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Sampled-Data Output Feedback Control of Uncertain Nonholonomic Systems in Chained Forms with Applications to Mobile Robots

Brandy Alger, Johnathan Votion, and Chunjiang Qian, PhD, *Member, IEEE*

***Abstract*—It is imperative to find a sampled-data controller for nonholonomic systems due to their implementation within digital computers. Nonholonomic systems in chained form are sufficiently important to research via the numerous real world applications, mobile robots being one of the biggest. Moreover, due to the presence of uncertain nonlinearities, most of the existing design methods are inapplicable to these systems. It has been proven that under a lower-triangular growth condition, a class of uncertain nonlinear systems can be globally stabilized by a sampled-data output feedback controller whose observer and control laws are discrete-time and linear. In this paper, using a change of coordinates and combining the recently developed sampled-data output feedback control method, we first design a sampled-data output feedback controller to stabilize the nonholonomic system with a single z-state. For nonholonomic systems with two-dimensional z-states, the output feedback control problem becomes much more challenging since the boundedness of the change of coordinates is not proved; we shall consider this in future works. Examples and computer simulations were conducted to show the effectiveness of the proposed controllers for a single z-state.**

***Index Terms*—nonholonomic, nonlinear, discrete, mobile robots.**

I. Introduction

Nonholonomic systems have been studied in depth to provide solutions for the controls problems. Nonholonomic systems are mechanical systems that contain constraints that are imposed and are not integrable, meaning they cannot be written as a time derivative of a function. Nonholonomic control systems are in particular interest to the controls field due to the fact that little literature has been made on the subject and therefore leaves room for a vast amount of growth. The main interest in nonlinear control problems is due to the fact that most of the real world is nonlinear and are not transformable into linear control problems, which then require nonlinear approaches to solve the control problem. Stabilization of nonholonomic systems are exceptionally challenging due to their inherent violation of Brocketts necessary condition, which is shows that nonholonomic systems cannot be controlled by any time-invariant continuous state feedback, even if it is controllable. This interesting dilemma has inspired many researchers to find a solution, of which many interesting results have been found, although many of these are continuous solutions. A sampled-data controller to stabilize the system becomes imperitive as more and more controllers are being implented via digital computers. Controller design for nonlinear nonholonomic systems becomes more challenging when under sampled-data implementation, also literature for this special case is lacking. Most of the current solutions only guarantee local or semi-global stability due to the invetitbility of approximation errors. Another popular method is to discretize a continuous-time controller and by carefully selecting a sample gain, can guarantee global stability. The problem with this method is that it assumes that all states are measurable. When not all of the states are measurable, a sampled-data output feedback controller is necessary. The issue with this problem is that most of the solutions require a very large gain and very small sampling period, which will only produce local or semi-global stability. One good approach is found in [3], which finds a linear observer and a linear output feedback control law without considering the nonlinearities. Then a scaling gain is introduced to both the observer and the control law, and by tuning the scaling gain can show global asymptotic stability. This paper goes further by using this method for nonholonomic systems by converting general nonholonomic systems to a generalized nonlinear system using a change of coordinates. This will allow us to successfully solve for the global output feedback stabilization in single-dimensional z-system. To solve the problem, we will need to delineate nonholonomic systems.

A nonholonomic mechanical system is a system that cannot move in an arbitrary direction in its configuration space, basically a system whose state depends on the path taken to achieve it. A nonholonomic system is described by a set of parameters subject to differential constraints. In robotics, a nonholonomic system is one such that the controllable degrees of freedom are less than the total degrees of freedom. The point is that a constraint is holonomic if it can be integrated and then applied to reduce the degrees of freedom in a system. Nonholonomic systems result from non-integrable constraints on the system. This is important due to the systems inherent nature of nonlinearity and cannot be solved using methods of linear control solutions, therefore a nonlinear approach is required. In the example of a unicycle, the motion of a unicyclist exhibits a nonholonomic constraint.

