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Robust Tracking and Regulation Control for Mobile Robots

W. E. Dixon, D. M. Dawson, E. Zergeroglu, and F. Zhang

Department of Electrical & Computer Engineering,

Clemson University, Clemson, SC 29634-0915

voice: (864) 656-5924; fax: (864) 656-7220 email: ddawson@eng.clemson.edu

Abstract - This paper presents the design of a differentiable, robust tracking controller for a mobile robot system. The controller provides robustness with regard to parametric uncertainty and additive bounded disturbances in the dynamic model. Through the use of a dynamic oscillator and a Lyapunov-based stability analysis, we demonstrate that the position/orientation tracking errors exponentially converge to a neighborhood about zero that can be made arbitrarily small (i.e., the controller ensures that the tracking error is globally uniformly ultimately bounded (GUUB)). In addition, we illustrate how the robust tracking controller can be reconfigured as a variable structure controller that ensures global exponential regulation to an arbitrary desired setpoint.

1 Introduction

The motion control problem of mechanical systems with nonholonomic constraints has been a heavily researched area due to both the challenging theoretical nature of the problem and its practical importance. One example of a nonholonomic system that has received a large amount of research activity is the wheeled mobile robot (WMR). In recent years, control researchers have targeted the problems of: *i)* regulating the position and orientation of the WMR to an arbitrary setpoint, *ii)* tracking a time-varying reference trajectory (which includes the *path following* problem as a special case [6]), and *iii)* incorporating the effects of the dynamic model during the control design to enhance robustness. With regard to the control of nonholonomic systems, one of the technical hurdles often cited is that the regulation problem cannot be solved via a smooth, time-invariant state feedback law due to the implications of Brockett's condition [4]. To deal with this obstacle, some researchers have proposed controllers that utilize discontinuous control laws, piecewise continuous control laws, smooth time-varying control laws, or hybrid controllers to achieve setpoint regulation (see [20], [22], and the references therein for an in-depth review of the previous work). Specifically, in [3], Bloch *et al.* developed a piecewise continuous control structure for locally regulating several different types of nonholonomic systems to a setpoint. In [5], Canudas de Wit *et al.* constructed a piecewise smooth controller to exponentially stabilize a WMR to a setpoint; however, due to the control structure, the orientation of the WMR is not arbitrary. One of the first smooth, time-varying feedback controllers that could be utilized to asymptotically regulate a WMR to a desired setpoint was proposed by Samson in [22]. Smooth, time-varying controllers were also developed for other classes of nonholonomic systems in [8], [21], and [25]. More recently, in [23], Samson developed global asymptotic feedback controllers for a general class of nonholonomic systems, and hence, provided a control solution that could be used to stabilize a WMR to a desired posture or a fixed reference-frame path. In order to overcome the *slower*, asymptotic response of the previous smooth, time varying controllers, Godhavn *et al.* [13] and McCloskey *et al.* [20] constructed control laws that locally ρ-exponentially (as well globally asymptotically) stabilized classes of nonholonomic systems. Under the assumption of exact model knowledge, McCloskey *et al.* [20] also illustrated how the dynamic model of a WMR could be included during the control design.

Several controllers have also been proposed for the reference robot tracking problem (*i.e.*, the desired time-varying linear/angular velocity are specified). Specifically, in [17], Kanayama *et al.* utilized a continuous feedback control law for a linearized kinematic model to obtain local asymptotic tracking; whereas, Walsh *et al.* [26] obtained local exponential stability results for a similar linearized model using a continuous, linear control law. In [14], Jiang *et al.* developed a global asymptotic tracking controller for a WMR; however, angular acceleration measurements were required. In [15] and [16], Jiang *et al.* provided semi-global and global asymptotic tracking solutions for the general chained system form, and hence, provided a solution for the WMR tracking problem that removed the need for angular acceleration measurements required in [14]. In [11], Escobar *et al.* illustrated how a field oriented induction motor controller can be redesigned to exponentially stabilize the nonholonomic double integrator control problem (*e.g.*, Heisenberg flywheel); however, the controller exhibited singularities. To compensate for parametric uncertainty in the dynamic model, Dong *et al.* [10] utilized the kinematic control proposed in [23] to construct an adaptive control solution for a class of nonholonomic systems that yielded global asymptotic tracking. We also note that several researchers (see [1], [6], and the references within) have proposed various controllers for the path following problem.

