Polynomial Fuzzy Modeling and Tracking Control of Wheeled Mobile Robots via Sum of Squares Approach

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***Abstract*-This paper proposes the polynomial fuzzy modeling and tracking control methods for wheeled mobile robots by using sum of squares (SOS) approach, which is developed as SOSTOOL under the Matlab environment. Due to the polynomial fuzzy modeling, we can obtain the linearized tracking error dynamics such that both LMI (Linear Matrix Inequality) and SOS approaches can be applied. Since SOS approach handles more nonlinear system characteristics than LMI, we can obtain the better tracking performance, which is demonstrated in the numerical simulations. The proposed method has advantages in that the control structure can be simplified and it can be further extended to accommodate the input saturation, disturbance compensation, etc., which is well developed for LMI control methods.**

I. INTRODUCTION

Mobile robot trajectory tracking problem can be applied to many applications. Mobile robots are typical nonholonomic system, and accordingly, there has been much research on that. It is well known that nonholonomic systems cannot be asymptotically stabilized around an equilibrium point using the smooth time-invariant feedback controller [1]. Thus, most research works have focused on solving the nonholonomic constraints. In [2], the feedback linearization controller is proposed to make the whole system linear by using the nonlinear control inputs. In [3, 4], the new state variables are introduced to avoid the constraints, and then, the chain structure is composed by using the relationship between the states of the system. However, this chain structure makes the system complex as the system order increases. Recently, [5] showed that the mobile robot system can be linearzed around the small error range. The control input is composed of feedforward and feedback terms such that the whole mobile robot system can be linearized. In this way, this method can change the trajectory tracking problem to the stability problem.

On the other hands, to solve the stability problem, the PDC (Parallel Distributed Compensation) algorithm for the fuzzy model using LMI method is proposed [6, 7]. This algorithm is, however, difficult to apply to the real mobile robot, in that the resulting control inputs may become so large to be used for the system. Furthermore, the states of the algorithm keep oscillating around the reference trajectory until the tracking error converges to zero.

In this paper, we employ the kinematic error model in [5] and solve the trajectory tracking control problem. Based on the linearized model, we fuzzified the linear and angular velocities and then we obtained the control inputs by using SOSTOOLS. In order to use the LMI method, the elements of the system matrix should be constants. Thus, in the case of LMI method, the system performance becomes much degraded due to the neglected nonlinear characteristics of the system. To solve this problem, we propose the new polynomial fuzzy modeling and trajectory tracking control method for wheeled mobile robots by representing the nonlinear terms of the mobile robot system as polynomial terms by using the Taylor expansion. The proposed method gives smoother trajectory results, and smaller control inputs, compared with the LMI method. Also, we verified that tracking errors become smaller from the numerical simulation. The advantages of this method are that i) the control structure can be simplified, and ii) it can be easily extended to the case where the system has the input constraints or uncertainties since many algorithms using LMI method developed to solve the input constraints and uncertainties [10, 11] can be employed.

This paper is organized as follows. First, we show the kinematics of the mobile robot system and then we transform the trajectory tracking problem to the stability problem. Section Ⅲ shows the polynomial fuzzy model and the method to obtain the control inputs. Section Ⅳ shows the simulation results of the method. Finally, we will conclude the paper in Section Ⅴ.

II. KINEMATICS OF WHEELED MOBILE ROBOTS

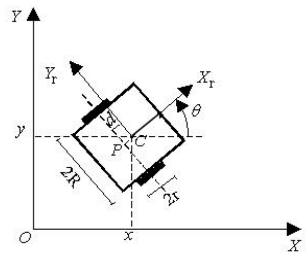


Fig. 1. Kinematic model of wheeled mobile robots

Wheeled mobile robot kinematic model in Fig. 1 is given as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \upsilon\cos\theta \\ \upsilon\sin\theta \\ \omega \end{bmatrix} \tag{1}$$

where x, y, θ are kinematic coordinates and v, w are the linear and angular velocities of the mobile robots. The error between the reference trajectory and mobile robot trajectory is defined as

$$\begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \theta_r - \theta \end{bmatrix}. \tag{2}$$

where x_r, y_r, θ_r are kinematic coordinates of the reference trajectory. Using (1) and (2), we can obtain the error dynamic equation (3) which will be used for the controller design of the mobile robot.

