State and output feedback stabilization for second-order nonholonomic chained systems based on sampled data control approach

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Abstract

This paper proposes new state and output feedback control systems based on sampled data control for a class of nonholonomic systems in second-order chained form. Our novel approach is to find a time-varying coordinate transformation by which the chained system discretized with the zero order hold and the sampler is transformed into a simple linear time-invariant discrete-time system. The contributions are as follows. Firstly, the problem of finding state feedback controllers to stabilize the chained systems is reduced to the well-known pole assignment problem for linear time-invariant discrete-time systems. As a result, a simple and explicit design method of the state feedback controller is obtained. Secondly, design problems of observers and dynamic output feedback controllers to stabilize the chained systems are posed and are solved. The problems are reduced to the well-known linear control problems. The proposed design methods are simple and straightforward.

1. Introduction

A chained form is a canonical form for a class of nonholonomic systems introduced by Murray and Sastry [15] and certain mechanical systems with nonholonomic constraints can be locally, or globally, converted to the chained form. A nonholonomic system described in the chained form is called chained system. The feedback stabilization problem for the chained systems has been studied by many researchers. The major obstruction to the stabilization problem was the fact that there exists no continuous time-invariant state feedback controller to stabilize the chained systems [3]. During the last few years, many interesting controller designs have been proposed to stabilize the chained systems, for example, the methods based on discontinuous feedback control [1], on time-varying feedback control [18,20] and on hybrid control [4,14,19,21]. For a good overview, see [7] and [13]. These studies were primarily limited to the stabilization of first-order chained systems.

This paper addresses state and output feedback stabilization problems for second-order chained systems. The second-order chained systems arise very often in the study of mechanical systems. Typical examples include redundant manipulators [11], underactuated systems [17] and so on. In [8], the result of [1] for first-order chained systems has been generalized to high-order generalized chained systems and simple discontinuous state feedback controllers have been presented to stabilize exponentially the systems. In [16], a control strategy based on a multirate digital control has been proposed and a deadbeat control problem for high-order chained systems has been solved. In [12], an adaptive and robust control scheme has been proposed to stabilize second-order chained systems in the presence of input uncertainty. These results are restricted to stabilization problems by means of state feedback. On the other hands, to our knowledge, the only papers that addressed dynamic output feedback problems are [2] that is based on a discontinuous control law, [6] where a backstepping approach is used and [9,10] that is based on the reduction to cascaded systems. In [6,9,10], design methods of observers and dynamic output feedback controllers for first-order chained systems have been presented. In [2], dynamic output feedback controllers to stabilize first-order chained systems with multiple input have been proposed.

This paper proposes a new state and dynamic output feedback controllers for the second-order chained systems based on sampled data control to achieve asymptotically stabilization to the origin. The key idea is to find a time-varying coordinate transformation by which the chained system discretized with the zero order hold and the sampler is transformed into a simple *linear time-invariant discrete-time system*. The contributions are as follows. Firstly, the problem of finding state feedback controllers to stabilize the chained systems is reduced to the well-known pole assignment problem for linear time-invariant discrete-time systems. As a result, a simple and explicit design method of the state feedback controller is obtained. This result is the extension of [22] to the class of second-order chained systems. Secondly, design problems of observers and output feedback controllers to stabilize the chained systems are posed and are solved. The problems are reduced to the well-known linear control problems. The proposed design methods are simple and straightforward. A simulation for the motion control of a planar rigid body demonstrates the effectiveness of the proposed method.

