

NCERT Question 10.5.2.5

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Question 10.5.2.5 : Find the number of terms in each of the following APs :

- (i) 7, 13, 19, ... 205
(ii) 18, $15\frac{1}{2}$, 13, ... -47

Solution :

Parameter	Used to denote	1 st AP	2 nd AP
$x(n)$	n^{th} term of series	$7 + 6n$	$18 - 2n\frac{1}{2}$
$x(0)$	First term of AP	7	18
d	Common difference	6	$-2\frac{1}{2}$
$X(z)$	z transform of $x(n)$	To find	To find
ROC	Region of Convergence	To find	To find

TABLE 0
PARAMETER TABLE

(i)

The n^{th} term of the Arithmetic progression ($x(n)$) is given as $x(0) + nd$.

The common difference of the AP is given by the difference between successive terms.

$$\text{Common difference } (d) = 13 - 7 = 6$$

$$\text{First term } (x(0)) = 7$$

If 205 is the n^{th} term of the series, we have :

$$\begin{aligned} 205 &= 7 + (n)6 & (1) \\ \Rightarrow 198 &= 6n & (2) \\ \Rightarrow 33 &= n & (3) \end{aligned}$$

$\therefore n$ goes from 0 to 33.

Answer : There are 34 elements in the series.

(ii) The n^{th} term of the Arithmetic progression ($x(n)$) is given as $x(0) + nd$

The common difference of the AP is given by the difference between successive terms.

$$\text{Common difference } (d) = 15\frac{1}{2} - 18 = -2\frac{1}{2}$$

$$\text{First term } (a) = 18$$

If -47 is the n^{th} term of the series, we have :

$$-47 = 18 + (n)\left(-2\left(\frac{1}{2}\right)\right) \quad (4)$$

$$\Rightarrow -65 = (n)\left(-2\left(\frac{1}{2}\right)\right) \quad (5)$$

$$\Rightarrow n = 26 \quad (6)$$

$\therefore n$ goes from 0 to 26.

Answer : There are 27 elements in the series.

Question : Express the n^{th} term in each case as $x(n)$ and find its z transform.

Solution :

(i) The n^{th} term of the Arithmetic progression is given as $x(0) + (n)d$. $x(0) = 7$ and $d = 6$.

$$x(n) = 7 + (n)6 \quad (7)$$

The Z transformation for $x[n]$ is given by :

$$X(z) = \left[\sum_{n=-\infty}^{\infty} x[n] z^{-n} \right] \quad (8)$$

However, $x(n)$ cannot be summed from $-\infty$ to zero as the number of terms cannot be negative due to which the n^{th} will not be defined for this range of n .

So, we will modify $x[n]$ by multiplying it with discrete unit step function $u(n)$ so we have the value as zero for $n < 0$.

$$x(n) = (7 + (n)6) u[n] \quad (9)$$

$$x(n) = \begin{cases} 0 & \text{for } n < 0 \\ 7 + (n)6 & \text{for } n \geq 0 \end{cases} \quad (10)$$

Now, using the above function in equation (8):

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \quad (11)$$

$$= \sum_{n=0}^{\infty} (x(0) + nd) z^{-n} \quad (12)$$

$$= \left[\sum_{n=0}^{\infty} x(0) z^{-n} \right] + \left[\sum_{n=0}^{\infty} nd z^{-n} \right] \quad (13)$$

$$= \left[\sum_{n=0}^{\infty} 7z^{-n} \right] + \left[\sum_{n=0}^{\infty} 6nz^{-n} \right] \quad (14)$$

$$= 7 \frac{z}{z-1} + 6 \left[\sum_{n=0}^{\infty} nz^{-n} \right] \quad (15)$$

Calculating the integral in the above expression :

$$S = \left[\sum_{n=0}^{\infty} nz^{-n} \right] \quad (16)$$

$$S = z^{-1} + 2z^{-2} + 3z^{-3} + \dots \quad (17)$$

$$S z^{-1} = z^{-2} + 2z^{-3} + 3z^{-4} + \dots \quad (18)$$

Subtracting (17) and (18)