The position of the unicyclist is given as a pair of coordinates (x, y) and the orientation of the unicycle is specified by angle θ above. For any position (x, y), the unicyclist can be pointed in any direction θ. Therefore $f(x, y, \theta, t) = 0$ has no constraint and there are three degrees of freedom.

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However, the unicyclist can only move (non-slipping) in the direction θ, which creates the constraint $\dot{x} = v(cos\theta\hat{x} + sin\theta\hat{y})$, resulting in a nonholonomic system due to its inability to be integrated into a holomonic system.

As seen in the previous example, nonholonomic systems arise as models which include equations that represent nonholonomic constraints in a standard form. Consider a simple nonholonomic system:

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= yu - xv \end{aligned} \tag{1}$$

It has been proved by Brockett [2] that there is no continuous state feedback stabilizer for (1). **Brocketts necessary condition:** Considering the general control system $\dot{x} = F(x, u)$, assuming F is continuously differentiable.

$$F : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n \qquad \text{and} \qquad F(0,0) = 0$$

A continuous control law exists only if the mapping $\dot{x} = F(x, u)$ is on to an open set of including 0 (zero). Then by Brocketts necessary condition, system (1) cannot be stabilized by any continuous controller. According to Brocketts necessary condition, letting

$$f = \begin{bmatrix} u \\ v \\ yu - xv \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \varepsilon \end{bmatrix} \tag{2}$$

implies that $yu - xv = \varepsilon$, $u = 0$,and $v = 0$. However, $u = 0$ and $v = 0$ which implies that $yu - xv \neq \varepsilon$, so the condition is not satisfied.

II. OUTPUT FEEDBACK STABILIZATION OF NONHOLONOMIC SYSTEM

A. Nonholonomic System

First, A general nonholonomic system:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= u_2 \\ \dot{x}_1 &= x_2 z_2 + f_1(z_1, z_2, x_1, d(t)) \\ &\vdots \\ \dot{x}_n &= u_1 + f_1(z_1, z_2, x_1, ..., x_n, d(t)) \\ Y &= (x_1, z_1)^T \end{aligned} \tag{3}$$

where Y is the output, u_1 and u_2 are control input, and $d(t)$ is a bounded disturbance.

Digital systems are much more realistic and necessary to the real world, such that a discrete-time solution must be found. The same continuous-time system under the same assumption can be globally stabilized using a sampled-data output feedback controller whose observer and control law are discrete time and linear, and hence can be easily implemented by computers [2]. The uncertain nonlinear system is described by:

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t) + \phi_i(t, x(t), u(t)) \\ \dot{x}_n(t) &= u(t) + \phi_n(t, x(t), u(t)) \\ y(t) &= x_i(t) \end{aligned} \tag{4}$$

where $i = 1, 2, ..., n-1$, $x(t) = (x_1(t), ..., x_n(t))^T \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}$ is the control input, $y(t) \in \mathbb{R}$ is the system output, and $\phi_i(t, x, u)$, $i = 1, ..., n$, are unknown continuous functions.

The control law is implemented in discrete time under a sampler and zero-order hold device:

$$u(t) = u(t_k) \qquad \forall \qquad t \in [t_k, t_{k+1}), \quad t_k = kT \tag{5}$$

for $k = 0, 1, 2, ...$, where the time instants t_k and t_{k+1} are the sampling points and T is the sampling period.

Many other references exist for the global stabilization problem stated in system (4), but are not as beneficial to the solution due the nonlinearities of the system being unknown, in which it is impossible to use the unknown states in the observer design.