2. Second-order chained form

The class of nonholonomic systems to be studied in this paper is described in the following second-order chained form

$$\begin{cases} x_0^{(2)}(t) = u_1(t) \\ x_1^{(2)}(t) = u_2(t) \\ x_i^{(2)}(t) = x_{i-1}(t)\, u_1(t), \quad i \in \{2, \cdots, n\}, \end{cases} \tag{2.1}$$

where $[x_0, \cdots, x_n]^T \in \boldsymbol{R}^{n+1}$ is the state to be regulated and $u_i \in \boldsymbol{R}$, $i = 1,2$ are two control inputs. Let

$$x_{ij}(t) = x_i^{(j)}(t), \quad i = 0, \cdots, n, \; j = 0, 1 \; .$$

Then the chained system of (2.1) can be represented as the

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following two subsystems of Σ_1 and Σ_2 ;

$$\Sigma_1 : \ \frac{d}{dt} z(t) = A_z\, z(t) + b_z\, u_1(t) \tag{2.2}$$

and

$$\Sigma_2 : \ \frac{d}{dt} x(t) = A(u_1(t))\, x(t) + b\, u_2(t) \ , \tag{2.3}$$

where

$$z = \begin{bmatrix} x_{00} \\ x_{01} \end{bmatrix} \in R^2, \ A_z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in R^{2\times 2}, \ b_z = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in R^2 \ .$$

$$x = \begin{bmatrix} x_{n0} \\ x_{n1} \\ \vdots \\ x_{20} \\ x_{21} \\ x_{10} \\ x_{11} \end{bmatrix} \in R^{2n}, \ A(u_1(t)) = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & u_1 & & & 0 & \\ & & 0 & \ddots & & & \\ & & & \ddots & 1 & & \\ & & & & 0 & u_1 & \\ & 0 & & & & 0 & 1 \\ & & & & & & 0 \end{bmatrix} \in R^{2n\times 2n} ,$$

$$b = [0 \ \cdots \ 0 \ \ 1]^T \in R^{2n} \ .$$

In the next section, we consider the stabilization problem by means of state feedback. In section 4, we give a design method of an observer to estimate the state from the output. Moreover, we solve the stabilization problem by means of output feedback.

3. State feedback stabilization

In this section, suppose that all the state variables are available as measurable outputs. Figure 1 shows the control system proposed in this section, which is based on the sampled data control. S_T is the sampler with the sampling period $T \in R$, and the states of both $z(t)$ and $x(t)$ are sampled at the sampling instants. H_T is the zero order hold with the sampling period $T \in R$ [5]. Note that S_T and H_T are synchronized. In the digital controllers, the control inputs $u_1[i]$ *and* $u_2[i]$ are determined from the sampled states, $z(iT)$ and $x(iT)$ at the sampling instants. Then the control inputs become piece-wise constants as follows:

$$u_j(t) = u_j[i] \in R \qquad (iT \le t < (i+1)T)$$

for $j = 1, 2$. In this section, we consider the following problem.

State feedback Stabilization Problem

Find state feedback controllers $u_j[i] = C_j(i, z(iT), x(iT))$, $j = 1, 2$ such that

$$\lim_{t\to\infty} \| z(t) \| = 0 \tag{3.1}$$

and

$$\lim_{t\to\infty} \| x(t) \| = 0 \tag{3.2}$$

for any $z(0) \neq (0,0)$ *and* $x(0) \in R^{2n}$, where $\| \ \|$ denotes the Euclidean norm.

The purpose of this section is to present controllers satisfying the stabilizing problem mentioned above. Under sampled data control of Fig.1, the subsystems of (2.2) and (2.3) can be written as

$$\Sigma_1 : \ \frac{d}{dt} z(t) = A_z\, z(t) + b_z\, u_1[i] \tag{3.3}$$

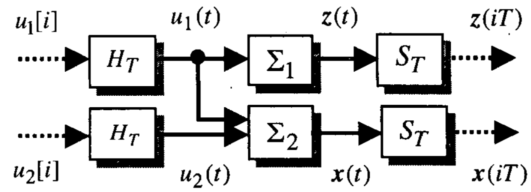

Fig.1 Proposed control system for the state feedback stabilization problem

and

$$\Sigma_2 : \ \frac{d}{dt} x(t) = A(u_1[i])\, x(t) + b\, u_2[i] \tag{3.4}$$

during the sampling intervals $I_i = [iT, (i+1)T)$, respectively. As can be seen, the subsystem of (3.4) is a controllable piece-wise linear system as long as $u_1[i]$ is nonzero. To begin with, we focus on the subsystem Σ_2 of (3.4). Then the response of the state $x(t)$ can be expressed as follows.

