Simultaneous Stabilization and Tracking of Nonholonomic Mobile Robots: A Lyapunov-Based Approach

Yaonan Wang, Zhiqiang Miao, Hang Zhong, and Qi Pan

***Abstract*— A smooth time-varying controller is proposed to simultaneously address the stabilization and tracking problems of nonholonomic mobile robots for most admissible reference trajectories without switching. The controller is developed with the aid of a delicately designed time-varying signal and Lyapunov method. Computational simplification and asymptotic convergence of regulation or tracking errors are achieved by the proposed controller. Our approach provides an interesting way to unify the existing results on point stabilization and trajectory tracking of mobile robots. The simulation and experimental results on a wheeled mobile robot are presented to demonstrate the effectiveness of the proposed controller.**

***Index Terms*— Lyapunov method, mobile robots, nonholonomic systems, stabilization and tracking, time-varying feedback.**

I. Introduction

As a benchmark problem for nonholonomic systems, the motion control of mobile robots has been extensively studied over the last two decades due to theoretical and practical interests. The nonholonomy of mobile robots turns out to be very useful since it allows us to control them with less control inputs. However, the price of this advantage is a substantial complication in control design. As illustrated by Brockett's theorem [1], it is impossible to asymptotically stabilize such systems by smooth or even continuous pure-state feedback. To circumvent this difficulty, different types of feedback laws have been developed for the stabilization problem of nonholonomic systems, including smooth time-varying feedback [2]–[11], discontinuous time-invariant feedback [12]–[17], and hybrid feedback [18]–[20]. An early survey on the developments in mobile robot control, and nonholonomic systems in general, can be found in [21], and some more recent ones can be seen in [22] and [23]. Specifically, the first explicit time-varying control law for the feedback stabilization of a mobile robot was proposed in [2]. This approach was further developed for a class of nonholonomic chained systems [3], where a heat function was exploited to excite the system in some sense. Motivated by Samson's work, the so-called concept of uniform delta persistency of excitation ($u\delta$-PE) was introduced in [9]–[11] to transparently interpret the time-varying stabilization mechanism for nonholonomic systems. Their work provided new insights into the stabilization problem of nonholonomic systems.

Another issue is related to the trajectory tracking problem. In general, this problem is easier than point stabilization, and various feedback solutions have been proposed. Solutions for trajectory tracking of mobile robot were introduced in [24]–[28], and [29]–[36] to cover a broader class of nonholonomic systems in chained form. To achieve asymptotic tracking, these controllers usually rely on some persistent excitation (PE) conditions upon the reference trajectory. In the literature, the detailed definition of the required PE condition may change from one reference to another, depending on the proposed controller structure [37]. Roughly speaking, the fulfillment of a PE condition implies that the desired reference trajectory is a moving trajectory, instead of a fixed set point [38]. These assumptions make it impossible for the tracking controllers being extended to the stabilization problem.

The point stabilization and trajectory tracking problems of nonholonomic systems are studied as two distinct subproblems. Consequently, switching between two different types of controller is necessary when the control problem of mobile robots is not known in advance. However, switching between controllers may be impractical when the mobile robot must operate in a fully autonomous mode and no priori information on the reference trajectory is available [39]. In practical applications, it is preferable to solve the stabilization problem and the tracking problem simultaneously using a single controller. This problem was first addressed in [40] for unicycle-modeled mobile robots using backstepping technique. However, the reference linear velocity of the robots in their work was assumed to be nonnegative, which may be quite restrictive. Similar ideas were used in [41] and [42], where time-varying output feedback controller and adaptive controller were presented to solve both stabilization and tracking for mobile robots. However, their controllers were quite complicated and computationally demanding, and some control parameters must be chosen very carefully to satisfy certain

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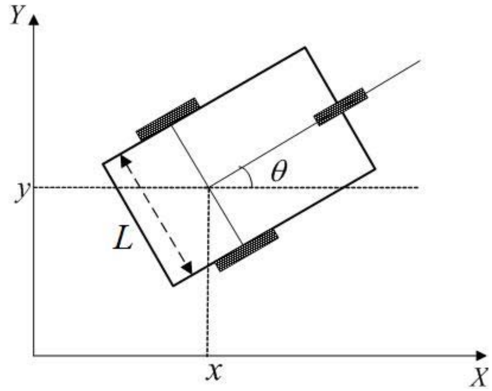


Fig. 1. Two-wheeled nonholonomic mobile robot.

constraints. The inflexibility of the parameter selection makes their approaches less practicable. In [43], a receding horizon controller was proposed for simultaneous regulation and tracking of a nonholonomic mobile robot. However, as the receding horizon control is data based, an open-loop optimization problem should be solved online, which may be very time consuming.

In this paper, we focus on smooth feedback solutions for simultaneous stabilization and tracking problem to make the extension to dynamic level possible. By fully exploiting the aforementioned separated results on stabilization and tracking of mobile robots, a relatively simple time-varying feedback controller is formulated to solve the stabilization and tracking problem simultaneously. It should be noted that we do not try to provide a solution to asymptotically stabilize any admissible trajectory, which was proved to be impossible in [37]. In the proposed controller, a time-varying signal is introduced to make the single controller be capable of adaptively and smoothly converting between stabilizer and tracker rather than switching between these two different types of controllers. The control is developed based on Lyapunov approach, and guarantees global asymptotic convergence of the regulation and tracking errors to the origin. Some simulation and experimental studies on a wheeled mobile robot will be presented to illustrate the effectiveness of the proposed controller.

The remainder of this paper is organized as follows. The problem is formulated in Section II. In Section III, the main results are stated. The simulation and experimental results for illustrating the effectiveness of the proposed controller are given in Section IV. Section V concludes this paper.

II. Problem Formulation

Consider the two-wheeled nonholonomic mobile robot shown in Fig. 1. The configuration of the mobile robot can be described by a vector of generalized coordinates

$$q = [x, y, \theta]^T \tag{1}$$

where (x, y) is the position of the mobile robot and θ is the heading angle. The pure rolling and nonslipping condition states that the robot can only move in the direction normal to the axis of the driving wheels. The nonholonomic constraint subjected to the mobile robot is given as

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0. \tag{2}$$

Under such a constraint condition, the kinematic model of the mobile robot can be described by

$$\dot{x} = \upsilon\cos\theta \quad \dot{y} = \upsilon\sin\theta \quad \dot{\theta} = \omega \tag{3}$$

where υ and ω represent the linear velocity and the angular velocity of the mobile robot, respectively.

Assume that the reference trajectory is admissible (or feasible), and generated by the following virtual robot:

$$\dot{x}_d = \upsilon_d\cos\theta_d \quad \dot{y}_d = \upsilon_d\sin\theta_d \quad \dot{\theta}_d = \omega_d \tag{4}$$

where $q_d = [x_d, y_d, \theta_d]^T$ denotes the position and orientation of the virtual robot, and (υ_d, ω_d) denotes the linear and angular velocities of the virtual robot, which is assumed to satisfy the following assumption.