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_\theta \end{bmatrix} = \begin{bmatrix} e_y w_c - v_c + v_r \cos e_\theta \\ -e_x w_c + v_r \sin e_\theta \\ w_r - w_c \end{bmatrix} \tag{3}$$

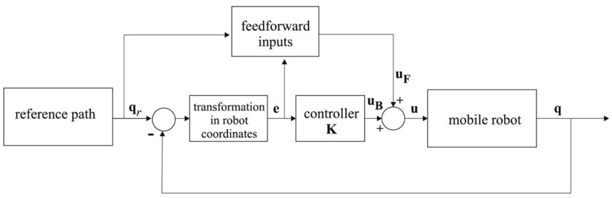


Fig. 2. The structure of the proposed control method.

The structure of the proposed method is described in Fig. 2. We can set up the control inputs as feedforward and feedback terms as

$$u = u_F + u_B = [v_r \cos e_\theta \;\; w_r]^T + [v_c \;\; w_c]^T. \tag{4}$$

Then, we can obtain the following equation from (3).

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_\theta \end{bmatrix} = \begin{bmatrix} 0 & w_r & 0 \\ -w_r & 0 & v_r \sin e_\theta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} + \begin{bmatrix} -1 & e_y \\ 0 & -e_x \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_c \\ w_c \end{bmatrix}. \tag{5}$$

To employ the LMI method, we need to obtain the linearized model around the small error range $(e_x = e_y = e_\theta = 0)$. That is, instead of (5), we have to use

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_\theta \end{bmatrix} = \begin{bmatrix} 0 & w_r & 0 \\ -w_r & 0 & v_r \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_c \\ w_c \end{bmatrix}. \tag{6}$$

Equation (6) has the form of $\dot{e} = Ae + Bu$, so we can employ the linear control or LMI methods. However, due to the modeling error between (5) and (6), the performance of the controller can be much degraded. Thus, we use the Taylor series expansion of the sine function in (5), and obtain

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_\theta \end{bmatrix} = \begin{bmatrix} 0 & w_r & 0 \\ -w_r & 0 & v_r\{1-(e_\theta{}^2)/6+RN)\} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} + \begin{bmatrix} -1 & e_y \\ 0 & -e_x \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_c \\ w_c \end{bmatrix} \tag{7}$$

where RN denotes the N-th higher order Taylor remainder term. Then, we can employ SOSTOOL to control the wheeled mobile robots.

III. Polynomial Fuzzy Model and Controller

In previous section, we converted the trajectory tracking problem to the stability problem and then we showed that the system can be obtained as a polynomial model. In this section we consider the polynomial fuzzy modeling and controller for the wheeled mobile robot system.

A. Polynomial Fuzzy Model

Polynomial fuzzy model is a new type of fuzzy model [6, 7]. Unlike the typical fuzzy model, consequent parts of polynomial fuzzy model are represented by polynomials. We can represent the polynomial fuzzy model as follows:

Rule i:

$$\begin{aligned} &If\; z_1(t)\; is\; M_1\; and\; \cdots\; and\; z_p(t)\; is\; M_p \\ &then\;\; \dot{x}(t) = A_i(x(t))\hat{x}(x(t)) + B_i(x(t))u(t)\;\; (i = 1,2,\cdots,r) \end{aligned} \tag{8}$$

where $z_i(t)\,(i = 1,2,\cdots,p)$ is a variable of membership function, $A_i(x(t))$ and $B_i(x(t))$ are polynomial matrices in $x(t)$, and $\hat{x}(x(t))$ is a column vector whose entries are all monomials in $x(t)$. The defuzzification process of model (8) can be represented as