The work in [3] proposed an output feedback domination design where the observer is designed first without the consideration of nonlinearities. A scaling gain would then be implemented to dominate the nonlinearities under the linear growth condition. In [3], the authors design a solution for output feedback stabilization for the system under the growth condition assumption:

Assumption 1: for $i = 1, ..., n$ there exists a constant $c \geq 0$ such that

$$|\phi_i(t, x(t), u(t))| \leq c(|x_1(t)| + ... + |x_i(t)|) \tag{6}$$

Under the use of this assumption, an output feedback stabilization for system (4) can be constructed by the using a change of coordinates to introduce a scaling gain with the purpose of dominating the nonlinearities in the system. After the introduction of a scaling gain, a discrete observer is designed disregarding the nonlinear terms, and then the control law will be designed using both the change of coordinates and the observer designed.

Linearizing the nonlinear system will make solving for a control law is an easier solution, but again only shows local stability and not GAS. Therefore, to solve for the control feedback law various conditions have been imposed. Using system (4) under Assumption 1, there exists an appropriate sampling period T and matrices $M \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{n \times 1}$, $K \in \mathbb{R}^{n \times 1}$, and $\zeta \in \mathbb{R}^{n \times 1}$ such that

$$\begin{aligned} \zeta(t_{k+1}) &= M\zeta(t_k) + Ny(t_k) \\ u(t) = u(t_k) &= K\zeta(t_k) \\ \forall \quad t \in [t_k, t_{k+1}), &\quad t_k = kT, \quad k = 0, 1, 2, ... \end{aligned} \tag{7}$$

and the linear sampled-data output feedback controller globally stabilizes the uncertain nonlinear system (4).

The method for solving for the control law that globally stabilizes the system is split into four steps.

1) Use change of coordinates to introduce a scaling gain L into the system.
2) A linear discrete-time observer is designed to estimate the unmeasurable states.
3) A linear sampled-data control law using the estimated states is designed.
4) The scaling gain L and sampling period T is fine tuned to render the system GAS.

The change of coordinates used to introduce the scaling gain L into the system is as follows:

$$\begin{aligned} z_i &= \frac{x_i}{L^{i-1}} \\ v &= \frac{u}{L^n} \end{aligned} \tag{8}$$

for $i = 1, \ldots, n$. System (4) then becomes:

$$\begin{aligned} \dot{z}_i(t) &= Lz_{i+1}(t) + \bar{\phi}_i(t, z(t), v(t)) \\ \dot{z}_n(t) &= Lv(t) + \bar{\phi}_n(t, z(t), v(t)) \\ y(t) &= z_1(t) \end{aligned} \tag{9}$$

where $i = 1, \ldots, n-1$ and

$$\bar{\phi}_i(t, z(t), v(t)) = \frac{\phi_i(t, x(t), u(t))}{L^{i-1}}$$

where $i = 1, 2, ..., n$.

Under Assumption 1 it can be verified that for $i = 1, \ldots, n$

$$\begin{aligned} |\bar{\phi}_i(t, z(t), v(t))| &\leq \frac{c}{L^{i-1}}(|z_1(t)| + ... + L^{i-1}|z_i(t)|) \\ &\leq c(|z_1(t)| + ... + |z_i(t)|) \end{aligned} \tag{10}$$

since $L \geq 1$.

We then define the system:

$$\begin{aligned} \dot{z}(t) &= LAz(t) + LBv(t) + \Phi(t, z(t), v(t)) \\ y(t) &= Cz(t) \end{aligned} \tag{11}$$

where

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^T \tag{12}$$

$$\Phi(t, z(t), v(t)) = \begin{bmatrix} \bar{\phi}_1(\cdot) \\ \bar{\phi}_2(\cdot) \\ \vdots \\ \bar{\phi}_n(\cdot) \end{bmatrix}$$

This is the most important design due to the universal use of not only nonlinear systems, but also digital computers. Solving the global output feedback stabilization problem for a nonlinear system using a sampled-data output feedback controller opens the door for a multitude of projects and applications. The immense use of this type of system and controller would make controllers cheaper and easily implementable in a copious amount of situations, ultimately creating faster, higher quality products.

