

Stability analysis of perturbed nonlinear systems applied to the tracking control of a wheeled mobile robot under lateral slip

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Abstract: For high-speed operation under adverse ground conditions, motion control of mobile robots is a difficult task, because the robots' wheels may be subject to different kinds of slip. For a differential drive wheeled mobile robot, the lateral slip is a challenging problem to deal with, since the wheels cannot directly produce movement in the lateral direction. This paper presents an application of a useful result from Lyapunov Theory for perturbed nonlinear systems, which can be viewed as a framework for analyzing the robustness of kinematic controllers for mobile robots under lateral slip. An example of such application is given by proposing a simple state feedback kinematic control law that solves the robot's trajectory tracking control problem and then showing that the solution of the robot's posture error dynamics is uniformly ultimately bounded. Numerical results illustrate the key aspects of the proposed approach.

Keywords: mobile robots, trajectory tracking, lateral slip, perturbed nonlinear systems, ultimate boundedness.

INTRODUCTION

Most of the applications of wheeled mobile robots involve the precise tracking of a given trajectory under adverse ground conditions. Poor ground conditions may be responsible for the occurrence of wheel slip, which can be categorized either as longitudinal slip, occurring in the forward direction of the robot, or lateral slip, occurring perpendicularly to that direction.

For nonholonomic differential drive wheeled mobile robots, which are commonly used due to its versatility, longitudinal slip can be countered by estimating a certain longitudinal slip parameter and correcting the rotation speed of the wheels using adaptive control techniques, as done in Gonzales et al. (2009), Iossaqui et al. (2011), Iossaqui and Camino (2013). Lateral slip, on the other hand, may have a worse influence on the trajectory tracking problem for this kind of robot. Since differential drive mobile robots lack actuation in the lateral direction, the lateral slip does not allow the robot posture (its position and orientation) tracking error to be canceled.

While some researchers have developed kinematic and dynamic controllers to deal with the lateral slip, such as Sidek and Sarkar (2008), Ryu and Agrawal (2011), our paper shows how a useful result from Lyapunov theory for perturbed nonlinear systems (see Khalil (2001)) can be used to analyze the robustness properties of a state feedback kinematic controller that has been designed to solve the robot's trajectory tracking control problem without considering the lateral slip. The main idea is to treat the lateral slip as a perturbation term in the nonlinear dynamics of the robot posture error and then show that a state feedback controller is able to make the evolution of the closed-loop dynamics uniformly ultimately bounded. The concept of ultimate boundedness has been explored in the context of control of robot manipulators in the past, as seen in Corless and Leitmann (1981), Qu and Dorsey (1991), Spong (1992), Koo and Kim (1994), but there has been few applications of this theory for analyzing the stability properties of mobile robot systems.

This paper is organized as follows. Section II presents the kinematic model of a differential drive wheeled mobile robot under lateral slip and the resulting posture error dynamics, when the robot is controlled by the proposed state feedback controller. In Section III, the posture error dynamics is analyzed, using a Lemma from Lyapunov theory, and its solution is shown to be (under some assumptions) uniformly ultimately bounded. Section IV presents numerical results that show the behavior of the controlled mobile robot under lateral slip. Finally, Section V contains some concluding remarks.

TRAJECTORY TRACKING PROBLEM FORMULATION

Kinematic model of a mobile robot under lateral slip

Figure 1 shows a schematic model of the robot. It is assumed that the robot is a rigid body with two independently actuated wheels. The robot posture $q_{\text{pos}} = (x_{\text{pos}}, y_{\text{pos}}, \theta_{\text{pos}})^T$ is given by the position and orientation of the robot local body-fixed frame $(x_1 y_1)$ with respect to an inertial frame $(x_0 y_0)$. The y_1 axis of the robot local frame is parallel to the axes of rotation of the wheels.

The robot translational speed v has components v_x and v_y in the robot local frame. Without slip, the lateral speed v_y is

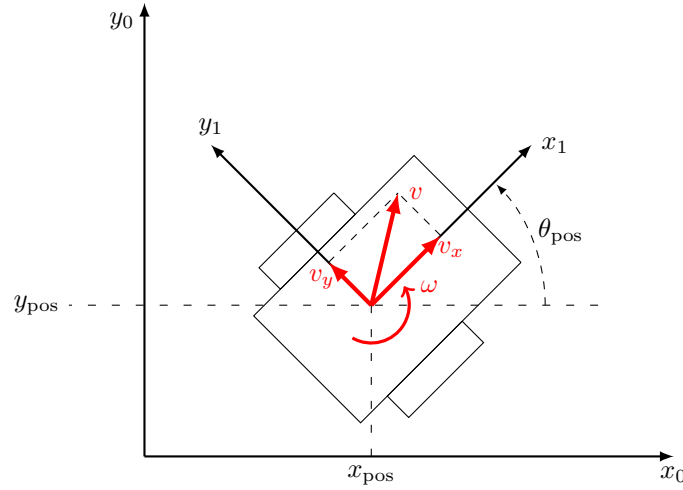


Figure 1 – Schematic representation of the robot on the plane.

zero and thus the translational speed is just $v = v_x$. The lateral slip is taken into consideration by assuming that a nonzero lateral speed v_y appears due to a perturbation term proportional to the forward speed v_x , that is,

$$v_y(t) = \sigma(t)v_x(t) \quad (1)$$

with $\sigma(t)$ being an unknown time-varying lateral slip parameter (see Wang and Low (2008)).

