NCERT Question 10.5.2.5

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Question 10.5.2.5 : Find the number of terms in each of the following APs :
(i) 7, 13, 19, ... 205
(ii) 18, $15\frac{1}{2}$, 13, ... -47

Solution :

(i) The n^{th} term of the Arithmetic progression is given as a + (n-1)d where a is the first term and d is the common difference.
The common difference of the AP is given by the difference between successive terms.

Common difference (d) = 13 - 7 = 6
First term (a) = 7

If 205 is the *nth* term of the series, we have :

$$205 = 7 + (n-1)6 \tag{1}$$
$$\implies 198 = (n-1)6 \tag{2}$$
$$\implies 33 = n-1 \tag{3}$$
$$\implies n = 34 \tag{4}$$

Answer : There are 34 elements in the series.

(ii) The n^{th} term of the Arithmetic progression is given as a + (n-1)*d where a is the first term and d is the common difference.
The common difference of the AP is given by the difference between successive terms.

Common difference (d) = $15\frac{1}{2}$ - 18 = $-2\frac{1}{2}$
First term (a) = 18

If -47 is the *nth* term of the series, we have :

$$-47 = 18 + (n-1)\left(-2\left(\frac{1}{2}\right)\right) \tag{5}$$
$$\implies -65 = (n-1)\left(-2\left(\frac{1}{2}\right)\right) \tag{6}$$
$$\implies 26 = n-1 \tag{7}$$
$$\implies n = 27 \tag{8}$$

Answer : There are 27 elements in the series.

Question : Express the n^{th} term in each case as x(n) and find its z transform.

Solution :

(i) The n^{th} term of the Arithmetic progression (T_n) is given as a + (n-1)d where a is the first term and d is the common difference.

$$\therefore T_n = 7 + (n-1)6 \tag{9}$$
$$x(n) = 7 + (n-1)6 \tag{10}$$

The Z transformation for x[n] is given by :

$$X(z) = \left[\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right] \tag{11}$$

However, x[n] cannot be summed from $-\infty$ to zero as the number of terms cannot be negative due to which the n^{th} will not be defined for this range of n.
So, we will modify x[n] by multiplying it with unit step function(u(n)) so we have the value as zero for n<1.

$$x(n) = (7 + (n-1)6)\, u(n) \tag{12}$$
$$x(n) = \begin{cases} 0 & \text{for } n < 1 \\ 7 + (n-1)6 & \text{for } n \geq 1 \end{cases} \tag{13}$$

Now, using the above function in equation (11):

$$X(z) = \sum_{n=1}^{\infty} x[n]z^{-n} \tag{14}$$
$$= \sum_{n=1}^{\infty}(a + (n-1)d)z^{-n} \tag{15}$$
$$= \left[\sum_{n=1}^{\infty} az^{-n}\right] + \left[\sum_{n=1}^{\infty}(nd)z^{-n}\right] - \left[\sum_{n=1}^{\infty} dz^{-n}\right] \tag{16}$$

$$= \left[\sum_{n=1}^{\infty} 7z^{-n}\right] + \left[\sum_{n=1}^{\infty}(6n)z^{-n}\right] - \left[\sum_{n=1}^{\infty} 6z^{-n}\right] \tag{17}$$

$$= \left[\sum_{n=1}^{\infty} z^{-n}\right] + \left[\sum_{n=1}^{\infty}(6n)z^{-n}\right] \tag{18}$$

$$= \frac{1}{z-1} + 6\left[\sum_{n=1}^{\infty}(n)z^{-n}\right] \tag{19}$$

Calculating the integral in the above expression :

$$S = \left[\sum_{n=1}^{\infty}(n)z^{-n}\right] \tag{20}$$

$$S = z^{-1} + 2z^{-2} + 3z^{-3} + ... \tag{21}$$

$$Sz^{-1} = z^{-2} + 2z^{-3} + 3z{-}4 + ... \tag{22}$$

Subtracting (22) and (21)

$$S(1 - z^{-1}) = z^{-1} + z^{-2} + z^{-3} + ... \tag{23}$$

$$= \frac{1}{(z-1)(1-z^{-1})} \tag{24}$$

$$= \frac{1}{z - 2 + z^{-1}} \tag{25}$$

$$\therefore S = \frac{z}{(z-1)^2} \tag{26}$$

Using equations (26) and (19) :

$$X(z) = \frac{1}{z-1} + \frac{6z}{(z-1)^2} \tag{27}$$

$$= \frac{7z-1}{(z-1)^2} \tag{28}$$

Answer : The z transformation for x(n) where x(n) is the n^{th} term of the AP is $\frac{7z-1}{(z-1)^2}$.