A general system for a one dimensional nonholonomic system is:

$$\begin{aligned} \dot{z} &= u_2 \\ \dot{x}_1 &= x_2 u_2 + f_1(z, x_1, d(t)) \\ \dot{x}_2 &= x_3 u_2 + f_2(z, x_1, x_2, d(t)) \\ &\vdots \\ \dot{x}_n &= u_1 + f_n(z, x_1, \cdots, x_n, d(t)) \\ Y &= (x_1, z)^T \end{aligned} \tag{13}$$

where $u = (u_1, u_2)^T \in \mathbb{R}^2$ and $Y \in \mathbb{R}^2$ are the input and output respectively. The mappings $f_i : \mathbb{R}^{i+2} \to \mathbb{R}^1$, $i = 1, 2, ..., n$, are smooth with $f_i(0, ..., 0, d(t))$, for any bounded disturbance. The control objective is to design a discrete-time observer and controller to stabilize system (13) The observer will only take the input $u(\cdot)$ and $Y(\cdot)$ and produce an estimate of the unknown states. The controller will use the information of the observer to stabilize the original system. Specifically, we are interested in the following control problem.

Problem of global regulation by output feedback for a class of uncertain nonlinear systems under sampled-data control: Find if possible, a linear output dynamic compensator:

$$\begin{aligned} u(t) &= u(t_k) = K\xi(t_k) \\ \xi(t_{k+1}) &= M\xi(t_k) + Ny(t_k) \qquad \forall \qquad t \in [t_k, t_{k+1}) \end{aligned} \tag{14}$$

where $t_k = kT$, $k = 0, 1, 2, ...$, and $\xi \in \mathbb{R}^{n \times 1}$, such that all the solutions of the closed loop system (13)-(14) are globally asymptotically stable. To solve the control problem stated above, the systems under consideration are assumed to satisfy the following condition:

Assumption 2: For $i = 0, 1, 2, ..., n$ there exists a function $\theta_i(z) \geq 0$ such that

$$|(f_i(z, x_1, ..., x_i, d(t))| \leq \theta_i(z)(|x_1(t)| + ... + |x_i|) \tag{15}$$

With the help of Assumption 1 and Assumption 2, one can combine a change of coordinates and the output feedback domination design together to obtain the following theorem.

Theorem 3.1: Under Assumption 2, system (13) is globally stabilized by an output feedback controller in the form of (14)

Proof:

The proof can be divided into two steps. Step 1 is used to change the coordinates of the original nonlinear system to transform it into a chain of integrators perturbed by an uncertain nonlinear vector field. Step 2 is used to design the observer and control law to globally stabilize the system by output feedback using the feedback domination design proposed in [3].

Step 1:

A nonzero constant controller u_2 is applied

$$u_2(t) = u_2(kT) = -z(kT) \qquad t \in [kT, (k+1)T) \tag{16}$$

A simple caclulation yields

$$z(t) = z(kT)(1 - (t - kT)) \tag{17}$$

At $t = T + kT$, the equation becomes

$$z(kT+T) = z(kT)(1-T) \tag{18}$$

Solving z(kT), we have

$$z(kT) = (1-T)^k z(0) \tag{19}$$

which implies that $z(k)$ is not zero if $z(0) \neq 0$. For $t \in [kT, (k+1)T)$, the system is transformed using the following change of coordinates:

$$\begin{aligned} y_0(t) &= z(t) \\ y_1(t) &= \frac{x_1(t)}{z^{n-1}(kT)} \\ y_2(t) &= -\frac{x_2(t)}{z^{n-2}(kT)} \\ &\vdots \\ y_n(t) &= (-1)^{n-1} x_n(t) \end{aligned} \tag{20}$$

