A Recursive Technique for Tracking Control of Nonholonomic Systems in Chained Form

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***Abstract*—In this paper the authors address the tracking problem for a class of nonholonomic chained form control systems. A recursive technique is proposed which appears to be an extension of the currently popular integrator backstepping idea to the tracking of nonholonomic control systems. Conditions are given under which the problems of semiglobal tracking and global path-following are solved for a nonholonomic system in chained form and its dynamic extension. Results on local exponential tracking are also obtained. Two physical examples of an articulated vehicle and a knife edge are provided to demonstrate the effectiveness of our algorithm through simulations.**

***Index Terms*— Chained form, integrator backstepping, nonholonomic systems, time-varying feedback laws, tracking control.**

I. Introduction

The CONTROL of nonholonomic dynamic systems has received considerable attention during the last few years. This particularly interesting class of nonlinear control systems arises from control problems related to mechanical systems with nonholonomic (or nonintegrable) constraints. See the survey paper [18] and references therein for many introductory examples.

The feedback stabilization problem has been investigated for nonholonomic control systems by many authors. The major obstruction to the asymptotic stabilization of some nonholonomic control systems was the uncontrollability of their first approximation and the nonexistence of a smooth (or even continuous) state-feedback control law of the kind $u = \mu(x)$ (see [2]). Several novel nonlinear control feedback designs have been proposed in the literature to achieve the asymptotic stabilization for such nonholonomic control systems. These methods include the use of smooth time-varying feedback of the form $u = \mu(t, x)$, discontinuous feedback techniques, and nonsmooth time-varying homogenenous feedback (see, for instance, [18], [5], and [31] for relevant references). Although the stabilization problem for nonholonomic control systems is now well understood, the tracking control problem has received less attention. As a matter of fact, it is not clear that the stabilization methodologies available now may be extended directly to tracking problems for nonholonomic systems. In [15], a linearization-based tracking control scheme was introduced for a mobile robot with two degrees of freedom. The scheme was recently extended in [8] to a simplified dynamic model of the mobile robot. A similar idea was independently examined by Murray *et al*. in [23] and Walsh *et al*. in [35]. The idea of input–output linearization was further explored by Oelen and van Amerongen in [25] for a two-degrees-of-freedom mobile robot. In [27] a nonholonomic model of a rolling disk is considered and the feedback design issue was addressed via a dynamic extension and input–output feedback linearization. Fliess *et al*. [9] looked at the trajectory stabilization problem via a differential flatness approach. All these papers solve the local tracking problem for some classes of nonholonomic systems. To our knowledge, the first global tracking control law was proposed by Samson in [30] for a two-wheel driven nonholonomic cart. Recently, Ortega and his coworkers [7] introduced a field-oriented control approach to the global tracking of the nonholonomic double integrator—the simplest chained form system. In our recent paper [11], we propose a backstepping-based tracking control method for the kinematic model of a two-wheel mobile robot and the simplified dynamic model of the two-wheel mobile robot. The local and global tracking problems were solved while keeping local exponential stability under suitable conditions. Notice that the terminology of integrator backstepping was invented by Kokotović [20] and the methodology turns out to be quite similar in several independent papers [17], [3], [34]. In recent papers [10], [13], [14], integrator backstepping was successfully exploited to tackle the problems of global asymptotic stabilization and adaptive control for some classes of nonholonomic systems.

The purpose of this paper is to develop a backstepping-based tracking control procedure for nonholonomic systems in chained form. The class of nonholonomic systems in chained form was introduced in Murray and Sastry [22] and has been studied as a benchmark example by several authors (see, e.g., [22], [26], [29], [32], [33], [4], and [10]). It is well known that many mechanical systems with nonholonomic constraints can be locally, or globally, converted to the chained form under coordinate change and state feedback. Interesting examples of such mechanical systems include tricycle-type mobile robots, cars towing several trailers, the knife edge, a vertical rolling wheel and a rigid spacecraft with two torque actuators (see, e.g., [22], [18], and [31]). As said above, we solved in our previous paper [11] the semiglobal and global tracking problems for a benchmark example of a mobile robot under a nonholonomic constraint which can be transformed into the simplest case of a third-order nonholonomic system in chained form after an appropriate change of coordinates and

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feedback. In this paper, we show that the backstepping-based tracking algorithm in [11] can be extended to a broader class of systems with nonholonomic constraints at the expense of more involved analysis. Of course, stronger conditions on the reference trajectory appear to be necessary.

The path-following problem was studied by several researchers (see, for instance, [24], [28], [29], and [31]). In [24], a local path-following control scheme was presented for a front-wheel driven autonomous vehicle. In [28], a time-scale transformation together with feedback linearization was used to design an approximate path-tracking controller for a vehicle towing a single trailer. In [29], Samson addressed the path-following problem for a car pulling several trailers. While smooth feedback laws were employed in [28] and [29], discontinuous feedback techniques were exploited in [31] to achieve path-following with an exponential rate of convergence for a wheeled mobile robot. A hybrid logic-based tracking scheme is presented in [19] for a class of cascaded nonlinear systems including nonholonomic systems in chained form. We show in this paper that our approach can be also extended to treat a global path-following problem.

The organization of this paper is as follows. Section II contains the system model and the problem statement. Section III describes the systematic tracking control scheme and presents the main semiglobal result. A global path-following problem is solved in Section IV based on the proposed recursive tracking algorithm. Section V shows how to develop the tracking control scheme for the dynamical extension of a chained form nonholonomic system. Section VI illustrates our recursive technique with the aid of two benchmark examples of a car towing a trailer and a knife edge. Some concluding remarks are offered in Section VII.

II. Statement of the Problem

The class of chained form nonholonomic systems to be studied in this paper is described by

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_2 u_1\\ &\vdots\\ \dot{x}_n &= x_{n-1} u_1\end{aligned}\tag{1}$$

where $x = (x_1, \cdots, x_n)$ is the state and u_1 and u_2 are two control inputs.

Assume that the desired trajectory $x_d = (x_{1d}, \cdots, x_{nd})$ is generated by the following equations:

$$\begin{aligned}\dot{x}_{1d} &= u_{1d}\\ \dot{x}_{2d} &= u_{2d}\\ \dot{x}_{3d} &= x_{2d} u_{1d}\\ &\vdots\\ \dot{x}_{nd} &= x_{(n-1)d} u_{1d}\end{aligned}\tag{2}$$

where $u_d = (u_{1d}, u_{2d})$ is the time-varying reference control.

For later use, denote the tracking error as $x_e := x - x_d$. It is directly checked that the x_e dynamics satisfy the following differential equations:

$$\begin{aligned}\dot{x}_{1e} &= u_1 - u_{1d}\\ \dot{x}_{2e} &= u_2 - u_{2d}\\ \dot{x}_{3e} &= x_{2e} u_{1d} + x_2(u_1 - u_{1d})\\ &\vdots\\ \dot{x}_{ne} &= x_{(n-1)e} u_{1d} + x_{n-1}(u_1 - u_{1d}).\end{aligned}\tag{3}$$

The following tracking control problems will be addressed in this paper.

Definition 1: The *tracking control problem* is said to be *semiglobally* solvable for system (1) if, given any compact set $S \in \mathbb{R}^n$ containing the origin, we can design appropriate Lipschitz continuous time-varying state-feedback controllers of the form

$$u_1 = \mu_1(t, x), \qquad u_2 = \mu_2(t, x)\tag{4}$$

such that, for any initial tracking error $x_e(0) = x(0) - x_d(0)$ in S, all the signals of the closed-loop system (3) and (4) are uniformly bounded over $[0, \infty)$. Furthermore

$$\lim_{t \to +\infty} |x_e(t)| = 0.\tag{5}$$

The *tracking control problem* is said to be *globally* solvable for system (1) if the above holds for any initial tracking error $x_e(0)$ in $\mathbb{R}^n$.

In Section V, we show that a similar tracking control problem can be solved for a simple dynamic extension of system (1), which can be viewed as a "prototype" for a dynamic model of certain mechanical systems.

In our previous paper [11], we have addressed the semiglobal and global tracking problems in the simplest case when $n = 3$. Clearly the solvability of the global tracking problem imposes stronger constraints on the reference trajectory than the semiglobal problem. In particular [11] deals with the kinematic model of a two-wheeled mobile robot since that model is feedback equivalent—after coordinate and feedback changes—to the model (1) with $n = 3$. It turns out that the analysis in case n larger than three becomes more involved although the results are of a similar nature.

III. Tracking Control of Chained Systems

A. Control Design Scheme

We propose in this section a systematic controller design procedure which yields a tracking controller that solves the above problem under suitable conditions on the reference trajectory. The stability analysis is given in Section III-B. Before we present the constructive method, we first introduce a change of coordinates and rearrange system (3) into a triangular-like form so that the integrator backstepping can be applied.

Denote $\tilde{x}_d := (x_{2d}, \cdots, x_{(n-1)d})$ and let $\Phi_1(\cdot; \tilde{x}_d)$: $\mathbb{R}^n \to \mathbb{R}^n$ be the mapping defined by

$$\begin{aligned}y_i &= x_{(n-i+1)e} - (x_{(n-i)e} + x_{(n-i)d})x_{1e},\\ &\qquad 1 \le i \le n-2\\ y_{n-1} &= x_{2e}\\ y_n &= x_{1e}.\end{aligned}\tag{6}$$

As it can be directly checked, $\Phi_1(\cdot;\tilde{x}_d)$ is a global diffeomorphism for each $\tilde{x}_d \in \mathbb{R}^{n-2}$ and its inverse $\Phi_1^{-1}(y;\tilde{x}_d)$ is given by

$$\begin{aligned} x_{1e} &= y_n \\ x_{2e} &= y_{n-1} \\ x_{ie} &= \sum_{k=1}^{i-2} x_{(i-k)d} y_n^k + \sum_{k=1}^{i-1} y_{n-i+k} y_n^{k-1}, \qquad 3 \le i \le n. \end{aligned} \tag{7}$$

In the new coordinates $y = (y_1, \cdots, y_n)$, system (3) is transformed into

$$\begin{aligned} \dot{y}_1 &= u_{1d} y_2 - x_{n-2}(u_1 - u_{1d}) y_n \\ &\vdots \\ \dot{y}_{n-3} &= u_{1d} y_{n-2} - x_2 (u_1 - u_{1d}) y_n \\ \dot{y}_{n-2} &= u_{1d} y_{n-1} - u_2 y_n \\ \dot{y}_{n-1} &= u_2 - u_{2d} \\ \dot{y}_n &= u_1 - u_{1d}. \end{aligned} \tag{8}$$

Hereafter, we consider system (8) as our starting point and formulate our backstepping design scheme for this new system (8).

Step 1: Starting with the y_1-subsystem of (8)

$$\dot{y}_1 = u_{1d} y_2 - x_{n-2}(u_1 - u_{1d}) y_n \tag{9}$$

we consider the variable y_2 as a virtual control input and the variables u_{1d} and y_n as time-varying functions.