Assumption 1: The reference signals $\upsilon_d, \omega_d, \dot{\upsilon}_d$, and $\dot{\omega}_d$ are bounded. In addition, one of the following conditions holds.

C1) There exist T, $\mu_1 > 0$ such that $\forall t \geq 0$

$$\int_t^{t+T} (|\upsilon_d(s)| + |\omega_d(s)|)\mathrm{d}s \geq \mu_1. \tag{5}$$

C2) There exists $\mu_2 > 0$ such that

$$\int_0^{\infty} (|\upsilon_d(s)| + |\omega_d(s)|)\mathrm{d}s \leq \mu_2. \tag{6}$$

The control objective considered in this paper is to design a (single) continuous feedback control law (υ, ω) for the mobile robot that simultaneously solves stabilization and tracking for a desirable reference trajectory, in particular

$$\lim_{t\to\infty} (q(t) - q_d(t)) = 0. \tag{7}$$

Remark 1: Recalling that an integrable function $f(t)$ is called persistently exciting if there exist $\delta, \varepsilon > 0$ such that for all $t \geq 0$: $\int_t^{t+\delta} |f(s)|\mathrm{d}s \geq \varepsilon$, and $f(t)$ is said to belong to L_1-space if $\int_0^{\infty} |f(s)|\mathrm{d}s < \infty$. Thus, C1 implies that υ_d or ω_d is PE, and C2 implies that both υ_d and ω_d belong to L_1 space. The problems of set-point regulation/stabilization, tracking a path approaching a set-point are included in C2, and tracking linear and circular paths belong to C1. It is worth noting that C1 is more general that PE conditions of other kinds used previously in [40]–[42] for studying the tracking problem of mobile robots.

Remark 2: It should be noted that asymptotic convergence of the regulation or tracking errors to the origin is achieved in this paper. Morin and Samson [39] proposed a controller that ensures practical stabilization of arbitrary reference trajectory [which does not necessarily satisfy (4)] based on transverse function approach. They also showed the possibility of asymptotic tracking for admissible trajectory under certain PE condition via the choice of adequate transverse function. Other works on practical stabilization can be found in [23] and [44], in which uniformly ultimately bounded of the tracking errors was achieved based on dynamic oscillators.

III. Main Results

A. Control Design

As often done in tracking control of mobile robots, consider the following tracking errors as $q_e = T(q)(q - q_d)$:

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_d \\ y - y_d \\ \theta - \theta_d \end{bmatrix}. \tag{8}$$

Then, the following error dynamics can be attained:

$$\begin{aligned} \dot{x}_e &= +\omega y_e + v - v_d \cos\theta_e \\ \dot{y}_e &= -\omega x_e + v_d \sin\theta_e \\ \dot{\theta}_e &= \omega - \omega_d. \end{aligned} \tag{9}$$

To facilitate the control design, we define a time-varying signal $\alpha = \alpha(t, x_e, y_e)$ as

$$\alpha = \rho(t) h(t, x_e, y_e) \tag{10}$$

with

$$\dot{\rho} = -(|v_d(t)| + |\omega_d(t)|)\rho, \quad \rho(0) = 1 \tag{11}$$

and $h(t, x_e, y_e)$ is assumed to satisfy the following assumption.

Assumption 2: $h(t, x_e, y_e)$ is a function of class C^2, with the first- and second-order successive partial derivatives uniformly bounded with respect to t, and is required to have the following three properties.

P1) $h(t, 0, 0) = 0$ and $h(t, x_e, y_e)$ satisfies

$$\frac{\partial h}{\partial x_e} y_e - \frac{\partial h}{\partial y_e} x_e = 0. \tag{12}$$

P2) $h(t, x_e, y_e)$ is uniformly bounded with respect to t, x_e, and y_e, i.e., there exists a constant $h_0 > 0$, such that

$$|h(t, x_e, y_e)| \le h_0 \ \forall t \ge 0 \ \forall (x_e, y_e) \in R^2. \tag{13}$$

P3) $\partial h / \partial t(t, 0, y_e)$ is $u\delta$-PE with respect to y_e, i.e., for each $\delta > 0$, there exist constants and $T > 0$ and $\mu > 0$ such that for all $t \ge 0$

$$\min_{s \in [t, t+T]} |y_e(s)| > \delta \Rightarrow \int_t^{t+T} \left| \frac{\partial h}{\partial t}(s, 0, y_e(s)) \right| \mathrm{d}s > \mu. \tag{14}$$

Remark 3: If the time-varying function $h = h(t, r)$, with $r = (x_e^2 + y_e^2)^{1/2}$, then it can be easily checked that $h(t, r)$ satisfies (12). The conditions imposed by Assumption 2 on $h(t, x_e, y_e)$ are not severe and can be easily met. For example, the following three functions all satisfy the properties:

$$\begin{aligned} h(t, r) &= h_0 \tanh(a r^b) \sin(ct) \\ h(t, r) &= \frac{2 h_0 a r^b}{1 + (a r^b)^2} \sin(ct) \\ h(t, r) &= \frac{2 h_0}{\pi} \arctan(a r^b) \sin(ct) \end{aligned}$$

with $a \ne 0, b > 0$, and $c \ne 0$.

Denote $\bar{\theta}_e = \theta_e - \alpha$, then we can rewrite the error model (9) as

$$\begin{aligned} \dot{x}_e &= \omega y_e + v - v_d \cos\theta_e \\ \dot{y}_e &= -\omega x_e + v_d(\sin\theta_e - \sin\alpha) + v_d \sin\alpha \\ \dot{\bar{\theta}}_e &= \omega - \omega_d - \dot{\alpha}. \end{aligned} \tag{15}$$

To design the control input, we consider the following Lyapunov function:

$$V_1 = \frac{1}{2} k_0 \left(x_e^2 + y_e^2\right) + \frac{1}{2} \bar{\theta}_e^2 \tag{16}$$

where k_0 is a positive constant. Differentiating V_1 along the trajectories of (15) leads to

$$\begin{aligned} \dot{V}_1 &= k_0(\omega y_e + v - v_d \cos\theta_e) x_e + (\omega - \omega_d - \dot{\alpha})\bar{\theta}_e \\ &\quad + k_0(-\omega x_e + v_d(\sin\theta_e - \sin\alpha) y_e + v_d \sin\alpha) y_e \\ &= k_0(v - v_d \cos\theta_e) x_e + k_0 v_d \sin\alpha \, y_e \\ &\quad + \left(\omega - \omega_d + k_0 v_d y_e \frac{\sin\theta_e - \sin\alpha}{\theta_e - \alpha} - \dot{\alpha}\right) \bar{\theta}_e. \end{aligned} \tag{17}$$

