,r} + m_{33}a_3\alpha^2c^4r_r^3k_2^2z_{1,e}^2z_{3,e}^2$$
$$+ m_{33}a_3\alpha c^2r_rv_ea_1k_2dz_{1,e}z_{3,e} + m_{33}a_3\alpha c^2r_rv_ea_1k_2dz_{3,e}z_{1,r}$$
$$+ m_{33}a_3\alpha c^3r_r^2v_ea_1k_1z_{2,e}z_{3,e} + m_{33}a_3\alpha c^3r_r^2v_ea_1k_1z_{3,e}z_{2,r}$$
$$+ m_{33}a_3\alpha c^2r_rv_ea_1k_2dz_{2,e} - m_{33}a_3\alpha c^2r_rv_ea_1k_2^2z_{3,e}z_{1,r}$$
$$- m_{33}a_3\alpha c^2r_rk_2^3z_{3,e}^2z_{1,r}^2a_1 + m_{33}a_3\alpha c^3r_rk_2^2z_{1,e}z_{3,e}^2du_r$$
$$+ m_{33}a_3\alpha c^2r_r^2k_2^2z_{1,e}z_{3,e}^2dz_{2,e} + m_{33}a_3\alpha c^2r_r^2k_2^2z_{1,e}z_{3,e}^2dz_{2,r}$$
$$+ m_{33}a_3\alpha c^3r_rk_2^2z_{1,e}z_{3,e}^2du_e + m_{33}a_3\alpha c^2r_rk_2^2z_{1,e}z_{3,e}dv_e$$
$$- m_{33}a_3\alpha c^3r_rk_2^3z_{1,e}z_{3,e}^2u_e - m_{33}a_3\alpha c^3r_rk_2^3z_{1,e}z_{3,e}^2u_r$$
$$+ m_{33}a_3\alpha^2c^6r_r^3u_r^2z_{3,e}^2 + m_{33}a_3\alpha c^4r_r^2u_rz_{3,e}^2a_1k_1z_{2,e}$$
$$- m_{33}a_3\alpha c^3r_r^2k_2z_{1,e}z_{3,e}a_1u_e - m_{33}a_3\alpha c^2r_rk_2^3z_{1,e}z_{3,e}v_e$$
$$- m_{33}a_3\alpha c^3r_rk_2^3z_{3,e}^2z_{1,r}u_e - m_{33}a_3\alpha c^3r_rk_2^3z_{3,e}^2z_{1,r}u_r$$
$$- m_{33}a_3\alpha c^2r_r^2k_2^3z_{3,e}^2z_{1,r}z_{2,e} - m_{33}a_3\alpha c^2r_r^2k_2^3z_{3,e}^2z_{1,r}z_{2,r}$$
$$- m_{33}a_3\alpha c^2r_r^2k_2^2z_{3,e}z_{1,r}dz_{1,e} + m_{33}a_3\alpha c^2r_r^2k_2^3z_{3,e}z_{1,r}z_{1,e}$$
$$- m_{33}a_3\alpha c^3r_r^3k_2z_{3,e}^2z_{1,r}^2k_1 + m_{33}a_3\alpha c^3r_rk_2^2z_{2,e}du_ez_{3,e}$$
$$- m_{33}a_3\alpha c^2r_r^2k_2^3z_{1,e}z_{3,e}^2z_{2,e} + m_{33}a_3\alpha c^2r_r^2v_ek_2u_r$$
$$+ m_{33}a_3\alpha c^3r_r^2k_2z_{3,e}^2z_{1,r}a_1k_1z_{2,e} - m_{33}a_3\alpha c^4r_ru_e^2z_{3,e}^2k_2^2$$
$$- m_{33}a_3\alpha c^4r_ru_r^2z_{3,e}^2k_2^2 - m_{33}a_3\alpha c^2r_r^2k_2^3z_{2,e}^2z_{3,e}$$
$$- 2m_{33}a_3r_ek_1cr_rk_2dz_{2,e}z_{1,e} - 2m_{33}a_3r_ek_1cr_rk_2dz_{1,e}z_{3,e}z_{1,r}$$
$$- m_{33}a_3r_ek_1cr_rk_2dz_{3,e}z_{1,r}^2 + m_{33}a_3r_ek_1cr_rk_2^2z_{2,e}z_{1,r}$$
$$+ m_{33}a_3r_ek_1cr_rk_2^2z_{1,e}^2z_{3,e} + 2m_{33}a_3r_ek_1cr_rk_2^2z_{1,e}z_{3,e}z_{1,r}$$
$$+ m_{33}a_3r_ek_1cr_rk_2^2z_{3,e}z_{1,r}^2 + m_{33}a_3r_ek_1c^2r_r^2u_ez_{1,e}$$
$$+ m_{33}a_3r_ek_1c^2r_r^2u_ez_{1,r} + m_{33}a_3r_ek_1^2c^2r_r^2z_{1,e}z_{1,r}$$
$$+ m_{33}a_3r_e\alpha k_2c^2r_r^2v_ez_{2,r} + m_{33}a_3r_e\alpha k_2c^3r_r^2u_ez_{3,e}z_{2,r}$$
$$+ m_{33}a_3r_e\alpha k_2c^3r_r^2u_rz_{3,e}z_{2,r} + m_{33}a_3r_e\alpha k_2^2c^2r_r^2z_{2,e}z_{2,r}$$
$$+ m_{33}a_3r_e\alpha k_2^2c^2r_r^2z_{1,e}z_{3,e}z_{2,r} + m_{33}a_3r_e\alpha k_2^2c^2r_r^2z_{3,e}z_{1,r}z_{2,r}$$
$$+ m_{33}a_3r_ek_1^2c^2r_r^2z_{1,e}^2 - m_{33}a_3\alpha c^3r_ru_rz_{3,e}^2a_1k_2^2z_{1,r}$$
$$+ m_{33}a_3\alpha c^3r_r^2u_r^2z_{3,e}k_2 - m_{33}a_3\alpha c^3r_ru_rz_{3,e}^2a_1k_2^2z_{1,e}$$
$$+ m_{33}a_3\alpha c^2r_rk_2^2z_{2,e}dv_e + 2m_{33}a_3\alpha^2c^5r_r^3u_rz_{3,e}^2k_2z_{1,r}$$
$$- m_{33}a_3\alpha c^3r_r^2k_2^2z_{2,e}u_e + m_{33}a_3\alpha c^3r_ru_rz_{3,e}^2a_1k_2dz_{1,e}$$
$$- m_{33}a_3\alpha c^4r_r^2u_rz_{3,e}a_1u_e + m_{33}a_3k_1cr_rk_2dz_{1,e}z_{3,e}v_e$$
$$- m_{33}a_3cd^2u_rk_2^2z_{3,e}z_{1,r} + 2m_{33}a_3cdu_rk_2^3z_{2,e}$$
$$+ 2m_{33}a_3\alpha c^2r_rk_2^2z_{2,e}a_1dz_{1,e}z_{3,e}$$
$$+ 2m_{33}a_3\alpha c^2r_rk_2^2z_{2,e}a_1dz_{3,e}z_{1,r}$$
$$+ m_{33}a_3\alpha c^3r_r^2k_2z_{2,e}a_1k_1z_{3,e}z_{2,r} - m_{33}a_3\alpha c^3r_r^2k_2z_{2,e}a_1k_1z_{1,e}$$
$$- 2m_{33}a_3\alpha c^2r_rk_2^3z_{2,e}a_1z_{3,e}z_{1,r} - 2m_{33}a_3\alpha c^2r_rk_2^3z_{2,e}a_1z_{1,e}z_{3,e}$$
$$- m_{33}a_3\alpha c^3r_r^2k_2z_{2,e}a_1u_e + m_{33}a_3\alpha c^3r_r^2k_2z_{2,e}du_e$$
$$- m_{33}a_3\alpha c^3r_r^2k_2^2z_{2,e}u_r + m_{33}a_3\alpha c^3r_r^2k_2z_{2,e}du_r$$
$$- m_{33}a_3\alpha c^3r_r^2k_2z_{2,e}k_1u_e - m_{33}a_3\alpha c^3r_r^3k_2z_{2,e}^2k_1$$
$$- m_{33}a_3\alpha c^3r_r^3k_2z_{1,e}^2z_{3,e}^2k_1 + m_{33}a_3\alpha c^2r_r^2k_2^3z_{1,e}^2z_{3,e}$$
$$+ m_{33}a_3\alpha c^3r_ru_rz_{3,e}^2a_1k_2dz_{1,r} + m_{33}a_3\alpha c^4r_r^2u_rz_{3,e}^2a_1k_1z_{2,r}$$

