SUBJECT CODE : 207003

As per Revised Syllabus of

SAVITRIBAI PHULE PUNE UNIVERSITY

Choice Based Credit System (CBCS)

S.E. (Comp / IT) Semester - IV

ENGINEERING MATHEMATICS - III

Dr. H. R. Bhapkar
M.Sc. SET, Ph.D. (Mathematics)
MIT Art, Design and Technology University's
MIT School of Engineering, Lonikalbhor, Pune.
Email : drhrbhapkar@gmail.com Mobile : 9011227141

Prof. A. R. Tambe
M.Sc. (Mathematics)
Associate Professor,
Amrut Vahini College of Engineering, Sangamner.

Dr. M. K. Nikam
M.Sc., M.Phil., Ph.D. SET
Associate Professor,
AISSMS College of Engineering, Pune.

Dr. S. N. Kadlag
M.Sc. (Mathematics), Ph.D.
Professor and Head of Science Dept.
K.K. Wagh Institute of Engineering Education
and Research, Nashik.

Dr. Vanita R. Daddi
M.Sc., M.Phil., SET Ph.D.
Assistant Professor,
Dr. D.Y. Patil Institute of Engineering,
Management and Research, Akurdi, Pune.

Prof. Shital A. Patil
M.Sc. (Mathematics) from IIT Bombay,
Assistant Professor,
Pimpri Chinchwad College of Engg. & Research, Ravet.

TECHNICAL® PUBLICATIONS
SINCE 1993
An Up-Thrust for Knowledge

ENGINEERING MATHEMATICS - III

Subject Code : 207003

S.E. (Computer Engineering / Information Technology) Semester - IV

Published by :

TECHNICAL PUBLICATIONS
SINCE 1993 An Up-Thrust for Knowledge

Amit Residency, Office No.1, 412, Shaniwar Peth,
Pune - 411030, M.S. INDIA Ph.: +91-020-24495496/97
Email : sales@technicalpublications.org Website : www.technicalpublications.org

Printer :
Yogiraj Printers & Binders
Sr.No. 10/1A,
Ghule Industrial Estate, Nanded Village Road,
Tal. - Haveli, Dist. - Pune - 411041.

ISBN 978-93-90450-65-7

9 789390 450657

SPPU 19

PREFACE

The importance of **Engineering Mathematics - III** is well known in various engineering fields. Overwhelming response to our books on various subjects inspired us to write this book. The book is structured to cover the key aspects of the subject **Engineering Mathematics - III.**

The book uses plain, lucid language to explain fundamentals of this subject. The book provides logical method of explaining various complicated concepts and stepwise methods to explain the important topics. Each chapter is well supported with necessary illustrations, practical examples and solved problems. All the chapters in the book are arranged in a proper sequence that permits each topic to build upon earlier studies. All care has been taken to make students comfortable in understanding the basic concepts of the subject.

The book not only covers the entire scope of the subject but explains the philosophy of the subject. This makes the understanding of this subject more clear and makes it more interesting. The book will be very useful not only to the students but also to the subject teachers. The students have to omit nothing and possibly have to cover nothing more.

We wish to express our profound thanks to all those who helped in making this book a reality. Much needed moral support and encouragement is provided on numerous occasions by our whole family. We wish to thank the **Publisher** and the entire team of **Technical Publications** who have taken immense pain to get this book in time with quality printing.

Any suggestion for the improvement of the book will be acknowledged and well appreciated.

Authors
Dr. H. R. Bhapkar
Prof. A. R. Tambe
Dr. M. K. Nikam
Dr. S. N. Kadlag
Dr. Vanita R. Daddi
Prof. Shital A. Patil

Dedicated to God

SYLLABUS

Engineering Mathematics - III (207003)

Credit Scheme	Examination Scheme and Marks
Theory : 03	**Mid_Semester(TH) :** 30 Marks
Tutorial : 01	**End_Semester(TH) :** 70 Marks
	Term Work : 25 Marks

Unit I Linear Differential Equations (LDE)

LDE of n^{th} order with constant coefficients, Complementary Function, Particular Integral, General method, Short methods, Method of variation of parameters, Cauchy's and Legendre's DE, Simultaneous and Symmetric simultaneous DE. **(Chapters - 1, 2)**

Unit II Transforms

Fourier Transform (FT) : Complex exponential form of Fourier series, Fourier integral theorem, Fourier Sine and Cosine integrals, Fourier transform, Fourier Sine and Cosine transforms and their inverses, Discrete Fourier Transform.

Z-Transform (ZT) : Introduction, Definition, Standard properties, ZT of standard sequences and their inverses. Solution of difference equations. **(Chapters - 3, 4)**

Unit III Statistics

Measures of central tendency, Measures of dispersion, Coefficient of variation, Moments, Skewness and Kurtosis, Curve fitting : fitting of straight line, parabola and related curves, Correlation and Regression, Reliability of Regression Estimates. **(Chapter - 5)**

Unit IV Probability and Probability Distributions

Probability, Theorems on Probability, Bayes theorem, Random variables, Mathematical Expectation, Probability density function, Probability distributions : Binomial, Poisson, Normal and Hypergeometric, Sampling distributions, Test of Hypothesis : Chi-Square test, t-test. **(Chapter - 6)**

Unit V Numerical Methods

Numerical Solution of Algebraic and Transcendental equations : Bisection, Secant, Regula-Falsi, Newton-Raphson and Successive Approximation Methods, Convergence and Stability. Numerical Solutions of System of linear equations : Gauss elimination, **LU Decomposition**, Cholesky, Jacobi and Gauss-Seidel Methods. **(Chapter - 7)**

Unit VI Numerical Methods

Interpolation : Finite Differences, Newton's and Lagrange's Interpolation formulae, Numerical Differentiation. Numerical Integration : Trapezoidal and Simpson's rules, Bound of truncation error. Solution of Ordinary differential equations : Euler's, Modified Euler's, Runge-Kutta 4^{th} order methods and Predictor-Corrector methods. **(Chapters - 8, 9)**

TABLE OF CONTENTS

Unit - III

Unit - IV

Notes

UNIT - I

1 Linear Differential Equations with Constant Coefficients

1.1 Linear Differential Equations with Constant Coefficients

An equation involving derivatives is known as the differential equation.

The general form of n^{th} order linear differential equation with constant coefficient is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + a_{n-1} \frac{dy}{dx} + a_n y = \phi(x) \; ; \; a_0 \neq 0 \qquad ... (1.1)$$

where $a_0, a_1, a_2 ... a_n$ are all constants and $\phi(x)$ is any function of x only.

Let $\quad D \equiv \frac{d}{dx}$ then equation becomes

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + ... + a_{n-1} Dy + a_n y = \phi(x)$$

$$(a_0 D^n + a_1 D^{n-1} + ... + a_{n-1} D + a_n) y = \phi(x)$$

$$f(D)\, y = \phi(x) \qquad ... (1.2)$$

where f(D) is a polynomial in D of degree n and $\phi(x)$ is any function of x.

f(D) y = 0 is known as Associated Differential equation, (Homogeneous Differential Equation) and f(D) = 0 is known as Auxiliary equation.

The solution of equation (1.2) involves two parts

1) Complementary Function (C.F.) (y_c)

2) Particular Integral (P.I.) (y_p)

y = C.F. + P.I. gives the complete solution of the differential equation.

1.2 Complementary Function

The complementary function is associated with the DE f(D)y = 0.

The complementary function is the **general solution** of a homogeneous linear differential equation, f(D)y = 0.

A general solution of DE is a solution in which the number of an arbitrary constants is equal to the order of the DE.

Thus, the number of an arbitrary constants in a complementary function is equal to the order of the DE.

Depending upon the nature of the roots of auxiliary equation, there are mainly four types of complementary functions, which are given below

Sr. No.	Nature of roots of A.E. f(D) = 0	Complementary functions (C.F.) y_c
1.	Roots are real and distinct a) $D = m_1$, $D = m_2$ are real and distinct roots b) $D = m_1$, $D = m_2$, $D = m_3$ are real and distinct roots	a) $y_c = C_1e^{m_1x} + C_2e^{m_2x}$ b) $y_c = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x}$
2.	Roots are real and repeated a) $D = m_1, m_1$ m_1 is real and repeated twice b) $D = m_1, m_1, m_1$ m_1 is real and repeated thrice	a) $y_c = (C_1 + C_2x)e^{m_1x}$ b) $y_c = (C_1 + C_2x + C_3x^2)e^{m_1x}$
3.	Roots are complex and distinct a) If $\alpha + i\beta$ is one root then $\alpha - i\beta$ is also root $\therefore$ $\alpha + i\beta$ and $\alpha - i\beta$ are complex and distinct roots b) If $\alpha \pm i\beta$ and $a \pm ib$ are complex and distinct roots	a) $y_c = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]$ b) $y_c = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x] + e^{\alpha x}[C_3 \cos bx + C_4 \sin bx]$
4.	Roots are complex and repeated a) $\alpha \pm i\beta$ repeated twice b) $\alpha \pm i\beta$ repeated thrice	a) $y_c = e^{\alpha x}[(C_1 + C_2x)\cos\beta x + (C_3 + C_4x)\sin\beta x]$ b) $y_c = e^{\alpha x}[(C_1 + C_2x + C_3x^2)\cos\beta x + (C_4 + C_5x + C_6x^2)\sin\beta x]$

Let us consider the following examples

Sr. No.	Nature of roots of A.E. f(D) = 0	Complementary function
1.	2, – 3	$y_c = C_1e^{2x} + C_2e^{-3x}$
2.	1, – 1, 5	$y_c = C_1e^{x} + C_2e^{-x} + C_3e^{5x}$
3.	3, 3	$y_c = (C_1 + C_2x)e^{3x}$
4.	– 2, – 2, – 2	$y_c = (C_1 + C_2x + C_3x^2)e^{-2x}$
5.	2 ± 3i	$y_c = e^{2x}[C_1 \cos 3x + C_2 \sin 3x]$
6.	2 ± 3i, 2 ± 3i	$y_c = e^{2x}[(C_1 + C_2x)\cos 3x + (C_3 + C_4x)\sin 3x]$
7.	2, – 3, – 3	$y_c = C_1e^{2x} + (C_2 + C_3x)e^{-3x}$
8.	1, – 2 ± 3i	$y_c = C_1e^{x} + e^{-2x}[C_2 \cos 3x + C_3 \sin 3x]$
9.	1, 2, 2, 1 ± i	$y_c = C_1e^{x} + (C_2 + C_3x)e^{2x} + e^{x}[C_4 \cos x + C_5 \sin x]$
10.	1, 2, 2, 2	$y_c = C_1e^{x} + (C_2 + C_3x + C_4x^2)e^{2x}$

Note :

1) The number of arbitrary constants present in the C.F. must be equal to the order of the differential equation.
2) If $(D - m_1)^2$ is present in f(D) then the root $D = m_1$ is repeated twice.
   If $(D - m_1)^3$ is present in f(D) then the root $D = m_1$ is repeated thrice.
3) If $aD^2 + bD + C = 0$
   i.e. If f(D) = 0 is quadratic polynomial then apply

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

4) For the cubic equation apply the synthetic division after finding one real non zero root.
5) For the 4^{th} power equation we may get all complex roots, in such a case synthetic division is not applicable. Here adjust the perfect square.
6) **Rules for synthetic division :**
   a) If sum of all coefficient in f(D) = 0 is zero then D = 1 is the root.
   b) If sums of alternate coefficients in f(D) = 0 are equal then D = – 1 is the root.
   c) If D = 1 and – 1 are not the roots then try by the divisors of the last number. i.e. constant term.
   d) If the constant term is absent in f(D) = 0, then D = 0 is one root of the A.E.
7) Synthetic division method is applicable only for finding real roots.

Find the C.F. of the following f(D) = 0

1) $D^3 - 7D - 6 = 0$

i.e. $D^3 + 0D^2 - 7D - 6 = 0$

The sum of the coefficients of I^{st} and III^{rd} degree terms is 1 – 7 = – 6 and II^{nd} and 0^{th} degree terms is – 6 + 0 = 6. Therefore, D = – 1 is one root of f(D) = 0

$\therefore$ Apply synthetic division method

– 1	1	0	– 7	– 6
		– 1	1	6
	1	– 1	– 6	$\boxed{0}$
Coefficient of	D^2	D	Constant term	

$\therefore$ The factors are D + 1, $D^2 - D - 6$

$\therefore (D+1)\ (D^2 - D - 6) = 0$

$\therefore\ (D+1)(D-3)(D+2) = 0$

$\therefore$ D = – 1, D = 3, D = – 2 are real roots.

As roots are real and distinct, we have

$$\therefore \quad \boxed{\text{C.F.} = C_1 e^{-x} + C_2 e^{3x} + C_2 e^{-2x}}$$

2) $$\underset{\uparrow}{D^3} + 7D^2 + \underset{\uparrow}{16D} + 10 = 0$$

Here coefficients of D and D^3 are 1 and 16 respectively and their sum is 17. And coefficients of D^2 and 1 are 7 and 10 respectively having sum 7 + 10 = 17.

Sums of alternates are equal $\therefore$ D = – 1.

$$\therefore \quad (D+1)\ (D^2 + 6D + 10) = 0$$

– 1	1	7	16	10
		– 1	– 6	– 10
	1	6	10	$\boxed{0}$
	D^2	D	Constant term	

Thus $\quad (D+1)(D^2 + 6D + 10) = 0$

D + 1 = 0, $D^2 + 6D + 10 = 0$

$$D = -1, \quad D = \frac{-6 \pm \sqrt{36-40}}{2}$$

$$D = \frac{-6 \pm \sqrt{-4}}{2}$$

$$= -3 \pm i = \alpha \pm i\beta$$

$$\therefore \quad \alpha = -3,\ \beta = 1$$

Thus D = – 1, D = – 3 + i, – 3 – i

$\therefore$ One root is real and two roots are complex.

$$\boxed{\text{C.F.} = C_1 e^{-x} + e^{-3x}(C_2 \cos x + C_3 \sin x)}$$

3) $D^4 - a^4 = 0$

$$(D^2 - a^2)\ (D^2 + a^2) = 0$$

$$(D+a)(D-a)(D^2 + a^2) = 0$$

$$D = -a,\ D = a$$

and $\quad D^2 = -a^2$

$$D = \pm ia = \alpha \pm i\beta$$

$$\alpha = 0,\ \beta = a$$

$\therefore$ Thus, roots are D = a, D = – a, D = – ia, D = ia

Two roots are real and two roots are complex.

C.F. = $C_1 e^{-ax} + C_2 e^{ax} + e^{0x}(C_3 \cos ax + C_4 \sin ax)$

4) $D^3 + 3D^2 + 3D + 1 = 0$

Sums of alternates are equal $\therefore$ D = – 1

– 1	1	3	3	1
		– 1	– 2	– 1
	1	2	1	$\boxed{0}$

$$(D+1)^3 = 0$$
$$(D+1)(D^2+2D+1) = 0$$
$$(D+1)(D+1)^2 = 0$$
$$D = -1, -1, -1$$
$$\text{C.F.} = \left(C_1+C_2x+C_3x^2\right)e^{-x}$$

5) $D^3+D^2+D+1 = 0$

Sums of alternates are equal $\therefore$ D = – 1

– 1	1	1	1	1
		– 1	0	– 1
	1	0	1	$\boxed{0}$

$$(D+1)(D^2+1) = 0$$
$$D = -1,\ D^2 = -1 \Rightarrow D = -1,\ D = \pm i$$

One root is real and two roots are complex.

$$\text{C.F.} = C_1e^{-x}+e^{0x}(C_2\cos x+C_3\sin x)$$

6) $D^6-6D^5+12D^4-6D^3-9D^2+12D-4 = 0$

Sum of all coefficients is zero $\therefore$ D = 1

1	1	– 6	12	– 6	– 9	12	– 4
		1	– 5	7	1	– 8	4
1	1	– 5	7	1	– 8	4	$\boxed{0}$
		1	– 4	3	4	– 4	
1	1	– 4	3	4	– 4	$\boxed{0}$	
		1	– 3	0	+ 4		
– 1	1	– 3	0	4	$\boxed{0}$		
		– 1	+ 4	– 4			
	1	– 4	+ 4	$\boxed{0}$			

$$(D-1)^3(D+1)(D^2-4D+4) = 0$$
$$(D-1)^3(D+1)(D-2)^2 = 0$$
$$D = 1, 1, 1, -1, 2, 2$$
$$\therefore \text{C.F.} = (C_1+C_2x+C_3x^2)e^x+C_4e^{-x}+(C_5+C_6x)e^{2x}$$

7) $D^3+D^2-D-1 = 0$

$$(D-1)(D^2+2D+1) = 0$$
$$(D-1)(D+1)^2 = 0$$

1	1	1	– 1	– 1
		1	2	1
	1	2	1	$\boxed{0}$

$$D = 1, -1, -1$$
$$\text{C.F.} = C_1e^x+(C_2+C_3x)e^{-x}$$

8) $D^3-5D^2+8D-4 = 0$

$$(D-1)(D^2-4D+4) = 0$$
$$(D-1)(D-2)^2 = 0$$

1	1	– 5	8	– 4
		1	– 4	4
	1	– 4	4	$\boxed{0}$

$$D = 1, 2, 2$$
$$\text{C.F.} = C_1e^x+(C_2+C_3x)e^{2x}$$

9) $D^4-1 = 0$

$$(D^2+1)(D^2-1) = 0$$
$$D^2 = -1,\quad D^2 = 1$$
$$D = \pm i \qquad D = \pm 1$$

Two roots are real and two roots are complex.

$$\therefore \quad \text{C.F.} = C_1e^x+C_2e^{-x}+C_3\cos x+C_4\sin x$$

10) $(D-1)^2(D^2+1)^2 = 0$

$$D = 1, 1,\ D^2 = -1$$
$$D = \pm i \text{ repeated}$$
$$\text{C.F.} = (C_1+C_2x)e^x+\left[(C_3+C_4x)\cos x+(C_5+C_6x)\sin x\right]$$

11) $D^4+8D^2+16 = 0$

$$(D^2+4)^2 = 0$$
$$D^2+4 = 0 \qquad \text{twice}$$
$$D^2 = -4$$
$$D = \pm 2i \qquad \text{repeated complex}$$
$$\text{C.F.} = e^{0x}\left[(C_1+C_2x)\cos 2x+(C_3+C_4x)\sin 2x\right]$$

12) $D^4-2D^3+D^2 = 0$

$$D^2(D^2-2D+1) = 0$$
$$D^2(D-1)^2 = 0$$
$$D^2 = 0,\ (D-1)^2 = 0$$
$$D = 0, 0, 1, 1$$
$$\text{C.F.} = (C_1+C_2x)e^{0x}+(C_3+C_4x)e^x$$

13) $D^3-2D+4 = 0$

$$(D+2)(D^2-2D+2) = 0$$

– 2	1	0	– 2	4
		– 2	4	– 4
	1	– 2	2	$\boxed{0}$

$$D = -2, \ D = \frac{+2 \pm \sqrt{4-8}}{2}$$

$$D = -2, \ D = 1 \pm i$$

$$\text{C.F.} = C_1 e^{-2x} + e^{+x} (C_2 \cos x + C_3 \sin x)$$

14) $D^5 - D = 0$

$$D(D^4 - 1) = 0$$

$$D = 0, \ (D^2 - 1)(D^2 + 1) = 0$$

$$(D-1)(D+1)(D^2+1) = 0$$

$$D = 0, D = 1, D = -1, D^2 = -1$$

$$\text{C.F.} = C_1 e^{0x} + C_2 e^x + C_3 e^{-x} + e^{0x} (C_4 \cos x + C_5 \sin x)$$

15) $D^3 - 3D^2 + 4$

– 1	1	– 3	0	4
		– 1	– 4	– 4
	1	– 4	4	$\boxed{0}$

$$(D-2)^2 (D+1) = 0$$

$$\text{C.F.} = C_1 e^{-x} + (C_2 + C_3 x) e^{2x}$$

16) $D^3 - D^2 + 3D + 5 = 0$

$$(D+1)(D^2 - 2D + 5) = 0$$

– 1	1	– 1	3	5
		– 1	2	– 5
	1	– 2	5	0

$$D = \frac{+2 \pm \sqrt{4-20}}{2}$$

$$D = -1, D = 1 \pm i2$$

$$\text{C.F.} = C_1 e^{-x} + e^x (C_2 \cos 2x + C_3 \sin 2x)$$

17) $D^3 + 7D^2 + 16D + 10 = 0$

$$(D+1)(D^2 + 6D + 10) = 0$$

– 1	1	7	16	10
		– 1	– 6	– 10
	1	6	10	0

$$D^2 + 6D + 10 = 0$$

$$D = \frac{-6 \pm \sqrt{36-40}}{2}$$

$$= -3 \pm i$$

$$\therefore \quad D = -1, \ D = -3 \pm i$$

$$\text{C.F.} = C_1 e^{-x} + e^{-3x} (C_2 \cos x + C_3 \sin x)$$

18) $D^3 - 3D^2 + 3D - 1 = 0$

$$(D-1)^3 = 0$$

$$D = 1, 1, 1$$

$$\text{C.F.} = (C_1 + C_2 x + C_3 x^2) e^{+x}$$

19) $D^4 - 7D^2 + 12 = 0$

$$(D^2 - 3)(D^2 - 4) = 0$$

$$(D + \sqrt{3})(D - \sqrt{3})(D-2)(D+2) = 0$$

$$\text{C.F.} = C_1 e^{-\sqrt{3}x} + C_2 e^{\sqrt{3}x} + C_3 e^{2x} + C_4 e^{-2x}$$

20) $D^4 - 10D^2 + 9 = 0$

$$(D^2 - 9)(D^2 - 1) = 0$$

$$(D-3)(D+3)(D-1)(D+1) = 0$$

$$D = 3, -3, 1, -1$$

$$\text{C.F.} = C_1 e^{-3x} + C_2 e^{3x} + C_3 e^x + C_4 e^{-x}$$

21) $D^4 - 2D^3 + 4D^2 + 2D - 5 = 0$

– 1	1	– 2	4	2	– 5
		– 1	3	– 7	5
+ 1	1	– 3	7	– 5	$\boxed{0}$
		+ 1	– 2	5	
	1	– 2	5	$\boxed{0}$	

$$(D-1)(D+1)(D^2 - 2D + 5) = 0$$

$$D = 1, -1, \quad D = \frac{2 \pm \sqrt{4-20}}{2}$$

$$D = 1, -1, \quad D = 1 \pm 2i$$

$$\text{C.F.} = C_1 e^x + C_2 e^{-x} + e^x (C_3 \cos 2x + C_4 \sin 2x)$$

Note : For problem numbers 22 to 33 as all the roots are complex synthetic division is not applicable so the roots are obtained by adjustment.

22) $D^4 + 10D^2 + 9 = 0$

$$(D^2 + 1)(D^2 + 9) = 0$$

$$D^2 = -1, \ D^2 = -9$$

$$D = \pm i \quad D = \pm 3i$$

All the roots are complex.

$$\text{C.F.} = C_1 \cos x + C_2 \sin x + C_3 \cos 3x + C_4 \sin 3x$$

23) $D^4+1 = 0$

$D^4+2D^2+1-2D^2 = 0$ ← Note this step.

$(D^2+1)^2-(\sqrt{2}\,D)^2 = 0$

$(D^2+1+\sqrt{2}\,D)(D^2+1-\sqrt{2}\,D) = 0$

$D^2+\sqrt{2}\,D+1 = 0, \quad D^2-\sqrt{2}\,D+1 = 0$

$\therefore \quad D = \dfrac{-\sqrt{2}\pm\sqrt{2-4}}{2} \quad D = \dfrac{\sqrt{2}\pm\sqrt{2-4}}{2}$

$\therefore \quad D = \dfrac{-1}{\sqrt{2}}\pm i\dfrac{1}{\sqrt{2}} \quad D = \dfrac{1}{\sqrt{2}}\pm i\dfrac{1}{\sqrt{2}}$

$$\text{C.F.} = e^{x/\sqrt{2}}\left[C_1\cos\frac{x}{\sqrt{2}}+C_2\sin\frac{x}{\sqrt{2}}\right]$$

$$+e^{-x/2}\left[C_3\cos\frac{x}{\sqrt{2}}+C_4\sin\frac{x}{\sqrt{2}}\right]$$

24) $D^4+8D^2+16 = 0$

$(D^2+4)^2 = 0$

$D^2 = -4$ repeated twice

$D = \pm 2i$

$$\text{C.F.} = e^{0x}\left[(C_1+C_2x)\cos 2x+(C_3+C_4x)\sin 2x\right]$$

25) $D^4+m^4 = 0$

$D^4+2m^2D^2+m^4-2m^2D^2 = 0$ ← note this step.

$(D^2+m^2)^2-(\sqrt{2}\,mD)^2 = 0$

$(D^2+m^2+\sqrt{2}\,m\,D)(D^2+m^2-\sqrt{2}\,m\,D) = 0$

$D^2+\sqrt{2}\,m\,D+m^2 = 0, \quad D^2-\sqrt{2}\,m\,D+m^2 = 0$

$D = \dfrac{-\sqrt{2}\,m\pm\sqrt{2m^2-4m^2}}{2}, \quad D = \dfrac{\sqrt{2}\,m\pm\sqrt{2m^2-4m^2}}{2}$

$D = -\dfrac{m}{\sqrt{2}}\pm i\dfrac{m}{\sqrt{2}}, \quad D = \dfrac{m}{\sqrt{2}}\pm i\dfrac{m}{\sqrt{2}}$

$$\text{C.F} = e^{-\frac{mx}{\sqrt{2}}}\left[C_1\cos\frac{mx}{\sqrt{2}}+C_2\sin\frac{mx}{\sqrt{2}}\right]$$

$$+e^{\frac{mx}{\sqrt{2}}}\left[C_3\cos\frac{mx}{\sqrt{2}}+C_4\sin\frac{mx}{\sqrt{2}}\right]$$

26) $D^4+D^2+1 = 0$

$D^4+2D^2+1-D^2 = 0$ ← note this step.

$(D^2+1)^2-D^2 = 0$

$(D^2+D+1)(D^2-D+1) = 0$

$D^2+D+1 = 0, \quad D^2-D+1 = 0$

$D = \dfrac{-1\pm\sqrt{1-4}}{2}, \quad D = \dfrac{1\pm\sqrt{1-4}}{2}$

$D = \dfrac{-1}{2}\pm i\dfrac{\sqrt{3}}{2}, \quad D = \dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2}$

$$\text{C.F.} = e^{-x/2}\left[C_1\cos\frac{\sqrt{3}x}{2}+C_2\sin\frac{\sqrt{3}x}{2}\right]$$

$$+e^{x/2}\left[C_3\cos\frac{\sqrt{3}x}{2}+C_4\sin\frac{\sqrt{3}x}{2}\right]$$

27) $D^4+5D^2+4 = 0$

$(D^2+1)(D^2+4) = 0$

$D^2 = -1, \quad D^2 = -4$

$D = \pm i \quad D = \pm 2i$

$$\text{C.F.} = e^{0x}(C_1\cos x+C_2\sin x)+e^{0x}(C_3\cos 2x+C_4\sin 2x)$$

28) $D^4+2D^3+3D^2+2D+1 = 0$

$(D^2)^2+2D^2D+D^2+2D^2+2D+1 = 0$

← note this step.

$(D^2+D)^2+2(D^2+D)+1 = 0$

$[(D^2+D)+1]^2 = 0$

$D = \dfrac{-1\pm i\sqrt{3}}{2}$ repeated twice

$$\text{C.F.} = e^{-x/2}\left[(C_1+C_2x)\cos\frac{\sqrt{3}x}{2}+(C_3+C_4x)\sin\frac{\sqrt{3}x}{2}\right]$$

29) $D^4-4D^3+8D^2-8D+4 = 0$

$(D^2)^2-2D^2\cdot 2D+(2D)^2+4D^2-8D+4 = 0$

← note this step.

$(D^2-2D)^2+4(D^2-2D)+4 = 0$

$[(D^2-2D)+2]^2 = 0$

$D = \dfrac{+2\pm\sqrt{4-8}}{2}$

$D = 1\pm i$

$$\text{C.F.} = e^x\left[(C_1+C_2x)\cos x+(C_3+C_4x)\sin x\right]$$

30) $D^5-D^4+2D^3-2D^2+D-1 = 0$

$D^4(D-1)+2D^2(D-1)+D-1 = 0$

$(D-1)(D^4+2D^2+1) = 0$

$(D-1)(D^2+1)^2 = 0$

$D = 1, \quad D^2 = -1$

$\therefore D = \pm i$ repeated twice.

$$\text{C.F.} = C_1e^x+e^{0x}\left[(C_2+C_3x)\cos x+(C_4+C_5x)\sin x\right]$$

1.3 Particular Integrals

If f(D)y = $\phi(x)$ is the linear differential equation with constant coefficients, then it's particular integral is

$$\text{P.I.} = y_p = \frac{1}{f(D)}[\phi(x)]$$

Particular integrant is any solution of a differential equation in which arbitrary constants are absent.

If f(D) is simple operator then $\frac{1}{f(D)}$ is the inverse operator of f(D).

If f(D) = D – a then

f(D) y = ϕ (x) gives

(D – a) y = ϕ (x)

$\therefore$ Particular integral $y = \frac{1}{D-a}\phi(x)$

$\therefore$ (D – a) y = ϕ (x)

where $D \equiv \frac{d}{dx}$

$\therefore \frac{dy}{dx} - ay = \phi(x)$

This is the linear differential equation of 1st order and 1st degree. It's general solution is given by

$$y\, e^{\int -a dx} = \int \phi(x)\, e^{\int -a dx}\, dx + C$$

$$y\, e^{-ax} = \int \phi(x)\, e^{-ax}\, dx + C$$

$$y = e^{ax}\int \phi(x)\, e^{-ax}\, dx + C\, e^{ax}$$

As all the arbitrary constants are involved in complementary function, particular integral will be independent of arbitrary constants.

$$\therefore \quad y = e^{ax}\int \phi(x)\, e^{-ax}\, dx$$

$$\therefore \quad \frac{1}{D-a}\phi(x) = e^{ax}\int \phi(x)\, e^{-ax}\, dx$$

Replacing a by – a

$$\frac{1}{D+a}\phi(x) = e^{-ax}\int \phi(x)\, e^{+ax}\, dx$$

Replacing a by 0

$$\frac{1}{D}\phi(x) = \int \phi(x)\, dx$$

Thus Particular Integral P.I. = $\frac{1}{f(D)}\phi(x)$ is given by

$$\boxed{\frac{1}{D}\phi(x) = \int \phi(x)\, dx}$$

$$\boxed{\frac{1}{D-a}\phi(x) = e^{ax}\int \phi(x)\, e^{-ax}\, dx\,,}$$

$$\boxed{\frac{1}{D+a}\phi(x) = e^{-ax}\int \phi(x)\, e^{ax}\, dx}$$

Example

$$\text{P.I.} = y_p = \frac{1}{D+2}(x)$$

$$= e^{-2x}\int x e^{2x} dx$$

$$= e^{-2x}\left[x\left(\frac{e^{2x}}{2}\right) - (1)\left(\frac{e^{2x}}{4}\right)\right]$$

$$= \frac{x}{2} - \frac{1}{4}$$

1.4 Shortcut Methods for Finding Particular Integral of Special Functions

1.4.1 Type 1 : ϕ (x) = e^{ax}, where a = Constant

1) As $D e^{ax} = a e^{ax}$, $D^2 e^{ax} = D\,[D e^{ax}] = D\,[a e^{ax}] = a^2 e^{ax}$ i.e. while finding derivative of e^{ax}, we replace D by a to get the answer. Generalising this for any polynomial f (D), we get formula F_1.

$$\boxed{\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, \quad \text{if } f(a) \neq 0}$$

2) **Case of failure** if f (a) = 0 i.e. we get zero in the denominator after the replacement then we apply the basic formula for Particular Integral.

i.e. $$\frac{1}{D-a}e^{ax} = e^{ax}\int e^{-ax}\cdot e^{ax}\, dx$$

$$= e^{ax}\cdot x$$

Similarly we can derive

$$\frac{1}{(D-a)^2}e^{ax} = \frac{x^2}{2!}e^{ax}$$

Generalizing this we get Formula F_2

$$\boxed{\frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}}$$

3) If Denominator is a combination of zero and non zero factors then use formula F_1 and F_2 separately.

$$\boxed{\frac{1}{\phi(D)\cdot(D-a)^r}e^{ax} = \frac{1}{\phi(a)}\cdot\frac{x^r}{r!}\cdot e^{ax};\ \phi(a) \neq 0}$$

4) If k = Constant = ke^{0x}. Thus for constant we can replace D by zero.

$$\boxed{\frac{1}{f(D)}k = k\cdot\frac{1}{f(D)}e^{0x} = \frac{k}{f(0)},\ f(0) \neq 0}$$

5) If $\phi(x) = a^x = e^{x\log a}$. Thus for a^x we can replace D by log a.

$$\boxed{\frac{1}{f(D)}a^x = \frac{1}{f(\log a)}a^x}$$

Note :

1) Here take the factors of f(D) for finding P.I.
2) Replace D by a only in the non zero factor.
3) For the zero factor use the formula (2).
4) If X = constant then replace D by zero.
5) If X = a^x then replace D by 'log a'.