Set $\overline{y}_1 = y_1$. Along the solutions of (8), the time derivative of the positive definite and proper function

$$V_1(\overline{y}_1) = \tfrac{1}{2}\,\overline{y}_1^2 \tag{10}$$

satisfies

$$\dot{V}_1 = u_{1d}\overline{y}_1 y_2 - \overline{y}_1 x_{n-2}(u_1 - u_{1d}) y_n. \tag{11}$$

Observe that $\alpha_1(y_1) = 0$ is a stabilizing function for system (9) whenever $y_n = 0$. We introduce a new variable $\overline{y}_2$ as follows:

$$\overline{y}_2 = y_2 - \alpha_1(y_1). \tag{12}$$

Then, (11) yields

$$\dot{V}_1 = u_{1d}\overline{y}_1\overline{y}_2 - \overline{y}_1 x_{n-2}(u_1 - u_{1d}) y_n. \tag{13}$$

Step i $(2 \le i \le n-3)$: Assume that after the $(i-1)$th step, for the $(y_1, \cdots, y_{i-1})$-subsystem of (8) with y_i viewed as virtual control input, we have designed smooth intermediate control functions α_j $(1 \le j \le i-1)$ such that the time derivative of the positive definite and proper function

$$V_{i-1}(\overline{y}_1, \cdots, \overline{y}_{i-1}) = \tfrac{1}{2}\,\overline{y}_1^2 + \cdots + \tfrac{1}{2}\,\overline{y}_{i-1}^2 \tag{14}$$

satisfies

$$\dot{V}_{i-1} = u_{1d}\overline{y}_{i-1}\overline{y}_i - \left(\sum_{j=1}^{i-1} \overline{y}_j x_{n-j-1} - \sum_{j=1}^{i-2}\sum_{k=1}^{j} \overline{y}_{j+1} \cdot \frac{\partial \alpha_j}{\partial y_k} x_{n-k-1} \right)(u_1 - u_{1d}) y_n \tag{15}$$

where $\overline{y}_1 = y_1$ and $\overline{y}_j = y_j - \alpha_{j-1}(y_1, \cdots, y_{j-1})$ for each $2 \le j \le i$.

We wish to prove that the above-mentioned properties also hold for the $(y_1, \cdots, y_i)$-subsystem of (8) when y_{i+1} is considered as a virtual control input. Toward this end, consider the positive definite and proper function

$$V_i(\overline{y}_1, \cdots, \overline{y}_i) = V_{i-1}(\overline{y}_1, \cdots, \overline{y}_{i-1}) + \tfrac{1}{2}\,\overline{y}_i^2. \tag{16}$$

Differentiating the function V_i along the solutions of (8) yields

$$\begin{aligned} \dot{V}_i = &-\left(\sum_{j=1}^{i-1} \overline{y}_j x_{n-j-1} - \sum_{j=1}^{i-2}\sum_{k=1}^{j} \overline{y}_{j+1} \frac{\partial \alpha_j}{\partial y_k} x_{n-k-1} \right) \\ &\cdot (u_1 - u_{1d}) y_n + \overline{y}_i \\ &\cdot \Bigg[u_{1d}\overline{y}_{i-1} + u_{1d} y_{i+1} \\ &\quad - x_{n-i-1}(u_1 - u_{1d}) y_n - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_k} \\ &\quad \cdot (u_{1d} y_{k+1} - x_{n-k-1}(u_1 - u_{1d}) y_n) \Bigg]. \end{aligned} \tag{17}$$

We obtain

$$\begin{aligned} \dot{V}_i = &-\left(\sum_{j=1}^{i} \overline{y}_j x_{n-j-1} - \sum_{j=1}^{i-1}\sum_{k=1}^{j} \overline{y}_{j+1} \frac{\partial \alpha_j}{\partial y_k} x_{n-k-1} \right) \\ &\cdot (u_1 - u_{1d}) y_n + u_{1d}\overline{y}_i \\ &\cdot \left[\overline{y}_{i-1} + y_{i+1} - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_k} y_{k+1} \right]. \end{aligned} \tag{18}$$

Setting

$$\alpha_i(y_1, \cdots, y_i) = -\overline{y}_{i-1} + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_k} y_{k+1} \tag{19}$$

$$\overline{y}_{i+1} = y_{i+1} - \alpha_i(y_1, \cdots, y_i), \tag{20}$$

it follows from (18) that

$$\dot{V}_i = u_{1d}\overline{y}_i\overline{y}_{i+1} - \left(\sum_{j=1}^{i} \overline{y}_j x_{n-j-1} - \sum_{j=1}^{i-1}\sum_{k=1}^{j} \overline{y}_{j+1} \cdot \frac{\partial \alpha_j}{\partial y_k} x_{n-k-1} \right)(u_1 - u_{1d}) y_n. \tag{21}$$

By construction, it is easy to see that the intermediate control function α_i as defined in (19) is smooth.

Step $n-2$: By induction, the time derivative of the following function:

$$V_{n-2}(\overline{y}_1, \cdots, \overline{y}_{n-2}) = V_{n-3}(\overline{y}_1, \cdots, \overline{y}_{n-3}) + \tfrac{1}{2}\,\overline{y}_{n-2}^2 \tag{22}$$

satisfies

$$\begin{aligned}\dot{V}_{n-2} = {} & u_{1d}\overline{y}_{n-3}\overline{y}_{n-2} - \left(\sum_{j=1}^{n-3}\overline{y}_j x_{n-j-1} - \sum_{j=1}^{n-4}\sum_{k=1}^{j}\overline{y}_{j+1} \cdot \frac{\partial\alpha_j}{\partial y_k}x_{n-k-1}\right)(u_1 - u_{1d})y_n \\ & + \overline{y}_{n-2}\left[u_{1d}y_{n-1} - u_2 y_n - \sum_{k=1}^{n-3}\frac{\partial\alpha_{n-3}}{\partial y_k} \cdot (u_{1d}y_{k+1} - x_{n-k-1}(u_1 - u_{1d})y_n)\right]. \end{aligned} \tag{23}$$

Then, we have

$$\begin{aligned}\dot{V}_{n-2} = {} & -\left(\sum_{i=1}^{n-3}\overline{y}_i x_{n-i-1} - \sum_{i=1}^{n-3}\sum_{k=1}^{i}\overline{y}_{i+1}\frac{\partial\alpha_i}{\partial y_k}x_{n-k-1}\right) \\ & \cdot (u_1 - u_{1d})y_n - \overline{y}_{n-2}u_2 y_n + u_{1d}\overline{y}_{n-2} \\ & \cdot \left[\overline{y}_{n-3} + y_{n-1} - \sum_{k=1}^{n-3}\frac{\partial\alpha_{n-3}}{\partial y_k}y_{k+1}\right]. \end{aligned} \tag{24}$$

Setting

$$\alpha_{n-2}(y_1, \cdots, y_{n-2}) = -\overline{y}_{n-3} + \sum_{k=1}^{n-3}\frac{\partial\alpha_{n-3}}{\partial y_k}y_{k+1} \tag{25}$$

$$\overline{y}_{n-1} = y_{n-1} - \alpha_{n-2}(y_1, \cdots, y_{n-2}) \tag{26}$$

from (24), it follows:

$$\begin{aligned}\dot{V}_{n-2} = {} & u_{1d}\overline{y}_{n-2}\overline{y}_{n-1} - \overline{y}_{n-2}u_2 y_n \\ & - \left(\sum_{i=1}^{n-3}\overline{y}_i x_{n-i-1} - \sum_{i=1}^{n-3}\sum_{k=1}^{i}\overline{y}_{i+1}\frac{\partial\alpha_i}{\partial y_k}x_{n-k-1}\right) \\ & \cdot (u_1 - u_{1d})y_n. \end{aligned} \tag{27}$$

We remark that the intermediate control function α_{n-2} as defined in (8) is smooth.

Step $n-1$: Consider the $(y_1, \cdots, y_{n-1})$-subsystem of (8) and the function

$$\begin{aligned} V_{n-1}(\overline{y}_1, \cdots, \overline{y}_{n-1}) & := V_{n-2}(\overline{y}_1, \cdots, \overline{y}_{n-2}) + \tfrac{1}{2}\,\overline{y}_{n-1}^2 \\ & = \sum_{i=1}^{n-1}\tfrac{1}{2}\,\overline{y}_i^2. \end{aligned} \tag{28}$$

In view of (8) and (27), the time derivative of V_{n-1} along the solutions of (8) satisfies

$$\begin{aligned}\dot{V}_{n-1} = {} & u_{1d}\overline{y}_{n-2}\overline{y}_{n-1} - \overline{y}_{n-2}u_2 y_n - \left(\sum_{i=1}^{n-3}\overline{y}_i x_{n-i-1}\right. \\ & \left. - \sum_{i=1}^{n-3}\sum_{k=1}^{i}\overline{y}_{i+1}\frac{\partial\alpha_i}{\partial y_k}x_{n-k-1}\right)(u_1 - u_{1d})y_n \\ & + \overline{y}_{n-1}\left[u_2 - u_{2d} - \sum_{i=1}^{n-3}\frac{\partial\alpha_{n-2}}{\partial y_i}(u_{1d}y_{i+1} - x_{n-i-1}(u_1 - u_{1d})y_n) - \frac{\partial\alpha_{n-2}}{\partial y_{n-2}} \cdot (u_{1d}y_{n-1} - u_2 y_n)\right] \\ = {} & \overline{y}_{n-1}\left(u_{1d}\overline{y}_{n-2} + u_2 - u_{2d} - \sum_{i=1}^{n-2}\frac{\partial\alpha_{n-2}}{\partial y_i}u_{1d}y_{i+1}\right) \\ & - \left(\overline{y}_{n-2} - \overline{y}_{n-1}\frac{\partial\alpha_{n-2}}{\partial y_{n-2}}\right)u_2 y_n \\ & - \left(\sum_{i=1}^{n-3}\overline{y}_i x_{n-i-1} - \sum_{i=1}^{n-3}\sum_{k=1}^{i}\overline{y}_{i+1}\frac{\partial\alpha_i}{\partial y_k}x_{n-k-1}\right. \\ & \left. - \sum_{i=1}^{n-3}\overline{y}_{n-1}\frac{\partial\alpha_{n-2}}{\partial y_i}x_{n-i-1}\right)(u_1 - u_{1d})y_n. \end{aligned} \tag{29}$$

Applying the following control law:

$$u_2 = u_{2d} - c_{n-1}\overline{y}_{n-1} - u_{1d}\overline{y}_{n-2} + \sum_{i=1}^{n-2}\frac{\partial\alpha_{n-2}}{\partial y_i}u_{1d}y_{i+1} \tag{30}$$

$$:= \alpha_{n-1}(y_1, \cdots, y_{n-1}, u_d) \tag{31}$$

with $c_{n-1} > 0$, we obtain

$$\begin{aligned}\dot{V}_{n-1} = {} & -c_{n-1}\overline{y}_{n-1}^2 - \left(\overline{y}_{n-2} - \overline{y}_{n-1}\frac{\partial\alpha_{n-2}}{\partial y_{n-2}}\right)u_2 y_n \\ & - \Delta_1(y_1, \cdots, y_{n-1}, \tilde{x}_d)(u_1 - u_{1d})y_n \end{aligned}$$

where Δ_1 is a smooth function given by

$$\begin{aligned}\Delta_1(y, \tilde{x}_d) = \sum_{i=1}^{n-3}\left(\overline{y}_i x_{n-i-1} - \sum_{k=1}^{i}\overline{y}_{i+1}\frac{\partial\alpha_i}{\partial y_k}x_{n-k-1}\right. \\ \left. - \overline{y}_{n-1}\frac{\partial\alpha_{n-2}}{\partial y_i}x_{n-i-1}\right). \end{aligned} \tag{32}$$

Recall that $y := (y_1, \cdots, y_n)$ and $\tilde{x}_d := (x_{2d}, \cdots, x_{(n-1)d})$.