$$x(t) = e^{A(u_1[i])(t-iT)} x(iT) + \int_0^{t-iT} e^{A(u_1[i])\sigma} b\, d\sigma\, u_2[i], \ \ t \in I_i \ .$$

In particular, at the sampling instants, we obtain a linear time-varying discrete-time state equation

$$x((i+1)T) = \tilde{A}(i)\, x(iT) + \tilde{b}(i) u_2[i] \ , \tag{3.5}$$

where

$$\tilde{A}(i) = e^{A(u_1[i])T} \in R^{2n\times 2n}, \ \ \tilde{b}(i) = \int_0^T e^{A(u_1[i])\sigma} b\, d\sigma \in R^{2n} \ .$$

It is interesting that these have the following decomposition:

$$\tilde{A}(i) = T[i]\, e^{JT}\, T[i]^{-1} \in R^{2n\times 2n},$$
$$\tilde{b}(i) = T[i] \int_0^T e^{J\sigma} b\, d\sigma \in R^{2n},$$

where

$$T[i] = diag \left[\underbrace{u_1[i]^{n-1}, u_1[i]^{n-1}}_{2}, \cdots, \underbrace{u_1[i], u_1[i]}_{2}, \underbrace{1, 1}_{2} \right] \in R^{2n\times 2n}$$

and

$$J = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \in R^{2n\times 2n} .$$

Let

$$\lambda_i = \frac{u_1[i+1]}{u_1[i]} \ . \tag{3.6}$$

Then

$$T[i+1] = T[i]\ diag \left[\underbrace{\lambda_i^{\,n-1}, \lambda_i^{\,n-1}}_{2}, \cdots, \underbrace{\lambda_i, \lambda_i}_{2}, \underbrace{1, 1}_{2} \right] .$$

We introduce a piece-wise constant of the coordinate transformation as follows

$$\bar{x}(t) = T[i]^{-1} x(t) \in R^{2n} \ , \ \ t \in I_i \ . \tag{3.7}$$

This transforms the time-varying discrete-time state equation of (3.5) into a time-invariant one as shown in the following lemma.

Lemma 3.1

Suppose that $\lambda_i = \lambda_{i_k} \in \boldsymbol{R},\ \forall i \geq i_k$ for a nonnegative integer i_k. Let $\lambda = \lambda_{i_k} \in \boldsymbol{R}$. Under the proposed sampled data control system, the state sequences $x(iT)$, $i \geq i_k$ of the chained system at the sampling instants can be expressed in both the following linear, time-invariant, discrete-time state equation;

$$\bar{x}((i+1)T) = \bar{A}\,\bar{x}(iT) + \bar{b}\,u_2[i] \tag{3.8}$$

and the following coordinate transformation

$$x(iT) = T[i]\,\bar{x}(iT), \tag{3.9}$$

where

$$\bar{A} = diag\left[\underbrace{\lambda^{-(n-1)}, \lambda^{-(n-1)}}_{2}, \cdots, \underbrace{\lambda^{-1}, \lambda^{-1}}_{2}, \underbrace{1, 1}_{2}\right] e^{JT} \in \boldsymbol{R}^{2n\times 2n}$$

$$\bar{b} = diag\left[\underbrace{\lambda^{-(n-1)}, \lambda^{-(n-1)}}_{2}, \cdots, \underbrace{\lambda^{-1}, \lambda^{-1}}_{2}, \underbrace{1, 1}_{2}\right] \int_0^T e^{J\sigma} b\, d\sigma \in \boldsymbol{R}^{2n}.$$

Moreover, the discrete-time state equation $(\bar{A}, \bar{b})$ of (3.8) is controllable for any sampling periods $T \in \boldsymbol{R}$ and $\lambda \neq 0, -1$.