Expressions	Particular integrals y_p
$\frac{1}{(D-a)^r}e^{ax}$	$\frac{x^r}{r!}e^{ax}$
$\frac{1}{(D+a)^r}e^{-ax}$	$\frac{x^r}{r!}e^{-ax}$
$\frac{1}{D-a}e^{ax}$	xe^{ax}
$\frac{1}{D+a}e^{-ax}$	xe^{-ax}
$\frac{1}{(D-a)^2}e^{ax}$	$\frac{x^2}{2!}e^{ax}$
$\frac{1}{(D-a)^3}e^{ax}$	$\frac{x^3}{3!}e^{ax}$

Procedure for Type 1

Step 1 : Use P.I. formula.

Step 2 : Use factors of f (D) for finding P.I. and if necessary use $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$.

Step 3 : Separate all the terms and consider PI_1, PI_2, PI_3 ....

For non zero factor	For mixed factors zero and non zero	For ϕ (x) = Constant	For ϕ (x) = a^x
Step 4 : Replace D by a only in non zero factor.	Step 4 : Replace D by a only in non zero factor.	Step 4 : $\phi(x)$ = Constant then replace D by zero.	Step 4 : $\phi(x) = a^x$ then replace D by log a.
Step 5 : Simplify.	Step 5 : Simplify.	Step 5 : Simplify.	Step 5 : Simplify.
	Step 6 : For zero factor use (2).		
	Step 7 : Simplify.		

1.5 Problems on Type 1

Solve the following D.E's.

Example 1.1 : *Solve* $(D^2-2D-3)y = 5+e^{2x}+3^x$

Solution : Step 1 : Find complementary function y_c

Auxilliary equation is $D^2-2D-3=0$

$$(D-3)(D+1) = 0$$

$$D = 3, -1$$

As roots of A.E. are real and distinct, the complementary function is

$$y_c = C_1e^{3x}+C_2e^{-x} \quad \text{...(1)}$$

Step 2 : Find particular integrals y_p

We have $$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^2-2D-3}[5+e^{2x}+3^x]$$

$$\therefore \quad y_p = \frac{1}{(D-3)(D+1)}[5e^{0x}+e^{2x}+3^x]$$

$$= \underset{D\to 0}{\frac{1}{(D-3)(D+1)}}[5e^{0x}]+\underset{D\to 2}{\frac{1}{(D-3)(D+1)}}[e^{2x}]+\underset{D\to \log 3}{\frac{1}{(D-3)(D+1)}}[3^x]$$

$$= \frac{1}{(0-3)(0+1)}5e^{0x}+\frac{1}{(2-3)(2+1)}e^{2x}+\frac{1}{(\log 3-3)(\log 3+1)}(3^x)$$

$$= \frac{-1}{3}[5]+\frac{1}{(-3)}e^{2x}+\frac{1}{(\log 3-3)(\log 3+1)}(3^x)$$

$$y_p = -\frac{5}{3}-\frac{1}{3}e^{2x}+\frac{1}{(\log 3-3)(\log 3+1)}(3^x) \quad \text{...(2)}$$

Step 3 : Complete solution

The complete solution is $y = y_c+y_p$

$$\therefore \quad y = C_1e^{3x}+C_2e^{-x}-\frac{5}{3}-\frac{1}{3}e^{2x}+\frac{1}{(\log 3-3)(\log 3+1)}(3^x)$$

Note :

1) If DE involves sinh x and/or cosh x then use $\sinh x = \frac{e^x-e^{-x}}{2}$ and $\cosh x = \frac{e^x+e^{-x}}{2}$.

2) If DE involves $e^{ax}\sinh x$ or $e^{bx}\cosh x$ then use above formulae and apply this type.

Example 1.2 : $(D^3+3D)y = \cosh 2x \sinh 3x$

Solution : Step 1 : Complementary function

Auxiliary equation is $D^3+3D=0$

Take D common.

$$D(D^2+3) = 0$$

$$D = 0 \qquad D^2+3=0$$

$$D = 0 \qquad D^2 = -3$$

Square of any real term is always positive.

$$\therefore \quad D = \pm i\sqrt{3} = 0 \pm i\sqrt{3}$$

$$= \alpha \pm i\beta$$

$$\Rightarrow \quad D = 0, \quad \alpha = 0, \beta = \sqrt{3}$$

As one root is real and two roots are complex, the C.F. is

$$\therefore \quad \boxed{\text{C.F.} = C_1 e^{0x} + e^{0x}(C_1 \cos x\sqrt{3} + C_2 \sin x\sqrt{3})}$$

Step 2 : Particular integral

We have

$$\text{P.I.} = \frac{1}{f(D)}\phi(x)$$

$$y_p = \frac{1}{(D^3+3D)} \cosh 2x \ \sinh 3x$$

$$= \frac{1}{D(D^2+3)}\left(\frac{e^{2x}+e^{-2x}}{2}\right)\left(\frac{e^{3x}-e^{-3x}}{2}\right)$$

$$= \frac{1}{D(D^2+3)}\frac{1}{4}[e^{5x}-e^{-5x}+e^{x}-e^{-x}]$$

$$= \frac{1}{4}\left\{\frac{1}{D(D^2+3)}e^{5x} - \frac{1}{D(D^2+3)}e^{-5x} + \underset{D\to a}{\frac{1}{D(D^2+3)}}e^{x} - \frac{1}{D(D^2+3)}e^{-x}\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{(5)(28)}e^{5x} - \frac{1}{(-5)(28)}e^{-5x} + \frac{1}{(1)(4)}e^{x} - \frac{1}{(-1)(4)}e^{-x}\right\}$$

$$= \frac{1}{4}\left\{\frac{e^{5x}+e^{-5x}}{14} + \frac{e^{x}+e^{-x}}{4}\right\}$$

$$= \frac{1}{280}\left(\frac{e^{5x}+e^{-5x}}{2}\right) + \frac{1}{8}\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$y_p = \frac{1}{280}\cosh 5x + \frac{1}{8}\cosh x$$

Step 3 : Complete solution

The complete solution is

$$y = \text{C.F.} + \text{P.I.} = y_c + y_p$$

$$\boxed{y = C_1 e^{0x} + C_2 \cos x\sqrt{3} + C_3 \sin x\sqrt{3} + \frac{1}{280}\cosh 5x + \frac{1}{8}\cos h\, x}$$

Example 1.3 : $(D^4-1)y = \cosh x \sinh x$

Solution : **Setp 1 :** Complementary function

Auxiliary equation is $D^4 - 1 = 0$

$$\text{i.e.} \quad (D^2-1)(D^2+1) = 0$$

$$\text{i.e.} \quad (D-1)(D+1)(D^2+1) = 0$$

$$\therefore \quad D = \pm 1, \quad D^2 = -1$$

$$D = \pm 1, \quad D = \pm i$$

As two roots are real and two roots are complex, the C.F. is,

$\therefore$ C.F $= C_1 e^{x} + C_2 e^{-x} + e^{0x}(C_3 \cos x + C_4 \sin x)$

Step 2 : Particular integral

$$y_p = \text{P.I.} = \frac{1}{D^4-1}\cosh x \sinh x$$

$$\text{P.I.} = \frac{1}{(D^2-1)(D^2+1)}\left(\frac{\sinh 2x}{2}\right)$$

$$\text{P.I.} = \frac{1}{2}\frac{1}{(D^2-1)(D^2+1)}\left(\frac{e^{2x}-e^{-2x}}{2}\right)$$

$$= \frac{1}{4}\left\{\frac{1}{(D^2-1)(D^2+1)}e^{2x} - \frac{1}{(D^2-1)(D^2+1)}e^{2x}\right\}$$

Replace D by a only in non zero factor.

$$= \frac{1}{4}\left\{\frac{1}{(4-1)(4+1)}e^{2x} - \frac{1}{(4-1)(4+1)}e^{-2x}\right\}$$

$$= \frac{1}{4}\left\{\frac{e^{2x}}{15} - \frac{e^{-2x}}{15}\right\}$$

$$= \frac{1}{30}\left(\frac{e^{2x}-e^{-2x}}{2}\right)$$

$$y_p = \frac{1}{30}\sinh 2x$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

$$\therefore \boxed{y = C_1 e^{x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x + \frac{1}{30}\sinh 2x}$$

Example 1.4 : $(D^3+D^2-D-1)y = \sinh x$

Solution : **Step 1 :** Complementary function

$$\text{C.F.} = C_1 e^{x} + (C_2 + C_3 x)e^{-x}$$

[Refer solved Problem No. 7 in problems on C.F. on page 1 - 4]

Step 2 : Particular integral

$$y_p = \text{P.I.} = \frac{1}{f(D)}\phi(x)$$

$$\text{P.I.} = \frac{1}{D^3+D^2-D-1}\sinh x$$

$$\text{P.I.} = \frac{1}{(D-1)(D+1)^2}\left(\frac{e^x-e^{-x}}{2}\right)$$

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{(D-1)(D+1)^2}e^x - \frac{1}{(D-1)(D+1)^2}e^{-x}\right]$$

Replace D by a only in non zero factor. Here a = 1 and a = – 1 respectively.

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{(D-1)(1+1)^2}e^x - \frac{1}{(-1-1)(D+1)^2}e^{-x}\right]$$

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{4}\cdot\frac{1}{D-1}e^x + \frac{1}{2}\cdot\frac{1}{(D+1)^2}e^{-x}\right]$$

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{4}\cdot x\,e^x + \frac{1}{2}\cdot\frac{x^2}{2!}e^{-x}\right]$$

$$\text{P.I.} = \frac{x}{8}e^x + \frac{x^2}{8}e^{-x}$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

Example 1.5 : $(D^3-5D^2+8D-4)y = e^{2x}+2e^x$

Solution : Step 1 : Complementary function

$$\text{C.F.} = C_1e^x + (C_2+C_3x)\,e^{2x}$$

[Refer solved Problem No. 8 in problems on C.F. on page 1-4]

$$\text{P.I.} = \frac{1}{f(D)}\phi(x)$$

Step 2 : Particular integral

$$= \frac{1}{D^3-5D^2+8D-4}\left(e^{2x}+2e^x\right)$$

$$= \frac{1}{(D-1)(D-2)^2}\left(e^{2x}+2e^x\right)$$

$$= \frac{1}{(D-2)^2(D-1)}e^{2x} + \frac{1}{(D-1)(D-2)^2}2e^x$$

Replace D by a only in non zero factor i.e. put D = 2 and D = 1 respectively.

$$y_p = \frac{1}{(2-1)(D-2)^2}e^{2x} + \frac{2}{(D-1)(1-2)^2}e^x$$

$$= \frac{1}{(D-2)^2}e^{2x} + \frac{2}{1}\cdot\frac{1}{D-1}e^x$$

$$y_p = \frac{x^2}{2!}e^{2x} + 2\cdot\frac{x}{1}e^x \qquad \ldots(2)$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

$$\therefore \quad \boxed{y = C_1e^x + (C_2+C_3x)e^{2x} + \frac{x^2}{2}e^{2x} + 2x\,e^x}$$

Example 1.6 : $(D^3+D^2+D+1)\,y = \cosh x$

Solution : Step 1 : Complementry function

Auxiliary equation i.e. $D^3+D^2+D+1 = 0$

$$\text{C.F.} = C_1e^{-x} + C_2\cos x + C_3\sin x$$

[Refer Problem No. 5 in problems on C.F. on page 1-4]

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^3+D^2+D+1}\cosh x$$

$$= \frac{1}{(D+1)(D^2+1)}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left\{\frac{1}{(D+1)(D^2+1)}e^x + \frac{1}{(D+1)(D^2+1)}e^{-x}\right\}$$

Replace D by a only in non zero factor i.e. D = 1 and D = – 1 respectively.

$$= \frac{1}{2}\left\{\frac{1}{(1+1)(1+1)}e^x + \frac{1}{(D+1)(1+1)}e^{-x}\right\}$$

$$= \frac{1}{2}\left\{\frac{e^x}{4} + \frac{1}{2}\frac{1}{(D+1)}e^{-x}\right\}$$

$$y_p = \frac{1}{2}\left\{\frac{e^x}{4} + \frac{1}{2}\cdot\frac{x^1}{1}e^{-x}\right\}$$

$$= \frac{e^x}{8} + \frac{x}{4}e^{-x}$$

Step 3 : Complete integral

$$y = \text{C.F + P.I.}$$

$$\boxed{y = C_1e^{-x} + C_2\cos x + C_3\sin x + \frac{e^x}{8} + \frac{x}{4}e^{-x}}$$

This is the complete solution of the Differential equation.

Mixed Problems

Note : In such a case find all the particular integrals separately PI_1, PI_2, PI_3, PI_4 and so on.

Example 1.7 : $(D^6 - 6D^5 + 12D^4 - 6D^3 - 9D^2 + 12D - 4)\ y = e^x + 1$

Solution : Step 1 : Complementary function

$$C.F. = (C_1 + C_2 x + C_3 x^2)\, e^x + C_4 e^{-x} + (C_5 + C_6 x)\, e^{2x}$$

[Refer Problem No. 6 in problems on C.F. on page 1 - 4]

Step 2 : Particular integral

$$P.I. = \frac{1}{f(D)} \phi(x)$$

$$P.I. = \frac{1}{(D-1)^3 (D+1)(D-2)^2} \left(e^x + 1\right)$$

$$= \left\{ \frac{1}{(D-1)^3 (D+1)(D-2)^2} e^x + \frac{1}{(D-1)^3 (D+1)(D-2)^2} 1 \right\}$$

$$= PI_1 + PI_2$$

Consider PI_1

$$PI_1 = \frac{1}{(D-1)^3 (D+1)(D-2)^2} e^x$$

Replace D by a in non zero factor. i.e. D = 1.

$$PI_1 = \frac{1}{(D-1)^3 (1+1)(1-2)^2} e^x$$

$$PI_1 = \frac{1}{2\,(D-1)^3} e^x$$

Consider PI_2

$$PI_2 = \frac{1}{(D-1)^3 (D+1)(D-2)^2} 1$$

As X is a constant then replace D by 0.

$$PI_2 = \frac{1}{(0-1)^3 (0+1)(0-2)^2}$$

$$PI_2 = \frac{1}{-4}$$

$$\therefore \quad PI_1 = \frac{1}{2} \cdot \frac{x^3}{3!} e^x$$

$$PI_1 = \frac{x^3}{12} e^x$$

Thus $P.I. = PI_1 + PI_2 = \frac{1}{12} x^3 e^x + \left(-\frac{1}{4}\right)$

Step 3 : Complete solution

$$y = C.F + P.I$$

This is the complete solution of the differential equation.

Example 1.8 : $(D^3 - 5D^2 + 8D - 4)$
$y = 4e^{2x} + e^x + 2^x + 3$

Solution : Step 1 : Complementary function

$$C.F. = C_1 e^x + (C_2 + C_3 x)\, e^{2x}$$

[Refer Problem No. 8 in problems on C.F. on page 1 - 4]

Step 2 : Particular integrals

$$P.I. = \frac{1}{f(D)} \phi(x)$$

$$\text{P.I.} = \frac{1}{(D-1)(D-2)^2}(4e^{2x}+e^x+2^x+3)$$

$$\text{P.I.} = \frac{1}{(D-1)(D-2)^2}4e^{2x} + \frac{1}{(D-1)(D-2)^2}e^x + \frac{1}{(D-1)(D-2)^2}2^x + \frac{1}{(D-1)(D-2)^2}3$$

Consider

$$PI_1 = \frac{1}{(D-1)(D-2)^2}4e^{2x}$$

Replace D by a in non zero factor. i.e. D = 2

$$PI_1 = \frac{1}{(2-1)(D-2)^2}4e^{2x}$$

$$= \frac{4}{1}\cdot\frac{1}{(D-2)^2}e^{2x} = \frac{4}{1}\frac{x^2}{2!}e^{2x}$$

$$PI_1 = 2x^2e^{2x}$$

Consider

$$PI_2 = \frac{1}{(D-1)(D-2)^2}e^x$$

Replace D by a in non zero factor i.e. D = 1.

$$PI_2 = \frac{1}{(D-1)(1-2)^2}e^x = \frac{1}{1\cdot(D-1)}e^x$$

$$PI_2 = \frac{x}{1}e^x$$

Consider

$$PI_3 = \frac{1}{(D-1)(D-2)^2}2^x$$

For a^x replace D by log a, (Here D = log 2)

$$PI_3 = \frac{1}{(\log 2-1)(\log 2-2)}2^x$$

Consider

$$PI_4 = \frac{1}{(D-1)(D-2)^2}3$$

For $\phi(x)$ = Constant replace D by 0.

$$PI_4 = \frac{1}{(0-1)(0-2)^2}3$$

$$PI_4 = -\frac{3}{4}$$

$$\text{Thus P.I.} = PI_1+PI_2+PI_3+PI_4$$

$$= 2x^2e^x + xe^x + \frac{2^x}{(\log 2-1)(\log 2-2)^2} - \frac{3}{4}$$

Step 3 : Complete solution

y = C.F + P.I is the complete solution.

Exercise 1.1

1. $(D^2+4D+4)y = e^{-2x}+2^x+3$

 [Ans. : $y = (C_1+C_2x)e^{-2x} + \frac{x^2}{2}e^{-2x} + \frac{2^x}{(\log 2+2)^2} + \frac{3}{4}$]

2. $(D^3-3D^2+4)y = \cosh 2x$

 [Ans. : $y = C_1e^{-x} + (C_2+C_3x)e^{2x} + \frac{x^2}{12}e^{2x} - \frac{e^{-2x}}{32}$]

3. $(D^2-4)y = (1+e^x)^2+3$

 [Ans. : $y = C_1e^{2x}+C_2e^{-2x} + \frac{xe^{2x}}{4} - \frac{2e^x}{3} - 1$]

4. $(D^3-5D^2+8D-4)y = e^{2x}+2e^x+3e^{-x}+2$

 [Ans. : $y = C_1e^x + (C_2+C_3x)e^{2x} + \frac{x^2}{2}e^{2x} + 2xe^x - \frac{e^{-x}}{6} - \frac{1}{2}$]

5. $(D^2+13D+36)y = e^{-4x}+\sinh x$

 [Ans. : $y = C_1e^{-4x}+C_2e^{-9x} + \frac{x}{5}e^{-4x} + \frac{e^x}{100} - \frac{e^{-x}}{48}$]

1.5.1 Type 2 : If $\phi(x)$ = sin (ax + b) or cos (ax+b), a and b are Constants

Let $\phi(x) = \cos(ax+b)$

We know

$$D\cos(ax+b) = -a\sin(ax+b)$$
$$D^2\cos(ax+b) = -a^2\cos(ax+b)$$
$$D^3\cos(ax+b) = +a^2\cdot a\sin(ax+b)$$
$$D^4\cos(ax+b) = a^4\cos(ax+b)$$

i.e. $(D^2)^2\cos(ax+b) = (-a^2)^2\cos(ax+b)$

Similarly $(D^2)^2\sin(ax+b) = (-a^2)^2\sin(ax+b)$

Generalizing this we get formula F_1.

$$\boxed{\frac{1}{f(D^2)}\,{}^{\sin}_{\cos}(ax+b) = \frac{1}{f(-a^2)}\,{}^{\sin}_{\cos}(ax+b) \text{ if } f(-a^2)\neq 0 \quad \ldots (1.3)}$$

i.e. For sin (ax + b) or cos (ax + b) replace $D^2 = -a^2$ if the denominator is non zero.

Case of failure : If $f(-a^2) = 0$

We know that $e^{iax} = \cos ax + i\sin ax$

$\therefore$ $\sin ax = \text{Img } e^{iax}$

$\cos ax = \text{Real } e^{iax}$

$$\therefore \quad \frac{1}{f(D^2)} \sin ax = \text{Img} \frac{1}{f(D^2)} e^{iax}$$

$$= \text{Img} \frac{x}{f'(D^2)} e^{iax}$$

Again put $D^2 = -a^2$

$$= \text{Img} \frac{x}{f'(-a^2)} e^{iax}$$

$$\frac{1}{f(D^2)} \sin ax = \frac{x}{f'(-a^2)} \sin ax$$

$$\text{if } f'(-a^2) \neq 0$$

i.e. For the zero factor multiply by x to the numerator and differentiate the denominator w.r.t. D and then put $D^2 = -a^2$ again.

Generalizing this we get formula F_2

$$\boxed{\frac{1}{(D^2+a^2)^r} \begin{matrix}\sin\\ \cos\end{matrix}(ax+b) = \left(-\frac{x}{2a}\right)^r \cdot \frac{1}{r!} \begin{matrix}\sin\\ \cos\end{matrix}\left(ax+b+\frac{r\pi}{2}\right) \quad \dots (1.4)}$$

For example :

A) $$y_p = \frac{1}{D^2+a^2} \sin ax$$

The denominator becomes zero if we put $D^2 = -a^2$
$\therefore$ Multiply by x to the numerator and differentiate the denominator.

$$= x \cdot \frac{1}{2D} \sin ax$$

$$\left\{\text{Use } \frac{1}{D} X = \int X\,dx = \int \sin ax\,dx = \frac{-\cos ax}{a}\right\}$$

$$y_p = x \cdot \frac{1}{2} \cdot \frac{-\cos ax}{a}$$

$$y_p = \frac{-x}{2a} \cos ax$$

B) $$\frac{1}{D^2+a^2} \cos ax = y_p$$

$D^2 = -a^2$ gives zero $\therefore$ Multiply by x to numerator and differentiate the denominator.

$$y_p = x \cdot \frac{1}{2D} \cos ax$$

$$\left\{\text{Use } \frac{1}{D} X = \int X\,dx = \int \cos ax\,dx = \frac{\sin ax}{a}\right\}$$

$$y_p = \frac{x}{2} \cdot \frac{\sin ax}{a}$$

$$y_p = \frac{x}{2a} \sin ax$$

C) $$\frac{1}{(D^2+a^2)^2} \sin ax = y_p$$

$D^2 = -a^2$ gives zero

$$\therefore \quad y_p = x \cdot \frac{1}{2(D^2+a^2) \cdot 2D} \cdot \sin ax$$

$$\left\{\text{Use } \frac{1}{D} X = \int X\,dx\right\}$$

$$y_p = \frac{x}{4} \cdot \frac{1}{D^2+a^2} \cdot \left(\frac{-\cos ax}{a}\right)$$

$$y_p = \frac{-x}{4a} \cdot \frac{1}{D^2+a^2} \cos ax$$

Using above formula (B).

$$y_p = \frac{-x}{4a} \cdot \frac{x}{2a} \sin ax$$

$$y_p = -\frac{x^2}{8a^2} \sin ax$$

D) $$\frac{1}{(D^2+a^2)^2} \cos ax = y_p$$

$D^2 = -a^2$ gives zero.

$$y_p = x \cdot \frac{1}{2(D^2+a^2)\,2D} \cos ax$$

$$\left\{\text{Use } \frac{1}{D} X = \int X\,dx\right\}$$

$$y_p = \frac{x}{4} \cdot \frac{1}{D^2+a^2} \cdot \frac{\sin ax}{a}$$

$$y_p = \frac{x}{4a} \frac{1}{D^2+a^2} \sin ax$$

Using above formula (A).

$$y_p = \frac{x}{4a} \cdot \frac{-x}{2a} \cos ax = \frac{-x^2}{8a^2} \cos ax$$

Thus for the zero factor.

A)	$\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$
B)	$\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$
C)	$\frac{1}{(D^2+a^2)^2} \sin ax = -\frac{x^2}{8a^2} \sin ax$
D)	$\frac{1}{(D^2+a^2)^2} \cos ax = -\frac{x^2}{8a^2} \cos ax$

Note : Here powers and multiplication of trigonometric terms are not allowed thus we should separate them using the following formulae.

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \; \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$2\sin A\cos B = \sin(A+B)+\sin(A-B)$$

$$2\cos A\cos B = \cos(A-B)+\cos(A+B)$$

$$2\sin A\sin B = \cos(A-B)-\cos(A+B)$$

Term	Its zero factor
sin x, cos x	D^2+1
sin 2x, cos 2x	D^2+4
sin 3x, cos 3x	D^2+9
sin ax, cos ax	D^2+a^2

Procedure for Type 2

Step 1 : Use P.I. formula.

Step 2 : Separate all the terms and consider PI_1, PI_2, PI_3 separately.

Step 3 : Observe the trigonometric ratio sin ax or cos ax and note the zero factor for that ratio. By using the factors which are present in complementary function we can notice that the zero factor is present in f(D) or not.

If the zero factor is not present then follow procedure (2A).

If the zero factor is present then follow procedure (2B).

Procedure (2A)

Step 4 : Don't take the factors if f(D) for finding P.I. (i.e. keep f(D) in polynomial form).

Step 5 : Replace $D^2 = -a^2$, $D^3 = D^2 \cdot D = -a^2 \cdot D$, $D^4 = (D^2)^2 = a^4$

(Don't replace D by $\sqrt{-a^2}$ it is wrong)

If the denominator reduces to a constant then P.I. is complete.

Step 6 : Simplify.

Step 7 : If the term involving D remains in the denominator then rationalise to get D^2 in denominator.

Step 8 : Simplify (i.e. multiply the two brackets in the denominator).

Step 9 : Put $D^2 = -a^2$ in denominator and simplify.

Step 10 : Open the bracket in the numerator and take the derivatives.

Step 11 : Simplify.

Examples

Example 1.9 : *Solve* $(D^2+4)y = \sin x + \cos 2x$

Solution : Step 1 : Complementary function

We have, auxilliary equation is

$$D^2+4=0 \Rightarrow D^2 = -4$$

$$D = \pm 2i\; ; \; \alpha = 0, \beta = 2$$

As roots are complex and distinct, the C.F. is

$$y_c = C_1\cos 2x + C_2 \sin 2x \quad \ldots(1)$$

Step 2 : Particular integrals

We have $$y_p = \frac{1}{D^2+4}[\sin x + \cos 2x]$$

$$= \frac{1}{D^2+4}\sin x + \frac{1}{D^2+4}\cos 2x$$

$$D^2 \to -1 \qquad D^2 \to -4,\ f(D^2)=0$$

$$= \frac{1}{-1+4}\sin x + \frac{x}{2(2)}\sin 2x$$

$$= \frac{1}{3}\sin x + \frac{x}{4}\sin 2x \quad \ldots(2)$$

Step 3 : Complete solution

The complete solution is $y = y_c + y_p$

$$y = C_1\cos 2x + C_2\sin 2x + \frac{1}{3}\sin x + \frac{x}{4}\sin 2x$$

Example 1.10 : *Find particular integral of* $(D^2+9)^4 y = \cos 2x + \sin 3x$

Solution : We have $$y_p = \frac{1}{(D^2+9)} + [\cos 2x + \sin 3x]$$

$$y_p = \frac{1}{(D^2+9)^4}[\cos 2x] + \frac{1}{(D^2+9)^4}[\sin 3x]$$

$$D^2 \to -4 \qquad D^2 \to -9,\ f(D^2)=0$$

$$= \frac{1}{(-4+9)^4}\cos 2x + \left(\frac{-x}{2(3)}\right)^4 \frac{1}{4!}\sin\left(3x+\frac{4\pi}{2}\right)$$

$$y_p = \frac{1}{625}\cos 2x + \frac{x^4}{6^4 4!}\sin 3x$$

Example 1.11 : *Find particular integral of* $(D^2+16)y = \cos^2 2x + e^{2x}$

Solution :

We have $y_p = \frac{1}{D^2+16}[\cos^2 2x + e^{2x}]$

$$y_p = \frac{1}{D^2+16}\left[\frac{1+\cos 4x}{2}\right] + \frac{1}{D^2+16}[e^{2x}]$$

$$= \frac{1}{D^2+16}\left[\frac{1}{2}+\frac{1}{2}\cos 4x\right] + \frac{1}{2^2+16}[e^{2x}]$$

$$= \frac{1}{D+16}\frac{1}{2} + \frac{1}{2}\frac{x}{2(4)}\sin 4x + \frac{1}{20}e^{2x}$$

$$= \frac{1}{32} + \frac{x}{16}\sin 4x + \frac{1}{20}e^{2x}$$

Example 1.12 : $(D^3 + D^2 - D - 1)y = 2 \sin x \cos x$

Solution : Step 1 : Complementary function

$$\text{C.F.} = C_1 e^x + (C_2 + C_3 x)\, e^{-x}$$

(Refer solved Problem No. 7 in problems on C.F. on page 1 - 4)

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 2 : Particular integrals

We have $\text{P.I.} = \frac{1}{D^3+D^2-D-1} 2 \sin x \cos x$

$$\text{P.I.} = \frac{1}{D^3+D^2-D-1} \sin 2x$$

The zero factor for $\sin 2x$ is $D^2 + 4$ which is not present in f(D)

Replace $D^2 = -a^2$, $D^3 = -a^2 D$, here a = 2

$\therefore$ $D^2 = -4$, $D^3 = -4D$

$$\text{P.I.} = \frac{1}{-4D-4-D-1} \sin 2x$$

$$\text{P.I.} = \frac{1}{-5(D+1)} \sin 2x$$

$$= \frac{(D-1)}{-5(D+1)(D-1)} \sin 2x$$

$$= \frac{(D-1)}{-5(D^2-1)} \sin 2x$$

Replace $D^2 = -a^2$ i.e. $D^2 = -4$ in denominator

$$= \frac{(D-1)}{-5(-4-1)} \sin 2x$$

$$= \frac{1}{25}(D-1) \sin 2x$$

Take derivatives

$$= \frac{1}{25}(2 \cos 2x - \sin 2x)$$

$$\{\because D \sin 2x = 2 \cos 2x\}$$

$$\text{P.I.} = \frac{1}{25}\{2 \cos 2x - \sin 2x\}$$

Step 3 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

i.e.
$$y = C_1 e^x + (C_2 + C_3 x) e^{-x} + \frac{1}{25}(2 \cos 2x - \sin 2x)$$

is the complete solution.

Example 1.13 : $(D^3 + D^2 + D + 1)y = \cos^2 x$

Hint : $\cos^2 x = \frac{1+\cos 2x}{2}$

Solution : Step 1 : Complementary function

$$\text{C.F.} = C_1 e^{-x} + C_2 \cos x + C_3 \sin x$$

(Refer Problem No. 5 in problems on C.F. on page 1 - 4).

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^3+D^2+D+1} \frac{1+\cos 2x}{2}$$

Separate PI_1, PI_2

$$= \frac{1}{2}\left\{\frac{1}{D^3+D^2+D+1} 1 + \frac{1}{D^3+D^2+D+1} \cos 2x\right\}$$

For constant put D = 0 and for cos 2x put $D^2 = -4$, $D^3 = -4D$.

$$= \frac{1}{2}\left\{\frac{1}{0+1} + \frac{1}{-4D-4+D+1} \cos 2x\right\}$$

$$= \frac{1}{2}\left\{1 + \frac{1}{-3(D+1)} \cos 2x\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{1}{3}\frac{D-1}{D^2-1} \cos 2x\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{1}{3}\frac{(D-1)}{(-4-1)} \cos 2x\right\}$$

$$= \frac{1}{2}\left\{1 + \frac{1}{15}(D-1) \cos 2x\right\}$$

$$= \frac{1}{2}\left\{1 + \frac{1}{15}(-2 \sin 2x - \cos 2x)\right\}$$

Step 3 : Complete solution

$$y = \text{C.F} + \text{P.I}$$

$$= C_1 e^x + C_2 \cos x + C_3 \sin x + \frac{1}{2} - \frac{1}{30}(2 \sin 2x + \cos 2x)$$

Example 1.14 : *Solve* $(D^4 + m^4)y = \sin mx$

Solution : **Step 1 :** Complementary function

$$\text{C.F.} = e^{mx/\sqrt{2}}\left[C_1\cos\frac{mx}{\sqrt{2}} + C_2\sin\frac{mx}{\sqrt{2}}\right] + e^{-mx/\sqrt{2}}\left[C_3\cos\frac{mx}{\sqrt{2}} + C_4\sin\frac{mx}{\sqrt{2}}\right]$$

[Refer solved Problem No. 25 in problems on C.F. on page 1 - 6]

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^4 + m^4}\sin mx$$

Put $D^2 = -m^2,\ D^4 = m^4$

$$\therefore \quad \text{P.I.} = \frac{1}{m^4 + m^4}\sin mx = \frac{1}{2m^4}\sin mx$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

$$\therefore \quad y = e^{mx/\sqrt{2}}\left[C_1\cos\frac{mx}{\sqrt{2}} + C_2\sin\frac{mx}{\sqrt{2}}\right] + e^{-mx/\sqrt{2}}\left[C_3\cos\frac{mx}{\sqrt{2}} + C_4\sin\frac{mx}{\sqrt{2}}\right] + \frac{1}{2m^4}\sin mx$$

Example 1.15 : $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ *where* $y = 3$, $\frac{dy}{dx} = 0$, *when* $x = 0$ *Find* y *when* $x = \frac{\pi}{2}$.

Solution : Let $D = \frac{d}{dx}$ $\therefore$ The equation becomes.

$$D^2y + 2Dy + 10y = -37\sin 3x$$

$$(D^2 + 2D + 10)\,y = -37\sin 3x$$

Step 1 : Complementary function

A.E. $D^2 + 2D + 10 = 0$

Use $aD^2 + bD + c = 0$ $\quad D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\therefore \quad D = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i = -1 \pm i3 = \alpha \pm i\beta$$

$$\therefore \quad \text{C.F.} = e^{-x}(C_1\cos 3x + C_2\sin 3x)$$

$$\text{P.I.} = \frac{1}{f(D)}X$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^2 + 2D + 10}(-37)\sin 3x$$

$D^2 + 9$ is the zero factor for sin 3x. Which is not present in f(D) $\therefore$ Don't take factors of f(D) for finding P.I.

