Using Computational Algebraic Geometry for Graph Coloring Problems

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Introduction to Graph Colorings

Definition

Let $G = (V, E)$ be a graph. The k-colorability problem is the problem of finding a function $c : V \to \{1, \ldots, k\}$ satisfying

$$c(u) \neq c(v) \forall \{u, v\} \in E.$$

If such a function exists, we call G k-colorable and c a (proper) k-coloring of G.

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Definition

The chromatic number $\chi(G)$ of a graph G is defined as the smallest number k such that G is k-colorable, i.e.,

$$\chi(G) := \min\{k \in \mathbb{N} : G \text{ is } k\text{-colorable}\}$$

Why Graph Coloring?

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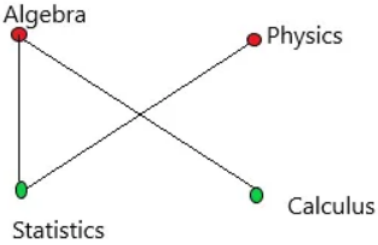
For example,
Exam Scheduling: Suppose algebra, physics, statistics and calculus are four courses of study in our college. If algebra and statistics have common students, then they can't be held on the same day. How do we schedule exams in minimum number of days so that courses having common students are not held on the same day?

Why Graph Coloring?

First draw a graph with courses as vertex and they are connected by edges if they have common students. Second color the graph such that no two adjacent vertices are assigned the same color as shown below



Look at the above graph. It solves our problem. We can conduct exam of courses on same day if they have same color.

Complexity of Graph Coloring

- Graph coloring is NP-complete for $k \geq 3$. This means that there is no known polynomial-time algorithm to determine if a graph is k-colorable for $k \geq 3$. The problem becomes increasingly difficult as k increases.
- The time complexity of finding a k-coloring for $k \geq 3$ is exponential in the number of vertices $|V|$ of the graph. Specifically, it is $O(k^{|V|})$.

Algebraic Perspective on Graph Colorings

Algebraic Perspective on Graph Colorings

Proposition

Let G = (V, E) be a graph, and let $k \in \mathbb{N}$, G is k-colorable if and only if the polynomial system of equations in $\mathbb{C}[x_1, \ldots, x_n]$

$$x_v^k - 1 = 0 \,\forall\, v \in V \quad \text{and} \quad \sum_{i=0}^{k-1} x_u^{k-1-i} x_v^i = 0 \,\forall\, \{u, v\} \in E$$

has a solution. We will denote this set of polynomials by $\mathcal{F}_G$, and the ideal spanned by its elements by $\mathcal{I}_G$.

Proof :

Notation : To simplify notation, we call

$$v_i := x_i^k - 1$$

the vertex polynomial for the vertex i and

$$e_{u,v} := \sum_{i=0}^{k-1} x_u^{k-1-i} x_v^i$$

the edge polynomial for the edge u, v.

$\Rightarrow$ Let $c : \{1, \ldots, n\} \to \{1, \ldots, k\}$ be a proper k-coloring. Set $x_i^* := \zeta^{c(i)} \forall i$, where $\zeta := e^{\frac{2\pi i}{k}}$ is the k-th root of unity. Then

- $v_i(x^*) = \left(\zeta^{c(i)}\right)^k - 1 = \left(\zeta^k\right)^{c(i)} - 1 = 1^k - 1 = 0$ and

$$e_{i,j}(x^*) = \frac{(x_i^*)^k - \left(x_j^*\right)^k}{x_i^* - x_j^*} = \frac{0}{x_i^* - x_j^*} = 0$$

for all $i \in V$ and $\{i, j\} \in E$. Therefore, $x^* \in \mathcal{V}(\mathcal{I}_G) \neq \emptyset$.
$\Leftarrow$ Let $\mathcal{V}(\mathcal{I}_G)$ be non-empty, and $x^* \in \mathcal{V}(\mathcal{I}_G)$. The equalities $v_i(x^*) = (x_i^*)^k - 1 = 0$ imply that the components of x^* are k-th roots of unity, that is,

$$\forall i \, \exists \, c_i \in \{1, \dots, k\} \text{ such that } x_i^* = \zeta^{c_i} \, .$$

Define $c : \{1, \dots, n\} \to \{1, \dots, k\}$ by $c(i) := c_i$. Then c is a coloring function for G, and

$$\begin{aligned} c_i = c_j \Rightarrow \quad & x_i^* = \zeta^{c_i} = \zeta^{c_j} = x_j^* \\ \Rightarrow \quad & e_{i,j}(x^*) = \sum_{l=1}^{k} (x_i^*)^k = k \cdot x_i^* \neq 0 \\ \Rightarrow \quad & e_{i,j} \notin \mathcal{I}_G \quad \Rightarrow \quad \{i, j\} \notin E, \end{aligned}$$

which means that c is a proper k-coloring of G.