Proof: The proof is similar to [21,22], and it is omitted. Q.E.D.

Next, we focus on the subsystem Σ_1 of (3.3). Consider the following state feedback controller

$$u_1[i] = \boldsymbol{k}_1 z(iT) \quad , \tag{3.10}$$

where $\boldsymbol{k}_1 \in \boldsymbol{R}^2$. We show a design method of the feedback gain $\boldsymbol{k}_1 \in \boldsymbol{R}^2$ satisfying (3.1) and

$$\frac{u_1[i+1]}{u_1[i]} = \lambda, \quad \forall i \geq i_k$$

for a nonnegative integer i_k. By using (3.10), the state sequences $z(iT)$ of the subsystem Σ_1 at the switching instants can be expressed as

$$z((i+1)T) = (\tilde{A}_z + \tilde{b}_z \boldsymbol{k}_1)z(iT) \quad , \tag{3.11}$$

where

$$\tilde{A}_z = e^{A_z T} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \in \boldsymbol{R}^{2\times 2}, \quad \tilde{b}_z = \int_0^T e^{A_z \sigma} b_z\, d\sigma = \begin{bmatrix} \frac{T^2}{2!} \\ T \end{bmatrix} \in \boldsymbol{R}^2.$$

Since $(\tilde{A}_z, \tilde{b}_z)$ of (3.11) is controllable, by an equivalence transformation of $\bar{z}(iT) = S\,z(iT)$, $(\tilde{A}_z, \tilde{b}_z)$ can be transformed into the following controllable canonical form:

$$\bar{z}((i+1)T) = (\bar{A}_z + \bar{b}_z \boldsymbol{k}_1 S^{-1})\bar{z}(iT) \quad ,$$

where

$$\bar{A}_z = S\tilde{A}_z S^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \in \boldsymbol{R}^{2\times 2}, \quad \bar{b}_z = S\tilde{b}_z = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \boldsymbol{R}^2.$$

Let

$$\boldsymbol{k}_1 = [1, -2+\lambda]S \in \boldsymbol{R}^2, \tag{3.12}$$

where $\lambda \in \boldsymbol{R}$ is arbitrary such that $0 < |\lambda| < 1$. Then since the eigenvalues of $(\bar{A}_z + \bar{b}_z \boldsymbol{k}_1 S^{-1})$ are 0 and $\lambda \in \boldsymbol{R}$, for any $z(0) \in \boldsymbol{R}^2$,

$$\lim_{i\to\infty} \left\| \bar{z}(iT) \right\| = \lim_{i\to\infty} \left\| z(iT) \right\| = 0 .$$

The response of the state $z(t)$ between the switching instants can be represented as follows.

$$z(t) = \left(e^{A_z(t-iT)} + \int_0^{t-iT} e^{A_z\sigma} b_z\, d\sigma \boldsymbol{k}_1\right) z(iT), \quad t \in I_i .$$

Note that there exists a positive constant $\bar{c}_1$ such that

$$\left\| e^{A_z(t-iT)} \right\| \leq \bar{c}_1, \quad \forall t \in I_i .$$

Accordingly,

$$\lim_{t\to\infty} \left\| z(t) \right\| = 0 .$$

Moreover, letting

$$\bar{z} = \begin{bmatrix} \bar{z}_2 \\ \bar{z}_1 \end{bmatrix} \in \boldsymbol{R}^2,$$

since

$$\bar{z}((i+1)T) = \begin{bmatrix} 0 & 1 \\ 0 & \lambda \end{bmatrix} \bar{z}(iT) \quad ,$$

it is easy to check that

$$\begin{aligned} u_1[i] &= \boldsymbol{k}_1 S^{-1}\,\bar{z}(iT) \\ &= \lambda^{i-1}(\lambda-1)^2 \bar{z}_1(0) . \end{aligned}$$

Therefore we obtain

$$\frac{u_1[i+1]}{u_1[i]} = \lambda, \quad \forall i \geq 1 \quad , \tag{3.13}$$

which means that the state feedback controller of (3.10) and (3.12) makes true the assumption of Lemma 3.1 on λ_i of (3.6). Accordingly, the stabilization problem of the chained systems is reduced to that of a linear time-invariant discrete-time system of (3.8). The following theorem presents a simple and explicit design method of the stabilizing controller.