(ii) The n^{th} term of the Arithmetic progression (T_n) is given as a + (n-1)d where a is the first term and d is the common difference.

$\therefore T_n = 18 + (n-1)\left(-2\frac{1}{2}\right)$

$x(n) = 18 + (n-1)\left(-2\frac{1}{2}\right)$

The Z transformation for x[n] is given by :

$$X(z) = \left[\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right] \tag{29}$$

However, x[n] cannot be summed from $-\infty$ to zero as the number of terms cannot be negative due to which the n^{th} will not be defined for this range of n.

So, we will modify x[n] by multiplying it with unit step function(u(n)) so we have the value as zero for n<1.

$$x(n) = \left(18 + (n-1)\left(-2\frac{1}{2}\right)\right)u(n) \tag{30}$$

$$x(n) = \begin{cases} 0 & \text{for } n < 1 \\ 18 + (n-1)\left(-2\frac{1}{2}\right) & \text{for } n \geq 1 \end{cases} \tag{31}$$

$$X(z) = \sum_{n=1}^{\infty} x[n]z^{-n} \tag{32}$$

$$= \sum_{n=1}^{\infty}(a + (n-1)d)z^{-n} \tag{33}$$

$$= \left[\sum_{n=1}^{\infty} az^{-n}\right] + \left[\sum_{n=1}^{\infty}(nd)z^{-n}\right] - \left[\sum_{n=1}^{\infty} dz^{-n}\right] \tag{34}$$

$$= \left[\sum_{n=1}^{\infty} 18z^{-n}\right] + \left[\sum_{n=1}^{\infty}(n)\left(-2\frac{1}{2}\right)z^{-n}\right] - \left[\sum_{n=1}^{\infty}\left(-2\frac{1}{2}\right)z^{-n}\right] \tag{35}$$

$$= \left[\sum_{n=1}^{\infty}\left(20\frac{1}{2}\right)z^{-n}\right] - \left[\sum_{n=1}^{\infty}\left(n\left(2\frac{1}{2}\right)\right)z^{-n}\right] \tag{36}$$

$$= \left(20\frac{1}{2}\right)\left[\sum_{n=1}^{\infty} z^{-n}\right] - \left[\sum_{n=1}^{\infty}\left(n\left(2\frac{1}{2}\right)\right)z^{-n}\right] \tag{37}$$

$$= \left(20\frac{1}{2}\right)\frac{1}{z-1} - \left(2\frac{1}{2}\right)\left[\sum_{n=1}^{\infty}(n)z^{-n}\right] \tag{38}$$

$$= \left(20\frac{1}{2}\right)\frac{1}{z-1} - \left(2\frac{1}{2}\right)S \tag{39}$$

Using equation (26) and (39) :

$$X(z) = \left(20\frac{1}{2}\right)\frac{1}{z-1} - \left(2\frac{1}{2}\right)\frac{z}{(z-1)^2} \tag{40}$$

$$= \frac{18z - \left(20\frac{1}{2}\right)}{(z-1)^2} \tag{41}$$

Answer : The z transformation for x(n) where x(n) is the n^{th} term of the AP is $\frac{18z-\left(20\frac{1}{2}\right)}{(z-1)^2}$.

Question : Plot the graph of x(n) and find the ROC of X(z) in each case.