Together with $u_2 = -z(kT)$, one can transform the system into

$$\begin{aligned} \dot{y}_0 &= \dot{z} = u_2 = -z = -y_0 \\ \dot{y}_1(t) &= \frac{\dot{x}_1(t)}{z^{n-1}(kT)} = \frac{x_2(t)u_2 + f_1(z, x_1, d(t))}{z^{n-1}(kT)} \\ &= \frac{-x_2 z(kT)}{z^{n-1}(kT)} + \frac{f_1(z, x_1, d(t))}{z^{n-1}(kT)} \\ &= y_2(t) + \Phi_1(y_0, y_1, d(t)) \\ \dot{y}_2(t) &= -\frac{\dot{x}_2(t)}{z^{n-2}(kT)} = -\frac{x_3(t)u_2 + f_2(z, x_1, x_2, d(t))}{z^{n-2}(kT)} \\ &= \frac{x_3 z(kT)}{z^{n-2}(kT)} + \frac{-f_2(z, x_1, x_2, d(t))}{z^{n-2}(kT)} \\ &= y_3(t) + \Phi_2(y_0, y_1, y_2, d(t)) \\ &\vdots \\ \dot{y}_{n-1} &= y_n(t) + \Phi_{n-1}(y_0, y_1, ..., y_{n-1}, d(t)) \\ \dot{y}_n &= v + \Phi_n(y_0, ..., y_n, d(t)) \\ v &= (-1)^{n-1} u_1 \end{aligned} \tag{21}$$

where

$$\Phi_i(y_0, ..., y_i, d(t)) = (-1)^{i-1} \frac{f_i(z, x_1, ..., x_i, d(t))}{z^{n-i}(kT)} \tag{22}$$

By Assumption 2

$$\begin{aligned} |\Phi_i| &\leq \frac{|f_i(z, x_1, x_2, ..., x_i, d(t))|}{|z^{n-i}(kT)|} \\ &\leq \frac{\theta_i(z)(|x_1| + ... + |x_i|)}{|z^{n-i}(kT)|} \end{aligned} \tag{23}$$

where $x_i = y_i(t) z^{n-i}(kT)$. Therefore the system transforms to:

$$\begin{aligned} |\Phi_i| &\leq \frac{\theta_i(z)(|z^{n-1}(kT)y_1| + ... + |z^{n-i}(kT)y_i|)}{|z^{n-i}(kT)|} \\ &\leq \frac{\theta_i(z)(|z^{n-1}(kT)||y_1| + ... + |z^{n-i}(kT)||y_i|)}{|z^{n-i}(kT)|} \\ &\leq \tilde{\theta}_i(z)(|y_1| + ... + |y_i|) \end{aligned} \tag{24}$$

for a function $\tilde{\theta}_i(z)$. Together with the fact that z is bounded, implies the existence of a constant C such that

$$|\Phi_i| \leq C(|y_1| + ... + |y_i|) \tag{25}$$

for $i = 1, 2, ..., n$.

Step 2: We can now design an output feedback controller for system. It can be seen that the $(y_1, ..., y_n)$ subsystem with measurable ouput $y_1 = \frac{x_1}{z^{n-1}(kT)}$ satisfies the condition in [3]. Therefore, we can use the feedback domination design proposed in [3] to explicitly construct an output feedback controller. The change of coordinates introduces a scaling gain L into the system. For a constant $L \geq 1$ to be determined later. Using the following change of coordinates:

$$\begin{aligned} \varsigma_i &= \frac{y_i}{L^{i-1}} \\ \nu &= \frac{u}{L^n} \end{aligned} \tag{26}$$

for $i = 1, 2, ..., n$. Therefore, the system (21) becomes:

$$\begin{aligned} \dot{\varsigma}_i &= L\varsigma_{i+1}(t) + \bar{\Phi}_i(t, \varsigma(t), \nu(t)) \\ \dot{\varsigma}_n &= L\nu(t) + \bar{\Phi}_n(t, \varsigma(t), \nu(t)) \\ y(t) &= \varsigma_1(t) \end{aligned} \tag{27}$$

where $i = 1, 2, ..., n-1$; and

$$\bar{\theta}_i(t, \varsigma(t), \nu(t)) = \frac{\Phi_i(t, x(t), u(t))}{L^{i-1}} \tag{28}$$

for $i = 1, 2, ..., n$.