$$
\begin{aligned}
&+ m_{33}a_3\alpha c^3 r_r u_r z_{3,e} a_1 k_2 d z_{2,e} - m_{33}a_3\alpha c^4 r_r^2 u_r z_{3,e} a_1 k_1 z_{1,e}\\
&- m_{33}a_3 k_1 c^2 r_r^2 u_e v_r - m_{33}a_3 k_1^2 c^2 r_r^2 z_{1,e} v_e\\
&- m_{33}a_3 k_1^2 c^2 r_r^2 z_{1,e} v_r + m_{33}a_3 k_1^2 c^2 r_r^2 z_{2,e} z_{3,e} v_e\\
&+ m_{33}a_3 k_1^2 c^2 r_r^2 z_{2,e} z_{3,e} v_r + m_{33}a_3 k_1^2 c^2 r_r^2 z_{3,e} z_{2,r} v_e\\
&+ m_{33}a_3 k_1^2 c^2 r_r^2 z_{3,e} z_{2,r} v_r + m_{33}a_3 k_1 c r_r k_2 d z_{2,e} v_e\\
&+ m_{33}a_3 k_1 c r_r k_2 d z_{2,e} v_r - m_{33}a_3 k_1 c r_r k_2^2 z_{2,e} v_e\\
&- 2m_{33}a_3 d u_e k_2^3 z_{2,e} + m_{33}a_3 d^2 u_r k_2^2 z_{2,e} - 2m_{33}a_3 d u_r k_2^3 z_{2,e}\\
&+ m_{33}a_3 k_2^4 u_e z_{1,e} z_{3,e} + m_{33}a_3 k_2^4 u_e z_{3,e} z_{1,r} + m_{33}a_3 k_2^4 u_r z_{1,e} z_{3,e}\\
&+ m_{33}a_3 k_2^4 u_r z_{3,e} z_{1,r} - m_{33}a_3 k_2^2 c^2 u_e^2 r_r - m_{33}a_3 k_2^4 c u_e z_{2,e}\\
&- m_{33}a_3 k_2^4 c u_r z_{2,e} + m_{33}a_3 k_2^2 u_e^2 c r_r + 2m_{33}a_3 c d u_r k_2^3 z_{1,e} z_{3,e}\\
&+ 2m_{33}a_3 c d u_r k_2^3 z_{3,e} z_{1,r} + m_{33}a_3 k_2^2 u_e k_1 c r_r z_{1,e}\\
&- m_{33}a_3 k_2^2 u_e k_1 c r_r z_{2,e} z_{3,e} - m_{33}a_3 k_2^2 u_e k_1 c r_r z_{3,e} z_{2,r}\\
&+ m_{33}a_3 k_2^2 u_r k_1 c r_r z_{1,e} - m_{33}a_3 k_2^2 u_r k_1 c r_r z_{2,e} z_{3,e}\\
&- m_{33}a_3 k_2^2 u_r k_1 c r_r z_{3,e} z_{2,r} - m_{33}a_3 k_2^2 c^2 u_e k_1 r_r z_{1,e}\\
&+ m_{33}a_3 k_2^2 c^2 u_e k_1 r_r z_{2,e} z_{3,e} + m_{33}a_3 k_2^2 c^2 u_e k_1 r_r z_{3,e} z_{2,r}\\
&- m_{33}a_3 k_2^4 c u_e z_{1,e} z_{3,e} - m_{33}a_3 k_2^4 c u_e z_{3,e} z_{1,r} - m_{33}a_3 k_2^2 c^2 u_r u_e r_r\\
&- m_{33}a_3 k_2^2 c^2 u_r k_1 r_r z_{1,e} + m_{33}a_3 k_2^2 c^2 u_r k_1 r_r z_{2,e} z_{3,e}\\
&+ m_{33}a_3 k_2^2 u_r u_e c r_r + m_{33}a_3 k_1 c r_r k_2 d z_{1,e} z_{3,e} v_r\\
&+ m_{33}a_3 k_1 c r_r k_2 d z_{3,e} z_{1,r} v_e + m_{33}a_3 k_1 c r_r k_2 d z_{3,e} z_{1,r} v_r\\
&- m_{33}a_3 k_1 c r_r k_2^2 z_{2,e} v_r - m_{33}a_3 k_1 c r_r k_2^2 z_{1,e} z_{3,e} v_e\\
&- m_{33}a_3 k_1 c r_r k_2^2 z_{1,e} z_{3,e} v_r - m_{33}a_3 k_1 c r_r k_2^2 z_{3,e} z_{1,r} v_e\\
&- m_{33}a_3 k_1 c r_r k_2^2 z_{3,e} z_{1,r} v_r + m_{33}a_3 d^2 u_r k_2^2 z_{1,e} z_{3,e}\\
&+ m_{33}a_3 d^2 u_r k_2^2 z_{3,e} z_{1,r} - 2m_{33}a_3 d u_r k_2^3 z_{1,e} z_{3,e}\\
&- 2m_{33}a_3 d u_r k_2^3 z_{3,e} z_{1,r} - m_{33}a_3 c d^2 u_e k_2^2 z_{2,e}\\
&- m_{33}a_3 c d^2 u_e k_2^2 z_{1,e} z_{3,e} - m_{33}a_3 c d^2 u_e k_2^2 z_{3,e} z_{1,r}\\
&+ 2m_{33}a_3 c d u_e k_2^3 z_{2,e} + 2m_{33}a_3 c d u_e k_2^3 z_{1,e} z_{3,e}\\
&+ 2m_{33}a_3 c d u_e k_2^3 z_{3,e} z_{1,r} - m_{33}a_3 k_2 c^2 d u_r k_1 r_r z_{3,e} z_{2,r}\\