Replace $D^2 = -a^2$ Here $a = 3$, $D^2 = -9$

$$\text{P.I.} = \frac{1}{-9 + 2D + 10}(-37)\sin 3x$$

$$= (-37)\cdot\frac{1}{2D+1}\sin 3x$$

$$= (-37)\frac{2D-1}{(2D+1)(2D-1)}\sin 3x$$

$$= (-37)\cdot\frac{2D-1}{4D^2 - 1}\sin 3x$$

Again D^2 = i.e. $D^2 = -9$

$$= -37\frac{(2D-1)}{(4(-9)-1)}\sin 3x$$

$$= (2D-1)\sin 3x$$

$$= 2(3\cos 3x) - \sin 3x$$

$$= 6\cos 3x - \sin 3x$$

Step 3 : Complete solution

$$\therefore \quad y = \text{C.F.} + \text{P.I.}$$

$$= e^{-x}(C_1\cos 3x + C_2\sin 3x) + 6\cos 3x - \sin 3x$$

The conditions $y = 3$, $\frac{dy}{dx} = 0$ at $x = 0$ are given to find the values of C_1 and C_2.

$\therefore$ We must find $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = -e^{-x}(C_1\cos 3x + C_2\sin 3x) + e^{-x}(-3C_1\sin 3x + 3C_2\cos 3x) - 18\sin 3x - 3\cos 3x$$

$y = 3$ when $x = 0$ gives

$$3 = e^0(C_1 + 0) + 6 - 0$$

$$\{\text{As } e^0 = 1, \cos 0 = 1, \sin 0 = 0\}$$

$\Rightarrow C_1 = -3$

Also $\frac{dy}{dx} = 0$ when $x = 0$ gives

$$0 = -e^0(C_1 + 0) + e^0(0 + 3C_2) + 0 - 3$$

$$0 = -C_1 + 3C_2 - 3$$

$$0 = 3 + 3C_2 - 3$$

$$\Rightarrow \quad 3C_2 = 0$$

$$\Rightarrow \quad C_2 = 0$$

Substituting C_1 and C_2 in y we get

$$y = e^{-x}(-3\cos 3x + 0) + 6\cos 3x - \sin 3x$$

at $x = \frac{\pi}{2}$

$$(y)_{\pi/2} = e^{-\pi/2}\left[-3\cos\frac{3\pi}{2}\right] + 6\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2}$$

We know $\left\{\cos\frac{3\pi}{2} = 0, \sin\frac{3\pi}{2} = -1\right\}$

$$\therefore \quad (y)_{x=\pi/2} = 0 + 0 - (-1)$$

$$= 1$$

Example 1.16 : $\frac{d^2x}{dt^2} + 9x = 4\cos\left(t + \frac{\pi}{3}\right)$

when $x = 0$ at $t = 0$ and $x = 2$ at $t = \frac{\pi}{6}$.

Solution : Let $D = \frac{d}{dt}$ $\therefore$ The equation becomes

$$(D^2 + 9)x = 4\cos\left(t + \frac{\pi}{3}\right)$$

Step 1 : Complementary function

A.E. $D^2 + 9 = 0$

$$D^2 = -9$$

$$D = \pm 3i$$

$$\text{C.F.} = C_1 \cos 3t + C_2 \sin 3t$$

As the independent variable is 't' $\therefore$C.F. will involve t.

Step 2 : Particular integral

Now $\quad \text{P.I.} = \frac{1}{D^2 + 9} 4\cos\left(t + \frac{\pi}{3}\right)$

Put $D^2 = -1$

$$= \frac{1}{-1+9} 4\cos\left(t + \frac{\pi}{3}\right)$$

$$= \frac{4}{8}\cos\left(t + \frac{\pi}{3}\right)$$

Step 3 : Complete solution

$$\therefore \quad x = \text{C.F.} + \text{P.I.}$$

$$x = C_1 \cos 3t + C_2 \sin 3t + \frac{1}{2}\cos\left(t + \frac{\pi}{3}\right) \qquad \text{... (1)}$$

Given $x = 0$, at $t = 0$ substituting in equation (1)

$$0 = C_1 + 0 + \frac{1}{2}\cos\left(\frac{\pi}{3}\right)$$

$$0 = C_1 + \frac{1}{2}\cdot\frac{1}{2} \Rightarrow C_1 = \frac{-1}{4}$$

Also $x = 2$ at $t = \frac{\pi}{6}$ substituting in equation (1).

$$2 = C_1 \cos\frac{3\pi}{6} + C_2 \sin\frac{3\pi}{6} + \frac{1}{2}\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$$

$$2 = C_1 \cos\frac{3\pi}{2} + C_2 \sin\frac{3\pi}{2} + \frac{1}{2}\cos\frac{\pi}{2}$$

$$2 = 0 + C_2(-1) + 0$$

$$\Rightarrow \quad C_2 = 2$$

Substituting C_1 and C_2 in equation (1).

$$x = -\frac{1}{4}\cos 3t + 2\sin 3t + \frac{1}{2}\cos\left(t + \frac{\pi}{3}\right)$$

Example 1.17 : $\operatorname{cosec} x \frac{d^4y}{dx^4} + (\operatorname{cosec} x)y = \sin 2x.$

Solution : Divide by cosec x

$\therefore$ we get $\frac{d^4y}{dx^4} + y = \sin 2x \cdot \sin x$

Let $\quad D = \frac{d}{dx}$

$$(D^4 + 1)y = \sin 2x \cdot \sin x$$

Now $\quad \sin 2x \sin x = \frac{1}{2}\cos(2x - x) + \cos(2x + x)$

$\{2\sin A \sin B = \cos(A - B) - \cos(A + B)\}$

$$= \frac{1}{2}[\cos x - \cos 3x]$$

$$(D^4 + 1)y = \frac{1}{2}[\cos x - \cos 3x]$$

Step 1 : Complementary function

A.E. is $\quad D^4 + 1 = 0$

[Refer Problem No. 23 in problems on C.F. on page 1 - 6]

$$\text{C.F.} = e^{x/\sqrt{2}}\left[C_1 \cos\frac{x}{\sqrt{2}} + C_2 \sin\frac{x}{\sqrt{2}}\right] + e^{-x/\sqrt{2}}\left[C_3 \cos\frac{x}{\sqrt{2}} + C_4 \sin\frac{x}{\sqrt{2}}\right]$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{f(D)} X$$

$$= \frac{1}{D^4 + 1}\frac{1}{2}[\cos x - \cos 3x]$$

$$= \frac{1}{2}\left\{\frac{1}{D^4 + 1}\cos x\right\} - \frac{1}{2}\left\{\frac{1}{D^4 + 1}\cos 3x\right\}$$

Put $D^2 = -a^2$, $D^4 = a^4$

$$= \frac{1}{2}\left\{\frac{1}{1+1}\cos x\right\} - \frac{1}{2}\left\{\frac{1}{89+1}\cos 3x\right\}$$

$$\text{P.I.} = \frac{1}{4}\cos x - \frac{1}{164}\cos 3x$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

Example 1.18 : $\frac{d^2x}{dt^2} + 2n\cos\alpha\frac{dx}{dt} + n^2x = a\cos nt$

given $t = 0, \frac{dx}{dt} = 0$ *at* $x = 0$.

Solution : Let $D = \frac{d}{dt}$

$$\therefore \quad (D^2 + 2n\cos\alpha\, D + n^2)\, x = a\cos nt$$

Step 1 : Complementary function

$$\text{A.E.} \quad D^2 + 2n\cos\alpha\, D + n^2 = 0$$

$$D = \frac{-2n\cos\alpha \pm \sqrt{4n^2\cos^2\alpha - 4n^2}}{2}$$

$$D = \frac{-2n\cos\alpha \pm 2ni\sqrt{1-\cos^2\alpha}}{2}$$

As $i = \sqrt{-1}$

$$D = -n\cos\alpha \pm i\, n\sin\alpha$$

$$\therefore \text{C.F.} = e^{(-n\cos\alpha)t}\left[C_1\cos(n\sin\alpha)t + C_2\sin(n\sin\alpha)t\right]$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^2 + 2n\cos\alpha D + n^2} a\cos nt$$

Put $D^2 = -n^2$

$$= \frac{1}{-n^2 + 2n\cos\alpha\, D + n^2} a\cos nt$$

$$= \frac{a}{2n\cos\alpha}\cdot\frac{1}{D}\cos nt$$

$$\left\{\text{Use } \frac{1}{D} X = \int X\, dx\right\}$$

$$= \frac{a}{2n\cos\alpha}\cdot\frac{\sin nt}{n}$$

Step 3 : Complete solution

$$\therefore \quad y = \text{C.F.} + \text{P.I.}$$

$$x = e^{(-n\cos\alpha)t}\left[C_1\cos(n\sin\alpha)t + C_2\sin(n\sin\alpha)t\right] + \frac{a}{2n^2\cos\alpha}\sin nt \quad \text{... (1)}$$

$$\frac{dx}{dt} = (-n\cos\alpha)\, e^{(-n\cos\alpha)t}\, [C_1\cos(n\sin\alpha)t + C_2\sin(n\sin\alpha)t] + e^{(-n\cos\alpha)t}(n\sin\alpha)[-C_1\sin(n\sin\alpha)t + C_2\cos(n\sin\alpha)t] + \frac{a\cos nt}{2n\cos\alpha} \quad \text{... (2)}$$

Given at $t = 0$, $x = 0$ substituting in equation (1).

$$0 = e^0[C_1 + 0] + 0 \Rightarrow C_1 = 0$$

Given at $t = 0$, $\frac{dx}{dt} = 0$, substituting in equation (2).

$$0 = (-n\cos\alpha)[C_1 + 0] + (n\sin\alpha)[0 + C_2] + \frac{a}{2n\cos\alpha}$$

$$0 = -n\cos\alpha\cdot C_1 + n\sin\alpha\cdot C_2 + \frac{a}{2n\cos\alpha}$$

$$\therefore \quad C_2 = \frac{-a}{2n^2\sin\alpha\cos\alpha} = \frac{-a}{n^2\sin 2\alpha}$$

Substituting C_1 and C_2 in equation (1)

$$x = e^{(-n\cos\alpha)t}\left[\frac{-a}{n^2\sin 2\alpha}\right] + \frac{a}{2n^2\cos\alpha}\sin nt$$

Example 1.19 : $(D^3 - D^2 + 3D + 5)y = 2\sin x\cos x$

Solution : Step 1 : Complementary function

$$\text{C.F.} = C_1 e^{-x} + e^x(C_2\cos 2x + C_3\sin 2x)$$

[Refer Problem No. 16 in problems on C.F. on page 1 - 5]

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^3 - D^2 + 3D + 5}\sin 2x$$

Put $D^2 = -4$, $D^3 = -4D$

$$\text{P.I.} = \frac{1}{-4D + 4 + 3D + 5}\sin 2x$$

$$\text{P.I.} = \frac{1}{(D+9)}\sin 2x$$

$$\text{P.I.} = \frac{(D-9)}{D^2 - 81}\sin 2x$$

$$\text{P.I.} = \frac{(D-9)}{-4-81}\sin 2x$$

$$\text{P.I.} = \frac{-1}{85}(2\cos 2x - 9\sin 2x)$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

$$y = C_1 e^{-x} + e^x(C_2\cos 2x + C_3\sin 2x) - \frac{1}{85}(2\cos 2x - 9\sin 2x)$$

Example 1.20 : $(D^4+10D^2+9)y=96\sin 2x\cos x$

Solution : $(D^2+9)(D^2+1)y = 48[2\sin 2x\cos x]$

$= 48[\sin 3x+\sin x]$

{As $2\sin A\cos B = \sin(A+B)+\sin(A-B)$}

Step 1 : Complementary function

A.E. $(D^2+9)(D^2+1) = 0$

$D^2 = -9, \quad D^2 = -1$

$D = \pm 3i, \quad D = \pm i$

$\therefore \quad C.F. = e^{0x}(C_1\cos 3x+C_2\sin 3x) + e^{0x}(C_3\cos x+C_4\sin x)$

Step 2 : Particular integral

$$P.I. = \frac{1}{f(D)}X$$

$$P.I. = \frac{1}{D^4+10D^2+9}48[\sin 3x+\sin x]$$

$$P.I. = 48\left\{\frac{1}{D^4+10D^2+9}\sin 3x+\frac{1}{D^4+10D^2+9}\sin x\right\}$$

$$PI_1 = 48\frac{1}{(D^2+9)(D^2+1)}\sin 3x$$

Put $D^2=-a^2$ in non zero factor. i.e. $D^2=-9$

$$PI_1 = 48\frac{1}{(D^2+9)(-9+1)}\sin 3x$$

$$PI_1 = \frac{48}{-8}\cdot\frac{1}{D^2+9}\sin 3x$$

$$\left\{\text{i.e.} \frac{1}{D^2+a^2}\sin ax = \frac{-x}{2a}\cos ax\right\}$$

$$PI_1 = (-6)\cdot\frac{-x}{2\cdot 3}\cos 3x$$

$$PI_1 = x\cos 3x$$

Also consider

$$PI_2 = 48\frac{1}{(D^2+9)(D^2+1)}\sin x$$

Replace $D^2=-a^2$ only in non zero factor. i.e. $D^2=-1$

$$PI_2 = 48\frac{1}{(-1+9)(D^2+1)}\sin x$$

$$PI_2 = \frac{48}{8}\cdot\frac{1}{D^2+1}\sin x$$

For zero factor use formula (A).

$$\left\{\text{i.e.} \frac{1}{D^2+a^2}\sin ax = \frac{-x}{2a}\cos ax\right\}$$

$$PI_2 = 6\cdot\left(\frac{-x}{2\cdot 1}\right)\cdot\cos x$$

$$PI_2 = -3x\cos x$$

Now

$$\therefore \quad PI = PI_1+PI_2$$

$$= x\cos 3x - 3x\cos x$$

Step 3 : Complete solution

Now y = C.F. + P.I. is the complete solution.

$\therefore$ i.e. $y = C_1\cos 3x+C_2\sin 3x+x\cos 3x-3x\cos x$

Example 1.21 : $(D^4+5D^2+4)y=\cos\frac{x}{2}\cos\frac{3x}{2}$

Solution : Step 1 : Complementary function

C.F. $= C_1\cos x+C_2\sin x+C_3\cos 2x+C_4\sin 2x$

[Refer Problem No. 27 in problems on C.F. on page 1 - 6]

Step 2 : Particular integral

$$P.I. = \frac{1}{D^4+5D^2+4}\cos\frac{x}{2}\cos\frac{3x}{2}$$

Now use $2\cos A\cos B = \cos(A-B)+\cos(A+B)$

i.e. $$\cos\frac{x}{2}\cos\frac{3x}{2} = \frac{1}{2}\left[\cos\left(\frac{x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{x}{2}+\frac{3x}{2}\right)\right]$$

$$= \frac{1}{2}[\cos(-x)+\cos 2x]$$

$$= \frac{1}{2}[\cos x+\cos 2x]$$

i.e. $$P.I. = \frac{1}{D^4+5D^2+4}\frac{1}{2}(\cos x+\cos 2x)$$

$$= \frac{1}{2}\left\{\frac{1}{D^4+5D^2+4}\cos x+\frac{1}{D^4+5D^2+4}\cos 2x\right\}$$

As the zero factors for $\cos x$ and $\cos 2x$ are D^2+1 and D^2+4 respectively. $\therefore$ Take the factors of f(D) for finding P.I.

$$= \frac{1}{2}\left\{\frac{1}{(D^2+1)(D^2+4)}\cos x+\frac{1}{(D^2+1)(D^2+4)}\cos 2x\right\}$$

Put $D^2=-a^2$ only in non zero factor. i.e. $D^2=-1$ and $D^2=-4$ respectively.

$$= \frac{1}{2}\left\{\frac{1}{(-1+4)(D^2+1)}\cos x+\frac{1}{(-4+1)(D^2+4)}\cos 2x\right\}$$

$$= \frac{1}{6}\left\{\frac{1}{D^2+1}\cos x - \frac{1}{D^2+4}\cos 2x\right\}$$

Use $\frac{1}{D^2+a^2}\cos ax = \frac{+x}{2a}\sin ax$

$$= \frac{1}{6}\left\{\frac{x}{2}\sin x - \frac{x}{2.2}\sin x\right\}$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

$$y = C_1\cos x + C_2\sin x + C_3\cos 2x + C_4\sin 2x + \frac{x}{12}\sin x - \frac{x}{24}\sin 2x.$$

Example 1.22 : $(D^2+4)y = \sin x\cos 3x$

Solution : Step 1 : Complementary function

A.E. $D^2+4 = 0$

$D^2 = -4$	As $\sin x\cos 3x$
$D = \pm 2i$	$= \frac{1}{2}[2\sin x\cos 3x]$
$\alpha \pm i\beta$	$= \frac{1}{2}[\sin 4x + \sin(-2x)]$
C.F. $= e^{0x}(C_1\cos 2x + C_2\sin 2x)$	$= \frac{1}{2}[\sin 4x - \sin 2x]$
P.I. $= \frac{1}{f(D)}X$	[2 sin A cos B $= \sin(A+B) + \sin(A-B)$]

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^2+4}\frac{1}{2}[\sin 4x - \sin 2x]$$

$$\text{P.I.} = \frac{1}{2}\left\{\frac{1}{D^2+4}\sin 4x - \frac{1}{D^2+4}\sin 2x\right\} = PI_1 + PI_2$$

$PI_1 = \frac{1}{2}\cdot\frac{1}{D^2+4}\sin 4x$	$PI_2 = \frac{1}{2}\cdot\frac{1}{D^2+4}\sin 2x$
$D^2 = -a^2$	
$PI_1 = \frac{1}{2}\cdot\frac{1}{-16+4}\sin 4x$	$PI_2 = \frac{1}{2}\cdot\left(\frac{-x}{2\cdot 2}\right)\cdot\cos 2x$
$PI_1 = \frac{1}{2}\cdot\frac{1}{-12}\sin 4x$	$PI_2 = \frac{-x}{8}\cos 2x$
$PI_1 = \frac{-1}{24}\sin 4x$	P.I. $= PI_1 + PI_2$

Step 3 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1\cos 2x + C_2\sin 2x - \frac{1}{24}\sin 4x - \frac{x}{8}\cos 2x$$

Example 1.23 : $(D^2+4)y = \cos x\cos 2x\cos 3x$

Hint : Use 2 cosA cosB = cos (A–B) + cos (A + B)

Solution : $(D^2+4)y = \frac{1}{2}[\cos(x-2x) + \cos(x+2x)]$

Consider

$$\cos x\cos 2x = \frac{1}{2}[\cos(-x) + \cos 3x]$$

$$= \frac{1}{2}[\cos x + \cos 3x]$$

As $\{\cos(-\theta) = \cos\theta\}$

Multiply both sides by cos 3x

$$\therefore \cos x\cos 2x\cos 3x = \frac{1}{2}[\cos x + \cos 3x]\cos 3x$$

$$= \frac{1}{2}[\cos 3x\cos x + \cos^2 3x]$$

Use the formula again and $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \frac{1}{2}\left\{\frac{1}{2}[\cos(2x) + \cos(4x)] + \frac{1+\cos 6x}{2}\right\}$$

$$= \frac{1}{4}\{\cos 2x + \cos 4x + 1 + \cos 6x\}$$

$$= \frac{1}{4}\{1 + \cos 2x + \cos 4x + \cos 6x\}$$

Step 1 : Complementary function

To find C.F. put $D^2+4 = 0$

i.e. $D^2 = -4$

$D = \pm 2i$

$\alpha = 0,\ \beta = 2$

C.F. $= e^{0x}(C_1\cos 2x + C_2\sin 2x)$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^2+4}\frac{1}{4}[1 + \cos 2x + \cos 4x + \cos 6x]$$

Separate PI_1, PI_2, PI_3 ....

$$= \frac{1}{4}\left\{\frac{1}{D^2+4}1 + \frac{1}{D^2+4}\cos 2x + \frac{1}{D^2+4}\cos 4x + \frac{1}{D^2+4}\cos 6x\right\}$$

Put $D^2 = -a^2$ in non zero factor.

{ For constant D = 0

For zero factor $\frac{1}{D^2+a^2}\cos ax = \frac{+x}{2a}\sin ax$ }

$$\text{P.I.} = \frac{1}{4}\left\{\frac{1}{0+4} + \frac{x}{2.2}\sin 2x + \frac{1}{-16+4}\cos 4x + \frac{1}{-36+4}\cos 6x\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{4}+\frac{x}{4}\sin 2x-\frac{1}{12}\cos 4x-\frac{1}{32}\cos 6x\right\}$$

Step 3 : Complete solution

y = C.F + P.I. is the complete solution.

$$y = C_1\cos 2x + C_2\sin 2x + \frac{1}{16}+\frac{x}{16}\sin 2x$$

$$-\frac{1}{48}\cos 4x - \frac{1}{128}\cos 6x$$

Example 1.24 : $(D-1)^2(D^2+1)^2 y = \cos^2 x/2$

Solution : Step 1 : Complementary function

A.E. $(D-1)^2(D^2+1)^2 = 0$

$$(D-1)^2 = 0,\quad (D^2+1)^2 = 0$$

$$D = 1, 1 \qquad D^2 = -1 \qquad \text{twice}$$

$$D = 1, 1, \qquad D = \pm i \qquad \text{twice}$$

∴ Roots are repeated real and repeated complex

$$\text{C.F.} = e^{0x}(C_1 + C_2x) + e^{0x}[(C_3+C_4x)\cos x + (C_5+C_6x)\sin x]$$

$$\text{P.I.} = \frac{1}{f(D)}\phi(x)$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{(D^2+1)^2(D-1)^2}\cos^2\frac{x}{2}$$

$$\text{P.I.} = \frac{1}{(D^2+1)^2(D-1)^2}\left(\frac{1+\cos x}{2}\right)$$

$$\text{P.I.} = \frac{1}{(D^2+1)^2(D^2-2D+1)}\left(\frac{1}{2}+\frac{\cos x}{2}\right)$$

Consider

$$PI_1 = \frac{1}{(D^2+1)^2(D-1)^2}\frac{1}{2}$$

$$PI_1 = \frac{1}{(D^2+1)^2(0-1)^2}\frac{1}{2}$$

$$PI_1 = \frac{1}{(1)(1)^2}\frac{1}{2}$$

$$= \frac{1}{2}$$

Consider

$$PI_2 = \frac{1}{(D^2+1)^2(D-1)^2}\frac{\cos x}{2}$$

Take the factors of f(D) for finding P.I. as the zero factor of cos x is present.

$$PI_2 = \frac{1}{2}\frac{1}{(D^2+1)^2(D^2-2D+1)}\cos x$$

Replace D^2 by $-a^2$ only in non zero term.

$$PI_2 = \frac{1}{2}\cdot\frac{1}{(D^2+1)^2(-1-2D+1)}\cos x$$

$$PI_2 = \frac{1}{2}\cdot\frac{1}{(D^2+1)^2}\cdot\frac{1}{-2D}\cdot\cos x$$

$$PI_2 = \frac{-1}{4}\frac{1}{(D^2+1)^2}\sin x$$

$$PI_2 = \frac{1}{-4}\left(-\frac{x^2}{8.1}\right)\sin x$$

$$= \frac{x^2}{32}\sin x$$

$$\therefore \quad PI = PI_1 + PI_2$$

$$= \frac{1}{2}+\frac{x^2}{32}\sin x$$

Step 3 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

Example 1.25 : $\left(D^4+8D^2+16\right)y = \cos^2 x$

Solution : Step 1 : Complementary function

A.E. $\left(D^2+4\right)^2 = 0 \qquad D^2+4 = \sqrt{0} = 0$

$$D^2 = -4$$

$$D = \pm 2i$$

$$\alpha \pm i\beta$$

$$\text{C.F.} = e^{0x}\left[(c_1+c_2x)\cos 2x + (c_3+c_4c)\sin 2x\right]$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{\left(D^2+4\right)^2}\cos^2 x$$

$$\text{P.I.} = \frac{1}{\left(D^2+4\right)^2}\frac{1+\cos 2x}{2}$$

$$= \frac{1}{\left(D^2+4\right)^2}\frac{1}{2}+\frac{1}{\left(D^2+4\right)^2}\frac{\cos 2x}{2}$$

Consider

$$PI_1 = \frac{1}{(D^2+4)^2}\frac{1}{2}$$

$$PI_1 = \frac{1}{(0+4)^2}\frac{1}{2}$$

$$PI_1 = \frac{1}{32}$$

Consider

$$PI_2 = \frac{1}{(D^2+4)^2}\frac{\cos 2x}{2}$$

$$PI_2 = \frac{1}{2}\cdot\frac{1}{(D^2+4)^2}\cos 2x$$

$$= \frac{-x^2}{8a^2}\cos ax$$

$$PI_2 = \frac{1}{2}\cdot\frac{-x^2}{8\cdot 4}\cdot\cos 2x$$

$$= \frac{-x^2}{64}\cos 2x$$

$\therefore$ $$\text{P.I.} = PI_1 + PI_2$$

$$= \frac{1}{32} - \frac{x^2}{64}\cos 2x$$

Step 3 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

Exercise 1.2

1. *Solve* $(D^4 + 2D^2 + 1)\, y = \cos x$

[Ans. : $y = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x - \frac{x^2}{8}\cos x$**]**

2. $(D^3 + D)\, y = \cos x]$

[Ans. : $y = C_1 + C_2 \cos x + C_3 \sin x - \frac{x}{2}\cos x$**]**

3. $\frac{d^2y}{dx^2} + n^2 y = h \sin px$

$y = a,\ \frac{dy}{dx} = b$ at $x = 0$

[Ans. : $y = a\cos nx + \left[\frac{b}{n} - \frac{ph}{n(n^2-p^2)}\right]\sin nx + \frac{h\sin px}{(n^2-p^2)}$**]**

4. $\frac{d^2x}{dt^2} + n^2 x = f\cdot\cos(nt+\alpha)$

[Ans. : $x = C_1 \cos nt + C_2 \sin nt + \frac{ft}{2n}\sin(nt+\alpha)$**]**

1.5.2 Type 3 : If $\phi(x)$ = cosh (ax + b) or sinh (ax + b)

If $\phi(x) = \cosh(ax+b)$ or $\sinh(ax+b)$. As earlier on the similar line, we have,

$$\frac{1}{f(D^2)}\cosh(ax+b) = \frac{1}{f(a^2)}\cosh(ax+b)\ ;\ f(a^2) \neq 0$$

and $$\frac{1}{f(D^2)}\sinh(ax+b) = \frac{1}{f(a^2)}\sinh(ax+b) : f(a^2) \neq 0$$

Case of failure :

if $f(D^2) = 0,\ \frac{1}{f(D^2)}\sinh(ax+b) = x\frac{1}{f'(a^2)}\sinh(ax+b);$ $f'(a^2) \neq 0$

$$\frac{1}{f(D^2)}\cosh(ax+b) = x\frac{1}{f'(a^2)}\cosh ax\ ;\ f'(a^2) \neq 0$$

Example 1.26 : *Solve* $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\cosh 2x$

Solution : Step 1 : Complementary function

A.E. is $D^3 - 4D = 0$

$$D(D^2-4) = 0$$

$$D(D-2)(D+2) = 0$$

$\Rightarrow$ D = 0, 2, – 2 which are real and distinct.

$$y_c = C_1 + C_2 e^{2x} + C_3 e^{-2x}$$

Step 2 : Particular integral

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{(D^2-4)}\frac{1}{D}2\cosh 2x$$

$$= \frac{1}{D^2-4}\int 2\cosh 2x\, dx = \frac{1}{D^2-4}(\sinh 2x)$$

$$= x\frac{1}{2D}\sinh 2x$$

$$= x\frac{D}{2D^2}\sinh 2x = \frac{x}{2(2)^2}D\sinh 2x$$

$$= \frac{2x}{8}(\cosh 2x) = \frac{x}{4}\cosh 2x$$

Step 3 : Complete solution

$$y = y_c + y_p$$

$\therefore$ $$y = C_1 + C_2 e^{2x} + C_3 e^{-2x} + \frac{x}{4}\cosh 2x$$

1.5.3 Type 4 : If $\phi(x)$ is Polynomial

$\phi(x) = x^p$; where p is positive integer then use the binomial series.

1) $(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \ldots$

2) $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$

3) $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots$

Note : If the DE involves the polynomial terms then use this method.