Under assumption 2 and the fact that $L \geq 1$, it can be verified that for $i = 1, 2, ..., n$

$$\begin{aligned} \bar{\theta}_i(t, \varsigma(t), \nu(t)) &\leq \frac{c}{L^{i-1}}(|\varsigma_1(t)| + ... + |L^{i-1}\varsigma_i(t)|) \\ &\leq c(|\varsigma_1(t)| + ... + |\varsigma_i(t)|) \end{aligned} \tag{29}$$

Matrices are now defined as

$$\begin{aligned} \varsigma(t) &= \begin{bmatrix} \varsigma_1(t) \\ \varsigma_2(t) \\ \vdots \\ \varsigma_n(t) \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^T \\ \Psi(t, \varsigma(t), \nu(t)) &= \begin{bmatrix} \bar{\phi}_1(\cdot) \\ \bar{\phi}_2(\cdot) \\ \vdots \\ \bar{\phi}_n(\cdot) \end{bmatrix} \end{aligned} \tag{30}$$

which can be rewritten as

$$\begin{aligned} \dot{\varsigma}(t) &= LAz(t) + LB\nu(t) + \Phi(t, \varsigma(t), \nu(t)) \\ y(t) &= C\varsigma(t) \end{aligned} \tag{31}$$

Since only $y = \varsigma_1$ is measurable at sampling points and the states $\varsigma_2, ..., \varsigma_n$ are not available, we must construct an observer to estimate the unmeasurable states. Based on the continuous-time high-gain observer, the following observer is designed. Continuous-time states over $[t_k, t_{k+1})$ and discrete time $\varsigma_1(t_k)$ and $\nu(t_k)$.

$$\begin{aligned}\dot{\hat\zeta}_i(t) &= L\hat\varsigma_{i+1}(t) + La_i(\varsigma_1(t_k) - \hat\varsigma_1(t))\\ \dot{\hat\zeta}_n(t) &= L\nu(t) + La_n(\varsigma_1(t_k) - \hat\varsigma_1(t))\\ \forall \quad t &\in [t_k, t_{k+1}), \quad i = 1, ..., n-1\end{aligned} \tag{32}$$

and where for a_j , $j = 1, 2, ..., n$, are the coefficients of the Hurwitz polynomial

$$p_1(s) = s^n + a_n s^{n-1} + ... + a_2 s + a_1 \tag{33}$$

With $\hat\varsigma(t) = [\hat\varsigma_1(t), ..., \hat\varsigma_n(t)]^T$, $H = [a_1, ..., a_n]$, and $\hat A = A - HC$. Then the observer can be rewritten as

$$\dot{\hat\varsigma}(t) = L\hat A\hat\varsigma(t) + LB\nu(t_k) + LH\varsigma_1(t_k) \;\forall \quad t \in [t_k, t_{k+1}) \tag{34}$$

It is common knowledge that the continuous-time observer with a sampler is equivalent to the following discrete-time system:

$$\begin{aligned}\hat\varsigma(t_{k+1}) &= e^{L\hat AT}\hat\varsigma(t_k) + \int_0^T e^{L\hat As} ds[LB\nu(t_k) + LH\varsigma_1(t_k)]\\ &:= F\hat\varsigma(t_k) + G\nu(t_k) + N\varsigma_1(t_k)\end{aligned} \tag{35}$$

where

$$\begin{aligned}F &= e^{L\hat AT}\\ G &= \int_0^T e^{L\hat As} ds[LB]\\ N &= \int_0^T e^{L\hat As} ds[LH]\end{aligned} \tag{36}$$