The time derivative of the robot posture $\dot{q}_{\text{pos}} = (\dot{x}_{\text{pos}}, \dot{y}_{\text{pos}}, \dot{\theta}_{\text{pos}})^T$ is obtained by writing $(v_x, v_y, \omega)^T$ in the inertial frame, i.e., by pre-multiplying $(v_x, v_y, \omega)^T$ by the usual rotation matrix. Performing this change of basis and using (1), the kinematic model of the robot under lateral slip is given by

$$\dot{q}_{\text{pos}} = \begin{pmatrix} v_x (\cos \theta_{\text{pos}} - \sigma \sin \theta_{\text{pos}}) \\ v_x (\sin \theta_{\text{pos}} + \sigma \cos \theta_{\text{pos}}) \\ \omega \end{pmatrix} \quad (2)$$

The reference posture $q_{\text{ref}} = (x_{\text{ref}}, y_{\text{ref}}, \theta_{\text{ref}})^T$, which the robot has to follow, written in the inertial frame, is defined using the same kinematic model without the lateral slip. Denoting by v_{ref} the forward reference speed and by ω_{ref} the angular reference speed, the kinematic model of the reference posture is given by

$$\dot{q}_{\text{ref}} = \begin{pmatrix} v_{\text{ref}} \cos \theta_{\text{ref}} \\ v_{\text{ref}} \sin \theta_{\text{ref}} \\ \omega_{\text{ref}} \end{pmatrix} \quad (3)$$

Kinematic control law for trajectory tracking

One of the main goals of the trajectory tracking control problem for wheeled mobile robots is to provide a kinematic control input (v_x, ω) such that

$$\lim_{t \rightarrow \infty} (q_{\text{ref}}(t) - q_{\text{pos}}(t)) = 0 \quad (4)$$

This problem is readily solved in the absence of slip. One such control law is presented in this section. However, whenever lateral slip occurs, this goal might no longer be achievable and the best one can hope is to ensure that the robot posture q_{pos} remains close, in some sense, to the reference posture q_{ref} .

To derive the posture error dynamics, which is used in our analysis, first the posture error $e(t)$, written in the robot local frame, is defined as the difference between the reference posture and the robot posture as follows:

$$e = \begin{pmatrix} \cos \theta_{\text{pos}} & \sin \theta_{\text{pos}} & 0 \\ -\sin \theta_{\text{pos}} & \cos \theta_{\text{pos}} & 0 \\ 0 & 0 & 1 \end{pmatrix} (q_{\text{ref}} - q_{\text{pos}}) \quad (5)$$

Now, differentiating (5) with respect to time, and using the kinematic models (2) and (3), the following error dynamics is obtained:

$$\dot{e} = \begin{pmatrix} \omega e_2 + v_{\text{ref}} \cos e_3 - v_x \\ -\omega e_1 + v_{\text{ref}} \sin e_3 - \sigma v_x \\ \omega_{\text{ref}} - \omega \end{pmatrix} \quad (6)$$

Neglecting the lateral slip, it will be seen that the kinematic control law (based on Fierro and Lewis (1997)) given by

$$\begin{aligned} v_x &= v_{\text{ref}} + k_1 e_1 \\ \omega &= \omega_{\text{ref}} + k_2 e_2 v_{\text{ref}} + k_3 e_3 \end{aligned} \quad (7)$$

with k_1, k_2 and k_3 positive constants, ensures that the zero equilibrium of the error dynamics is exponentially stable, meaning that (4) holds. However, in the presence of lateral slip, which is the focus of our analysis, the limit (4) no longer holds, as shown in the next section.

The closed-loop system, given by the dynamics (6) and the control law (7), becomes

$$\dot{e}(t) = f(t, e) + g(t, e) \quad (8)$$

with

$$f(t, e) = \begin{pmatrix} -k_1 e_1 + e_2(\omega_{\text{ref}}(t) + k_2 e_2 v_{\text{ref}}(t) + k_3 e_3) + v_{\text{ref}}(t)(\cos e_3 - 1) \\ -e_1(\omega_{\text{ref}}(t) + k_2 e_2 v_{\text{ref}}(t) + k_3 e_3 + k_1 \sigma(t)) + v_{\text{ref}}(t) \sin e_3 \\ -k_2 e_2 v_{\text{ref}}(t) - k_3 e_3 \end{pmatrix} \quad (9)$$

and

$$g(t, e) = \begin{pmatrix} 0 \\ -v_{\text{ref}}(t)\sigma(t) \\ 0 \end{pmatrix} \quad (10)$$

Note that the origin $e = 0$ is not an equilibrium point of the closed-loop nonlinear system (8) because the perturbation term $g(t, e) = g(t)$ does not necessarily vanish at the origin for every σ and v_{ref} . For this reason, $g(t)$ is called a nonvanishing perturbation of the nominal system given by $\dot{e} = f(t, e)$. When a nonlinear system is perturbed by a nonvanishing term, the stability of the origin of the perturbed system cannot be analyzed as an equilibrium point and several theorems from Lyapunov theory are not directly applicable.

ANALYSIS OF THE CLOSED-LOOP PERTURBED SYSTEM

Uniform ultimate boundedness

Although the origin of the closed-loop error dynamics, given by (8), is not an equilibrium point, it is still possible to show whether its solution $e(t)$ is bounded in a neighborhood of the origin. A stronger result than merely showing that a solution is bounded is to show it is *uniformly ultimately bounded*, as defined below

Definition 1 *The solution $e(t)$ of the nonlinear system (8) is uniformly ultimately bounded, with ultimate bound u_b , if there exist positive constants u_b and c , independent of t_0 , and, for every $e_0 \in (0, c)$, there is $T = T(e_0, u_b) \geq 0$, independent of t_0 , such that*

$$\|e(t_0)\| \leq e_0 \implies \|e(t)\| \leq u_b, \quad \forall t \geq t_0 + T$$

According to Definition 1, if it is shown that the posture error $e(t)$ is uniformly ultimately bounded, then it is guaranteed that it will remain within a certain distance of the origin after a certain amount of time has passed (provided that the initial condition is sufficiently small). Therefore, from (5), the robot posture q_{pos} will remain within a certain distance of the reference posture q_{ref} . Although the limit in (4) may not be achieved, the knowledge that q_{pos} will not diverge from q_{ref} when the robot is affected by lateral slip is very useful since it demonstrates that the nominal control law (7) has a certain degree of robustness with respect to the lateral slip.