Example 1.27 : *Find particular integrals of the following*

a) $(1+D)y = x^3$,

b) $(1-D^2)y = x^4+1$,

c) $\frac{1}{(1+D^2)y} = x^3+x$

d) $(1+D^2)^2 y = x^2+1$

Solution :

a) We have
$$y_p = \text{P.I.} = \frac{1}{1+D}(x^3)$$
$$= (1-D+D^2-D^3+D^4-\ldots)(x^3)$$
$$= x^3-3x^2+6x-6+0$$
$$= x^3-3x^2+6x-6$$

b)
$$y_p = \frac{1}{1-D^2}(x^4+1)$$
$$= [1+D^2+D^4+D^6+\ldots](x^4+1)$$
$$= (x^4+12x^2+24+0)+1+0$$
$$= x^4+12x^2+25$$

c) We have
$$y_p = (1+D^2)^2(x^3+x)$$
$$= \left[1+2D^2+\frac{2(2-1)}{2!}D^4\right](x^3+x)$$
$$= [1+2D^2+D^4](x^3+x)$$
$$= (x^3+2(6x)+0)+x$$
$$= x^3+13x$$

d)
$$y_p = \frac{1}{(1+D^2)^2}(x^2+1)$$
$$= \frac{1}{1+2D^2+D^4}(x^2+1)$$
$$= \frac{1}{1+(2D^2+D^4)}(x^2+1)$$
$$= [1-(2D^2+D^4)+(2D^2+D^4)^2-\ldots](x^2+1)$$
$$= x^2-2(2)+0+1$$
$$= x^2-3$$

Example 1.28 : $(D^4-2D^3+D^2)y = x^3$

Solution : Step 1 : Complementary function

A.E.
$$D^2(D^2-2D+1) = 0$$
$$D^2 = 0, \quad (D-1)^2 = 0$$
$$D^2 = 0 \qquad D = 1, 1$$
$$\text{C.F.} = (C_1+C_2x)\,e^{0x}+(C_3+C_4x)\,e^x$$

Step 2 : Particular integral
$$\text{P.I.} = \frac{1}{(D^4-2D^3+D^2)}x^3$$
$$\text{P.I.} = \frac{1}{D^2}\cdot\frac{1}{(D-1)^2}x^3$$
$$\text{P.I.} = \frac{1}{D^2}\cdot\frac{1}{(-1)^2}\cdot\frac{-1}{(1-D)^2}x^3$$

Write f(D) in the form $(1+z)^n$, $z = -D$, $n = -2$
$$\text{P.I.} = \frac{1}{D^2}[1+(-D)]^{-2}\,x^3$$

Use the binomial series $(1+z)^n$.
$$\text{P.I.} = \frac{1}{D^2}\left\{\begin{matrix}1+(-2)(-D)+\frac{(-2)(-3)}{2!}(-D)^2 \\ +\frac{(-2)(-3)(-4)}{3!}(-D)^3\end{matrix}\right\}x^3$$
$$\text{P.I.} = \frac{1}{D^2}\{1+2D+3D^2+4D^3\}\,x^3$$
$$\text{P.I.} = \frac{1}{D^2}\{x^3+2Dx^3+3D^2x^3+4D^3x^3\}$$
$$\text{P.I.} = \frac{1}{D^2}\left\{x^3+2(3x^2)+3\cdot(6x)+4(6)\right\}$$
$$\text{P.I.} = \frac{1}{D}\left\{\frac{x^4}{4}+2\cdot3\cdot\frac{x^3}{3}+3\cdot6\cdot\frac{x^2}{2}+4\cdot6\cdot x\right\}$$
$$\text{P.I.} = \left\{\frac{x^5}{4\cdot5}+2\cdot\frac{x^4}{4}+3\cdot3\cdot\frac{x^3}{3}+4\cdot6\cdot\frac{x^2}{2}\right\}$$
$$\text{P.I.} = \left\{\frac{x^5}{20}+\frac{x^4}{2}+3x^3+12x^2\right\}$$

Step 3 : Complete solution
$$y = \text{C.F.} + \text{P.I.}$$

i.e.
$$y = C_1+C_2x+(C_3+C_4x)\,e^x + \frac{x^5}{20}+\frac{x^4}{2}+3x^3+12x^2$$

Example 1.29 : *Solve* $(D^3-6D^2+12D-8)y = x^2+1$

Solution : Step 1 : Complementary function

A.E.
$$D^3-6D^2+12D-8 = 0$$
$$(D-2)^3 = 0$$
$$D = 2, 2, 2$$

2	1	– 6	12	– 8
		2	– 8	8
	1	– 4	4	0

$$(D-2)\,(D^2-4D+4) = 0$$
$$(D-2)\,(D-2)^2 = 0$$
$$(D-2)^3 = 0$$
$$\text{C.F.} = (C_1+C_2x+C_3x^2)\,e^{2x}$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^3-6D^2+12D-8}(x^2+1)$$
$$\text{P.I.} = \frac{1}{(D-2)^3}(x^2+1)$$
$$\text{P.I.} = \frac{1}{(-2)^3}\cdot\frac{1}{\left(-\frac{D}{2}+1\right)^3}(x^2+1)$$

Write f(D) in the form $(1+z)^n$, $z = \frac{-D}{2}$,

$$n = -3$$
$$\text{P.I.} = \frac{1}{-8}\left[1+\left(\frac{-D}{2}\right)\right]^{-3}(x^2+1)$$
$$(1+z)^n = \left\{1+nz+\frac{n(n-1)}{2!}z^2 \ldots\right\}$$
$$= \frac{1}{-8}\left\{1+(-3)\left(\frac{-D}{2}\right)+\frac{(-3)(-4)}{2!}\left(\frac{-D}{2}\right)^2 \ldots\right\}(x^2+1)$$
$$= \frac{-1}{8}\left\{1+\frac{3}{2}D+\frac{3}{2}D^2\right\}(x^2+1)$$
$$= \frac{-1}{8}\left\{(x^2+1)+\frac{3}{2}D(x^2+1)+\frac{3}{2}D^2(x^2+1) \ldots\right\}$$
$$= \frac{-1}{8}\left\{(x^2+1)+\frac{3}{2}(2x)+\frac{3}{2}(2)\right\}$$
$$= \frac{-1}{8}\{x^2+3x+4\}$$

Step 3 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

Example 1.30 : *Solve* $(D^3+2D^2+D)y = e^{2x}+x^2+x$

Solution : Step 1 : Complementary function

A.E. $D\,(D^2+2D+1) = 0$

$$D\,(D+1)^2 = 0$$
$$D = 0,\quad D = -1, -1$$
$$\text{C.F.} = C_1e^{0x}+(C_2+C_3x)\,e^{-x}$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^3+2D^2+D}(e^{2x}+x^2+x)$$

Consider
$$PI_1 = \frac{1}{D\,(D+1)^2}e^{2x}$$
$$= \frac{1}{2\,(2+1)^2}e^{2x}$$
$$PI_1 = \frac{e^{2x}}{18}$$

Also consider
$$PI_2 = \frac{1}{D^3+2D^2+D}(x^2+x)$$
$$PI_2 = \frac{1}{D\,(D+1)^2}(x^2+x)$$
$$PI_2 = \frac{1}{D}\cdot\frac{1}{(D+1)^2}(x^2+x)$$
$$PI_2 = \frac{1}{D}(1+D)^{-2}(x^2+x)$$
$$(1+z)^n = \left\{1+nz+\frac{n(n-1)}{2!}z^2 \ldots\right\}$$
$$PI_2 = \frac{1}{D}\left\{1+(-2)(D)+\frac{(-2)(-3)}{2!}(D)^2 \ldots\right\}(x^2+x)$$
$$PI_2 = \frac{1}{D}\{1-2D+3D^2 \ldots\}(x^2+x)$$
$$PI_2 = \frac{1}{D}\{(x^2+x)-2D(x^2+x)+3D^2(x^2+x) \ldots\}$$
$$PI_2 = \frac{1}{D}\{x^2+x-2(2x+1)+3(2)\}$$
$$PI_2 = \frac{1}{D}\{x^2+3x+4\}$$
$$PI_2 = \left\{\frac{x^3}{3}+3\frac{x^2}{2}+4x\right\}$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

Example 1.31 : $(D^3-2D+4)y = 3x^2-5x+2$

Solution : Step 1 : Complementary function

C.F. = $C_1e^{-2x}+e^{-x}(C_2\cos x+C_3\sin x)$

[Refer Problem No. 13 in problems on C.F. on page 1 - 4]

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{f(D)}\phi(x)$$

$$\text{P.I.} = \frac{1}{D^3-2D+4}(3x^2-5x+2)$$

$$\text{P.I.} = \frac{1}{4}\cdot\frac{1}{\left[1+\dfrac{D^3-2D}{4}\right]}(3x^2-5x+2)$$

$$z = \frac{D^3-2D}{4}.$$

$$\frac{1}{1+z} = \left\{1-z+z^2....\right\}$$

$$\text{P.I.} = \frac{1}{4}\left\{1-\left(\frac{D^3-2D}{4}\right)+\left(\frac{D^3-2D}{4}\right)^2...\right\}(3x^2-5x+2)$$

$$\text{P.I.} = \frac{1}{4}\left\{1-\frac{D^3}{4}+\frac{D}{2}+\left(\frac{D^6-4D^4+4D^2}{16}\right)\right\}(3x^2+5x+2)$$

$$\text{P.I.} = \frac{1}{4}\left\{\begin{array}{l}(3x^2-5x+2)-\dfrac{D^3}{4}(3x^2-5x+2)\\ +\dfrac{D}{2}(3x^2-5x+2)\\ +\dfrac{1}{16}\left[\begin{array}{l}D^6(3x^2-5x+2)-4D^4(3x^2-5x+2)\\ +4D^2(3x^2-5x+2)\end{array}\right]\end{array}\right\}$$

$$\textbf{P.I.} = \frac{1}{4}\left\{(3x^2-5x+2)-0+\frac{1}{2}(6x-5)+\frac{1}{16}[0-0+4(6)]\right\}$$

$$\text{P.I.} = \frac{1}{4}\{3x^2-2x+1\}$$

Step 3 : Complete solution

y = C.F + P.I. is the complete solution.

Example 1.32 : $(D^3-3D+2)y=(x^3-4x^2+2)$

Solution : Step 1 : Complementary function

A.E. $D^3-3D+2 = 0$

1	1	0	– 3	2
		1	1	– 2
	1	1	– 2	0

$(D-1)(D^2+D-2)$

$D = 1$

$$D = \frac{-1\pm\sqrt{1-8}}{2}$$

$$D = -\frac{1}{2}\pm i\frac{\sqrt{7}}{2}$$

$$\text{C.F.} = C_1e^x+e^{-x/2}\left[C_2\cos\frac{\sqrt{7}x}{2}+C_2\sin\frac{\sqrt{7}x}{2}\right]$$

Step 2 : Particular integral P.I. $= \dfrac{1}{f(D)}\phi(x)$

$$\text{P.I.} = \frac{1}{D^3-3D+2}(x^3-4x^2+2)$$

$$\text{P.I.} = \frac{1}{2}\,\frac{1}{1+\left(\dfrac{D^3-3D}{2}\right)}(x^3-4x^2+2)$$

$$\frac{1}{1+z} = \{1-z+z^2-z^3....\}$$

$$\text{P.I.} = \frac{1}{2}\left\{1-\left(\frac{D^3-3D}{2}\right)+\left(\frac{D^3-3D}{2}\right)^2-\left(\frac{D^3-3D}{2}\right)^3\right\}(x^3-4x^2+2)$$

As x^3 is involved ∴ Every term after D^3 will be zero ∴ Write only significant terms.

$$\text{P.I.} = \frac{1}{2}\left\{1-\left(\frac{D^3-3D}{2}\right)+\left(\frac{D^6-6D^4+9D^2}{4}\right)-\left(\frac{27}{8}D^3\right)\right\}(x^3-4x^2+2)$$

$$\text{P.I.} = \frac{1}{2}\left\{\begin{array}{l}(x^3-4x^2+2)-\left(\dfrac{6-3(3x^2-8x)}{2}\right)\\ +\dfrac{0-0+9(6x-8)}{4}-\dfrac{27}{8}(6)\end{array}\right\}$$

$$\text{P.I.} = \frac{1}{2}\left\{\begin{array}{l}x^3+\left(-4+\dfrac{9}{2}\right)x^2+\left(-12+\dfrac{27}{2}\right)x\\ +\left(2-3-18-\dfrac{81}{4}\right)\end{array}\right\}$$

$$\text{P.I.} = \frac{1}{2}\left\{x^3+\frac{x^2}{2}+\frac{3}{2}x+\frac{5}{4}\right\}$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

Exercise 1.3

1. $(D^3+6D^2+12D+8)y=e^{-2x}+x^2$

[Ans. : $\mathbf{y = (C_1+C_2x+C_3x^2)\,e^{-2x}+e^{-2x}\cdot\frac{x^3}{6}+\frac{1}{8}(x^2-3x+3)}$**]**

2. $(D^2+6D+9)y=\cos 3x+3^x+x^2+e^{-3x}$

[Ans. : $\mathbf{y = (C_1+C_2x)\,e^{-3x}+\frac{\sin 3x}{18}+\frac{3^x}{(\log 3+3)^2}+\frac{1}{9}\left(x^2-\frac{4}{3}x+\frac{2}{3}\right)+\frac{x^2}{2}e^{-3x}}$**]**

3. $(D^4-a^4)y=x^4$

[Ans. : $\mathbf{y = C_1\cos ax+C_2\sin ax+C_3e^{ax}+C_4e^{-ax}-\frac{1}{a^4}\left(x^4+\frac{24}{a^4}\right)}$**]**

4. $(D^2+2D+2)y = x^2+\cos x$

$$\left[\text{Ans. : } y = e^{-x}(C_1\cos x + C_2 \sin x) + \frac{1}{2}(x^2-2x+1) + \frac{1}{5}(2\sin x + \cos x)\right]$$

5. $(D^4+D^2+1)y = 53x^2+17$ $\quad$ $[\text{Ans. : P.I.} = 53x^2 - 89]$

6. $(D^5 - D)y = 12e^x + 85mx + 2^x$

$$\left[\text{Ans. : P.I.} = 3x\,e^x - \frac{85mx^2}{2} + \frac{2^x}{(\log 2)^5 - (\log 2)}\right]$$

7. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 10y = 50x$

at $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 0$

$$[\text{Ans. : } y = e^{-3x}[3\cos x + 5\sin x] + 5x - 3]$$

1.5.4 Type 5 : If $\phi(x) = e^{ax}\ V$

If $\phi(x) = e^{ax}\ V$ where

V is any function of x.

We know that

$$D(e^{ax}\ V) = e^{ax}\ DV + V\,e^{ax}\ a$$
$$= e^{ax}\ (DV + aV)$$
$$D(e^{ax}\ V) = e^{ax}\ (D+a)\,V$$

i.e. Derivative of e^{ax} V is obtained by taking e^{ax} outside and replacing D by D + a.

Similarly we can show that

$$f(D)\,e^{ax}\ V = e^{ax}\,f(D+a)\,V$$

and $$\boxed{\frac{1}{f(D)}e^{ax}\,V = e^{ax}\left[\frac{1}{f(D+a)}V\right]}$$

Note : 1) To operate $\dfrac{1}{f(D)}$ on a product $(e^{ax}V)$ where V is any function of x.

Take e^{ax} to the left side as a multiple replace D by (D + a) in f(D) and operate $\dfrac{1}{f(D+a)}$ on V.

2) If V is of the form x^p then the problem reduces to type 4.

If V is sin ax or cos ax then the problem reduces to type 2.

Example 1.33 : *Find particular integrals of the following*

a) $(D+1)y = e^{2x}x$

b) $(D^2+D)y = e^x x$

c) $(-4D+D^2)y = e^{2x}\sin 5x$

Solution :

a) We have

$$\text{P.I.} = y_p = \frac{1}{1+D}e^{2x}x$$
$$= e^{2x}\frac{1}{1+(D+2)}x$$
$$= e^{2x}\frac{1}{D+3}x$$
$$= e^{2x}\frac{1}{3}\frac{1}{\left(1+\frac{D}{3}\right)}(x)$$
$$= \frac{e^{2x}}{3}\left[1-\frac{D}{3}+\frac{D^2}{9}-\dots\right](x)$$
$$= \frac{e^{2x}}{3}\left(x-\frac{1}{3}\right)$$

b) We have

$$\text{P.I.} = \frac{1}{D+D^2}(e^x x)$$
$$= e^x\frac{1}{(D+1)+(D+1)^2}x$$
$$= e^x\frac{1}{D^2+3D+2}(x)$$
$$= \frac{e^x}{2}\frac{1}{\left(1+\frac{D^2+3D}{2}\right)}(x)$$
$$= \frac{e^x}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\dots\right](x)$$
$$= \frac{e^x}{2}\left[x-\frac{3}{2}\right]$$

c) We have

$$\text{P.I.} = \frac{1}{(-4D+D^2)}(e^{2x}\sin 5x)$$
$$= e^{2x}\frac{1}{-4(D+2)+(D+2)^2}[\sin 5x]$$
$$= e^{2x}\frac{1}{D^2-8}(\sin 5x)$$
$$= e^{2x}\frac{1}{-25-8}\sin 5x$$
$$= \frac{-e^{2x}\sin 5x}{33}$$

Example 1.34 : $(D^3-7D-6)y = e^{2x}(1+x)$

Solution : **Step 1 :** Complementary function

A.E. $D^3 - 7D - 6 = 0$

$$\text{C.F.} = C_1e^{-x} + C_2e^{3x} + C_3e^{-2x}$$

[Refer Problem No. 1 in problems on C.F. on page 1 - 3]

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^3 - 7D - 6} e^{2x}(1+x)$$

Take out e^{ax} and replace D by (D + a) i.e. D + 2

$$\text{P.I.} = e^{2x} \frac{1}{(D+2)^3 - 7(D+2) - 6}(1+x)$$

$$= e^{2x} \frac{1}{D^3 + 6D^2 + 5D - 12}(1+x)$$

$$= \frac{e^{2x}}{-12} \frac{1}{\left[1 - \left(\frac{D^3 + 6D^2 + 5D}{12}\right)\right]}(1+x)$$

Here $z = \frac{D^3 + 6D^2 + 5D}{12}$ use binomial series.

$$\frac{1}{1+z} = 1 + z + z^2 \ldots$$

$$\text{P.I.} = \frac{e^{2x}}{-12}\left\{1 + \left(\frac{D^3 + 6D^2 + 5D}{12}\right) \ldots\right\}(1+x)$$

$$= \frac{e^{2x}}{-12}\left\{(1+x) + \frac{D^3(1+x) + 6D^2(1+x) + 5D(1+x)}{12}\right\}$$

$$= \frac{e^{2x}}{-12}\left\{1 + x + \frac{0+0+5}{12}\right\}$$

$$\text{P.I.} = \frac{e^{2x}}{-12}\left\{x + \frac{17}{12}\right\}$$

Step 3 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

Exercise 1.4

1. $(D^3 - 7D - 6)\,y = e^{2x}(1+x^2)$

[Ans. : P.I. $= \frac{e^{2x}}{-12}\left\{x^2 + \frac{5x}{6} + \frac{169}{72}\right\}$**]**

2. $(D^2 + 5D + 6)\,y = 4x^2e^x$

[Ans. : $y = C_1e^{-2x} + C_2e^{-3x} + \frac{e^x}{3}\left(x^2 - \frac{7x}{6} + \frac{37}{72}\right)$**]**

3. $(D^2 - 1)\,y = e^x(1+x^2)$

[Ans. : $y = C_1e^x + C_2e^{-x} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2}\right]$**]**

4. $(D^2 + 6D + 9)\,y = \frac{1}{x^3}e^{-3x}$

[Ans. : $y = (C_1 + C_2x)\,e^{-3x} + \frac{e^{3x}}{2x}$**]**

5. $(D^2 - D - 1)\,y = e^{-x}x^3$

[Ans. : P.I. $= e^{-x}(x^3 + 9x^2 + 48x + 126)$**]**

6. $\left(D^4 - 2D^3 - 3D^2 + 4D + 4\right)y = x^2e^x$

[Ans. : $y = (C_1 + C_2x)\,e^{-x} + (C_3 + C_4x)\,e^{2x} + \frac{e^x}{4}\left(x^2 + 2x + \frac{7}{2}\right)$**]**

7. $\left(D^3 - 6D^2 + 11D - 6\right)y = xe^x$

[Ans. : $y = C_1e^x + C_2e^{2x} + C_3e^{3x} + \frac{e^x}{2}\left(\frac{x^2}{2} + \frac{3x}{2}\right)$**]**

8. $\left(D^3 - 3D + 2\right)y = x^2e^x$

[Ans. : $y = (C_1 + C_2x)\,e^x + C_3e^{-x} + \frac{e^xx^2}{108}\left(3x^2 - 4x + 4\right)$**]**

Example 1.35 : $(D^4 - 1)\,y = \cos x \cosh x$

Solution : **Step 1 :** Complementary function

$$\text{C.F.} = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x$$

[Refer Problem No. 9 in problems on C.F. on page 1 - 4)

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^4 - 1}\cos x \cosh x$$

Use $\cosh x = \left(\frac{e^x + e^{-x}}{2}\right)$

$$\text{P.I.} = \frac{1}{D^4 - 1}\cos x\left(\frac{e^x + e^{-x}}{2}\right)$$

$$\text{P.I.} = \frac{1}{2}\left\{\frac{1}{D^4 - 1}e^x\cos x + \frac{1}{D^4 - 1}e^{-x}\cos x\right\}$$

Take out e^{ax} and replace D by D + a.

$$\text{P.I.} = \frac{1}{2}\left\{e^x\frac{1}{(D+1)^4 - 1}\cos x + e^{-x}\frac{1}{(D-1)^4 - 1}\cos x\right\}$$

$$\text{P.I.} = \frac{1}{2}\left\{e^x \cdot \frac{1}{D^4 + 4D^3 + 6D^2 + 4D + 1 - 1}\cos x\right.$$

$$\left. + e^{-x} \cdot \frac{1}{D^4 - 4D^3 + 6D^2 - 4D + 1 - 1}\cos x\right\}$$

Replace $D^2 = -a^2$, $D^3 = -a^2D$, $D^4 = a^4$ Here a = 1.

$$\text{P.I.} = \frac{1}{2}\left\{e^x\frac{1}{1 - 4D - 6 + 4D}\cos x + e^{-x}\frac{1}{1 + 4D - 6 - 4D}\cos x\right\}$$

$$\text{P.I.} = \frac{1}{2}\left\{e^x \frac{1}{-5}\cos x + e^{-x}\frac{1}{-5}\cos x\right\}$$

$$= \frac{1}{-5}\left(\frac{e^x + e^{-x}}{2}\right)\cos x$$

$$= \frac{-1}{5}\cosh x \cos x$$

Step 3 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

Example 1.36 : $(D^4+1)y = e^x \cos 2x$

Solution : Step 1 : Complementary function

A.E. $D^4 + 1 = 0$

[Refer Problem No. 23 in problems on C.F.]

$$\text{C.F.} = e^{-x/\sqrt{2}}\left[C_1\cos\frac{x}{\sqrt{2}} + C_2\sin\frac{x}{\sqrt{2}}\right] + e^{x/\sqrt{2}}\left[C_3\cos\frac{x}{\sqrt{2}} + C_4\sin\frac{x}{\sqrt{2}}\right]$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^4+1}e^x\cos 2x$$

Take e^{ax} outside replace D by D+a, here a = 1.

$$\text{P.I.} = e^x\frac{1}{(D+1)^4+1}\cos 2x$$

Simplify

$$\text{P.I.} = e^x\frac{1}{D^4+4D^3+6D^2+4D+1+1}\cos 2x$$

Replace $D^2 = -4$, $D^3 = -4D$, $D^4 = 16$.

$$\text{P.I.} = e^x\frac{1}{16+4(-4D)+6(-4)+4D+1+1}\cos 2x$$

$$\text{P.I.} = e^x\frac{1}{-12D-6}\cos 2x$$

$$\text{P.I.} = \frac{e^x}{-6}\cdot\frac{1}{2D+1}\cdot\frac{2D-1}{2D-1}\cos 2x$$

$$\text{P.I.} = \frac{e^x}{-6}\cdot\frac{2D-1}{4D^2-1}\cos 2x$$

Again $D^2 = -4$.

$$\text{P.I.} = \frac{e^x}{-6}\cdot\frac{2D-1}{-16-1}\cos 2x$$

$$\text{P.I.} = \frac{e^x}{102}(2D-1)\cos 2x$$

$$\text{P.I.} = \frac{e^x}{102}[2D\cos 2x - \cos 2x]$$

$$\text{P.I.} = \frac{e^x}{102}[-4\sin 2x - \cos 2x]$$

Step 3 : Complete solution

y = C.F + P.I is the complete solution.

Exercise 1.5

1. $(D^4+D^2+1)y = ax^2 + be^{-x}\sin 2x$

$$\left[\textbf{Ans. : } y = e^{-x/2}\left[C_1\cos\frac{x\sqrt{3}}{2} + C_2\sin\frac{x\sqrt{3}}{2}\right] + e^{x/2}\left[C_3\cos\frac{x\sqrt{3}}{2} + C_4\sin\frac{x\sqrt{3}}{2}\right] + a(x^2-2) + \frac{be^{-x}}{-481}(20\cos 2x + 9\sin 2x)\right]$$

2. $(D^2+D-6)y = e^{-2x}\sin 3x$

$$\left[\textbf{Ans. : } y = C_1e^{-3x} + C_2e^{2x} + \frac{e^{-2x}}{250}[9\cos 3x - 13\sin 3x]\right]$$

3. $(D^3+3D)y = \sinh x \sin 3x$

$$\left[\textbf{Ans. : } y = C_1 + C_2\cos\sqrt{3}x + C_2\sin\sqrt{3}x + \frac{1}{610}\{9\cos 3x\sinh x - 23\cosh x\sin 3x\}\right]$$

4. $(D^3-D^2+3D+5)y = e^x\cos 3x$

$$\left[\textbf{Ans. : } y = C_1e^{-x} + e^x[C_2\cos 2x + C_3\sin 3x] - \frac{e^x}{65}[3\sin 3x + 2\cos 3x]\right]$$

5. $(D^2-1)y = \cosh x \cos x$

$$\left[\textbf{Ans. : } y = C_1e^x + C_2e^{-x} + \frac{1}{5}[2\sinh x \sin x - \cosh x \cos x]\right]$$

6. $(D^2-6D+13)y = 8e^{3x}\sin 4x$

$$\left[\textbf{Ans. : } y = e^{3x}[C_1\cos 2x + C_2\sin 2x] - \frac{2}{3}e^{3x}\sin 4x\right]$$

7. $(D^2+5D+6)y = e^{-2x}\sin 2x + 4x^2e^x$

$$\left[\textbf{Ans. : } y = C_1e^{-2x} + C_2e^{-3x} + \frac{e^{-2x}}{-10}(\cos 2x + 2\sin 2x) + \frac{e^x}{3}\left(x^2 - \frac{7x}{6} + \frac{37}{72}\right)\right]$$

8. $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$

$$\left[\textbf{Ans. : P.I.} = \frac{x\,e^{-x/2}}{4\sqrt{3}}\left[\sin\frac{\sqrt{3}x}{2} + \sqrt{3}\cos\frac{\sqrt{3}x}{2}\right]\right]$$

9. $(D^2-2D+2)y = e^{-x}\sin x$

$$\left[\textbf{Ans. : } y = e^{-x}(C_1\cos x + C_2\sin x) - \frac{x}{2}e^{-x}\cos x\right]$$

10. $\left(D^2-4D+8\right)y = e^{2x}\sin(2x+5)$

$$\left[\text{Ans. : } y = e^{2x}(C_1\cos 2x + C_2\sin 2x) - \frac{x}{4}e^{2x}\cos(2x+5)\right]$$

11. $\left(D^3-7D-6\right)y = \cosh x\cos x$

$$\left[\text{Ans. : } y = C_1e^{-x}+C_2e^{-2x}+C_3e^{3x} - \frac{e^x}{100}(\sin x + 3\cos x) - \frac{e^{-x}}{68}(5\sin x - 3\cos x)\right]$$

12. $\left(D^2-1\right)y = \cosh x\sin x\cos x$

$$\left[\text{Ans. : } y = C_1e^{x}+C_2e^{-x} - \frac{1}{16}\sinh x\cos 2x - \frac{1}{16}\cosh x\sin 2x\right]$$

1.5.5 Type 6 : If $\phi(x) = xV$

Let $\phi(x) = x\,V$ where V is any function of x

$$\therefore \quad D(xV) = xDV + V$$

$$\therefore \quad D^2(xV) = xD^2V + 2DV$$

$$D^3(xV) = xD^3V + 3D^2V$$

Similarly $\quad D^n(xV) = xD^nV + nD^{n-1}V$

$$\therefore \quad f(D)(xV) = xf(D)V + f'(D)V$$

Let $\quad f(D)V = V_1$ i.e. $V = \dfrac{1}{f(D)}V_1$

$$\therefore \quad f(D)\left[x\frac{1}{f(D)}V_1\right] = xV_1 + f'(D)\cdot\frac{1}{f(D)}V_1$$

$$\left[x\cdot\frac{1}{f(D)}V_1\right] = \frac{1}{f(D)}xV_1 + \frac{f'(D)}{[f(D)]^2}V_1$$

$$\therefore \quad \frac{1}{f(D)}xV_1 = x\cdot\frac{1}{f(D)}V_1 - \frac{f'(D)}{[f(D)]^2}V_1$$

$$\frac{1}{f(D)}xV_1 = \left\{x - \frac{f'(D)}{f(D)}\right\}\frac{1}{f(D)}V_1$$

But here V_1 is a function of x hence

$$\boxed{\frac{1}{f(D)}xV = \left\{x - \frac{f'(D)}{f(D)}\right\}\frac{1}{f(D)}V}$$

Note :

1) Here V is any function of x mostly trigonometric.
2) Power of x must be one.
3) If V is sin ax or cos ax and f(D) involves the factor (D^2+a^2) [i.e. the zero factor of sin ax or cos ax] then don't use type 6 use type 7.

i.e. for (1) $\dfrac{1}{D^2+a^2}x\sin ax$ and

(2) $\dfrac{1}{D^2+a^2}x\cos ax$ don't use type 6 use type 7.

Example 1.37 : *Find particular integrals of the following a) $(D+1)\,y = xe^{2x}$*
b) $(1-D^2)y = x\sin 3x$

Solution :

a) We have

$$\text{P.I.} = y_p = \frac{1}{D+1}(xe^{2x})$$

$$= \left\{x - \frac{1}{D+1}\right\}\frac{1}{D+1}e^{2x}$$

$$= x\frac{1}{D+1}e^{2x} - \frac{1}{(D+1)^2}e^{2x}$$

$$= x\frac{1}{2+1}e^{2x} - \frac{1}{(2+1)^2}e^{2x}$$

$$= e^{2x}\left(\frac{x}{3}-\frac{1}{9}\right)$$

b) We have

$$\text{P.I.} = y_p = \frac{1}{1-D^2}x\sin 3x$$

$$= \left\{x - \frac{(-2D)}{1-D^2}\right\}\frac{1}{1-D^2}\sin 3x$$

$$= \left\{x + \frac{2D}{1-D^2}\right\}\frac{1}{1-D^2}\sin 3x$$

$$= \left\{x + \frac{2D}{1-D^2}\right\}\frac{1}{1-9}\sin 3x$$

$$= \left\{x + \frac{2D}{1-D^2}\right\}\left(-\frac{1}{8}\sin 3x\right)$$

$$= -\frac{1}{8}x\sin 3x + \left(-\frac{1}{8}\right)2D\frac{1}{1-9}\sin 3x$$

$$= -\frac{1}{8}x\sin 3x + \frac{1}{32}D(\sin 3x)$$

$$= -\frac{1}{8}x\sin 3x + \frac{1}{32}(3)\cos 3x$$

Example 1.38 : $(D^2+3D+2)\,y = xe^{-x}\sin x$

Solution : Step 1 : Complementary function

A.E. $(D+1)(D+2) = 0$

$$D = -1, \quad D = -2$$

$$\text{C.F.} = C_1e^{-x} + C_2e^{-2x}$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^2+3D+2} x e^{-x} \sin x$$

As e^{-x} is present use

$$\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V \text{ i.e. Type 5.}$$

$$\text{P.I.} = e^{-x} \frac{1}{(D-1)^2+3(D-1)+2} x \sin x$$

$$\text{P.I.} = e^{-x} \left\{ \frac{1}{D^2+D} x \sin x \right\}$$

We have

$$\frac{1}{f(D)} xV = \left\{ x - \frac{f'(D)}{f(D)} \right\} \frac{1}{f(D)} V$$

$$\text{P.I.} = e^{-x} \left\{ x - \frac{2D+1}{D^2+D} \right\} \frac{1}{D^2+D} \sin x$$

Replace $D^2 = -a^2$ outside as well as inside the bracket i.e. $D^2 = -1$

$$= e^{-x} \left\{ x - \frac{2D+1}{-1+D} \right\} \frac{1}{-1+D} \sin x$$

$$= e^{-x} \left\{ x - \frac{(2D+1)(D+1)}{(D-1)(D+1)} \right\} \frac{D+1}{(D-1)(D+1)} \sin x$$

$$= e^{-x} \left\{ x - \frac{2D^2+3D+1}{D^2-1} \right\} \frac{D+1}{D^2-1} \sin x$$

$$= e^{-x} \left\{ x - \frac{-2+3D+1}{-1-1} \right\} \frac{D+1}{-1-1} \sin x$$

$$= \frac{e^{-x}}{-2} \left\{ x + \frac{(3D-1)}{2} \right\} (D+1) \sin x$$

$$= \frac{e^{-x}}{-2} \left\{ x(D+1) \sin x + \frac{1}{2}(3D-1)(D+1) \sin x \right\}$$

$$= \frac{e^{-x}}{-2} \left\{ x(D+1) \sin x + \frac{1}{2}(3D^2+2D-1) \sin x \right\}$$

$D^2 = -a^2$ in 2^{nd} term.

$$= \frac{e^{-x}}{-2} \left\{ x(D+1) \sin x + \frac{1}{2}(-3+2D-1) \sin x \right\}$$

$$= \frac{e^{-x}}{-2} \left\{ x(D+1) \sin x + \frac{1}{2}(2D-4) \sin x \right\}$$

$$= \frac{e^{-x}}{-2} \{ x(\cos x + \sin x) + (\cos x - 2\sin x) \}$$

$$\text{P.I.} = \frac{e^{-x}}{-2} \{ (x+1)\cos x + (x-2)\sin x \}$$

Step 3 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

Exercise 1.6

1. $(D^2-1)y = xe^x \sin x$

[Ans. : $y = C_1 e^x + C_2 e^{-x} - \frac{e^x}{25} \{(10x+2)\cos x + (5x-14)\sin x\}$**]**

2. $(D^2-2D+1)y = xe^x \sin x$

[Ans. : $y = (C_1 + C_2 x) e^x - e^x [x \sin x - 2\cos x]$**]**

3. $(D^2+2D+1)y = x\cos x$

[Ans. : $y = (C_1 + C_2 x) e^{-x} + \frac{1}{2}[(x-1)\sin x + \cos x]$**]**

4. $(D^3-2D+2)y = x\cos x$

[Ans. : $\text{P.I.} = \frac{1}{5}\left\{ \left(x + \frac{2}{5}\right)\cos x - \left(2x + \frac{14}{5}\right)\sin x \right\}$**]**

5. $(D^3+3D+2)y = x\cos 2x$

[Ans. : $\text{P.I.} = \frac{1}{20}\left\{ \left(3x - \frac{7}{20}\right)\sin 2x + \left(\frac{12}{5} - x\right)\cos 2x \right\}$**]**

6. $(D^2+4)y = x\sin x$

[Ans. : $y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{9}(3x \sin x - 2\cos x)$**]**

7. $(D^2-1)y = x\sin x + e^x(1+x^2)$

[Ans. : $\text{P.I.} = \frac{-1}{2}(x\sin x + \cos x) + \frac{xe^x}{12}(2x^2 - 3x + 9)$**]**

8. $(D^2+3D+2)y = x\sin 2x$

[Ans. : $\text{P.I.} = \frac{-1}{200}\{(10x-24)\sin 2x + (30x-7)\cos 2x\}$**]**

1.5.6 Type 7 : If $\phi(x) = x^n \sin ax$ or $x^n \cos ax$

If $\phi(x) = x^n \sin ax$ or $x^n \cos ax$

We know that $e^{iax} = \cos ax + i \sin ax$

$$\therefore \quad \frac{1}{f(D)} x^n [\cos ax + i \sin ax] = \frac{1}{f(D)} x^n \cdot e^{iax}$$

Now use type 5

$$= e^{iax} \frac{1}{f(D+ai)} x^n$$

Now this can be evaluated by type 4 and then equating real and imaginary parts we get the result.

$$\boxed{\begin{aligned} \frac{1}{f(D)} x^n \sin ax &= \text{Img } \frac{1}{f(D)} x^n e^{iax} \\ \frac{1}{f(D)} x^n \cos ax &= \text{Real } \frac{1}{f(D)} x^n e^{iax} \end{aligned}}$$

Example 1.39 : $[(D^2+2D+5)^2 y]\, e^x = x \cos 2x$,

Solution : Step 1 : Complementary function

$$(D^2+2D+5)^2 y = e^{-x} \cdot x \cos 2x$$

A.E. $(D^2+2D+5)^2 = 0$

$$D = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm 2i \quad \text{repeated complex}$$

C.F. $= e^{-x}\left[(C_1+C_2x)\cos 2x + (C_3+C_4x)\sin 2x\right]$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{(D^2+2D+5)^2} e^{-x} x \cos 2x$$

Use type 5 i.e. $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$

$$= e^{-x} \frac{1}{\left((D-1)^2+2(D-1)+5\right)^2} x \cos 2x$$

$$= e^{-x} \frac{1}{(D^2+4)^2} x \cos 2x$$

Here power of x is one but D^2+4 is the zero factor for cos 2x $\therefore$ Don't use type 6 use type 7.

$$= e^{-x} \text{ Real } \frac{1}{(D^2+4)^2} x\, e^{i2x}$$

Use type 5 $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V.$

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{[(D+2i)^2+4]^2} x$$

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{[D^2+4iD-4+4]^2} x$$

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{\left[D^2+4iD\right]^2} x$$

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{(4iD)^2} \cdot \frac{1}{\left[1+\frac{D}{4i}\right]^2} x$$

Write f(D) in the form $(1+z)^n$

{Note : $i^2 = -1$}

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{-16 D^2}\left[1+\frac{D}{4i}\right]^{-2} x$$

We have binomial series

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \ldots$$

$$\text{P.I.} = e^{-x} \text{ Real } e^{i2x} \frac{1}{-16 D^2}\left[1+(-2)\left(\frac{D}{4i}\right)+\ldots\right] x$$

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{-16 D^2}\left[x - \frac{1}{2i}\right]$$

$\left(\text{Use } \frac{1}{i} = -i\right)$

$$= e^{-x} \text{ Real } e^{i2x} \frac{1}{-16 D^2}\left\{x + \frac{i}{2}\right\}$$

$$= e^{-x} \text{ Real } e^{-i2x} \frac{1}{-16}\left\{\frac{x^3}{2 \cdot 3} + \frac{i}{2} \cdot \frac{x^2}{2}\right\}$$

$\left(\text{Use } e^{iax} = (\cos ax + i \sin ax)\right)$

$$= \frac{e^{-x}}{-16} \text{Real } (\cos 2x + i \sin 2x)\left(\frac{x^3}{6} + \frac{ix^2}{4}\right)$$

(Use (a + ib)(c + id) = (ac – bd)+ i (ad + bc)).

$$= \text{Real} \frac{e^{-x}}{-16}\left\{\left(\frac{x^3}{6}\cos 2x - \frac{x^2}{4}\sin 2x\right) + i\left(\frac{x^3}{6}\sin 2x + \frac{x^2}{4}\cos 2x\right)\right\}$$

$$\text{P.I.} = \frac{e^{-x}}{-16}\left(\frac{x^3}{6}\cos 2x - \frac{x^2}{4}\sin 2x\right)$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

Example 1.40 : $(D^2-2D+4)^2 = x e^x \cos(\sqrt{3}x+\alpha)$

Solution : Step 1 : Complementary function

A.E. is $(D^2-2D+4)^2 = 0$

$$D = \frac{2 \pm \sqrt{4-16}}{2}$$

$$D = \frac{2 \pm \sqrt{-12}}{2}$$

$$D = \frac{2 \pm i2\sqrt{3}}{2}$$

$$D = 1 \pm i\sqrt{3} \quad \text{repeated complex}$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{(D^2-2D+4)^2} x\, e^x \cos(\sqrt{3}x+\alpha)$$

$$= e^x \frac{1}{[(D+1)^2-2(D+1)+4]^2} x \cos(\sqrt{3}x+\alpha)$$

$$= e^x \frac{1}{(D^2+3)^2} x \cos(\sqrt{3}x+\alpha)$$

Here power of x is one but D^2+3 is the zero factor for $\cos(\sqrt{3}x+\alpha)$ $\therefore$ use type 7.

$$= e^x \text{ Real } \frac{1}{(D^2+3)^2}\, x\, e^{i(\sqrt{3}x+\alpha)}$$

$$= e^x \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{[(D+i\sqrt{3})^2+3]^2}\, x$$

$$= e^x \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{[D^2+2i\sqrt{3}D-3+3]^2}\, x$$

$$= e^x \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{[D^2+2i\sqrt{3}D]^2}\, x$$

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{(i\sqrt{3}D)^2} \frac{1}{\left[1+\frac{D}{2i\sqrt{3}}\right]^2}\, x$$

Write f (D) in the form $(1+z)^n$.

