NCERT Question 10.5.2.5

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Question 10.5.2.5 : Find the number of terms in each of the following APs :

(*i*) 7, 13, 19, ... 205

(*ii*) 18, $15\frac{1}{2}$, 13, ... -47

Solution : (*i*)

Parameter	Used to denote	Values
$x_1(n)$	n^{th} term of 1^{st} series	7 + 6n
$x_2(n)$	n^{th} term of 2^{nd} series	18 $-2\frac{1}{2}$n
$x_1(0)$	First term of 1^{st} AP	7
$x_2(0)$	First term of 2^{nd} AP	18
d_1	Common difference of 1^{st} AP	6
d_2	Common difference of 2^{nd}	$-2\frac{1}{2}$
$X_i(z)$	z transform of $x_i(n)$, i= 1 or 2	$\sum_{n=-\infty}^{\infty} x(n) z^{-n}$
u(n)	unit step function	$\begin{cases} 1 \text{ for } n \geq 0 \\ 0 \text{ for } n < 0 \end{cases}$
U(z)	z transform of u(n)	$\sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$
ROC	Region of Convergence	To find

TABLE 0
PARAMETER TABLE

$x_1(n) = x_1(0) + nd_1$

The common difference of the AP is given by the difference between successive terms.

$\therefore$ Common difference $(d_1) = 6$

First term $x_1(0) = 7$

If 205 is the *nth* term of the series, we have :

$$205 = 7 + (n)\,6 \tag{1}$$

$$\implies 198 = 6n \tag{2}$$

$$\implies 33 = n \tag{3}$$

$\therefore$ n goes from 0 to 33.

Answer : There are 34 elements in the series.

(*ii*)

$x_2(0) = x_2(0) + nd_2$

The common difference of the AP is given by the difference between successive terms.

$\therefore$Common difference $d_2 = -2\frac{1}{2}$

First term $x_2(0) = 18$

If -47 is the n^{th} term of the series, we have :

$$-47 = 18 + (n)\left(-2\left(\frac{1}{2}\right)\right) \tag{4}$$

$$\implies -65 = (n)\left(-2\left(\frac{1}{2}\right)\right) \tag{5}$$

$$\implies n = 26 \tag{6}$$

$\therefore$ n goes from 0 to 26.

Answer : There are 27 elements in the series.

Question : Express the n^{th} term in each case as x(n) and find its z transform.

Solution :

(*i*)

$$x_1(n) = x_1(0) + nd_1 \tag{7}$$

$$x_1(n) = 7 + (n)\,6 \tag{8}$$

The Z transformation for x(n) is given by :

$$X(z) = \left[\sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\right] \tag{9}$$

However, $x_1(n)$ cannot be summed from $-\infty$ to zero as the number of terms cannot be negative due to which the n^{th} will not be defined for this range of n.

So, we will modify $x_1(n)$ by multiplying it with unit step function u(n) so we have the value as zero for n<0.

$$x_1(n) = (7 + (n)\,6)\,u(n) \tag{10}$$

$$x_1(n) = \begin{cases} 0 & \text{for } n < 0 \\ 7 + (n)\,6 & \text{for } n \geq 0 \end{cases} \tag{11}$$

Now, using the above function in equation (9):

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n)\, u(n)\, z^{-n} \tag{12}$$

$$= \sum_{n=0}^{\infty} \left(x_1(0) + nd_1\right) z^{-n} \tag{13}$$

$$= \left[\sum_{n=0}^{\infty} 7z^{-n}\right] + \left[\sum_{n=0}^{\infty} 6nz^{-n}\right] \tag{14}$$

$$\text{using U}(z) \text{ from } Table\ 0 \tag{15}$$

$$= 7\frac{1}{1-z^{-1}} + 6\left[\sum_{n=0}^{\infty} nz^{-n}\right] \tag{16}$$

Calculating the integral in the above expression :

$$S = \sum_{n=0}^{\infty} nz^{-n} \tag{17}$$

$$\text{The z transform of nu}(n) \text{ is -z}\frac{dU(z)}{dz} \tag{18}$$

$$\therefore\ S = \frac{z^{-1}}{\left(1-z^{-1}\right)^2} \tag{19}$$

Using equations (16) and (19) :

$$X_1(z) = \frac{7}{1-z^{-1}} + \frac{6z^{-1}}{\left(1-z^{-1}\right)^2} \tag{20}$$

$$= \frac{7-z^{-1}}{\left(1-z^{-1}\right)^2} \tag{21}$$

Answer : The z transformation for x(n) where x(n) is the n^{th} term of the AP is $\frac{7-z^{-1}}{\left(1-z^{-1}\right)^2}$.