Step n: At this last step, we consider the following positive definite and proper function V_n which serves as a candidate Lyapunov function for the whole system (8)

$$\begin{aligned} V_n(\overline{y}) & = V_{n-1}(\overline{y}_1, \cdots, \overline{y}_{n-1}) + \frac{\lambda}{2}y_n^2 \\ & = \frac{1}{2}\overline{y}_1^2 + \cdots + \frac{1}{2}\overline{y}_{n-1}^2 + \frac{\lambda}{2}y_n^2 \end{aligned} \tag{33}$$

where $\lambda > 0$ is a design parameter and $\overline{y} = (\overline{y}_1, \cdots, \overline{y}_{n-1}, y_n)$.

By virtue of (32), differentiating V_n along the solutions of (8) gives

$$\begin{aligned}\dot{V}_n = -c_{n-1}\overline{y}_{n-1}^2 + y_n\left[(\lambda - \Delta_1(y, \tilde{x}_d))(u_1 - u_{1d})\right. \\ \left. - \left(\overline{y}_{n-2} - \overline{y}_{n-1}\frac{\partial\alpha_{n-2}}{\partial y_{n-2}}\right)u_2\right]. \end{aligned} \tag{34}$$

By choosing the following control law:

$$
\begin{aligned}
u_1 = & u_{1d} + (\lambda - \Delta_1(y, \tilde{x}_d))^{-1} \\
& \cdot \left[-c_n y_n + \left(\overline{y}_{n-2} - \overline{y}_{n-1} \frac{\partial \alpha_{n-2}}{\partial y_{n-2}} \right) u_2 \right] \qquad (35)\\
:= & \alpha_n(y_1, \cdots, y_{n-1}, y_n, u_2, u_d, \tilde{x}_d) \qquad (36)
\end{aligned}
$$

with $c_n > 0$, we arrive at

$$
\dot{V}_n(\overline{y}) = -c_{n-1}\overline{y}_{n-1}^2 - c_n y_n^2. \qquad (37)
$$

We will prove in the next section that, under appropriate conditions on the reference control functions u_{1d} and u_{2d} and initial tracking errors $x_e(0)$, with a good choice of λ, the control law u_1 in (35) is well-defined for every solution of the closed-loop system (8), (30), and (35).

For future use, let $\overline{y} = (\overline{y}_1, \cdots, \overline{y}_{n-1}, y_n)^T$ and Φ_2 be the mapping defined by $\Phi_2(y) = \overline{y}$. As it can be easily proved, Φ_2 is a global diffeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$ which preserves the origin.

The resulting closed-loop system in $\overline{y}$-coordinates is described by

$$
\begin{aligned}
\dot{\overline{y}}_1 = & u_{1d}\overline{y}_2 - x_{n-2}(u_1 - u_{1d})y_n \\
\dot{\overline{y}}_2 = & -u_{1d}\overline{y}_1 + u_{1d}\overline{y}_3 - x_{n-3}(u_1 - u_{1d})y_n \\
& \vdots \\
\dot{\overline{y}}_i = & -u_{1d}\overline{y}_{i-1} + u_{1d}\overline{y}_{i+1} \\
& - \left(x_{n-i-1} - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial y_k} x_{n-k-1} \right) \\
& \cdot (u_1 - u_{1d})y_n \\
& \vdots \\
\dot{\overline{y}}_{n-2} = & -u_{1d}\overline{y}_{n-3} + u_{1d}\overline{y}_{n-1} - u_2 y_n \\
& + \sum_{k=1}^{n-3} \frac{\partial \alpha_{n-3}}{\partial y_k} x_{n-k-1}(u_1 - u_{1d})y_n \\
\dot{\overline{y}}_{n-1} = & -u_{1d}\overline{y}_{n-2} - c_{n-1}\overline{y}_{n-1} \\
& + \sum_{k=1}^{n-3} \frac{\partial \alpha_{n-2}}{\partial y_k} x_{n-k-1}(u_1 - u_{1d})y_n \\
& + \frac{\partial \alpha_{n-2}}{\partial y_{n-2}} u_2 y_n \\
\dot{y}_n = & (\lambda - \Delta_1(y, \tilde{x}_d))^{-1} \\
& \cdot \left[-c_n y_n + \left(\overline{y}_{n-2} - \overline{y}_{n-1} \frac{\partial \alpha_{n-2}}{\partial y_{n-2}} \right) u_2 \right]. \qquad (38)
\end{aligned}
$$

Remark 1: The tracking technique presented above is quite general and can be applied *mutatis mutandis* to other systems with a similar structure to (8). In particular, it was shown in [12] that, under appropriate assumptions on the reference inputs u_d, the choice of *time-varying* and/or *nonlinear* functions α_i at the above design steps, instead of time-invariant linear functions α_i in the variable $(y_1, \cdots, y_i)$, is useful to enhance stability properties for the closed-loop system.

B. Stability Analysis

In this section, we state and prove the main result for the semiglobal tracking control problem for system (1).

Proposition 1: Assume that $x_{id}(2 \leq i \leq n-1), u_d$, and $\dot{u}_{1d}$ are bounded over $[0, \infty)$. Then, the tracking control problem is semiglobally solvable for system (1). In particular, performing the coordinates transformation (6) and applying the design procedure in the above section to system (8), given any compact neighborhood S of the origin in $\mathbb{R}^n$, we can find a sufficiently large $\lambda > 0$ so that, for any initial tracking conditions $x_e(0)$ in S, all the solutions of the closed-loop system (3), (30), and (35) are uniformly bounded. Furthermore, if $u_{1d}(t)$ does not converge to zero

$$
\lim_{t \to +\infty} |x(t) - x_d(t)| = 0. \qquad (39)
$$

Before proving Proposition 1, we introduce a technical lemma which was frequently used in [29].

Lemma 1 [21]: For any differentiable function $f\colon \mathbb{R}_+ \to \mathbb{R}$, if $f(t)$ converges to zero as $t \to +\infty$ and its derivative satisfies

$$
\dot{f}(t) = f_0(t) + \eta(t), \qquad \forall t \geq 0 \qquad (40)
$$

where f_0 is a uniformly continuous function and $\eta(t)$ tends to zero as $t \to +\infty$, then, $\dot{f}(t)$ and $f_0(t)$ tend to zero as $t \to +\infty$.

Proof of Proposition 1: Recall that, along the solutions of the resulting system (38), the time derivative of the positive definite and proper function V_n as defined in (33) satisfies

$$
\dot{V}_n(\overline{y}) = -c_{n-1}\overline{y}_{n-1}^2 - c_n y_n^2. \qquad (41)
$$

Note that the function $\dot{V}_n(\overline{y})$ is only negative semidefinite.

We first prove that for any given compact set S of $\mathbb{R}^n$ whose interior set contains the origin, there exists a sufficiently large λ such that for any initial condition $x_e(0) \in S$, the feedback law (35) for u_1 is well defined on $[0, T)$ and the maximal interval of definition of the corresponding solution $x_e(t)$.

Define a finite real number $c^* \geq 0$ by

$$
c^* = \sup_{t \geq 0, x \in S} V_n(\Phi(x; \tilde{x}_d(t))) \qquad (42)
$$

where $\tilde{x}_d(t) := (x_{2d}(t), \cdots, x_{(n-1)d}(t))$ and Φ is the composed function between Φ_1 and Φ_2, i.e., $\Phi = \Phi_2 \circ \Phi_1$.

Pick a sufficiently large $\lambda > 0$ so that the following property holds:

$$
\begin{aligned}
\Omega := & \{x \in \mathbb{R}^n\colon V_n(\Phi(x; \tilde{x}_d(t))) \leq c^* \quad \forall t \geq 0\} \\
& \subset \{x \in \mathbb{R}^n\colon \Delta_2(x; \tilde{x}_d(t)) < \lambda \quad \forall t \geq 0\} \qquad (43)
\end{aligned}
$$

where $\Delta_2(\Phi_1^{-1}(y; \tilde{x}_d(t))) = \Delta_1(y, \tilde{x}_d)$ and Δ_1 is defined as in (32).

From (41), $V_n(\overline{y}(t))$ is decreasing along the solution $x_e(t)$ of the closed-loop system and therefore $x_e(t)$ remains in the set Ω. Then, the control law $u_1(t)$ exists for each $t \in [0, T)$. Moreover, since $V_n(\overline{y}(t))$ is bounded, by (33), $\overline{y}(t)$ is bounded over $[0, T)$. As a result, from the definition of $\overline{y}$ and y in (12), (20), (26), and in (6), respectively, it follows that $y(t)$ and $x_e(t)$ are bounded over $[0, T)$. Thus, $T = +\infty$.

It remains to prove the convergence property (39). By virtue of Barbălat's lemma [16, Lemma 4.2], from (41), it follows that $\overline{y}_{n-1}(t)$ and $y_n(t)$ converge to zero as $t \to \infty$. Consider the $\overline{y}_{n-1}$-equation of (38) in closed loop

$$\dot{\overline{y}}_{n-1} = -u_{1d}y_{n-2} - \left[c_{n-1}\overline{y}_{n-1} - \sum_{k=1}^{n-3}\frac{\partial\alpha_{n-2}}{\partial y_k}x_{n-k-1} \cdot (u_1 - u_{1d})y_n - \frac{\partial\alpha_{n-2}}{\partial y_{n-2}}u_2 y_n\right]. \quad (44)$$

As a direct application of Lemma 1, it follows:

$$\lim_{t\to\infty} u_{1d}(t)\overline{y}_{n-2}(t) = 0. \quad (45)$$

Next, consider the variable $u_{1d}^2\overline{y}_{n-2}$ which satisfies

$$\frac{d}{dt}(u_{1d}^2\overline{y}_{n-2}) = -u_{1d}^3\overline{y}_{n-3} + \delta_1(t) \quad (46)$$

where

$$\delta_1(t) = 2\dot{u}_{1d}u_{1d}\overline{y}_{n-2} + u_{1d}^2\left(u_{1d}\overline{y}_{n-1} - u_2 y_n + \sum_{k=1}^{n-3}\frac{\partial\alpha_{n-3}}{\partial y_k} \cdot x_{n-k-1}(u_1 - u_{1d})y_n\right). \quad (47)$$

Clearly, $\delta_1(t)$ tends to zero as $t \to \infty$. Consequently, a direct application of Lemma 1 yields

$$\lim_{t\to\infty} u_{1d}(t)\overline{y}_{n-3}(t) = 0. \quad (48)$$

As in [29], by induction, we can prove that, for all $2 \le i \le n-2$

$$\frac{d}{dt}(u_{1d}^2\overline{y}_i) = -u_{1d}^3\overline{y}_{i-1} + \delta_{n-1-i}(t) \quad (49)$$

where $\delta_{n-1-i}(t)$ is a bounded time-varying function that converges to zero as $t \to \infty$.

