Harmonic Functions and Collision Probabilities

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Abstract

This paper describes a relationship between harmonic functions – which have recently been proposed for path planning – and recent work on randomization and learning in robotics. In short, the hitting probabilities for random walks can be cast as a Dirichlet problem for harmonic functions, in much the same way as in path planning. This equivalence has implications both for uncertainty in motion planning and in some robot learning problems.

1 Introduction

Some recent work in robotics has focussed on the use of harmonic functions for robot path planning [1, 2, 3, 4, 5, 6, 7, 8]. This approach resembles a potential field appro⁻ ⁻h, except that the field is usually computed in a global manner (that is, over the entire region of interest). In particular, the method as described in [3] uses a grid-based technique which results in a potential such that, at each point, the potential function is the *average* of the potentials at neighboring points. These functions can be computed easily either by relaxation [3] or resistive grids [6].

This paper explores the relationship between this work, its probabilistic interpretation, and recent work on uncertainty and task planning. The probabilistic interpretation of harmonic functions is described in detail in [9], and in [10]. In the following discussion, the equivalence is described in terms of lattices – since this is a representation compatible with grid-based approaches – and is used to draw some inferences about its application in robotics. Specifically, harmonic functions can be directly interpreted as collision probabilities. They can be used to introduce an appropriate drift in randomization, and can also be used to guarantee some useful properties for robot learning systems.

2 Basic Equivalence

In this discussion, all problems are assumed to be defined on a lattice (grid) of points in Z^n – this lattice corresponds to the configuration space of the robot. Functions will be defined on this lattice which have specific values at each lattice point. For both Laplace's equation and the theory of Markov processes, the following discussion also extends to the continuous case (see Kemeny, et al. [10]).

2.1 Laplace's Equation

Harmonic functions are solutions to Laplace's equation:

$$\nabla^2 \phi = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} = 0 \tag{1}$$

The expression ϕ_x will be used subsequently to denote the derivative of ϕ with respect to x. The discrete form of Laplace's equation can be derived by taking the Taylor series expansion about a point x [11]. Denoting the harmonic function by ϕ, and using h as a small increment, the central difference approximations to each second directional derivative can be computed using Taylor series expansions in the appropriate variable:

$$\begin{aligned}
\phi(x+h) &= \phi(x) + hphi_x(x) + \frac{h^2}{2}\phi_{xx}(x) + \\
&\quad \frac{h^3}{6}\phi_{xxx}(x) + \frac{h^4}{24}\phi_{xxxx}(\xi_+) \\
\phi(x-h) &= \phi(x) - hphi_x(x) + \frac{h^2}{2}\phi_{xx}(x) - \\
&\quad \frac{h^3}{6}\phi_{xxx}(x) + \frac{h^4}{24}\phi_{xxxx}(\xi_-)
\end{aligned}$$

$$\phi(x+h) - 2\phi(x) + \phi(x-h) = \frac{h^2}{2}\phi_{xx}(x) +$$

$$\frac{h^4}{24}\phi_{xxxx}(\xi)$$

Since $h = 1$ on the lattice, we have in two dimensions:

$$\phi(x+1,y) + \phi(x-1,y) + \phi(x,y+1) + \phi(x,y-1) \\ -4\phi(x) \approx \nabla^2\phi = 0$$

Therefore, up to truncation error, the value of ϕ at a point in the lattice is the average of the values at the points in the (manhattan) neighborhood. This is known as the mean-value property, and applies in both the discrete and continuous cases [12]. It also holds for any number of dimensions.

2.2 Random Walks

Consider a Markov chain on this lattice. Define some points as absorbing states, with the caveat that certain absorbing states are obstacles and certain other absorbing states are goals. The hitting probability $p(x)$ at a point x is the probability that a random walk starting at x will be absorbed by an obstacle before being absorbed by the goal. Since the walk is random, the probabilities of transition from the current state to any (manhattan) neighbor state are all equal ($\frac{1}{2n}$, where n is the number of dimensions in the lattice). The probability $p(x)$ can then be defined in two dimensions as:

$$p(x) = \frac{1}{4}(\quad p(N|x \to N) + p(E|x \to E) \\ +p(W|x \to W) + p(S|x \to S))$$

where N, E, W, S are neighbor states and $x \to N$ denotes a transition from x to N. Since the process is Markov, probabilities do not depend on prior time steps, so that $p(N|x \to N) = p(N)$, and likewise for the other neighbors. Thus, the hitting probability at any lattice point is the average of the hitting probabilities at the (manhattan) neighbors. These probabilities are therefore harmonic (see Doyle and Snell [9] for a detailed description of this property).

There is a simple interpretation of this equivalence in the context of harmonic function path planning: The Dirichlet form for obstacles is assumed (as described in [2]), where obstacle boundaries are fixed at $\phi(x) = 1$, and goal regions are fixed at $\phi(x) = 0$. In this case, the resulting harmonic function ϕ is the collision probability function over the workspace. Therefore, gradient descent of ϕ minimizes at each step the probability of collisions with obstacles.

3 Uncertainty in Motion

Several authors have examined robot task execution in the presence of uncertainty in environmental modeling, robot motion, and obstacle motion (e.g., [13, 14, 15, 16]). This discussion centers around Erdmann's work [16]. Erdmann argues for the utility of randomization in robotic tasks as a way of overcoming uncertainty. The approach used by Erdmann relies on a randomization step in cases where sensing uncertainty prevents a reliable measurement of effector position. Bounds can be computed on the expected time-to-completion for the randomization process. However, the approach does not explicitly take estimated obstacle positions into account.