&- m_{33}a_3 k_2 d u_e k_1 c r_r z_{1,e} + m_{33}a_3 k_2 d u_e k_1 c r_r z_{2,e} z_{3,e}\\
&+ m_{33}a_3 k_2 d u_e k_1 c r_r z_{3,e} z_{2,r} - m_{33}a_3 k_2 d u_r u_e c r_r\\
&- m_{33}a_3 k_2 d u_r k_1 c r_r z_{1,e} - m_{33}a_3 k_2 d u_e^2 c r_r\\
&+ m_{33}a_3 k_2 c^2 d u_e^2 r_r + m_{33}a_3 k_2 d u_r k_1 c r_r z_{2,e} z_{3,e}\\
&+ m_{33}a_3 k_2 d u_r k_1 c r_r z_{3,e} z_{2,r} + m_{33}a_3 k_2 c^2 d u_e k_1 r_r z_{1,e}\\
&- m_{33}a_3 k_2 c^2 d u_e k_1 r_r z_{2,e} z_{3,e} - m_{33}a_3 k_2 c^2 d u_e k_1 r_r z_{3,e} z_{2,r}\\
&+ m_{33}a_3 k_2^2 c^2 u_r k_1 r_r z_{3,e} z_{2,r} + m_{33}a_3 d^2 u_e k_2^2 z_{2,e}\\
&+ 2m_{33}a_3\alpha^2 c^5 r_r^3 u_e z_{3,e}^2 k_2 z_{1,e} + 2m_{33}a_3\alpha^2 c^5 r_r^3 u_e z_{3,e}^2 k_2 z_{1,r}\\
&+ 2m_{33}a_3\alpha^2 c^6 r_r^3 u_e z_{3,e}^2 u_r + m_{33}a_3\alpha c^3 r_r u_e z_{3,e}^2 a_1 k_2 d z_{1,e}\\
&+ m_{33}a_3\alpha c^3 r_r u_e z_{3,e}^2 a_1 k_2 d z_{1,r} + m_{33}a_3\alpha^2 c^4 r_r^3 v_e^2\\
&+ m_{33}a_3 k_2 c^2 d u_r k_1 r_r z_{1,e} - m_{33}a_3 k_2 c^2 d u_r k_1 r_r z_{2,e} z_{3,e}\\
&- m_{33}a_3 k_2^4 c u_r z_{1,e} z_{3,e} - m_{33}a_3 k_2^4 c u_r z_{3,e} z_{1,r}\\
&+ m_{33}a_3 d^2 u_e k_2^2 z_{1,e} z_{3,e} + m_{33}a_3 d^2 u_e k_2^2 z_{3,e} z_{1,r}\\
&- 2m_{33}a_3 d u_e k_2^3 z_{1,e} z_{3,e} + 2m_{33}a_3\alpha^2 c^5 r_r^3 u_e z_{3,e} k_2 z_{2,e}\\
&- 2m_{33}a_3 d u_e k_2^3 z_{3,e} z_{1,r} + m_{33}a_3 k_2 c^2 d u_r u_e r_r\\
&- m_{33}a_3 k_1 c^2 r_r^2 u_e v_e - m_{33}a_3\alpha c^3 r_r k_2^3 z_{2,e} u_r z_{3,e}\\
&- m_{33}a_3\alpha c^3 r_r u_r z_{3,e} a_1 k_2^2 z_{2,e} + 2m_{33}a_3\alpha^2 c^4 r_r^3 k_2^2 z_{1,e} z_{3,e}^2 z_{1,r}\\
&- m_{33}a_3\alpha c^2 r_r^2 k_2^3 z_{2,e} z_{3,e} z_{2,r} - m_{33}a_3\alpha c^2 r_r^2 k_2^2 z_{2,e} d z_{1,e}\\
&- 2m_{33}a_3\alpha c^3 r_r^3 k_2 z_{2,e} k_1 z_{3,e} z_{1,r} + 2m_{33}a_3\alpha^2 c^4 r_r^3 k_2^2 z_{2,e} z_{1,e} z_{3,e}\\
&+ 2m_{33}a_3\alpha^2 c^4 r_r^3 k_2^2 z_{2,e} z_{3,e} z_{1,r} + m_{33}a_3\alpha c^3 r_r^2 k_2 z_{2,e}^2 a_1 k_1 z_{3,e}\\
&- m_{33}a_3\alpha c^4 r_r^2 u_r^2 z_{3,e} k_2 + m_{33}a_3\alpha c^4 r_r^2 u_r^2 z_{3,e} d\\
&- m_{33}a_3\alpha c^4 r_r^2 u_r z_{3,e} k_1 u_e - m_{33}a_3\alpha c^4 r_r^3 u_r z_{3,e} k_1 z_{2,e}\\
&- 2m_{33}a_3\alpha c^3 r_r^3 k_2 z_{2,e} k_1 z_{1,e} z_{3,e} + m_{33}a_3\alpha c^3 r_r k_2^2 z_{2,e} d u_r z_{3,e}\\
&+ m_{33}a_3\alpha c^2 r_r^2 k_2^2 z_{2,e}^2 d z_{3,e} + m_{33}a_3\alpha c^2 r_r^2 k_2^2 z_{2,e} d z_{3,e} z_{2,r}\\
&- m_{33}a_3\alpha c^3 r_r k_2^3 z_{2,e} u_e z_{3,e} + 2m_{33}a_3\alpha^2 c^4 r_r^3 v_e k_2 z_{1,e} z_{3,e}\\