From a review of the literature, it seems that we can make the following observations for the previously developed kinematic controllers: *i)* the tracking controllers do not solve the regulation problem (*i.e.*, restrictions on the reference model trajectory signals prohibit extension to the regulation problem), *ii)* the stability results for the differentiable, kinematic controllers tend to be global asymptotic instead of global exponential, *iii)* the heavy reliance of Barbalat's Lemma and its extensions during the kinematic stability analysis prohibit the use of robust nonlinear controllers [7] for rejection of uncertainty associated with the dynamic model (*i.e.*, the Lyapunov derivative is negative semi-definite in the system states as opposed to negative definite), and *iv)* some of the kinematic controllers are not differentiable (*e.g.*, see the kinematic controller developed in [20]), and unfortunately, the standard backstepping procedure, often used for incorporating the mechanical dynamics, requires that the kinematic controller be differentiable (see the discussion in [20]). In an attempt to address the above issues, we present the design of a new, differentiable kinematic control law that achieves global uniformly ultimately bounded (GUUB) tracking control for a WMR. That is, the position and orientation tracking errors globally exponentially converge to a neighborhood about zero that can be made arbitrarily small. Since the kinematic tracking controller does not restrict the reference model in any way, the proposed kinematic controller can also be used for the regulation problem; hence, we present a unified control framework for both the tracking and regulation problem. Moreover, since the proposed kinematic controller is differentiable, we illustrate how standard backstepping techniques can be used to design a nonlinear robust controller that compensates for uncertainty associated with the dynamic model (*i.e.*, parameter uncertainty and additive bounded disturbances). We note that the proposed kinematic controller does not utilize explicit sinusoidal terms in the feedback controller; rather, a damped dynamic oscillator with a tunable frequency of oscillation is constructed. Roughly speaking, the frequency of oscillation is used as an auxiliary control input to cancel odious terms during the Lyapunov analysis. It should be noted that the proposed solution to the kinematic problem is crucial for developing the proposed robust controller for the dynamic model (*i.e.*, the Lyapunov derivative is negative definite in the system states as opposed to negative semi-definite).

The paper is organized as follows. In Section 2, we transform the kinematic and dynamic models of the WMR into a form which facilitates the subsequent control development. In Section 3, we present the control law and the corresponding closed-loop error system. In Section 4, we provide a Lyapunov based stability analysis that illustrates global uniformly ultimately bounded tracking. In Section 5, we reconfigure the controller as a variable structure controller that globally exponentially stabilizes the WMR to an arbitrary desired setpoint. In Section 6, we present some concluding remarks.

2 Problem Formulation

2.1 WMR Kinematic Model

The kinematic model for the so-called kinematic wheel under *pure rolling* and *non slipping* conditions is given as follows [20]

$$\dot{q} = S(q)v \tag{1}$$

where $q(t)$, $\dot{q}(t) \in \Re^3$ are defined as

$$q = [x_c \quad y_c \quad \theta]^T \qquad \dot{q} = \begin{bmatrix}\dot{x}_c & \dot{y}_c & \dot{\theta}\end{bmatrix}^T \tag{2}$$

$x_c(t)$, $y_c(t)$, and $\theta(t) \in \Re^1$ denote the linear position and orientation, respectively, of the COM, $\dot{x}_c(t)$, $\dot{y}_c(t)$, and $\dot{\theta}(t) \in \Re^1$ denote the linear and angular velocity, respectively, of the center of mass (COM), the matrix $S(q) \in \Re^{3\times 2}$ is defined as follows

$$S(q) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}, \tag{3}$$

and the velocity vector $v(t) \in \Re^2$ is defined as follows

$$v = [v_1 \quad v_2]^T = \begin{bmatrix} v_l & \dot{\theta}\end{bmatrix}^T \tag{4}$$

with $v_l(t) \in \Re^1$ and $\dot{\theta}(t) \in \Re^1$ denoting the linear and angular WMR velocities.

2.2 Control Objective

As defined in previous work (*e.g.*, see [14] and [17]), the reference robot is assumed to move according the following dynamic trajectory

$$\dot{q}_r = S(q_r)v_r \tag{5}$$

where $S(\cdot)$ was defined in (3), $q_r(t) = \begin{bmatrix} x_{rc}(t) & y_{rc}(t) & \theta_r(t)\end{bmatrix}^T \in \Re^3$ denotes the desired time-varying position/orientation trajectory, and $v_r(t) = \begin{bmatrix} v_{r1}(t) & v_{r2}(t)\end{bmatrix}^T \in \Re^2$ denotes the reference time-varying linear/angular trajectory. With regard to (5), it is assumed that the signal $v_r(t)$ is constructed to produce the desired motion and that $v_r(t)$, $\dot{v}_r(t)$, $q_r(t)$, and $\dot{q}_r(t)$ are bounded for all time.