Using again Lemma 1, with (49), it follows:

$$\lim_{t\to\infty} u_{1d}(t)\overline{y}_i(t) = 0, \qquad \forall 1 \le i \le n-3. \quad (50)$$

Hence, by definition of $V_n, u_{1d}(t)^2 V_n(t)$ goes to zero as $t \to \infty$. Since $V_n(t)$ is decreasing along the solutions of (38), $V_n(t)$ converges to a nonnegative constant denoted as V_*. Due to the fact that u_{1d} does not converge to zero, V_* must be equal to zero. Therefore, all $\overline{y}_i$ $(1 \le i \le n-1)$ and y_n tend to zero. The property (39) follows readily. □

Remark 2: It is of interest to note that an estimate of the region of attraction for the closed-loop system can be obtained from (43). Obviously, the size of the domain is proportional to the value of λ. On the other hand, from a practical point of view, to avoid poor performances due to the choice of large λ, we need to choose large design parameters c_{n-1} and c_n in the tracking control laws (35) and (30). This can be justified in virtue of (33), (37), and (56).

Remark 3: If the lower limit of $|u_{1d}|$ as $t \to \infty$ is positive, i.e., $\liminf |u_{1d}(t)| > 0$, then the condition that u_{1d} has a bounded derivative is not needed to conclude (39). Indeed, from (45), we can prove that $\overline{y}_{n-2}(t)$ tends to zero as $t \to \infty$. Back tracking to the $\overline{y}_{n-3}$-equation in (38) and applying Lemma 1, one sees that $u_{1d}(t)\overline{y}_{n-3}(t)$ converges to zero and therefore $\overline{y}_{n-3}(t)$ converges to zero. Repeating this procedure, we finally prove that $\overline{y}_i(t)$ tends to zero for all $1 \le i \le n-3$, as desired.

The above tracking algorithm does not in general lead to the convergence property (39) if u_{1d} converges to zero. In the following, we discuss some special situations where we can obtain a similar convergence result although u_{1d} goes to zero. Particularly, in the case when $n = 3$, our result recovers part of the results in our previous paper [11].

Corollary 1: Under the conditions of Proposition 1, if u_{1d} converges to zero but u_{2d} does not converge to zero and if $n = 3$, we have

$$\lim_{t\to+\infty} |x(t) - x_d(t)| = 0. \quad (51)$$

Proof: Since $n = 3$, as shown in the above, $\overline{y}_{n-1}(t) = \overline{y}_2(t)$ and $y_n(t) = y_3(t)$ tend to zero as t goes to ∞. Now consider the y_3-equation in (38), i.e.,

$$\dot{y}_3 = (\lambda - \Delta_1(y, \tilde{x}_d))^{-1}\left[-c_3 y_3 + \left(\overline{y}_1 - \overline{y}_2\frac{\partial\alpha_1}{\partial y_1}\right)u_2\right]. \quad (52)$$

From Lemma 1, it follows that $\overline{y}_1(t)u_2(t)$ tends to zero. That is

$$\lim_{t\to+\infty} u_{2d}(t)\overline{y}_1(t) = 0. \quad (53)$$

Since $n = 3$ and $V_3(\overline{y}) = \frac{1}{2}\,\overline{y}_1^2 + \frac{1}{2}\,\overline{y}_2^2 + \frac{1}{2}\,y_3^2$, with (53), it holds that $u_{2d}(t)V_3(\overline{y}(t))$ converges to zero as $t \to \infty$. Since $u_{2d}(t)$ does not converge to zero and $V_3(\overline{y}(t)) \ge 0$ is nonincreasing, $V_3(\overline{y}(t))$ converges to a finite number which must be zero. As a result, $y(t) = \overline{y}(t)$ (in the present case of $n = 3$) converges to zero. By (7), the property (51) follows readily. □

Remark 4 (Point Stabilization): If $u_{1d} = u_{2d} = 0$, the tracking control problem is globally solvable for system (1).

Indeed, it is sufficient to notice that, in the present case, the error system (3) can be transformed into another system in chained form

$$\begin{aligned}\dot{\xi}_1 &= u_1\\ \dot{\xi}_2 &= u_2\\ \dot{\xi}_3 &= \xi_2 u_1\\ &\vdots\\ \dot{\xi}_n &= \xi_{n-1}u_1\end{aligned} \quad (54)$$

via the following state transformation:

$$\xi_1 = x_{1e}, \quad \xi_2 = x_{2e},$$
$$\xi_i = x_{ie} - \sum_{k=1}^{i-2}\frac{1}{k!}x_{(i-k)d}x_{1e}^k, \qquad \forall 3 \le i \le n. \quad (55)$$

Then either the backstepping-based stabilization method in [10] or any other smooth stabilization scheme in the literature (see, e.g., [26] and [29]) allows us to find a desired Lipschitz continuous time-varying feedback law.

C. Exponential Stability

As a by-product of our systematic control design procedure, we give a corollary to Proposition 1 about the exponential stability of the closed-loop system (3), (30), and (35). Some earlier results for dynamic systems with nonholonomic constraints along this line of thought were proposed in [15], [23], and [35] via Taylor linearization. Unlike in [11] and [12] where exponential stability was proved by means of Lyapunov's direct method, the following result is based on Lyapunov's indirect method and the stability theory of slowly varying systems (cf. [16, Sec. 5.7]).

We need the following additional assumption on the reference input signals u_d.

Assumption 1: There exist a finite $t^* \geq 0$ and two compact intervals $\mathcal{I}_1, \mathcal{I}_2 \subset \mathbb{R}$ such that $0 \notin \mathcal{I}_1$ and $u_{id}(t) \in \mathcal{I}_i$ for $i = 1, 2$ and all $t \geq t^*$.

To check if the equilibrium point $x_e = 0$ of the closed-loop system (3), (30), and (35) is exponentially stable, we consider the linearization of system (38) around $\overline{y} = 0$

$$\begin{aligned} \dot{w}_1 &= u_{1d} w_2 \\ \dot{w}_i &= -u_{1d} w_{i-1} + u_{1d} w_{i+1}, \qquad 2 \leq i \leq n-3 \\ \dot{w}_{n-2} &= -u_{1d} w_{n-3} + u_{1d} w_{n-1} - u_{2d} w_n \\ \dot{w}_{n-1} &= -u_{1d} w_{n-2} - c_{n-1} w_{n-1} \\ \dot{w}_n &= -\frac{c_n}{\lambda} w_n + \frac{1}{\lambda} u_{2d} w_{n-2} \end{aligned} \tag{56}$$

where we used the property $\partial \alpha_{n-2}/\partial y_{n-2} \equiv 0$.

In more compact form, system (56) is rewritten as

$$\dot{w} = A(u_d) w. \tag{57}$$

Under the Assumption 1, it is easy to prove that $A(u_d)$ is Hurwitz for each *frozen* $u_d \in \mathcal{I}_1 \times \mathcal{I}_2$. One way of doing this is to establish the exponential stability of the equilibrium $w = 0$ of (56) by means of the quadratic storage function

$$V_L(w) = \frac{1}{2} w_1^2 + \cdots + \frac{1}{2} w_{n-1}^2 + \frac{\lambda}{2} w_n^2 \tag{58}$$

whose time derivative satisfies

$$\dot{V}_L = -c_{n-1} w_{n-1}^2 - c_n w_n^2. \tag{59}$$

Consequently, a direct application of [16, Lemma 5.12] gives the following.

Lemma 2: The parameterized Lyapunov equation

$$P A(u_d) + A^T(u_d) P = -I \tag{60}$$

has a unique positive definite C^1 solution $P(u_d)$ for every fixed $u_d \in \mathcal{I}_1 \times \mathcal{I}_2$. So, there exist four positive constants p_i $(1 \leq i \leq 4)$ such that

$$\left.\begin{aligned} p_3 |w|^2 \leq w^T P(u_d) w \leq p_4 |w|^2 \\ \left| \frac{\partial P}{\partial u_{id}}(u_d) \right| \leq p_i, \qquad i = 1, 2 \end{aligned}\right\} \\ \forall (w, u_d) \in \mathbb{R}^n \times \mathcal{I}_1 \times \mathcal{I}_2. \tag{61}$$

We are now ready to present our result on the exponential stability of the equilibrium $x_e = 0$ of the closed-loop system (3), (30), and (35).

Corollary 2: Under the conditions of Proposition 1, if Assumption 1 holds with

$$\sup_{t \geq t^*} |\dot{u}_d(t)| < \frac{1}{\sqrt{p_1^2 + p_2^2}} \tag{62}$$

where p_1 and p_2 are positive constants given in (56), then the trivial solution $x_e = 0$ of the closed-loop system (3), (30), and (35) is exponentially stable.

Proof: We first prove that the linearized system (56) is exponentially stable at the equilibrium point $w = 0$. Notice that (59) implies the uniform stability of the time-varying linear system (56) at $w = 0$. In particular, every solution $w(t)$ of system (56) satisfies

$$|w(t)| \leq |w(0)|, \qquad \forall t \geq 0. \tag{63}$$

Then, by Lemma 2, differentiating the Lyapunov function candidate

$$U(t, w) = w^T P(u_d) w \tag{64}$$

yields

$$\begin{aligned} \dot{U}(t, w) &= w^T (P(u_d) A + A^T P(u_d)) w \\ &\quad + w^T \left(\frac{\partial P}{\partial u_{1d}}(u_d) \dot{u}_{1d} + \frac{\partial P}{\partial u_{2d}}(u_d) \dot{u}_{2d} \right) w \\ &\leq -(1 - \sqrt{p_1^2 + p_2^2} |\dot{u}_d(t)|) |w|^2. \end{aligned} \tag{65}$$

Letting $\delta := 1 - \sqrt{p_1^2 + p_2^2} \sup_{t \geq t^*} |\dot{u}_d(t)|$ which is a positive constant, it follows from Lemma 2 and (65) that

$$\dot{U}(t, w) \leq -\frac{\delta}{p_4} U(t, w), \qquad \forall t \geq t^*. \tag{66}$$

Applying the Gronwall–Bellman inequality [16, Lemma 2.1], we obtain

$$U(t, w(t)) \leq e^{-(\delta/p_4)(t - t_0)} U(t_0, w(t_0)), \qquad \forall t \geq t_0 \geq t^* \tag{67}$$

which implies

$$|w(t)| \leq \sqrt{\frac{p_4}{p_3}} e^{-(\delta/2p_4)(t - t_0)} |w(t_0)|, \qquad \forall t \geq t_0 \geq t^*. \tag{68}$$

Combining (63) and (68), we establish

$$|w(t)| \leq \sqrt{\frac{p_4}{p_3}} e^{(\delta/2p_4)t^*} e^{-(\delta/2p_4)(t - t_0)} |w(t_0)|, \\ \forall t \geq t_0 \geq 0. \tag{69}$$

Thus, the equilibrium $w = 0$ of the linearized system (56) is exponentially stable. By [16, Th. 3.11], the time-varying nonlinear system (38) is also exponentially stable at the equilibrium $\overline{y} = 0$, that is, there exist positive constants k_1, k_2 such that all solutions $\overline{y}(t)$ of (38) starting from a small neighborhood of $\overline{y} = 0$ satisfy

$$|\overline{y}(t)| \leq k_1 e^{-k_2 (t - t_0)} |\overline{y}(t_0)|, \qquad \forall t \geq t_0 \geq 0. \tag{70}$$

Finally, since the mappings $x_e \mapsto y$ and $y \mapsto \overline{y}$ as introduced in Section III-A are global diffeomorphisms from $\mathbb{R}^n$ onto $\mathbb{R}^n$, the exponential stability of the equilibrium $x_e = 0$ of the original system (3), (30), and (35) follows readily. □

IV. Global Path Following

In the previous section, we solved the semiglobal tracking problem for a nonholonomic chained system of the form (1). In this section, we will discuss some particular situations where we can treat the global tracking control problem, that is, we want to construct a tracking controller so that all the tracking errors $x_e(t)$ asymptotically converge to zero for *arbitrary* initial tracking errors $x_e(0)$.