$$S(1 - z^{-1}) = z^{-1} + z^{-2} + z^{-3} + \dots \quad (19)$$

$$= \frac{1}{(z-1)(1-z^{-1})} \quad (20)$$

$$= \frac{1}{z-2+z^{-1}} \quad (21)$$

$$\therefore S = \frac{z}{(z-1)^2} \quad (22)$$

Using equations (15) and (22) :

$$X(z) = \frac{7z}{z-1} + \frac{6z}{(z-1)^2} \quad (23)$$

$$= \frac{7z^2 - z}{(z-1)^2} \quad (24)$$

Answer : The z transformation for $x(n)$ where $x(n)$ is the n^{th} term of the AP is $\frac{7z^2 - z}{(z-1)^2}$.

(ii) The n^{th} term of the Arithmetic progression is given as $x(0) + nd$. $x(0) = 18$ and $d = -2\frac{1}{2}$

$$x(n) = 18 + n\left(-2\frac{1}{2}\right)$$

The Z transformation for $x[n]$ is given by :

$$X(z) = \left[\sum_{n=-\infty}^{\infty} x[n] z^{-n} \right] \quad (25)$$

However, $x[n]$ cannot be summed from $-\infty$ to zero as the number of terms cannot be negative due to which the n^{th} will not be defined for this range of n .

So, we will modify $x[n]$ by multiplying it with unit step function $u[n]$ so we have the value as zero for $n < 0$.

$$x(n) = \left(18 + n\left(-2\frac{1}{2}\right) \right) u(n) \quad (26)$$

$$x(n) = \begin{cases} 0 & \text{for } n < 0 \\ 18 + n\left(-2\frac{1}{2}\right) & \text{for } n \geq 0 \end{cases} \quad (27)$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \quad (28)$$

$$= \sum_{n=0}^{\infty} (x(0) + nd) z^{-n} \quad (29)$$

$$= \left[\sum_{n=0}^{\infty} x(0) z^{-n} \right] + \left[\sum_{n=0}^{\infty} nd z^{-n} \right] \quad (30)$$

$$= \left[\sum_{n=0}^{\infty} 18z^{-n} \right] + \left[\sum_{n=0}^{\infty} n\left(-2\frac{1}{2}\right) z^{-n} \right] \quad (31)$$

$$= \left[\sum_{n=0}^{\infty} 18z^{-n} \right] - \left[\sum_{n=0}^{\infty} \left(n\left(2\frac{1}{2}\right) \right) z^{-n} \right] \quad (32)$$

$$= 18 \left[\sum_{n=0}^{\infty} z^{-n} \right] - \left[\sum_{n=0}^{\infty} \left(n\left(2\frac{1}{2}\right) \right) z^{-n} \right] \quad (33)$$

$$= 18 \frac{z}{z-1} - \left(2\frac{1}{2} \right) \left[\sum_{n=0}^{\infty} nz^{-n} \right] \quad (34)$$

$$= 18 \frac{z}{z-1} - \left(2\frac{1}{2} \right) S \quad (35)$$

Using equation (22) and (35) :

$$X(z) = \frac{18z}{z-1} - \left(2\frac{1}{2} \right) \frac{z}{(z-1)^2} \quad (36)$$

$$= \frac{18z^2 - (20\frac{1}{2})z}{(z-1)^2} \quad (37)$$

Answer : The z transformation for $x(n)$ where $x(n)$ is the n^{th} term of the AP is $\frac{18z^2 - (20\frac{1}{2})z}{(z-1)^2}$.

Question : Plot the graph of $x(n)$ and find the ROC of $X(z)$ in each case.