$$\begin{aligned} \dot{x}(t) &= \frac{\sum_{i=1}^{r} w_i(z(t))\{A_i(x(t))\hat{x}(x(t)) + B_i(x(t))u(t)\}}{\sum_{i=1}^{r} w_i(z(t))} \\ &= \sum_{i=1}^{r} h_i(z(t))\{A_i(x(t)) + B_i(x(t))u(x(t))\}\hat{x}(x(t)). \end{aligned} \tag{9}$$

where $w_i(z(t)) = \prod_{j=1}^{p} M_{ij}(z_j(t))$. It should be noted from the properties of membership functions that the following relations hold.

$$\sum_{i=1}^{r} w_i(z(t)) > 0, \quad w_i(z(t)) \geq 0 \quad i = 1,2,\cdots,r$$

Hence,

$$h_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))} \geq 0, \quad \sum_{i=1}^{r} h_i(z(t)) = 1$$

In the polynomial fuzzy model, the fuzzy controller with polynomial rule consequence can be represented as

$$\begin{array}{l} \textit{Controller } i: \\ \quad \textit{If } z_1(t) \textit{ is } M_1 \textit{ and } \cdots \textit{ and } z_p(t) \textit{ is } M_p \ . \\ \quad \textit{then } u(t) = -F_i(x(t))\hat{x}(x(t)) \ \ i = 1,2,\cdots,r \end{array} \tag{10}$$

Then, the overall fuzzy controller can be calculated by

$$u(t) = -\sum_{i=1}^{r} h_i(z(t)) F_i(x(t)) \hat{x}(x(t)). \tag{11}$$

From (8) through (11), the closed-loop system can be represented as

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \times \{A_i(x(t)) - B_i(x(t)) F_j(x(t))\} \hat{x}(x(t)). \tag{12}$$

B. Polynomial Fuzzy Controller

In this subsection, we design the control inputs by using SOSTOOL based on the polynomial fuzzy system. Polynomial $f(x(t))$ (where $x(t) \in \Re^n$) is a *Sum of Squares (SOS)* [8,9] if there exist polynomials $f_1(x(t)), \cdots, f_m(x(t))$ such that

$$f(x(t)) = \sum_{i=1}^{m} f_i^{\,2}(x(t)). \tag{13}$$

It is clear that $f(x(t))$ being an SOS naturally implies $f(x(t)) > 0$ for all $x(t) \in \Re^n$. Let $f(x(t))$ be a polynomial in $x(t) \in \Re^n$ of degree $2d$. In addition, let $\hat{x}(x(t))$ be a column vector whose entries are all monomials in $x(t)$ with degree no greater than d . Then $f(x(t))$ is a sum of squares if and only if there exists a positive semidefinite matrix P such that

$$f(x(t)) = \hat{x}(x(t)) P \hat{x}(x(t)). \tag{14}$$

Expressing $f(x(t))$ as a SOS polynomial using a quadratic form as in (14) has also been referred to as the *Gram Matrix method* [6, 7]. Then, we can design the tracking controller as in the following theorem.

Theorem 1: The control system consisting of (11) and (12) is stable if there exists a symmetric polynomial matrix $X(\widetilde{x}) \in \Re^{N \times N}$ and a polynomial matrix $M_i(x) \in \Re^{n \times N}$ such that (15) and (16) are satisfied, where $\varepsilon_1(x) > 0$ for $x \neq 0$ and $\varepsilon_{2ij}(x) > 0$ for all x .