&+ 2m_{33}a_3\alpha^2 c^4 r_r^3 v_e k_2 z_{3,e} z_{1,r} + 2m_{33}a_3\alpha^2 c^5 r_r^3 v_e u_r z_{3,e}\\
&- m_{33}a_3\alpha c^3 r_r^2 v_e a_1 k_1 z_{1,e} + m_{33}a_3\alpha c^2 r_r^2 k_2^3 z_{2,e} z_{1,e}\\
&+ m_{33}a_3\alpha c^2 r_r k_2^2 z_{2,e}^2 a_1 d + m_{33}a_3\alpha c^2 r_r^2 k_2^2 z_{2,e} u_e\\
&- m_{22}u_r^2 v_r \alpha k_2 c^3 r_r^2 z_{3,e} z_{2,r} - m_{22}u_r v_r \alpha k_2^2 c^2 r_r^2 z_{2,e} z_{2,r}\\
&- m_{22}u_r v_r \alpha k_2^2 c^2 r_r^2 z_{1,e} z_{3,e} z_{2,r} - m_{22}u_r v_r \alpha k_2^2 c^2 r_r^2 z_{3,e} z_{1,r} z_{2,r}\\
&+ m_{22}u_r v_r k_1^2 c^2 r_r^2 z_{2,e} z_{3,e} z_{1,e} + m_{33}a_3\alpha c^2 r_r^2 v_e k_2 d z_{3,e} z_{2,r}\\
&+ 2m_{33}a_3\alpha^2 c^4 r_r^3 v_e k_2 z_{2,e} + m_{33}a_3\alpha c^2 r_r^2 k_2^2 z_{2,e} u_r\\
&- m_{33}a_3\alpha c^2 r_r k_2^3 z_{1,e}^2 z_{3,e}^2 a_1 - m_{33}a_3\alpha c^2 r_r k_2^3 z_{3,e} z_{1,r} v_e\\
&+ m_{33}a_3\alpha c^2 r_r^2 v_e k_2^2 z_{1,e} - m_{33}a_3\alpha c^2 r_r v_e a_1 k_2^2 z_{2,e}\\
&+ m_{33}a_3\alpha c^2 r_r^2 v_e k_2 u_e - m_{33}a_3\alpha c^3 r_r u_e z_{3,e} a_1 k_2^2 z_{2,e}\\
&- m_{33}a_3\alpha c^3 r_r u_e z_{3,e}^2 a_1 k_2^2 z_{1,r} + m_{33}a_3\alpha c^3 r_r^2 u_e^2 z_{3,e} k_2\\
&+ 2m_{33}a_3\alpha c^3 r_r^2 u_e z_{3,e} k_2 u_r - m_{33}a_3\alpha c^3 r_r u_e z_{3,e}^2 a_1 k_2^2 z_{1,e}\\
&- m_{33}a_3\alpha c^4 r_r^2 u_e^2 z_{3,e} a_1 + m_{33}a_3\alpha c^4 r_r^2 u_e^2 z_{3,e} d\\
&- m_{33}a_3\alpha c^4 r_r^2 u_e^2 z_{3,e} k_2 - 2m_{33}a_3\alpha c^4 r_r^2 u_e z_{3,e} k_2 u_r\\
&+ 2m_{33}a_3\alpha c^4 r_r^2 u_e z_{3,e} d u_r - m_{33}a_3\alpha c^4 r_r^2 u_e^2 z_{3,e} k_1\\
&- m_{33}a_3\alpha c^4 r_r^3 u_e z_{3,e} k_1 z_{2,e} - m_{33}a_3\alpha c^4 r_r^3 u_r z_{3,e}^2 k_1 z_{1,e}\\
&- m_{33}a_3\alpha c^4 r_r^3 u_e z_{3,e}^2 k_1 z_{1,r} - m_{33}a_3\alpha c^4 r_r^3 u_e z_{3,e}^2 k_1 z_{1,e}\\
&+ 2m_{33}a_3\alpha c^4 r_r u_e z_{3,e}^2 k_2 d u_r + m_{33}a_3\alpha c^3 r_r^2 u_e z_{3,e}^2 k_2 d z_{2,e}\\
&+ m_{33}a_3\alpha c^3 r_r^2 u_e z_{3,e}^2 k_2 d z_{2,r} + m_{33}a_3\alpha c^4 r_r u_e z_{3,e}^2 k_2 d\\
&- 2m_{33}a_3\alpha c^4 r_r u_e z_{3,e}^2 k_2^2 u_r - m_{33}a_3\alpha c^3 r_r^2 u_e z_{3,e}^2 k_2^2 z_{2,e}\\
&- m_{33}a_3\alpha c^3 r_r^2 u_e z_{3,e}^2 k_2^2 z_{2,r} - m_{33}a_3\alpha c^3 r_r^2 v_e a_1 u_e\\
&+ m_{33}a_3\alpha c^3 r_r^2 v_e d u_e - m_{33}a_3\alpha c^3 r_r^2 v_e k_2 u_e\\
&- m_{33}a_3\alpha c^3 r_r^2 v_e k_2 u_r + m_{33}a_3\alpha c^3 r_r^2 v_e d u_r - m_{22}u_r v_r \lambda c^2 r_r^2\\
&- m_{33}a_3 c d^2 u_r k_2^2 z_{2,e} + m_{22}u_r v_r k_1 c r_r k_2 d z_{2,e} z_{1,r}\\
&+ m_{22}u_r v_r k_1 c r_r k_2 d z_{1,e}^2 z_{3,e} + 2m_{22}u_r v_r k_1 c r_r k_2 d z_{1,e} z_{3,e} z_{1,r}\\
