Q-1. Solve the integral equation (May 2019)

$$\int_0^\infty f(x).\cos(\lambda x).\,dx = 1-\lambda \quad , \quad 0 \le \lambda < 1$$
$$= 0 \quad , \quad \lambda \ge 1$$

$\rightarrow$ $$F(\lambda) = \int_0^\infty f(u).\cos(\lambda u).\,du = \begin{cases} 1-\lambda & , \ 0 \le \lambda < 1 \\ 0 & , \ \lambda \ge 1. \end{cases}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda).\cos(\lambda x).\,d\lambda.$$
$$= \frac{2}{\pi}\left[\int_0^1 (1-\lambda)\cos(\lambda x).\,d\lambda + \int_1^\infty (0).\cos(\lambda x).\,d\lambda\right]$$
$$= \frac{2}{\pi}\left[(1-\lambda)\left(\frac{\sin(\lambda x)}{x}\right) - (-1)\left(\frac{-\cos(\lambda x)}{x^2}\right)\right]_0^1$$
$$= \frac{2}{\pi}\left[-\frac{\cos x}{x^2} + \frac{1}{x^2}\right]$$
$$= \frac{2}{\pi}\left[\frac{1-\cos x}{x^2}\right]$$

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$$\therefore \boxed{f(x) = \frac{2}{\pi}\left[\frac{1-\cos x}{x^2}\right]}$$

Q-2. Find F.T of $f(x) = \begin{cases} 1-x^2 & , |x| \leq 1 \\ 0 & , \text{otherwise} \end{cases}$ (May 2018)

$\rightarrow$ Here, $f(x) = \begin{cases} 1-x^2 & , -1 \leq x \leq 1 \\ 0 & , \text{otherwise.} \end{cases}$

F.T for $f(x)$ is given by

$$F(\lambda) = \int_{-\infty}^{\infty} f(u).\, e^{-i\lambda u}.\, du$$

$$= \int_{-\infty}^{-1} f(u).\, e^{-i\lambda u}.\, du + \int_{-1}^{1} f(u).\, e^{-i\lambda u}.\, du + \int_{1}^{\infty} f(u) e^{-i\lambda u}.\, du$$

$$= \int_{-\infty}^{-1} 0.\, du + \int_{-1}^{1} (1-u^2).\, e^{-i\lambda u}.\, du + \int_{1}^{\infty} 0.\, du.$$

$$= \int_{-1}^{1} (1-u^2)(\cos\lambda u - i\sin\lambda u).\, du$$

$$= \int_{-1}^{1} \underbrace{(1-u^2)}_{\text{even}} \underbrace{(\cos\lambda u)}_{\text{even}}.\, du - i\int_{-1}^{1} \underbrace{(1-u^2)}_{\text{even}}.\underbrace{(\sin\lambda u)}_{\text{odd}}.\, du$$

$$= 2\int_{0}^{1} (1-u^2)\cos(\lambda u).\, du - i(0)$$

$$= 2\left[\frac{(1-u^2)\sin\lambda u}{\lambda} - \frac{2u\cos\lambda u}{\lambda^2} + \frac{2\sin\lambda u}{\lambda^3}\right]_0^1$$

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$$= 2\left[\frac{-2\cos\lambda}{\lambda^2} + \frac{2\sin\lambda}{\lambda^3}\right]$$

$$\therefore \boxed{\text{F.T of } f(x) = \frac{4\sin\lambda}{\lambda^3} - \frac{4\cos\lambda}{\lambda^2}}$$

Q-3. Find FIR of $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ [May 2017]

Hence evaluate $\int_0^\infty \frac{\sin\lambda}{\lambda} \cdot \cos\lambda x \cdot dx \cdot d\lambda$ and $\int_0^\infty \frac{\sin\lambda}{\lambda} \cdot d\lambda$

$\longrightarrow$ $f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$F(\lambda) = \int_{-\infty}^{\infty} f(u) \cdot e^{-i\lambda u} \cdot du$$

$$= \int_{-\infty}^{-1} f(u) \cdot e^{-i\lambda u} du + \int_{-1}^{1} f(u) \cdot e^{-i\lambda u} \cdot du + \int_{1}^{\infty} f(u) \cdot e^{-i\lambda u} \cdot du.$$

$$= 0 + \int_{-1}^{1} 1 \cdot e^{-i\lambda u} \cdot du + 0$$

$$= \int_{-1}^{1} (\cos\lambda u - i\sin\lambda u) \cdot du$$

$$= \int_{-1}^{1} \underbrace{\cos\lambda u}_{eve} \cdot du - i \int_{-1}^{1} \underbrace{\sin\lambda u}_{odd} \cdot du$$

$$= 2\left[\frac{\sin\lambda u}{\lambda}\right]_0^1$$

$$\boxed{F(\lambda) = \frac{2\sin\lambda}{\lambda}}$$

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For FIR, take inverse F.T of $F(\lambda)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda).e^{-i\lambda x}.d\lambda$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin\lambda}{\lambda}.e^{-i\lambda x}.d\lambda$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \underbrace{\frac{\sin\lambda.\cos\lambda x}{\lambda}}_{\text{even}\ \ \text{even}}.d\lambda + i\int_{-\infty}^{\infty}\underbrace{\frac{\sin\lambda.\sin\lambda x}{\lambda}}_{\text{even}\ \ \text{odd}}.d\lambda\right]$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\lambda}{\lambda}.\cos\lambda x.d\lambda$$