Theorem 3.2

Consider the following state feedback controllers;

$$u_1[i] = \boldsymbol{k}_1 z(iT) \quad , \tag{3.14}$$

$$u_2[i] = \boldsymbol{k}_2(i, z(iT))\, x(iT) \quad , \tag{3.15}$$

where $\boldsymbol{k}_1 \in \boldsymbol{R}^2$ is given by (3.12) and $\lambda \in \boldsymbol{R}$ is arbitrary such that $0 < |\lambda| < 1$.

$$\boldsymbol{k}_2(i, z(iT)) = \bar{\boldsymbol{k}}_2 T[i]^{-1} \in \boldsymbol{R}^{2n} \quad , \tag{3.16}$$

and $\bar{\boldsymbol{k}}_2 \in \boldsymbol{R}^{2n}$ is a constant state feedback gain such that all eigenvalues of $(\bar{A} + \bar{b}\bar{\boldsymbol{k}}_2)$ lie in the open unit disk, i.e.,

$$\max_j \left| \lambda_j(\bar{A} + \bar{b}\,\bar{\boldsymbol{k}}_2) \right| < 1 \quad . \tag{3.17}$$

Then (3.1) and (3.2) are satisfied for any $x(0) \in \boldsymbol{R}^{2n}$ and

$z(0) \neq (0,0)^T$.

Proof: The remaining of the proof is to show (3.2) is satisfied for any $x(0) \in \boldsymbol{R}^{2n}$ and $z(0) \neq (0,0)^T$ on the subsystem of Σ_2. Note that (3.13) holds. By using (3.15) and (3.16), from Lemma 3.1, the state sequences $\bar{x}(iT)$ of (3.8) at the switching instants can be expressed as

$$\bar{x}((i+1)T) = (\bar{A} + \bar{b}\,\bar{k}_2)\,\bar{x}(iT) \ .$$

Then the intersampling response of the state $x(t)$ can be expressed as follows.

$$\begin{aligned} x(t) &= T[i]\left(e^{J(t-iT)} + \int_0^{t-iT} e^{J\sigma} b\, d\sigma \bar{k}_2\right)\bar{x}(iT) \\ &= T[i]\left(e^{J(t-iT)} + \int_0^{t-iT} e^{J\sigma} b\, d\sigma \bar{k}_2\right)(\bar{A} + \bar{b}\,\bar{k}_2)^i T[0]^{-1} x(0), \quad t \in I_i . \end{aligned}$$

There exists a positive constant $\bar{c}_1$ such that

$$\left\| e^{J(t-iT)} \right\| \leq \bar{c}_1 \quad \forall t \in I_i$$

and

$$\lim_{i \to \infty} \left\| T[i] \right\| = 1 \ .$$

Therefore (3.2) is satisfied for any $x(0) \in \boldsymbol{R}^{2n}$ and $z(0) \neq (0,0)^T$. The proof is completed. Q.E.D.

Remark 1: From Lemma 3.1, since the discrete-time state equation $(\bar{A}, \bar{b})$ is controllable, the design method of finding $\bar{k}_2 \in \boldsymbol{R}^n$ satisfying (3.17) is provided easily by the well-known conventional linear control theory, for example [5].

Remark 2: In this theorem, it is revealed that the stabilization problem of the chained system is reduced to that of a linear time-invariant discrete-time system of (3.8). One advantage of our approach is that it easily allows extending to some interesting control problems, for example, deadbeat control and so on.