&+ m_{22}u_r v_r k_1 c r_r k_2 d z_{3,e} z_{1,r}^2 - 2m_{22}u_r v_r k_1 c r_r k_2^2 z_{2,e} z_{1,e}\\
&- m_{22}u_r v_r k_1 c r_r k_2^2 z_{2,e} z_{1,r} - m_{22}u_r v_r k_1 c r_r k_2^2 z_{1,e}^2 z_{3,e}\\
&- 2m_{22}u_r v_r k_1 c r_r k_2^2 z_{1,e} z_{3,e} z_{1,r} - m_{22}u_r v_r k_1 c r_r k_2^2 z_{3,e} z_{1,r}^2\\
&- m_{22}u_r v_r k_1 c^2 r_r^2 u_e z_{1,e} - m_{22}u_r v_r k_1 c^2 r_r^2 u_e z_{1,r}\\
&- m_{22}u_r v_r k_1^2 c^2 r_r^2 z_{1,e} z_{1,r} - m_{22}u_r v_r k_1^2 c^2 r_r^2 z_{1,e}^2\\
&+ 2m_{22}u_r v_r k_2^2 k_1 c r_r z_{2,e} z_{3,e} z_{2,r} - m_{22}u_r v_r k_2 d k_1 c r_r z_{3,e} z_{2,r}^2\\
&- m_{22}u_r v_r k_2 d k_1 c r_r z_{2,e}^2 z_{3,e} + m_{22}u_r v_r k_2 d c r_r u_e z_{2,r}\\
&+ m_{22}u_r v_r k_2 d k_1 c r_r z_{1,e} z_{2,r} - 2m_{22}u_r v_r k_2 d k_1 c r_r z_{2,e} z_{3,e} z_{2,r}\\
&- m_{22}u_r v_r k_2^2 d^2 z_{2,e}^2 + 2m_{22}u_r v_r d k_2^3 z_{2,e}^2 - m_{22}u_r v_r k_2^4 z_{2,e} z_{2,r}\\
&+ m_{22}u_r v_r k_2^2 k_1 c r_r z_{2,e}^2 z_{3,e} + m_{22}u_r v_r k_2^2 k_1 c r_r z_{3,e} z_{2,r}^2\\
&- m_{22}u_r v_r k_2^2 d^2 z_{1,e} z_{3,e} z_{2,e} - m_{22}u_r v_r k_2^2 d^2 z_{1,e} z_{3,e} z_{2,r}\\
&- m_{22}u_r v_r k_2^2 d^2 z_{3,e} z_{1,r} z_{2,e} - m_{22}u_r v_r k_2^2 d^2 z_{2,e} z_{2,r}\\
&+ 2m_{22}u_r v_r d k_2^3 z_{2,e} z_{2,r} - m_{22}u_r v_r k_2^4 z_{1,e} z_{3,e} z_{2,e}\\
&- m_{22}u_r v_r k_2^4 z_{1,e} z_{3,e} z_{2,r} - m_{22}u_r v_r k_2^4 z_{3,e} z_{1,r} z_{2,e}\\
&- m_{22}u_r v_r k_2^4 z_{3,e} z_{1,r} z_{2,r} + m_{22}u_r v_r k_1^2 c^2 r_r^2 z_{2,e} z_{3,e} z_{1,r}\\
&+ m_{22}u_r v_r k_1^2 c^2 r_r^2 z_{3,e} z_{2,r} z_{1,e} + m_{22}u_r v_r k_1^2 c^2 r_r^2 z_{3,e} z_{2,r} z_{1,r}\\
&- m_{22}u_r v_r k_2^4 z_{2,e}^2 + m_{33}a_3 r_e k_2^4 z_{2,e}^2 + m_{11}u_e^2 v_e c^3 z_{3,e}\alpha r_r^2 k_2 z_{2,e}\\
&+ 2m_{11}u_e v_e c^3 u_r z_{3,e}\alpha r_r^2 k_2 z_{2,e} + m_{11}u_e v_e^2 c^2 \alpha r_r^2 k_2 z_{2,e}\\
&+ m_{11}u_e v_e k_2^2 z_{2,e}^2 c^2 \alpha r_r^2 + m_{11}u_e v_e k_2^2 z_{2,e} c^2 \alpha r_r^2 z_{1,e} z_{3,e}\\
&+ m_{11}u_e v_e k_2^2 z_{2,e} c^2 \alpha r_r^2 z_{3,e} z_{1,r} - m_{11}u_e^2 v_e k_2 z_{2,e} d c r_r\\
&+ m_{11}u_e^2 v_e k_2^2 z_{2,e} r_r c + m_{11}u_e v_e k_2^2 d^2 z_{3,e} z_{1,r} z_{2,r} + m_{11}u_e v_e k_2^4 z_{2,e}^2\\
&+ m_{33}a_3\alpha^2 c^6 r_r^3 u_e^2 z_{3,e}^2 + 2m_{33}a_3\alpha^2 c^5 r_r^3 u_r z_{3,e}^2 k_2 z_{1,e}\\
&- m_{33}a_3\alpha c^2 r_r k_2^2 v_e^2 + m_{33}a_3 r_e k_2^2 d^2 z_{1,e} z_{3,e} z_{2,e}\\
&+ 2m_{33}a_3\alpha c^3 r_r v_e k_2 d u_e z_{3,e} - 2m_{33}a_3\alpha c^3 r_r v_e k_2^2 u_r z_{3,e}\\