To facilitate the subsequent control synthesis and the corresponding stability proof, we define the following transformation

$$\begin{bmatrix} w \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -\tilde{\theta}\cos\theta + 2\sin\theta & -\tilde{\theta}\sin\theta - 2\cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\theta} \end{bmatrix} \tag{6}$$

where $\tilde{x}(t)$, $\tilde{y}(t)$, $\tilde{\theta}(t) \in \Re^1$ denote the difference between the actual Cartesian position/orientation of the COM and the desired position/orientation of the COM as follows

$$\tilde{x} = x_c - x_{rc} \qquad \tilde{y} = y_c - y_{rc} \qquad \tilde{\theta} = \theta - \theta_r. \tag{7}$$

After taking the time derivative of (6), and using (1), (2), (3), (4), (6), and (7), we can rewrite the kinematic tracking error dynamics in terms of the new variables defined in (6) as follows

$$\begin{aligned} \dot{w} &= u^T J^T z + f \\ \dot{z} &= u \end{aligned} \tag{8}$$

where $J \in \Re^{2\times 2}$ is defined as

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{9}$$

$f(z, v_r, t) \in \Re^1$ is defined as

$$f = 2\left(v_{r2}z_2 - v_{r1}\sin z_1\right), \tag{10}$$

the auxiliary variable $u(t) = \begin{bmatrix} u_1(t) & u_2(t)\end{bmatrix}^T \in \Re^2$ is defined in terms of the WMR position/orientation, the WMR linear velocities, and the desired trajectory as follows

$$\begin{aligned} u &= T^{-1}v - \begin{bmatrix} v_{r2} \\ v_{r1}\cos\tilde{\theta} \end{bmatrix} \\ v &= Tu + \begin{bmatrix} v_{r1}\cos\tilde{\theta} + v_{r2}\left(\tilde{x}\sin\theta - \tilde{y}\cos\theta\right) \\ v_{r2} \end{bmatrix} \end{aligned} \tag{11}$$

where the matrix $T \in \Re^{2\times 2}$ is defined as follows

$$T = \begin{bmatrix} (\tilde{x}\sin\theta - \tilde{y}\cos\theta) & 1 \\ 1 & 0 \end{bmatrix}. \tag{12}$$

2.3 WMR Dynamics

The dynamic model for the kinematic wheel can be easily expressed in the following form

$$M\dot{v} + F(\dot{q}) + T_d = B\tau \tag{13}$$

where $\dot{v}(t) \in \Re^2$ denotes the time derivative of $v(t)$ defined in (4), $M \in \Re^{2\times 2}$ represents the constant inertia matrix $F(\dot{q}) \in \Re^2$ represents the friction effects, $T_d(t) \in \Re^2$ represents a vector of unknown, bounded disturbances, $\tau(t) \in \Re^2$ represents the torque input vector, and $B \in \Re^{2\times 2}$ represents an input matrix that governs torque transmission.

To facilitate the subsequent control design, we premultiply (13) by T^T and substitute (11) and (12) for $v(t)$ to obtain the following convenient dynamic model

$$\bar{M}\dot{u} + \bar{N} = \bar{B}\tau \tag{14}$$

where $\bar{M} = T^T M T$, $\bar{N} = T^T \left(M\dot{T}u + F(\dot{q}) + T_d + M\Pi\right)$, $\bar{B} = T^T B$, and $\Pi \in \Re^2$ is given by

$$\Pi = \begin{bmatrix} \dot{v}_{r1}\cos\tilde{\theta} - v_{r1}\,\dot{\tilde{\theta}}\sin\tilde{\theta} + \dot{v}_{r2}\left(\tilde{x}\sin\theta - \tilde{y}\cos\theta\right) \\ +v_{r2}\left(\tilde{x}\dot{\theta}\cos\theta + \dot{\tilde{x}}\sin\theta + \tilde{y}\dot{\theta}\sin\theta - \dot{\tilde{y}}\cos\theta\right) \\ \\ \dot{v}_{r2} \end{bmatrix}. \tag{15}$$

The dynamic equation of (14) exhibits the following property which will be employed during the subsequent control development and stability analysis.