(ii)

Similar to how we wrote equation (30) in the previous case,

$$x_2(n) = \left(18 + n\left(-2\frac{1}{2}\right)\right) u(n) \tag{22}$$

$$x_2(n) = \begin{cases} 0 & \text{for } n < 0 \\ 18 + n\left(-2\frac{1}{2}\right) & \text{for } n \geq 0 \end{cases} \tag{23}$$

using equation (9) for z transfrom and Table 0,

$$X(z) = \sum_{n=0}^{\infty} x_2(n)\, u(n)\, z^{-n} \tag{24}$$

$$= \sum_{n=0}^{\infty} \left(x_2(0) + nd_2\right) z^{-n} \tag{25}$$

$$= 18\left[\sum_{n=0}^{\infty} z^{-n}\right] + \left(-2\frac{1}{2}\left[\sum_{n=0}^{\infty} nz^{-n}\right]\right) \tag{26}$$

Using equation (19) and U(z),

$$X(z) = 18\frac{1}{1-z^{-1}} - \left(2\frac{1}{2}\right) S \tag{27}$$

$$= \frac{18}{1-z^{-1}} - \left(2\frac{1}{2}\right)\frac{z^{-1}}{\left(1-z^{-1}\right)^2} \tag{28}$$

$$= \frac{18 - \left(20\frac{1}{2}\right) z^{-1}}{\left(1-z^{-1}\right)^2} \tag{29}$$

Answer : The z transformation for x(n) where x(n) is the n^{th} term of the AP is $\frac{18-\left(20\frac{1}{2}\right)z^{-1}}{\left(1-z^{-1}\right)^2}$.

Question : Plot the graph of x(n) and find the ROC of X(z) in each case.

Solution :

(i) The graph of $x_1(n)$ is :

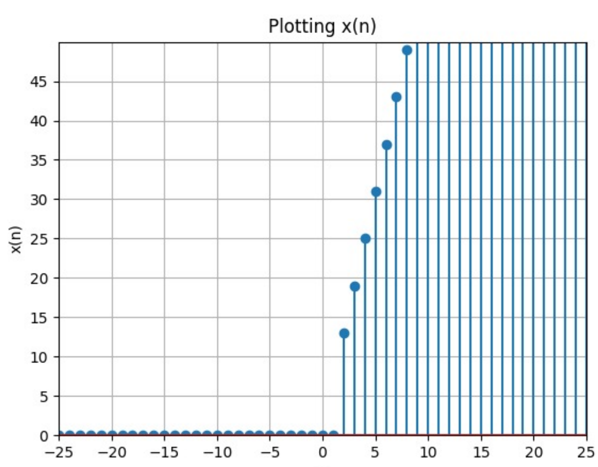

Fig. 0. Plot of x(n)

The ROC of x(n) is defined as the range of values of z for which X(z) will converge where

X(z) is the z transform of x(n).

$$|X(z)| = \sum_{n=-\infty}^{\infty} |x(n) z^{-n}| < \infty \tag{30}$$

By equation (16) :

$$X(z) = 7\left[\sum_{n=0}^{\infty} z^{-n}\right] + 6\left[\sum_{n=0}^{\infty} nz^{-n}\right] \tag{31}$$

The sum ($\sum_{n=0}^{\infty} z^{-n}$) will converge only if z is not zero and $|z^{-1}| < 1$ as it is forming an infinite GP with common difference z^{-1}.

Similarly for $\sum_{n=0}^{\infty} nz^{-n}$ to converge, we must have $|z^{-1}| < 1$

Hence, for $X_1(z)$ to converge, we must have $|z^{-1}| < 1$ or $|z| > 1$.

Answer : The ROC of $X_1(z)$ is $|z| > 1$

(ii) The graph of $x_2(n)$ is :

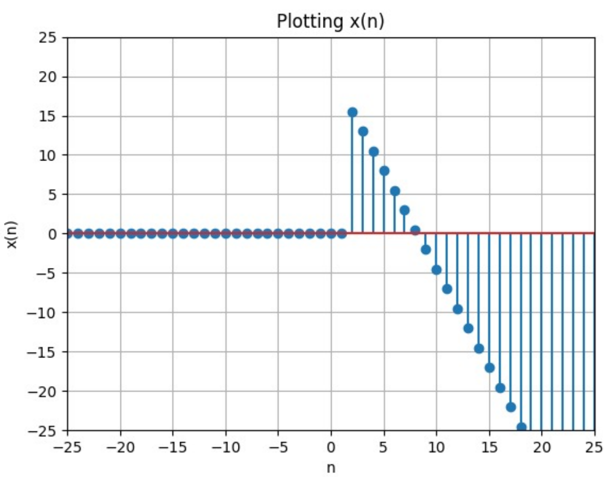

Fig. 1. Plot of x(n)

By equation (26) :

$$X(z) = 18\left[\sum_{n=0}^{\infty} z^{-n}\right] - \left(2\frac{1}{2}\right)\left[\sum_{n=0}^{\infty} nz^{-n}\right] \tag{32}$$

ROC is given by equation (30)

The sum $[\sum_{n=0}^{\infty} z^{-n}]$ will converge only if z is not zero and $\left|z^{-1}\right| < 1$ as it is forming an infinite GP with common difference z^{-1}.

Similarly for $\sum_{n=0}^{\infty} nz^{-n}$ to converge, we must have $|z^{-1}| < 1$

Hence, for $X_2(z)$ to converge, we must have $|z^{-1} < 1|$ or $|z| > 1$.

Answer : The ROC of $X_2(z)$ is $|z| > 1$