&- m_{33}a_3\alpha c^2 r_r^2 v_e k_2^2 z_{2,e} z_{3,e} - m_{33}a_3\alpha c^2 r_r^2 v_e k_2^2 z_{3,e} z_{2,r}\\
&- m_{33}a_3\alpha c^2 r_r^2 v_e k_2 d z_{1,e} - m_{33}a_3\alpha c^3 r_r^3 v_e k_1 z_{3,e} z_{1,r}\\
&+ 2m_{33}a_3\alpha^2 c^5 r_r^3 v_e u_e z_{3,e})/(\gamma c^2 r_r^2 z_{1,e}^2 + \gamma c^2 r_r^2 z_{2,e}^2 + \gamma c^2 r_r^2 z_{2,e} z_{2,r}\\
&+ \gamma c^2 r_r^2 z_{1,e} z_{1,r} + c^3 u_e z_{3,e}\alpha r_r^2 k_2 z_{2,e} + c^3 u_r z_{3,e}\alpha r_r^2 k_2 z_{2,e}\\
&+ v_e c^2 \alpha r_r^2 k_2 z_{2,e} + k_2^2 z_{2,e}^2 c^2 \alpha r_r^2 + k_2^2 z_{2,e} c^2 \alpha r_r^2 z_{1,e} z_{3,e}\\
&+ k_2^2 z_{2,e} c^2 \alpha r_r^2 z_{3,e} z_{1,r} - k_2 z_{2,e} d c r_r u_e + k_2^2 z_{2,e} r_r c u_e\\
&+ k_2^2 d^2 z_{3,e} z_{1,r} z_{2,r} - 2d k_2^3 z_{1,e} z_{3,e} z_{2,e} - 2d k_2^3 z_{1,e} z_{3,e} z_{2,r}\\
&- 2d k_2^3 z_{3,e} z_{1,r} z_{2,e} - 2d k_2^3 z_{3,e} z_{1,r} z_{2,r} + k_2^2 r_r c u_e z_{2,r}\\
&+ k_2^2 k_1 c r_r z_{1,e} z_{2,r} - 2k_2^2 k_1 c r_r z_{2,e} z_{3,e} z_{2,r} + k_2 d k_1 c r_r z_{3,e} z_{2,r}^2\\
&+ k_2 d k_1 c r_r z_{2,e}^2 z_{3,e} - k_2 d c r_r u_e z_{2,r} - k_2 d k_1 c r_r z_{1,e} z_{2,r}\\
&+ 2k_2 d k_1 c r_r z_{2,e} z_{3,e} z_{2,r} + k_2^2 d^2 z_{2,e}^2 - 2d k_2^3 z_{2,e}^2 + k_2^4 z_{2,e} z_{2,r}\\
&- k_2^2 k_1 c r_r z_{2,e}^2 z_{3,e} - k_2^2 k_1 c r_r z_{3,e} z_{2,r}^2 + k_2^2 d^2 z_{1,e} z_{3,e} z_{2,e}\\
&+ k_2^2 d^2 z_{1,e} z_{3,e} z_{2,r} + k_2^2 d^2 z_{3,e} z_{1,r} z_{2,e} + k_2^2 d^2 z_{2,e} z_{2,r} - 2d k_2^3 z_{2,e} z_{2,r}\\
&+ k_2^4 z_{1,e} z_{3,e} z_{2,e} + k_2^4 z_{1,e} z_{3,e} z_{2,r} + k_2^4 z_{3,e} z_{1,r} z_{2,e} + k_2^4 z_{3,e} z_{1,r} z_{2,r}\\
&- k_1^2 c^2 r_r^2 z_{2,e} z_{3,e} z_{1,r} - k_1^2 c^2 r_r^2 z_{3,e} z_{2,r} z_{1,e} - k_1^2 c^2 r_r^2 z_{3,e} z_{2,r} z_{1,r}\\
&- 2k_1 c r_r k_2 d z_{2,e} z_{1,e} - k_1 c r_r k_2 d z_{2,e} z_{1,r} - k_1 c r_r k_2 d z_{1,e}^2 z_{3,e}\\
&- 2k_1 c r_r k_2 d z_{1,e} z_{3,e} z_{1,r} - k_1 c r_r k_2 d z_{3,e} z_{1,r}^2 + 2k_1 c r_r k_2^2 z_{2,e} z_{1,e}\\
&+ k_1 c r_r k_2^2 z_{2,e} z_{1,r} + k_1 c r_r k_2^2 z_{1,e}^2 z_{3,e} + 2k_1 c r_r k_2^2 z_{1,e} z_{3,e} z_{1,r}\\
&+ k_1 c r_r k_2^2 z_{3,e} z_{1,r}^2 + k_1 c^2 r_r^2 u_e z_{1,e} + k_1 c^2 r_r^2 u_e z_{1,r}\\
&+ k_1^2 c^2 r_r^2 z_{1,e} z_{1,r} + \alpha k_2 c^2 r_r^2 v_e z_{2,r} + \alpha k_2 c^3 r_r^2 u_e z_{3,e} z_{2,r}\\
&+ \alpha k_2 c^3 r_r^2 u_r z_{3,e} z_{2,r} + \alpha k_2^2 c^2 r_r^2 z_{2,e} z_{2,r} + \alpha k_2^2 c^2 r_r^2 z_{1,e} z_{3,e} z_{2,r}\\
&+ \alpha k_2^2 c^2 r_r^2 z_{3,e} z_{1,r} z_{2,r} + k_1^2 c^2 r_r^2 z_{1,e}^2 - k_1^2 c^2 r_r^2 z_{2,e} z_{3,e} z_{1,e} + k_2^4 z_{2,e}^2\\
&+ \lambda c^2 r_r^2)
\end{aligned}
$$