The relationship between collision probabilities and harmonic functions can be used here to provide some information to the randomization technique described in [16]. Let $p_c(x)$ be the probability that a random walk starting at x collides with an obstacle before reaching the goal. As noted in [16], a prerequisite to any randomizing step is that $p_c(x) < 1$ (i.e., that there is a nonzero probability of reaching the goal before hitting an obstacle). If O denotes the obstacle set, and G the goal set, then these probabilities can be computed directly by setting the following boundary conditions:

$$\begin{aligned} p_c(x) &= 1 \qquad x \in O \\ p_c(x) &= 0 \qquad x \in G \end{aligned}$$

and then computing the harmonic function which satisfies these constraints (see section 2.1). If $p_c(x) = 1$, then the goal is blocked, and some other strategy must be employed.

In [16], the randomization process is modeled as a diffusion, with drift $\bar{\mu}$ (a vector) and variance σ.[1] A label function ℓ is used to analyze expected velocity of the process. If we let $\ell(x) = p_c(x)$ (a harmonic function), then a drift velocity can be computed which biases the randomization toward the goal. If L is the diffusion operator:

$$\begin{aligned} v(x) &= (L\ell)(x) \\ &= \frac{1}{2}\sigma^2\nabla^2\ell + \bar{\mu}\nabla\ell \end{aligned}$$

but if $\ell(x) = p_c(x)$ then it is harmonic, and the first term vanishes:

$$v(x) = \bar{\mu}\nabla p_c(x)$$

[1] For brownian motion, drift is zero and the variance is a scalar [17]. Here, the drift is allowed to be nonzero.

The vector $\bar{\mu}$ is the drift. By allowing μ to be a function of configuration, and then adopting Erdmann's convention that the expected infinitesimal velocity $v(x) = -1$ (with respect to the label function p_c), then $\bar{\mu}(x)$ must be a vector in the opposite half-plane with respect to $\nabla p_c(x)$, i.e., the drift should be in the direction of decreasing collision probabilities. Thus, the same framework which is used for coarse motion planning in [3] can also be used for determining the drift in randomization. This allows randomization to take into account the estimated locations of obstacles in the workspace.

4 Learning in Robotics

Several authors have applied learning techniques to the robot navigation problem (e.g., [18, 19, 20]). In each case, the learning system tries to incorporate environmental information to learn the appropriate robot motion. The equivalence between harmonic functions and collision probabilities suggests two useful strategies for such learning systems:

1. learning potential functions (as in [19]) can be reduced to the problem of acquiring or computing collision probabilities
2. various properties of these probabilities (e.g., prevention of local minima, smoothness) can be preserved by restricting the types of operations that can be performed on the collision probabilities.

In some cases, the learning process can also be greatly accelerated by only learning obstacle positions and computing collision probabilities directly (as harmonic functions). This can be illustrated by considering a robot in an environment with obstacles and goals. Suppose the robot performs random walks (a popular training technique), storing collision statistics[2] at every starting point. This operation is essentially a Monte-Carlo solution for a harmonic function [9]. Note that this process is a good deal slower than a direct solution employing relaxation or a resistive network. In this pure case, the learning task can be reduced to learning obstacle and goal configurations, and then *computing* the collision probabilities directly (see section 2.1).

Mel [18] describes a connectionist model for vision-based planning and control of reaching movements. Although the motivation in [18] is to gain insight into human motor control, it has some characteristics in common with related work on robot learning. In [18], the planning phase consists of (simulated) random movements which are either accepted if they are "safe", or rejected if they exited the workspace or caused collisions. This system therefore attempts to reduce the chance of collision [18], and hence encodes these probabilities in the system. The probability of collision, however, can be computed by using a harmonic function (with obstacle and goal position estimates). This suggests a more direct method for performing the planning phase of a reach (see, for example [8]).

Likewise, Bachrach [21] (see also Dayan [22]) uses random walks in the environment to train a network to perform obstacle-avoiding movements. The resulting network has current state as input, and desired action as output. The network learns potential fields, and adjusts these to avoid local minima and produce efficient trajectories. Since collision probabilities are harmonic, they exhibit no local minima. Therefore, this adjustment phase can be eliminated if collision probabilities (or estimates thereof) are used directly, rather than implicitly coded within the network.

Systems which implicitly use collision probabilities (e.g., by way of a random training phase) should be able to take advantage of certain properties of harmonic functions: i.e., smoothness and the min-max principle (no "local minima"). The following operations preserve the harmonic properties of a function ϕ (c is a constant):

- dilation $\phi'(x) = \phi(cx)$
- translation $\phi'(x) = \phi(x + c)$
- scaling $\phi'(x) = c\phi(x)$
- bias $\phi'(x) = c + \phi(x)$

These properties can all be verified by applying the Laplacian to ϕ'. Obviously, any linear combination of harmonic functions is also harmonic. Thus, when collision probabilities can be isolated, their properties can be preserved by using the above operations (see also [12]). The application of learning techniques to robot navigation can be simplified in some cases by preserving these properties.

5 Conclusion

The equivalence of harmonic functions and collision probabilities has been applied to certain robotics problems. In general, techniques which rely on randomiza-

[2] i.e., whether an obstacle was encountered before the goal.

tion to achieve their results can benefit from the probabilistic interpretation of harmonic functions. This connection is applied here to motion uncertainty, and in particular to Erdmann's randomisation method. The result is an explicit incorporation of obstacle information into the randomisation phase, and an interpretation of harmonic functions in terms of goal-reaching probabilities and drift in the random motion. This also illustrates the interpretation of harmonic functions as an extrapolation of obstacle geometry [3]. In addition, collision probabilities are usually implicitly (and sometimes explicitly) used in many navigation-learning systems. These probabilities are harmonic, and can be exploited to avoid local minima and enforce smooth, safe behavior of the system. In some cases, this can lead to a considerable improvement in performance (e.g., by substituting a direct solution for a Monte-Carlo simulation).

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