Kaustubh S. Kabra. SE-Comp-I, A, 20.

Page No.
Date

* Tutorial Number - 7 (z-Transform).

Q-> Find Z transform of the following:-

1) 3^k, $k<0$

-> $f(k)=3^k$, $k<0$

$$z(3^k) = \sum_{-\infty}^{-1}(3^k \cdot z^{-k}) = \sum_{-\infty}^{-1}(z/3)^{-k}$$

$$= \left[\left(\frac{z}{3}\right)^1 + \left(\frac{z}{3}\right)^2 + \ldots\right]$$

$$= \left[-1 + 1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \ldots\right]$$

$$= \left[-1 + \frac{1}{1-(z/3)}\right]$$

$$= \left[-1 + \frac{3}{3-z}\right]$$

$$= \left[\frac{-3+z+3}{3-z}\right]$$

$$\boxed{z(3^k) = \frac{z}{3-z}, \quad |z|<3.}$$

2) 3^k, $k>0$

-> $f(k)=3^k$, $k>0$

$$z(3^k) = \sum_{1}^{\infty}(3^k \cdot z^{-k}) = \sum_{1}^{\infty}(3/z)^k$$

$$= \left[\left(\frac{3}{z}\right)^1 + \left(\frac{3}{z}\right)^2 + \ldots\right]$$

$$= \left[-1 + 1 + \left(\frac{3}{z}\right)^1 + \left(\frac{3}{z}\right)^2 + \ldots\right]$$

$$= \left[-1 + \frac{1}{1-(3/z)} \right]$$

$$= \left[-1 + \frac{z}{z-3} \right]$$

$$F(z) = \left[\frac{3}{z-3} \right]$$

$$\boxed{\underset{k>0}{Z}(3^k) = \frac{3}{z-3}, \ |z|>|3|}$$

3) 3^k, $k \le 0$

$\rightarrow$ $f(k) = 3^k$, $k \le 0$

$$\underset{k\le 0}{Z}(3^k) = \sum_{-\infty}^{0} 3^k . z^{-k} = \sum_{-\infty}^{0} (z/3)^{-k}$$

$$= \left[\left(\frac{z}{3}\right)^0 + \left(\frac{z}{3}\right)^1 + \left(\frac{z}{3}\right)^2 + \cdots \right]$$

$$= \left[1 + \left(\frac{z}{3}\right)^1 + \left(\frac{z}{3}\right)^2 + \cdots \right]$$

$$= \left[\frac{1}{1-(z/3)} \right]$$

$$F(z) = \frac{3}{3-z}$$

$$\boxed{\underset{k\le 0}{Z}(3^k) = \frac{3}{3-z}, \ |z|<|3|}$$

4) $3^k, k \geq 0$.

$\rightarrow f(k) = 3^k, k \geq 0$.

$$\underset{k\geq 0}{Z}(3^k) = \sum_{0}^{\infty} (3^k) . z^{-k} = \sum_{0}^{\infty} (3/z)^k$$

$$= \left[\left(\frac{3}{z}\right)^0 + \left(\frac{3}{z}\right)^1 + \left(\frac{3}{z}\right)^2 + \dots \right]$$

$$= \left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \dots \right]$$

$$= \left[\frac{1}{1-(3/z)} \right]$$

$$F(z) = \left[\frac{z}{z-3} \right]$$

$$\boxed{\underset{k\geq 0}{Z}\left[3^k\right] = \frac{z}{z-3}, \quad |z| > |3)}$$

5) $3^{-k}, k < 0$.

$\rightarrow f(k) = 3^{-k}, k < 0$

$$\underset{k<0}{Z}(3^{-k}) = \sum_{-\infty}^{-1} 3^{-k} . z^{-k} = \sum_{-\infty}^{-1} (3z)^{-k}$$

$$= \left[(3z)^1 + (3z)^2 + \dots \right]$$

$$= \left[-1 + 1 + 3z + (3z)^2 + \dots \right]$$

$$= \left[-1 + \frac{1}{1-3z} \right]$$

$$F(z) = \left[\frac{3z}{1-3z} \right] \quad |1-3z| < 0.$$

$$\boxed{\underset{k<0}{Z}(3^{-k}) = \left[\frac{3z}{1-3z}\right], \quad |z| < \frac{1}{3}}$$

