NCERT Question 10.5.2.5

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Question 10.5.2.5 : Find the number of terms in each of the following APs :

(i) 7, 13, 19, ... 205

(ii) 18, $15\frac{1}{2}$, 13, ... -47

Solution :

Parameter	Used to denote
a	First term of AP
d	Common difference
x(n)	n^{th} term of series
X(z)	z transform of x(n)
u $[n]$	discrete unit step function
S	$[\sum_{n=0}^{\infty} nz^{-n}]$
ROC	Region of Convergence

TABLE 0
PARAMETER TABLE

(i)

The n^{th} term of the Arithmetic progression is given as a + nd.

The common difference of the AP is given by the difference between successive terms.

Common difference (d) = 13 - 7 = 6
First term (a) = 7

If 205 is the nth term of the series, we have :

$$205 = 7 + (n)\,6 \tag{1}$$
$$\implies 198 = 6n \tag{2}$$
$$\implies 33 = n \tag{3}$$

$\therefore$ n goes from 0 to 33.
Answer : There are 34 elements in the series.

(ii) The n^{th} term of the Arithmetic progression is given as a + nd.

The common difference of the AP is given by the difference between successive terms.

Common difference (d) = $15\frac{1}{2}$ - 18 = -$2\frac{1}{2}$
First term (a) = 18

If -47 is the n^{th} term of the series, we have :

$$-47 = 18 + (n)\left(-2\left(\frac{1}{2}\right)\right) \tag{4}$$
$$\implies -65 = (n)\left(-2\left(\frac{1}{2}\right)\right) \tag{5}$$
$$\implies n = 26 \tag{6}$$

$\therefore$ n goes from 0 to 26.
Answer : There are 27 elements in the series.

Question : Express the n^{th} term in each case as x(n) and find its z transform.

Solution :

(i) The n^{th} term of the Arithmetic progression is given as a + (n)d .

$$x(n) = 7 + (n)\,6 \tag{7}$$

The Z transformation for x $[n]$ is given by :

$$X(z) = \left[\sum_{n=-\infty}^{\infty} x[n]\, z^{-n}\right] \tag{8}$$

However, x(n) cannot be summed from $-\infty$ to zero as the number of terms cannot be negative due to which the n^{th} will not be defined for this range of n.
So, we will modify x $[n]$ by multiplying it with discrete unit step function u(n) so we have the value as zero for n<0.

$$\implies x(n) = (7 + (n)\,6)\, u[n] \tag{9}$$
$$x(n) = \begin{cases} 0 & \text{for } n < 0 \\ 7 + (n)\,6 & \text{for } n \geq 0 \end{cases} \tag{10}$$

Now, using the above function in equation (8):

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} \tag{11}$$
$$= \sum_{n=0}^{\infty} (a + nd)\, z^{-n} \tag{12}$$
$$= \left[\sum_{n=0}^{\infty} az^{-n}\right] + \left[\sum_{n=0}^{\infty} ndz^{-n}\right] \tag{13}$$
$$= \left[\sum_{n=0}^{\infty} 7z^{-n}\right] + \left[\sum_{n=0}^{\infty} 6nz^{-n}\right] \tag{14}$$
$$= 7\frac{z}{z-1} + 6\left[\sum_{n=0}^{\infty} nz^{-n}\right] \tag{15}$$

Calculating the integral in the above expression :

$$S = \left[\sum_{n=0}^{\infty} nz^{-n}\right] \tag{16}$$
$$S = z^{-1} + 2z^{-2} + 3z^{-3} + ... \tag{17}$$
$$S z^{-1} = z^{-2} + 2z^{-3} + 3z{-4} + ... \tag{18}$$

Subtracting (17) and (18)

$$S\left(1 - z^{-1}\right) = z^{-1} + z^{-2} + z^{-3} + ... \tag{19}$$
$$= \frac{1}{(z-1)(1-z^{-1})} \tag{20}$$
$$= \frac{1}{z - 2 + z^{-1}} \tag{21}$$
$$\therefore S = \frac{z}{(z-1)^2} \tag{22}$$

Using equations (15) and (22) :

$$X(z) = \frac{7z}{z-1} + \frac{6z}{(z-1)^2} \tag{23}$$
$$= \frac{7z^2 - z}{(z-1)^2} \tag{24}$$

Answer : The z transformation for x(n) where x(n) is the n^{th} term of the AP is $\frac{7z^2-z}{(z-1)^2}$.

(ii) The n^{th} term of the Arithmetic progression is given as a + nd.

$x(n) = 18 + n\left(-2\frac{1}{2}\right)$

The Z transformation for x $[n]$ is given by :

$$X(z) = \left[\sum_{n=-\infty}^{\infty} x[n]\, z^{-n}\right] \tag{25}$$

However, x $[n]$ cannot be summed from $-\infty$ to zero as the number of terms cannot be negative due to which the n^{th} will not be defined for this range of n.

So, we will modify x $[n]$ by multiplying it with unit step function u $[n]$ so we have the value as zero for n<0.

$$x(n) = \left(18 + n\left(-2\frac{1}{2}\right)\right) u(n) \tag{26}$$

$$x(n) = \begin{cases} 0 & \text{for } n < 0 \\ 18 + n\left(-2\frac{1}{2}\right) & \text{for } n \geq 0 \end{cases} \tag{27}$$