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{(-12D^2)}\left[1+\frac{D}{i\sqrt{3}}\right]^{-2} x$$

Use binomial series

$$(1+z)^n = 1+nz+\frac{n(n-1)}{2!}z^2+....$$

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{(-12D^2)}\left[1+(-2)\left(\frac{D}{2i\sqrt{3}}\right)...\right] x$$

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{-12D^2}\left[x-\frac{2}{2i\sqrt{3}}\right]$$

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{-12D^2}\left[x+\frac{i\,2}{2\sqrt{3}}\right]$$

$$= e^{-x} \text{ Real } e^{i(\sqrt{3}x+\alpha)} \frac{1}{-12}\left[\frac{x^3}{2\cdot 3}+\frac{i\,2}{2\sqrt{3}}\cdot\frac{x^2}{2}\right]$$

(Use $e^{iax}=(\cos ax + i \sin ax)$)

$$= \frac{e^{-x}}{-12} \text{ Real}[\cos(\sqrt{3}x+\alpha)+i\sin(\sqrt{3}x+\alpha)]\left[\frac{x^3}{6}+\frac{i\,x^2}{2\sqrt{3}}\right]$$

(Use (a + ib)(c + id) = (ac – bd) + i (ad + bc))

$$= \frac{e^{-x}}{-12} \text{ Real}\left[\frac{x^3}{6}\cos(\sqrt{3}x+\alpha)-\frac{x^2}{2\sqrt{3}}\sin(\sqrt{3}x+\alpha)\right]$$

$$+ i\left[\frac{x^2}{2\sqrt{3}}\cos(\sqrt{3}x+\alpha)+\frac{x^3}{6}\sin(\sqrt{3}x+\alpha)\right]$$

$$= \frac{e^{-x}}{-12}\left[\frac{x^3}{6}\cos(\sqrt{3}x+\alpha)-\frac{x^3}{2\sqrt{3}}\sin(\sqrt{3}x+\alpha)\right]$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

Example 1.41 : $(D^2-1)y = x^2 \sin 3x$

Solution : Step 1 : Complementary function

A.E. $D^2-1=0$

$$(D-1)\,(D+1) = 0$$

$$D = 1,\ D = -1$$

$$\text{C.F.} = C_1e^{-x}+C_2e^x$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^2-1}x^2 \sin 3x$$

$$\text{P.I.} = \text{Img } \frac{1}{D^2-1}x^2e^{i3x}$$

Use type 5 $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$

$$\text{P.I.} = \text{Img } e^{i3x}\frac{1}{(D+i3)^2-1}x^2$$

$$= \text{Img } e^{i3x}\frac{1}{D^2+6iD-9-1}x^2$$

$$= \text{Img } e^{i3x}\frac{1}{D^2+6iD-10}x^2$$

$$= \text{Img } e^{i3x}\frac{1}{-10}\frac{1}{1-\left(\frac{D^2+6iD}{10}\right)}x^2$$

Use binomial series $\frac{1}{1-z}=1+z+z^2....$

$$\text{P.I.} = \text{Img } e^{i3x}\frac{1}{-10}\left[1+\left(\frac{D^2+6iD}{10}\right)+\left(\frac{D^2+6iD}{10}\right)^2...\right]x^2$$

$$= \text{Img } \frac{e^{i3x}}{-10}\left[-1+\frac{D^2+6iD}{10}+\frac{D^4+12iD^3-36D^2}{100}\right]x^2$$

$$= \text{Img } \frac{e^{i3x}}{-10}\left[x^2+\frac{2+6i(2x)}{10}+\frac{0+0-36(2)}{100}\right]$$

$$= \text{Img } \frac{e^{i3x}}{-10}\left[x^2+\frac{2}{10}+\frac{12xi}{10}-\frac{72}{100}\right]$$

$$= \text{Img } \frac{e^{i3x}}{-10}\left[\left(x^2-\frac{13}{25}\right)+\frac{6x}{5}i\right]$$

$$= \text{Img } \left(\frac{\cos 3x + i \sin 3x}{-10}\right)\left[\left(x^2-\frac{13}{25}\right)+\frac{6x}{5}i\right]$$

$$= \frac{1}{-10} \text{ Img } \left[\left(x^2-\frac{13}{25}\right)\cos 3x - \frac{6x}{5}\sin 3x\right]$$

$$+ i\left[\frac{6x}{5}\cos 3x+\left(x^2-\frac{13}{25}\right)\sin 3x\right]$$

$$\text{P.I.} = \frac{1}{-10}\left[\frac{6x}{5}\cos 3x+\left(x^2-\frac{13}{25}\right)\sin 3x\right]$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

Exercise 1.7

1. $(D^2+4)y = x\sin^2 x$

$$\left[\text{Ans.}: y = C_1\cos 2x + C_2\sin 2x + \frac{x}{8} - \frac{1}{32}(x\cos 2x + 2x^2\sin 2x)\right]$$

2. $(D^2-4D+4)y = 8x^2 e^{2x}\sin 2x$

$$\left[\text{Ans.}: y = e^{2x}\left[(C_1+C_2x) + (3-2x^2)\sin 2x - 4x\cos 2x\right]\right]$$

3. $(D^2-1)y = x^2\cos x$

$$\left[\text{Ans.}: y = C_1 e^x + C_2 e^{-x} - \frac{1}{2}\left[(x^2-1)\cos x - 2x\sin x\right]\right]$$

4. $(D^4+2D^2+1)y = x^2\cos x$

$$\left[\text{Ans.}: \text{P.I.} = \frac{x^3}{12}\sin x - \left(\frac{x^4-9x^2}{48}\right)\cos x\right]$$

5. $(D^4+8D^2+16)y = x\cos 2x$

$$\left[\text{Ans.}: \text{P.I.} = \frac{-x^2}{192}[2x\cos 2x - 3\sin 2x]\right]$$

1.5.7 Type 8 : General Method

If it is not possible to apply any of above types then use type 8 i.e. if $\phi(x)$ involves tan ax, sec ax, cosec ax, cot ax, $\sin e^x$, $\cos e^x$, e^{e^x}, $\frac{1}{1+e^x}$, $\frac{1}{1+e^{-x}}$, log x then use type 8 i.e. the general method for finding P.I.

1) $$\boxed{\frac{1}{D-a}\phi(x) = e^{ax}\int \phi(x)\, e^{-ax}\, dx}$$

2) $$\boxed{\frac{1}{D+a}\phi(x) = e^{-ax}\int \phi(x)\, e^{ax}\, dx}$$

Example 1.42 : *Solve* $(D^2+3D+2)y = e^{e^x}$

Solution : Step 1 : Complementary function

A.E. $(D+2)(D+1) = 0$

$$D = -2,\ D = -1$$

$$\text{C.F.} = C_1 e^{-2x} + C_2 e^{-x}$$

Step 2 : Particular integrals

$$\text{P.I.} = \frac{1}{(D+2)(D+1)} e^{e^x}$$

$$\left(\text{Use } \frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a} - \frac{1}{D-b}\right]\right)$$

$$\text{P.I.} = \left[\frac{1}{D+1} - \frac{1}{D+2}\right] e^{e^x}$$

$$= \left[\frac{1}{D+1} e^{e^x} - \frac{1}{D+2} e^{e^x}\right]$$

$$= \left\{e^{-x}\int e^x e^{e^x} dx - e^{-2x}\int e^{2x} e^{e^x} dx\right\}$$

$$= \left\{e^{-x}\int e^{e^x} e^x dx - e^{-2x}\int e^x e^{e^x} e^x dx\right\}$$

Put $e^x = t$

$\therefore$ $e^x dx = dt$

$$= \{e^{-x}\int e^t dt - e^{-2x}\int t e^t dt\}$$

Integrate 1st term and for 2nd term use generalised rule of integration by parts i.e. $\int uv dx = uv_1 - u'v_2 \ldots$

$$= \{e^{-x}(e^t) - e^{-2x}[t(e^t) - (1)(e^t)]\}$$

$$\qquad\qquad u v_1 \quad u' v_2$$

$$= e^t\{e^{-x} - e^{-2x}\cdot t + e^{-2x}\}$$

Put $t = e^x$

$$= e^{e^x}\{e^{-x} - e^{-2x}\cdot e^x + e^{-2x}\}$$

$$= e^{e^x}\{e^{-x} - e^{-x} + e^{-2x}\}$$

$$\text{P.I.} = e^{e^x}\cdot e^{-2x}$$

Step 3 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

Example 1.43 : $(D^2-1)y = e^{-x}\sin e^{-x} + \cos e^{-x}$

Solution : Step 1 : Complementary function

A.E. $(D-1)(D+1) = 0$

$$D = 1, -1$$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^x$$

Step 2 : Particular integrals

$$\text{P.I.} = \frac{1}{(D-1)(D+1)}(e^{-x}\sin e^{-x} + \cos e^{-x})$$

$$\left(\text{Use } \frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a} - \frac{1}{D-b}\right]\right)$$

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right]\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)$$

$$\text{P.I.} = \frac{1}{2}\left\{\begin{array}{l}\frac{1}{D-1}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right) \\ -\frac{1}{D+1}\left(e^{-x}\sin e^{-x} + \cos e^{-x}\right)\end{array}\right\}$$

Consider

$$PI_1 = \frac{1}{2}\left\{\frac{1}{D-1} e^{-x} \sin e^{-x} + \cos e^{-x}\right\}$$

Use $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$

$$= \frac{1}{2} e^{x} \int \left(e^{-x} \sin e^{-x} + \cos e^{-x}\right) dx$$

$$= \frac{1}{2} e^{x} \int \left(e^{-x} \sin e^{-x} + \cos e^{-x}\right) e^{-x} dx$$

Put $e^{-x} = t \therefore -e^{-x} dx = dt$ i.e. $e^{-x} dx = -dt$

$$= \frac{1}{2} e^{x} \int (t \sin t + \cos t)(-dt)$$

$$= \frac{-1}{2} e^{x} \left[\int t \sin t \, dt + \int \cos t \, dt\right]$$

For $\int t \sin t \, dt$ use general integration by parts $\int uv dx = uv_1 - u'v_2 ...$

$\therefore \int \underset{u}{t} \, \sin t \, dt = (t)\underset{v_1}{(-\cos t)} - \underset{u'}{(1)}\underset{v_2}{(-\sin t)}$

$$= \frac{-e^{x}}{2}\{[(t)(-\cos t) - (1)(-\sin t)] + \sin t\}$$

$$= \frac{-e^{x}}{2}\{-t \cos t + 2 \sin t\}$$

Put $t = e^{-x}$

$$= \frac{-e^{x}}{2}\left\{-e^{-x} \cos e^{-x} + 2 \sin e^{-x}\right\}$$

$$PI_1 = \frac{1}{2} \cos e^{-x} - e^{x} \sin e^{-x}$$

Consider

$$PI_2 = \frac{1}{2}\frac{1}{D+1}\left(e^{-x} \sin e^{-x} + \cos e^{-x}\right)$$

$$\frac{1}{D+a} X = e^{-ax} \int e^{ax} \, X \, dx$$

$$= \frac{1}{2} e^{-x} \int -e^{x}\left(\cos e^{-x} + e^{-x} \sin e^{-x}\right) dx$$

Use $\int e^{x} [f(x) + f'(x)] dx = e^{x} f(x)$

Here $f(x) = \cos e^{-x} \therefore f'(x) = -\sin e^{-x} \, (-e^{-x})$

$$= e^{-x} \sin e^{-x}$$

$$= \frac{1}{2} e^{-x} \cdot e^{x} \cos e^{-x}$$

$$PI_2 = \frac{1}{2} \cos e^{-x}$$

Thus

$$PI = PI_1 - PI_2$$

$$= -e^{x} \sin e^{-x}$$

Step 3 : Complete solution

$$\therefore \quad y = C.F. + P.I.$$

$$= C_1 e^{-x} + C_2 e^{x} - e^{x} \sin e^{-x}$$

Example 1.44 : $(D^2 - D - 2)y = 2 \log x + \frac{1}{x} + \frac{1}{x^2}$

Solution : Step 1 : Complementary function

$$D^2 - D - 2 = 0$$

$$(D-2)(D+1) = 0$$

$$D = 2, -1$$

$$C.F. = C_1 e^{2x} + C_2 e^{-x}$$

Step 2 : Particular integral

$$= \frac{1}{(D-2)(D+1)}\left(2 \log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$\left(\text{Use } \frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a} - \frac{1}{D-b}\right]\right)$$

$$= \frac{1}{-3}\left[\frac{1}{D+1} - \frac{1}{D-2}\right]\left(2 \log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

Consider

$$PI_1 = \frac{1}{D+1}\left(2 \log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

Use P.I. $= \frac{1}{D+1} X = e^{-x} \int e^{x} \, X \, dx$

$$= e^{-x} \int e^{x}\left(2 \log x + \frac{1}{x} + \frac{1}{x^2}\right) dx$$

Write $\frac{1}{x} = \frac{-1}{x} + \frac{2}{x}$

$$= e^{-x} \int e^{x}\left[\left(2 \log x - \frac{1}{x}\right) + \left(\frac{2}{x} + \frac{1}{x^2}\right)\right] dx$$

Use $\int e^{x} [f(x) + f'(x)] dx = e^{x} f(x)$

$$= e^{-x} \cdot e^{x} \cdot \left(2 \log x - \frac{1}{x}\right)$$

$$P.I. = 2 \log x - \frac{1}{x}$$

Consider

$$PI_2 = \frac{1}{D-2}\left(2 \log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= e^{2x} \int e^{-2x}\left(2 \log x + \frac{1}{x} + \frac{1}{x^2}\right) dx$$

Put $-2x = t \therefore x = \frac{-t}{2} \therefore dx = \frac{-dt}{2}$

$$= e^{2x} \int e^{t}\left[2 \log\left(\frac{t}{-2}\right) - \frac{2}{t} + \frac{4}{t^2}\right] \frac{dt}{-2}$$

Write $\frac{-2}{t} = \frac{-4}{t} + \frac{2}{t}$

$$= \frac{e^{2x}}{-2} \int e^t \left[\left(2\log\left(\frac{t}{-2}\right) - \frac{4}{t}\right) + \left(\frac{2}{t} + \frac{4}{t^2}\right)\right] dt$$

Use $\int e^t [f(t) + f'(t)]\, dt = e^t f(t)$

$$= \frac{e^{2x}}{-2} \cdot e^t \left[2\log\left(\frac{t}{-2}\right) - \frac{4}{t}\right]$$

Put $t = -2x$

$$= \frac{e^{2x}}{-2} \cdot e^{-2x} \left[2\log\left(\frac{-2x}{2}\right) - \left(\frac{4}{-2x}\right)\right]$$

$$= \frac{1}{-2}\left[2\log x + \frac{2}{x}\right]$$

$$= -\log x - \frac{1}{x}$$

Thus

$$\text{P.I.} = -\frac{1}{3}[PI_1 - PI_2]$$

$$= -\frac{1}{3}[3\log x]$$

$$= -\log x$$

Step 3 : Complete solution

$$\therefore \quad y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2x} + C_2 e^{-x} - \log x$$

Example 1.45 : $(D^2 + 5D + 6)y = e^{-2x} \sec^2 x(1 + 2\tan x)$

Solution : Step 1 : Complementary function

$$D^2 + 5D + 6 = 0$$

$$(D+3)(D+2) = 0$$

$$D = -3, -2$$

$$\text{C.F.} = C_1 e^{-3x} + C_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{f(D)} X$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D^2 + 5D + 6} e^{-2x} \sec^2 x(1 + 2\tan x)$$

$$\left(\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+2)} V\right)$$

$$= e^{-2x} \frac{1}{(D-2)^2 + 5(D-2) + 6} \sec^2 x(1 + 2\tan x)$$

$$= e^{-2x} \frac{1}{D^2 + D} \sec^2 x\,(1 + 2\tan x)$$

$$= e^{-2x} \frac{1}{D(D+1)} \sec^2 x(1 + 2\tan x)$$

$$\left(\text{Use} \frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a} - \frac{1}{D-b}\right]\right)$$

$$= e^{-2x}\left[\frac{1}{D} - \frac{1}{D+1}\right] \sec^2 x(1 + 2\tan x)$$

Consider

$$PI_1 = \frac{1}{D}\sec^2 x(1 + 2\tan x)$$

$$= \int \sec^2 x(1 + 2\tan x)\, dx$$

Put $\tan x = t$, $\sec^2 x\, dx = dt$

$$= \int (1 + 2t)\, dt$$

$$= t + 2 \cdot \frac{t^2}{2}$$

$$PI_1 = \tan x + \tan^2 x$$

Also consider

$$PI_2 = \frac{1}{D+1}\sec^2 x(1 + 2\tan x)$$

Use P.I. formula $\frac{1}{D+a} X = e^{ax} \int X e^{-ax}\, dx$

$$= e^{-x} \int e^x \left(\sec^2 x + 2\tan x \sec^2 x\right) dx$$

Use $\int e^x [f(x) + f'(x)]\, dx = e^x f(x)$

$$= e^{-x} \cdot (e^x \cdot \sec^2 x)$$

$$= \sec^2 x$$

$$\therefore \quad \text{P.I.} = e^{-2x}[PI_1 - PI_2]$$

$$= e^{-2x}[\tan x + \tan^2 x - \sec^2 x]$$

Use $\sec^2 x - \tan^2 x = 1$

$$\therefore \quad \text{P.I.} = e^{-2x}[\tan x - 1]$$

Step 3 : Complete solution

Thus $\quad y = \text{C.F.} + \text{P.I.}$

$$= C_1 e^{-3x} + C_2 e^{-2x} + e^{-2x}(\tan x - 1)$$

Example 1.46 : $(D^2 + D)\, y = \frac{1}{1 + e^x}$

Solution : Step 1 : Complementary function

A.E. $\quad D(D+1) = 0$

$$D = 0, \; D = -1 \text{ real roots.}$$

$$\text{C.F.} = C_1 e^{0x} + C_2 e^{-x}$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{D(D+1)}\frac{1}{1+e^x}$$

$$\left(\text{Use } \frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a}-\frac{1}{D-b}\right]\right)$$

$$\text{P.I.} = \left[\frac{1}{D}-\frac{1}{D+1}\right]\frac{1}{1+e^x}$$

Separate PI_1 and PI_2.

Consider $$PI_1 = \frac{1}{D}\frac{1}{1+e^x}$$

$$= \int \frac{1}{1+e^x}dx$$

As the derivative of denominator term is not present in the numerator write $e^x = \frac{1}{e^{-x}}$.

$$= \int \frac{1}{1+\frac{1}{e^{-x}}}dx$$

$$= \int \frac{e^{-x}}{e^{-x}+1}dx$$

Put $e^{-x}+1 = t \quad -e^{-x}dx = dt$

$$= \int \frac{-dt}{t}$$

$$= -\log t$$

Put $t = e^{-x}+1$

$$= -\log(e^{-x}+1)$$

$$PI_1 = -\log\left(\frac{1}{e^x}+1\right)$$

$$= -\log\left(\frac{1+e^x}{e^x}\right)$$

(Use $\log(a/b) = \log a - \log b$)

$$= -[\log(1+e^x)-\log e^x]$$

$$= \log e^x - \log(e^x+1)$$

Use $$\log e^x = x$$

$$PI_1 = x-\log(1+e^x)$$

Consider

$$PI_2 = \frac{1}{D+1}\frac{1}{1+e^x}$$

$$= e^{-x}\int e^x \cdot \frac{1}{1+e^x}dx$$

As the derivative of denominator is present in numerator put $1+e^x = t$

$\therefore e^x dx = dt.$

$$= e^{-x}\int \frac{dt}{t}$$

$$= e^{-x}\log t$$

Put $$t = 1+e^x$$

$$PI_2 = e^{-x}\log(1+e^x)$$

Now,

$\therefore$ $$\text{P.I.} = PI_1 - PI_2$$

$$= x-\log(1+e^x)-e^{-x}\log(1+e^x)$$

Take $\log(1+e^x)$ common from 2^{nd} and 3^{rd} term.

$$= x-(1+e^{-x})\log(1+e^x)$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

Example 1.47 : $(D^2-1)y = \frac{1}{(1+e^{-x})^2}$

Solution : Step 1 : Complementary function

A.E. $$(D^2-1) = 0$$

$$(D-1)(D+1) = 0$$

$$D = 1, -1 \quad \text{real roots}$$

$$\text{C.F.} = C_1e^x + C_2e^{-x}$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{(D-1)(D+1)}\frac{1}{(1+e^{-x})^2}$$

$$= \frac{1}{2}\left[\frac{1}{D-1}-\frac{1}{D+1}\right]\frac{1}{(1+e^{-x})^2}$$

Consider $$PI_1 = \frac{1}{D-1}\frac{1}{(1+e^{-x})^2}$$

$$= e^x\int e^{-x}\frac{1}{(1+e^{-x})^2}dx$$

Derivative of denominator term is present in numerator.

Put $e^{-x}+1 = t \quad -e^{-x}dx = dt$

$$PI_1 = e^x\int \frac{-dt}{t^2}$$

$$= e^x\left[\frac{1}{t}\right]$$

Put $$t = 1+e^{-x}$$

$$PI_1 = e^x\frac{1}{1+e^{-x}}$$

$$= e^x \frac{1}{1+\frac{1}{e^x}}$$

$$PI_1 = \frac{e^{2x}}{e^x+1}$$

Consider $$PI_2 = \frac{1}{D+1}\frac{1}{(1+e^{-x})^2}$$

$$= e^{-x}\int e^x \frac{1}{(1+e^{-x})^2}dx$$

$$= e^{-x}\int e^x \frac{1}{\left(1+\frac{1}{e^x}\right)^2}dx$$

$$= e^{-x}\int \frac{e^x \cdot e^{2x}}{(e^x+1)^2}dx$$

Put $e^x+1 = t$, $e^x = t-1$, $e^x dx = dt$

$$= e^{-x}\int \frac{(t-1)^2\, dt}{t^2}$$

$\left(\text{Use } (a+b)^2 = a^2-2ab+b^2\right)$

$$= e^{-x}\int \frac{t^2-2t+1}{t^2}dt$$

$$PI_2 = e^{-x}\int \left(1-\frac{2}{t}+\frac{1}{t^2}\right)dt$$

$$= e^{-x}\left[t-2\log t-\frac{1}{t}\right]$$

Put $t = 1+e^x$

$$= e^{-x}\left[(1+e^x)-2\log(1+e^x)-\frac{1}{1+e^x}\right]$$

$$= -2e^{-x}\log(1+e^x)-\frac{e^{-x}}{1+e^x}+e^{-x}(1+e^x)$$

Now $$\text{P.I.} = \frac{1}{2}[PI_1 - PI_2]$$

Substitute PI_1 and PI_2.

$$\text{P.I.} = \frac{1}{2}\left[\frac{e^{2x}}{e^x+1}+2e^{-x}\log(1+e^x)+\frac{e^{-x}}{1+e^x}-e^{-x}(1+e^x)\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x)+\frac{e^{2x}+e^{-x}-e^{-x}(1+e^x)^2}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x)+\frac{e^{2x}+e^{-x}-e^{-x}(1+2e^x+e^{2x})}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x)+\frac{e^{2x}+e^{-x}-e^{-x}-2-e^x}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x)+\frac{e^{2x}-2-e^x}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x)+\frac{e^{2x}-1-1-e^x}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x)+\frac{(e^x-1)(e^x+1)-(1+e^x)}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x)+\frac{[e^x-1-1]\,(e^x+1)}{1+e^x}\right]$$

$$= \frac{1}{2}\left[2e^{-x}\log(1+e^x)+\frac{(e^x-2)\,(1+e^x)}{(1+e^x)}\right]$$

$$= \frac{1}{2}[2e^{-x}\log(1+e^x)+e^x-2]$$

$$= e^{-x}\log(1+e^x)+\frac{e^x}{2}-1$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

Example 1.48 : $(D^3+D)y = \operatorname{cosec} x$

Solution : Step 1 : Complementary function

A.E. $$D(D^2+1) = 0$$

$$D(D+i)\,(D-i) = 0$$

$$D = 0,\ D = \pm i \quad \text{real complex}$$

$$\text{C.F.} = C_1e^{0x}+e^{0x}\,[C_2\cos x + C_3\sin x]$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{f(D)}\,\phi(x)$$

$$= \frac{1}{D\,(D+i)\,(D-i)}\operatorname{cosec} x$$

Consider $\frac{1}{D\,(D+i)\,(D-i)}$ use partial fractions.

$$= \frac{A}{D}+\frac{B}{D+i}+\frac{C}{D-i}$$

$$A = \lim_{D\to 0} D\cdot\left[\frac{1}{D\,(D+i)\,(D-i)}\right] = \frac{1}{1}$$

$$B = \lim_{D\to -i}(D+i)\left[\frac{1}{D\,(D+i)\,(D-i)}\right] = \frac{1}{(-i)\,(-2i)} = \frac{-1}{2}$$

$$C = \lim_{D\to i}(D-i)\left[\frac{1}{D\,(D+i)\,(D-i)}\right] = \frac{1}{(i)\,(2i)} = \frac{-1}{2}$$

Step 3 : Substitute A, B, C.

$$\therefore \quad = \frac{1}{D}+\frac{-1/2}{D+i}+\frac{-1/2}{D-i}$$

$$\text{P.I.} = \left[\frac{1}{D}+\frac{-1/2}{D+i}+\frac{-1/2}{D-i}\right]\operatorname{cosec} x$$

Separate PI_1, PI_2, PI_3 and consider PI_1.

$$PI_1 = \frac{1}{D}\operatorname{cosec} x = \int \operatorname{cosec} x\, dx$$

$$= \log(\operatorname{cosec} x - \cot x)$$

Consider PI_2

$$PI_2 = \frac{-1/2}{D+i}\operatorname{cosec} x$$

$$= \frac{-1}{2}e^{-ix}\int e^{ix}\operatorname{cosec} x\, dx$$

(Use $e^{ia} = \cos\theta + i\sin\theta$)

$$= \frac{-1}{2}e^{-ix}\int (\cos x + i\sin x)\operatorname{cosec} x\, dx$$

Multiply by cosec x inside the bracket

$\operatorname{cosec} x = \frac{1}{\sin x}$.

$$= \frac{-1}{2}e^{-ix}\int (\cot x + i)\, dx$$

$$= \frac{-1}{2}e^{-ix}[\log\sin x + ix]$$

Replacing i by – i we get.

$$PI_3 = \frac{-1/2}{D-i}\operatorname{cosec} x = \frac{-i}{2}e^{ix}[\log\sin x - ix]$$

Consider $PI_2 + PI_3$.

$$= \frac{-1}{2}\{e^{-ix}(\log\sin x) + e^{-ix}(ix) + e^{ix}(\log\sin x) - e^{ix}(ix)\}$$

Take log sin x common from 1^{st} and 3^{rd} term and (– ix) common from 2^{nd} and 4^{th} term.

$$= \frac{-1}{2}\{\log\sin x\,(e^{ix}+e^{-ix}) - ix\,(e^{ix}-e^{ix})\}$$

Use $e^{i\theta}+e^{-i\theta} = 2\cos\theta$ and $e^{i\theta}-e^{-i\theta} = 2i\sin\theta$

$$= \frac{-1}{2}\{\log\sin x\,(2\cos x) - ix\,(2i\sin x)\}$$

Simplify $i^2 = -1$.

$$= \{-\cos x\log\sin x - x\sin x\}$$

$$\therefore \text{P.I.} = PI_1 + PI_2 + PI_3 = \log(\operatorname{cosec} x - \cot x) - x\sin x - \cos x\log\sin x$$

Step 3 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

Example 1.49 : $(D^2+a^2)y = \tan ax.$ *for* $a = 2$.

Solution : Step 1 : Complementary function

A.E. $\quad D^2+a^2 = 0$

$$D^2 = -a^2$$

$$D = \pm ai$$

$$\text{C.F.} = C_1\cos ax + C_2\sin ax$$

Step 2 : Particular integral

$D^2+a^2 = (D+ai)(D-ai)$.

$$\text{P.I.} = \frac{1}{(D+ai)(D-ai)}\tan ax$$

$$\left(\text{Use } \frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a}-\frac{1}{D-b}\right]\right)$$

$$PI_1 = \frac{1}{2ai}\left[\frac{1}{D-ai}-\frac{1}{D+ai}\right]\tan ax$$

$$PI_1 = \frac{1}{D-ai}\tan ax$$

$$= e^{iax}\int e^{-aix}\tan ax\, dx$$

(Use $e^{-i\theta} = \cos\theta - i\sin\theta$)

$$= e^{aix}\int (\cos ax - i\sin ax)\tan ax\, dx$$

Multiply tan ax inside the bracket

$\tan ax = \frac{\sin ax}{\cos ax}$.

$$PI_1 = e^{aix}\int \sin ax - i\frac{\sin^2 ax}{\cos ax}\, dx$$

Write $\sin^2\theta = 1-\cos^2\theta$.

$$PI_1 = e^{aix}\int \sin ax - i\frac{1-\cos^2 ax}{\cos ax}\, dx$$

$$= e^{aix}\int \sin ax - i\left(\frac{1}{\cos ax} - \cos ax\right) dx$$

$$= e^{aix}\int (\sin ax - i\sec ax + i\cos ax)\, dx$$

$$PI_1 = e^{aix}\int (\sin ax + i\cos ax - i\sec ax)\, dx$$

$$= e^{aix}\left[-\frac{\cos ax}{a}+\frac{i}{a}\sin ax - \frac{i}{a}\log(\sec ax + \tan ax)\right]$$

$$= \frac{e^{aix}}{a}[-(\cos ax - i\sin ax) - i\log(\sec ax + \tan ax)]$$

(Use $e^{-iax} = \cos ax - i\sin ax$).

$$= \frac{e^{aix}}{a}[-e^{-aix} - i\log(\sec ax + \tan ax)]$$

Multiply e^{aix} outside.

$$PI_1 = \frac{1}{a}[-1 - ie^{aix}\log(\sec ax + \tan ax)]$$

Replacing i by – i we get.

$$PI_2 = \frac{1}{D+ai}\tan ax$$

$$= \frac{1}{a}[-1+i\,e^{-aix}\log(\sec ax+\tan ax)]$$

Now $\quad P.I. = \frac{1}{2ai}[PI_1 - PI_2]$

Substitute PI_1 and PI_2.

$$P.I. = \frac{1}{2ai}\cdot\frac{1}{a}\{[-1-i\,e^{aix}\log(\sec ax+\tan ax)]$$

$$-[-1+i\,e^{-aix}\log(\sec ax+\tan ax)]\}$$

Take log (sec ax + tan ax) common.

$$= \frac{1}{2a^2i}\{(-i)\log(\sec ax+\tan ax)\,(e^{aix}+e^{-aix})\}$$

Use $e^{i\theta}+e^{-i\theta} = 2\cos\theta$.

$$= \frac{1}{2a^2i}\{-i\,2\cos ax\cdot\log(\sec ax+\tan ax)\}$$

$$= \frac{-1}{a^2}\cos ax\log(\sec ax+\tan ax)$$

Step 3 : Complete solution

y = C.F. + P.I. is the complete solution.