Property 1: The transformed inertia matrix $\bar{M}$ is symmetric, positive definite, and satisfies the following inequalities [19]

$$m_1 \|\xi\|^2 \leq \xi^T \bar{M}\xi \leq m_2(z,w)\|\xi\|^2 \qquad \forall \xi \in \Re^2 \tag{16}$$

where m_1 is a known positive constant, $m_2(z,w) \in \Re^1$ is a known, positive bounding function which is assumed to be bounded provided its arguments are bounded, and $\|\cdot\|$ is the standard Euclidean norm. Based on the fact that $\bar{M}$ is symmetric and positive definite, we can use (16) to show that the inverse of $\bar{M}$ satisfies the following inequality

$$\frac{1}{m_2(z,w)}\|\xi\|^2 \leq \xi^T \bar{M}^{-1}\xi \leq \frac{1}{m_1}\|\xi\|^2 \qquad \forall \xi \in \Re^2. \tag{17}$$

3 Control Development

Our control objective is to design a robust[1] controller for the transformed WMR model given by (14). To this end, we define an auxiliary error signal $\tilde{z}(t) \in \Re^2$ as the difference between the subsequently designed auxiliary signal $z_d(t) \in \Re^2$ and the transformed variable $z(t)$, defined in (6), as follows

$$\tilde{z} = z_d - z. \tag{18}$$

In addition, we define an auxiliary error signal $\eta(t) \in \Re^2$ as the difference between the subsequently designed auxiliary signal $u_d(t) \in \Re^2$ and the auxiliary signal $u(t)$, defined in (11), as shown below

$$\eta = u_d - u. \tag{19}$$

3.1 Control Formulation

Based on the kinematic equations given in (8) and the subsequent stability analysis, we design the auxiliary signal $u_d(t)$ as follows

$$u_d = u_a - kz \tag{20}$$

where $k \in \Re^1$ is a positive, constant control gain, the auxiliary control term $u_a(t) \in \Re^2$ is defined as

$$u_a = \left(\frac{kw+f}{\delta_d^2}\right)Jz_d + \Omega_1 z_d, \tag{21}$$

the auxiliary signal $z_d(t)$ is defined by the following oscillator-like relationship

$$\dot{z}_d = \frac{\dot{\delta}_d}{\delta_d}z_d + \left(\frac{kw+f}{\delta_d^2} + w\Omega_1\right)Jz_d \qquad z_d^T(0)z_d(0) = \delta_d^2(0), \tag{22}$$

[1] Roughly speaking, the controller will be designed to reject parametric uncertainty and additive bounded disturbances.

the auxiliary terms $\Omega_1(t) \in \Re^1$ and $\delta_d(t) \in \Re^1$ are defined as

$$\Omega_1 = k + \frac{\dot{\delta}_d}{\delta_d} + \frac{kw^2 + wf}{\delta_d^2} \tag{23}$$

and

$$\delta_d = \alpha_0 \exp(-\alpha_1 t) + \varepsilon_1 \tag{24}$$

respectively, α_0, α_1, $\varepsilon_1 \in \Re^1$ are positive, constant parameters, and $f(z, v_r, t)$ was defined in (10)

Based on the transformed dynamic model given by (14) and the subsequent stability analysis, we design the control torque input $\tau(t)$ as follows

$$\tau = \left(\bar{B}\right)^{-1}\left(\hat{\kappa} + km_2(z,w)\eta + v_R\right) \tag{25}$$

where $\hat{\kappa}(\eta, w, z_d, z, t) \in \Re^2$ is a best guess estimate of $\kappa(\eta, w, z_d, z, t) \in \Re^2$ which is explicitly defined below

$$\kappa = \bar{M}\dot{u}_d + \bar{N}, \tag{26}$$

the auxiliary robust control term $v_R(\eta, w, z_d, z, t) \in \Re^2$ is defined as follows

$$v_R = \frac{m_2(z,w)\eta\rho^2}{\|\eta\|\rho + \varepsilon_2}, \tag{27}$$

$\varepsilon_2 \in \Re^1$ is a positive, constant control gain, and the bounding function $\rho(\eta, w, z_d, z, t) \in \Re^1$ is constructed to satisfy the following inequality

$$\rho \geq \left\|\bar{M}^{-1}\left(\kappa - \hat{\kappa} + \bar{M}Jzw + \bar{M}\dot{\tilde{z}}\right)\right\|. \tag{28}$$

Remark 1 *Motivation for the structure of (22) is obtained by taking the time derivative of $z_d^T z_d$ as follows*

$$\frac{d}{dt}\left(z_d^T z_d\right) = 2z_d^T\left(\frac{\dot{\delta}_d}{\delta_d}z_d + \left(\frac{kw+f}{\delta_d^2} + w\Omega_1\right)Jz_d\right) \tag{29}$$

where (22) has been utilized. After noting that the matrix J of (9) is skew symmetric, we can rewrite (29) as follows

$$\frac{d}{dt}\left(z_d^T z_d\right) = 2\frac{\dot{\delta}_d}{\delta_d}z_d^T z_d. \tag{30}$$

As result of the selection of the initial conditions given in (22), it is easy to verify that

$$z_d^T z_d = \|z_d\|^2 = \delta_d^2 \tag{31}$$

is a unique solution to the differential equation given in (30). The relationship given by (31) will be used during the subsequent error system development and stability analysis.

