MAS115: SEMESTER 1 MINI PROJECT

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ANNIE THE ANT

This problem is about determining the vertex at which Annie ends up after walking seven steps around a pentagon and covering each side of the pentagon in just one step. Further, Annie is very moody and so she can reverse her direction of the movement after stopping at any vertex. She always starts her journey from vertex 0.

Program to simulate Annie's journey:

```
import random
random.seed()
repetitions=1000000
steps=7
sides=5
#(A row matrix that will measure the number of times Annie stops at each vertex.)
frequency=[0]*sides

# (The following loop generates 1 million simulations of the journey.)
for i in range(repetitions):
  vertex_int=0
  #(The following loop generates numbers 1 and −1 randomly and adds those random numbers.)
  for j in range(steps):
    a=random.randrange(−1,3,2)
    vertex_int=vertex_int+a

  vertex=vertex_int%sides #(Taking modulo to the number of sides)
  frequency[vertex]=frequency[vertex]+1

print(frequency)
for i in range(sides):
  print("The percentage of total times Annie is stopping at vertex", i, "is", (frequency[i]/repetitions)*100,"%")
```

Here, 1 million simulations of the journey of Annie around the pentagon, each time taking 7 steps, have been created. The matrix 'frequency' records

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the number of times Annie stops at each vertex at the end. We are following the convention that each anti-clockwise step is denoted by -1 and each clockwise step is denoted by +1. The variable 'vertex_int' stores the sum of the random numbers. This variable is actually tracking the movement of Annie and her each clockwise and anti-clockwise movement is being stored here until she has covered 7 steps. The resultant sum is her net movement.

Taking modulo to the number of sides then gives us the vertex at which Annie finally stops after taking 7 steps.

Output:

```
[108558, 273590, 172321, 171706, 273825]
The percentage of total times Annie is stopping at vertex 0 is
    10.8558 %
The percentage of total times Annie is stopping at vertex 1 is
    27.359 %
The percentage of total times Annie is stopping at vertex 2 is
    17.2321 %
The percentage of total times Annie is stopping at vertex 3 is
    17.1706 %
The percentage of total times Annie is stopping at vertex 4 is
    27.3825 %
```

Discussion 1: *Annie stops at vertices 1 and 4 more often as compared to other vertices*

Let X be the random variable that denotes the vertex at which Annie ends up. Now, X in this case can take all the values in the set $\{0, 1, 2, 3, 4\}$. Also, let $p_X(x)$ be the probability mass function of the random variable X.

It can be clearly seen that X=x only when the total sum of each digit of the string made up of 1's and -1's modulo the number of sides equals x.

For example the string '1111111' denotes that Annie took seven clockwise steps and the string '1(-1)111(-1)1' denotes that Annie first took a step in the clockwise direction, then a step in the anti-clockwise direction, then 3 steps in clockwise direction, then a step in the anti-clockwise direction followed by a step in the clockwise direction. Note also that the resultant sum of the digits of this string: 1+1+1+1+1+(-1)+(-1)=3 is the number of the vertex at which Annie stops after completing her journey.

Now assume that Annie stops at vertex x after seven steps. If the 7-digit string contains u 1's and v (-1)'s, then u,v will satisfy the following 2 equations:

$$u - v = x \tag{1}$$

$$u + v = 7 \tag{2}$$

Solving the above 2 equations simultaneously we get $u = \frac{x+7}{2}$ and $v = \frac{7-x}{2}$.

Now a problem occurs if we consider $x = 0$ or any even number. But in this case, we can always express an even numbered vertex as an odd numbered vertex by subtracting '5' from it since subtracting by 5 would mean that Annie takes a round in anti-clockwise direction starting from that vertex and ends at that vertex again. Hence, if x is odd then the string consists of $\frac{x+7}{2}$ 1's and $\frac{7-x}{2}$ (-1)'s. But if x is even then the string consists of $\frac{(x-5)+7}{2} = \frac{x+2}{2}$ 1's and $\frac{7-(x-5)}{2} = \frac{12-x}{2}$ (-1)'s.

Now,

$$p_X(x) = \frac{\text{Number of 7-digit strings in which the sum of the digits is } x}{\text{Total number of 7-digit strings possible}}$$

If x is odd then number of 7-digit strings in which the sum of the digits is x = Number of strings with $\frac{x+7}{2}$ 1's and $\frac{7-x}{2}$ (-1)'s $=^7 C_{\frac{x+7}{2}}$.

Similarly if x is even then number of strings in which the sum of the digits is x=Number of strings of length 7 with $\frac{x+2}{2}$ 1's and $\frac{12-x}{2}$ (-1)'s $={}^7C_{\frac{x+2}{2}}$.

As we have 2 choices for each place in the string of length 7, 1 and -1, so the total number of strings possible = 2.2.2.2.2.2.2=2^7.

Hence, if x is odd then $p_X(x) = \frac{{}^7C_{\frac{x+7}{2}}}{2^7}$ and if x is even then $p_X(x) = \frac{{}^7C_{\frac{x+2}{2}}}{2^7}$.

The p.m.f $p_X(x)$ is maximum when the binomial coefficient ${}^7C_{\frac{x+7}{2}}$ or ${}^7C_{\frac{x+2}{2}}$ is maximum. If ${}^nC_k$ is any binomial coefficient then ${}^nC_k$ assumes the maximum value at $k = \lceil \frac{n}{2} \rceil$ or $k = \lfloor \frac{n}{2} \rfloor$. Here n=7 so k=3 or 4. Hence, $p_X(x)$ is maximum when $\frac{x+7}{2} = 3$ or 4 and $\frac{x+2}{2} = 3$ or 4. Solving both these cases we get that $p_X(x)$ takes the maximum value at $x = 1$ and $x = 4$. Hence, Annie tends to stop at vertices 1 and 4 more often.