The desired trajectory x_d is now a straight line. Without loss of generality, assume that $x_{2d} = \cdots = x_{nd} = 0$ and $u_{2d} = 0$. Then, the tracking error $x_e = (x_{1e}, x_2 \cdots, x_n)$ satisfies

$$\begin{aligned} \dot{x}_{1e} &= u_1 - u_{1d} \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_{1d} + x_2(u_1 - u_{1d}) \\ &\vdots \\ \dot{x}_n &= x_{n-1}u_{1d} + x_{n-1}(u_1 - u_{1d}). \end{aligned} \tag{71}$$

As in Section III, we first introduce an appropriate transformation of coordinates which brings system (71) into a triangular-like form so that the integrator backstepping can be applied.

For this purpose, we consider the following change of coordinates, instead of (6):

$$\begin{aligned} z_i &= x_{n-i+1} + \sum_{j=1}^{n-i-1} \frac{(-1)^j}{j!} x_{n-i+1-j} x_{1e}^j, \\ &\qquad 1 \le i \le n-2 \\ z_{n-1} &= x_2 \\ z_n &= x_{1e} \end{aligned} \tag{72}$$

so that system (71) is transformed into

$$\begin{aligned} \dot{z}_i &= u_{1d} z_{i+1} + \frac{(-1)^{n-i-1}}{(n-i-1)!} u_2 z_n^{n-i-1}, \\ &\qquad 1 \le i \le n-2 \\ \dot{z}_{n-1} &= u_2 \\ \dot{z}_n &= u_1 - u_{1d}. \end{aligned} \tag{73}$$

Remark 5: Note that, when $u_{1d} = 0$, system (73) is akin to the so-called *power form* system introduced in [33]. As it can be directly verified, the inverse of the global diffeomorphism Ψ: $x_e \mapsto z$ as defined in (72) is given by $x_{1e} = z_n, x_2 = z_{n-1}$, and $x_i = z_{n-i+1} + \Sigma_{j=1}^{i-2} (1/j!) z_{n-i+1+j} z_n^j$ $(3 \le i \le n)$.

In the sequel, we show how the control design procedure presented in Section III-A can be applied to system (73) and builds up a desired global tracking controller for u.

By applying the systematic control design scheme in Section III-A to the $(z_1, \cdots, z_{n-1})$-subsystem of (73) with u_2 considered as control input and terms related to $u_2 z_n^j$ $(1 \le j \le n-2)$ as time-varying disturbances, we obtain a new change of coordinates defined by $\overline{z}_1 = z_1$ and $\overline{z}_i = z_i - \alpha_{i-1}(z_1, \cdots, z_{i-1})$ $(2 \le i \le n-1)$ together with a quadratic function $W_{n-1} = \Sigma_{i=1}^{n-1} (1/2) \overline{z}_i^2$. In particular, when we apply the control law $u_2 = \alpha_{n-1}(z_1, \cdots, z_{n-1}, u_d)$ as given by (31), the time derivative of W_{n-1} along the solutions of (73) satisfies

$$\dot{W}_{n-1} = -c_{n-1}\overline{z}_{n-1}^2 + \Delta_3(z_1, \cdots, z_n, u_d) z_n \tag{74}$$

where $c_{n-1} > 0$, and Δ_3 is defined by

$$\begin{aligned} \Delta_3 = \sum_{i=1}^{n-2} \Bigg(&\overline{z}_i \frac{(-1)^{n-i-1}}{(n-i-1)!} u_2 z_n^{n-i-2} \\ &- \overline{z}_{i+1} \sum_{k=1}^{i} \frac{\partial \alpha_i}{\partial z_k} \frac{(-1)^{n-k-1}}{(n-k-1)!} u_2 z_n^{n-k-2} \Bigg). \end{aligned} \tag{75}$$

Then, consider the Lyapunov function candidate $W_n = W_{n-1}(\overline{z}_1, \cdots, \overline{z}_{n-1}) + \frac{1}{2} z_n^2$ for the overall system (73). With (74), differentiating W_n along the solutions of system (73) gives

$$\dot{W}_n = -c_{n-1}\overline{z}_{n-1}^2 + z_n[\Delta_3(z_1, \cdots, z_n, u_d) + u_1 - u_{1d}]. \tag{76}$$

By choosing the following control law for u_1:

$$u_1 = u_{1d} - c_n z_n - \Delta_3(z_1, \cdots, z_n, u_d) \tag{77}$$

with $c_n > 0$, we establish

$$\dot{W}_n = -c_{n-1}\overline{z}_{n-1}^2 - c_n z_n^2. \tag{78}$$

We are in a position to state the global tracking result.

Proposition 2: Assume that u_{1d} and its derivative $\dot{u}_{1d}$ are bounded over $[0, \infty)$. Then all the solutions of the resulting closed-loop system are uniformly bounded. Furthermore, if $u_{1d}(t)$ does not converge to zero

$$\lim_{t \to +\infty} (|x_1(t) - x_{1d}(t)| + |x_2(t)| + \cdots + |x_n(t)|) = 0. \tag{79}$$

Proof: The proof follows from Barbălat's lemma and a similar reasoning as in the proof of Proposition 1. □

Remark 6: Of course, as in Corollary 2, under Assumption 1, we can deduce the exponential convergence of the tracking errors after a considerable period of time.

Remark 7: It should be noted that a local (global when the reference trajectory is a straight line) path-following result was established in [29] for a car with multiple trailers, and the latter can be put into a form of (1). Nonetheless, the construction of our tracking controller is based on an inverse design spirit. Specifically, thanks to the use of integrator backstepping, the designs of the control inputs u_1 and u_2 are totally separated and rely on a recursive procedure. A Lyapunov function is obtained after the control design procedure is completed. In earlier tracking work, we first find a Lyapunov function and then design a tracking control law to make the derivative of this function nonpositive (see, e.g., [15], [30], and [35]), whereas in [29] the designs of u_1 and u_2 are mixed.

V. Extension to Dynamic Models

The tracking problems were addressed in previous sections using a chained system of form (1) which reflects the kinematic model of a mechanical control system such as wheeled mobile robot and cars with trailers. The purpose of this section is to propose a dynamical extension of system (1) so that the tracking control problem can also be solved using mechanical torques. It also demonstrates that the recursive design extends to a class of systems which contain a nonzero drift term.

More precisely, we consider system (1) with the dynamic extension

$$\dot{u}_1 = v_1, \qquad \dot{u}_2 = v_2 \tag{80}$$

where v_1 and v_2 are the control inputs of the dynamic model.

The control task is now to design Lipschitz continuous time-varying state-feedback controllers of the form

$$v_1 = \nu_1(t, X), \qquad v_2 = \nu_2(t, X) \tag{81}$$

such that $x(t)$ tracks the desired trajectory $x_d(t)$ of the reference system (2).

Using the same notations as in Section III, it is easily seen that the tracking error dynamics $(x_e, u - u_d)$ are described by (8) and (80). Then, in order to design time-varying feedback laws v_1 and v_2 to force y and $u - u_d$ to zero (which in turn implies that $x_e(t)$ tends to zero), we invoke integrator backstepping and Proposition 1.

Indeed, as shown in Section III-A, under the action of the control laws (31) and (36), the time derivative of the function V_n as defined in (33) satisfies

$$\dot{V}_n = -c_{n-1}\overline{y}_{n-1}^2 - c_n y_n^2. \tag{82}$$

However, since u_1 and u_2 are not the true control inputs for the dynamic model (1)–(80) under study, the control laws (31) and (36) cannot be used. We introduce the new variables $\overline{u}_1$ and $\overline{u}_2$ as

$$\overline{u}_1 = u_1 - \alpha_n(y_1, \cdots, y_{n-1}, y_n, u_2, u_d, \tilde{x}_d) \tag{83}$$

$$\overline{u}_2 = u_2 - \alpha_{n-1}(y_1, \cdots, y_{n-1}, u_d). \tag{84}$$

Then, by (29) and (34), it is direct to see that the time derivative of V_n along the solutions of (8)–(80) satisfies, instead of (82)

$$\dot{V}_n = -c_{n-1}\overline{y}_{n-1}^2 - c_n y_n^2 + \overline{y}_{n-1}\overline{u}_2 + y_n(\lambda - \Delta_1(y, \tilde{x}_d))\overline{u}_1. \tag{85}$$

Let $\overline{u} = (\overline{u}_1, \overline{u}_2)^T$ and consider the Lyapunov function candidate

$$V_{n+1}(\overline{y}, \overline{u}) = V_n(\overline{y}) + \tfrac{1}{2}\,\overline{u}_1^2 + \tfrac{1}{2}\,\overline{u}_2^2 \tag{86}$$

which is positive definite and proper. From (85), it follows:

$$\begin{aligned}\dot{V}_{n+1} = &-c_{n-1}\overline{y}_{n-1}^2 - c_n y_n^2 + \overline{u}_2(\overline{y}_{n-1} + v_2 - \dot{\alpha}_{n-1}) \\ &+ \overline{u}_1[y_n(\lambda - \Delta_1(y, \tilde{x}_d)) + v_1 - \dot{\alpha}_n].\end{aligned}$$

Therefore, choosing the following control laws:

$$v_1 = -y_n(\lambda - \Delta_1(y, \tilde{x}_d)) + \dot{\alpha}_n - c_{v_1}\overline{u}_1 \tag{87}$$

$$v_2 = -\overline{y}_{n-1} + \dot{\alpha}_{n-1} - c_{v_2}\overline{u}_2 \tag{88}$$

with $c_{v_1}, c_{v_2} > 0$, we establish

$$\dot{V}_{n+1} = -c_{n-1}\overline{y}_{n-1}^2 - c_n y_n^2 - c_{v_1}\overline{u}_1^2 - c_{v_2}\overline{u}_2^2. \tag{89}$$

We are now ready to state our solution to the tracking control problem for the dynamic system (1)–(80).

Proposition 3: Assume that $x_{id}(t)$ $(2 \le i \le n-1)$, $u_d(t)$, $\dot{u}_d(t)$, and $\ddot{u}_{1d}(t)$ are bounded over the time interval $[0, \infty)$. Then, the tracking control problem is semiglobally solvable for system (80). Specifically, given any compact neighborhood S_e of the origin in $\mathbb{R}^{n+2}$, we can find a sufficiently large $\lambda > 0$ so that, for any initial tracking conditions $(x_e(0), \overline{u}(0))$ in S_e, all the solutions of the resulting closed-loop system (8), (80), (87), and (88) are uniformly bounded. Furthermore, if $u_{1d}(t)$ does not converge to zero

$$\begin{aligned}\lim_{t\to+\infty}(&|x(t) - x_d(t)| + |u_1(t) - u_{1d}(t)| \\ &+ |u_2(t) - u_{2d}(t)|) = 0.\end{aligned} \tag{90}$$

Proof: The proof follows from (89) and Barbălat's lemma along similar lines of the proof of Proposition 1. □

Corollary 3: Under the conditions of Proposition 3 and Corollary 2, the equilibrium $(y, \overline{u}) = (0, 0)$ of system (8), (80), (87), and (88) is exponentially stable if c_{v_1} and c_{v_2} are sufficiently large.