Remark 2 *One method for creating $\hat{\kappa}(\cdot)$ and $\rho(\cdot)$ used in (25) and (27) is to note that part of $\kappa(\cdot)$ can be linear parameterized as follows*

$$\bar{M}\dot{u}_d + T^T\left(M\dot{T}u + F(\dot{q}) + M\Pi\right) = Y_d\theta \tag{32}$$

where $\theta \in \Re^p$ contains the unknown constant system parameters, and the desired regression matrix $Y_d(\eta, w, z_d, z, t) \in \Re^{n\times p}$ contains known functions. Hence, $\hat{\kappa}(\cdot)$ could be constructed as follows

$$\hat{\kappa}(\cdot) = Y_d\hat{\theta} \tag{33}$$

where $\hat{\theta}(t) \in \Re^p$ denotes the constant, best-guess parameter estimate vector. To satisfy (28), it would be an easy matter to use upper bounds of the maximum parameter error and the additive bounded disturbance to construct $\rho(\cdot)$ as follows

$$\rho \geq \left\|\bar{M}^{-1}\left(Y_d\tilde{\theta} + \bar{T}_d + \bar{M}Jzw + \bar{M}\dot{\tilde{z}}\right)\right\| \tag{34}$$

where $\tilde{\theta}(t) \in \Re^2$ is defined as shown below

$$\tilde{\theta} = \theta - \hat{\theta}. \tag{35}$$

3.2 Error System Development

To facilitate the closed-loop error system development, we inject the auxiliary control input $u_d(t)$ into the open-loop dynamics of $w(t)$ given by (8) by adding and subtracting the term $u_d^T J^T z$ to the right-side of (8) and utilizing (19) to obtain the following expression

$$\dot{w} = u_d^T J^T z - \eta^T J^T z + f. \tag{36}$$

After substituting (20) for $u_d(t)$, adding and subtracting $u_a^T J^T z_d$ to the resulting expression, utilizing (18), and exploiting the skew symmetry of J defined in (9), we can rewrite the dynamics for $w(t)$ as follows

$$\dot{w} = -u_a^T J z_d + u_a^T J \tilde{z} + \eta^T J z + f. \tag{37}$$

Finally, by substituting (21) for only the first occurrence of $u_a(t)$ in (37) and then utilizing the equality given by (31), the skew symmetry of J defined in (9), and the fact that $J^T J = I_2$ (Note that I_2 denotes the standard 2×2 identity matrix), we can obtain the final expression for the closed-loop error system for $w(t)$ as follows

$$\dot{w} = -kw + u_a^T J \tilde{z} + \eta^T J z. \tag{38}$$

To determine the closed-loop error system for $\tilde{z}(t)$, we take the time derivative of (18), substitute (22) for $\dot{z}_d(t)$, and then substitute (8) for $\dot{z}(t)$ to obtain

$$\dot{\tilde{z}} = \frac{\dot{\delta}_d}{\delta_d} z_d + \left(\frac{kw + f}{\delta_d^2} + w\Omega_1 \right) J z_d - u + u_d - u_d \tag{39}$$

where the auxiliary control input $u_d(t)$ was injected by adding and subtracting $u_d(t)$ to the right-side of (39). After utilizing (19), substituting (20) for only the last occurrence of $u_d(t)$, and then substituting (21) for $u_a(t)$ in the resulting expression, we can rewrite the expression given by (39) as follows

$$\dot{\tilde{z}} = \frac{\dot{\delta}_d}{\delta_d} z_d + w\Omega_1 J z_d - \Omega_1 z_d + kz + \eta. \tag{40}$$