Algorithm

Graph $G = (V, E)$

Ideal $\mathcal{I}_G$

$$\left\langle \begin{matrix} x_i^k - 1 \; \forall i \in V \\ \sum_{i=0}^{k-1} x_u^{k-1-i} x_v^i \; \forall (u,v) \in E \end{matrix} \right\rangle$$

Gröbner basis $\mathcal{G}_G$

$$\begin{gathered} \{x_1 + x_2 + x_3, \\ x_2^2 + x_2x_3 + x_3x_4 + x_4^2, \\ x_3^2 + x_3x_4 + x_4^2, x_4^3 + 1\} \end{gathered}$$

Is G k-colorable?

$$\begin{cases} \text{Yes} & \text{if } \mathcal{G}_G \neq \{1\} \\ \text{No} & \text{otherwise} \end{cases}$$

Fig. 0.1: Overall approach for graph coloring

One Step Closer

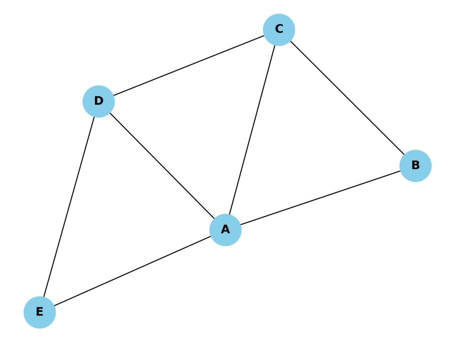
We have special cases of the graph coloring problem that can be solved in polynomial time.

Preliminaries

Definition

A chordal graph is one in which all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.

Example :

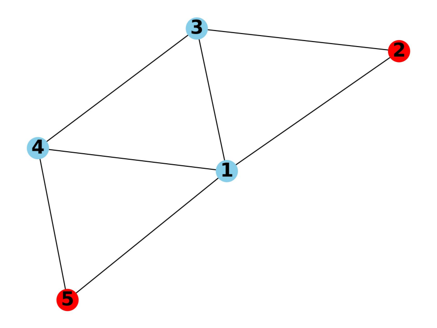


Preliminaries

Definition

A simplicial vertex is one whose neighbors form a clique.

Example:

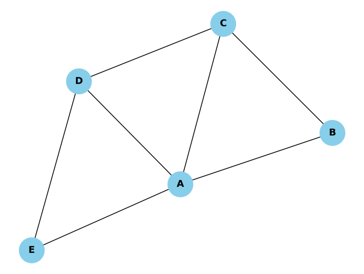


Preliminaries

Definition

A graph is recursively simplicial if it contains a simplicial vertex v which can be removed such that the remaining subgraph is again recursively simplicial.

Example :



Preliminaries

Theorem

A graph is chordal if and only if it is recursively simplicial.

Preliminaries

Definition

Let $G = (V, E)$ be a graph, and let $V = \{1, \ldots, n\}$.The vertex order is called a perfect elimination ordering if $\mathcal{N}(v) \cap \{1, \ldots, v-1\}$ is a complete graph $\forall v \in V$.

Lemma

A vertex order on a graph G is a perfect elimination order if and only if it can be used to show that G is recursively simplicial.

Preliminaries

Lemma

A vertex order on a graph G is a perfect elimination order if and only if it can be used to show that G is recursively simplicial.

Corollary

A graph is chordal if and only if it has a perfect elimination ordering.