$\hookrightarrow$ FIR of $f(x)$.

Hence, $f(x) = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\lambda}{\lambda}.\cos\lambda x.d\lambda$

$$\int_{0}^{\infty}\frac{\sin\lambda}{\lambda}.\cos\lambda x.d\lambda = \frac{\pi}{2}.f(x)$$

$$= \frac{\pi}{2}.\begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

$$= \begin{cases} \pi/2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

put $x = 0$, $\therefore \int_{0}^{\infty}\frac{\sin\lambda}{\lambda}.d\lambda = \boxed{\frac{\pi}{2}}$

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Q-4. Find $F(z) = \frac{1}{(z-3)(z-4)}$, $|z|>4$ [Nov. 2017]

$\rightarrow$ $F(z) = \frac{1}{(z-3)(z-4)}$

$= \frac{-1}{(z-3)} + \frac{1}{(z-4)}$

$z^{-1}(F(z)) = z^{-1}\left(\frac{-1}{(z-3)}\right) + z^{-1}\left(\frac{1}{(z-4)}\right)$

$= \frac{1}{3} z^{-1}\left[\frac{-3}{z-3}\right] + \frac{1}{4} \cdot z^{-1}\left[\frac{4}{z-4}\right]$

$|z|>3$ $\quad$ $|z|>4.$

$= \frac{-1}{3} \cdot 3^{k} \quad k>0 \quad + \frac{1}{4} \cdot 4^{k} \quad k>0$

$\therefore z^{-1}\left\{\frac{1}{(z-3)(z-4)}\right\} = \begin{cases} 0 & , k \le 0 \\ 4^{k-1} - 3^{k-1} & , k > 0 \end{cases}$

Q-5. Solve the following difference equation:-

$12 f(k+2) - 7 f(k+1) + f(k) = 0, \quad k \ge 0, \quad f(0) = 0, \quad f(1) = 3$

$\rightarrow$ Given DE $\rightarrow$ $12 f(k+2) - 7 f(k+1) + f(k) = 0.$

Applying z-transform, we get

$$12\, z\{f(k+2)\} - 7z\{f(k+1) + z\{f(k) = z\{0\}$$

$$z\{f(k+2)\} = z^2 F(z) - z^2 f(0) - z f(1) = z^2 F(z) - 3z$$

$$z\{f(k+1)\} = z F(z) - z f(0) = z F(z)$$

$$\therefore\ 12[z^2 F(z) - 3z] - 7[zF(z) + F(z)] = 0$$

$$F(z) = \frac{36z}{12z^2 - 7z + 1}$$

$$= \frac{36z}{(4z-1)(3z-1)}$$

$$= \frac{36z}{4(z-1/4)\,3(z-1/3)}$$

$$= \frac{3z}{(z-1/4)(z-1/3)} \Rightarrow \text{Poles are } 1/4 \text{ and } 1/3$$

$$F(z) z^{k-1} = \frac{3z}{(z-1/4)(z-1/3)} \cdot z^{k-1} = \frac{3z^k}{(z-1/4)(z-1/3)}$$

Residue of $F(z).z^{k-1}$ at pole $z = 1/4$ is

$$= \frac{3.\ (1/4)^k}{\left[\frac{1}{4} - \frac{1}{3}\right]} = \frac{-36}{4^k} = \frac{-9}{4^{k-1}}$$

Residue of $F(z).z^{k-1}$ at pole $z = 1/3$ is

$$\left[(z - 1/3).F(z).z^{k-1}\right]_{z=1/3} = \frac{3(1/3)^k}{\left[\frac{1}{3} - \frac{1}{4}\right]} = \frac{36}{3^k} = \frac{4}{3^{k-2}}$$

$\therefore z^{-1}\{F(z)\}$ = Sum of residue

$$= \frac{36}{3^k} - \frac{36}{4^k}$$

$$= 36\left[\left(\frac{1}{3}\right)^k - \left(\frac{1}{4}\right)^k\right]$$

$$\boxed{f(k) = 36\left[\left(\frac{1}{3}\right)^k - \left(\frac{1}{4}\right)^k\right]}$$