6) 3^{-k}, $k>0$

$\rightarrow$ $f(k) = 3^{-k}$, $k>0$

$$\underset{k>0}{z(3^{-k})} = \sum_{1}^{\infty}(3^{-k}).\,z^{-k} = \sum_{1}^{\infty}(1/3z)^{k}$$

$$= \left[\left(\frac{1}{3z}\right)^{1} + \left(\frac{1}{3z}\right)^{2} + \cdots\right]$$

$$= \left[-1 + 1 + \frac{1}{3z} + \left(\frac{1}{3z}\right)^{2} + \cdots\right]$$

$$= \left[-1 + \frac{1}{1-(1/3z)}\right]$$

$$= \left[-1 + \frac{3z}{(3z)-1}\right]$$

$$F(z) = \left[\frac{1}{3z-1}\right] \qquad |3z-1| \leq 0 \qquad |z| \leq \frac{1}{3}$$

$$\boxed{\underset{k>0}{z(3^{-k})} = \frac{1}{3z-1}, \quad |z| \leq \frac{1}{3}}$$

7) 3^{-k}, $k \leq 0$

$\rightarrow$ $f(k) = 3^{-k}$, $k \leq 0$

$$\underset{k\leq 0}{z(3^{-k})} = \sum_{-\infty}^{0} 3^{-k}, z^{-k} = \sum_{-\infty}^{0}(3z)^{-k}$$

$$= \left[(3z)^{0} + (3z)^{1} + (3z)^{2} + \cdots\right]$$

$$= \left[1 + 3z + (3z)^{2} + \cdots\right]$$

$$F(z) = \left[\frac{1}{1-3z}\right]$$

$$\boxed{\underset{k\leq 0}{z(3^{-k})} = \frac{1}{1-3z}, \quad |z| > \frac{1}{3}}$$

8) $3^{-k}, \ k \geq 0$

$\rightarrow f(k) = 3^{-k}, \ k \geq 0$

$$\underset{k\geq 0}{Z}\left(3^{-k}\right) = \sum_{0}^{\infty} \left(3^{-k}\right) . z^{-k} = \sum_{0}^{\infty} \left(1/3z\right)^{k}$$

$$= \left[\left(\frac{1}{3z}\right)^{0} + \left(\frac{1}{3z}\right)^{1} + \left(\frac{1}{3z}\right)^{2} + \dots \right]$$

$$= \left[1 + \frac{1}{3z} + \left(\frac{1}{3z}\right)^{2} + \dots \right]$$

$$= \left[\frac{1}{1 - (1/3z)} \right]$$

$$F(z) = \left[\frac{3z}{3z - 1} \right]$$

$$\boxed{\underset{k\geq 0}{Z}\left(3^{-k}\right) = \frac{3z}{3z-1} , \ |z| < \frac{1}{3}}$$

9) $3^{k} + 4^{k}, \ k \geq 0$

$\rightarrow f(k) = 3^{k} + 4^{k}, \ k \geq 0$

$$\underset{k\geq 0}{Z}\left(3^{k} + 4^{k}\right) = \underset{k\geq 0}{Z}\left(3^{k}\right) + \underset{k\geq 0}{Z}\left(4^{k}\right) = \sum_{0}^{\infty} \left(3^{k}\right) . z^{-k} + \sum_{0}^{\infty} \left(4^{k}\right) . z^{-k}$$

$$= \left[\left(\frac{3}{z}\right)^{0} + \left(\frac{3}{z}\right)^{1} + \left(\frac{3}{z}\right)^{2} + \dots \right]$$

$$+ \left[\left(\frac{4}{z}\right)^{0} + \left(\frac{4}{z}\right)^{1} + \left(\frac{4}{z}\right)^{2} + \dots \right]$$

$$= \left[\frac{1}{1 - (3/z)} + \frac{1}{1 - (4/z)} \right]$$

$$\boxed{\underset{k\geq 0}{Z}\left(3^{k} + 4^{k}\right) = \left[\frac{z}{z-3} + \frac{z}{z-4} \right] , \ |z| > 3}$$

10) $3^{-|k|}$ for all k.