Preliminaries

Definition

The *k-th elementary symmetric polynomial* $\sigma_k(x_1, \dots, x_n)$ over n variables is

$$\sigma_k(x_1, \dots, x_n) := \sum_{1 \le j_1 < \cdots < j_k} x_{j_1} \cdots x_{j_k}$$

The *k-th complete homogeneous symmetric polynomial* $S_k(x_1, \dots, x_n)$ over n variables is

$$S_k(x_1, \dots, x_n) := \sum_{1 \le j_1 \le \cdots \le j_k \le n} x_{j_1} \cdots x_{j_k} \quad .$$

Preliminaries

Lemma

For any $k, r \in \mathbb{N}$, where $k > r$, and for any partial assignment $\{x_{c_1} = \zeta_1, \ldots, x_{c_r} = \zeta_r\}$ of r distinct k-th roots of unity to variables in $\mathbb{K}[x_1, \ldots, x_n]$, there is a polynomial $p \in \mathbb{K}[x_1, \ldots, x_{n+1}]$ which is homogeneous of degree $k - r$, such that

$$p(x_1, \ldots, x_{n+1}) = 0 \iff x_{n+1} \in R_k \setminus \{\zeta_1, \ldots, \zeta_r\},$$

i.e., the solutions of $x_n + 1$ in p are exactly the $k - r$ other roots of unity.

Preliminaries

Proof : Without loss of generality, assume that $c_i = i \ \forall i \in \{1, \ldots, r\}$, and set $x := x_n + 1$. We show that the complete homogeneous symmetric polynomial $p := S_k - r(x_1, \ldots, x_r, x)$ is the polynomial we are looking for. First, note that it suffices to prove that

$$S_{k-r}(x_1, \ldots, x_r, x) \cdot (x - x_1) \cdots (x - x_r) = x^k - 1$$

Preliminaries

Hence, we have found $k - r$ distinct roots of a $(k - r)$-dimensional univariate polynomial and this ensures that there are no other roots of p. Now, consider the degree d-homogeneous polynomial $\sigma_i(x_1, \ldots, x_r)S_{d-i}(x_1, \ldots, x_r)$. For every monomial x^α with $|\alpha| = d$ and $|\sup(\alpha)| = m$, its coefficient is the number of square-free factors of degree i, that is, $\binom{m}{i}$. Summing up these coefficients over d with alternating signs gives that the coefficient of x^α in

$$\sum_{i=0}^{d}(-1)^{d-i}\sigma_i(x_1, \ldots, x_r)S_{d-i}(x_1, \ldots, x_r)$$

is

$$\sum_{i=0}^{m}(-1)^{d-i}\binom{m}{i} = 0 \quad .$$

Preliminaries

Therefore,

$$\sum_{i=0}^{d}(-1)^{d-i}\sigma_i(x_1,\dots,x_r)S_{d-i}(x_1,\dots,x_r) = 0 \;\forall\; d \in \{0,\dots,k-1\}\,.$$

Using the specific values of the assignment for $x_1,\dots,x_k$, we see moreover that

$$\sum_{i=0}^{d}\sigma_i(\zeta_1,\dots,\zeta_r)\sigma_{d-i}(\zeta_{r+1},\dots,\zeta_k) = \sigma_d(\zeta_1,\dots,\zeta_k) = 0$$

$\forall d \in \{1,\dots,k-1\}$

Equating these sums for $d = 1$ and using the fact that $\sigma_0(\zeta_{r+1}, \ldots, \zeta_k) = 1 = S_0(x_1, \ldots, x_r)$ gives

$$(-1)^d S_d(\zeta_1, \ldots, \zeta_r) = \sigma_d(\zeta_{r+1}, \ldots, \zeta_k)$$

for the case $d = 1$. Now we increase d by 1 and insert the last equation to yield the same equality for $d = 2$, and so on up to $d = k - 1$.

$$\begin{aligned}
&S_{k-r}(\zeta_1, \ldots, \zeta_r, x) \cdot \prod_{i=1}^{r}(x - \zeta_i) \\
&= \sum_{d=0}^{k-r} S_d(\zeta_1, \ldots, \zeta_r) x^{k-r-d} \cdot \prod_{i=1}^{r}(x - \zeta_i) \\
&= \sum_{d=0}^{k-r} (-1)^d \sigma_d(\zeta_{r+1}, \ldots, \zeta_k) x^{k-r-d} \cdot \prod_{i=1}^{r}(x - \zeta_i)
\end{aligned}$$