Example 1.50 : $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y = e^{-x}\sec^3 x$

Solution : Step 1 : Complementary function

Let $\quad D = \frac{d}{dx}$

$\therefore$ The equation becomes

$$D^2y+2Dy+2y = e^{-x}\sec^3 x$$

$$(D^2+2D+2)\,y = e^{-x}\sec^3 x$$

A.E. $\quad D^2+2D+2 = 0 \quad$ Use $D = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

a = 1, b = 2, c = 2 complex roots.

$$D = \frac{-2\pm\sqrt{4-18}}{2}$$

$$D = -1\pm i$$

$$C.F. = e^{-x}[C_1\cos x + C_2\sin x]$$

Step 2 : Particular integral

$$P.I. = \frac{1}{D^2+2D+2}e^{-x}\sec^3 x$$

$$= e^{-x}\frac{1}{(D-1)^2+2(D-1)+2}\sec^3 x$$

$$= e^{-x}\frac{1}{D^2+1}\sec^3 x$$

$$= e^{-x}\frac{1}{(D-i)(D+i)}\sec^3 x$$

$$\left(\text{Use } \frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a}-\frac{1}{D-b}\right]\right)$$

$$P.I. = e^{-x}\frac{1}{2i}\left[\frac{1}{D-i}-\frac{1}{D+i}\right]\sec^3 x$$

Consider $\quad PI_1 = \frac{1}{D-i}\sec^3 x$

$$= e^{ix}\int e^{-ix}\sec^3 x\,dx$$

$\left(\text{Use } e^{-ix} = \cos x - i\sin x\right)$

$$= e^{ix}\int(\cos x - i\sin x)\sec^3 x\,dx$$

Multiply $\sec^3 x$ inside the bracket.

$$= e^{ix}\int(\sec^2 x - i\tan x\sec^2 x)\,dx$$

Integrate use $\int[f(x)]\cdot f'(x)\,dx = \frac{f(x)^2}{2}$ for 2^{nd} term.

$$PI_1 = e^{ix}\left(\tan x - i\frac{\tan^2 x}{2}\right)$$

Replacing i by – i we get PI_2.

$$PI_2 = \frac{1}{D+i}\sec^3 x = e^{-ix}\left(\tan x + i\frac{\tan^2 x}{2}\right)$$

$$P.I. = \frac{e^{-x}}{2i}[PI_1 - PI_2]$$

$$\text{Now P.I.} = \frac{e^{-x}}{2i}\left[\begin{array}{l} e^{ix}\left(\tan x - i\frac{\tan^2 x}{2}\right) \\ -e^{-ix}\left(\tan x + i\frac{\tan^2 x}{2}\right)\end{array}\right]$$

Take tanx and $\frac{\tan^2 x}{2}$ common.

$$= \frac{e^{-x}}{2i}\left\{\tan x\,(e^{ix}-e^{-ix}) - \frac{i}{2}\tan^2 x\,(e^{ix}+e^{-ix})\right\}$$

Use $e^{ix}-e^{-ix} = 2i\sin x$, $e^{ix}+e^{-ix} = 2\cos x$.

$$= \frac{e^{-x}}{2i}\left\{\tan x\,(2i\sin x) - \frac{i}{2}\tan^2 x\,(2\cos x)\right\}$$

Simplify write $\tan x = \frac{\sin x}{\cos x}$ in 2^{nd} term.

$$= \frac{e^{-x}}{2i}\left\{2i\sin x\tan x - i\tan x\cdot\frac{\sin x}{\cos x}\cdot\cos x\right\}$$

Simplify.

$$= \frac{e^{-x}}{2i}\{2i \sin x \tan x - i \sin x \tan x\}$$

$$= \frac{e^{-x}}{2} \sin x \tan x$$

Step 3 : Complete solution

$$\therefore \quad y = \text{C.F.} + \text{P.I.}$$

Example 1.51 : $(D^2+2D+1)y = e^{-x} \log x.$

Solution : Step 1 : Complementary function

A.E. $\quad (D+1)^2 = 0$

$$D = -1, -1$$

$$\text{C.F.} = (C_1 + C_2 x)\, e^{-x} \text{ repeated roots.}$$

Step 2 : Particular integral

$$\text{P.I.} = \frac{1}{(D+1)^2} e^{-x} \log x$$

$$= e^{-x} \frac{1}{(D-1+1)^2} \log x$$

$$= e^{-x} \frac{1}{D^2} \log x = e^{-x} \frac{1}{D}\frac{1}{D}(\log x)$$

$$= e^{-x} \frac{1}{D}[x \log x - x]$$

$$= e^{-x}\left[\int x \log x\, dx - \frac{x^2}{2}\right]$$

$$\int uv dx = u\int v dx - \int \left(\frac{dy}{dx}\int v dx\right) dx$$

$$= e^{-x}\left[\log x\left(\frac{x^2}{2}\right) - \int \frac{1}{x}\cdot\frac{x^2}{2} dx - \frac{x^2}{2}\right]$$

$$= e^{-x}\left[\frac{x^2}{2}\log x - \frac{1}{2}\cdot\frac{x^2}{2} - \frac{x^2}{2}\right]$$

$$= e^{-x}\left[\frac{x^2}{2}\log x - \frac{3}{4}x^2\right]$$

Step 3 : Complete solution

$y = \text{C.F.} + \text{P.I.}$ is the complete solution.

Exercise 1.8

1. $(D^2+3D+2)y = \sin e^x$

[Ans. : $y = C_1 e^{-2x} + C_2 e^{-x} - e^{-2x} \sin e^x$]

2. $(D^2-3D+2)y = \frac{1}{e^{e^{-x}}} + \cos\left(\frac{1}{e^x}\right)$

[Ans. : $y = C_1 e^{2x} + C_2 e^x + e^{2x}\left[\frac{1}{e^{e^{-x}}} - \cos\left(\frac{1}{e^x}\right)\right]$]

3. $(D^2-9D+18)y = e^{e^{-3x}}$

[Ans. : $y = C_1 e^{3x} + C_2 e^{6x} + \frac{1}{9} e^{6x} e^{e^{-3x}}$]

4. $\left(D^2-2D-3\right)y = 3e^{-3x}\sin\left(e^{-3x}\right) + \cos\left(e^{-3x}\right)$

[Ans. : $y = C_1 e^{3x} + C_2 e^{-x} - \frac{e^{3x}}{3}\sin e^{-3x}$]

5. $\left(D^2+5D+6\right)y = e^{e^x}$

[Ans. : $y = C_1 e^{-2x} + C_2 e^{-3x} + \left(e^{-2x} - 2e^{-3x}\right) e^{e^x}$]

6. $\left(D^2+2D+1\right)y = \frac{e^{-x}}{x+2}$

[Ans. : $y = (C_1 + C_2 x)\, e^{-x} + e^{-x}\,\{x \log(x+2) - 2\log(x+2) - x\}$]

7. $\left(D^2+9\right)y = \operatorname{cosec} 3x$ **[Ans. : $y = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{9}\sin 3x \log \sin 3x - \frac{1}{3} x \cos 3x$]**

8. $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

[Ans. : $y = C_1 \cos x + C_2 \sin x + \sin \log \sin x - x \cos x$]

1.6 Legendre's Differential Equations

General form for 3rd order differential equation is

$$a_0\,(ax+b)^3 \frac{d^3y}{dx^3} + a_1(ax+b)^2 \frac{d^2y}{dx^2}$$

$$+ a_2(ax+b)\frac{dy}{dx} + a_3 y = X$$

Here the coefficients are not constants so we can't solve such example by using previous methods.

$\therefore$ We must reduce the above equation to the linear differential equation with constant coefficients.

$\therefore$ Put $(ax+b) = e^z$ i.e. $z = \log(ax+b)$

We know that

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{a}{ax+b}$$

$$\therefore \quad (ax+b)\frac{dy}{dx} = a\cdot\frac{dy}{dz}$$

Let $\quad D = \frac{d}{dz}$

$$\therefore \quad (ax+b)\frac{dy}{dx} = a\,Dy$$

$$\underset{\text{Variable coefficient}}{\downarrow} \qquad \underset{\text{Constant coefficient}}{\downarrow}$$

Thus the substitution $ax+b = e^z$ reduces the variable coefficient to the constant coefficient similarly we can show that

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)\,y$$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)\,y \text{ and so on.}$$

1.6.1 Cauchy's Differential Equation or Homogeneous Differential Equations

It is the particular case of Legendres differential equation if we put a = 1 and b = 0 in Legendre's then we get Cauchy's D.E. Thus the general form of 3rd order differential equation is

$$a_0x^3\frac{d^3y}{dx^3}+a_1x^2\frac{d^2y}{dx^2}+a_2x\frac{dy}{dx}+a_3y = X$$

Put $x = e^z$ i.e. $z = \log x$

$$\therefore \quad x\frac{dy}{dx} = D\,y$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)\,y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)\,y$$

where $\quad D \equiv \dfrac{d}{dz}$

1.6.2 Procedure to Solve Legendre's or Cauchy's Differential Equation

Step 1 : Put $(ax + b) = e^z$ or $z = \log(ax + b)$

$$\therefore \quad (ax+b)\frac{dy}{dx} = a\,Dy$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)\,y$$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)\,y \;...$$

and so on where $D = \dfrac{d}{dz}$

Step 2 : Substituting in the given equation we get a linear differential equation with constant coefficients in y and z.

Step 3 : Solve by previous methods.

$$\therefore \quad y = \text{C.F.} + \text{P.I.}$$

As $D = \dfrac{d}{dz}$ here C.F and P.I will be in terms of z.

Step 4 : Resubstitute for $z = \log(ax + b)$ or $e^z = (ax + b)$ we get y in terms of x.

Example 1.52 : $(2x+1)^2\dfrac{d^2y}{dx^2}-2(2x+1)\dfrac{dy}{dx}-12y = 6x.$

Solution : Step 1 : Legendre's D.E. with a = 2, b = 1

$\therefore$ Put $2x+1 = e^z$ i.e. $z = \log(2x + 1)$

$$\therefore \quad x = \frac{e^z-1}{2}$$

Thus $(2x+1)\dfrac{dy}{dx} = 2\,Dy$

$$(2x+1)^2\frac{d^2y}{dx^2} = 4D(D-1)\,y \quad \text{where } D = \frac{d}{dz}$$

Step 2 : The equation becomes

$$4D(D-1)\,y-2\cdot 2\cdot D\,y-12y = 6\left(\frac{e^z-1}{2}\right)$$

$$4[D^2-D-D-3]\,y = 3(e^z-1)$$

$$(D^2-2D-3)\,y = \frac{3}{4}(e^z-1)$$

The above equation is linear differential equation with constant coefficient, which we can solve by previous methods. Note that $D = \dfrac{d}{dz}$.

Step 3 : Solution

A.E. $\quad D^2-2D-3 = 0$

$$(D-3)(D+1) = 0$$

$$D = 3 \quad D = -1$$

$$\text{C.F.} = C_1e^{3z}+C_2e^{-1z}$$

$$\text{P.I.} = \frac{1}{f(D)}Z$$

$$\text{P.I.} = \frac{1}{(D-3)(D+1)}\,\frac{3}{4}(e^z-1)$$

$$= \frac{3}{4}\left\{\frac{1}{(1-3)(1+1)}e^z-\frac{1}{-3}1\right\}$$

$$= \frac{3}{4}\left\{\frac{1}{(D-3)(D+1)}e^z-\frac{1}{(D-3)(D+1)}1\right\}$$

$$= \frac{3}{4}\left\{\frac{e^z}{-4}+\frac{1}{3}\right\}$$

$$= \frac{-3e^z}{16}+\frac{1}{4}$$

y = C.F. + P.I.

$$\therefore \quad y = C_1e^{3z}+C_2e^{-z}-\frac{3}{16}e^z+\frac{1}{4}$$

Step 4 : Put $e^z = 2x+1$

$$y = C_1(2x+1)^3+C_2(2x+1)^{-1}-\frac{3}{16}(2x+1)+\frac{1}{4}$$

Example 1.53 : $(3x+2)^2 \frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y = 3x^2+4x+1$

Solution : Step 1 : Legendre's differential equation with a = 3, b = 2.

Put $3x + 2 = e^z$ i.e. $z = \log(3x+2)$

$$(3x+2)\frac{dy}{dx} = 3\,D\,y$$

$$(3x+2)^2\frac{d^2y}{dx^2} = 9\,D\,(D-1)\,y$$

Step 2 : Thus the equation becomes

$$9D(D-1)\,y+3\cdot 3\,Dy-36y = 3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1$$

$$9\,[D^2-D+D-4]\,y = 3\frac{(e^{2z}-4e^z+4)}{9}+\frac{4\,(e^z-2)}{3}+\frac{3}{3}$$

$$= \frac{e^{2z}-4e^z+4+4e^z-8+3}{3}$$

$$9\,(D^2-4)\,y = \frac{e^{2z}-1}{3}$$

$$(D^2-4)\,y = \frac{1}{27}(e^{2z}-1)$$

Which is linear differential equation with constant coefficient.

Step 3 : Solution

A.E. $\quad D^2-4 = 0$

$$(D-2)(D+2) = 0$$

$$D = 2, -2$$

$$\text{C.F.} = C_1e^{2z}+C_2e^{-2z}$$

$$\text{P.I.} = \frac{1}{f(D)}Z$$

$$\text{P.I.} = \frac{1}{(D-2)(D+2)}\frac{1}{27}(e^{2z}-1)$$

$$= \frac{1}{27}\left\{\frac{1}{(D-2)(D+2)}e^{2z}-\frac{1}{(D-2)(D+2)}1\right\}$$

Replace D by a only in non-zero factor. For constant D = 0.

$$= \frac{1}{27}\left\{\frac{1}{(D-2)}\cdot\frac{1}{(2+2)}e^{2z}-\frac{1}{(0-2)(0+2)}\right\}$$

$$= \frac{1}{27}\left\{\frac{1}{4}\cdot\frac{1}{D-2}e^{2z}+\frac{1}{4}\right\}$$

$$F_2\ \frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}.$$

$$= \frac{1}{108}\left\{\frac{z}{1}e^{2z}+1\right\}$$

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1e^{2z}+C_2e^{-2z}+\frac{1}{108}\{z\,e^{2z}+1\}$$

Step 4 : Now $e^z = 3x+2 = C_1(3x+2)^2+C_2(3x+2)^{-2}$ $+\frac{1}{108}\{(3x+2)^2\log(3x+2)+1\}$

Example 1.54 : $(5+2x)^2\frac{d^2y}{dx^2}-6(5+2x)\frac{dy}{dx}+8y = 5\log(5+2x)$

Solution : Step 1 : Legendre's with a = 2, b = 5

Put $5 + 2x = e^z$, $z = \log(5+2x)$

$$(5+2x)\frac{dy}{dx} = 2\,Dy$$

$$(5+2x)^2\frac{d^2y}{dx^2} = 4\,D\,(D-1)\,y$$

Step 2 : The equation becomes.

$$4\,D\,(D-1)\,y-6\cdot 2\cdot Dy+8y = 5z$$

$$4\,[D^2-D-3D+2]\,y = 5z$$

$$(D^2-4D+2)\,y = \frac{5}{4}z$$

This is linear differential equation with constant coefficient.

Step 3 : Solution : A.E. is

$$D^2-4D+2 = 0$$

$$D = \frac{+4\pm\sqrt{16-8}}{2}$$

$$D = \frac{+4\pm\sqrt{8}}{2}$$

$$D = +2 \pm \sqrt{2}$$

Which are real roots.

$$\text{C.F.} = C_1 e^{(+2+\sqrt{2})z} + C_2 e^{(2-\sqrt{2})z}$$

Use P.I. formula.

$$\text{P.I.} = \frac{1}{D^2 - 4D + 2} \frac{5}{4} z$$

$$= \frac{5}{2}\left[\frac{1}{1+\left(\frac{D^2-4D}{2}\right)}\right]\frac{z}{4}$$

Here f(D) takes the form $\frac{1}{1+z} = 1 - z + z^2 ...$

$$= \frac{5}{2}\left\{1 - \left(\frac{D^2-4D}{2}\right)...\right\}\frac{1}{4}z$$

$$= \frac{5}{2}\cdot\frac{1}{4}\left\{z - \frac{0-4}{2}\right\}$$

$$= \frac{5}{8}\{z+2\}$$

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1 e^{(2+\sqrt{2})z} + C_2 e^{(2-\sqrt{2})z} + \frac{5}{8}(z+2)$$

Step 4 : Put $e^z = (5+2x)$ i.e. $z = \log(5+2x)$.

$$\therefore \quad y = C_1 (5+2x)^{2+\sqrt{2}} + C_2 (5+2x)^{2-\sqrt{2}} + \frac{5}{8}[\log(5+2x)+2]$$

Example 1.55 : *The radial displacement u in a rotating disc at a distance 'r' from axis is given by* $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + kr = 0$. *Find the displacement if* $u = 0$ *for* $r = 0$, $r = a$.

Solution : $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + kr = 0$

Step 1 : Multiple by r^2.

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u + kr^3 = 0$$

Take kr^3 to R.H.S.

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -kr^3$$

Which is homogeneous in u and r.

$\therefore$ Put $\quad r = e^z$

$$\therefore \quad r\frac{du}{dr} = Du$$

and $r^2\frac{d^2u}{dr^2} = D(D-1)u$ where $D = \frac{d}{dz}$

Step 2 : The equation becomes.

$$D(D-1)u + Du - u = -ke^{3z}$$

$$(D^2 - D + D - 1)u = -ke^{3z}$$

$$(D^2 - 1)u = -ke^{3z}$$

Step 3 : Solution

A.E. $\quad D^2 - 1 = 0$

$$(D-1)(D+1) = 0$$

$$D = 1, -1$$

$$\text{C.F.} = C_1 e^z + C_2 e^{-z}$$

$$\text{P.I.} = \frac{1}{f(D)} Z$$

$$\text{P.I.} = \frac{1}{D^2 - 1} - ke^{3z}$$

$$\text{P.I.} = \frac{1}{(3)^2 - 1} - ke^{3z}$$

$$\text{P.I.} = \frac{-k}{8} e^{3z}$$

u = C.F. + P.I. is the complete solution.

$$u = C_1 e^z + C_2 e^{-z} - \frac{k}{8} e^{3z}$$

Step 4 : Put $e^z = r$

$$u = C_1 r + \frac{C_2}{r} - \frac{k}{8} r^3 \qquad ... (1)$$

Here the conditions are given to find the values of C_1 and C_2.

Given u = 0 for r = 0 substitute in (1).

$$0 = C_1(0) + \frac{C_2}{(0)} - \frac{k}{8}(0)^3$$

If C_2 is non zero then R.H.S becomes infinity which is not possible $\therefore C_2$ must be zero.

Thus equation (1) becomes

$$u = C_1 r + 0 - \frac{k}{8} r^3 \qquad ... (2)$$

Also given u = 0 for r = a substitute in (2).

$$0 = C_1 a - \frac{k}{8} a^3$$

$$C_1 = \frac{k}{8} a^2$$

Thus equation (2) becomes

$$u = \frac{k}{8}a^2 r - \frac{k}{8}r^3$$

$$\therefore \quad u = \frac{k}{8} r\,(a^2 - r^2)$$

Example 1.56 : $u = r\frac{d}{dr}\left(r\frac{du}{dr}\right) + ar^3$

Solution : Hint : $u = r\left[\frac{d}{dr}\left(r\frac{du}{dr}\right)\right] + ar^3$ differentiate w.r.t. r using product rule

$$u = r\left[r\frac{d^2u}{dr^2} + 1\frac{du}{dr}\right] + ar^3$$

$$\therefore \quad r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -ar^3$$

which is same as that of previous problem.

Example 1.57 : $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = x^2$

Solution : Step 1 : Separate the operator.

$$\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)y = x^2 \quad \text{... (1)}$$

Let $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)y = V$ ... (2)

$\therefore$ The equation (1) becomes.

$$\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right) V = x^2$$

$$\left(x^2\frac{d^2}{dx^2} - 2\right) V = x^4$$

$$x^2\frac{d^2v}{dx^2} - 2V = x^4$$

Which is homogeneous in V and x.

$\therefore$ Put $x = e^z$

$\therefore x^2\frac{d^2v}{dx^2} = D\,(D-1)\,V$ where $D = \frac{d}{dz}$

Step 2 :

Thus the equation becomes.

$$D\,(D-1)\,V - 2V = e^{4z}$$

$$(D^2 - D - 2)\,V = e^{4z}$$

which is the differential equation in V and x $\therefore$ Solve for V.

Step 3 : Solution

A.E. $D^2 - D - 2 = 0$

$$(D-2)(D+1) = 0$$

$$D = 2, \; D = -1$$

$$\text{C.F.} = C_1e^{2z} + C_2e^{-z}$$

$$\text{P.I.} = \frac{1}{D^2 - D - 2}e^{4z}$$

$$\text{P.I.} = \frac{1}{16-4-2}e^{4z}$$

$$\text{P.I.} = \frac{e^{4z}}{10}$$

As the C.F. and P.I. is for V thus

V = C.F. + P.I.

$$\therefore \quad V = C_1e^{2z} + C_2e^{-z} + \frac{e^{4z}}{10}$$

Put $e^z = x$

$$V = C_1x^2 + \frac{C_2}{x} + \frac{x^4}{10}$$

Substitute the value of V in equation (2).

$$\therefore \quad \left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)y = C_1x^2 + \frac{C_2}{x} + \frac{x^4}{10}$$

Multiply x^2 and open bracket on L.H.S.

$$x^2\frac{d^2y}{dx^2} - 2y = C_1x^4 + C_2x + \frac{x^6}{10}$$

Which is the homogeneous differential equation in y and x $\therefore$ Solve for y.

Put $x = e^z$

$$\therefore \; x^2\frac{d^2y}{dx^2} = D\,(D-1)\,y$$

$\therefore$ The equation becomes

$$D\,(D-1)\,y - 2y = C_1e^{4z} + C_2e^{z} + \frac{e^{6z}}{10}$$

$$(D^2 - D - 2)\,y = C_1e^{4z} + C_2e^{-z} + \frac{e^{6z}}{10}$$

A.E. $D^2 - D - 2 = 0$

$$(D-2)(D+1) = 0$$

$$D = 2, \; D = -1$$

$$\text{C.F.} = C_3e^{2z} + C_4e^{-z}$$

Use P.I. formula for y.

$$\text{P.I.} = \frac{1}{D^2-D-2}C_1e^{4z}+C_2e^z+\frac{e^{6z}}{10}$$

Take factors of f (D) and separate all the terms as PI_1, PI_2, PI_3 ...

$$\text{P.I.} = \underset{\substack{\uparrow\\ D=4}}{\frac{1}{(D-2)(D+1)}}C_1e^{4z}+\underset{\substack{\uparrow\\ D=1}}{\frac{1}{(D-2)(D+1)}}C_2e^z$$

Put

$$+\underset{\substack{\uparrow\\ D=6}}{\frac{1}{(D-2)(D+1)}}\frac{e^{6z}}{10}$$

Replace D by a only in non zero factor.

$$\text{P.I.} = \frac{C_1e^{4z}}{10}+\frac{C_2e^z}{-2}+\frac{e^{6z}}{280}$$

y = C.F. + P.I.

$$y = C_3e^{2z}+C_4e^{-z}+\frac{C_1e^{4z}}{10}-\frac{C_2}{2}e^z+\frac{e^{6z}}{280}$$

Put $x = e^z$

$$y = C_3x^2+\frac{C_4}{x}+\frac{C_1}{10}x^4-\frac{C_2}{2}x+\frac{x^6}{280}$$

Example 1.58 : *Find the equation of curve which satisfies the Differential Equation* $4x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+y=0$ *and crosses x axis at an angle of 60° at x = 1.*

Solution :

Step 1 :

$$4x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+y = 0$$

Which is homogenous in y and x

$\therefore$ Put $x = e^z$

$$\therefore\ x\frac{dy}{dx} = D\,y$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)\,y \quad \text{where } D = \frac{d}{dz}$$

$$4D(D-1)y-4Dy+y = 0$$

$$(4D^2-8D+1)y = 0$$

Step 3 : Solution

A.E. $4D^2-8D+1 = 0$

$$D = \frac{8\pm\sqrt{64-16}}{2\cdot 4} = \frac{8\pm\sqrt{48}}{8} = \frac{8\pm4\sqrt{3}}{8} = 1\pm\frac{\sqrt{3}}{2}$$

Both the roots are real.

$$\therefore\quad \text{C.F.} = C_1e^{\left(1+\frac{\sqrt{3}}{2}\right)z}+C_2e^{\left(1-\frac{\sqrt{3}}{2}\right)z}$$

As R.H.S. = 0 $\therefore$ P.I. = 0.

$$\therefore\quad y = \text{C.F.} + \text{P.I.}$$

$$y = C_1e^{\left(1+\frac{\sqrt{3}}{2}\right)z}+C_2e^{\left(1-\frac{\sqrt{3}}{2}\right)z}$$

Put $e^z = x$

$$y = C_1x^{\left(1+\frac{\sqrt{3}}{2}\right)}+C_2x^{\left(1-\frac{\sqrt{3}}{2}\right)} \quad \text{... (1)}$$

Given that curve crosses x axis at x = 1 means y = 0 at x = 1

Substitute in equation (1)

$$0 = C_1+C_2 \quad \text{... (2)}$$

Also given that curve crosses x axis at an angle of 60° at x = 1.

Means at x = 1, $\frac{dy}{dx} = \tan 60 = \sqrt{3}$

$\therefore$ Find $\frac{dy}{dx}$ from equation (1).

$$\frac{dy}{dx} = C_1\left(1+\frac{\sqrt{3}}{2}\right)x^{\sqrt{3}/2}+C_2\left(1-\frac{\sqrt{3}}{2}\right)x^{-\sqrt{3}/2}$$

Substitute x = 1, $\frac{dy}{dx} = \sqrt{3}$

$$\sqrt{3} = C_1\left(1+\frac{\sqrt{3}}{2}\right)+C_2\left(1-\frac{\sqrt{3}}{2}\right)$$

Simplify.

$$\sqrt{3} = C_1+C_2+\frac{\sqrt{3}}{2}(C_1-C_2)$$

$$\sqrt{3} = 0+\frac{\sqrt{3}}{2}(C_1-C_2)\ \{\text{As } C_1+C_2 = 0 \text{ from 2}\}$$

$$2 = C_1-C_2 \quad \text{... (3)}$$

Solve equations (2) and (3) for C_1 and C_2.

$$0 = C_1+C_2$$

$$2 = C_1-C_2$$

$$\Rightarrow\quad C_1 = 1 \quad C_2 = -1$$

Substitute C_1 and C_2 in equation (1)

$$y = x^{\left(1+\frac{\sqrt{3}}{2}\right)}-x^{\left(1-\frac{\sqrt{3}}{2}\right)}$$

Example 1.59 : $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=\dfrac{x^3}{1+x^2}$

Solution : Step 1 : Which is homogeneous in y and x

Put $x = e^z$

$\therefore\ x\dfrac{dy}{dx} = Dy$

$x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

Step 2 : $\therefore$ The equation becomes

$$D(D-1)y+Dy-y = \frac{e^{3z}}{1+e^{2z}}$$

$$(D^2-D+D-1)y = \frac{e^{3z}}{1+e^{2z}}$$

$$(D^2-1)y = \frac{e^{3z}}{1+e^{2z}}$$

Step 3 : Solution

A.E. is $D^2-1 = 0$

$(D-1)(D+1) = 0$

$D = 1,\ D = -1$ real roots.

$\therefore$ C.F. $= C_1e^z+C_2e^{-z}$

$$= \frac{1}{D^2-1}\,\frac{e^{3z}}{1+e^{2z}}$$

$$= \frac{1}{(D-1)(D+1)}\,\frac{e^{3z}}{1+e^{2z}}$$

$$= \frac{1}{2}\left[\frac{1}{D-1}-\frac{1}{D+1}\right]\frac{e^{3z}}{1+e^{2z}}$$

$$= \frac{1}{2}\left\{e^z\int e^{-z}\frac{e^{3z}}{1+e^{2z}}dz-e^{-z}\int e^z\frac{e^{3z}}{1+e^{2z}}dz\right\}$$

$$= \frac{1}{2}\left\{e^z\int\frac{e^{2z}}{1+e^{2z}}dz-e^{-z}\int\frac{e^{2z}\cdot e^{2z}}{1+e^{2z}}dz\right\}$$

Put $1+e^{2z}=t,\ 2e^{2z}dz=dt,\ e^{2z}dz=\dfrac{dt}{2},\ e^{2z}=t-1.$

$$\therefore\ = \frac{1}{2}\left\{e^z\int\frac{dt/2}{t}-e^{-z}\int\frac{(t-1)\,dt/2}{t}\right\}$$

$$= \frac{1}{4}\left\{e^z\int\frac{dt}{t}-e^{-z}\int\left(1-\frac{1}{t}\right)dt\right\}$$

$$= \frac{1}{4}\{e^z\log t-e^{-z}(t-\log t)\}$$

$$= \frac{1}{4}\{e^z\log(1+e^{2z})-e^{-z}[(1+e^{2z})-\log(1+e^{2z})]\}$$

$$= \frac{1}{4}\{(e^z+e^{-z})\log(1+e^{2z})-(e^z+e^{-z})\}$$

$$= \frac{1}{4}\{\log(1+e^{2z})-1\}\,(e^z+e^{-z})$$

y = C.F. + P.I.

$$y = C_1e^z+C_2e^{-z}+\frac{1}{4}(e^z+e^{-z})\{\log(1+e^{2z})-1\}$$

Step 4 : Put $e^z = x$

$$y = C_1x+\frac{C_2}{x}+\frac{1}{4}\left(x+\frac{1}{x}\right)\{\log(1+x^2)-1\}$$

Example 1.60 : $(x^2D^2+3xD+1)y=(1-x)^{-2}$

Solution :

Step 1 : Here $D = \dfrac{d}{dx}$ $\therefore$ Substituting we get

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y = \frac{1}{(1-x)^2}$$

Which is homogeneous in y and x

$\therefore$ Put $x = e^z$

$\therefore\ x\dfrac{dy}{dx} = Dy$

$x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

Step 2 : The equation becomes

$$D(D-1)y+3Dy+y = \frac{1}{(1-e^z)^2}$$

$$(D^2+2D+1)y = \frac{1}{(1-e^z)^2}$$

Step 3 : Consider A.E.

$D^2+2D+1 = 0$

$(D+1)^2 = 0$

$D = -1$ twice

$\therefore$ C.F. $= (C_1+C_2z)\,e^{-z}$

P.I. $= \dfrac{1}{(D+1)^2}\,\dfrac{1}{(1-e^z)^2}$

$\therefore$ Consider $\dfrac{1}{D+1}\,\dfrac{1}{(1-e^z)^2}$

$$= e^{-z}\int e^z\frac{1}{(1-e^z)^2}dz$$

Put $1-e^z = t$

$\therefore\ -e^zdz = dt$

$$\therefore\ = e^{-z}\int\frac{-dt}{t^2}$$

$$= e^{-z}\left[\frac{1}{t}\right]$$

$$\frac{1}{D+1}\frac{1}{(1-e^z)^2} = e^{-z}\frac{1}{(1-e^z)}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}\frac{1}{(1-e^z)^2}$$

$$= \frac{1}{D+1}\left[\frac{1}{D+1}\frac{1}{(1-e^z)^2}\right]$$

$$\therefore \quad \text{P.I.} = \frac{1}{D+1}\left[\frac{e^{-z}}{1-e^z}\right]$$

$$\text{P.I.} = e^{-z}\int e^z\cdot\frac{e^{-z}}{1-e^z}\,dz$$

$$\text{P.I.} = e^{-z}\int\frac{1}{1-e^z}\,dz$$

$$\text{P.I.} = e^{-z}\int\frac{1}{1-\frac{1}{e^{-z}}}\,dz$$

$$\text{P.I.} = e^{-z}\int\frac{e^{-z}}{e^{-z}-1}\,dz$$

Put $e^{-z}-1=t$

$\therefore -e^{-z}dz = dt$

$\therefore e^{-z}dz = -dt$

$$\text{P.I.} = e^{-z}\int\frac{-dt}{t}$$

$$\text{P.I.} = e^z(-\log t)$$

$$\text{P.I.} = -e^{-z}\log(e^{-z}-1)$$

y = C.F. + P.I.

$$y = (C_1+C_2z)e^{-z} - e^{-z}\log(e^{-z}-1)$$

Take e^{-z} common and write $e^{-z} = \frac{1}{e^z}$.

$$y = \frac{1}{e^z}\left\{(C_1+C_2z)-\log\left(\frac{1}{e^z}-1\right)\right\}$$

Put $e^z = x$ i.e. $z = \log x$.

$$y = \frac{1}{x}\left\{1+C_2\log x-\log\left(\frac{1}{x}-1\right)\right\}$$

$$y = \frac{1}{x}\left\{C_1+C_2\log x-\log\left(\frac{1-x}{x}\right)\right\}$$

$$y = \frac{1}{x}\{C_1+C_2\log x-[\log(1-x)-\log x]\}$$

$$y = \frac{1}{x}\{C_1+C_2\log x-\log(1-x)+\log x\}$$

Example 1.61 : $x^3\dfrac{d^2y}{dx^2}+3x^2\dfrac{dy}{dx}+xy = \sin(\log x)$

Solution :

Step 1 : Here the coefficient of $\frac{d^2y}{dx^2}$ is x^3.