In order to show that the solution of the posture error dynamics (8) is uniformly ultimately bounded, the following Lemma will be used.

Lemma 1 (Khalil (2001, Lemma 9.2)) *Let $e = 0$ be an exponentially stable equilibrium point of the nominal system $\dot{e} = f(t, e)$. Let $D = \{e \in \mathbb{R}^n \mid \|e\| < d\}$ and suppose that $V(t, e) : [0, \infty) \times D$ is a Lyapunov function for the nominal system that satisfies the conditions*

$$c_1 \|e(t)\|^2 \leq V(t, e) \leq c_2 \|e(t)\|^2 \quad (11)$$

$$\frac{\partial V(t, e)}{\partial t} + \frac{\partial V(t, e)}{\partial e} f(t, e) \leq -c_3 \|e(t)\|^2 \quad (12)$$

$$\left\| \frac{\partial V(t, e)}{\partial e} \right\| \leq c_4 \|e(t)\| \quad (13)$$

Suppose the perturbation term $g(t, e)$ satisfies

$$\|g(t, e)\| \leq \delta < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \beta d \quad (14)$$

for all $t \geq 0$, all $e \in D$, and some positive constant $\beta < 1$. Then, for all $\|e(t_0)\| < \sqrt{c_1/c_2} d$, the solution $e(t)$ of the perturbed system (8) satisfies

$$\begin{aligned} \|e(t)\| &\leq \alpha \exp(-\gamma(t - t_0)) \|e(t_0)\|, \quad \forall t_0 \leq t < t_0 + T, \\ \|e(t)\| &\leq u_b, \quad \forall t \geq t_0 + T, \end{aligned}$$

for some finite T , where

$$\alpha = \sqrt{\frac{c_2}{c_1}}, \quad \gamma = \frac{(1 - \beta)c_3}{2c_2}, \quad u_b = \frac{c_4}{c_3} \sqrt{\frac{c_2}{c_1}} \frac{\delta}{\beta}$$

Note that Lemma 1 not only guarantees that an ultimate bound exists for $e(t)$, but also that $e(t)$ will enter this bounded region with an exponential decay rate.

Stability of the nominal system

For the application of Lemma 1, the origin $e = 0$ must be an exponentially stable equilibrium point of the nominal system $\dot{e} = f(t, e)$. Clearly, the origin is an equilibrium point since $f(t, 0) = 0$ for all $t \geq 0$. To show that it is an exponentially stable equilibrium point, Lyapunov's linearization method (see (Khalil, 2001, Theorem 4.15)) can be applied, provided that the Jacobian matrix $\partial f / \partial e$ is bounded and Lipschitz on some neighborhood around the origin, which is the case since $v_{\text{ref}}(t)$, $\omega_{\text{ref}}(t)$ and $\sigma(t)$ are bounded by hypothesis. It is assumed that their finite bounds are

$$M_v \geq |v_{\text{ref}}(t)|, \quad M_\omega \geq |\omega_{\text{ref}}(t)|, \quad M_\sigma \geq |\sigma(t)|$$

According to Lyapunov's linearization method, the origin $e = 0$ is an exponentially stable equilibrium point of the nonlinear nominal system $\dot{e} = f(t, e)$ if it is an exponentially stable equilibrium point of the linearized nominal system $\dot{e} = A(t)e$, with $A(t)$ given by

$$A(t) = \left. \frac{\partial f}{\partial e}(t, e) \right|_{e=0} = \begin{pmatrix} -k_1 & \omega_{\text{ref}} & 0 \\ -k_1\sigma - \omega_{\text{ref}} & 0 & v_{\text{ref}} \\ 0 & -k_2v_{\text{ref}} & -k_3 \end{pmatrix} \quad (15)$$

Following the ideas in Iossaquí and Camino (2013), the origin of the linearized system $\dot{e} = A(t)e$ can be proven to be exponentially stable using Theorem 1 from Rosenbrock (1963) and the Liénard and Chipart stability criterion (see Gantmacher (1960)).

Lemma 2 (Rosenbrock (1963, Theorem 1)) *Let $\dot{e} = A(t)e$, where for all $t \geq t_0$ every element $a_{ij}(t)$ of $A(t)$ is differentiable and satisfies $|a_{ij}| \leq a$ and every eigenvalue λ of $A(t)$ satisfies*

$$\text{Re}[\lambda(A(t))] \leq -\epsilon < 0.$$

Then, there is some $\rho > 0$ (independent of t) such that if every $|\dot{a}_{ij}| \leq \rho$, the origin $e = 0$ is uniformly asymptotically stable, which, for linear systems, is equivalent to exponential stability (Khalil (2001)).

