* Tutorial Number - 4

Solve the following on method of variation of parameters:-

1) $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$

$\rightarrow$ Compare the above DE with $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

we get $a=1,\ b=0,\ c=1,\quad f(x) = \operatorname{cosec} x.$

$\phi(D) = D^2 + 1.$

$\hookrightarrow$ roots are $\pm i.$

$\therefore \boxed{y_c = c_1 \cos x + c_2 \sin x.}$

Comparing y_c with $c_1y_1 + c_2y_2$, we get.

$y_1 = \cos x. \qquad y_1' = -\sin x$

$y_2 = \sin x \qquad y_2' = \cos x$

$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x$
$= 1$

$\therefore u = \int \frac{-y_2}{W} \cdot f(x) \cdot dx = \int -\frac{\sin x}{1} \cdot \operatorname{cosec} x \cdot dx = \int -1\ dx$
$= -x.$

$v = \int \frac{y_1}{W} \cdot f(x) \cdot dx = \int \frac{\cos x}{1} \cdot \operatorname{cosec} x. = \int \cot x\ dx$
$= \ln|\sin x|$

$\therefore y_p = u.y_1 + vy_2$
$= -x.\cos x + \ln|\sin x| . \sin x.$

$\therefore y = y_c + y_p$

$\boxed{y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x . \ln|\sin x|}$

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2) $(D^2-2D)y = e^x \sin x.$

$\rightarrow$ Compare above DE with $\phi(D)y = f(x)$, we get.

$\phi(D) = D^2-2D$, $f(x) = e^x.\sin x.$

$= D(D-2)$

$\therefore y_c = c_1.e^{0x} + c_2 e^{2x}$

$= c_1 + c_2 e^{2x}.$

Comparing y_c with $c_1 y_1 + c_2 y_2$, we get

$y_1 = 1$, $y_1' = 0$

$y_2 = e^{2x}$ $y_2' = 2e^{2x}.$

$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$

$u = \int \frac{-y_2}{W}.f(x).dx = \int \frac{-\cancel{e^{2x}}}{2\cancel{e^{2x}}}.e^x.\sin x.dx$

$= \frac{-1}{2}\int e^x.\sin x.dx.$

$= \frac{-1}{2}.\frac{1}{2}e^x(\sin x - \cos x)$

$= \frac{e^x}{4}(\cos x - \sin x).$

$v = \int \frac{y_1}{W}.f(x).dx = \int \frac{1}{2e^{2x}}.e^x.\sin x.dx$

$= \frac{1}{2}\int e^{-x}.\sin x.dx$

$= \frac{1}{2}\left[\frac{-1}{2}.e^{-x}(\sin x + \cos x)\right]$

$= -\frac{e^{-x}}{4}(\sin x + \cos x)$

$$y_p = u y_1 + v y_2$$

$$= \frac{e^x}{4}(\cos x - \sin x)\,1 - \frac{e^{-x}}{4}(\sin x + \cos x)\,e^{2x}$$

$$= \frac{e^x}{4}\left[\cancel{\cos x} - \sin x - \sin x - \cancel{\cos x}\right]$$

$$= -\frac{e^x \cdot \sin x}{2}$$

$$\therefore\ y = y_c + y_p$$

$$\boxed{y = C_1 + C_2 e^{2x} - \frac{e^x \cdot \sin x}{2}}$$

3) $(D^2 - 4D + 4)y = e^{2x} x^{-2}$

→ Compare above DE with $\phi(D)y = f(x)$, we get.

