Kaustubh Shrikant Kabra, SE Comp shift I, 20

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Date 01 02 2021

* Tutorial Number - 5 (Legendre and Cauchy's DE)

1) $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$

$\rightarrow$ Given DE is legendre DE,

Here $ax+b = 1+x \Rightarrow a=1,\ b=1$

Let $1+x = e^z$ — ①

$x = e^z - 1$ — ②

And Let $D = \frac{d}{dx}$.

$\therefore (ax+b)\frac{dy}{dx} = aDy \qquad \Rightarrow (x+1)\frac{dy}{dx} = Dy$.

$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y \Rightarrow (x+1)^2\frac{d^2y}{dx^2} = D(D-1)y$.

$\therefore$ Given DE becomes $\Rightarrow D(D-1)y + Dy + y = 4\cos[z]$

$\Rightarrow [D^2 - D + D + 1]y = 4\cos[(z)]$

$\therefore \phi(D) = D^2+1$
$= (D+i)(D-i)$, $f(z) = 4\cos[(z)]$

$\therefore \boxed{y_c = C_1\cos x + C_2 \sin x}$

$y_p = \frac{1}{\phi(D)} \cdot f(z)$

$= \frac{1}{D^2+1} \cdot 4\cos[(z)]$

$= 4^2\left[\frac{-z}{2(+1)} \cdot \frac{1}{1!} \cos(+z+\pi/2)\right]$

$= -2z\left[\cos z \cdot \cancel{\cos\pi/2}^{\,0} - \sin z \cdot \cancel{\sin\pi/2}^{\,1}\right]$

$= 2z\sin z$

$\boxed{y_p = 2\log(1+x) \cdot \sin[\log(1+x)]}$

2) $(x+2)^2 \dfrac{d^2y}{dx^2} + 3(x+2)\dfrac{dy}{dx} + y = \cos[\log(x+2)]$

→ Given DE is Legendre DE,

Here $ax+b = x+2 \Rightarrow a=1, b=2$

Let $x+2 = e^z$

$x = e^z - 2$

Let $D = \dfrac{d}{dx}$, $(ax+b)\dfrac{dy}{dx} = aDy$, $(ax+b)^2\dfrac{d^2y}{dx^2} = a^2D(D-1)y$

$(x+2)\dfrac{dy}{dx} = Dy$, $(x+2)^2\dfrac{d^2y}{dx^2} = D(D-1)y$

Given DE becomes

$$D(D-1)y + 3Dy + y = \cos z$$

$$[D^2 - D + 3D + 1]y = \cos z$$

$\therefore \phi(D) = D^2 + 2D + 1$

$= (D+1)^2$, $f(z) = \cos z$

$y_c = (C_1 + xC_2)e^{-z}$

$\boxed{y_c = (C_1 + xC_2)(x+2)^{-1}}$

$y_p = \dfrac{1}{\phi(D)} \cdot f(z)$

$= \dfrac{1}{(D+1)^2} \cdot \cos z$

$= \dfrac{1}{D^2+2D+1} \cdot \cos z$

$a=1, D^2 \to -1$

$= \dfrac{1}{\cancel{-1}+2D+\cancel{1}} \cdot \cos z$

$$= \frac{D}{2D^2} \cdot \cos z$$

$a = 1, D^2 \to -1$

$$= \frac{D \cos z}{2(-1)}$$

$$= \frac{-D \cos z}{2}$$

$$= \frac{-(-\sin z)}{2}$$

$$= \frac{\sin z}{2}$$

$$y_p = \frac{\sin[\log(x+2)]}{2}$$

$$y = y_c + y_p = \frac{(C_1 + xC_2)}{(x+2)} + \frac{\sin[\log(x+2)]}{2}.$$

3) $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^2 + 2 \log x.$

→ Give DE is Legendre DE,
Here $ax + b = x$. $\Rightarrow a = 1, b = 0$.
Let $x = e^z$
and $D = \frac{d}{dx}$, $(ax+b)\frac{dy}{dx} = aDy$, $(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y.$

$$x\frac{dy}{dx} = Dy. \qquad x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