All elements of $A(t)$, given by (15), are clearly bounded. To show that the eigenvalues of $A(t)$ respect the condition stated in Lemma 2, namely, that their real parts are bounded above by a negative constant, consider the characteristic polynomial of $A(t)$, for frozen values of t , given by

$$\alpha_0 \lambda^3 + \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3 = 0$$

with

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= k_1 + k_3 \\ \alpha_2 &= k_2 v_{\text{ref}}^2 + \omega_{\text{ref}}^2 + k_1(k_3 + \sigma \omega_{\text{ref}}) \\ \alpha_3 &= k_1 k_2 v_{\text{ref}}^2 + k_3(\omega_{\text{ref}}^2 + k_1 \sigma \omega_{\text{ref}}) \end{aligned}$$

The Liénard and Chipart stability criterion guarantees that the real parts of the eigenvalues of $A(t)$ will be negative if the following conditions hold:

$$\begin{aligned} i) \quad &0 < \varepsilon_\ell \leq \alpha_\ell, \quad \ell = 1, 2, 3 \\ ii) \quad &0 < \varepsilon_4 \leq \alpha_1 \alpha_2 - \alpha_0 \alpha_3 = k_2 k_3 v_{\text{ref}}^2 + k_1^2(k_3 + \sigma \omega_{\text{ref}}) + k_1(k_3^2 + \omega_{\text{ref}}^2) \end{aligned} \quad (16)$$

It is assumed at this point that v_{ref} is bounded below by a positive constant μ , i.e., $0 < \mu \leq v_{\text{ref}}(t)$. When the gains of the control law (7) are chosen such that

$$\begin{aligned} k_1 &> 0 \\ k_2 &> \begin{cases} k_1 k_3 M_\sigma^2 / (4\mu^2), & \text{if } k_1 \leq 2M_\omega / M_\sigma \\ k_3 M_\omega M_\sigma / \mu^2, & \text{if } k_1 > 2M_\omega / M_\sigma \end{cases} \\ k_3 &> M_\sigma M_\omega \end{aligned} \quad (17)$$

then, conditions *i* and *ii* given by (16) hold. To see this, note first that the proposed choice of k_3 guarantees that the term $(k_3 + \sigma\omega_{\text{ref}})$ is always positive and thus condition *i*, for $\ell = 1, 2$, and condition *ii* are satisfied. Then, knowing that

$$\min_{\sigma, \omega_{\text{ref}}} \omega_{\text{ref}}^2 + k_1 \sigma \omega_{\text{ref}} = \begin{cases} -k_1^2 M_\sigma^2 / 4, & \text{if } k_1 \leq 2M_\omega / M_\sigma \\ M_\omega (M_\omega - k_1 M_\sigma), & \text{if } k_1 > 2M_\omega / M_\sigma \end{cases}$$

the proposed choice of k_2 , based on these minima, ensures that $0 < \varepsilon_3 \leq \alpha_3$. Thus, by the Liénard and Chipart stability criterion, the real parts of the eigenvalues of $A(t)$ are bounded above by a negative constant. Consequently, by Lemma 2, there is some $\rho > 0$ such that if every $|\dot{a}_{ij}| \leq \rho$, the origin $e = 0$ is an exponentially stable equilibrium point of the linearized nominal system $\dot{e} = A(t)e$. Although ρ is not known, by a proper choice of the design parameters in (15), i.e., $k_1, k_2, k_3, v_{\text{ref}}$ and ω_{ref} , the terms $|\dot{a}_{ij}|$ can be made arbitrarily small, provided that $\dot{\sigma}(t)$ is bounded.

With all these considerations, Lemma 2 leads to the conclusion that $e = 0$ is an exponentially stable equilibrium point of the linearized nominal system $\dot{e} = A(t)e$. Therefore, by Lyapunov's linearization method, the origin $e = 0$ is also an exponentially stable equilibrium point of the nonlinear nominal system $\dot{e} = f(t, e)$.

Ultimate boundedness of the posture error

Since it has been proven that the origin $e = 0$ is an exponentially stable equilibrium point of the nominal system $\dot{e} = f(t, e)$, with $f(t, e)$ given by (9), the Converse Lyapunov Theorem (Khalil, 2001, Theorem 4.14) guarantees there exists a Lyapunov function $V(t, e)$ for the nominal system satisfying conditions (11) to (13) of Lemma 1. Although the existence of some $V(t, e)$ is ensured, the values of c_i , in (11) to (13), can only be computed if a valid $V(t, e)$ is in fact provided. Nevertheless, the mere knowledge of the existence of $V(t, e)$ is enough to apply Lemma 1. To do so, first observe that the norm of the nonvanishing perturbation $g(t)$ from (10) obeys the inequality

$$\|g(t)\| \leq M_v M_\sigma$$

As the values of c_i in the inequality (14) are not known, it is not possible to determine exactly what values of $\delta = M_v M_\sigma$ would satisfy this inequality. But, for sufficiently small $M_v M_\sigma$, inequality (14) can always be satisfied. When this is the case, Lemma 1 leads to the conclusion that the solution $e(t)$ of the perturbed closed-loop system (8) is uniformly ultimately bounded, with the ultimate bound u_b given by

$$u_b = \frac{c_4}{c_3} \sqrt{\frac{c_2}{c_1} \frac{M_v M_\sigma}{\beta}}$$

It should be noticed that whenever the maximum magnitude of the lateral slip σ is expected to be important, the maximum reference speed v_{ref} needs to be low. This contributes not only to ensure that the posture error $e(t)$ will be uniformly ultimately bounded, but also that the ultimate bound will be small.

NUMERICAL RESULTS

In this section, five numerical simulations are presented for a differential drive mobile robot controlled by the kinematic law (7). In the first four simulations, the robot must follow the same reference trajectory but with different lateral slip affecting its wheels. The last simulation is used to show the advantage of reducing the maximum v_{ref} under moderate slip conditions.

The reference trajectory for the first four simulations is generated by numerically integrating the kinematic model (3) with the initial condition $q_{\text{ref}}(0) = (0, 0, 0)^T$ and the reference inputs $v_{\text{ref}}(t)$ and $\omega_{\text{ref}}(t)$ (adapted from Kanayama et al.