$$\phi(D) = (D^2 - 4D + 4) \quad , \quad f(x) = e^{2x} \cdot x^{-2}$$

$$= (D-2)^2$$

$$\therefore\ y_c = (C_1 + xC_2)e^{2x}$$

$$= C_1 e^{2x} + e^{2x} x C_2$$

Comparing y_c with $C_1 y_1 + C_2 y_2$, we get

$$y_1 = e^{2x} \quad , \quad y_1' = 2e^{2x}$$

$$y_2 = xe^{2x} \qquad y_2' = x.(2e^{2x}) + e^{2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}(2x+1) \end{vmatrix}$$

$$= e^{4x^2}(2x+1) - 2x e^{4x^2}$$

$$= e^{4x^2}$$

$$u = \int \frac{-y_2}{W} \cdot f(x)\, dx = \int -\frac{x \cdot \cancel{e^{2x}}}{\cancel{e^{4x^2}}} \cdot \cancel{e^{2x}} \cdot x^{-2} \cdot dx = -\int \frac{1}{x} \cdot dx$$

$$= -\ln x$$

$$V = \int \frac{y_1}{W} \cdot f(x) \cdot dx = \int \frac{e^{2x}}{e^{4x}} \cdot e^{2x} \cdot x^{-2} \cdot dx$$

$$= \int x^{-2} \cdot dx$$

$$= -\frac{1}{x}$$

$$\therefore y_p = u y_1 + v y_2$$

$$y_p = \ln x \cdot e^{2x} + \left(\frac{-1}{x}\right) \cdot x \cdot e^{2x}$$

$$= e^{2x} [\ln(x) - 1]$$

$$\therefore y = y_c + y_p$$

$$\boxed{y = (c_1 + x c_2) e^{2x} + e^{2x} [\ln x + 1]}$$

4) $\frac{d^2y}{dx^2} + y = \sec x \cdot \tan x \rightarrow D^2 y + y = \sec x \cdot \tan x$

→ Compare above DE with $\phi(D) y = f(x)$, we get

$\phi(D) = (D^2 + 1)$, $f(x) = \sec x \cdot \tan x$

↳ roots are $\pm i$.

$$\therefore y_c = (c_1 \cdot \cos x + c_2 \cdot \sin x)$$

Comparing y_c with $c_1 y_1 + c_2 y_2$, we get.

$y_1 = \cos x$ $\quad\quad$ $y_1' = -\sin x$

$y_2 = \sin x$ $\quad\quad$ $y_2' = \cos x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

$$v = \int \frac{y_1}{W} f(x)\, dx = \int \frac{\cos x}{1} \sec x . \tan x = \int \tan x . dx$$

$$= -\ln|\cos x|$$

$$u = \int \frac{-y_2}{W} f(x) . dx = \int -\sin x . \sec x . \tan x . dx .$$

$$= -\int \sin x \frac{1}{\cos x} . \frac{\sin x}{\cos x} dx$$

$$= -\int \tan^2 x \, dx$$

$$= -\int (\sec^2 x - 1) . dx .$$

$$= -\left[\int \sec^2 x . dx - \int 1 . dx \right]$$

$$= -\left[\tan x - x \right]$$

$$= [x - \tan x]$$

$$y_p = u y_1 + v y_2$$

$$= [x - \tan x] \cos x . + [-\ln \cos x] \sin x .$$

$$y_p = x . \cos x - \sin x - \sin x . \ln|\cos x|$$

$$\therefore y = y_c + y_p$$

$$\boxed{y = [C_1 \cos x + C_2 \sin x] + x . \cos x . - \sin x - \sin x . \ln|\cos x|}$$

* Solve the following on method of general :-

1) $(D^2 + 3D + 2) y = e^{e^x}$

$\rightarrow \phi(D) = D^2 + 3D + 2$, $f(x) = e^{e^x}$.