Since $\nu(t_k) = -K\hat\varsigma(t_k)$, with K being defined in step 3, the system can be rewritten as:

$$\begin{aligned}\hat\varsigma(t_{k+1}) &= (F - GK)\hat\varsigma(t_k) + N\varsigma_1(t_k)\\ &:= M\hat\varsigma(t_k) + Ny(t_k)\end{aligned} \tag{37}$$

where M and N depend on the sampling period T. It is clear to see that the discrete time observer and the continuous time observer will produce the same estimate of $\hat\varsigma(t_k)$. Since states $\varsigma_2, ..., \varsigma_n$ are not measurable, a sampled-data control law using the observer designed in (35) is constructed as follows:

$$\nu(t) = \nu(t_k) = K\hat\varsigma(t_k) \quad \forall \quad t \in [t_k, t_{k+1}) \tag{38}$$

where $K = [k_1, ..., k_n]$, and $k_j > 0$ for $j = 1, 2, ..., n$ are the coeffecients of the Hurwitz polynomial

$$p_2(s) = s^n + k_n s^{n-1} + ... + k_2 s + k_1 \tag{39}$$

Which yields the following closed loop system in the time interval $[t_k, t_{k+1})$:

$$\begin{aligned}\begin{bmatrix}\dot\varsigma(t)\\ \dot{\hat\varsigma}(t)\end{bmatrix} &= L\begin{bmatrix}A & 0\\ 0 & \hat A\end{bmatrix}\begin{bmatrix}\varsigma(t)\\ \hat\varsigma(t)\end{bmatrix} - L\begin{bmatrix}B\\ B\end{bmatrix}K\hat\varsigma(t_k)\cdots\\ &\cdots + L\begin{bmatrix}0\\ H\end{bmatrix}\varsigma_1(t_k) + \begin{bmatrix}\Phi(t, \varsigma_1(t), \nu(t))\\ 0\end{bmatrix}\end{aligned} \tag{40}$$

According to [3] there exists a large enough L and a smaller T such that the system is GAS.

III. Example

Example 1:

Consider the following system:

$$\begin{aligned}\dot z &= u_2\\ \dot x_1 &= x_1 u_2\\ \dot x_2 &= u_1\end{aligned} \tag{41}$$

where z and x_1 are measurable and $f_i(\cdot) = 0$ which implies that Assumption 2 holds. Choose $u_2 = -z(t_k)$ for $t \in [t_k, t_{k+1})$. With the same change of coordinates, the system can be transformed into:

$$\begin{aligned}\dot y_0 &= \dot z = u_2 = -z(t_k)\\ \dot y_1 &= \frac{\dot x_1}{z(kT)} = \frac{x_2 u_2}{z(kT)}\\ &= \frac{x_2 z(kT)}{z(kT)} = x_2 = y_2\\ \dot y_2 &= \dot x_2 = u_1\end{aligned} \tag{42}$$

Therefore the system becomes:

$$\begin{aligned}\dot y_0 &= -y_0(t_k)\\ \dot y_1 &= y_2\\ \dot y_2 &= u_1\end{aligned} \tag{43}$$

which is in the form described in system (14) and can therefore be applied to the design approach described above.Implentation of this results in:

$$\hat z(t_{k+1}) = M\hat z(t_k) + Ny(t_k) \tag{44}$$

where $M = e^{L\hat AT} - \int_0^T e^{L\hat As} ds \cdot K$ and $N = \int_0^T e^{L\hat As} ds \cdot LH$, which renders the following sampled-data control law:

$$\nu(t) = \nu(t_k) = -K\hat z_1(t_k) \quad \forall \quad t \in [t_k, t_{k+1}) \tag{45}$$

Therefore,

$$\nu(t) = -[1, 1]\hat\varsigma(t_k) \tag{46}$$

The system can now be simulated to show that this sampled-data output feedback control law will work for Example 1.

As seen in Figures 2, 3, and 4, Example 1 is rendered globally asymptotically stable. The Z-system and the x-systems approach zero. In the following figures, an L of 4 and a T of 0.02 was used to simulate the solution.