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Summary

The subject of this thesis is the design of tracking controllers for certain classes of mechanical systems. The thesis consists of two parts. In the first part an accurate mathematical model of the mechanical system under consideration is assumed to be given. The goal is to follow a certain specified trajectory. Therefore, a feasible reference trajectory is assumed to be given i.e., a trajectory that can be realized for the system under consideration. The tracking error at each time is defined as the difference between where the system is and where it should be. The problem now is to design a controller for the system which is such that the tracking error converges to zero, no matter where the system is initialized nor at which time-instant. A new design methodology is presented, based on the theory of cascaded systems, i.e., systems that can be seen as a special interconnection of two stable subsystems. This new approach is applied to three different models. In Chapter 4 the kinematic model of a mobile car is considered. Chapter 5 is concerned with systems in chained form. A large class of interesting mechanical systems can be transformed to the chained form, including a mobile robot, a car towing multiple trailers, a knife edge, a vertical rolling wheel and a rigid spacecraft with two torque actuators. In Chapter 6 the tracking problem for an under-actuated ship is dealt with, i.e., a ship with only two controls is considered, whereas it has three degrees of freedom. All systems under consideration happen to have two inputs.

In the cascaded design approach, first one of the inputs is used to stabilize a subsystem of the tracking error dynamics. Next, it is assumed for the remaining dynamics that the stabilization of the first subsystem has worked out, i.e., it is assumed that for the first subsystem the state equals the reference state and the input equals the reference input. This assumption simplifies the remaining dynamics considerably. Next, the remaining input is used for establishing asymptotic stability of the simplified remaining dynamics. Having found in this manner control laws for the two inputs, the resulting closed-loop tracking error dynamics is considered. Due to the design this closed-loop tracking error dynamics has a cascaded structure. By means of the theory of cascaded systems global uniform asymptotic stability of the tracking error dynamics is shown, i.e., it is shown that the tracking error converges to zero, no matter where the system is initialized nor at which time-instant.

This design strategy is applied to the three models mentioned above. The behavior of the resulting controllers is illustrated by means of numerical simulations and in case of the ship by means of experiments on a scale model of an offshore supply vessel.

In the second part of the thesis some uncertainties are taken into account, concerning the models of the mechanical systems under consideration. First the set-point control problem

for a rigid robot manipulator is studied in case the vector of gravitational forces is unknown. Since compensation for this vector is needed to achieve perfect regulation, it is common practice to use a PID controller instead of a PD-controller with gravity compensation. It is shown that this approach leads to global asymptotic stability of the error dynamics provided that the integral action is not activated from the beginning, but only after some period of time.

Secondly, the visual servoing problem for a rigid robot manipulator is considered. Imagine that a rigid robot manipulator is moving in the horizontal plane and a camera is placed at the ceiling to watch the manipulator from above. The output of this camera is displayed at a screen. An operator determines a spot on the screen to which the tip of the manipulator should move and a controller has to be found which makes the manipulator do so. One of the major difficulties is that some of the camera parameters are unknown. The fact that also the orientation angle of the camera is unknown leads to designing an adaptive controller to solve this problem.

Thirdly, the tracking problem of Part I is considered again, but this time it is assumed that certain system parameters are unknown. By means of an example it is shown that the formulation of the adaptive tracking control problem is far from trivial. This is due to the fact that entirely specifying the reference trajectory is in conflict with not knowing certain parameters. For this example of a four-wheel mobile robot with unknown length a formulation of the adaptive tracking problem is presented and also solved.

The thesis ends with conclusions and recommendations for further research.

Samenvatting

Dit proefschrift gaat over het ontwerpen van regelaars voor het volg-probleem voor bepaalde klassen van mechanische regelsystemen. Het proefschrift bestaat uit twee delen. In het eerste deel wordt een wiskundig model van het mechanische regelsysteem met twee ingangen bekend verondersteld. Het doel is om een gespecificeerd traject te volgen, waarbij wordt aangenomen dat dat gewenste traject daadwerkelijk door het regelsysteem te volgen is. Op elk tijdstip wordt de volgfout gedefinieerd als het verschil tussen waar het regelsysteem is en waar het zou moeten zijn (gezien het gewenste traject). Het probleem is nu om voor het regelsysteem een regelaar te ontwerpen die er voor zorgt dat de volgfout naar nul convergeert, ongeacht waar of op welk tijdstip het regelsysteem wordt geïnitialiseerd. Er wordt een nieuwe ontwerpmethode gepresenteerd die gebaseerd is op de theorie van cascade systemen, dat wil zeggen, systemen die gezien kunnen worden als een bijzondere verbinding van twee stabiele systemen. Deze nieuwe aanpak wordt toegepast op drie verschillende modellen. In hoofdstuk 4 wordt het kinematische model van een mobiele robot beschouwd. Hoofdstuk 5 gaat over regelsystemen in “chained form”. Een grote klasse van interessante mechanische regelsystemen kan worden getransformeerd naar de “chained form”, waaronder een robot karretje, een wagen die een aantal opleggers trekt, een mesblad, een verticaal rollend wiel en een ruimteschip met twee aandrijfmogelijkheden. In hoofdstuk 6 wordt het volg-probleem behandeld voor een schip met slechts twee stuurmiddelen, terwijl het drie vrijheidsgraden heeft.