Solution :

(i) The graph of $x(n)$ is :

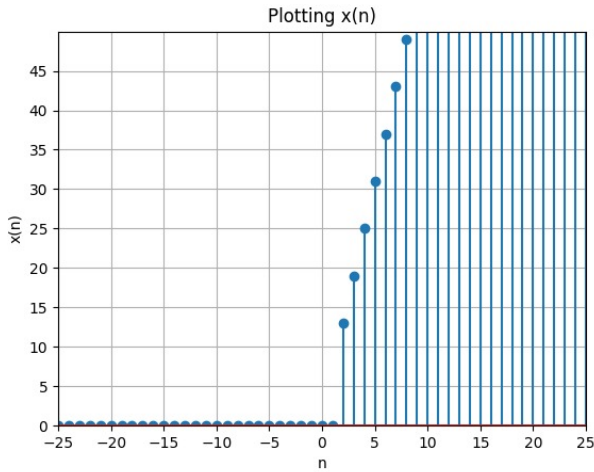


Fig. 0. Plot of $x(n)$

The ROC of $x(n)$ is defined as the range of values of z for which $X(z)$ will converge where $X(z)$ is the z transform of $x(n)$.

$$|X(z)| = \sum_{n=-\infty}^{\infty} |x(n) z^{-n}| < \infty \quad (38)$$

By equation (15) :

$$X(z) = 7 \left[\sum_{n=0}^{\infty} z^{-n} \right] + 6 \left[\sum_{n=0}^{\infty} n z^{-n} \right] \quad (39)$$

The sum $(\sum_{n=0}^{\infty} z^{-n})$ will converge only if z is not zero and $|z^{-1}| < 1$ as it is forming an infinite GP with common difference z^{-1} .

Observe the second part of the equation (39) or (15) is $6S$ which was calculated using equation (22) :

$$S(1 - z^{-1}) = z^{-1} + z^{-2} + z^{-3} + \dots \quad (40)$$

The sum S will converge only if z is not 0 and $|z^{-1}| < 1$ or $|z| > 1$ as again we obtain an infinite GP with common difference z^{-1} . For both the parts of the equation (39), the ROC is same.

Answer : The ROC of $X(z)$ is $|z| > 1$

(ii) The graph of $x(n)$ is :

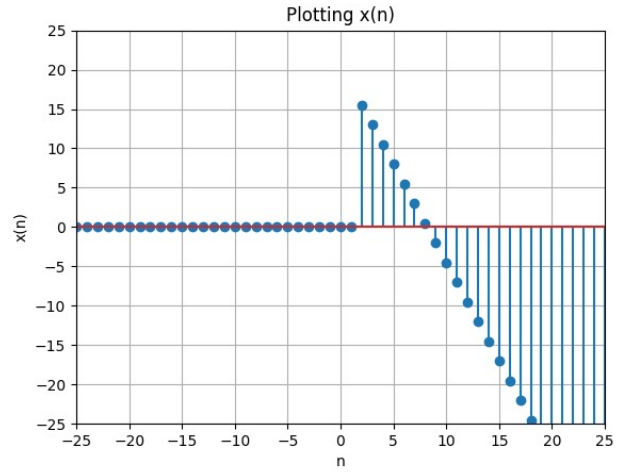


Fig. 1. Plot of $x(n)$

The ROC of $x(n)$ is defined as the range of values of z for which $X(z)$ will converge where $X(z)$ is the z transform of $x(n)$.

$$|X(z)| = \sum_{n=-\infty}^{\infty} |x(n) z^{-n}| < \infty \quad (41)$$

By equation (33) :

$$X(z) = 18 \left[\sum_{n=0}^{\infty} z^{-n} \right] - \left(2\frac{1}{2} \right) \left[\sum_{n=0}^{\infty} n z^{-n} \right] \quad (42)$$

The sum $[\sum_{n=0}^{\infty} z^{-n}]$ will converge only if z is not zero and $|z^{-1}| < 1$ as it is forming an infinite GP with common difference z^{-1} .

Observe the second part of the equation (33) or (42) is $-2\frac{1}{2}S$ which was calculated using equation (22) :

$$S(1 - z^{-1}) = z^{-1} + z^{-2} + z^{-3} + \dots \quad (43)$$

The sum S will converge only if z is not 0 and $|z^{-1}| < 1$ or $|z| > 1$ as again we obtain an infinite GP with common difference z^{-1} . For both the parts of the equation (42), the ROC is same.

Answer : The ROC of $X(z)$ is $|z| > 1$