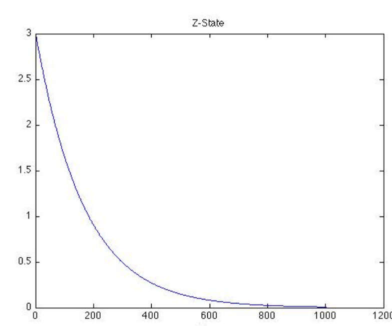

Fig. 1. z-state

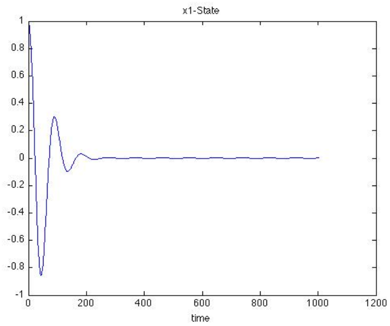

Fig. 2. x_1 state

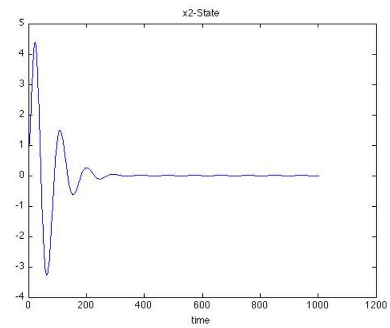

Fig. 3. x_2 state

As seen in Figures 2, 3, and 4, example 1 is rendered globally asymptotically stable. The z-system and x-systems approach zero. In the above figures an L of 4 and a T of 0.02 were used to simulate the solution. Example 2: Consider Example 1 with an added nonlinearity.

$$\begin{aligned} \dot{z}_1 &= u_2 \\ \dot{x}_1 &= x_2 u_2 \\ \dot{x}_2 &= u_1 + sin(x_2) \end{aligned} \tag{47}$$

As shown in Figure 5,6 and 7, even with the nonlinear term

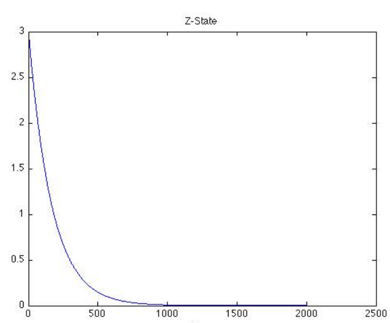

Fig. 4. z-state

added into the x-system the whole system remains globally asymptotically stable.

IV. CONCLUSION

As shown from figures 2-7, the uncertain nonholonomic system in an unchained form can be controlled through the

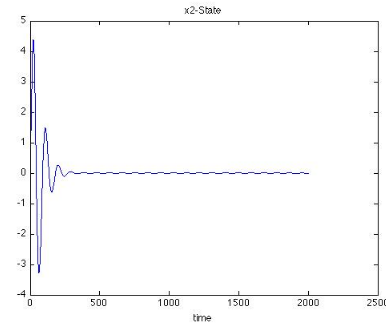

Fig. 5. x_1 state

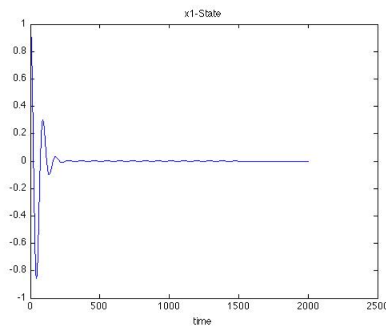

Fig. 6. x_2 state

use of a sampled-data output feedback controller developed in this paper. Through the use of a scaling gain, L, a linear discrete-time observer (for unmeasurable states), and a linear sampled-data control law, one can control these nonholonomic systems for single Z states. When a two-dimensional Z state is introduced, the problem is immensely more challenging due to the boundedness of the change of coordinates. This issue will be addressed in a future journal.

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