$\rightarrow$ $$f(k) = \begin{cases} 3^{-k} , & k \geq 0 \\ 3^{k} , & k < 0 \end{cases}$$

$$\underset{\text{all } k}{Z}\left(3^{-|k|}\right) = \sum_{-\infty}^{-1} 3^{k} . z^{-k} + \sum_{0}^{\infty} 3^{-k} . z^{-k}$$

$$= \sum_{-\infty}^{-1} (z/3)^{-k} + \sum_{0}^{\infty} (1/3z)^{k}$$

$$= \left[\left(\frac{z}{3}\right)^{1} + \left(\frac{z}{3}\right)^{2} + \cdots\right] + \left[\left(\frac{1}{3z}\right)^{0} + \left(\frac{1}{3z}\right)^{1} + \left(\frac{1}{3z}\right)^{2} + \cdots\right]$$

$$= \left[-1 + 1 + \frac{z}{3} + \left(\frac{z}{3}\right)^{2} + \cdots\right] + \left[1 + \frac{1}{3z} + \left(\frac{1}{3z}\right)^{2} + \cdots\right]$$

$$= \left[-1 + \frac{1}{1-(z/3)}\right] + \left[\frac{1}{1-(1/3z)}\right]$$

$$\boxed{\underset{\text{all } k}{Z}\left(3^{-|k|}\right) = \left[\frac{z}{3-z} + \frac{3z}{3z-1}\right] , \quad \left|\frac{1}{3}\right| < |z| < |3|}$$

11) $\left(\frac{1}{5}\right)^{|k|}$ for all k.

$\rightarrow$ $$f(k) = \begin{cases} (1/5)^{k} , & k \geq 0 \\ (1/5)^{-k} , & k < 0 \end{cases}$$

$$\underset{\text{all } k}{Z}\left[\left(\frac{1}{5}\right)^{|k|}\right] = \sum_{-\infty}^{-1} \left(\frac{1}{5}\right)^{-k} . z^{-k} + \sum_{0}^{\infty} \left(\frac{1}{5}\right)^{k} . z^{-k}$$

$$= \sum_{\infty}^{-1} \left(\frac{z}{5}\right)^{-k} + \sum_{0}^{\infty} \left(\frac{1}{5z}\right)^{k}$$

$$= \left[\frac{z}{5} + \left(\frac{z}{5}\right)^{2} + \cdots\right] + \left[1 + \frac{1}{5z} + \left(\frac{1}{5z}\right)^{2} + \cdots\right]$$

$$= \left[-1 + 1 + \frac{z}{5} + \left(\frac{z}{5}\right)^{2} + \cdots\right] + \left[1 + \frac{1}{5z} + \left(\frac{1}{5z}\right)^{2} + \cdots\right]$$

$$F(z) = \left[-1 + \frac{1}{1-(z/5)} \right] + \left[\frac{1}{1-(1/5z)} \right]$$

$$\left|\frac{z}{5}\right| < 1 \qquad \left|\frac{1}{5z}\right| < 1$$

$$\boxed{Z\left[\left(\frac{1}{5}\right)^{|k|}\right]_{\text{all } k} = \left[\frac{z}{5-z} + \frac{5z}{5z-1}\right], |z| < 5, |z| > \frac{1}{5}}$$

12) $e^{-ak}, k \geq 0$

$\rightarrow$ $f(k) = e^{-ak}, \quad k \geq 0$

$$Z\left[(e^{-a})^k\right]_{k \geq 0} = \sum_0^{\infty} (e^{-a})^k . (z)^{-k} = \sum_0^{\infty} \left(e^{-a}/z\right)^k$$

$$= \left[1 + \frac{e^{-a}}{z} + \left(\frac{e^{-a}}{z}\right)^2 + \cdots\right]$$

$$= \frac{1}{1-(e^{-a}/z)}, \quad \frac{e^{-a}}{z} > 1$$

$$\boxed{Z\left[(e^{-a})^k\right]_{k \geq 0} = \frac{z}{z-e^{-a}}, \quad |z| > e^{-a}}$$

13) $e^{ak}, k \geq 0$

$\rightarrow$ $f(k) = e^{ak}, k \geq 0$

$$Z\left[e^{ak}\right]_{k \geq 0} = \sum_0^{\infty} (e^{a})^k . (z)^{-k} = \sum_0^{\infty} \left(e^{a}/z\right)^k$$

$$= \left[1 + \frac{e^{a}}{z} + \left(\frac{e^{a}}{z}\right)^2 + \cdots\right]$$

$$= \left[\frac{1}{1-\left(\frac{e^{a}}{z}\right)}\right], \quad \frac{e^{a}}{z} < 1$$

$$F(z) = \left[\frac{z}{z-e^{a}}\right], \ |z| > e^{a}$$

$$\boxed{Z\left[e^{ak}\right]_{k\geq 0} = \left[\frac{z}{z-e^{a}}\right], \ |z| > e^{a}}$$

14) $e^{ak} + e^{-ak}$, $k \geq 0$

$\rightarrow$ $f(k) = e^{ak} + e^{-ak}$, $k \geq 0$.