Proof: The resulting closed-loop system can be rewritten in more compact form

$$\begin{aligned}\dot{y} &= \overline{f}(t, y) + \overline{g}(t, y, \overline{u}) \\ \dot{\overline{u}} &= \overline{A}\overline{u} + \overline{h}(t, y)\end{aligned} \tag{91}$$

where $\overline{f}, \overline{g}, \overline{A}$, and $\overline{h}$ readily follow from (8), (80), (87), and (88).

In the light of the proof of Corollary 2, we know that the equilibrium $y = 0$ of $\dot{y} = \overline{f}(t, y)$ is exponentially stable. By the Converse Lyapunov Theorem [16, Th. 3.12], there is a function V_c satisfying the inequalities

$$\begin{aligned}\epsilon_1|y|^2 \le V_c(t, y) &\le \epsilon_2|y|^2 \\ \frac{\partial V_c}{\partial t} + \frac{\partial V_c}{\partial y}\overline{f}(t, y) &\le -\epsilon_3|y|^2 \\ \left|\frac{\partial V_c}{\partial y}\right| &\le \epsilon_4|y|\end{aligned} \tag{92}$$

for some positive constants ϵ_i $(1 \le i \le 4)$ and all $(t, y) \in [0, \infty) \times D_0, D_0$ being a bounded neighborhood of $y = 0$.

Consider the function

$$\overline{V}_c(t, y, \overline{u}) = V_c(t, y) + \tfrac{1}{2}\,|\overline{u}|^2. \tag{93}$$

Then, the time derivative of $\overline{V}_c$ along the solutions of system (91) satisfies

$$\begin{aligned}\dot{\overline{V}}_c \le &-\epsilon_3|y|^2 + \epsilon_4|y| \cdot |\overline{g}(t, y, \overline{u})| - c_{v_1}|\overline{u}_1|^2 - c_{v_2}|\overline{u}_2|^2 \\ &+ |\overline{u}| \cdot |\overline{h}(t, y)|.\end{aligned} \tag{94}$$

Noticing that $\overline{g}(t, y, 0) = 0$ and $\overline{h}(t, 0) = 0$, there are positive constants ϵ_5 and ϵ_6 such that

$$|\overline{g}(t, y, \overline{u})| \le \epsilon_5|\overline{u}|, \qquad |\overline{h}(t, y)| \le \epsilon_6|y| \tag{95}$$

for all $(t, y, \overline{u}) \in [0, \infty) \times D_1 \times \mathbb{R}^2$, with $D_1 := D_0 \cap \{y \in \mathbb{R}^n\colon\ \Delta_1(y, \tilde{x}_d(t)) \le 0.5\lambda\ \forall t \ge 0\}$.

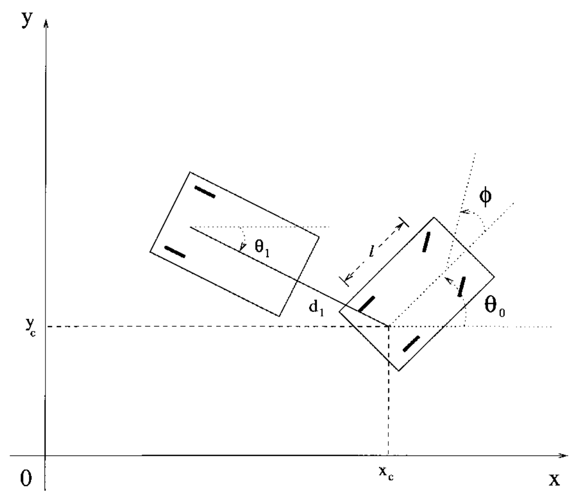

Fig. 1. Kinematic model of a car with a single trailer [22] where the control inputs are the forward velocity ν and the steering velocity ω of the tow car.

Hence

$$\dot{\overline{V}}_c \le -(|y|\,|\overline{u}|)M\begin{pmatrix}|y|\\|\overline{u}|\end{pmatrix}$$

where

$$M=\begin{bmatrix}\epsilon_3 & -\dfrac{\epsilon_4\epsilon_5+\epsilon_6}{2}\\ -\dfrac{\epsilon_4\epsilon_5+\epsilon_6}{2} & \min\{c_{v_1},c_{v_2}\}\end{bmatrix}. \tag{96}$$

Obviously, M is a positive definite matrix if c_{v_1} and c_{v_2} are chosen sufficiently large. Therefore, the proof is completed with the help of [16, Corollary 3.4] together with the fact that $\overline{V}_c$ as defined in (93) satisfies a property similar to (92). □

Remark 8: Analogously, the global path-following problem as addressed in Section IV can be solved for the dynamic model (1), (80); see Section VI-B for an illustration.

VI. Application to Mechanical Systems

In this section, we illustrate our recursive tracking methodology with the help of two benchmark mechanical systems under nonholonomic constraints: an articulated vehicle and a knife edge.

A. Articulated Vehicles

An articulated vehicle, which is a car pulling a single semi-trailer as depicted in Fig. 1, has been served as a benchmark nonholonomic example in several studies [22], [29], [31], [28].

As usual, we assume that the wheels of the articulated vehicle are allowed to roll and spin but not slip. Under this assumption, the kinematic motion of the articulated vehicle is described by [22]

$$\begin{aligned}\dot{x}_c &= \nu\cos\theta_0\\ \dot{y}_c &= \nu\sin\theta_0\\ \dot{\phi} &= \omega\\ \dot{\theta}_0 &= \frac{1}{l}\nu\tan\phi\\ \dot{\theta}_1 &= \frac{1}{d_1}\nu\sin(\theta_0-\theta_1)\end{aligned} \tag{97}$$

where l is the wheelbase of the tow car, d_1 is the distance from the wheels of the trailer to the wheels of the car, and ν and ω represent the driving velocity and the steering velocity of the tow car respectively.

Although system (97) is not in chained form, it can be transformed into (1) via a change of coordinates and state feedback. A general algorithm for (locally) converting the kinematics of a car pulling *multiple* trailers into a chained system was proposed by Sørdalen [31]. However, in the present case of a car towing a single trailer, we invoke the simpler conversion algorithm due to Murray and Sastry [22].

According to [22, Proposition 7], introduce the following local change of coordinates and feedback:

$$\begin{aligned}x_1 &= x_c\\ x_2 &= \frac{1}{ld_1}\sec^3\theta_0\sec\theta_1\tan\phi+\frac{1}{d_1^2}\sec^2\theta_0\sin\theta_1\\ &\quad\cdot\sec^3\theta_1\sin^2(\theta_0-\theta_1)-\frac{1}{d_1^2}\sec\theta_0\sec^3\theta_1\\ &\quad\cdot\sin(\theta_0-\theta_1)\\ x_3 &= \frac{1}{d_1}\sin(\theta_0-\theta_1)\sec^2\theta_1\sec\theta_0\\ x_4 &= \tan\theta_1\\ x_5 &= y_c-d_1\log\left(\frac{1+\sin\theta_1}{\cos\theta_1}\right)\\ \nu &= \sec(\theta_0)u_1\\ \omega &= \beta_1(\phi,\theta_0,\theta_1)u_1+\beta_2(\phi,\theta_0,\theta_1)u_2\end{aligned} \tag{98}$$

where

$$\begin{aligned}\beta_1 = &-\frac{3}{l}\sec^2\theta_0\sin\theta_0\sin^2\phi\\ &-\left(\frac{1}{2}+\frac{1}{d_1}\sin\theta_0\sec\theta_1\sin(\theta_0-\theta_1)\right)\\ &\cdot\sec\theta_0\tan\theta_1\sin(\theta_0-\theta_1)\sin(2\phi)\\ &-\left(\frac{1}{2d_1}\sin(2\phi)-\frac{l}{d_1^2}\sin(\theta_0-\theta_1)\cos^2\phi\right)\\ &\cdot\sin\theta_1\sec^2\theta_1\sin(2\theta_0-2\theta_1)-\frac{l}{d_1^2}\Big(\sin(\theta_0-\theta_1)\\ &+3\sin(\theta_0-\theta_1)\tan^2\theta_1-3\cos\theta_0\tan\theta_1\sec\theta_1\Big)\\ &\cdot\sec\theta_1\sin^2(\theta_0-\theta_1)\cos^2\phi+\frac{1}{2d_1}\sec^2\theta_1\cos\theta_1)\\ &\cdot\sin(2\phi)-\frac{l}{2d_1^2}\cos\theta_0\sec^2\theta_1\sin(2\theta_0-2\theta_1)\cos^2\phi\end{aligned} \tag{99}$$

$$\beta_2 = ld_1\cos^3\theta_0\cos\theta_1\cos^2\phi. \tag{100}$$

Now, the transformed version of system (97) is in chained form, i.e.,

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_2u_1\\ \dot{x}_4 &= x_3u_1\\ \dot{x}_5 &= x_4u_1.\end{aligned} \tag{101}$$

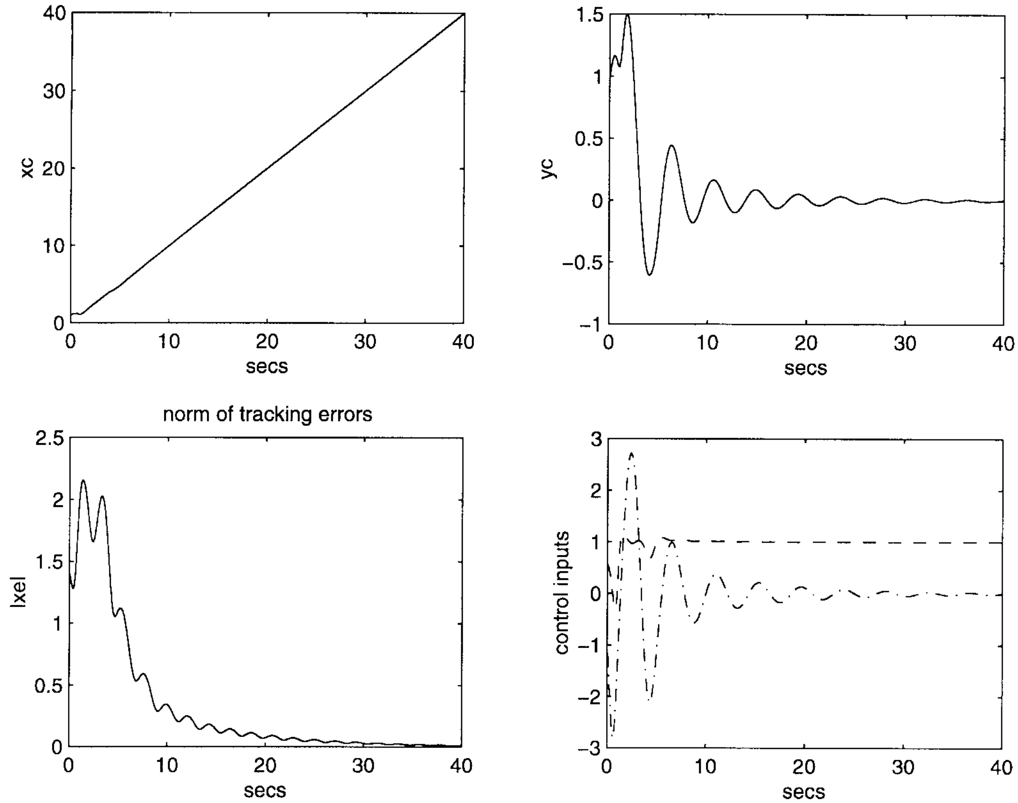

Fig. 2. Kinematic model of the articulated vehicle. Time histories of the vehicle motion x_c and y_c "-." Plot of the norm of the tracking errors x_e versus time "-" and control performance with the tracking control laws u_1 (103) "- -" and u_2 (102) "- ·."