Given DE becomes

$$D(D-1)y - 2Dy - 4y = x^2 + 2\log x.$$

$$(D^2 - D - 2D - 4)y = \cancel{x^2 + 2\log x}\ e^{2z} + 2z.$$

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$$\phi(D) = D^2 - 3D - 4 \qquad , \quad f(z) = e^{2z} + 2z$$
$$(D-4)(D+1)$$

$$y_c = c_1 e^{-z} + c_2 e^{4z}$$

$$y_c = c_1 e^{-\log x} + c_2 e^{4\log x}$$

$$\boxed{y_c = c_1 \cdot x^{-1} + c_2 x^4}$$

$$y_p = \frac{1}{\phi(D)} \cdot f(z)$$

$$= \frac{1}{(D-4)(D+1)} \cdot \left[e^{2z} + 2z\right]$$

$$= \frac{1}{(D-4)(D+1)} \cdot e^{2z} + \frac{1}{(D-4)(D+1)} \cdot 2z$$

$a = 2$

$$= \frac{1}{(2-4)(2+1)} \cdot e^{2z} + \frac{2}{(D+1)}\left(\frac{-1}{4}\right)\left[\frac{1}{(1-(D/4))} \cdot z\right] \rightarrow t = D/4$$

$$= \frac{e^{2z}}{-6} - \frac{1}{2(D+1)}\left[(1) + (D/4) + (D/4)^2\right] z$$

$$= \frac{e^{2z}}{-6} - \frac{1}{2(D+1)}\left[z + \frac{1}{4}\right] \rightarrow t = D$$

$$= \frac{e^{2z}}{-6} - \frac{1}{2}\left[(1) - (D)\right]\left[z + \frac{1}{4}\right]$$

$$= \frac{e^{2z}}{-6} - \frac{1}{2}\left[z + \frac{1}{4} - 1\right]$$

$$= \frac{e^{2\log x}}{-6} - \frac{1}{2}\left[\log x - \frac{3}{4}\right]$$

$$\boxed{y_p = -\frac{x^2}{6} - \frac{\log x}{2} + \frac{3}{8}}$$

$$\boxed{y = y_c + y_p = \frac{c_1}{x} + c_2 x^4 - \frac{x^2}{6} - \frac{\log x}{2} + \frac{3}{8}}$$

4) $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = x + \frac{1}{x}$.

→ Given DE is Legendre DE,
Here $ax+b = x \Rightarrow a=1, b=0$.
Let $x = e^z$.
Let $D = \frac{d}{dx}$, $(ax+b)\frac{dy}{dx} = aDy$, $(x)\frac{dy}{dx} = Dy$.

$(ax+b)^2 \frac{d^2y}{dx^2} = a^2D(D-1)y$, $x^2\frac{d^2y}{dx^2} = D(D-1)y$

$(ax+b)^3 \frac{d^3y}{dx^2} = a^3D(D-1)(D-2)y$, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$.

∴ Given DE becomes

$$D(D-1)(D-2)y + 2D(D-1)y + 2y = e^z + \frac{1}{e^z}$$

$$[D^3 - 3D^2 + \cancel{2D} + 2D^2 - \cancel{2D} + 2]y = e^z + e^{-z}$$

$$[D^3 - D^2 + 2]y = e^z + e^{-z}$$

$\phi(D) = D^3 - D^2 + 2$, $f(z) = e^z + e^{-z}$.

roots:- -1, $1+i$, $1-i$

$$\therefore y_c = C_1 e^{-z} + e^{z}[C_2 \cos z + C_3 \sin z]$$
$$= C_1 x^{-1} + x[C_2 \cos(\log x) + C_3 \sin(\log x)]$$

$$y_p = \frac{1}{\phi(D)} \cdot f(z)$$

$$= \left[\frac{1}{D^3-D^2+2} \cdot e^z\right] + \left[\frac{1}{D^3-D^2+2} \cdot e^{-z}\right]$$

$a=1, D=1$.

$$= \left[\frac{1}{1-1+2} \cdot e^z\right] + \left[\frac{1}{(D+1)(D^2-2D+2)} \cdot e^{-z}\right]$$

$$= \frac{e^3}{2} + \left[\frac{3}{1!} \cdot \frac{e^{-3}}{[(-1)^2 - 2(-1) + 2]} \right]$$

$$= \frac{e^3}{2} + \left[\frac{3 \cdot e^{-3}}{5} \right]$$

$$\boxed{y_p = \frac{x}{2} + \frac{\log x}{5x}}$$

$$y = y_c + y_p$$

$$\boxed{y = \frac{C_1}{x} + x[C_2 \cos(\log x) + C_3 \sin(\log x)] + \frac{x}{2} + \frac{\log x}{5x}}$$

5) $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 4y = \cos(\log x) + x \sin(\log x)$

→ Given DE is Legendre DE,
Here $ax+b = x$, ⇒ $a=1$, $b=0$.
Let $x = e^z$ ⇒ $\log x = z$
Let $\frac{d}{dx} = D$, $x\frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$.