Because the function h satisfies (12), we have

$$\begin{aligned} \dot{\alpha} &= \frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial x_e} \dot{x}_e + \frac{\partial \alpha}{\partial y_e} \dot{y}_e \\ &= \frac{\partial \alpha}{\partial t} + \rho \left(\frac{\partial h}{\partial x_e} \dot{x}_e + \frac{\partial h}{\partial y_e} \dot{y}_e \right) \\ &= \frac{\partial \alpha}{\partial t} + \rho \left(\frac{\partial h}{\partial x_e} (v - v_d \cos\theta_e) + \frac{\partial h}{\partial y_e} v_d \sin\theta_e \right) \\ &\quad + \rho \left(\frac{\partial h}{\partial x_e} y_e - \frac{\partial h}{\partial y_e} x_e \right) \omega \\ &= \frac{\partial \alpha}{\partial t} + \rho \left(\frac{\partial h}{\partial x_e} (v - v_d \cos\theta_e) + \frac{\partial h}{\partial y_e} v_d \sin\theta_e \right) \end{aligned} \tag{18}$$

where $\partial \alpha / \partial t$ is defined by

$$\frac{\partial \alpha}{\partial t} = -\left(|v_d(t)| + |\omega_d(t)|\right)\alpha + \rho \frac{\partial h}{\partial t}. \tag{19}$$

Following (18), we know that $\dot{\alpha}$ is independent of the control input ω, hence we can safely choose the control as

$$\begin{aligned} v &= -k_1 x_e + v_d \cos\theta_e \\ \omega &= -k_2 \bar{\theta}_e + \omega_d - k_0 v_d y_e f_1 + \dot{\alpha} \end{aligned} \tag{20}$$

where k_1 and k_2 are positive control gains; $\dot{\alpha}$ is given by (18), and f_1 is defined as

$$\begin{aligned} f_1 &= \frac{\sin\theta_e - \sin\alpha}{\theta_e - \alpha} \\ &= \frac{\sin\bar{\theta}_e \cos\alpha + (\cos\bar{\theta}_e - 1)\sin\alpha}{\bar{\theta}_e}. \end{aligned} \tag{21}$$

As $\sin\bar{\theta}_e / \bar{\theta}_e = \int_0^1 \cos(s\bar{\theta}_e)\mathrm{d}s$ and $(1 - \cos\bar{\theta}_e)/\bar{\theta}_e = \int_0^1 \sin(s\bar{\theta}_e)\mathrm{d}s$, f_1 is a smooth and bounded function in $\bar{\theta}_e$.

Remark 4: It can be noted that the control inputs (v, ω) depend on time explicitly via the function α. The delicately designed time-varying signal $\rho(t)(0 \le \rho(t) \le 1)$ plays a very essential role in the controller. Thanks to the time-varying signal $\rho(t)$, the controller (v, ω) can be considered as an interesting combination of time-varying stabilization and tracking controllers proposed in the literature. In fact, setting $\rho(t) = 0$ results in (v, ω) similar to the tracking control law proposed in [26] or [27]. On the other hand, setting $\rho(t) = 1$ results in (v, ω) similar to the stabilization control law proposed in [3].

Based on the above analysis, the error dynamics (15) is transformed into

$$\begin{aligned}\dot{x}_e &= -k_1 x_e + \omega y_e \\ \dot{y}_e &= -\omega x_e + v_d(\sin\theta_e - \sin\alpha) + v_d \sin\alpha \\ \dot{\bar{\theta}}_e &= -k_2\bar{\theta}_e - k_0 v_d y_e f_1.\end{aligned} \tag{22}$$

In the following, we will give the stability analysis of the closed-loop system (22).

B. Stability Analysis

Before giving the stability analysis of closed-loop system (22), a technical lemma is first presented.

Lemma 1: Let $V : R^+ \to R^+$ be continuously differentiable and $W : R^+ \to R^+$ uniformly continuous satisfying that for each $t \geq 0$

$$\dot{V}(t) \leq -W(t) + p_1(t)V(t) + p_2(t)\sqrt{V(t)} \tag{23}$$

with both $p_1(t)$ and $p_2(t)$ are nonnegative and belong to L_1 space. Then, $V(t)$ is bounded, and there exists a constant c, such that $W(t) \to 0$ and $V(t) \to c$ as $t \to \infty$.

Proof: First, we prove that $V(t)$ is bounded. According to (23), we have

$$\dot{V}(t) \leq p_1(t)V(t) + p_2(t)\sqrt{V(t)} \tag{24}$$

which implies the following inequality:

$$\frac{\mathrm{d}(\sqrt{V(t)})}{\mathrm{d}t} \leq \frac{p_1(t)}{2}\sqrt{V(t)} + \frac{p_2(t)}{2}. \tag{25}$$

Integrating the inequality from 0 to t, we have

$$\sqrt{V(t)} \leq \int_0^t \frac{p_1(s)}{2}\sqrt{V(s)}\mathrm{d}s + \left(\sqrt{V(0)} + \int_0^t \frac{p_2(s)}{2}\mathrm{d}s\right). \tag{26}$$

The application of the Gronwall–Bellman inequality [45] to the function $\sqrt{(V(t))}$ results in

$$\sqrt{V(t)} \leq \left(\sqrt{V(0)} + \int_0^t \frac{p_2(s)}{2}\mathrm{d}s\right)\exp\left(\int_0^t \frac{p_1(s)}{2}\mathrm{d}s\right). \tag{27}$$

Since both $p_1(t)$ and $p_2(t)$ belong to L_1 space, $V(t)$ is bounded. Hence, there exists a positive constant δ, such that for each $r_0 > 0$

$$\sqrt{V(t)} \leq \delta \quad \forall\sqrt{V(0)} \leq r_0. \tag{28}$$

Then, from (23), for $\forall\sqrt{V(0)} \leq r_0$

$$\dot{V}(t) \leq -W(t) + \delta^2 p_1(t) + \delta p_2(t) \tag{29}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(V(t) - \delta^2\int_0^t p_1(s)\mathrm{d}s - \delta\int_0^t p_2(s)\mathrm{d}s\right) \leq 0. \tag{30}$$

It follows that $(V(t) - \delta^2\int_0^t p_1(s)\mathrm{d}s - \delta\int_0^t p_2(s)\mathrm{d}s)$ is nonincreasing. Since $V(t)$ is bounded from below by zero, $V(t)$ tends to a finite nonnegative constant.

On the other hand, it follows from (23) that

$$\begin{aligned}V(t) + \int_0^t W(s)\mathrm{d}s \leq V(0) + \delta^2\int_0^t p_1(s)\mathrm{d}s \\ +\delta\int_0^t p_2(s)\mathrm{d}s < \infty.\end{aligned} \tag{31}$$

The above inequality implies that $W(t)$ belongs to L_1 space. Thus, by Barbalat's lemma, $W(t)$ asymptotically converges to zero. This completes the proof. ■

We now use Lemma 1 to investigate the stability properties of the closed-loop system (22), and the result is stated in the following theorem.