4. Dynamic output feedback stabilization with State Observer

In this section, assume that the available variables as the measured outputs are $z(t) \in \boldsymbol{R}^2$ *and* $x_{n0}(t) \in \boldsymbol{R}$ and let

$$y(t) = c\, x(t) \ , \tag{4.1}$$

where

$$c = [1, 0, \cdots, 0] \in \boldsymbol{R}^{2n} \ .$$

This assumption is the same as [6,10]. Figure 2 shows the control system proposed in this section, which is based on the sampled data control. The outputs of both $z(t)$ and $y(t)$ are sampled at the sampling instants. The control inputs $u_1[i]$ *and* $u_2[i]$ are determined from the sampled outputs, $z(iT)$ and $y(iT)$ at the sampling instants. Then the control inputs are piece-wise constants as follows:

$$u_j(t) = u_j[i] \in \boldsymbol{R}, \quad t \in I_i$$

for $j = 1, 2$. In this section, we consider the following problem.

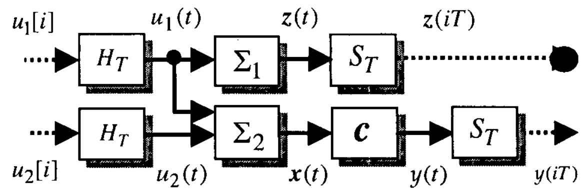

Fig.2 Proposed control system for the output feedback stabilization problem

Output feedback Stabilization Problem

Design an observer to estimate the state $x(iT) \in \boldsymbol{R}^{2n}$ at the sampling instants from the available inputs of u_1 *and* u_2 and the outputs of y *and* z. Moreover, by using the observer, find output feedback controllers $u_j[i] = C_j(i, z(iT), y(iT)),\ j = 1, 2$ such that

$$\lim_{t \to \infty} \left\| z(t) \right\| = 0 \tag{4.2}$$

and

$$\lim_{t \to \infty} \left\| x(t) \right\| = 0 \tag{4.3}$$

for any $z(0) \neq (0,0)^T$ *and* $x(0) \in \boldsymbol{R}^{2n}$.

To begin with, we show an observer to estimate the state $x(iT) \in \boldsymbol{R}^{2n}$ at the sampling instants. Under sampled data control of Fig.2, the system of (2.2), (2.3) and (4.1) can be written as

$$\Sigma_1 : \ \frac{d}{dt} z(t) = A_z\, z(t) + b_z\, u_1[i], \quad t \in I_i$$

and

$$\hat{\Sigma}_2 : \begin{cases} \frac{d}{dt} x(t) = A(u_1[i])\, x(t) + b\, u_2[i] \\ y(t) = c\, x(t) \end{cases}, \quad t \in I_i \ . \tag{4.4}$$

We focus on the subsystem $\hat{\Sigma}_2$ of (4.4). By using the similar discussion as section 3, we obtain the following lemma.

Lemma 4.1

Consider the feedback controller of (3.10) and (3.12). Under the proposed sampled data control system, the output sequences $y(iT)$, $i \geq 1$ of the chained system at the sampling instants can be expressed in both the following linear, time-invariant, discrete-time state equation;

$$\begin{cases} \bar{x}((i+1)T) = \bar{A}\,\bar{x}(iT) + \bar{b}\, u_2[i] \\ \bar{y}(iT) = c\,\bar{x}(iT) \end{cases} \tag{4.5}$$

and the following coordinate transformation

$$x(iT) = T[i]\ \bar{x}(iT),$$

and

$$y(iT) = u_1[i]^{\,n-1}\ \bar{y}(iT) \ ,$$

where $\bar{A}$ *and* $\bar{b}$ are given by (3.9). Moreover, the discrete-time state equation $(\bar{A}, \bar{b}, c)$ of (4.5) is controllable and observable for any sampling periods $T \in \boldsymbol{R}$ and $\lambda \neq 0, -1, 1$.

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Proof: Omitted. Q.E.D.

Note that $u_1[i]$ *and* $T[i]$ are available. The design problem of the observer is reduced to that of an observer to estimate $\bar{x}(iT) \in R^{2n}$ for the linear time-invariant discrete-time system of (4.5). The following theorem presents a simple and explicit design method of the observer.