$$v^T (X(\widetilde{x}) - \varepsilon_1(x) I) v \ \text{ is SOS} \tag{15}$$

$$\begin{aligned} & -v^T (T(x)A_i(x)X(\widetilde{x}) - T(x)B_i(x)M_j(x) \\ & \quad + X(\widetilde{x})A_i^{\,T}(x)T^T(x) - M_j^{\,T}(x)B_i^{\,T}(x)T^T(x) \\ & \quad + T(x)A_j(x)X(\widetilde{x}) - T(x)B_j(x)M_i(x) \\ & \quad + X(\widetilde{x})A_j^{\,T}(x)T^T(x) - M_i^{\,T}(x)B_j^{\,T}(x)T^T(x) \\ & -\sum_{k \in K} \frac{\partial X}{\partial x_k}(\widetilde{x}) A_i^{\,k}(x)\hat{x}(x) \\ & -\sum_{k \in K} \frac{\partial X}{\partial x_k}(\widetilde{x}) A_j^{\,k}(x)\hat{x}(x) + \varepsilon_{2ij}(x) I) v \\ & \textit{is} \quad \textit{SOS} \quad (\, i \le j) \end{aligned} \tag{16}$$

where $v \in \Re^N$ is a vector which is independent of x and $T(x) \in \Re^{N \times n}$ is a polynomial matrix whose (i, j) th entry is given by

$$T^{ij}(x) = \frac{\partial \hat{x}_i}{\partial x_j}(x) \ . \tag{17}$$

A stabilizing feedback gain $F_i(x)$ in (11) can be obtained using $X(\widetilde{x})$ and $M_i(x)$ as

$$F_i(x) = M_i(x) X^{-1}(\widetilde{x}) \ . \tag{18}$$

The proof is shown in [8, 9].

IV. Simulation Results

This section shows the numerical simulation results obtained by using the proposed method. We used the reference linear and angular velocities as fuzzy input variables to apply (15) and (16) to (7) and normalized the velocities as follows.

$$z_1(t) = \frac{v_r(t)}{\max(v_r(t))} \tag{19}$$

$$z_2(t) = \frac{\omega_r(t)}{\left|\max(\omega_r(t))\right|} . \tag{20}$$

Normalized linear velocities exist in the range [0, 1] and angular velocities in the range [-1, 1]. Fuzzy membership functions are set as Fig. 3.

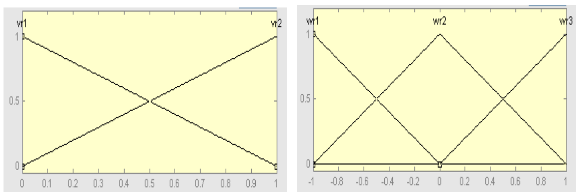


Fig. 3. Membership functions for z_1 and z_2 .

Figs. 4-6 show the numerical simulation results. Initial point of the robot in the simulation is $q(0)=[-1(m),-1(m),0(\deg)]^T$ and that of the reference trajectory is $q_r(0)=[0(m),0(m),45(\deg)]^T$. Reference linear and angular velocities are $v_r(t)=1+5\exp(-2t)$ [m/s], and $\omega_r(t)=10\sin(0.01t)$ [rad/s]. Using the proposed method, wheeled mobile robots can follow the reference trajectory well. Compared with LMI results, the errors of the proposed method are smaller and smoother than those of the LMI method. Fig. 5 shows that the tracking errors converge to zero more quickly by using the proposed method. Also, Fig. 6 shows that the linear and angular velocities of the LMI method are much larger than those of the proposed method. It means that the mobile robot needs the large energy using control inputs in the case of the LMI method. In particular, control inputs of the LMI method are too large to be used for the control of the mobile robots. The averages of the linear and angular velocities of two methods are shown in Table 1.

TABLE I
AVERAGES OF THE LINEAR AND ANGULAR VELOCITIES OF METHODS

	LMI	Proposed method (SOS)
Linear velocity[m/s]	1.4616	1.4060
Angular velocity[rad/s]	0.5793	0.5786

As the velocities are fuzzified by using (19) and (20), the proposed method can be applied to the reference trajectories with time-varying velocities. In addition, it can be seen from these results that the proposed SOS approach can be effectively used for the nonlinear systems, and can be extended to use the previously developed LMI control methods for input/state saturation, disturbance accommodation, etc. in the future work.