(1990)) given by

$$\begin{aligned}
 0 \text{ s} \leq t < 5 \text{ s} : & \quad v_{\text{ref}}(t) = 0.1 + 0.25(1 - \cos(\pi t/5)) & \text{and} \quad \omega_{\text{ref}}(t) = 0 \\
 5 \text{ s} \leq t < 20 \text{ s} : & \quad v_{\text{ref}}(t) = 0.6 & \text{and} \quad \omega_{\text{ref}}(t) = 0 \\
 20 \text{ s} \leq t < 25 \text{ s} : & \quad v_{\text{ref}}(t) = 0.1 + 0.25(1 + \cos(\pi t/5)) & \text{and} \quad \omega_{\text{ref}}(t) = 0 \\
 25 \text{ s} \leq t < 30 \text{ s} : & \quad v_{\text{ref}}(t) = 0.1 + 0.15\pi(1 - \cos(2\pi t/5)) & \text{and} \quad \omega_{\text{ref}}(t) = -0.1\pi(1 - \cos(2\pi t/5)) \\
 30 \text{ s} \leq t < 35 \text{ s} : & \quad v_{\text{ref}}(t) = 0.1 + 0.15\pi(1 - \cos(2\pi t/5)) & \text{and} \quad \omega_{\text{ref}}(t) = 0.1\pi(1 - \cos(2\pi t/5)) \\
 35 \text{ s} \leq t < 40 \text{ s} : & \quad v_{\text{ref}}(t) = 0.1 + 0.15\pi(1 - \cos(2\pi t/5)) & \text{and} \quad \omega_{\text{ref}}(t) = -0.1\pi(1 - \cos(2\pi t/5)) \\
 40 \text{ s} \leq t < 45 \text{ s} : & \quad v_{\text{ref}}(t) = 0.1 + 0.15\pi(1 - \cos(2\pi t/5)) & \text{and} \quad \omega_{\text{ref}}(t) = 0.1\pi(1 - \cos(2\pi t/5)) \\
 45 \text{ s} \leq t < 50 \text{ s} : & \quad v_{\text{ref}}(t) = 0.1 + 0.25(1 + \cos(\pi t/5)) & \text{and} \quad \omega_{\text{ref}}(t) = 0 \\
 50 \text{ s} \leq t : & \quad v_{\text{ref}}(t) = 0.6 & \text{and} \quad \omega_{\text{ref}}(t) = 0
 \end{aligned} \tag{18}$$

The reference inputs $v_{\text{ref}}(t)$ and $\omega_{\text{ref}}(t)$, given by (18), are shown in Figure 2. The reference trajectory for the fifth simulation is a straight line generated by integrating the kinematic model (3) with the initial condition $q_{\text{ref}}(0) = (0, 0, 0)^T$ and the reference inputs $v_{\text{ref}}(t) = 0.3$ and $\omega_{\text{ref}}(t) = 0$.

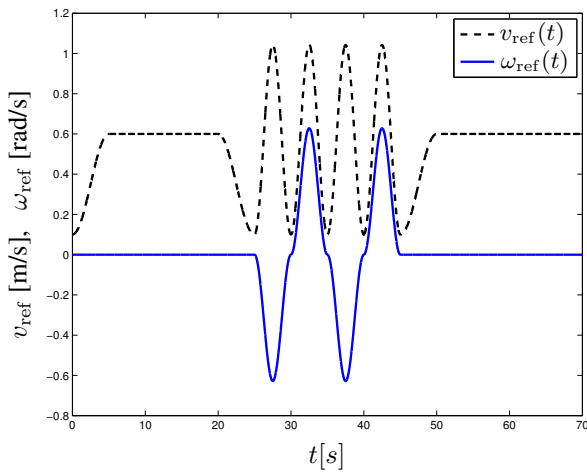


Figure 2 – Forward and angular reference speed.

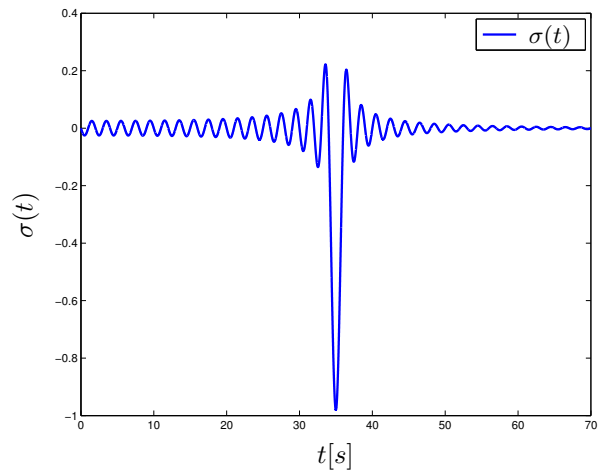


Figure 3 – Time-varying lateral slip.

The lateral slip parameters used for all simulations are given in Table 1. In simulation I, the robot is subject to the time-varying lateral slip shown in Figure 3. For simulations II through V, the lateral slip is constant. The upper bound M_σ on $|\sigma(t)|$ is taken as $M_\sigma = 1.0$ for all simulations.

Table 1 – Parameters used in each simulation.