$= (D+1)(D+2)$

$$y_c = c_1 e^{-x} + c_2 e^{-2x} . \quad - (1)$$

Applying general method for y_p

$$\frac{1}{D-m} \cdot f(x) = e^{mx} \int e^{-mx} \cdot f(x) \cdot dx$$

$$\therefore \quad y_p = \frac{1}{(D+2)(D+1)} \cdot e^{e^x}$$

$$= \frac{1}{(D+2)} \left[\frac{1}{D-(-1)} \cdot e^{e^x} \right]$$

$m = -1$

$$= \frac{1}{(D+2)} \left[e^{-x} \int e^x \cdot e^{e^x} \cdot dx \right]$$

$e^x = t \quad , \quad e^x \cdot dx = dt$

$$= \frac{1}{(D+2)} \left[e^{-x} \int e^t \, dt \right]$$

$$= \frac{1}{(D+2)} \left[e^{-x} \cdot e^t \right]$$

$$= \left[\frac{1}{(D+2)} \cdot e^{-x} \cdot e^{e^x} \right]$$

$m = -2$

$$= \left[e^{-2x} \int e^{2x} \cdot e^{-x} \cdot e^{e^x} \right]$$

$$= \left[e^{-2x} \int e^x \cdot e^{e^x} \cdot dx \right]$$

$e^x = t$

$$y_p = e^{-2x} \cdot e^{e^x} \quad — (2)$$

$$\therefore \; y = y_c + y_p$$

$$\boxed{y = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x} \cdot e^{e^x}} \quad — (3)$$

2) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{1+e^x}$

$\rightarrow$ Let $\frac{d}{dx} = D \quad \therefore \quad D^2y + Dy = \frac{1}{1+e^x}$

$$(D^2+D)y = \frac{1}{1+e^x}$$

$\therefore \quad \phi(D) = D^2 + D \quad , \quad f(x) = \frac{1}{1+e^x}$

$= D(D+1)$

$$y_c = C_1 + e^{-x}C_2 \qquad - (1)$$

Applying general method for y_p.

$$y_p = \frac{1}{(D-m)} \cdot f(x)$$

$$= \frac{1}{D(D+1)} \cdot \frac{1}{1+e^x} \longrightarrow m = -1.$$

$$= \frac{1}{D}\left[e^{-x}\int e^x \cdot \frac{1}{(1+e^x)} \cdot dx\right]$$

$e^x = t, \quad e^x\, dx = dt$

$$= \frac{1}{D}\left[e^{-x}\int \frac{1}{1+t} \cdot dt\right]$$

$$= \frac{1}{D}\left[e^{-x} \cdot \ln|1+t|\right]$$

$$= \frac{1}{D}\left[e^{-x} \cdot \ln|1+e^x|\right] = \int e^{-x} \cdot \ln|1+e^x|$$

$$= -\log(1+e^x) \cdot e^{-x} + \int \frac{1}{1+e^x} \cdot e^{-x} \cdot e^x$$

$$= -\log(1+e^x) \cdot e^{-x} + \int 1 - \frac{e^x}{1+e^x} \rightarrow t.$$

$$\boxed{y_p = -\log(1+e^x) \cdot e^{-x} + x - \log(1+e^x)}$$

3) $(D^2+3D+2)y = e^{e^x} + \cos(e^x)$

$\rightarrow$ $\phi(D) = D^2+3D+2$, $f(x) = e^{e^x} + \cos(e^x)$

$= (D+1)(D+2)$

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

Apply general method for y_p

$$y_p = \frac{1}{(D-m)} \cdot f(x)$$

$$= \frac{1}{(D+2)}\left[\frac{1}{D-(-1)} \cdot e^{e^x}\right] + \frac{1}{(D+2)}\left[\frac{1}{(D-(-1))} \cdot \cos e^x\right]$$

$m = -1$ $m = 1$

$$= \frac{1}{(D+2}\left[e^{-x}\int e^x \cdot e^{e^x} \cdot dx\right] + \frac{1}{(D+2)}\left[e^{-x}\int e^x \cdot \cos(e^x)\, dx\right]$$

$e^x = t,\ e^x \cdot dx = dt$ $e^x = t,\ e^x \cdot dx = dt$

$$= \frac{1}{(D+2)}\left[e^{-x} \cdot e^{e^x}\right] + \frac{1}{(D+2)}\left[e^{-x} \cdot \sin(e^x)\right]$$

$m = -2$ $m = -2$

$$= \left[e^{-2x}\int e^{2x} \cdot e^{-x} \cdot e^{e^x} \cdot dx\right] + \left[e^{-2x}\int e^{2x} \cdot e^{-x} \sin(e^x) \cdot dx\right]$$

$$= \left[e^{-2x}\int e^x \cdot e^{e^x} \, dx\right] + \left[e^{-2x}\int e^x \sin(e^x) \cdot dx\right]$$