We need x^2 $\therefore$ Divide by x

$$\therefore x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y = \frac{1}{x}\sin(\log x)$$

Which is homogeneous in y and x.

$\therefore$ Put $x = e^z$ or $z = \log x$

$$\therefore x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)\,y$$

Step 2 : $\therefore$ The equation becomes

$$D(D-1)\,y+3\,Dy+y = \frac{1}{e^z}\sin(z)$$

$$(D^2+2D+1)\,y = e^{-z}\sin z$$

Step 3 : Consider A.E.

$$D^2+2D+1 = 0$$

$$(D+1)^2 = 0$$

$$D = -1, -1 \quad \text{real repeated}$$

$$\text{C.F.} = (C_1+C_2z)\,e^{-z}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}e^{-z}\sin z$$

$$\text{P.I.} = e^{-z}\frac{1}{(D-1+1)^2}\sin z$$

$$= e^{-z}\frac{1}{D^2}\sin z$$

$$\text{P.I.} = e^{-z}\frac{1}{-1}\sin z$$

$$= -e^{-z}\sin z$$

y = C.F. + P.I.

$$y = (C_1+C_2z)e^{-z}-e^{-z}\sin z$$

Take e^{-z} common write $e^{-z} = \frac{1}{e^z}$.

$$y = \frac{1}{e^z}[C_1+C_2z-\sin z]$$

Step 4 : Put $e^z = x$, $z = \log x$.

$$y = \frac{1}{x}[C_1+C_2\log x-\sin(\log x)]$$

Exercise 1.9 (Problems Reducible to Type 1)

1. $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x+4$

[Ans. : $y = (x+2)\{C_1 + C_2 \log(x+2)\} + \frac{3}{2}(x+2)[\log(x+2)]^2 - 2$]

2. $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 5x+6$

[Ans. : P.I. $= \frac{5}{2}[\log(x+2)]^2 \cdot (x+2) - 4$]

3. $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

[Ans. : $y = \frac{C_1}{x} + x\,[C_2 \cos(\log x) + C_3 \sin(\log x)] + 5x + \frac{2}{x}\log x$]

4. $(x^3D^3 + x^2D^2 - 2)\,y = x + \frac{1}{x^3}$

[Ans. : $y = C_1 x^2 + C_2 \cos(\log x) + C_3 \sin(\log x) - \frac{x}{2} - \frac{1}{50}x^3$]

Hint : Here given D is $\frac{d}{dx}$ thus the equation is

$$x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} - 2y = x + \frac{1}{x^3}$$

5. $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 3x - 7$

[Ans. :
$y = C_1 x + x^{-1/2}\left[C_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + C_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + x\log x + 7$]

6. $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = 0$

[Ans. : $y = C_1 x^4 + C_2 x^2 + C_3 x + \frac{C_4}{x}$]

7. $\left(\frac{d}{dx} + \frac{1}{x}\right)^2 y = \frac{1}{x^4}$

[Ans. : $y = C_1 + \frac{C_2}{x} + \frac{1}{2x^2}$]

Problems Reducible to Type 2 and 4

8. $(x+2)^2 \frac{d^2y}{dx^2} + (x+2)\frac{dy}{dx} + y = 2\sin\log(x+2)$

[Ans. : $y = C_1 \cos z + C_2 \sin z - z\cos z$ where $z = \log(x+2)$]

9. $(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4\cos[\log(x+1)]$

[Ans. : $y = C_1 \cos z + C_2 \sin z + 2z\sin z$ where $z = \log(x+1)$]

10. $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$

[Ans. : $y = e^z[C_1 \cos\sqrt{3}z + C_2 \sin\sqrt{3}z] + \frac{e^z}{2}\sin z + \frac{1}{13}(3\cos z - 2\sin z)$ where $z = \log x$]

11. $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = A + B\log x$

[Ans. : $y = C_1 + C_2 \log x + \frac{Ax^2}{4} + \frac{Bx^2}{4}(\log x - 1)$]

12. $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \log x$

[Ans. : $y = x^2\{C_1 \cos(\log x) + C_2 \sin(\log x) + \log x\}$]

13. $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x\log x$

[Ans. : $y = x\left[C_1 + C_2 \log x + \frac{1}{6}(\log x)^3\right]$]

14. $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$

[Ans. : $y = C_1 x^4 + \frac{C_2}{x} - \frac{x^2}{6} - \frac{1}{2}\log x + \frac{3}{8}$]

15. $(2x+1)^2 \frac{d^2y}{dx^2} + 2(2x+1)\frac{dy}{dx} + 4y = 4\sin[2\log(2x+1)]$

[Ans. : $C_1 \cos\log(2x+1) + C_2 \sin\log(2x+1) - \frac{1}{3}\sin[2\log(2x+1)]$]

16. $(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} - y = 2\log(x+1) + (x-1)$

[Ans. : $y = C_1(x+1) + \frac{C_2}{x+1} + \frac{1}{3}(x-3)\log(x+1) + 2$]

17. $(4x+1)^2 \frac{d^2y}{dx^2} + 2(4x+1)\frac{dy}{dx} + y = 2x+1$

[Ans. : $y = [C_1 + C_2 \log(4x+1)]\,(4x+1)^{1/4} + \left(\frac{2x+5}{9}\right)$]

18. $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \sin(\log x)$

[Ans. : $y = x^2[C_1 \cos\log x + C_2 \sin\log x] - \frac{x^2}{2}\log x\cos\log x$]

19. $(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)$

[Ans. : $y = C_1 + C_2 \log(x+1) + (x+1)^2 + 6(x+1) + [\log(x+1)]^2$]

20. $\left(x^3D^3 + x^2D^2 - 2\right)y = x^2 + x^{-3}$

[Ans. : $y = C_1 x^2 + C_2 \cos\log x + C_3 \sin\log x - \frac{1}{50x^3} + \frac{x^2}{5}\log x$]

21. $\left(x^2D^2+xD-4\right)y=\left(x+\frac{1}{x}\right)^2$

[Ans. : $y=C_1x^2+\frac{C_2}{x^2}+\frac{1}{2}$ [log x sinh (2 log x) − 1]]

22. *Find the equation of the curve that satisfies the differential equation* $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+x=0$ *and crosses x axis at an angle of 45° at (1, 0).* **[Ans. : $y=\frac{3}{2}\log x+\frac{1}{4x^2}-\frac{1}{4}$]**

1.7 Lagrange's Method of Variation of Parameters

The method of variation of parameters was first sketched by the Swiss Mathematician Leonhard Euler (1707 – 1783) and later completed by Joseph-Louis Lagrange. It is also known as variation of constants.

Consider the linear second order differential equation with constant coefficients.

$$f(D)\,y = \phi(x) \quad \text{... (i)}$$

Let its C.F be $\quad y = C_1y_1+C_2y_2 \quad$ ... (ii)

Assume $\quad$ P.I $= uy_1+vy_2$

As P.I satisfies equation (i)

$\therefore \quad y = uy_1+vy_2$

Differentiate w.r.t. x

$$y' = u'y_1+v'y_2+uy_1'+vy_2'$$

Assume $u'y_1+v'y_2 = 0 \quad$ ... (iii)

Thus $\quad y' = uy_1'+vy_2' \quad$ ... (iv)

Again $\quad y'' = uy_1''+vy_2''+u'y_1'+v'y_2' \quad$ ... (v)

Substituting y, y′, y″ in equation (i) we get

$$u'y_1'+v'y_2' = \phi(x) \quad \text{... (vi)}$$

Thus we have two equations in u′ and v′ i.e. equation (iii) and (vi).

$$u'y_1+v'y_2 = 0$$

$$u'y_1'+v'y_2' = \phi(x)$$

Solving these two equations by Cramer's rule we get

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ \phi(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\Delta u}{\Delta}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \phi(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\Delta v}{\Delta}$$

Thus $\quad u = \int \frac{\Delta u}{\Delta}\,dx$ and $v = \int \frac{\Delta v}{\Delta}\,dx$

Method of variation of parameters to find particular integral

Procedure 1 :

Step 1 : If the differential equation is of order two then the complementary function will be of the form C.F. $= C_1y_1+C_2y_2$

Step 2 : Assume

$$\text{P.I.} = uy_1+vy_2$$

Step 3 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \Delta u = \begin{vmatrix} 0 & y_2 \\ \phi(x) & y_2' \end{vmatrix} \quad \Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & \phi(x) \end{vmatrix}$$

where y_1' and y_2' are derivatives of y_1 and y_2 also $\phi(x)$ is the R.H.S. of equation. [i.e. f (D) y = $\phi(x)$]

Step 4 : Then

$$u = \int \frac{\Delta u}{\Delta}\,dx$$

$$v = \int \frac{\Delta v}{\Delta}\,dx$$

Step 5 : Substitute u and v in P.I. $= uy_1+vy_2$

Procedure 2 :

Step 1 : If the differential equation is of order three then C.F. will be

C.F. $= C_1y_1+C_2y_2+C_3y_3$

Step 2 : ∴ Assume

$$\text{P.I.} = uy_1+vy_2+wy_3$$

$$\Delta = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ \phi(x) & y_2'' & y_3'' \end{vmatrix}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & \phi(x) & y_3'' \end{vmatrix}$$

$$\Delta w = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & \phi(x) \end{vmatrix}$$

Step 3 : Find $u = \int \frac{\Delta u}{\Delta} dx$, $v = \int \frac{\Delta v}{\Delta} dx$,

$w = \int \frac{\Delta w}{\Delta} dx$

Step 4 : P.I. $= uy_1 + vy_2 + wy_3$

Step 5 : Complete solution

$$y = \text{C.F.} + \text{P.I.}$$

Example 1.62 : $(D^3 + D)y = cosec\,x$

Solution : Step 1 : Consider A.E. $D^3 + D = 0$

$$D(D^2 + 1) = 0$$

$$D = 0,\ D^2 = -1$$

$$D = 0,\ D = \pm i,\ \alpha = 0,\ \beta = 1 \text{ complex}$$

$\therefore$ $$\text{C.F.} = C_1 e^{0x} + e^{0x}(C_2 \cos x + C_3 \sin x)$$

$$\text{C.F.} = C_1 + C_2 \cos x + C_3 \sin x$$

Comparing with C.F. $= C_1 y_1 + C_2 y_2 + C_3 y_3$ we get.

$$y_1 = 1,\ y_2 = \cos x,\ y_3 = \sin x$$

Step 2 : Assume P.I. $= uy_1 + vy_2 + wy_3$. To find u, v, w find derivatives of $y_1\ y_2\ y_3$.

$$y_1 = 1 \quad y_2 = \cos x \quad y_3 = \sin x$$

$$y_1' = 0 \quad y_2' = -\sin x \quad y_3' = +\cos x$$

$$y_1'' = 0 \quad y_2'' = -\cos x \quad y_3'' = -\sin x$$

Step 3 : Find Δ, Δu, Δv, Δw

$$\Delta = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$\Delta = 1(\sin^2 x + \cos^2 x) - \cos x(0-0) + \sin x(0-0)$$

$$= 1$$

Find Δu.

$$\Delta u = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ X & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ cosec\,x & -\cos x & -\sin x \end{vmatrix}$$

$$= 0(\sin^2 x + \cos^2 x) - \cos x(0 - \cos x\, cosec\,x)$$

$$+ \sin x(0 + \sin x \cdot cosec\,x)$$

$$= 0 + \frac{\cos^2 x}{\sin x} + \sin x(1)$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x}$$

$$= \frac{1}{\sin x} = cosec\,x$$

Find Δv.

$$\Delta v = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & X & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & cosec\,x & -\sin x \end{vmatrix}$$

$$= 1(0 - cosec\,x\ \cos x) - (0-0) + \sin x(0-0)$$

$$= -\cot x$$

Find Δw.

$$\Delta w = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & X \end{vmatrix} = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & cosec\,x \end{vmatrix}$$

$$= 1(-\sin x\ cosec\,x - 0) - \cos x(0-0) + 0\,(0-0)$$

$$= -1$$

Step 4 : Find u, v, w.

$$u = \int \frac{\Delta u}{\Delta} dx = \int cosec\,x\, dx = \log(cosec\,x - \cot x)$$

$$v = \int \frac{\Delta v}{\Delta} dx = \int -\cot x\, dx = -\log \sin x$$

$$w = \int \frac{\Delta w}{\Delta} dx = \int -1\, dx = -x$$

Step 5 : Substitute u, v, w y_1, y_2, y_3 in P.I. $= uy_1 + vy_2 + wy_3$

$\therefore$ $$\text{P.I.} = \log(cosec\,x - \cot x)$$

$$= \cos x \log \sin x - x \sin x$$

Step 6 : y = C.F. + P.I. is the complete solution of D.E.

Example 1.63 : $(D^2 - 4)y = 2\,sech\,2x$

Solution : Step 1 : Consider A.E. $D^2 - 4 = 0$

$$(D-2)(D+2) = 0$$

$$D = 2, -2$$

$$\text{C.F.} = C_1' e^{2x} + C_2' e^{-2x}$$

Note : In this problem as the R.H.S is hyperbolic. $\therefore$ If the C.F. is hyperbolic then the integration becomes easy.

$\therefore$ Consider $\quad$ C.F. $= C_1 \cosh 2x + C_2 \sinh 2x$

Comparing with C.F. $= C_1 y_1 + C_2 y_2$

$$y_1 = \cosh 2x \quad y_2 = \sinh 2x$$

Step 2 : Assume P.I. $= uy_1 + vy_2$

Find derivatives of y_1, y_2.

$$y_1 = \cosh 2x \quad y_2 = \sinh 2x$$

$$y_1' = 2 \sinh 2x \quad y_2' = 2 \cosh 2x$$

Step 3 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cosh 2x & \sinh 2x \\ 2\sinh 2x & 2\cosh 2x \end{vmatrix}$$

$$= 2(\cosh^2 2x - \sinh^2 2x)$$

$$= 2(1) \qquad \because \cosh^2\theta - \sinh^2\theta = 1$$

$$= 2$$

Find Δu.

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sinh 2x \\ 2\text{sech } 2x & 2\cosh 2x \end{vmatrix}$$

$$= -2 \cdot \frac{\sinh 2x}{\cosh 2x}$$

Find Δv.

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = \begin{vmatrix} \cosh 2x & 0 \\ 2\sinh 2x & 2\text{sech } 2x \end{vmatrix}$$

$$= 2 \cdot \cosh 2x \cdot \text{sech } 2x = 2$$

Step 4 : Find u and v :

$$u = \int \frac{\Delta u}{\Delta} dx \qquad v = \int \frac{\Delta v}{\Delta} dx$$

$$u = \int -\frac{2\sinh 2x}{2\cosh 2x} dx, \quad v = \int \frac{2}{2} dx$$

$$u = -\frac{1}{2}\log(\cosh 2x), \qquad v = x$$

$$\because \int \frac{f'(x)}{f(x)} dx = \log f(x)$$

Step 5 : P.I. $= uy_1 + vy_2$

$$\text{P.I.} = -\frac{1}{2}[\log \cosh 2x] \cosh 2x + x \sinh 2x$$

Step 6 : y = C.F. + P.I. is the complete solution.

Example 1.64 : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x$

Solution : $D^2 - 2D + 2 = 0$

Step 1 : C.F. $= e^x (C_1 \cos x + C_2 \sin x)$

Comparing with C.F. $= C_1 y_1 + C_2 y_2$

$$y_1 = e^x \cos x, \quad y_2 = e^x \sin x$$

Step 2 : Assume P.I. $= u\, y_1 + v\, y_2$

Find derivatives of y_1, y_2

$y_1' = e^x (\cos x - \sin x)$, $y_2' = e^x (\sin x + \cos x)$

Step 3 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$= e^x \cos e^x (\sin x + \cos x)$$

$$-e^x \sin x\, e^x (\cos x - \sin x)$$

$$= e^{2x} (\cos^2 x + \sin^2 x) = e^{2x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = -X\, y_2$$

$$= -e^x \tan x \cdot e^x \sin x = -e^{2x} \cdot \frac{\sin^2 x}{\cos x}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = X\, y_1$$

$$= e^x \tan x \cdot e^x \cos x = e^{2x} \sin x$$

Step 4 : $u = \int \frac{\Delta u}{\Delta} dx = \int \frac{-\sin^2 x}{\cos x} dx$

$$= \int -\frac{(1-\cos^2 x)}{\cos x} dx$$

$$= \int -\sec x + \cos x\, dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$v = \int \frac{\Delta v}{\Delta} dx = \int \sin x\, dx$$

$$= -\cos x$$

Step 5 : P.I. $= u\, y_1 + v\, y_2$

$$= e^x \cos x[-\log(\sec x + \tan x) + \sin x]$$

$$+ e^x \sin x[-\cos x]$$

$$= -e^x \cos x \log(\sec x + \tan x)$$

Step 6 : y = C.F. + P.I. is the complete solution.

$$y = e^x (C_1 \cos x + C_2 \sin x) - e^x \cos x$$

$$-\log(\sec x + \tan x)$$

Example 1.65 : $(D^2+1)\, y = \frac{1}{1+\sin x}$

Solution :

Step 1 : $D^2 + 1 = 0$

$$D^2 = -1$$

$$D = \pm i$$

$\therefore$ C.F $= C_1 \cos x + C_2 \sin x$

Compare C.F with C.F $= C_1 y_1 + C_2 y_2$

Ste 2 : Assume P.I $= u\, y_1 + v\, y_2$

Find derivatives of y_1, y_2

$\therefore$ $y_1 = \cos x \quad y_2 = \sin x$

$y_1' = -\sin x \quad y_2' = \cos x$

Step 3 : Find Δ, Δu, Δv.

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$= \cos^2 x + \sin^2 x$$
$$= 1$$
$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = -X y_2 = -\frac{1}{1+\sin x} \sin x$$
$$= -\frac{\sin x}{1+\sin x}$$
$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = X y_1 = \frac{1}{1+\sin x} \cdot \cos x$$
$$= \frac{\cos x}{1+\sin x}$$

Step 4 : $u = \int \frac{\Delta u}{\Delta} dx,\ v = \int \frac{\Delta v}{\Delta} dx$

$$u = \int -\frac{\sin x}{1+\sin x} dx$$
$$= \int -\frac{\sin x\,(1-\sin x)}{1-\sin^2 x} dx \quad \text{(Rationalise)}$$
$$= \int \frac{-\sin x + \sin^2 x}{\cos^2 x} dx$$
$$= \int \left(\frac{-\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\right) dx$$
$$= \int (-\sec x \tan x + \tan^2 x)\, dx$$
$$= \int (-\sec x \tan x + \sec^2 x - 1)\, dx$$
$$u = -\sec x + \tan x - x$$
$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{\cos x}{1+\sin x} dx$$
$$= \log(1 + \sin x)$$

Step 5 : P.I. $= u y_1 + v y_2$

$$\text{P.I} = \cos x\,(-\sec x + \tan x - x)$$
$$+ \sin x\,(\log(1+\sin x))$$
$$= -1 + \sin x - x\cos x + \sin x \log(1+\sin x)$$

Step 6 : y = C.F + P.I is the complete solution.

$$y = C_1 \cos x + C_2 \sin x - 1 + \sin x - x\cos x$$
$$+ \sin x \log(1+\sin x)$$

Example 1.66 : $(D^2 - 4D + 4)\, y = e^{2x} \sec^2 x$

Solution :

Step 1 : $D^2 - 4D + 4 = 0$

$$(D-2)^2 = 0$$
$$D = 2, 2$$
$$\text{C.F} = (C_1 + C_2 x)\, e^{2x}$$

Compare with C.F $= C_1 y_1 + C_2 y_2$

Step 2 : Assume P.I $= u y_1 + v y_2$

Find derivatives of y_1, y_2.

$$y_1 = e^{2x} \qquad y_2 = x e^{2x}$$
$$y_1' = 2e^{2x} \qquad y_2' = (2x+1)\, e^{2x}$$

Step 3 : Find Δ, Δu, Δv

$$\Delta = y_1 y_2' - y_1' y_2$$
$$= e^{2x}\,(2x+1)\, e^{2x} - 2e^{2x} \cdot x e^{2x}$$
$$= e^{4x}$$
$$\Delta u = -X y_2 = -e^{2x} \sec^2 x \cdot x\, e^{2x}$$
$$= -x \sec^2 x\, e^{4x}$$
$$\Delta v = X y_1 = e^{2x} \sec^2 x \cdot e^{2x}$$
$$= e^{4x} \sec^2 x$$

Step 4 : $u = \int \frac{\Delta u}{\Delta} = \int \frac{-x \sec^2 x\, e^{4x}}{e^{4x}} dx$

$$= -\int x \sec^2 x\, dx$$
$$= -\{x \cdot \tan x - (1)(\log \sec x)\}$$
$$= -x \tan x + \log \sec x$$
$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{e^{4x} \sec^2 x}{e^{4x}} dx$$
$$= \int \sec^2 x\, dx$$
$$= \tan x$$

Step 5 : P.I. $= u y_1 + v y_2$

$$= e^{2x}\,(-x \tan x + \log \sec x)$$
$$+ e^{2x} \cdot \tan x \cdot x$$
$$= e^{2x} \log \sec x$$

Step 6 : y = C.F. + P.I.

$$y = (C_1 + C_2 x)\, e^{2x} + e^{2x} \log \sec x$$

is the complete solution.

Example 1.67 : $(D^2 + 1)\, y = x \sin x$

Solution : $D^2 + 1 = 0$

$$D^2 = -1$$
$$D = \pm i$$
$$\text{C.F.} = C_1 \cos x + C_2 \sin x$$

Compare with C.F. $= C_1 y_1 + C_2 y_2$

Step 2 : Assume P.I. $= u y_1 + v y_2$

Find derivatives of y_1, y_2

$$y_1 = \cos x \qquad y_2 = \sin x$$

$$y_1' = -\sin x \quad y_2' = \cos x$$

Step 3 : Find Δ, Δu, Δv.

$$\Delta = y_1 y_2' - y_1' y_2$$
$$= \cos x \cdot \cos x + \sin x \sin x = 1$$
$$\Delta u = -X y_2 = -x \sin x \cdot \sin x$$
$$= -x \sin^2 x = \frac{-x}{2}(1-\cos 2x)$$
$$\Delta v = X y_1 = x \sin x \cos x$$
$$= \frac{x}{2}\sin 2x$$

Step 4 : $u = \int \frac{\Delta u}{\Delta} dx,\ v = \int \frac{\Delta v}{\Delta} dx$

$$u = \int \left(-\frac{x}{2} + \frac{x}{2}\cos 2x\right) dx$$
$$= -\frac{x^2}{4} + \frac{1}{2}\int x \cos 2x \, dx$$
$$= -\frac{x^2}{4} + \frac{1}{2}\left\{x\left(\frac{\sin 2x}{2}\right) - (1)\left(\frac{-\cos 2x}{4}\right)\right\}$$
$$= -\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x$$
$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{x}{2}\sin 2x \, dx$$
$$= \frac{1}{2}\left\{x\left(\frac{-\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right\}$$
$$= -\frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x$$

Step 5 : P.I. $= u y_1 + v y_2$

$$\text{P.I} = \left(-\frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x\right)\cos x$$
$$+\left(-\frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x\right)\sin x$$
$$= \frac{-x^2}{4}\cos x + \frac{x}{4}(\sin 2x \cos x - \cos 2x \sin x)$$
$$+\frac{1}{8}(\cos 2x \cos x + \sin 2x \sin x)$$
$$= -\frac{x^2}{4}\cos x + \frac{x}{4}\sin x + \frac{1}{8}\cos x$$

Step 6 : $y = \text{C.F.} + \text{P.I.}$

$$= C_1 \cos x + C_2 \sin x - \frac{x^2}{4}\cos x$$
$$+\frac{x}{4}\sin x + \frac{1}{8}\cos x$$

is the complete solution of the equation.

Example 1.68 : $(D^2+1)y = 3x - 8\cot x$ **SPPU : Dec.-16**

Solution : Step 1 : C.F. $= C_1 \cos x + C_2 \sin x$

Step 2 : Assume P.I. $= u y_1 + v y_2$

Find y_1', y_2'

$$y_1 = \cos x \quad y_2 = \sin x$$
$$y_1' = -\sin x \quad y_2' = \cos x$$

Step 3 : Find Δ, Δu, Δv

$$\Delta = y_1 y_2' - y_1' y_2$$
$$= \cos^2 x + \sin^2 x = 1$$
$$\Delta u = -X y_2 = -\sin x (3x - 8\cot x)$$
$$= -3x \sin x + 8\cos x$$
$$\Delta v = X y_1 = \cos x (3x - 8\cot x)$$
$$= 3x \cos x - 8\frac{\cos^2 x}{\sin x}$$
$$= 3x\cos x - 8\frac{(1-\sin^2 x)}{\sin x}$$
$$= 3x \cos x - 8\,\text{cosec}\, x + 8 \sin x$$

Step 4 : Find u, v.

$$u = \int \frac{\Delta u \, dx}{\Delta} = \int (-3x\sin x + 8\cos x)\, dx$$
$$= -3\{x(-\cos x) - (1)(-\sin x)\} + 8\sin x$$
$$= 3x\cos x - 3\sin x + 8\sin x$$
$$= 3x\cos x + 5\sin x$$
$$v = \int \frac{\Delta v \, dx}{\Delta} = \int (3x\cos x - 8\,\text{cosec}\, x + 8\sin x)\, dx$$
$$= 3[x\sin x + \cos x] - 8\log(\text{cosec}\, x - \cot x) - 8\cos x$$
$$= 3x\sin x - 5\cos x - 8\log(\text{cosec}\, x - \cot x)$$

Step 5 : $\therefore$ P.I. $= u y_1 + v y_2$

$$= \cos x\,[3x\cos x + 5\sin x]$$
$$+ \sin x[3x\sin x - 5\cos x - 8\log(\text{cosec}\, x - \cot x)]$$
$$= 3x(\cos^2 x + \sin^2 x) - 8\sin x \log(\text{cosec}\, x - \cot x)$$
$$= 3x - 8\sin x \log(\text{cosec}\, x - \cot x)$$

Step 6 : $y = \text{C.F.} + \text{P.I.}$

$$= C_1\cos x + C_2\sin x + 3x$$
$$-8\sin x \log(\text{cosec}\, x - \cot x)$$

is the complete solution.

Example 1.69 : $(D^2+1)y = cosec\,x$

SPPU : May-19

Solution : Step 1 : C.F. $= C_1 \cos x + C_2 \sin x$

Step 2 : Let P.I. $= u\,y_1 + v\,y_2$

Find derivatives of y_1, y_2

$$y_1 = \cos x \quad y_2 = \sin x$$
$$y_1' = -\sin x \quad y_2' = \cos x$$

Step 3 : Find Δ, Δu, Δv

$$\Delta = y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$$
$$\Delta u = -x\,y_2 = -\,\text{cosec}\,x \sin x = -1$$
$$\Delta v = x\,y_1 = \text{cosec}\,x \cos x = \cot x$$

Step 4 : Find u, v

$$u = \int \frac{\Delta u}{\Delta}\,dx \quad v = \int \frac{\Delta v}{\Delta}\,dx$$
$$= \int -1\,dx, \quad = \int \cot x\,dx$$
$$= -x, \quad = \log \sin x$$

Step 5 : P.I. $= u\,y_1 + v\,y_2$

$$= -x\cos x + \sin x \log \sin x$$

Step 6 : y = C.F. + P.I.

$y = C_1 \cos x + C_2 \sin x - x\cos x + \sin x \log \sin x$

is the complete solution.

Example 1.70 : $(D^2+2D+5)y = 4e^{-x}\tan 2x + 5e^x$

Solution :

Step 1 : $D^2+2D+5 = 0$

$$D = \frac{-2 \pm \sqrt{4-20}}{2}$$
$$= -1 \pm 2i$$
$$\text{C.F} = e^{-x}(C_1 \cos 2x + C_2 \sin 2x)$$

Compare with C.F. $= C_1 y_1 + C_2 y_2$

Step 2 : Assume P.I $= u\,y_1 + v\,y_2$

Write $y_1\ y_2$ and find y_1', y_2'

$$y_1 = e^{-x}\cos 2x, \quad y_2 = e^{-x}\sin 2x$$
$$y_1' = e^{-x}(-\cos 2x - 2\sin 2x)$$
$$y_2' = e^{-x}(-\sin 2x + 2\cos 2x)$$

Step 3 : Find Δ, Δu, Δv

$$\Delta = y_1 y_2' - y_1' y_2$$
$$= e^{-2x}(-\sin 2x\cos 2x + 2\cos^2 2x$$
$$+\sin 2x \cos 2x + 2\sin^2 2x)$$
$$= e^{-2x}\cdot 2$$

$$\Delta u = -X\,y_2$$
$$= -e^{-x}\sin 2x\left(4e^{-x}\tan 2x + 5e^x\right)$$
$$= -4e^{-2x}\frac{\sin^2 2x}{\cos 2x} - 5\sin 2x$$

$$\Delta v = X\,y_1$$
$$= e^{-x}\cos 2x\left(4e^{-x}\tan 2x + 5e^x\right)$$
$$= 4e^{-2x}\sin 2x + 5\cos 2x$$

Step 4 : Find u, v

$$u = \int \frac{\Delta u}{\Delta}\,dx$$
$$= \int \left[\frac{-2\left(1-\cos^2 2x\right)}{\cos 2x} + \frac{5\sin 2x}{2e^{-2x}}\right]dx$$
$$= -2\int \sec 2x\,dx + 2\int \cos 2x\,dx + \frac{5}{2}\int e^{2x}\sin 2x\,dx$$
$$= -\log(\sec 2x + \tan 2x) + \sin 2x + \frac{5}{2}\cdot\frac{e^{2x}}{4+4}[2\sin 2x - 2\cos 2x]$$
$$= -\log(\sec 2x + \tan 2x) + \sin 2x + \frac{5e^{2x}}{8}(\sin 2x - \cos 2x)$$

$$v = \int \frac{\Delta v}{\Delta}\,dx$$
$$= \int \left(2\sin 2x + \frac{5}{2}\,\frac{\cos 2x}{2e^{-2x}}\right)dx$$
$$= -\cos 2x + \int \frac{5}{2}e^{2x}\cos 2x\,dx$$
$$= -\cos 2x + \frac{5}{2}\,\frac{e^{2x}}{4+4}(2\cos 2x + 2\sin 2x)$$
$$= -\cos 2x + \frac{e^{2x}}{8}(\cos 2x + \sin 2x)$$

Step 5 : P.I. $= u\,y_1 + v\,y_2$

$$\text{P.I.} = e^{-x}\cos 2x\begin{bmatrix}-\log(\sec 2x + \tan 2x) + \sin 2x \\ +\frac{5}{8}e^{2x}(\sin 2x - \cos 2x)\end{bmatrix}$$
$$+ e^{-x}\sin 2x\left[-\cos 2x + \frac{e^{2x}}{8}(\cos 2x + \sin 2x)\right]$$
$$= -e^{-x}\cos 2x\left[\log(\sec 2x + \tan 2x)\right] + \frac{5}{8}e^x$$

Step 6 : y = C.F. + P.I.

$$y = e^{-x}(c_1\cos 2x + c_2 \sin 2x)$$

$- e^{-x} \cos 2x\left[\log(\sec 2x + \tan 2x)\right] + \frac{5}{8} e^{x}$

is the complete solution

Exercise 1.10

Solve by method of variation of parameters.