Simulation	Reference inputs	Lateral slip parameter
I	v_{ref} and ω_{ref} given by (18)	$\sigma(t) = -2.8 e^{-0.03t} \text{sinc}(t - 35)$
II	v_{ref} and ω_{ref} given by (18)	$\sigma = 0.1$
III	v_{ref} and ω_{ref} given by (18)	$\sigma = 0.5$
IV	v_{ref} and ω_{ref} given by (18)	$\sigma = 1.0$
V	$v_{\text{ref}} = 0.3$ and $\omega_{\text{ref}} = 0$	$\sigma = 0.5$

From the reference inputs (18), it is possible to determine the lower bound $\mu = 0.1$ on $v_{\text{ref}}(t)$ and the upper bound $M_\omega = 0.7$ on $|\omega_{\text{ref}}(t)|$. Using the inequalities (17) and the chosen values for M_σ , M_ω and μ , it is now possible to select the controller gains for which the origin of the nominal system is stable. These gains will be the same for all simulations. For simplicity, k_1 is set to $k_1 = 1$ and the gains k_2 and k_3 are chosen such that they remain roughly 10% above the lower bounds enforced by (17), resulting in $k_2 = 22$ and $k_3 = 0.8$. The simulations are performed by numerically integrating the robot kinematic model (2) with v_x and ω given by the control law (7) with the above-mentioned gains and reference inputs. The simulations were run until $t = 800$ seconds. The robot posture initial condition is $q_{\text{pos}}(0) = (0.5, -0.75, -\pi/6)^T$. The relative and absolute tolerance used for the integration is 10^{-12} .

Figures 4 to 11 show the robot's trajectory (solid line) and the reference trajectory (dashed line). Figures 4 and 5, for simulation I, show that the robot is able to follow the given trajectory under the time-varying lateral slip. The peak slip

at 35 seconds, seen in Figure 3, occurs when the robot is turning, momentarily taking it out of its course, but the robot manages to recover well. Figures 6 and 7, for simulation II, in which the slip $\sigma = 0.1$ is small, show that the performance of the closed-loop system is quite satisfactory. However, as shown in Figures 8 and 9, for simulation III, when the lateral slip grows to $\sigma = 0.5$, the performance deteriorates and the robot begins to constantly oscillate around the reference trajectory. Finally, when the lateral slip is $\sigma = 1.0$ for simulation IV, the robot spins away from the reference trajectory, as shown in Figures 10 and 11.

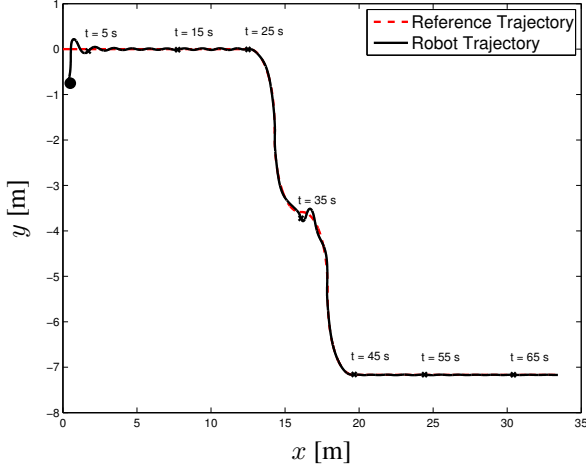


Figure 4 – Simulation I: x,y trajectory, time varying $\sigma(t)$.

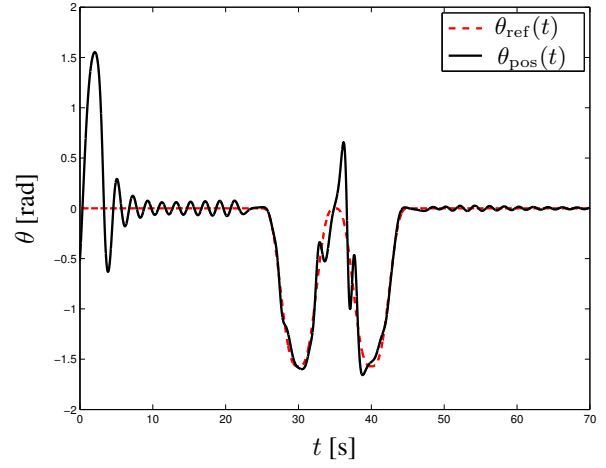


Figure 5 – Simulation I: θ trajectory, time varying $\sigma(t)$.

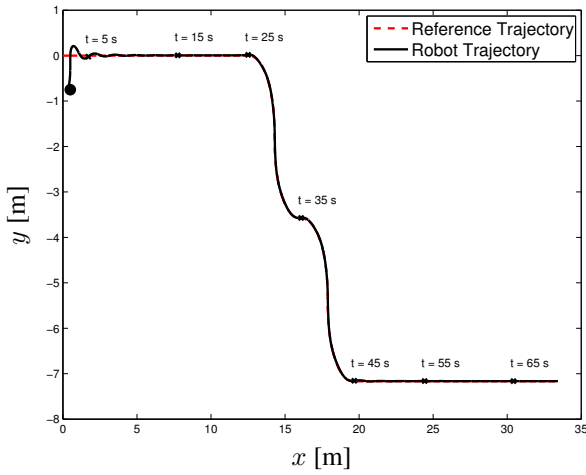


Figure 6 – Simulation II: x,y trajectory, $\sigma = 0.1$.

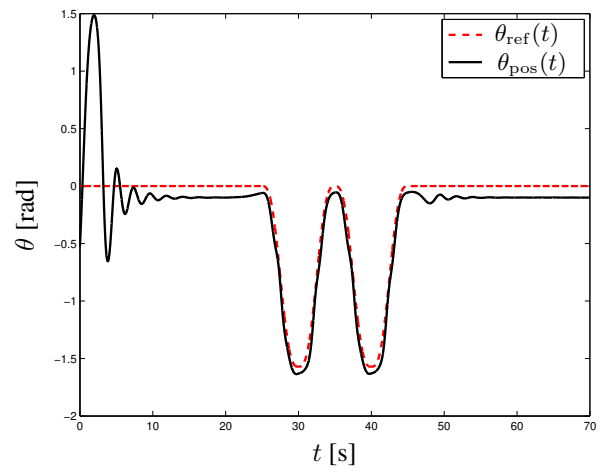


Figure 7 – Simulation II: θ trajectory, $\sigma = 0.1$.