$e^x = t$ $e^x = t$

$$= \left[e^{-2x} \cdot e^{e^x}\right] + \left[e^{-2x} \cdot (-\cos(e^x))\right]$$

$$y_p = e^{-2x}\left[e^{e^x} - \cos(e^x)\right]$$

$$\boxed{y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x}\left[e^{e^x} - \cos(e^x)\right]}$$

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4) $(D^2+2D+1)y = \dfrac{e^{-x}}{x+2}$.

$\rightarrow$ $\phi(D) = D^2+2D+1$ $\quad , f(x) = \dfrac{e^{-x}}{x+2}$
$= (D+1)^2$

$y_c = (C_1 + xC_2)e^{-x}$.

Applying general method for y_p

$$y_p = \frac{1}{(D+1)}\left[\frac{1}{(D+1)} \cdot \frac{e^{-x}}{x+2}\right]$$

$m = +1$

$$= \frac{1}{(D+1)}\left[e^{-x}\int \cancel{e^{x}} \cdot \frac{\cancel{e^{-x}}}{x+2} \cdot dx\right]$$

$$= \frac{1}{(D+1)}\left[e^{-x} \ln(x+2)\right]$$

$m = -1$.

$$= \left[e^{-x}\int \cancel{e^{x}} \cdot \cancel{e^{-x}} \cdot \ln(x+2)\right]$$

$$= \left[e^{-x} \int \ln(x+2)\right]$$

$$y_p = e^{-x}\left[\log(x+2) \cdot [x+2] - x\right]$$

$$y = y_c + y_p$$

$$\boxed{y = (C_1 + xC_2)e^{-x} + e^{-x}\left[\log(x+2) \cdot \{x+2\} - x\right]}$$

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6) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$.

$\rightarrow$ Let $\frac{d}{dx} = D$ $\therefore$ $D^2y - Dy - 2y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$

$\phi(D) = D^2 - D - 2$
$= (D+1)(D-2)$. , $f(x) = 2\log x + \frac{1}{x} + \frac{1}{x^2}$

$y_c = C_1 e^{-x} + C_2 e^{+2x}$.

Apply general method for y_p

$$y_p = \frac{1}{(D-2)}\left[\frac{1}{D-(-1)} . 2\log x + \frac{1}{x} + \frac{1}{x^2}\right]$$
$m = -1$

$$= \frac{1}{(D-2)}\left[e^{-x}\int e^{x} . \left[2\log x + \frac{2}{x} - \frac{1}{x} + \frac{1}{x^2}\right]\right]$$

$$= \frac{1}{(D-2}\left[e^{-x}\int e^{x}\left[2\log x + \frac{2}{x}\right] . dx + e^{-x}\int e^{x}\left[\frac{-1}{x} + \frac{1}{x^2}\right] . dx\right]$$

$$= \frac{1}{(D-2)}\left[e^{-x} . \left[e^{x} . 2\log x + e^{x}\left(\frac{-1}{x}\right)\right]\right]$$

$$= \frac{1}{D-2}\left[2\log x - \frac{1}{x}\right]$$
$m = 2$

$$= \left[e^{+2x}\int e^{-2x} . 2\log x - \frac{1}{x} dx\right]$$

$$= \left[e^{2x} . 2\int e^{-2x}\log x\right] - \left[e^{2x}\int e^{-2x} \frac{1}{x}\right]$$

$$= e^{2x}\left[\cancel{2\int e^{-2x}\log x} - e^{-2x}\log x - \cancel{\int -2e^{-2x}\log x}\right]$$

$$= e^{2x} . \left[e^{-2x}\log x\right] \qquad \therefore \boxed{y_p = \log x}$$

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$y = y_c + y_p$

$$\boxed{y = c_1 e^{-x} + c_2 e^{2x} - \log x}$$