Theorem 1: Under Assumptions 1 and 2, the closed-loop system (22) is globally asymptotically convergent to zero. Thus, the control law given by (20) makes (7) holds.

Proof: Solving the differential equation (11), we have

$$\rho(t) = \exp\left(-\int_0^t (|v_d(s)| + |\omega_d(s)|)\mathrm{d}s\right). \tag{32}$$

We can see that $0 \leq \rho(t) \leq 1$. If C1 holds, then based on the stability results of linear time-varying systems, $\rho(t)$ is exponentially convergent to zero, and $\rho(t) \in L_1$. If C2 holds, then $0 < \exp(-\mu_2) < \rho(t) \leq 1$.

We first investigate the convergence of $x_e(t)$ and $\bar{\theta}_e(t)$ via Lemma 1. Consider the Lyapunov function presented by (16). Substituting (20) into (17), the time derivative of V_1 is given as

$$\dot{V}_1 = -k_0 k_1 x_e^2 - k_2\bar{\theta}_e^2 + k_0 v_d(\sin\alpha)y_e. \tag{33}$$

Because $|y_e| \leq \sqrt{2V_1/k_0}$, which is implied by (16), we have

$$\dot{V}_1 \leq -k_0 k_1 x_e^2 - k_2\bar{\theta}_e^2 + |v_d \sin\alpha|\sqrt{2k_0 V_1}. \tag{34}$$

Considering $|v_d \sin\alpha| \leq |v_d\alpha| \leq h_0|v_d\rho|$, and v_d and ρ are bounded, then under Assumption 1 and using the fact $\rho(t) \in L_1$ for case C1, it can be easily checked that for both cases C1 and C2, $v_d\rho \in L_1$ and hence $v_d \sin\alpha \in L_1$

$$\int_0^t |v_d(s)\sin(\alpha(s))|\,\mathrm{d}s < \infty. \tag{35}$$

Equation (34) can be formulated as the form of (23), with $W = k_0 k_1 x_e^2 + k_2\bar{\theta}_e^2$, $p_1 = 0$, and $p_2 = \sqrt{2k_0}|v_d \sin\alpha|$. Following (34) and (35), we have x_e and $\bar{\theta}_e$ tend to zero, and y_e tends to a constant as $t \to \infty$ by means of Lemma 1.

In the following, the convergence of y_e to zero is proved with the aid of the extended version of Barbalat's lemma [3]. Because $\lim_{t\to\infty}\bar{\theta}_e(t) = 0$, applying the extended version of Barbalat's lemma to the last equation of (22) yields

$$\lim_{t\to\infty}(v_d y_e f_1)(t) = 0 \tag{36}$$

where the limit of $f_1(t)$ satisfies

$$\lim_{t\to\infty} f_1(t) = \lim_{\theta_e\to\alpha}\frac{\sin\theta_e - \sin\alpha}{\theta_e - \alpha} = \lim_{t\to\infty}(\cos\alpha)(t). \tag{37}$$

Similarly, since $\lim_{t\to\infty} x_e(t) = 0$, applying the extended version of Barbalat's lemma to the first equation of (22) yields

$$\lim_{t\to\infty}\omega(t)y_e(t) = 0. \tag{38}$$

Because $\lim_{t\to\infty}(v - v_d\cos\theta_e)(t) = 0$ and $\lim_{t\to\infty}(v_d\sin\theta_e)(t) = \lim_{t\to\infty}(v_d\sin\alpha)(t) = 0$, then from (18), we have

$$\lim_{t\to\infty}\left(\dot{\alpha} - \frac{\partial\alpha}{\partial t}\right)(t) = 0. \tag{39}$$

Substituting the expression of ω in (20) into (38), and using (36) and (39), as well as the fact $\bar{\theta}_e$ tends to zero, we have

$$\lim_{t\to\infty}\left(\omega_d+\frac{\partial\alpha}{\partial t}\right)y_e(t)=0. \tag{40}$$

Under (40), we then will show that y_e tends to zero for cases C1 and C2, respectively.

1) If C1 holds, ρ as well as α tend to zero, then $\lim_{t\to\infty}(\partial\alpha/\partial t)(t) = 0$ following (19). Thus, (40) implies

$$\lim_{t\to\infty}\omega_d(t)y_e(t)=0. \tag{41}$$

Because α tends to zero, we have $\lim_{t\to\infty} f_1(t) = \lim_{t\to\infty}(\cos\alpha)(t) = 1$ using (37), then (36) gives

$$\lim_{t\to\infty}v_d(t)y_e(t)=0. \tag{42}$$

Combining (41) and (42), we have

$$\lim_{t\to\infty}(|v_d(t)|+|\omega_d(t)|)y_e(t)=0. \tag{43}$$

Following (43), it can be easily proved that $\lim_{t\to\infty} y_e(t) = 0$ by contradiction.

2) If C2 holds, v_d and ω_d tend to zero, and $0 < \exp(-\mu_2) < \rho \le 1$. From (40), we arrive at

$$\lim_{t\to\infty}\frac{\partial h}{\partial t}(t,0,y_e(t))y_e(t)=0. \tag{44}$$

Similarly, we can obtain y_e tends to zero by contradiction. Assume that $y_e(t)$ does not tend to zero, then $\lim_{t\to\infty}(\partial h/\partial t)(t,0,y_e(t)) = 0$; this is certainly not compatible with property P3 in Assumption 2.

Because $h(t,x_e,y_e)$ satisfies $h(t,0,0) = 0$, we have $\alpha(t,0,0) = 0$. By uniformly continuity of α and $\alpha(t,0,0)=0$, $x_e(t)$, $y_e(t)$, and $\bar{\theta}_e(t)$ converge to zero implies $x_e(t)$, $y_e(t)$, and $\theta_e(t)$ converge to zero. This completes the proof of Theorem 1. ■

IV. Simulation and Experimental Results

In this section, some simulation and experimental studies will be performed to investigate the effectiveness of the controller developed in the previous section. The proposed controller is first verified with some computer simulations on the kinematic model of the mobile robot. The following four cases are simulated.

Case 1: Set-point stabilization: $v_d = 0, \omega_d = 0$.
Case 2: Approach to a point: $v_d = 3e^{-0.2t}, \omega_d = e^{-0.6t}$.
Case 3: Tracking a line path: $v_d = 2, \omega_d = 0$.
Case 4: Tracking a circle: $v_d = 2, \omega_d = 1$.

The reference trajectory $q_d(t) = [x_d(t), y_d(t), \theta_d(t)]^T$ is generated by the desired velocities $v_d(t)$ and $\omega_d(t)$ with the initial conditions $q_d(0) = [0,0,0]^T$.