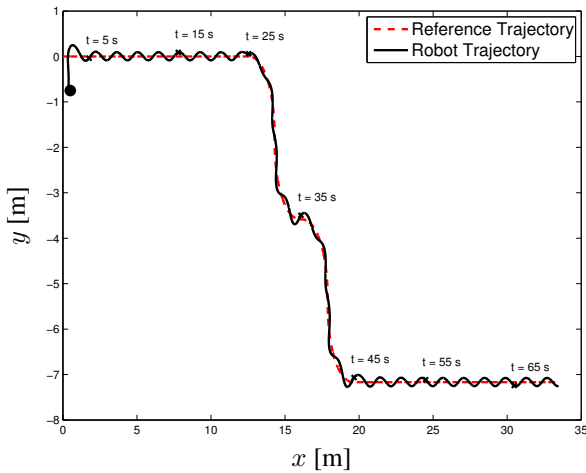


Figure 8 – Simulation III: x,y trajectory, $\sigma = 0.5$.

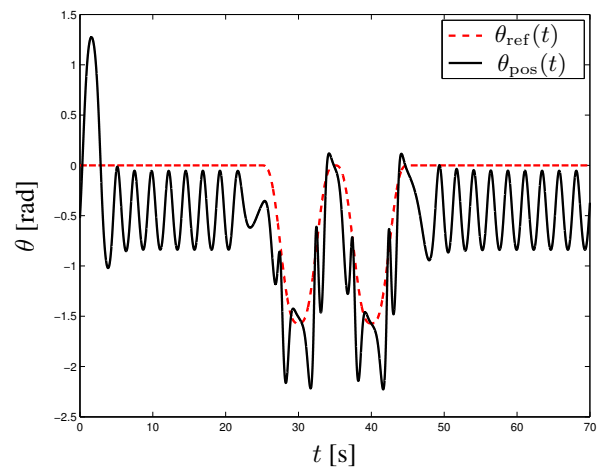
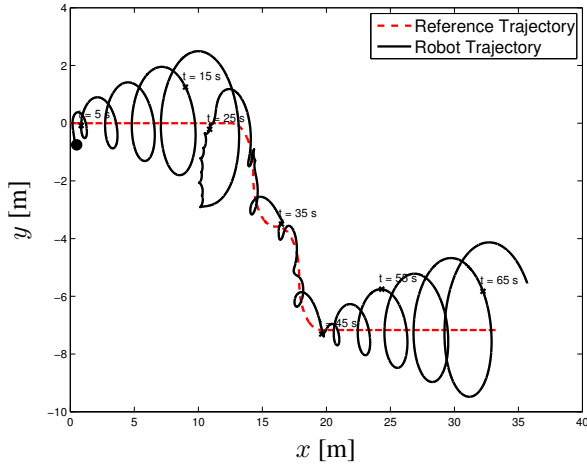
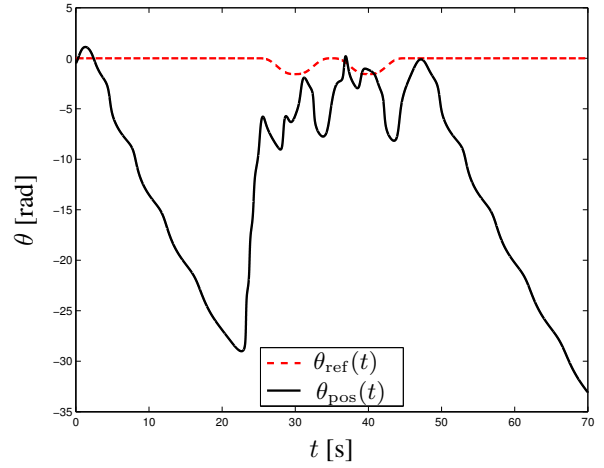


Figure 9 – Simulation III: θ trajectory, $\sigma = 0.5$.

A quantitative analysis can be made by investigating the steady-state behavior of the posture error. Table 2 shows the steady-state values, computed during the interval of time $t \in [770, 800]$ seconds, for each one of the error variables. It

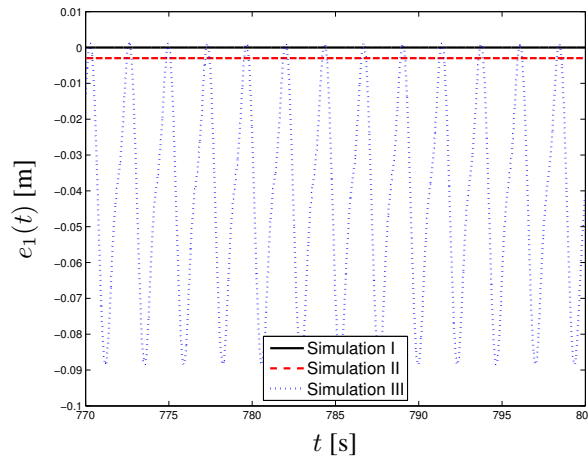

 Figure 10 – Simulation IV: x, y trajectory, $\sigma = 1.0$.

 Figure 11 – Simulation IV: θ trajectory, $\sigma = 1.0$.

can be seen that the steady-state error for simulation I is smaller than 10^{-12} , showing that the robot follows the reference trajectory perfectly whenever the slip is not present (the time-varying lateral slip $\sigma(t)$ is negligible for $t \geq 150$ seconds). As expected, there exists a steady-state error for simulation II since the lateral slip is now a constant value given by $\sigma = 0.1$. Note that the controller manages to keep a low error e_1 and e_2 with an angular error e_3 of approximately 0.1 radians (5.7 degrees). Thus, under small slip, the controller compensates the unexpected sideways motion by turning the robot toward the direction of the reference trajectory.