Using the same notations as in Section III, as a direct application of the recursive synthesis procedure presented in Section III-A, a semiglobal tracking control law for (101) [but local for the original model (97)] is given by

$$u_2 = u_{2d} - c_4 y_4 - 2c_4 y_2 - u_{1d}(3y_3 + y_1) := \alpha_4 \tag{102}$$

$$u_1 = u_{1d} + \frac{1}{\lambda - (y_3 + 2y_1)x_3 - (2y_4 + 5y_2)x_2} \cdot (-c_5 y_5 + (y_1 + y_3)u_2) := \alpha_5 \tag{103}$$

where $\lambda > 0, c_4 > 0$, and $c_5 > 0$ are design parameters.

For simulation use, we consider the reference signal of the form (2) characterized by

$$u_{1d} = 1, \qquad u_{2d} = 0, \qquad x_{id}(0) = 0, \qquad \forall 1 \le i \le 5. \tag{104}$$

Fig. 2 demonstrates the evolution of the norm of the tracking errors $x_e(t)$ as well as the controller performance based on the following choice of design parameters and initial conditions:

$$\lambda = 5, \qquad c_4 = c_5 = 2, \qquad l = d_1 = 1$$
$$x_e(0) = (1, 0.5, 0.5, 0.5, 0.5). \tag{105}$$

With the choice of (104), the reference trajectory $x_d(t)$ to be tracked is a straight line since $x_{1d}(t) = t$ and $x_{id} \equiv 0$ for all $2 \le i \le 5$. In the original coordinates $(x_c, y_c, \phi, \theta_0, \theta_1)$, the reference trajectory x_d is the x-axis because $x_c(t) = t$ and $(y_c(t), \phi(t), \theta_0(t), \theta_1(t)) = (0, 0, 0, 0)$. In this case, using the same notations in Section IV, a direct application of the design procedure in Section IV yields a global tracking controller for the transformed system (101) [but local for the original model (97)]

$$u_2 = -c_4 z_4 - 2c_4 z_2 - u_{1d}(3z_3 + z_1) \tag{106}$$

$$u_1 = u_{1d} - c_5 z_5 + u_2\left(z_1 + z_3 - \tfrac{5}{2} z_2 z_5 - z_4 z_5 + \tfrac{1}{3} z_1 z_5^2 + \tfrac{1}{6} z_3 z_5^2\right). \tag{107}$$

In Fig. 3 where the top picture is drawn with the coordinates (x_c, y_c) of the tow car, we see that the articulated car eventually approaches the desired x-axis, the reference trajectory in our case.

With (102) and (103) as the tracking controllers for the kinematic model (101), applying the design procedure in Section V yields a semiglobal tracking control law for the dynamic extension of system (101)

$$\begin{aligned} v_2 = &-c_{v2}(u_2 - \alpha_4) - 2y_2 - y_4 - c_4 u_2 - 2c_4 u_{1d} y_3 y_5 \\ &+ 2c_4 x_2 (u_1 - u_{1d}) y_5 - 3u_{1d}^2 y_4 + 3u_{1d} u_2 y_5 \\ &- u_{1d}^2 y_2 + x_3 u_{1d}(u_1 - u_{1d}) y_5 \end{aligned} \tag{108}$$

$$\begin{aligned} v_1 = &-c_{v1}(u_1 - \alpha_5) - y_5(\lambda - (y_3 + 2y_1)x_3 \\ &- (2y_4 + 5y_2)x_2 + \frac{1}{\lambda - (y_3 + 2y_1)x_3 - (2y_4 + 5y_2)x_2} \\ &\cdot [-c_5(u_1 - u_{1d}) + (y_2 + x_3(u_1 - u_{1d})y_5 + u_{1d} y_4 \\ &- u_2 y_5) u_2 + (y_1 + y_3) v_2] \\ &+ \frac{1}{(\lambda - (y_3 + 2y_1)x_3 - (2y_4 + 5y_2)x_2)^2} \\ &\cdot (-c_5 y_5 + y_1 u_2 + y_3 u_2) \times [x_3(u_{1d} y_4 - u_2 y_5 \\ &+ 2u_{1d} y_2 - 2x_3(u_1 - u_{1d}) y_5) + (y_3 + 2y_1) x_2 u_1 \\ &+ x_2(2u_2 + 5u_{1d} y_3 - 5x_2(u_1 - u_{1d}) y_5) \\ &+ (2y_4 + 5y_2) u_2] \end{aligned} \tag{109}$$

where $c_{v_1} > 0$ and $c_{v_2} > 0$ are design parameters.

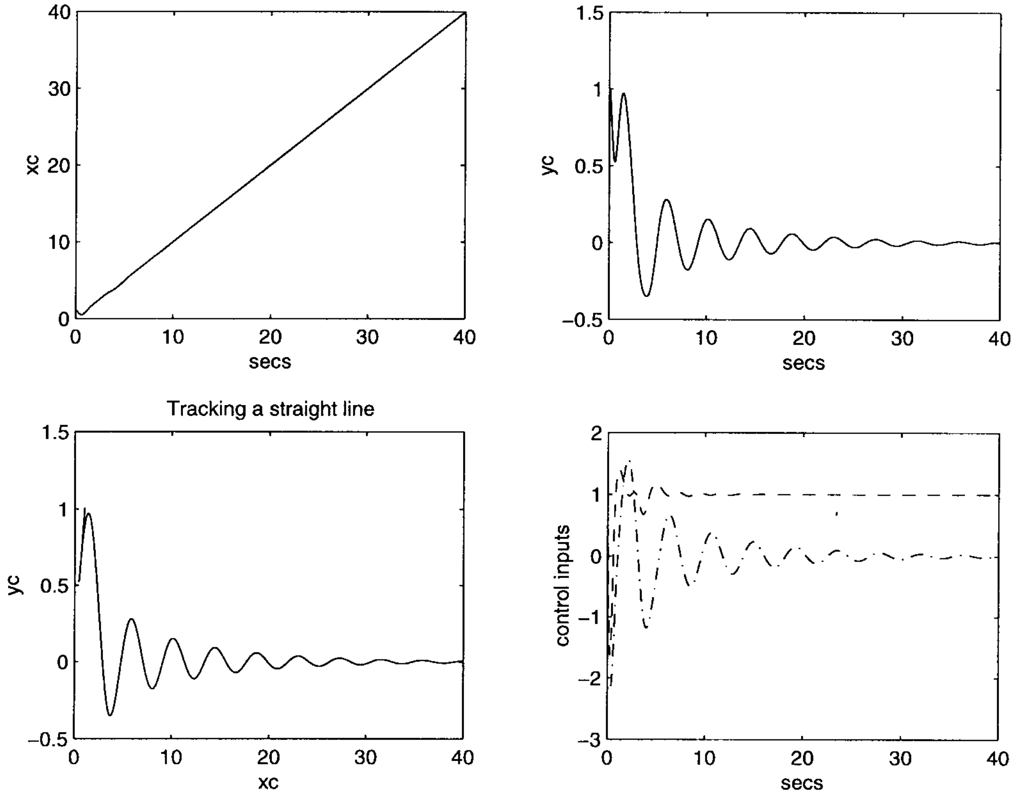

Fig. 3. Kinematic model of the articulated vehicle. Time histories of the vehicle motion x_c and y_c "-." Graph of the vertical position y_c versus the horizontal position x_c "-" and control performance with the tracking control laws u_1 (107) "- -" and u_2 (106) "-·."

For simulation use, we pick $c_{v_1} = c_{v_2} = 2$ and the initial conditions (105) and $u_1(0) = u_2(0) = 0.5$. Fig. 4 plots the norm of the tracking errors $(x_e, u - u_d)$ versus time and the control performance of the tracking controllers v_1 (109) and v_2 (108).

We observe in Figs. 2–4 that the articulated vehicle exhibits some awkward motion. This is mainly because after the highly nonlinear transformations (98)–(100), the new states except x_1 have lost any physical meaning. This should initiate the search of other more structure-based tracking methods.

It should be noted that because of the local nature of the state and feedback transformations (98) the tracking feedback laws designed for the transformed system do not guarantee semiglobal or global stability properties for the original model of the articulated vehicle. Indeed, since the coordinate transformation and state feedback are well defined within a domain where the original coordinates are such that $(\phi, \theta_0, \theta_1, x_c, y_c)$ belong to $(-\pi/2, \pi/2) \times (-\pi/2, \pi/2) \times (-\pi/2, \pi/2) \times \mathbb{R} \times \mathbb{R}$, we have that only within such a domain can we obtain (semi-) "global" stability. On the other hand, it follows from Corollary 2 that the proposed controller (106) and (107) is locally exponentially stable.

In our second example of a knife edge moving on the plane, we show that certain global feedback transformation exists to bring the knife-edge system into a dynamic chained form so that semiglobal or global properties hold for both systems.

B. A Knife Edge

The simple nonholonomic example of a knife edge moving on the plane was introduced in [1] and has recently been used to illustrate the hybrid stabilization and tracking methods in [19]. The motion of the knife-edge dynamics is described by the following differential equations [19]:

$$\begin{aligned} \ddot{x}_c &= \frac{\gamma}{m}\sin\phi + \frac{\tau_1}{m}\cos\phi \\ \ddot{y}_c &= -\frac{\gamma}{m}\cos\phi + \frac{\tau_1}{m}\sin\phi \\ \ddot{\phi} &= \frac{\tau_2}{I_c} \\ \dot{x}_c\sin\phi &= \dot{y}_c\cos\phi \end{aligned} \tag{110}$$

where (x_c, y_c) denotes the coordinates of the center of mass of the knife edge, ϕ denotes the heading angle measured from the x-axis, and τ_1 is the pushing force in the direction of the heading angle, τ_2 is the steering torque about the vertical axis through the center of mass. The constants (m, I_c) are the mass and the moment of inertia of the knife edge, respectively, and γ is the scalar constrain multiplier. Note that the fourth equation in (110) represents the nonholonomic constraint on the knife-edge system.