In the simulation, the initial positions and velocities of the wheeled robot are picked as $q(0) = [2,-2,-1]^T$ and $[v(0),\omega(0)]^T = [0,0]^T$. The control parameters are set as $k_0 = 1$, $k_1 = 6$, and $k_2 = 5$. Controllers with two different candidates of the nonlinear time-varying function h are simulated.

Controller 1: $h = 8\tanh(x_e^2+y_e^2)\sin(t)$.
Controller 2: $h = 6\arctan(x_e^2+y_e^2)\sin(t)$.

The simulation results are shown in Figs. 2–5, where the mean-square-error (MSE) control performance shown in Figs. 2(b)–5(b) is in the form of $(x_e^2+y_e^2+\theta_e^2)^{1/2}$. It can be seen from the figures that the tracking errors of the two controllers all approach zero with similar control performance. With our approach, mobile robots can follow paths specified by a straight line, a circle, a path approaching to a set point, or just a set point. The results of the simulation demonstrate the effectiveness of the proposed approach.

We also test the proposed controller on a service robot developed by the Robotics Laboratory in Hunan University based on the Pioneer 2DX mobile robot platform. The hardware configuration of the mobile robot control system is shown in Fig. 6. The motion of the robot is controlled by adjusting the velocities of the left and right wheels via a motion control card. The onboard industrial Personal Computer (PC) (EVOC IPC POS-1811LNA) only needs to send the velocity commands to the motion control card, which manages the velocity servo control. Communication between the PC and the motion control card is through an RS-232 serial port. The whole control system has a two-level control architecture, in which the high-level control algorithms are written in C++ programming language, and the low-level control layer is in charge of the execution of the high-level velocity commands.

The position estimation of the robot is reconstructed using an odometer, which samples the left and right wheel velocities v_l and v_r to calculate the current robot configuration. The formula for this pose estimator is

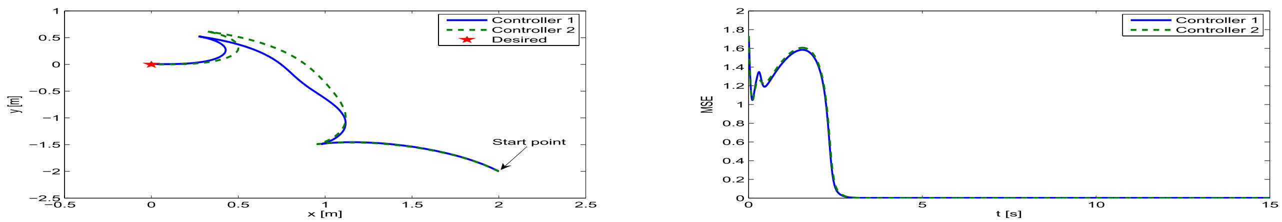
$$v=\frac{v_l+v_r}{2},\quad \omega=\frac{v_l-v_r}{L} \tag{45}$$

$$\begin{aligned} x_k &= x_{k-1}+v\cos(\theta_{k-1})\Delta T\\ y_k &= y_{k-1}+v\sin(\theta_{k-1})\Delta T\\ \theta_k &= \theta_{k-1}+\omega\Delta T \end{aligned} \tag{46}$$

where $q_k = [x_k, y_k, \theta_k]^T$ is the posture at time $t_k = k\Delta T$ and $\Delta T = 100$ ms is the sampling period. $L = 0.33$ m is the distance between two wheels. The block diagram of the mobile robot control system is shown in Fig. 7.

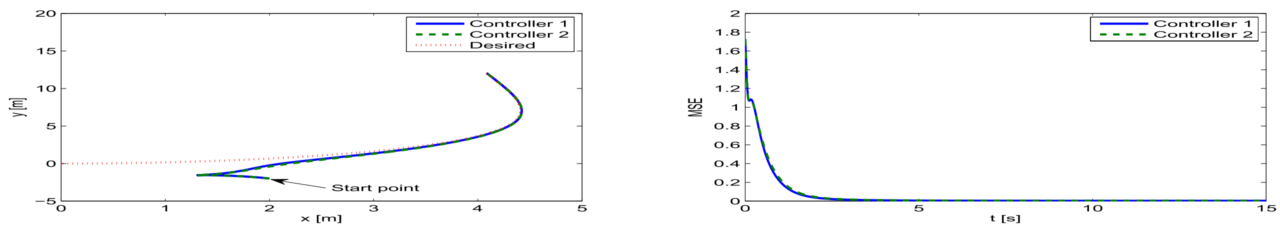
For the experimental purpose, the reference velocities are chosen as follows:

$$\begin{aligned} 0\le t<10:\ v_d &= 0.43\sin(\pi t/30)\\ \omega_d &= 0\\ 10\le t<20:\ v_d &= 0.43\sin(\pi t/30)\\ \omega_d &= -\pi^2/20\sin(\pi t/10)\\ 20\le t<30:\ v_d &= 0.43\sin(\pi t/30)\\ \omega_d &= 0\\ 30\le t:\ v_d &= 0\\ \omega_d &= 0. \end{aligned} \tag{47}$$



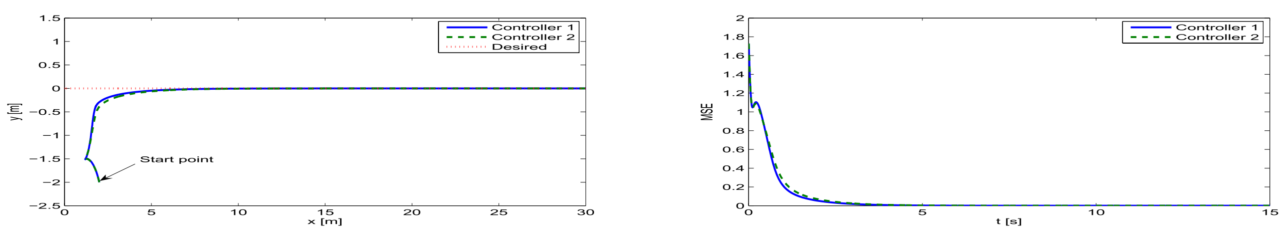
(a) (b)

Fig. 2. Simulation results of Case 1. (a) Robot position in the (x, y) plane. (b) MSE control performance.



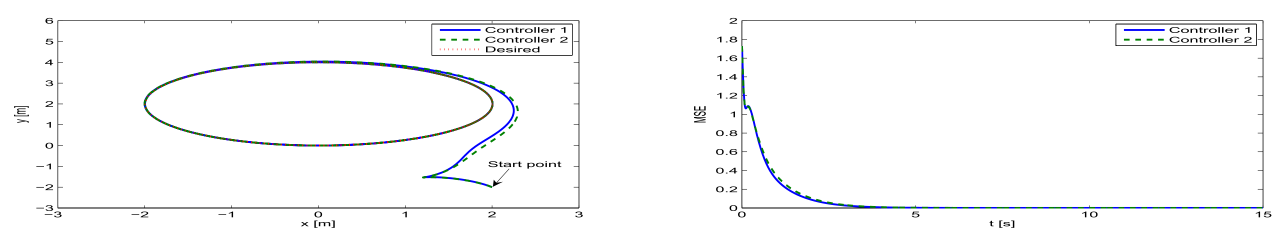
(a) (b)

Fig. 3. Simulation results of Case 2. (a) Robot position in the (x, y) plane. (b) MSE control performance.