In de cascade ontwerpaanpak wordt eerst een van de ingangen gebruikt om een deelsysteem van de volgfout-dynamica te stabilizeren. Daarna wordt voor de resterende dynamica aangenomen dat de stabilisatie van het eerste deelsysteem gelukt is. Dat betekent dat verondersteld wordt dat voor het eerste deelsysteem de toestand gelijk is aan de referentietoestand en de ingang gelijk is aan de referentie-ingang. Deze aanname vereenvoudigt de resterende dynamica aanzienlijk. Vervolgens wordt de ingang die nog over is gebruikt om asymptotische stabiliteit van de vereenvoudigde resterende dynamica te bewerkstelligen. Nu er op deze manier regelwetten voor de twee ingangen gevonden zijn, wordt de resulterende volgfout-dynamica in gesloten lus bekeken. Dankzij de ontwerpaanpak heeft de volgfout-dynamica een cascade structuur. Met behulp van de theorie van cascade systemen wordt vervolgens globale uniforme asymptotische stabiliteit van de volgfout-dynamica aangetoond, wat wil zeggen dat de volgfout naar nul convergeert, ongeacht waar of op welk tijdstip het regelsysteem geïnitialiseerd wordt.

Deze ontwerpaanpak wordt toegepast op de drie eerder genoemde modellen. Het gedrag van de resulterende regelaars wordt geïllustreerd met behulp van numerieke simulaties danwel,

in geval van het schip, met behulp van experimenten op een schaalmodel van een zeewaardig bevoorradingsschip.

In het tweede deel van het proefschrift wordt rekening gehouden met een aantal onzekerheden in de modellen van de mechanische regelsystemen. Als eerste wordt gekeken naar het probleem om een stijve robotarm naar een vast punt te sturen waarbij de zwaartekrachtsvector onbekend verondersteld wordt. Aangezien deze vector gecompenseerd moet worden om perfecte positionering te krijgen is het gebruikelijk om een PID-regelaar te gebruiken in plaats van een PD-regelaar met compensatie voor de zwaartekracht. Er wordt aangetoond dat deze aanpak leidt tot globale asymptotische stabiliteit van de foutdynamica, onder de voorwaarde dat de integrerende actie niet van het begin af aan, maar pas na enige tijd wordt geactiveerd.

Als tweede wordt het 'visual servoing' probleem voor stijve robotarmen bekeken. Neem aan dat een stijve robotarm in een horizontaal vlak beweegt en dat een camera aan het plafond de robotarm van boven registreert. Op een beeldscherm wordt weergegeven wat die camera ziet. Het doel is om een regelaar te ontwerpen die er voor zorgt dat de robotarm zich beweegt naar de plek die door iemand op het scherm is aangewezen. Een van de problemen die hierbij een rol speelt, is dat enkele van de parameters van de camera niet bekend zijn. Het feit dat bovendien de orientatiehoek van de camera onbekend is, heeft er toe geleid een adaptieve regelaar te ontwerpen om dit probleem op te lossen.

Als derde wordt het volg-probleem van deel I opnieuw bekeken, maar deze keer wordt aangenomen dat enkele parameters van het regelsysteem onbekend zijn. Door middel van een voorbeeld wordt aangetoond dat het formuleren van een adaptief volg-probleem verre van triviaal is. Dit komt doordat het volledig specificeren van het referentie-traject en het niet kennen van enkele parameters met elkaar in conflict is. Voor dit voorbeeld van een vier-wielige mobiele robot met onbekende lengte wordt het adaptieve volg-probleem zowel geformuleerd als opgelost.

Het proefschrift wordt afgesloten met conclusies en aanbevelingen voor verder onderzoek.

Curriculum Vitae

Erjen Lefeber was born on the 7th of December 1972 in Beverwijk. In 1990 he finished the "Christelijk Gymnasium Sorghvliet" in The Hague. In the summer of that same year he was one of the six members of the team that represented The Netherlands at the International Mathematical Olympiad in Beijing, China. After the summer of 1990 he started his studies at the Faculty of Mathematical Sciences at the University of Twente. During the final years of his study he was involved in the optimal control of a solar-car and he graduated in April 1996 on the "(adaptive) control of chaotic and robot systems via bounded controls", which was awarded the KIvI-regeltechniekprijs 1996.

In May 1996 he started a research project on "(adaptive) control of nonlinear mechanical systems with nonholonomic constraints", supervised by Prof.dr. H. Nijmeijer. The results of this project can be found in this thesis.

He started January 1st, 2000 at the Department of Mechanical Engineering of Eindhoven Univerity of Technology. He is with the group of Systems Engineering of Prof.dr.ir. J.E. Rooda. His research subject is control of industrial systems.

During his studies he was an active member of the Mathematical Student Society Abacus. He was student-member of the VSNU-committee that was concerned with the educational visitation of the mathematics departments in The Netherlands. He also was dedicated to popularizing mathematics by participating in VIERKANT voor Wiskunde (leading mathematics summer camps, developing material, doing mathematics markets) and by being member of the editorial staff of Pythagoras.

During his last two years in Enschede he was elder of the Oosterkerk in Enschede.

Deze pagina hoort er niet meer bij!

Snij-lijnen zijn aangegeven voor formaat van $17 \times 24,5$cm.