∴ Given DE becomes,

$$D(D-1)y - Dy + 4y = \cos(z) + e^z \cdot \sin(z)$$

$$[D^2 - D - D + 4]y = \cos(z) + e^z \cdot \sin(z)$$

$\phi(D) = D^2 - 2D + 4$, $f(z) = \cos(z) + e^z \cdot \sin(z)$
roots are $1 + \sqrt{3}i$, $1 - \sqrt{3}i$

$$y_c = e^z [C_1 \cos(\sqrt{3}z) + C_2 \sin(\sqrt{3}z)]$$

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$$\boxed{y_c = x\left[C_1 \cos(\sqrt{3}\log x) + C_2 \sin(\sqrt{3}\log x)\right]}$$

$$y_p = \frac{1}{\phi(D)} \cdot f(z)$$

$$= \frac{1}{(D^2-2D+4)} \cdot \cos(z) + \frac{1}{(D^2-2D+4)} \cdot e^3 \cdot \sin(z)$$

$a=1,\ D^2 \to -1.$ $\qquad a=1,\ V=\sin z,\ D \to (D+1)$

$$= \frac{1}{(-1-2D+4)} \cdot \cos(z) + e^3\left[\frac{1}{(D+1)^2 - 2(D+1)+4} \cdot \sin z\right]$$

$$= \frac{1}{(3-2D)} \cdot \cos(z) + e^3\left[\frac{1}{D^2+\cancel{2D}+1-\cancel{2D}-2+4} \cdot \sin z\right]$$

$$= \frac{1(3+2D)}{(3-2D)(3+2D)} \cdot \cos(z) + e^3\left[\frac{1}{D^2+3} \cdot \sin z\right]$$

$a=1,\ D^2 \to -1.$

$$= \frac{(3+2D)\cos z}{9-4D^2} + e^3\left[\frac{1}{-1+3} \cdot \sin z\right]$$

$a=1,\ D^2 \to -1$

$$= \frac{(3+2D)\cos(z)}{9-4(-1)} + e^3\left[\frac{\sin 3}{2}\right]$$

$$= \frac{1}{13}\left[3\cos z - 2\sin z\right] + e^3\,\frac{\sin 3}{2}$$

$$\boxed{y_p = \left[\frac{3\cos(\log x) - 2\sin(\log x)}{13}\right] + \frac{x.\sin(\log x)}{2}}$$

$$y = y_c + y_p$$

$$\boxed{y = x\left[C_1\cos(\sqrt{3}\log x) + C_2\sin(\sqrt{3}\log x)\right] + \left[\frac{3\cos(\log x) - 2\sin(\log x)}{13}\right] + \frac{x.\sin(\log x)}{2}}$$

6) $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x^5.$

$\rightarrow$ Given DE is Legendre DE,

Here $ax+b = 1x \Rightarrow a=1, b=0.$

Let $x = e^z$

$\log x = z.$

Let $\dfrac{d}{dx} = D$, $x\dfrac{dy}{dx} = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y.$

Given DE becomes

$$D(D-1)y - 4Dy + 4y = e^{5z}.$$

$$[D^2 - D - 4D + 4]y = e^{5z}.$$

$$\phi(D) = D^2 - 5D + 4 \quad , \quad f(z) = e^{5z}$$
$$= (D-1)(D-4).$$

$$y_c = C_1 e^{z} + C_2 e^{4z}$$

$$y_c = C_1 e^{\log x} + C_2 e^{4\log x}.$$

$$\boxed{y_c = xC_1 + x^4 C_2}$$

$$y_p = \frac{1}{\phi(D)} \cdot f(z).$$

$$= \frac{1}{(D-1)(D-4)} \cdot e^{5z}$$

$$= \frac{1}{(5-1)(5-4)} \cdot e^{5z}$$

$$= \frac{1}{4.1} \cdot e^{5z}$$

$$= \frac{e^{5z}}{4}$$

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$$\boxed{y_p = \frac{x^5}{4}}$$

$$\therefore y = y_c + y_p$$

$$\boxed{y = xC_1 + x^4C_2 + \frac{x^5}{4}}$$