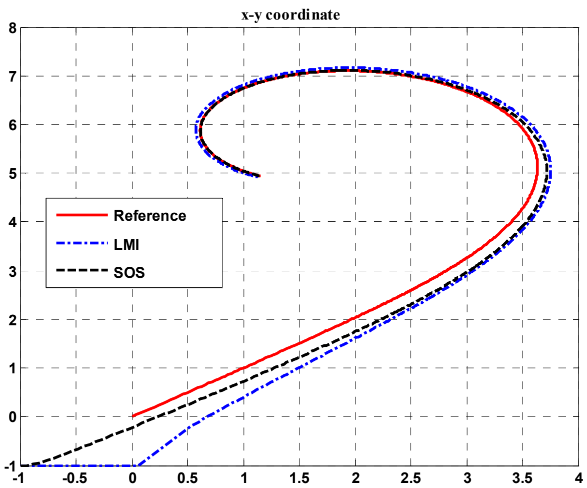
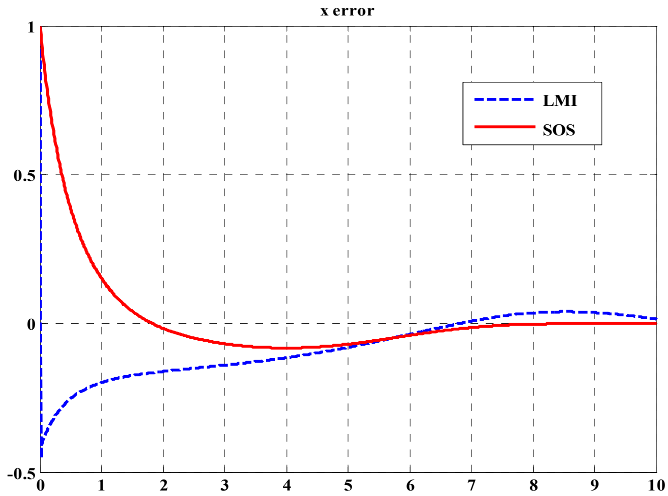
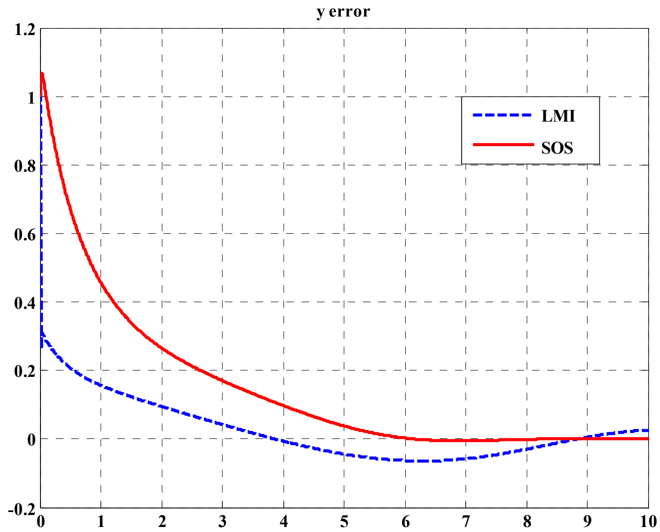