After substituting (23) for only the second occurrence of $\Omega_1(t)$ in (40) and using the fact that $JJ = -I_2$, we can cancel common terms and rearrange the resulting expression to obtain

$$\dot{\tilde{z}} = -k\tilde{z} + wJ \left[\left(\frac{kw + f}{\delta_d^2} \right) J z_d + \Omega_1 z_d \right] + \eta \tag{41}$$

where (18) has been utilized. Finally, since the bracketed term in (41) is equal to $u_a(t)$ defined in (21), we can obtain the final expression for the closed-loop error system for $\tilde{z}\,(t)$ as follows

$$\dot{\tilde{z}} = -k\tilde{z} + wJu_a + \eta. \tag{42}$$

In order to develop the closed-loop error system for $\eta(t)$, we develop the open-loop error dynamics by taking the time derivative of (19), substituting for $\dot{u}(t)$ from (14), substituting for $\dot{u}_d(t)$ from (26), and then rearranging the resulting expression to obtain the following expression

$$\dot{\eta} = \bar{M}^{-1} \left(\kappa - \bar{B}\tau \right). \tag{43}$$

After substituting for the control torque input $\tau(t)$ defined in (25) into (43), we obtain the following expression for the closed-loop error system for $\eta(t)$

$$\begin{aligned} \dot{\eta} \;=\; & -km_2(z,w)\bar{M}^{-1}\eta - \bar{M}^{-1}v_R - Jzw - \tilde{z} \\ & + \bar{M}^{-1}\left(\kappa - \hat{\kappa} + \bar{M}Jzw + \bar{M}\tilde{z} \right) \end{aligned} \tag{44}$$

where $Jzw + \tilde{z}$ has been added and subtracted to the right-hand side of (44) to facilitate the following stability analysis.

4 Stability Analysis

Theorem 1 *Given the closed-loop system of (38), (42), and (44), and that the control term $v_R(t)$ has been selected according to (27), the position/orientation tracking errors defined in (7) are GUUB in the sense that*

$$\begin{aligned} |\tilde{x}(t)|, |\tilde{y}(t)|, \left|\tilde{\theta}(t)\right| \;\leq\; & \sqrt{\beta_0 \exp(-\gamma_0 t) + \varepsilon_2 \beta_1} \\ & + \beta_2 \exp(-\gamma_1 t) + \beta_3 \varepsilon_1 \end{aligned} \tag{45}$$

for some positive scalar constants β_0, β_1, β_2, β_3, γ_0, and γ_1. Note that the control parameters ε_1 and ε_2 were originally defined in (24) and (27), respectively.

Proof: To prove *Theorem 1*, we define the following non-negative, scalar function denoted by $V(t) \in \Re^1$ as follows

$$V(t) = \frac{1}{2}w^2 + \frac{1}{2}\eta^T\eta + \frac{1}{2}\tilde{z}^T\tilde{z}. \tag{46}$$

After taking the time derivative of (46) and making the appropriate substitutions from (38), (42), and (44), we obtain the following expression

$$\begin{aligned} \dot{V} \;=\; & w\left[-kw + u_a^T J\tilde{z} + \eta^T Jz \right] \\ & + \tilde{z}^T\left[-k\tilde{z} + wJu_a + \eta \right] \\ & + \eta^T\left[-km_2(z,w)\bar{M}^{-1}\eta - \bar{M}^{-1}v_R - Jzw - \tilde{z} \right. \\ & \left. + \bar{M}^{-1}\left(\kappa - \hat{\kappa} + \bar{M}Jzw + \bar{M}\tilde{z} \right) \right]. \end{aligned} \tag{47}$$

After cancelling common terms, utilizing (28), and substituting (27) for $v_R(t)$, we obtain the following upper bound for $\dot{V}(t)$ of (47) as follows

$$\begin{aligned} \dot{V} \;\leq\; & -k\left(w^2 + \tilde{z}^T\tilde{z} \right) - km_2(z,w)\eta^T\bar{M}^{-1}\eta \\ & + \left[\|\eta\|\rho - m_2(z,w)\frac{\eta^T\bar{M}^{-1}\eta\rho^2}{\|\eta\|\rho + \varepsilon_2} \right]. \end{aligned} \tag{48}$$

Then by utilizing (17), we note that $\dot{V}(t)$ of (47) can be upper bounded as follows

$$\dot{V} \leq -k\left(w^2 + \eta^T\eta + \tilde{z}^T\tilde{z} \right) + \left[\|\eta\|\rho - \frac{\|\eta\|^2\rho^2}{\|\eta\|\rho + \varepsilon_2} \right]. \tag{49}$$