In general, if we have s sides and Annie takes n steps then the sum of the digits of the string of length n will be x when there are $\frac{n+x+2m.s}{2}$ 1's and $\frac{n-x-2m.s}{2}$ (-1)'s $\forall m \in \mathbb{Z}$ such that $\frac{-(n+x)}{2s} \leq m \leq \frac{n-x}{2s}$ if n and x have the same parity. If n and x have different parity then the string will consist of $\frac{n+x-(2p-1)s}{2}$ 1's and $\frac{n-x+(2p-1)s}{2}$ (-1)'s $\forall p \in \mathbb{Z}$ such that $\frac{s-n-x}{2s} \leq p \leq \frac{n-x+s}{2s}$.

Discussion 2: *Annie doesn't stop at any even-numbered vertex when she takes odd number of steps around a polygon having even number of sides.*

Let s be even, n be odd and let x be an even number i.e. assume Annie stops at an even numbered vertex after taking odd number of steps around a polygon having even number of sides. Clearly, x and n have different parity so the string will consist of $\frac{n+x-(2p-1)s}{2}$ 1's and $\frac{n-x+(2p-1)s}{2}$ (-1)'s. The number $n + x - (2p - 1)s$ is odd and so is $n - x + (2p - 1)s$. Hence, $\frac{n+x-(2p-1)s}{2}$ and $\frac{n-x+(2p-1)s}{2}$ won't be integers. Hence, it is not possible for the n-digit string to have $\frac{n+x-(2p-1)s}{2}$ 1's and $\frac{n-x+(2p-1)s}{2}$ (-1)'s. So, it is not possible for x to be even and Annie to stop at an even numbered vertex.

Discussion 3: *Annie doesn't stop at any odd-numbered vertex when she takes even number of steps around a polygon having even number of sides.*

Following the same conventions, we get that the parity of n and x are different again as n is even and x is odd in this case. Now, $n+x-(2p-1)s$ and $n-x+(2p-1)s$ are odd again as $(2p-1)s$ is even and and so it is impossible to have a string with $\frac{n+x-(2p-1)s}{2}$ 1's and $\frac{n-x+(2p-1)s}{2}$ (-1)'s. Hence, Annie doesn't stop at any odd-numbered vertex in this case.

Discussion 4: *When Annie takes a large number of steps around the pentagon, she has the same probability to stop at each vertex.*

If $n \to \infty$, then for any $r \in \mathbb{N}$ and $r < n$

$$\frac{\sum_{m=0}^{\lfloor \frac{n}{r} \rfloor} {}^nC_{mr}}{\sum_{m=0}^{\lfloor \frac{n}{r} \rfloor} {}^nC_{mr+1}} = \frac{\sum_{m=0}^{\lfloor \frac{n}{r} \rfloor} {}^nC_{mr+1}}{\sum_{m=0}^{\lfloor \frac{n}{r} \rfloor} {}^nC_{mr+2}} = \ldots\ldots = \frac{\sum_{m=0}^{\lfloor \frac{n}{r} \rfloor} {}^nC_{mr+r-2}}{\sum_{m=0}^{\lfloor \frac{n}{r} \rfloor} {}^nC_{mr+r-1}} = 1 \tag{3}$$

and hence, when n is large and r is small:

$$\sum_{m=0}^{\lfloor \frac{n}{r} \rfloor} {}^nC_{mr} \approx \sum_{m=0}^{\lfloor \frac{n}{r} \rfloor} {}^nC_{mr+1} \approx \sum_{m=0}^{\lfloor \frac{n}{r} \rfloor} {}^nC_{mr+2} \approx \ldots\ldots \approx \sum_{m=0}^{\lfloor \frac{n}{r} \rfloor} {}^nC_{mr+r-1} \tag{4}$$

In this discussion we are considering that Annie takes a large number of steps around the pentagon. Let the number of steps be 99. Hence, n=99.

$$\text{As } p_X(x) = \frac{\text{Number of n-digit strings in which the sum of the digits is } x}{\text{Total number of n-digits strings possible using only 1 and -1}}$$

and total number of n-digit strings made of only 1 and -1 is 2^n so

x	0	1	2	3	4
$p_X(x)$	$\frac{\sum_{m=0}^{19} {}^{99}C_{5m+2}}{2^{99}}$	$\frac{\sum_{m=0}^{19} {}^{99}C_{5m}}{2^{99}}$	$\frac{\sum_{m=0}^{19} {}^{99}C_{5m+3}}{2^{99}}$	$\frac{\sum_{m=0}^{19} {}^{99}C_{5m+1}}{2^{99}}$	$\frac{\sum_{m=0}^{19} {}^{99}C_{5m+4}}{2^{99}}$

Clearly the p.m.f is similar to equation (4) and therefore

$$\sum_{m=0}^{19} {}^{99}C_{5m} \approx \sum_{m=0}^{19} {}^{99}C_{5m+2} \approx \sum_{m=0}^{19} {}^{99}C_{5m+3} \approx \sum_{m=0}^{19} {}^{99}C_{5m+1} \approx \sum_{m=0}^{19} {}^{99}C_{5m+4}.$$

and so $p_X(0) \approx p_X(1) \approx p_X(2) \approx p_X(3) \approx p_X(4)$. Hence, when we carry out a large number of simulations of Annie's journey around the pentagon in which Annie takes a large number of steps in each journey, we observe that Annie stops at each vertex almost equal number of times.

This is also true for any polygon having odd or even number of sides but in case of a polygon having even numbers of sides we have to consider the conditions discussed in Discussion 2 and 3.