Theorem 4.2

Consider the feedback controller of (3.10) and (3.12). Under the proposed sampled data control system, the observer to estimate the state $x(iT) \in R^{2n}$ from the available inputs u_1 *and* u_2 and the output y is given by

$$\begin{cases} \hat{\bar{x}}((i+1)T) = \bar{A}\hat{\bar{x}}(iT) + \bar{b}u_2[i] - l\left(\dfrac{y(iT)}{u_1[i]^{n-1}} - c\hat{\bar{x}}(iT)\right) \\ \hat{x}(iT) = T[i]\hat{\bar{x}}(iT) \end{cases}, \quad (4.6)$$

where $l \in R^{2n}$ is a constant observer gain such that all eigenvalues of $(\bar{A} + lc)$ lie in the open unit disk, i.e.,

$$\max_j \left| \lambda_j(\bar{A} + lc) \right| < 1 . \quad (4.7)$$

Then

$$\lim_{i \to \infty} \left\| x(iT) - \hat{x}(iT) \right\| = 0$$

for any $x(0) \in R^{2n}$ and $z(0) \neq (0,0)^T$.

Proof: The proof is straightforward from Lemma 4.1 and [5] and *it is omitted.* Q.E.D.

Next, we consider the output feedback stabilization problem. The following theorem provides a simple solution of the problem.

Theorem 4.3

Consider the following output feedback controllers

$$u_1[i] = k_1 z(iT) \quad ,$$

$$u_2[i] = k_2(i, z(iT))\hat{x}(iT) \quad ,$$

where $k_1 \in R^2$ is given by (3.12) and $\lambda \in R$ is arbitrary such that $0 < |\lambda| < 1$.

$$k_2(i, z(iT)) = \bar{k}_2 T[i]^{-1} \in R^{2n} \quad ,$$

where $\bar{k}_2 \in R^{2n}$ is a constant state feedback gain such that all eigenvalues of $(\bar{A} + \bar{b}\bar{k}_2)$ lie in the open unit disk, i.e.,

$$\max_j \left| \lambda_j(\bar{A} + \bar{b}\bar{k}_2) \right| < 1 . \quad (4.8)$$

$\hat{x}(iT) \in R^{2n}$ is the output of the observer of (4.6). Then (4.2) and (4.3) are satisfied for any $x(0) \in R^{2n}$ and $z(0) \neq (0,0)^T$.

Proof: The proof is similar to that of Theorem 3.2 and it is omitted. Q.E.D.

Remark : From the well-known *separation principle*, the state feedback gain and the observer gain can be determined independently, as they do not influence each other. Moreover From Lemma 4.1, since the discrete-time state equation $(\bar{A}, \bar{b}, c)$ is minimal, the design methods of finding $l \in R^{2n}$ and $\bar{k}_2 \in R^{2n}$ satisfying (4.7) and (4.8), respectively, are provided easily by the well-known conventional linear control theory, for example [5].

5. Numerical example

Consider a free rigid body system moving on horizontal plane as shown in Fig.3 [12]. m and J are the mass and the moment of inertia, respectively. (x, y) denotes the position of the center of the rigid body and θ is the orientation to x-axis. An external force f and torque τ are acting on the rigid body. The direction of f is assumed to be parallel to the body's axis. The motion of the body is described as follows.

$$\begin{cases} m\ddot{x} = f\cos\theta \\ m\ddot{y} = f\sin\theta \\ J\ddot{\theta} = \tau \end{cases} . \quad (5.1)$$

By using the input transformations given by

$$\begin{bmatrix} f \\ \tau \end{bmatrix} = \begin{bmatrix} \dfrac{m}{\cos\theta} & 0 \\ 0 & J\cos^2\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2J\dot{\theta}^2\tan\theta \end{bmatrix},$$

the system (5.1) is transformed into the second-order chained form (2.1) for $n = 2$, where

$$\begin{cases} x_0 = x \\ x_1 = \tan\theta \\ x_2 = y \end{cases} .$$

The sampling period is given by $T = 0.2$ (s). In simulations, the initial states of the vehicle are given by $[x(0), \dot{x}(0), y(0), \dot{y}(0), \theta(0), \dot{\theta}(0)] = [0, 5, 2, 0, 0, 0]$ and the desired states are zero.