(a) (b)

Fig. 4. Simulation results of Case 3. (a) Robot position in the (x, y) plane. (b) MSE control performance.



(a) (b)

Fig. 5. Simulation results of Case 4. (a) Robot position in the (x, y) plane. (b) MSE control performance.

The initial position and orientation of reference robot is $q_d(0) = [1, 0.5, 0]^T$. From the expression in (47), we know that the reference velocity is in the C1 case if $0 \leq t < 30$, otherwise it is in the C2 case. The reference velocities generate a U-shaped trajectory in the (x, y) plane with start point $(1, 0.5)$ and end point $(1, 2.5)$, as shown in Fig. 8(a). In the experimental implementation, the controller parameters are set as $k_0 = 10$, $k_1 = 1$, and $k_2 = 3$, and the nonlinear time-varying function is chosen as $h = 1.2 \tanh(x_e^2 + y_e^2) \sin(t)$. The experimental results are shown in Fig. 8, where the dotted lines and the solid lines in Fig. 8(a), (c), and (d) represent the reference signals and the corresponding actual experimental signals, respectively. Fig. 9 shows the motion sequences of the mobile robot during the experiment, in which the reference trajectory is marked in black. As shown in Fig. 8(a) and (b), the tracking errors all approach zero and the robot follows the desired trajectory with satisfactory performance. In addition, the linear velocity and the angular velocity of the robot shown in Fig. 8(c) and (d) are within the maximum velocity limits of the mobile robot: $v_{\max} = 1.6$ m/s and $\omega_{\max} = 300°$/s.

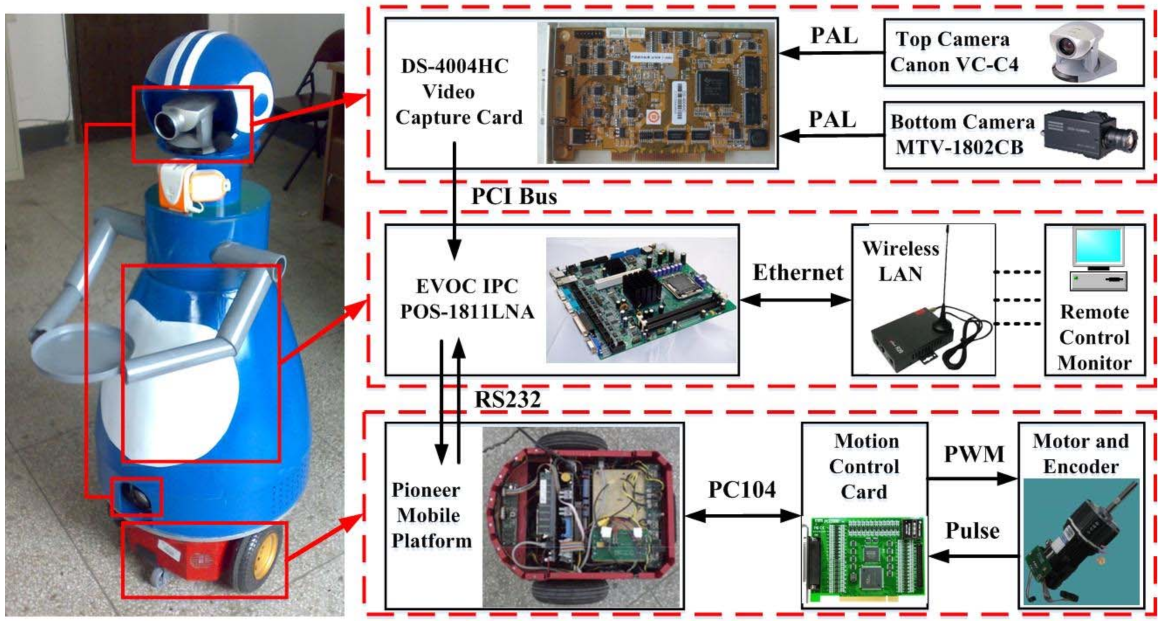


Fig. 6. Hardware configuration of the mobile robot control system.

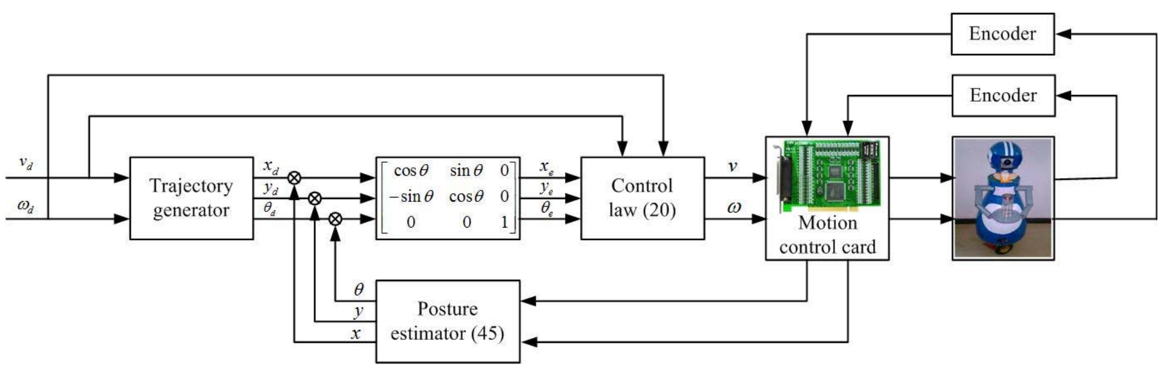


Fig. 7. Block diagram of the mobile robot control system.

Finally, for the purpose of comparison, some comparative experiments are executed under the same conditions. In the comparative study, two different types of tasks are introduced to verify the performance.

Task A: Stabilization: $v_d = 0$ and $\omega_d = 0$.

Task B: Tracking

$$\begin{aligned} 0 \le t < 5: \; v_d &= -0.5 \\ \omega_d &= 0 \\ 5 \le t < 10: \; v_d &= 0.1(t-10) \\ \omega_d &= 0 \\ 10 \le t < 15: \; v_d &= 0.1(t-10) \\ \omega_d &= 0.1(t-10) \\ 15 \le t: \; v_d &= 0.5 \\ \omega_d &= 0.5. \end{aligned}$$

The reference linear velocity and the angular velocity of the robot for Task B are shown in Fig. 10. The initial conditions for all tasks are $q_d(0) = [1, 0.5, 0]^T$. The performance of the proposed controller is compared with the other two published controllers in [40] and [41] that also claimed to simultaneously solve the stabilization and tracking problems.