Preliminaries

$$
\begin{aligned}
S_{k-r}(\zeta_1, \ldots, \zeta_r, x) \cdot \prod_{i=1}^{r}(x - \zeta_i) &= \prod_{i=r+1}^{k}(x - \zeta_i) \cdot \prod_{i=1}^{r}(x - \zeta_i) \\
&= \prod_{i=1}^{k}(x - \zeta_i) \\
&= x^k - 1,
\end{aligned}
$$

which is what we claimed

Algorithm

Algorithm 1: Gröbner Basis of a Chordal Graph

Input : Chordal graph $G = (V, E)$, coloring number k
Output: Gröbner basis $\mathcal{G}$ of size $|V|$
Function BUILDGROBNERBASIS(G, k)**:**

```
n ← |V|;
G_n ← G;
𝒢 ← {v_n};
for i ← n − 1 to 1 do
    for all v ∈ V_{i+1} do
        if IsSIMPLICIAL(v) then
            v_i ← v;
            C_i ← 𝒩(v);
            G_i ← G_{i+1} − v;
            𝒢 ← 𝒢 ∪ {S_{k−|C_i|}(C_i, v_i)};
            exit for;
        end
    end
end
return 𝒢;
end
```

Algorithm

Algorithm 2: Testing a vertex for simpliciality

Input : Graph $G = (V, E)$, vertex $v \in V$
Output: **true** if v is simplicial in G, else **false**
Function $\texttt{IsSIMPLICIAL}(G, v)$**:**

```
    d ← deg(v);
    for each n ∈ 𝒩(v) do
        if |𝒩(v) ∩ 𝒩(n)| < d − 1 then
            return false;
        end
    end
    return true;
end
```

Correctness

Theorem

Let G be a graph on n vertices, and let $\succ$ be a term order. Let $C = \{c_1, \ldots, c_r\}$ be an r-clique in G, and choose a Gröbner basis $\mathcal{G}$ of $\mathcal{I}_G$. Then, with

$$p = S_{k-r}(x_{c_1}, \ldots, x_{c_r}, x_{n+1}),$$

$$\langle \mathcal{G}, p \rangle = \langle \mathcal{G}, v_{n+1}, e_{c_1,n+1}, \ldots, e_{c_r,n+1} \rangle = \mathcal{I}_{G+C} \ .$$

Correctness

Proof : We show that $\langle \mathcal{G}, p \rangle$ is a radical ideal, and that both ideals generate the same variety. Then the claim follows from the bijection between varieties and radical ideals.

- Since $p = \prod_{i=r+1}^{k}(x - x_i)$ is square-free, we know that $\langle p \rangle$ is a radical ideal. The same holds for $\langle \mathcal{G} \rangle$ as the coloring ideal of a graph. But then

  $$\mathrm{rad}(\langle \mathcal{G}, p \rangle) = \mathrm{rad}(\langle \mathcal{G} \rangle \cap \langle p \rangle) = \mathrm{rad}(\langle \mathcal{G} \rangle) \cap \mathrm{rad}(\langle p \rangle) = \langle \mathcal{G} \rangle \cap \langle p \rangle = \langle \mathcal{G}, p \rangle$$

  as claimed.
- Let $x = (x_1, \ldots, x_{n+1}) \in \mathcal{V}(\langle \mathcal{G}, p \rangle)$. By definition, $x_n + 1$ is a k-th root of unity, and therefore $v_n + 1 = 0$. Moreover, $x_n + 1 \neq x_{c_i} \forall i \in \{1, \ldots, r\}$, which implies that $e_{c_i, n+1} = 0$. In sum, $x \in \mathcal{V}(\mathcal{I}_{G+C})$.
  Let now $x = (x_1, \ldots, x_{n+1}) \in \mathcal{V}(\mathcal{I}_{G+C})$. Then the generator polynomials $v_{n+1, c_1, n+1}, \ldots, e_{c_r, n+1}$ ensure that $x_{n+1} \in R_k \setminus \{x_{c_1}, \ldots, x_{c_r}\}$, hence $p(x) = 0$ and $x \in \mathcal{V}(\langle \mathcal{G}, p \rangle)$.