$$Z\left[e^{ak} + e^{-ak}\right]_{k\geq 0} = Z\left[e^{ak}\right]_{k\geq 0} + Z\left[e^{-ak}\right]_{k\geq 0}$$

$$= \sum_{0}^{\infty}\left[(e^{a})^{k}\right] z^{-k} + \sum_{0}^{\infty}\left[(e^{-a})^{k}\right] z^{-k}$$

$$= \left[1 + \frac{e^{a}}{z} + \left(\frac{e^{a}}{z}\right)^{2} + \cdots\right] + \left[1 + \frac{e^{-a}}{z} + \left(\frac{e^{-a}}{z}\right)^{2} + \cdots\right]$$

$$= \left[\frac{1}{1-\frac{e^{a}}{z}}\right] + \left[\frac{1}{1-\frac{e^{-a}}{z}}\right]$$

$$F(z) = \left[\frac{z}{z-e^{a}} + \frac{z}{z-e^{-a}}\right], \ \left|\frac{e^{a}}{z}\right| < 1, \ \left|\frac{e^{-a}}{z}\right| < 1.$$

$$\boxed{Z\left[e^{ak} + e^{-ak}\right]_{k\geq 0} = \left[\frac{z}{z-e^{a}} + \frac{z}{z-e^{-a}}\right], \ |z| > e^{a}, \ |z| > e^{-a}}$$

Scanned by TapScanner

15) $\sin \alpha k \, , \, k \geq 0$

$\rightarrow \sin \alpha k = \dfrac{e^{i\alpha k} - e^{-i\alpha k}}{2i}$

$f(k) = \dfrac{e^{i\alpha k}}{2i} - \dfrac{e^{-i\alpha k}}{2i}, \; k \geq 0$

$$Z\left[\frac{1}{2i}\left[(e^{i\alpha})^k - (e^{-i\alpha})^k\right]\right] = \frac{1}{2i} \cdot Z\left[(e^{i\alpha})^k - (e^{-i\alpha})^k\right]$$

$$= \frac{1}{2i}\left[\underset{k \geq 0}{Z\left[(e^{i\alpha})^k\right]} - \underset{k \geq 0}{Z\left[(e^{-i\alpha})^k\right]} \right]$$

$$= \frac{1}{2i}\left[\sum_0^{\infty} \left[e^{i\alpha}\right]^k . z^{-k} - \sum_0^{\infty} \left[e^{-i\alpha}\right]^k . z^{-k} \right]$$

$$= \frac{1}{2i}\left[\sum_0^{\infty} \left(\frac{e^{i\alpha}}{z}\right)^k - \sum_0^{\infty} \left(\frac{e^{-i\alpha}}{z}\right)^k \right]$$

$$= \frac{1}{2i}\left[\left\{1 + \frac{e^{i\alpha}}{z} + \left(\frac{e^{i\alpha}}{z}\right)^2 + \dots\right\} - \left\{1 + \frac{e^{-i\alpha}}{z} + \left(\frac{e^{-i\alpha}}{z}\right)^2 \dots\right\}\right]$$

$$= \frac{1}{2i}\left[\left\{\frac{1}{1 - \left(\frac{e^{i\alpha}}{z}\right)}\right\} - \left\{\frac{1}{1 - \left(\frac{e^{-i\alpha}}{z}\right)}\right\}\right]$$

$$e^{i\alpha} < |z| \;, \quad e^{-i\alpha} < |z|$$

$$\boxed{\underset{k \geq 0}{Z\left[\sin \alpha k\right]} = \frac{1}{2i}\left[\frac{z}{z - e^{i\alpha}} - \frac{z}{z - e^{-i\alpha}}\right], \; |z| > e^{i\alpha}, \; |z| > e^{-i\alpha}}$$

Scanned by TapScanner

16) $\cos ak,\ k \geq 0$

$\rightarrow$ $\cos ak = \dfrac{e^{iak} + e^{-iak}}{2}$

$$f(k) = \frac{e^{iak}}{2} + \frac{e^{-iak}}{2},\ k \geq 0$$

$$Z\left[\frac{1}{2}(e^{ia})^k + \frac{1}{2}(e^{-ia})^k\right] = \frac{1}{2}\left[\underset{k\geq 0}{Z}\left[(e^{ia})^k\right] + \underset{k\geq 0}{Z}\left[(e^{-ia})^k\right]\right]$$