A slight modification of the state and feedback transformations in [19] leads to a global change of coordinates and feedback

$$\begin{aligned} x_1 &= \phi \\ x_2 &= x_c\cos\phi + y_c\sin\phi \\ x_3 &= x_c\sin\phi - y_c\cos\phi \\ x_4 &= \dot{\phi} \\ x_5 &= \dot{x}_c\cos\phi + \dot{y}_c\sin\phi \\ &\quad + \dot{\phi}(-x_c\sin\phi + y_c\cos\phi) \end{aligned}$$

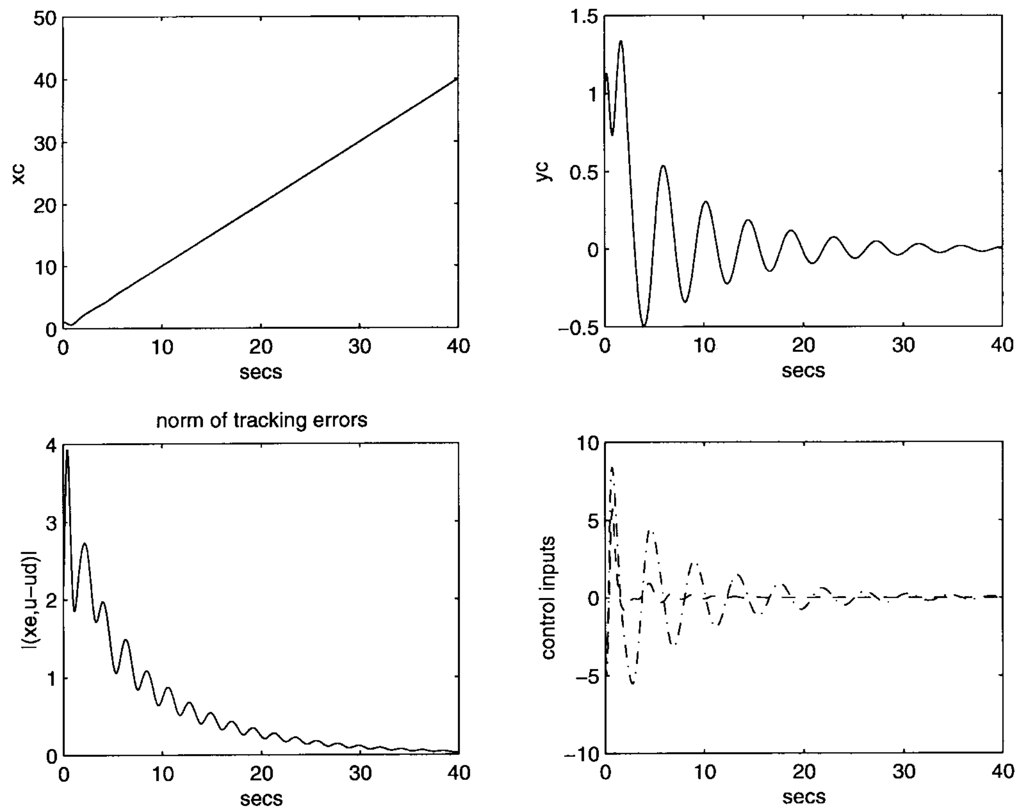

Fig. 4. Dynamic model of the articulated vehicle. Time histories of the vehicle motion x_c and y_c "-." Plot of the norm of the tracking errors $(x_e, u - u_d)$ versus time "-" and control performance with the tracking control laws v_1 (109) "- -" and v_2 (108) "-·."

$$\begin{aligned} v_1 &= \frac{\tau_2}{I_c} \\ v_2 &= \frac{\tau_1}{m} + \frac{\tau_2}{I_c}(-x_c \sin\phi + y_c \cos\phi) \\ &\quad - \dot{\phi}^2(x_c \cos\phi + y_c \sin\phi). \end{aligned} \tag{111}$$

In the new coordinates, the dynamic model of the knife edge (110) is put in extended chained form (80), that is

$$\begin{aligned} \dot{x}_1 &= x_4 \\ \dot{x}_2 &= x_5 \\ \dot{x}_3 &= x_2 x_4 \\ \dot{x}_4 &= v_1 \\ \dot{x}_5 &= v_2. \end{aligned} \tag{112}$$

In other words, the transformed model of the knife edge can be seen as a cascaded interconnection of a system in chained form

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \end{aligned} \tag{113}$$

and two integrators

$$\begin{aligned} \dot{u}_1 &= v_1 \\ \dot{u}_2 &= v_2. \end{aligned} \tag{114}$$

As in [19], consider the following reference trajectory:

$$\begin{aligned} \phi^{\text{ref}}(t) &= t, \qquad x_c^{\text{ref}}(t) = \sin t \\ y_c^{\text{ref}}(t) &= -\cos t \qquad \forall t \geq 0 \end{aligned} \tag{115}$$

which corresponds to the center of mass of the knife edge moving along the circle centered at the origin of unit radius with uniform angular rate.

In the transformed x-coordinates, the desired trajectory is

$$\begin{aligned} x_{1d}(t) &= t, \qquad x_{2d}(t) = 0, \qquad x_{3d}(t) = 1 \\ x_{4d}(t) &= u_{1d}(t) = 1, \qquad x_{5d}(t) = u_{2d}(t) = 0. \end{aligned} \tag{116}$$

It is interesting to note that the desired circular path (115) in the original coordinates now becomes a straight line in the transformed coordinates. Also note that the tracking errors satisfy

$$\begin{aligned} \dot{x}_{1e} &= u_1 - u_{1d} \\ \dot{x}_2 &= u_2 \\ \dot{x}_{3e} &= x_2 u_{1d} + x_2(u_1 - u_{1d}) \\ \dot{u}_1 &= v_1 \\ \dot{u}_2 &= v_2 \end{aligned} \tag{117}$$

where $x_{1e}(t) = x_1(t) - t$ and $x_{3e}(t) = x_3(t) - 1$.

A combined use of the tracking methods in Sections IV and V gives a global tracking control law for both the transformed system (112) and the original model (110)

$$v_2 = -2\overline{u}_2 - 2z_2 - 2x_5 + z_3 x_5 \tag{118}$$

$$\begin{aligned} v_1 &= -2\overline{u}_1 - z_3 + z_2 x_5 - z_1 z_3 x_5^2 + z_1 v_2 \\ &\quad - 2(x_4 - 1) \end{aligned} \tag{119}$$

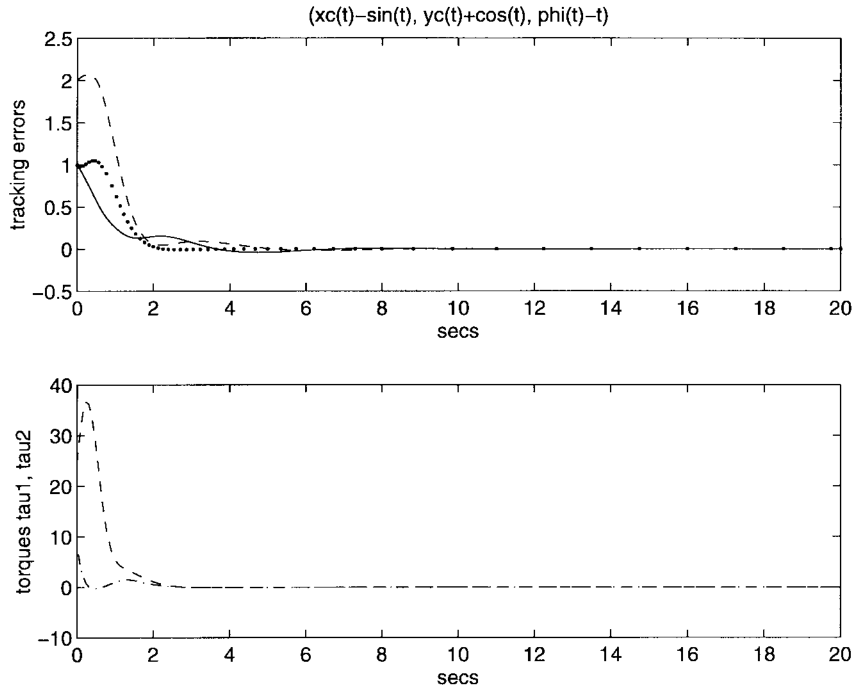

Fig. 5. The knife edge. Plot of the tracking errors $(x_c(t) - \sin t)$ "-," $(y_c(t) + \cos t)$ "- -," $(\phi(t) - t)$ "· · ·" versus time and control performance with the global tracking control laws τ_1 (119)–(111) "- -" and τ_2 (118)–(111) "- ·."

where $z_1 = x_{3e} - x_2 x_{1e}, z_2 = x_2, z_3 = x_{1e}$ and

$$\overline{u}_1 = x_4 - 1 - z_1 x_5 + 2z_3, \qquad \overline{u}_2 = x_5 + 2z_2 + z_1.$$

For simulation use, take $m = I_c = 1$. Under the following choice of initial conditions:

$$(\phi(0), x_c(0), y_c(0), \dot{\phi}(0), \dot{x}_c(0), \dot{y}_c(0)) = (1, 1, 1, 0.5, 0.5, 0.5).$$

Fig. 5 illustrates the time histories of the tracking errors $(x_c(t) - \sin t, y_c(t) + \cos t, \phi(t) - t)$ as well as the control performance for the torques τ_1, τ_2 required for the knife edge. Note that the tracking errors quickly converge to zero in a few seconds.

For the desired trajectory in (115), both our *Lipschitz continuous* tracking feedback laws and the hybrid stabilizing laws of [19] achieve the global asymptotic stability with exponential convergence for the resulting error system. However, our tracking strategy brings an additional property of *exponential stability* (in the sense of Lyapunov) for the closed-loop error system (see Corollary 3). Comparing Fig. 5 with [19, Fig. 4], it turns out that our controller (118) and (119) yields better performance than the hybrid controller of [19].

VII. Conclusion

A recursive technique was proposed for the tracking control of a class of nonholonomic chained systems. On the one hand, we have broadened the domain of applicability of integrator backstepping to nonholonomic control systems. On the other hand, a semiglobal tracking control law was derived on the basis of a stepwise controller design procedure. It is important to note that the proposed tracking technique is analytically simple and produces continuous tracking feedbacks. Under additional conditions on the reference inputs, the convergence rate is guaranteed to be exponential after a (considerable) period of time. We have also discussed some special cases where the tracking problem can be globally solved. More interestingly, we showed that our tracking design procedure can be extended directly to a dynamical extension of the chained form system, i.e., the chained system appended with two integrators. The recursive design for tracking is illustrated in two benchmark examples of chained form nonholonomic systems, the pulling car and the knife edge.

Last but not the least the design strategy described in this paper is different to earlier tracking methods in [8], [15], [19], [23], [28]–[30], and [35]. A linearization, or feedback linearization viewpoint, was adopted in [15], [23], [35], [8], and [28] to design tracking control laws for nonholonomic mobile robots without or with one trailer. In [30], a Lyapunov function was found to construct a global tracking controller for a nonholonomic wheeled cart without trailer. This Lyapunov direct method was extended in [29] to derive a (generally local) tracking solution for a nonholonomic car towing multiple trailers. Our aim was not only to give a (first) semiglobal tracking solution for general nonholonomic dynamical systems in chained form but to build up an inverse design method, that is, the desired control law is designed via recursive steps and a Lyapunov function is found after the control design is completed. It seems therefore very challenging to compare the controllers proposed here on an experimental setup with those given in, e.g., [19], [23], [29], and [35].

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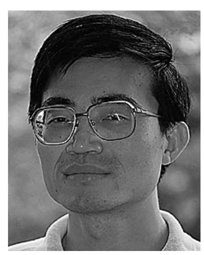

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