Correctness

Theorem

For every Gröbner basis $\mathcal{G}$ of $\mathcal{I}_G$ with respect to term order $\succ$, $\mathcal{G} \cup \{p\}$ is a Gröbner basis of $\mathcal{I}_{G+C}$ with respect to an extended term order $\succ'$, where $p = S_{k-r}(x_{c_1}, \ldots, x_{c_r}, x_{n+1})$

Proof : Previous lemma shows that $\langle \mathcal{G}, p \rangle = \mathcal{I}_{G+C}$. Hence, it is left to show that all S- polynomials in $\mathcal{G} \cup \{p\}$ reduce to 0. We only have to consider S-pairs that involve the new polynomial p.
By definition of $\succ'$, we have that $LM_{\succ'}(p) = x_{n+1}^{k-r}$, which is relatively prime to all $g \in \mathcal{G}$, since $x_n + 1$ does not appear in this basis. Therefore,

$$S(g, p) \rightarrow_{\mathcal{G} \cup \{p\}} 0 \quad \forall g \in \mathcal{G}$$

by Lemma 6.
This is suffcient for $\mathcal{G}' := \mathcal{G} \cup \{p\}$ to be a Gröbner Basis.

Correctness

Theorem

Upon termination of BUILDGROBNERBASIS(G), the set $\mathcal{G}$ is a Gröbner basis for $\mathcal{I}_G$ under the Lex order, where the vertices are ordered in the perfect elimination order that was established in the algorithm.

Proof. Note that $\{p_1 := v_n\}$ is a Gröbner basis for G_1. By Previous Lemma, this basis can be extended in $n-1$ steps by adding p_i as constructed in the algorithm. Therefore, $\mathcal{G} = \{p_1, \ldots, p_n\}$ is a Gröbner basis of $G_n = G$ with respect to the extended vertex order, which concludes the proof.

Complexity

Theorem

As we have seen above, exactly one polynomial is added to $\mathcal{G}$ for every vertex of G. But what is the degree and length of these polynomials? From the defnition of $p := S_k(x_1, \ldots, x_n)$, we see that len$(p) = \binom{k+n-1}{n-1}$ and $\deg(p) = k$. Therefore, we add polynomials S_i with len$(S_i) = \binom{k}{|C_i|}$ and $\deg(S_i) = k - |C_i|$. Note that, for a fixed number k of colors, both numbers are polynomials.

Running Time

Theorem

The function IsSIMPLICIAL consists of an outer loop with exactly n iterations, in each of which the intersection of two subsets of V is formed. Such an intersection can be computed in linear time, therefore the function runs in time $\mathcal{O}(n^2)$.
In the main function BUIILDGROBNERBASIS, the two nested for-loops are traversed $\mathcal{O}(n)$ times each, and every time IsSIMPLICIAL is called. The main part of the if-case is the assignment of B. Building the polynomial $S_{k-|C_i|}(C_i, v_i)$ takes $(k-r)\cdot\binom{k}{r}$ steps, which is clearly in $\mathcal{O}(k \cdot k!)$. The remaining statements in the loop have running time $\mathcal{O}(n^2)$. Putting the pieces together, we obtain a total running time of

$$\mathcal{O}\left(n^2(k \cdot k! + n^2)\right) \quad ,$$

which is polynomial in n for fixed k.

Applications of Graph Coloring

- **Map Coloring:** Coloring regions of a map (countries, states) with different colors such that adjacent regions have different colors.
- **Job Scheduling:** Scheduling tasks or processes without conflicts (tasks represented as vertices, conflicts as edges).
- **Frequency Assignment:** Assigning frequencies to transmitters in a wireless network to avoid interference.
- **Register Allocation:** Allocating registers in a compiler to variables in code to minimize register usage.
- **Timetable Scheduling:** Scheduling classes or events to minimize conflicts in a school timetable.
- **Resource Allocation:** Allocating resources (like bandwidth) in networks or systems to avoid overlaps.
- **Sudoku and Puzzles:** Solving puzzles that involve constraints on placement of elements (numbers, symbols).

Using Graph Coloring to Solve Sudoku

	2				8			7
				1			4	
4	7					1		9
	5			2				
			9					
		6			5			
1		3						
					6			
			7			8		

- **Sudoku Representation**
  - Sudoku can be represented as a graph where each cell is a vertex.
  - Constraints (row, column, and block constraints) create edges between vertices.
- **Graph Coloring Approach**
  - Use graph coloring to assign numbers (colors) to vertices (cells) such that no two adjacent vertices have the same color (number).
  - Each color (number) represents a unique value (1 to 9 for standard Sudoku).

Thank You

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Any questions?