$$= \frac{1}{2}\left[\sum_0^\infty (e^{ia})^k . z^{-k} + \sum_0^\infty (e^{-ia})^k . z^{-k}\right]$$

$$= \frac{1}{2}\left[\sum_0^\infty \left(\frac{e^{ia}}{z}\right)^k + \sum_0^\infty \left(\frac{e^{-ia}}{z}\right)^k\right]$$

$$= \frac{1}{2}\left[\left\{1 + \frac{e^{ia}}{z} + \left(\frac{e^{ia}}{z}\right)^2 + \cdots\right\} + \left\{1 + \frac{e^{-ia}}{z} + \left(\frac{e^{-ia}}{z}\right)^2 + \cdots\right\}\right]$$

$$= \frac{1}{2}\left[\left\{\frac{1}{1 - \frac{e^{ia}}{z}}\right\} + \left\{\frac{1}{1 - \frac{e^{-ia}}{z}}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{ia}} + \frac{z}{z - e^{-ia}}\right]$$

$$\boxed{\underset{k\geq 0}{Z}[\cos ak] = \frac{1}{2}\left[\frac{z}{z - e^{ia}} + \frac{z}{z - e^{-ia}}\right],\ |z| > e^{ia},\ |z| > e^{-ia}}$$

17) $\frac{2}{3}$, $k \geq 0$

$\rightarrow$ $f(k) = \frac{2}{3}$, $k \geq 0$.

$$Z\left[\frac{2}{3}\right] = \frac{2}{3} \cdot Z\underset{k\geq 0}{[1]} = \frac{2}{3} \cdot \sum_{0}^{\infty} 1 . z^{-k}$$

$$= \frac{2}{3}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots\right]$$

$$= \frac{2}{3}\left[\frac{1}{1-(1/z)}\right], \quad \frac{1}{z} < 1$$

$$= \frac{2}{3}\left[\frac{z}{z-1}\right]$$

$$\boxed{Z\underset{k\geq 0}{\left[\frac{2}{3}\right]} = \frac{2z}{3(z-1)}, \quad |z| > 1}$$

Q $\rightarrow$ Multiple choice Questions on z-transform :-

1) $z\{1\} = ?$ for $k \geq 0$.

$\rightarrow$ $\underset{k\geq 0}{Z\{1\}} = \sum_{0}^{\infty} 1 . z^{-k} = 1\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots\right] = \left[\frac{1}{1-(1/z)}\right] = \left[\frac{z}{z-1}\right], |z| > 1$ (B)

2) $z\{3\} = ?$ for k

$\rightarrow$ $\underset{k\geq 0}{z\{3\}} = \sum_{0}^{\infty} 3 . z^{-k} = 3\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots\right] = 3\left[\frac{1}{1-(1/z)}\right] = \frac{3z}{z-1}, |z| > 1$ (B)

$\frac{1}{z} < 1$

3) $z\{5^k\} = ?$ for $k \geq 0$.

$\rightarrow$ $\underset{k\geq 0}{z\{5^k\}} = \sum_{0}^{\infty} 5^k . z^{-k} = \left[1 + \frac{5}{z} + \left(\frac{5}{z}\right)^2 + \cdots\right] = \left[\frac{1}{1-(5/z)}\right] = \frac{z}{z-5}, |z| > 5$ (A)

$\frac{5}{z} < 1$

4) $Z\{4^k\} = ?$ for $k \le 0$.

$\rightarrow$ $\underset{k\le 0}{Z\{4^k\}} = \sum_{-\infty}^{0} 4^k \cdot z^{-k} = \sum_{-\infty}^{0} \left(\frac{z}{4}\right)^{-k} = \left[1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \cdots\right]$

$$= \frac{1}{1-\frac{z}{4}} , \quad \frac{z}{4} < 1$$

$$= \frac{4}{4-z} , \quad |z| < 4$$

D

5) $Z\{5^{k+1}\} = ?$ for $k \ge 0$.

$\rightarrow$ $\underset{k\ge 0}{Z\{5^{k+1}\}} = 5Z\{5^k\} = 5\sum_{0}^{\infty} [5^k] z^{-k} = 5 \cdot \sum_{0}^{\infty} \left[\frac{5}{z}\right]^k$

$$= 5\left[1 + \frac{5}{z} + \left(\frac{5}{z}\right)^2 + \cdots\right]$$

$$= 5\left[\frac{1}{1-5/z}\right] , \quad \frac{5}{z} < 1$$

$$= \frac{5z}{z-5} , \quad |z| > 5$$

A.