The stabilization and tracking results of the aforementioned three controllers are illustrated in Figs. 11 and 12, respectively, to compare the performance. It is observed from these figures that satisfactory performances are obtained for both tasks using our proposed controller and the controller in [41], while the controller proposed in [40] only achieves good result for Task A. A robot with a controller in [40] cannot follow the desired trajectory in Task B over a period of time, though eventually the tracking errors tend to zero. The poor performance of the controller in [41] for Task B can be attributable to the nonnegative restriction on the reference linear velocity. It is observed from Fig. 10 that the reference linear velocity for Task B is negative for the first 10 s, hence the tracking errors of robot with the controller in [41] do not go to zero at the beginning. However, the robot

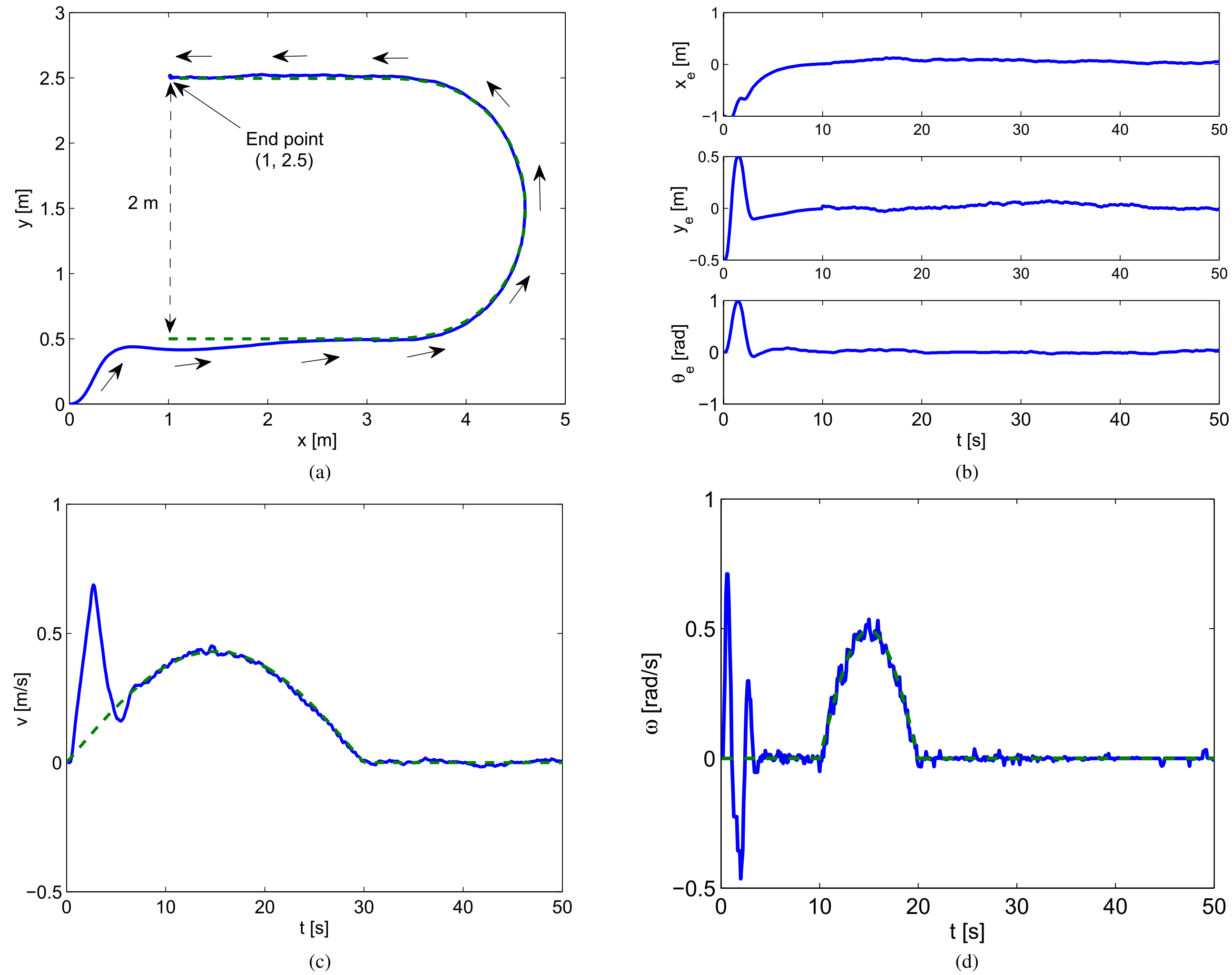


Fig. 8. Experimental results on the mobile robot. (a) Robot position in the (x, y) plane. (b) Tracking errors. (c) Linear velocity of the mobile robot. (d) Angular velocity of the mobile robot.

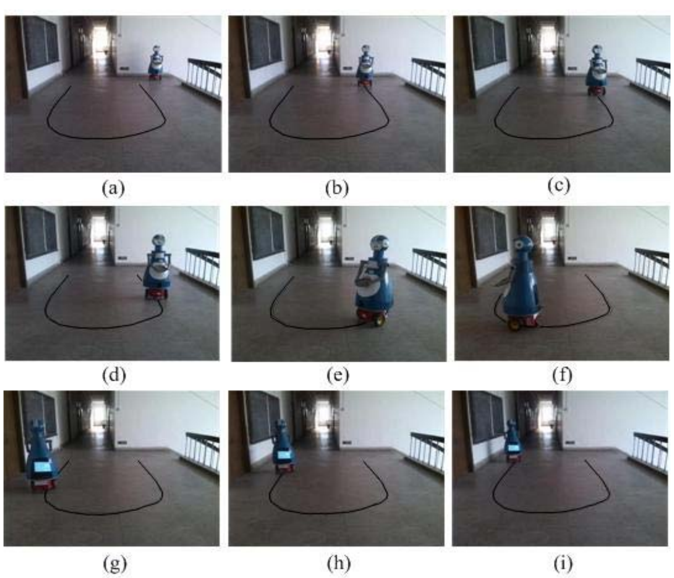


Fig. 9. Motion sequences of the mobile robot during the experiment.

eventually follows the desired trajectory when the reference linear velocity becomes positive thereafter. Another comment on the experimental comparisons is that although the performance of controller in [41] is comparable with our controller, the controller in [41] is more complicated, and some design constants must be carefully selected to meet some constraints like [41, (43) and (46)].