Table 2 – Steady-state posture error.

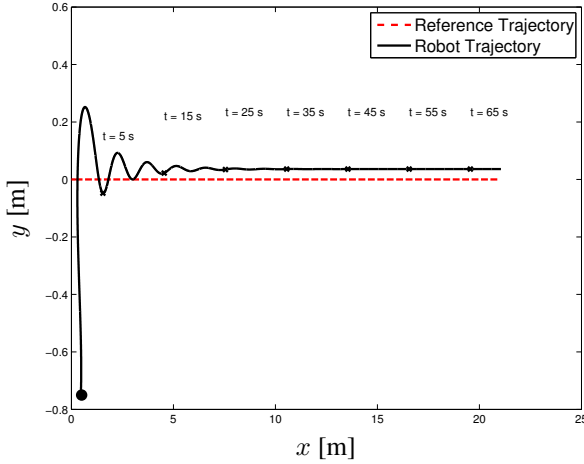
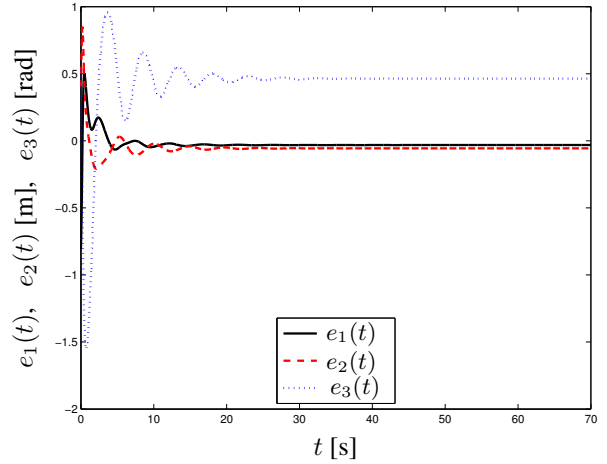
Simulation	$e_1(800)$	$e_2(800)$	$e_3(800)$
I	3.23×10^{-23}	-8.28×10^{-13}	4.53×10^{-13}
II	-2.98×10^{-3}	-6.04×10^{-3}	9.97×10^{-2}
III	oscillatory behavior min: -8.84×10^{-2} max: 1.04×10^{-3}	oscillatory behavior min: -1.00×10^{-1} max: 6.13×10^{-2}	oscillatory behavior min: 6.30×10^{-2} max: 8.30×10^{-1}
V	-3.16×10^{-2}	-5.62×10^{-2}	4.63×10^{-1}

For simulation III, in which the robot is affected by a moderate lateral slip ($\sigma = 0.5$), the performance deteriorates significantly, causing an oscillatory behavior in the posture error, as seen in Figures 8 and 9. Figure 12 show the oscillatory behavior of $e_1(t)$ at steady-state, i.e., during the interval of time $t \in [770, 800]$ seconds. The figures for $e_2(t)$ and $e_3(t)$ are omitted since they have a similar pattern. The min and max values shown in the third line of Table 2 denote, respectively, the lowest and highest amplitude values for the oscillatory posture error at steady-state.


 Figure 12 – Steady-state value of $e_1(t)$ for simulations I, II, and III.

According to the theory presented in the previous section, by reducing the upper bound M_v on the reference speed v_{ref} , the ultimate bound u_b decreases. Thus, it is expected that the oscillations that appear in simulation III at steady-state

can be reduced by decreasing v_{ref} . With that in mind, simulation V was performed using the same slip as simulation III, $\sigma = 0.5$, and the reference speed given by $v_{\text{ref}} = 0.3$, which is half the steady-state speed used for simulation III (notice that $v_{\text{ref}}(t) = 0.6$ and $\omega_{\text{ref}}(t) = 0$ for $t \geq 50$ seconds for the reference inputs given by (18)). Figures 13 and 14 show, respectively, the robot behavior on the x_0y_0 plane and the posture error for simulation V. The steady-state values are shown in the fourth line of Table 2. Note that the steady-state oscillatory behavior of the posture error does not occur for simulation V. Moreover, the magnitude of each component $|e_i(800)|$ of the posture error for simulation V is about half the magnitude of the worst case deviation from equilibrium, $\max_t |e_i(t)|$ with $t \in [770, 800]$, for simulation III. Thus, for these simulations, the region around the origin in which the error $e(t)$ remains enclosed is smaller when the reference speed v_r is lower.

Figure 13 – Simulation V: x, y trajectory, $\sigma = 0.5$.Figure 14 – Simulation V: posture error $e(t)$.

CONCLUSION

This paper presents a framework for analyzing the robustness property of a state feedback control law with respect to a perturbation introduced by the lateral slip in the robot's wheels. It was shown that even though the control design did not take into account the lateral slip, the control law could ensure that the robot's posture error was uniformly ultimately bounded, provided that the maximum reference speed was sufficiently small. In practice, as seen in the numerical section, this means that the robot, under lateral slip, can still follow the trajectory, in the sense that it remains in a certain neighborhood of the origin. The same framework can be used to analyze the robustness of other types of mobile robot control laws with respect to the lateral slip.

In applications where it is desirable to know a numeric value for the maximum allowed upper bounds M_v and M_σ , a valid Lyapunov function $V(t, e)$ needs to be found for the nominal system, so that the values of c_i , from (11) to (13), can be computed. Since many Lyapunov functions for a given system might exist, the values of M_v and M_σ may not be unique and some can be more conservative than others.

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