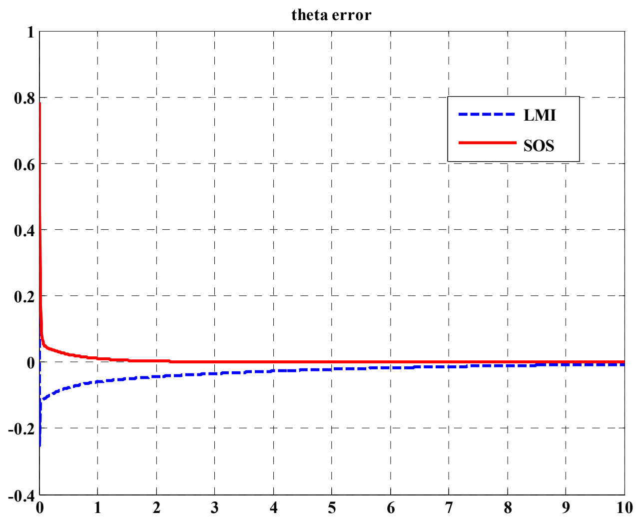
Fig. 4. Trajectory tracking performance.
(solid : reference, dash-dotted : LMI, dotted : SOS)



(a) x error(dotted : LMI, solid : SOS)

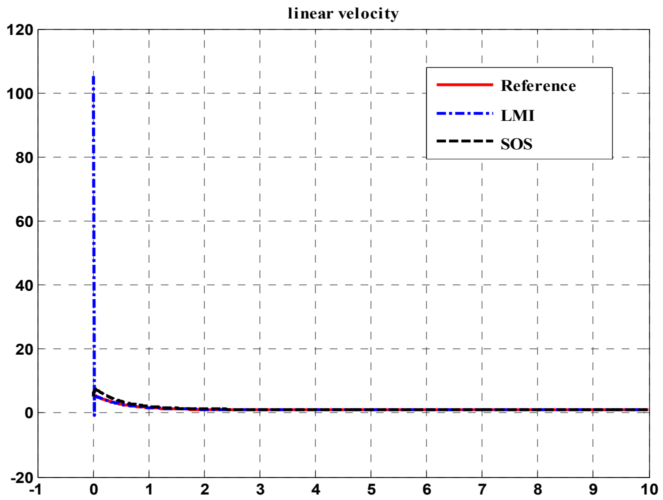


(b) y error(dotted : LMI, solid : SOS)

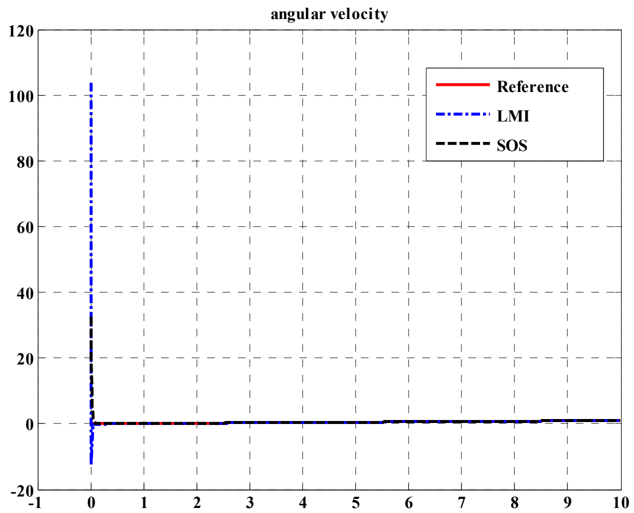


(c) θ error(dotted : LMI, solid : SOS)

Fig. 5. Tracking errors.



(a) linear velocity
(solid : reference, dash-dotted : LMI, dotted : SOS)



(b) angular velocity
(solid : reference, dash-dotted : LMI, dotted : SOS)
Fig. 6. Control inputs.

V. CONCLUSION

We considered the trajectory tracking problem using polynomial fuzzy modeling and SOS approach. First, we converted the trajectory tracking problem to the stability problem and then we obtained the polynomial fuzzy model. Using SOSTOOL, we can obtain the control inputs numerically. From the numerical simulation results, mobile robot can track the reference trajectory well. Compared with LMI method, the proposed method can give the smaller control inputs and better tracking control performance. Also, the trajectory is smoother than using the LMI method.

As proposed method has linear form, and there are many algorithms using LMI, this method can be extended easily for the systems with the input constraints or uncertainties, which is very difficult in the case of the nonlinear control.

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