We design the observer and output feedback controller by using Theorem 4.2 and 4.3 for $(\lambda_j(\bar{A} + \bar{b}\bar{k}_2), j = 1,2) = (0.5, 0.5)$, $\lambda = 0.9$ and $\lambda_j(\bar{A} + lc) = 0, j = 1,2$, which means a deadbeat observer. Figures 4 and 5 illustrate the responses of the states for the case of output feedback. These figures demonstrate that the proposed control strategy provides excellent regulations of the states to the origin. Figure 6 shows the comparison between the real state and the estimated one on x_{10} in the proposed observer. It is demonstrated that the estimated error at the sampling instance is given by

$$\left| x_{10}(iT) - \hat{x}_{10}(iT) \right| = 0, \ \forall i \geq 5 \ ,$$

and the proposed observer provides fast estimation of the state.

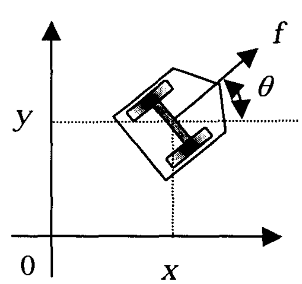

Fig.3 Coordinates of planar rigid body system

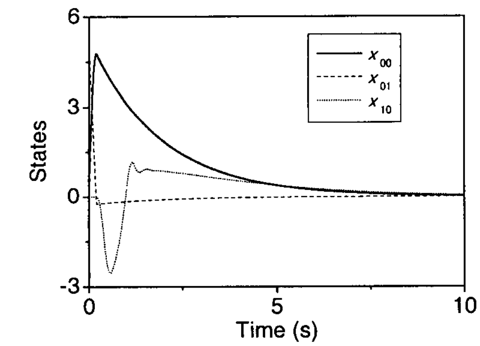

Fig.4 Responses of the states

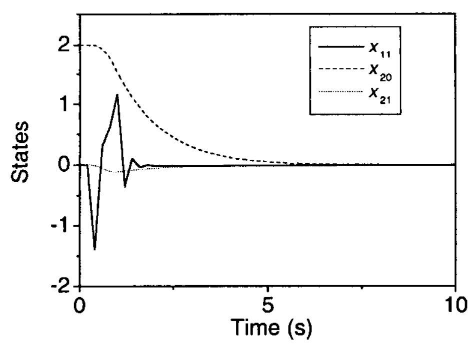

Fig.5 Responses of the states

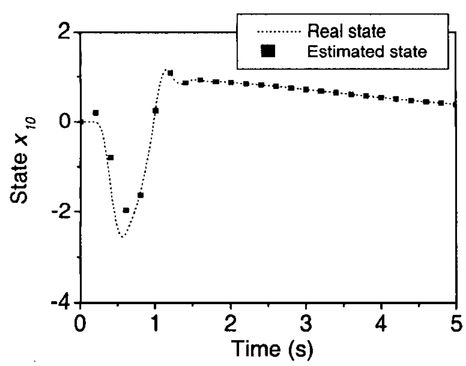

Fig.6 Responses of the state and the estimated one

6. Conclusion

This paper has presented a simple and explicit design method of state and dynamic output feedback controllers for the second-order chained systems based on sampled data control to achieve asymptotically stabilization to the origin as shown in Theorems 3.2 and 4.3.

The advantage of our approach is to allow easily extending to many interesting control problems for chained systems, for example, deadbeat control, adaptive control and so on. Future study will focus on the extension to high-order generalized chained form and the challenge to some optimal control problems.

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