Furthermore, after noting that the bracketed term in (49) is less than or equal to ε_2, we can use (46) to obtain the following new upper bound for $\dot{V}(t)$

$$\dot{V} \leq -2kV + \varepsilon_2. \tag{50}$$

Standard arguments can now be employed to solve the differential inequality given in (50) as follows

$$V \leq \exp(-2kt)V(0) + \frac{\varepsilon_2}{2k}(1 - \exp(-2kt)). \tag{51}$$

Finally, we can utilize (46) to rewrite the inequality given by (51) as follows

$$\|\Psi\| \leq \sqrt{\exp(-2kt)\left\|\Psi(0)\right\|^2 + \frac{\varepsilon_2}{k}(1 - \exp(-2kt))} \tag{52}$$

where the vector $\Psi(t) \in \Re^5$ is defined as

$$\Psi = \begin{bmatrix} w & \eta^T & \tilde{z}^T \end{bmatrix}^T. \tag{53}$$

Based on (52) and (53), it is straightforward to see that $w(t)$, $\eta(t)$, $\tilde{z}(t) \in \mathcal{L}_\infty$. After utilizing (18), (31), and the fact that $\tilde{z}(t), \delta_d(t) \in \mathcal{L}_\infty$, we can conclude that $z(t)$, $z_d(t) \in \mathcal{L}_\infty$. From (8), (19), (20), (21), (22), (23), and (24), we can sho that $u_d(t)$, $u_a(t)$, $\dot{z}_d(t)$, $\Omega_1(t)$, $u(t) \in \mathcal{L}_\infty$. Now, in order to illustrate that the Cartesian position/orientation signals defined in (1) are bounded, we calculate the inverse transformation of (6) as follows

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\left(\tilde{\theta}\sin\theta + 2\cos\theta\right) & \frac{1}{2}\sin\theta \\ 0 & -\frac{1}{2}\left(\tilde{\theta}\cos\theta - 2\sin\theta\right) & -\frac{1}{2}\cos\theta \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ w \end{bmatrix}. \tag{54}$$

Since $z(t) \in \mathcal{L}_\infty$, it is clear from (54) that $\tilde{\theta}(t)$, $\theta(t) \in \mathcal{L}_\infty$ (*i.e.*, see (7)). From (54) and the fact that $w(t)$, $z(t)$, $\tilde{\theta}(t) \in \mathcal{L}_\infty$, we can conclude that $\tilde{x}(t)$, $\tilde{y}(t)$, $x_c(t)$, $y_c(t) \in \mathcal{L}_\infty$ (*i.e.*, see (7)). We can utilize (11), the boundedness restrictions on the desired trajectory, and the fact that $u(t)$, $\tilde{x}(t)$, $\tilde{y}(t) \in \mathcal{L}_\infty$, to show that $v(t) \in \mathcal{L}_\infty$; therefore, it follows from (1), (2), (3), and (4) that $\dot{\theta}(t)$, $\dot{x}_c(t)$, $\dot{y}_c(t) \in \mathcal{L}_\infty$. We can now employ standard signal chasing arguments to conclude that all of the remaining signals in the control and the system remain bounded during closed-loop operation.

In order to prove (45), we first show that $z(t)$ defined in (6) goes to zero exponentially fast by applying the triangle inequality to (18) to obtain the following exponential bound for $z(t)$

$$\begin{aligned} \|z\| &\leq \|\tilde{z}\| + \|z_d\| \\ &\leq \sqrt{\exp(-2kt)\|\Psi(0)\|^2 + \frac{\varepsilon_2}{k}(1-\exp(-2kt))} \\ &\qquad + \alpha_0\exp(-\alpha_1 t) + \varepsilon_1 \end{aligned} \tag{55}$$

where (24), (31), and (52) have been utilized. The main result given by (45) can now be directly obtained from (52), (53), (54), and (55). □

5 Setpoint Control Extension

In this section, we illustrate how the proposed controller given in the previous section can be reformulated as a variable structure controller which attains global exponential setpoint regulation. Since this new control objective is now targeted at the position/orientation setpoint control problem, the desired position/orientation vector, denoted by $q_r = \begin{bmatrix} x_{rc} & y_{rc} & \theta_r \end{bmatrix}^T \in \Re^3$ and originally defined in (5), is now assumed to be an arbitrary constant vector. Based on the fact that q_r is now defined as a constant vector, it is easy to show that the transformed kinematic and dynamic models are almost exactly the same as those given in (8) and (14), respectively, with the exception being that v_{r1} and v_{r2} given in (5), and consequently $f(z, v_r, t)$ and Π defined in (10) and (15), respectively, are all set to zero. We also note that the auxiliary variable $u(t)$ originally defined in (11), is now defined as follows

$$u = T^{-1}v \qquad v = Tu \tag{56}$$

where matrix T was defined in (12). Furthermore, to obtain global exponential setpoint control, we also set the control parameters ε_1 and ε_2, introduced in (24) and (27), respectively, to zero. Based on these design modifications, we present the following theorem.