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} \tag{28}$$
$$= \sum_{n=0}^{\infty} (a + nd)\, z^{-n} \tag{29}$$
$$= \left[\sum_{n=0}^{\infty} az^{-n}\right] + \left[\sum_{n=0}^{\infty} ndz^{-n}\right] \tag{30}$$
$$= \left[\sum_{n=0}^{\infty} 18z^{-n}\right] + \left[\sum_{n=0}^{\infty} n\left(-2\frac{1}{2}\right) z^{-n}\right] \tag{31}$$
$$= \left[\sum_{n=0}^{\infty} 18z^{-n}\right] - \left[\sum_{n=0}^{\infty} \left(n\left(2\frac{1}{2}\right)\right) z^{-n}\right] \tag{32}$$
$$= 18\left[\sum_{n=0}^{\infty} z^{-n}\right] - \left[\sum_{n=0}^{\infty} \left(n\left(2\frac{1}{2}\right)\right) z^{-n}\right] \tag{33}$$
$$= 18\frac{z}{z-1} - \left(2\frac{1}{2}\right)\left[\sum_{n=0}^{\infty} nz^{-n}\right] \tag{34}$$
$$= 18\frac{z}{z-1} - \left(2\frac{1}{2}\right) S \tag{35}$$

Using equation (22) and (35) :

$$X(z) = \frac{18z}{z-1} - \left(2\frac{1}{2}\right)\frac{z}{(z-1)^2} \tag{36}$$
$$= \frac{18z^2 - \left(20z\frac{1}{2}\right)}{(z-1)^2} \tag{37}$$

Answer : The z transformation for x(n) where x(n) is the n^{th} term of the AP is $\frac{18z^2-\left(20\frac{1}{2}\right)z}{(z-1)^2}$.

Question : Plot the graph of x(n) and find the ROC of X(z) in each case.

Solution :

(i) The graph of x(n) is :

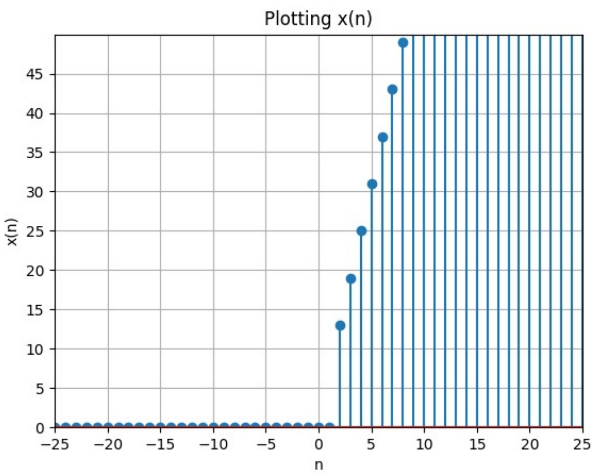

Fig. 0. Plot of x(n)

The ROC of x(n) is defined as the range of values of z for which X(z) will converge where X(z) is the z transform of x(n).

$$|X(z)| = \sum_{n=-\infty}^{\infty} |x(n) z^{-n}| < \infty \tag{38}$$

By equation (15) :

$$X(z) = 7\left[\sum_{n=0}^{\infty} z^{-n}\right] + 6\left[\sum_{n=0}^{\infty} nz^{-n}\right] \tag{39}$$

The sum $(\sum_{n=0}^{\infty} z^{-n})$ will converge only if z is not zero and $|z^{-1}| < 1$ as it is forming an infinite GP with common difference z^{-1}.

Observe the second part of the equation (39) or (15) is 6S which was calculated using equation (22) :

$$S\left(1 - z^{-1}\right) = z^{-1} + z^{-2} + z^{-3} + ... \tag{40}$$

The sum S will converge only if z is not 0 and $|z^{-1}| < 1$ or $|z| > 1$ as again we obtain an infinite GP with common difference z^{-1}.For both the parts of the equation (39), the ROC is same.

Answer : The ROC of X(z) is $|z| > 1$

(ii) The graph of x(n) is :

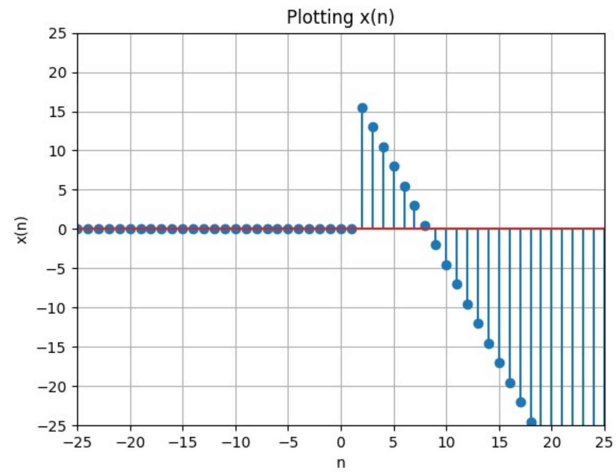

Fig. 1. Plot of x(n)

The ROC of x(n) is defined as the range of values of z for which X(z) will converge where X(z) is the z transform of x(n).

$$|X(z)| = \sum_{n=-\infty}^{\infty} |x(n) z^{-n}| < \infty \tag{41}$$

By equation (33) :

$$X(z) = 18\left[\sum_{n=0}^{\infty} z^{-n}\right] - \left(2\frac{1}{2}\right)\left[\sum_{n=0}^{\infty} nz^{-n}\right] \tag{42}$$

The sum $[\sum_{n=0}^{\infty} z^{-n}]$ will converge only if z is not zero and $\left|z^{-1}\right| < 1$ as it is forming an infinite GP with common difference z^{-1}.

Observe the second part of the equation (33) or (42) is $-2\frac{1}{2}$S which was calculated using equation (22) :

$$S\left(1 - z^{-1}\right) = z^{-1} + z^{-2} + z^{-3} + ... \tag{43}$$

The sum S will converge only if z is not 0 and $|z^{-1}| < 1$ or $|z| > 1$ as again we obtain an infinite GP with common difference z^{-1}.For both the parts of the equation (42), the ROC is same.

Answer : The ROC of X(z) is $|z| > 1$