Solution :

(i) The graph of x(n) is :

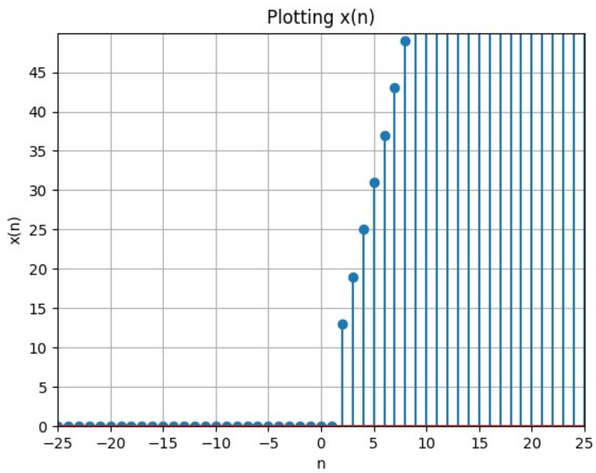

Fig. 0. Plot of x(n)

The ROC (Region of Convergence) of x(n) is defined as the range of values of z for which X(z) will converge where X(z) is the z transform of x(n).

$$|X(z)| = \sum_{n=-\infty}^{\infty} \left|x(n)z^{-n}\right| < \infty \tag{42}$$

By equation (18) :

$$X(z) = \left[\sum_{n=1}^{\infty} z^{-n}\right] + 6\left[\sum_{n=1}^{\infty} (n)z^{-n}\right] \tag{43}$$

The sum $[\sum_{n=1}^{\infty} z^{-n}]$ will converge only if z is not zero and $\left|z^{-1}\right| < 1$ (or $|z| > 1$) as it is forming an infinite GP with common difference z^{-1}.

Observe the second part of the equation (43)(or 18) is 6S which was calculated using equation (23) :

$$S(1 - z^{-1}) = z^{-1} + z^{-2} + z^{-3} + ... \tag{44}$$

The sum S will converge only if z is not 0 and $\left|z^{-1}\right| < 1$ as again we obtain an infinite GP with common difference z^{-1}.For both the parts of the equation (45), the ROC is same.

Answer : The ROC of X(z) is $|z| > 1$

(ii) The graph of x(n) is :

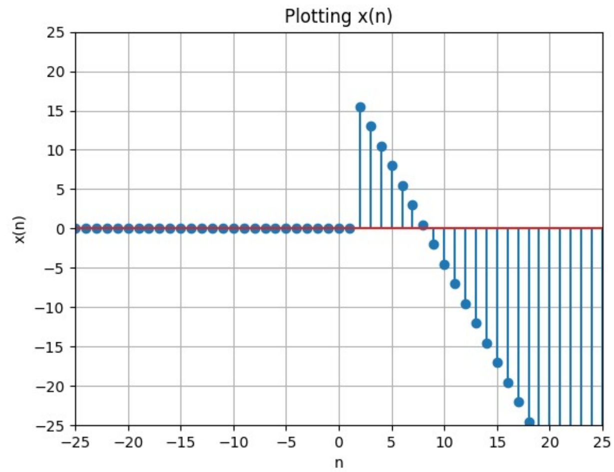

Fig. 1. Plot of x(n)

The ROC (Region of Convergence) of x(n) is defined as the range of values of z for which X(z) will converge where X(z) is the z transform of x(n).

$$|X(z)| = \sum_{n=-\infty}^{\infty} \left|x(n)z^{-n}\right| < \infty \tag{45}$$

By equation (37) :

$$X(z) = \left(20\frac{1}{2}\right)\left[\sum_{n=1}^{\infty} z^{-n}\right] - \left(2\frac{1}{2}\right)\left[\sum_{n=1}^{\infty} (n)\, z^{-n}\right] \tag{46}$$

The sum $[\sum_{n=1}^{\infty} z^{-n}]$ will converge only if z is not zero and $\left|z^{-1}\right| < 1$ (or $|z| > 1$) as it is forming an infinite GP with common difference z^{-1}.

Observe the second part of the equation (46)(or 37) is $-2\frac{1}{2}$S which was calculated using equation (23) :

$$S(1 - z^{-1}) = z^{-1} + z^{-2} + z^{-3} + ... \tag{47}$$

The sum S will converge only if z is not 0 and $|z| < 1$ as again we obtain an infinite GP with common difference z^{-1}.For both the parts of the equation (47), the ROC is same.

Answer : The ROC of X(z) is $|z| > 1$