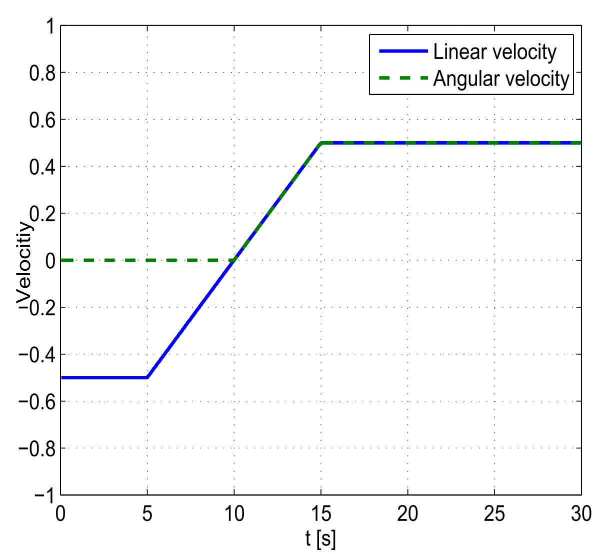
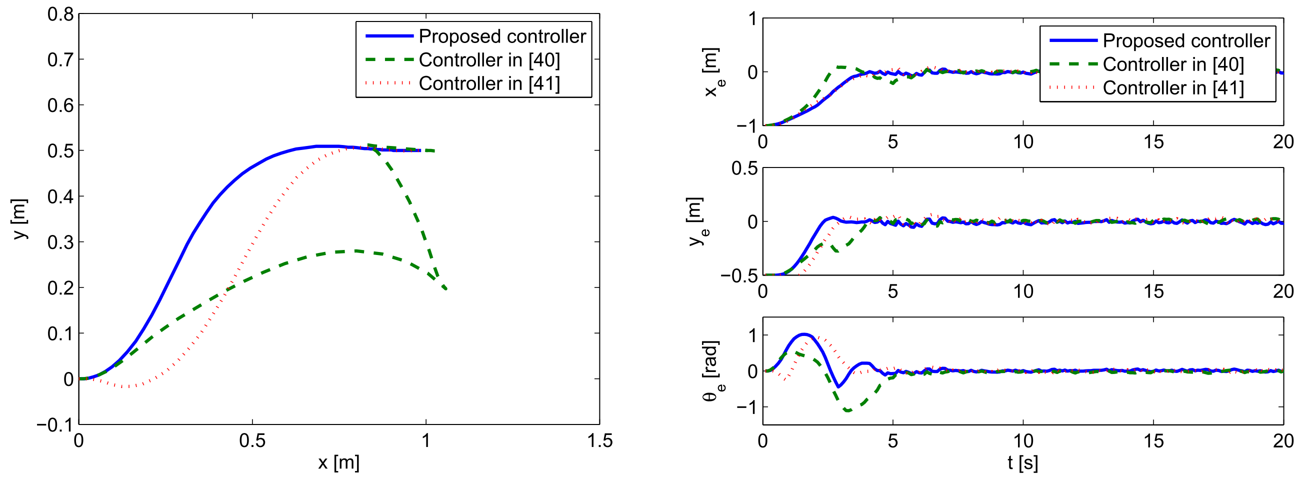
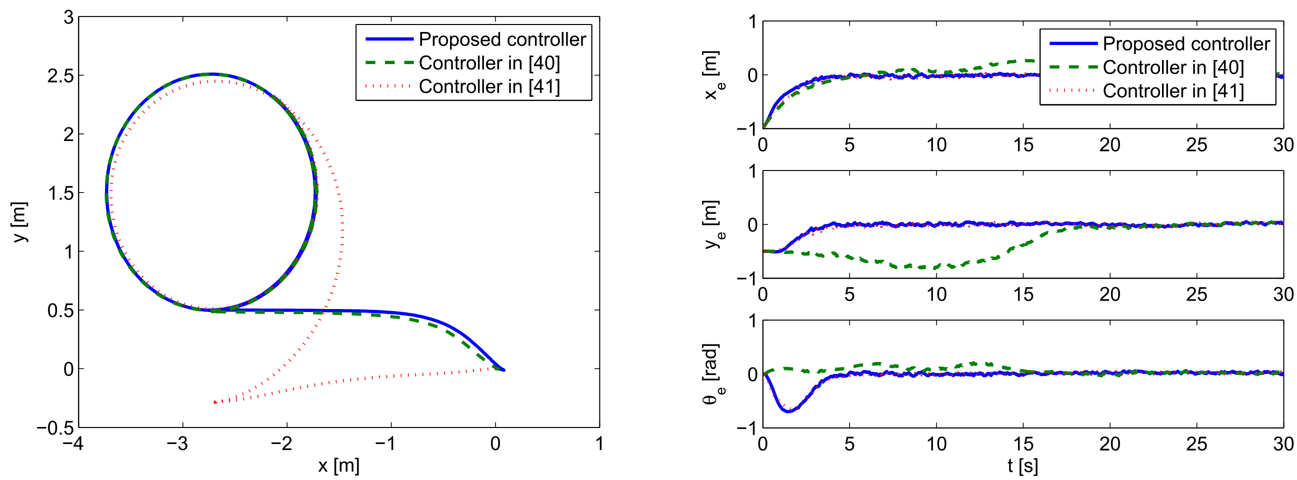


Fig. 10. Reference velocities for Task B.



(a) (b)

Fig. 11. Experimental comparison results for Task A. (a) Robot position in the (x, y) plane. (b) Stabilization errors.



(a) (b)

Fig. 12. Experimental comparison results for Task B. (a) Robot position in the (x, y) plane. (b) Tracking errors.

V. Conclusion

A single time-varying controller was developed with the aid of Lyapunov method to simultaneously solve the asymptotic regulation and tracking problems of a mobile robot. The simulation and experimental results verified the effectiveness of the proposed controller. The outstanding feature of the proposed controller is computationally simple due to its full use of the existing results on stabilization and tracking for wheeled mobile robots. It is also interesting to investigate how to extend the results in this paper to more general driftless systems, e.g., the nonholonomic chained form systems. In [46], we showed that the idea used in this paper can be applied to any mechanical system that can be transformed into the chained form of order three. Future works may to extend the proposed methodology to mobile robots with mechanical dynamics and input saturations [47], [48].

Appendix

The extended version of Barbalat's lemma [3]: if a differentiable function $f(t) : R^+ \to R$ converges to a limit value as $t \to \infty$, and if its derivative $(\mathrm{d}f/\mathrm{d}t)$ is the sum of two terms, one being uniformly continuous and another tending to zero as $t \to \infty$, then $(\mathrm{d}f/\mathrm{d}t) \to 0$ when $t \to \infty$.

Gronwall–Bellman inequality [45]: let μ: [a, b] $\to R$ be continuous and nonnegative, $c(t)$ is nondecreasing. If a continuous function y: [a, b] $\to R$ satisfies $y(t) \le c(t) + \int_a^t \mu(s)y(s)\mathrm{d}s$ for $a \le t \le b$, then in the same interval: $y(t) \le c(t)\exp(\int_a^t \mu(s)\mathrm{d}s)$.

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