Theorem 2 *Given the closed-loop system of (38), (42), and (44), if the control parameters ε_1 and ε_2 are set to zero in (24) and (27), respectively, and the control parameters α_1 and k are selected as follows*

$$k > 2\alpha_1, \tag{57}$$

then the position/orientation setpoint errors defined in (7) are globally exponentially stable in the sense that

$$|\tilde{x}(t)|, |\tilde{y}(t)|, \left|\tilde{\theta}(t)\right| \leq \beta_4\exp(-\gamma_2 t) + \beta_5\exp(-\gamma_3 t) \tag{58}$$

for some positive scalar constants β_4, β_5, γ_2, and γ_3.

Proof:
To prove *Theorem 2*, we can follow the proof of *Theorem 1* up to (52) to obtain

$$\|\Psi\| \leq \exp(-kt)\|\Psi(0)\| \tag{59}$$

where we have utilized the fact that ε_2 has been set to zero. As in the prove of *Theorem 1*, it is now easy to see that $w(t)$, $\eta(t)$, $\tilde{z}(t)$ $z(t)$, $z_d(t) \in \mathcal{L}_\infty$. However, since ε_1 and ε_2 have been set to zero, there appear to be potential singularities in the auxiliary terms given by (21), (22), (23), and in the time derivative of (20). That is, since $\delta_d(t)$ of (24) goes to zero exponentially fast, the terms contained in (21), (22), and (23), which are given below

$$\frac{kw}{\delta_d^2}Jz_d, \qquad \frac{kw^2}{\delta_d^2}z_d, \qquad \frac{kw^3}{\delta_d^2}Jz_d, \tag{60}$$

appear to be unbounded as $t \to \infty$; however, since $w(t)$ is driven to zero within the exponential envelope given in (59), it can be clearly seen that if the sufficient condition given in (57) holds then the potential singularities depicted in (60) are always avoided. Based on this fact, we can now use standard signal chasing arguments to show that $u_d(t)$, $u_a(t)$, $\dot{z}_d(t)$, $\Omega_1(t)$, $u(t) \in \mathcal{L}_\infty$; hence, from the boundedness of the previous signals, it is straightforward that $\dot{u}_d(t) \in \mathcal{L}_\infty$. Due to the fact that $\eta(t)$, $w(t)$, $\tilde{z}(t)$, $u(t)$, $\dot{u}_d(t) \in \mathcal{L}_\infty$, we can now show that $\tau(t) \in \mathcal{L}_\infty$. Standard signal chasing arguments can now be used to show that all of the remaining signals in the control and the system are bounded during closed-loop operation.

In order to prove (58), we first show that $z(t)$ defined in (6) goes to zero exponential fast by applying the triangle inequality to (18) to obtain the following exponential bound for $z(t)$

$$\|z\| \leq \|\tilde{z}(t)\| + \|z_d\| \leq \|\Psi(0)\|\exp(-kt) + \alpha_0\exp(-\alpha_1 t) \tag{61}$$

where (24), (31), and (59) have been utilized. The main result given by (58) now directly follows (53), (54), (59), and (61). □

Remark 3 *We note that if ε_1 and ε_2 were not set to zero in the previous setpoint control development and stability analysis, then we could obtain the same result as given in (45) for the setpoint problem. That is, in contrast to the above variable structure controller, we can modify the purposed setpoint controller to construct a smooth, torque input that exponentially drives the position/orientation setpoint errors to an arbitrarily small neighborhood about zero.*

Remark 4 *If exact model knowledge of the system dynamics is available and the additive bounded disturbance in (13) is set to zero, we can redesign the controller given by (25) as the following smooth, torque controller*

$$\tau = \left(\bar{B}\right)^{-1}\left(\kappa + k\bar{M}\eta + \bar{M}Jzw + \bar{M}\tilde{z}\right) \tag{62}$$

where $\kappa(\cdot)$ was defined in (26). With all of the other previous definitions being exactly the same, it is easy to use the proof of Theorem 1 to show that the smooth torque controller given by (62) yields the same result as that given by (58).

6 Conclusion

In this paper, we designed a robust tracking controller for a mobile robot system. Through the use of a Lyapunov-based stability analysis, we have demonstrated that: *i)* the position and orientation tracking errors exponentially converge to a neighborhood about zero that can be made arbitrarily small, and *ii)* the controller provides robustness with regard to parametric uncertainty and additive bounded disturbances in the dynamic model. In addition, we illustrated how the robust tracking controller can be reconfigured as a variable structure controller that ensures global exponential regulation to an arbitrary desired setpoint. It should also be noted that in addition to the WMR problem, the kinematic portion of the proposed controller can be applied to other nonholonomic systems (see [3] for examples). Future work will involve experimental trials on a modified Cybermotion K2A mobile robot system.

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