1. $(D^3 + D)y = \cot x$

 [Ans. : $y = C_1 + C_2 \cos x + C_3 \sin x + \log \sin x - \cos x \log(\operatorname{cosec} x - \cot x)$]

2. $(D^3 + D)y = \sec x$

 [Ans. : P.I. = $\log(\sec x + \tan x) + x \cos x + \sin x \log \cos x$]

3. $(D^2 + 9)y = \frac{1}{1 + \sin 3x}$

 [Ans. : $y = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{9}[\sin 3x - 1 - 3x \cos 3x + \sin 3x \log(1 + \sin 3x)]$]

4. $(D^3 + D)y = \sec x \tan x$

 [Ans. : P.I = $\cos \log \cos x + x \sin x + \cos x$]

5. $(D^3 + D)y = \operatorname{cosec}^2 x$ [Ans. : P.I = $-2 \cot x - \tan x \sin x$]

6. $(D^3 + D)y = \tan x$

 [Ans. : P.I = $\log \sec x - 1 - \sin x \log(\sec x + \tan x)$]

7. $(D^2 + 4)y = 4\sec^2 2x$

 [Ans. : $y = C_1 \cos 2x + C_2 \sin 2x - 1 + \sin 2x \log(\sec 2x + \tan 2x)$]

8. $(D^2 - 1)y = (1 + e^{-x})^2$

 [Ans. : $y = C_1 e^{x} + C_2 e^{-x} - 1 + e^{-x} \log(1 + e^{x})$]

9. $(D^2 - 1)y = \frac{2}{1 + e^{x}}$

 [Ans. : $y = C_1 e^{x} + C_2 e^{-x} + e^{x} \log(1 + e^{x}) - e^{-x} \log(1 + e^{x})$]

10. $(D^2 + 3D + 2)y = e^{e^{x}}$ [Ans. : $C_1 e^{-x} + C_2 e^{-2x} + e^{-2x} e^{e^{x}}$]

11. $(D^2 - 1)y = e^{-x} \sin e^{-x} + \cos e^{-x}$

 [Ans. : $y = C_1 e^{x} + C_2 e^{-x} - e^{-x} \sin e^{-x}$]

12. $(D^2 + a^2)y = \tan ax$

 [Ans. : $y = C_1 \cos ax + C_2 \sin ax - \frac{1}{a^2} \log(\sec ax + \tan ax) \cos ax$]

13. $(D^2 + D)y = \frac{1}{1 + e^{x}}$

 [Ans. : $y = C_1 e^{0x} + C_2 e^{-x} + x - (1 + e^{-x}) \log(1 + e^{x})$]

14. $(D^2 + 4)y = \sec 2x$

 [Ans. : $y = C_1 \cos 2x + C_2 \sin 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x \log \cos 2x$]

15. $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

 [Ans. : $y = (C_1 + C_2 x) e^{3x} - e^{3x}[1 + \log x]$]

16. $(D^2 - 4D + 4)y = e^{2x} \sec^2 x$

 [Ans. : $y = [C_1 + C_2 x + \log \sec x] e^{2x}$]

17. $(D^2 - 2D)y = e^{x} \sin x$ [Ans. : $y = C_1 + C_2 e^{2x} - \frac{e^{x}}{2} \sin x$]

18. $(D^2 + 9)y = \operatorname{cosec} 3x$

 [Ans. : $y = C_1 \cos 3x + C_2 \sin 3x - \frac{x}{3} \cos 3x + \frac{1}{9} \sin 3x \log \sin 3x$]

19. $(D^2 + 1)y = x + \tan x$

 [Ans. : $y = C_1 \cos x + C_2 \sin x + x - \cos x \log(\sec x + \tan x)$]

University Solved Examples

Example 1.71 : $(D^3 + 3D)y = 2 \cosh 2x \sinh 2x$

Solution :

Step 1 : A.E. is $D^3 + 3D = 0$

$\therefore$ $D(D^2 + 3) = 0$

$D = 0,\ D = \pm i\sqrt{3}$

Step 2 : y_c = C.F. = $C_1 + C_2 \cos\sqrt{3}\, x + C_3 \sin\sqrt{3}\, x$

Step 3 : The particular integral is

$$\text{P.I.} = y_p = \frac{1}{D(D^2 + 3)} (\sinh 4x)$$

$$= \frac{1}{D(-16 + 3)} (\sinh 4x)$$

$$= -\frac{1}{13} \frac{1}{D} \sinh 4x$$

$$= -\frac{1}{13}\frac{\cosh 4x}{4}$$

$$= -\frac{1}{52}\cosh 4x$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

$$y = C_1 + C_2\cos\sqrt{3}\,x + C_3\sin\sqrt{3}\,x - \frac{1}{52}\cosh 4x$$

Example 1.72 : $(D-1)^3 y = e^x + 5^x - 1$

Solution :

Step 1 : A.E. is $(D-1)^3 = 0$

$\Rightarrow \quad D = 1, 1, 1$

Step 2 : The complementary function is

$$y_c = (C_1 + C_2x + C_3x^2)\,e^x$$

Step 3 : The particular integral is

$$\text{P.I.} = y_p = \frac{1}{(D-1)^3}[e^x + 5^x - 1]$$

$$= \frac{1}{(D-1)^3}e^x + \frac{1}{(D-1)^3}5^x + \frac{1}{(D-1)^3}(-1)$$

$$= \frac{x^3}{3!}e^x + \frac{1}{(\log 5-1)^3}5^x + \frac{1}{(-1)^3}(-1)$$

$$y_p = \frac{x^3}{6}e^x + \frac{1}{(\log 5-1)^3}5^x + 1$$

Step 4 : The complete solution is

$$y = y_c + y_p = (C_1 + C_2x + C_3x^2)\,e^x + \frac{x^3}{6}e^x + \frac{1}{(\log 5-1)^3}5^x + 1$$

Example 1.73 : $(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x}$

Solution :

Step 1 : A.E. is $\quad D^2 - 6D + 9 = 0$

$$(D-3)(D-3) = 0$$

$$D = 3, 3$$

Step 2 : The C.F. is $\quad y_c = (C_1 + C_2x)\,e^{3x}$

Step 3 : The P.I. is

$$y_p = \frac{1}{(D-3)^2}[6e^{3x} + 7e^{-2x}]$$

$$= 6\frac{x^2}{2!}e^{3x} + 7\frac{1}{(-2-3)^2}e^{-2x}$$

$$= 3x^2e^{3x} + \frac{7}{25}e^{-2x}$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

$$= (C_1 + C_2x)\,e^{3x} + 3x^2e^{3x} + \frac{7}{25}e^{-2x}$$

Example 1.74 : $(D^4 + 6D^2 + 8)y = 2\sin^2 x$

Solution :

Step 1 : A.E. is $\quad D^4 + 6D^2 + 8 = 0$

$$(D^2+4)(D^2+2) = 0$$

$$D = \pm 2i,\ D = \pm i\sqrt{2}$$

Step 2 : The C.F. is

$$y_c = (C_1\cos 2x + C_2\sin 2x) + (C_3\cos\sqrt{2}x + C_4\sin\sqrt{2}\,x)$$

Step 3 : The P.I. is

$$y_p = \frac{1}{(D^2+4)(D^2+2)}[1-\cos 2x]$$

$$= \frac{1}{(0+4)(0+2)} - \frac{1}{(D^2+4)(D^2+2)}(\cos 2x)$$

$$= \frac{1}{8} - \frac{1}{(D^2+4)}\left[\frac{1}{-4+2}\cos 2x\right]$$

$$= \frac{1}{8} + \frac{1}{2(D^2+4)}[\cos 2x]$$

$$= \frac{1}{8} + \frac{x}{2(2)}\sin 2x = \frac{1}{8} + \frac{x\sin 2x}{4}$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.75 : $(D^2 - 2D + 5)y = 10\sin x$

Solution :

Step 1 : A.E. is $\quad D^2 - 2D + 5 = 0$

$$D = \frac{-(-2)\pm\sqrt{4-4(5)}}{2} = \frac{2\pm\sqrt{-16}}{2}$$

$$D = \frac{2\pm 4i}{2} = 1\pm 2i$$

Step 2 : The C.F. is $y_c = e^x(C_1\cos 2x + C_2\sin 2x)$

Step 3 : The P.I. is $y_p = \dfrac{1}{D^2-2D+5}(10\sin x)$

$$= (10)\frac{1}{-1-2D+5}\sin x = (10)\frac{1}{4-2D}\sin x$$

$$= 10\frac{4+2D}{16-4D^2}\sin x$$

$$= 10\frac{(4+2D)}{16-4(-1)}\sin x$$

$$= \frac{1}{2}(4\sin x + 2\cos x)$$

$$y_p = 2\sin x + \cos x$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.76 : *Solve* $\dfrac{d^2y}{dx^2} + a^2 y = \dfrac{a^2 R}{P}(l - x)$ *where a, R, P, l are constants, subject to the conditions* $y = \dfrac{dy}{dx} = 0$ *at* $x = 0$.

Solution :

Step 1 : A.E. is $D^2 + a^2 = 0$

$\Rightarrow$ $D = \pm a i$

Step 2 : The C.F. is

$$y = C_1 \cos ax + C_2 \sin ax$$

Step 3 : The P.I. is

$$y_p = \frac{1}{(D^2+a^2)}\left(\frac{a^2 R}{P}(l-x)\right)$$

$$= \frac{a^2 R}{P}\left[\frac{l}{a^2} - \frac{1}{D^2+a^2}x\right]$$

$$= \frac{Rl}{P} - \frac{R}{P}\left(1+\frac{D^2}{a^2}\right)^{-1} x$$

$$y_p = \frac{Rl}{P} - \frac{Rx}{P} = \frac{R}{P}(l - x)$$

The complete solution is $y = y_c + y_p$

Example 1.77 : $(D^3 + 8)y = x^4 + 2x + 1$

Solution :

Step 1 : A.E. is $D^3 + 8 = 0$

By synthetic division method, we get,

$$(D+2)(D^2 - 2D + 4) = 0$$

$\therefore$ $D = -2,\ D = 1 \pm i\sqrt{3}$

Step 2 : The C.F. is

$$y_c = C_1 e^{-2x} + e^{x}[C_2 \cos\sqrt{3}\,x + C_3 \sin\sqrt{3}\,x]$$

Step 3 : The P.I. is

$$y_p = \frac{1}{D^3+8}[x^4 + 2x + 1]$$

$$= \frac{1}{8}\frac{1}{\left(1+\frac{D^3}{8}\right)}[x^4 + 2x + 1]$$

$$= \frac{1}{8}\left[1 - \frac{D^3}{8} + \left(\frac{D^3}{8}\right)^2 \cdots\right][x^4 + 2x + 1]$$

$$= \frac{1}{8}\left[x^4 + 2x + 1 - \frac{1}{8}(24\,x) + 0\right]$$

$$= \frac{1}{8}[x^4 - x + 1]$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.78 : $(D^3 - D^2 - 6D)y = 1 + x^2$

Solution :

Step 1 : A.E. is

$$D^3 - D^2 - 6D = 0$$

$$D(D^2 - D - 6) = 0$$

$$D(D-3)(D+2) = 0$$

$$D = 0, 3, -2$$

Step 2 : The complementary function is

$$y_c = C_1 + C_2 e^{3x} + C_3 e^{-2x}$$

Step 3 : The P.I. is

$$y_p = \frac{1}{D(D-3)(D+2)}(x^2+1)$$

$$= \frac{1}{(D-3)(D+2)}\left[\frac{x^3}{3} + x\right]$$

$$= \left(\frac{\frac{1}{5}}{D-3} - \frac{\frac{1}{5}}{D+2}\right)\left(\frac{x^3}{3} + x\right)$$

$$= \left[-\frac{1}{15}\left(1-\frac{D}{3}\right)^{-1} - \frac{1}{10}\left(1+\frac{D}{2}\right)^{-1}\right]\left[\frac{x^3}{3} = x\right]$$

$$= -\frac{1}{15}\left[1 + \frac{D}{3} + \left(\frac{D}{3}\right)^2 + \left(\frac{D}{3}\right)^3 + \cdots\right]$$

$$-\frac{1}{10}\left[1 + \frac{D}{2} + \left(\frac{D}{2}\right)^2 + \left(\frac{D}{2}\right)^3\right]\left[\frac{x^3}{3} + x\right]$$

$$y_p = -\frac{1}{15}\left[\frac{x^3}{3}+x+\frac{x^2}{3}+1+\frac{2x}{9}+\frac{2}{27}\right]$$

$$-\frac{1}{10}\left[\frac{x^3}{3}+x+\frac{x^2}{2}+1+\frac{2x}{4}+\frac{2}{8}\right]$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.79 : $(D^2-4D+3)y = x^3e^{2x}$ **SPPU : Dec.-17**

Solution : **Step 1 :** A.E. is $D^2-4D+3 = 0$

$$(D-1)(D-3) = 0$$

$$D = 1, 3$$

Step 2 : The C.F. is $y_c = C_1 e^x + C_2 e^{3x}$

Step 3 :

The P.I. is $y_p = \frac{1}{D^2-4D+3}(x^3e^{2x})$

$$= e^{2x}\frac{1}{(D+2)^2-4(D+2)+3}(x^3)$$

$$= e^{2x}\frac{1}{D^2-1}(x^3)$$

$$= e^{2x}(-1)(1-D^2)^{-1}(x^3)$$

$$= -e^{2x}[1+D^2+D^4+D^8+\cdots](x^3)$$

$$= -e^{2x}[x^3+6x+0]$$

$$y_p = -e^{2x}(x^3+6x)$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.80 : $(D^2+D-6)y = e^{2x}\sin 3x$

Solution : **Step 1 :** The A.E is $D^2+D-6 = 0$

$$(D+3)(D-2) = 0$$

$$D = 3, 2$$

Step 2 : The C.F is $y_c = C_1 e^{2x} + C_2 e^{-3x}$

Step 3 : The P.I. is $y_p = \frac{1}{D^2+D-6}e^{2x}\sin 3x$

$$= e^{2x}\frac{1}{(D+2)^2+(D+2)-6}\sin 3x$$

$$= e^{2x}\frac{1}{D^2+5D}\sin 3x$$

$$= e^{2x}\frac{1}{-9+5D}\sin 3x$$

$$= e^{2x}\frac{5D+9}{25D^2-81}\sin 3x$$

$$= e^{2x}\frac{5D+9}{25(-9)-81}\sin 3x$$

$$y_p = \frac{e^{2x}}{-306}[15\cos 3x+9\sin 3x]$$

Step 4 : The complete solution is $y = y_c + y_p$.

Example 1.81 : $(D^2-1)y = \sin x \sin 4x$

Solution :

Step 1 : A.E. is $D^2-1=0 \Rightarrow D=\pm 1$

Step 2 : The C.F. is $y = C_1 e^x + C_2 e^{-x}$

Step 3 : The P.I. is $y_p = \frac{1}{D^2-1}[\sin x \sinh x]$

$$y_p = \frac{1}{D^2-1}\left[\left(\frac{e^x-e^{-x}}{2}\right)\sin x\right]$$

$$= \frac{1}{2(D^2-1)}e^x\sin x - \frac{1}{2}\frac{1}{D^2-1}e^{-x}\sin x$$

$$= \frac{1}{2}e^x\frac{1}{D^2+2D}\sin x - \frac{1}{2}e^{-x}\frac{1}{D^2-2D}\sin x$$

$$= \frac{1}{2}e^x\frac{1}{2D-1}\sin x - \frac{e^x}{2}\frac{1}{-1-2D}\sin x$$

$$= \frac{1}{2}e^x\frac{2D+1}{4D^2-1}\sin x + \frac{e^{-x}}{2}\frac{1-2D}{1-4D^2}\sin x$$

$$= \frac{e^x}{2}\left(-\frac{1}{5}\right)(2D+1)\sin x + \frac{e^{-x}}{2}\left(\frac{1}{5}\right)(1-2D)\sin x$$

$$y_p = -\frac{e^x}{10}[2\cos x+\sin x] + \frac{e^{-x}}{10}[\sin x - 2\cos x]$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.82 : $(D^3-D^2+3D+5)y = e^x\sin 3x$

Solution : **Step 1 :** The A.E. is $D^3-D^2+3D+5 = 0$

$$D = -1, \pm 2i$$

Step 2 : The C.F is $y_c = C_1e^{-x}+C_2\cos 2x + C_3\sin 2x$

Step 3 : The P.I. is $y_p = \frac{1}{D^3-D^2+3D+5}e^x\sin 3x$

$$= e^x\frac{1}{D^3+2D^2-2D+8}\sin 3x$$

$$= e^x\frac{1}{-9D-18-2D+8}\sin 3x \qquad (\because D^2=-9)$$

$$= e^x\frac{1}{-11D-10}\sin 3x$$

$$= -e^x\frac{11D-10}{121D^2-100}\sin 3x = -e^x\frac{11D-10}{-1189}\sin 3x$$

$$y_p = \frac{e^x}{1189}[33\cos 3x - 10\sin 3x]$$

Step 4 : The required complete integral is

$$y = y_c + y_p$$

Example 1.83 : $(D^2 + D - 6)y = e^{-2x}\sin 3x$

Solution : Step 1 : A.E. is $D^2 + D - 6 = 0$

$D = 2, -3$

Step 2 : The C.F. is $y_c = C_1 e^{2x} + C_2 e^{-3x}$

Step 3 : The P.I. is $y_p = \dfrac{1}{D^2 + D - 6} e^{-2x}\sin 3x$

$$y_p = e^{-2x}\frac{1}{(D-2)^2 + (D-2) - 6}\sin 3x$$

$$= e^{-2x}\frac{1}{D^2 - 3D - 4}\sin 3x$$

$$= e^{-2x}\frac{1}{-9 - 3D - 4}\sin 3x$$

$$= -e^{-2x}\frac{1}{3D + 13}\sin 3x$$

$$= -e^{-2x}\frac{3D - 13}{9D^2 - 169}\sin 3x$$

$$= \frac{-e^{-2x}}{-81 - 169}(3D - 13)\sin 3x$$

$$y_p = \frac{e^{-2x}}{250}(9\cos 3x - 13\sin 3x)$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.84 : $(D^2 + 3D + 2)y = x\sin 2x$

Solution : Step 1 : A.E. is $D^2 + 3D + 2 = 0$

$(D + 2)(D + 1) = 1$

$D = -1, -2$

Step 2 : The C.F. is $y_c = C_1 e^{-x} + C_2 e^{-2x}$

Step 3 : The P.I. is $y_p = \dfrac{1}{D^2 + 3D + 2} x \sin 2x$

$$= \left[x - \frac{2D + 3}{D^2 + 3D + 2}\right]\frac{1}{D^2 + 3D + 2}\sin 2x$$

$$= x\frac{1}{D^2 + 3D + 2}\sin 2x - \frac{2D + 3}{(D^2 + 3D + 2)^2}\sin 2x$$

$$= x\frac{1}{3D - 2}\sin 2x - \frac{2D + 3}{(3D - 2)^2}\sin 2x$$

$$= x\frac{3D + 2}{9D^2 - 4}\sin 2x - \frac{2D + 3}{9D^2 - 12D + 4}\sin 2x$$

$$= \frac{x}{-40}(6\cos 2x + 2\sin 2x) - \frac{2D + 3}{-12D - 32}\sin 2x$$

$$= -\frac{x}{20}[3\cos 2x + \sin 2x] + \frac{1}{4}\left(\frac{2D + 3}{3D + 8}\right)\left(\frac{3D - 8}{3D - 8}\right)\sin 2x$$

$$= -\frac{x}{20}[3\cos 2x + \sin 2x] + \frac{1}{4}\frac{6D^2 - 7D - 24}{9D^2 - 64}\sin 2x$$

$$= -\frac{x}{20}[3\cos 2x + \sin 2x] + \frac{1}{4}\frac{1}{(-100)}(6D^2 - 7D - 24)\sin 2x$$

$$y_p = -\frac{x}{20}[3\cos 2x + \sin 2x] - \frac{1}{400}[-24\sin 2x - 14\cos 2x - 24\sin 2x]$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.85 : $(D^2 - 4)y = x\sinh x$

Solution : Step 1 : A.E. is $D^2 - 4 = 0$

$\Rightarrow \quad D = \pm 2$

Step 2 : The C.F. is $y_c = C_1 e^{2x} + C_2 e^{-2x}$

Step 3 : The P.I. is $y_p = \dfrac{1}{D^2 - 4} x \sinh x$

$$= \left[x - \frac{2D}{D^2 - 4}\right]\frac{1}{D^2 - 4}\sinh x$$

$$= \left[x - \frac{2D}{D^2 - 4}\right]\left(\frac{1}{1 - 4}\sinh x\right)$$

$$= \left[x - \frac{2D}{D^2 - 4}\right]\left(-\frac{1}{3}\sinh x\right)$$

$$= -\frac{1}{3}x\sinh x + \frac{2D}{3}\left(-\frac{1}{3}\sinh x\right)$$

$$y_p = -\frac{1}{3}x\sinh x - \frac{2}{9}\cosh x$$

Step 4 : The complete solution is $y = y_c + y_p$

Example 1.86 : $(D^2-4)y = x\ sinh\ 2x$

Solution : **Step 1 :** $y_c = C_1e^{2x} + C_2e^{-2x}$

Step 2 : The P.I. is $y_p = \frac{1}{D^2-4} x \sinh 2x$

$$= \frac{1}{D^2-4} x\left(\frac{e^{2x}-e^{-2x}}{2}\right)$$

$$= \frac{1}{2}e^{2x}\frac{1}{(D+2)^2-4}x - \frac{1}{2}e^{-2x}\frac{1}{(D-2)^2-4}x$$

$$= \frac{e^{2x}}{2}\frac{1}{D^2+4D}x - \frac{e^{-2x}}{2}\frac{1}{D^2-4D}x$$

$$= \frac{e^{2x}}{2}\frac{1}{4D\left(1+\frac{D}{4}\right)}x + \frac{e^{-2x}}{2}\frac{1}{4D\left(1-\frac{D}{4}\right)}x$$

$$= \frac{e^{2x}}{2}\frac{1}{4D}\left[1-\frac{D}{4}+\ldots\right]x$$

$$+\frac{e^{-2x}}{2}\frac{1}{4D}\left[1+\frac{D}{4}+\ldots\right]x$$

$$= \frac{e^{2x}}{2}\frac{1}{4D}\left[x-\frac{1}{4}\right] + \frac{e^{-2x}}{2}\frac{1}{4D}\left[x+\frac{1}{4}\right]$$

$$= \frac{e^{+2x}}{2}\frac{1}{4}\left(\frac{x^2}{2}-\frac{1}{4}x\right) + \frac{e^{-2x}}{2}\frac{1}{4}\left(\frac{x^2}{2}+\frac{1}{4}x\right)$$

$$y_p = \frac{e^{2x}}{8}\left(\frac{x^2}{2}-\frac{1}{4}x\right) + \frac{e^{-2x}}{8}\left(\frac{x^2}{2}+\frac{1}{4}x\right)$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Example 1.87 : $(D^2+4)y = x\sin^2 x$

Solution : **Step 1 :** A.E. is $D^2+4 = 0$

$$D = \pm 2i$$

Step 2 : The C.F. is $y_c = C_1 \cos 2x + C_2 \sin 2x$

Step 3 : The P.I. is $y_p = \frac{1}{D^2+4} x \sin^2 x$

$$= \frac{1}{D^2+4} x\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\frac{1}{D^2+4}x - \frac{1}{2}\frac{1}{D^2+4}x\cos 2x$$

$$= y_{p1} + y_{p2}$$

where $y_{p1} = \frac{1}{2}\frac{1}{D^2+4}x = \frac{1}{8}\frac{1}{1+\frac{D^2}{4}}x = \frac{x}{8}$

$$y_{p2} = -\frac{1}{2}\frac{1}{D^2+4}x\cos 2x$$

$$= \text{Real part of}\left(-\frac{1}{2}\right)\frac{1}{D^2+4}x\,e^{2ix}$$

$$= \text{Real part of}\left(-\frac{1}{2}\right)e^{i2x}\frac{1}{(D+2i)^2+4}x$$

$$= \text{Real part of}\left(-\frac{1}{2}e^{i2x}\right)\frac{1}{D^2+4iD}x$$

$$= \text{Real part of}\left(-\frac{1}{2}e^{+i2x}\right)\frac{1}{4iD}\left(\frac{1}{1-\frac{iD}{4}}\right)x$$

$$= \text{Real part of}\left(-\frac{1}{2}e^{i2x}\right)\frac{1}{4iD}\left[1+\frac{iD}{4}\right]x$$

$$= \text{Real part of}\left(-\frac{1}{2}e^{i2x}\right)\left(\frac{1}{4iD}\right)\left(x+\frac{i}{4}\right)$$

$$= \text{Real part of}\left(-\frac{e^{i2x}i}{4}\right)\left(\frac{x^2}{2}+\frac{ix}{4}\right)$$

$$= \text{Real part of}\ \frac{1}{16}(\cos 2x + i\sin 2x)$$

$$(-x+i\,2x^2)$$

$$y_{p2} = -\frac{1}{16}(-x\cos 2x - 2x^2\sin 2x)$$

$$\therefore \quad y_p = y_{p1} + y_{p2}$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.88 : $(D^2+4)y = x\sin x$

Solution : **Step 1 :** A.E. is $D^2+4 = 0$

$$D = \pm 2i$$

Step 2 : The C.F. is $y_c = C_1 \cos 2x + C_2 \sin 2x$

Step 3 : The P.I. is $y_p = \frac{1}{D^2+4} x \sin x$

$$= \left[x - \frac{2D}{D^2+4}\right]\frac{1}{D^2+4}\sin x$$

$$= \left[x - \frac{2D}{D^2+4}\right]\left[\frac{1}{3}\sin x\right]$$

$$= \frac{1}{3}x\sin x - \frac{2D}{3(D^2+4)}\sin x$$

$$= \frac{1}{3}x\sin x - \frac{2}{3}\frac{D}{3}\sin x$$

$$y_p = \frac{1}{3}x\sin x - \frac{2}{9}\cos x$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.89 : $(D^2+1)y = x^2 \sin 2x$

Solution : Step 1 : A.E. is $D^2+1=0$

$$D = \pm i$$

Step 2 : The C.F. is $y_c = C_1 \cos x + C_2 \sin x$

Step 3 : The P.I. is $y_p = \frac{1}{D^2+1} x^2 \sin 2x$

$$= \text{I.P of } \frac{1}{D^2+1} x^2 e^{i2x}$$

$$= \text{I.P of } e^{i2x} \frac{1}{(D+2i)^2+1}(x^2)$$

$$= \text{I.P of } e^{i2x} \frac{1}{D^2+4iD-3}(x^2)$$

$$= \text{I.P of } \frac{e^{i2x}}{-3} \frac{1}{\left[1-\frac{1}{3}(4iD+D^2)\right]} x^2$$

$$= \text{I.P. of } \frac{e^{i2x}}{-3}\left[1+\frac{1}{3}(4iD+D^2) + \frac{1}{9}(-16D^2+8iD^3+D^4)+\ldots\right]x^2$$

$$= \text{I.P. of } \frac{e^{i2x}}{(-3)}\left[x^2-\frac{26}{9}+i\frac{8}{3}x\right]$$

$$= \text{I.P. of } \left[\frac{\cos 2x + i \sin 2x}{-3}\right]\left(x^2-\frac{26}{9}+i\frac{8}{3}x\right)$$

$$y_p = -\frac{1}{3}\left(x^2-\frac{26}{9}\right)\sin 2x - \frac{8}{9}x\cos 2x$$

Step 4 : The complete integral is

$$y = y_c + y_p$$

Example 1.90 : $(D^5-D)y = 12e^x + 8\sin x - 2x$

Solution : Step 1 : A.E. is $D^5 - D = 0$

$$D(D^4-1) = 0$$

$$D = 0, \pm 1, \pm i$$

Step 2 : The C.F. is

$$y_c = C_1 + C_2 e^x + C_3 e^{-x} + C_4 \cos x + C_5 \sin x$$

Step 3 : The P.I. is

$$y_p = \frac{1}{D^5-D}[12e^x + 8\sin x - 2x]$$

$$= 12\frac{1}{D^5-D}e^x + 8\frac{1}{D^5-D}\sin x - 2\frac{1}{D^5-D}x$$

$$= 12\left[\frac{x}{5D^4-1}e^x\right] + 8\left[\frac{x}{5D^4-1}\sin x\right] + \frac{2}{D}\frac{1}{1-D^4}x$$

$$= 12\left[\frac{x}{5-1}\right]e^x + 8\left[\frac{x}{5(-1)^2-1}\sin x\right] + \frac{2}{D}x$$

$$y_p = 3x e^x + 2x \sin x + x^2$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.91 : $(D^2-1)y = (1+\lambda+\cos x)e^x$

Solution : Step 1 : A.E. is $D^2-1=0$

$$D = \pm 1$$

Step 2 : The C.F. is $y_c = C_1 e^x + C_2 e^{-x}$

Step 3 : The P.I. is $y_p = \frac{1}{D^2-1}(1+x+\cos x)e^x$

$$y_p = e^x \frac{1}{(D+1)^2-1}(1+x+\cos x)$$

$$= e^x \frac{1}{D^2+2D}(1+x+\sin x)$$

$$= e^x\left\{\frac{1}{D^2+2D}1 + \frac{1}{2D}\left(1+\frac{D^2}{2D}\right)^{-1}x + \frac{1}{2D-1}\cos x\right\}$$

$$= e^x\left\{\frac{x}{2D+2}(1) + \frac{1}{2D}\left(x-\frac{1}{2}\right) - \frac{2D+1}{5}\cos x\right\}$$

$$= \frac{xe^x}{2} + \frac{e^x}{2}\left(\frac{x^2}{2}-\frac{x}{2}\right) + \frac{2\sin x}{5} - \frac{1}{5}\cos x$$

$$y_p = \frac{x^2e^x}{4} + \frac{xe^x}{4} + \frac{2\sin x}{5} - \frac{\cos x}{5}$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.92 : $(D^3-3D^2+3D-1)y = \sqrt{x}\, e^x$

Solution : Step 1 : A.E. is $D^3-3D^2+3D-1=0$

$$(D-1)^3 = 0$$

$$D = 1, 1, 1$$

Step 2 : The C.F. is $y_c = (C_1 + C_2 x + C_3 x^2)e^x$

Step 3 : The P.I. is $y_p = \frac{1}{(D-1)^3}\sqrt{x}\, e^x$

$$y_p = e^x \frac{1}{(D+1-1)^3}\sqrt{x} = e^x \frac{1}{D^3}\sqrt{x}$$

$$= e^x \frac{1}{D^2}\int \sqrt{x}\, dx = e^x \frac{1}{D^2}\frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \frac{2}{3} e^x \frac{1}{D} \int x^{\frac{3}{2}} dx = \frac{2}{3} e^x \frac{1}{D} \frac{x^{\frac{5}{2}}}{\frac{5}{2}}$$

$$= \frac{4}{15} e^x \int x^{\frac{5}{2}} dx = \frac{4}{15} e^x \frac{x^{\frac{7}{2}}}{\frac{7}{2}}$$

$$y_p = \frac{4}{15} \times \frac{2}{7} e^x x^{\frac{7}{2}} = \frac{8}{105} e^x x^{\frac{7}{2}}$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.93 : $(D^2 - 2D + 1)y = xe^x \sin x$

Solution : Step 1 : The A.E. is $D^2 - 2D + 1 = 0$
$(D-1)^2 = 0 \Rightarrow D = 1, 1$

Step 2 : The C.F. is $y_c = (C_1 + C_2 x) e^x$

Step 3 : The P.I. is $y_p = \frac{1}{(D-1)^2} x e^x \sin x$

$$= e^x \frac{1}{[D+1-1]^2} x \sin x$$

$$y_p = e^x \frac{1}{D^2} x \sin x$$

$$= e^x \left[x - \frac{2D}{D^2}\right] \frac{1}{D^2} \sin x$$

$$= e^x \left[x - \frac{2D}{D^2}\right] (-\sin x)$$

$$= -e^x \left[x - \frac{2D}{D^2}\right] \sin x$$

$$y_p = -e^x [x \sin x + 2 \cos x]$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.94 : $(D^2 + 13D + 36)y = e^{-4x} + \sin h x$

Solution : Step 1 : A.E. is $D^2 + 13D + 36 = 0$

$$(D+9)(D+4) = 0$$

$$D = -9, -4$$

Step 2 : The C.F. is $y_c = C_1 e^{-4x} + C_2 e^{-9x}$

Step 3 : The P.I. is $y_p = \frac{1}{D^2 + 13D + 36} (e^{-4x} + \sinh x)$

$$= \frac{1}{(D+9)(D+4)} e^{-4x} + \frac{1}{D^2 + 13D + 36} \sinh x$$

$$= x \frac{1}{2D + 13} e^{-4x} + \frac{1}{13D + 37} \sinh x$$

$$= x \frac{1}{-8+13} e^{-4x} + \frac{37 - 13D}{(37)^2 - 169D^2} \sinh x$$

$$y_p = \frac{xe^{-4x}}{5} + \frac{1}{1200} (37 \sinh x - 13 \cosh x)$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.95 : $(D^2 - 1)y = x \sin x + e^x (1 + x^2)$

Solution : Step 1 : A.E. is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

Step 2 : The C.F. is $y_c = C_1 e^x + C_2 e^{-x}$

Step 3 : The P.I. is $y_p = \frac{1}{D^2 - 1} [x \sin x + e^x (1 + x^2)]$

$$= \left[x - \frac{2D}{D^2 - 1}\right] \frac{1}{D^2 - 1} \sin x + e^x \frac{1}{D^2 + 2D} (1 + x^2)$$

$$= \left[x - \frac{2D}{D^2 - 1}\right] \left(-\frac{1}{2} \sin x\right) + e^x \frac{1}{2D} \left(1 + \frac{D}{2}\right)^{-1} (1 + x^2)$$

$$= -\frac{1}{2} \sin x - \frac{1}{4} 2 \cos x + e^x \frac{1}{2D} \left[1 + x^2 + x + \frac{1}{2}\right]$$

$$y_p = -\frac{1}{2} \sin x - \frac{1}{2} \cos x + \frac{e^x}{2} \left[x + \frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{2} x\right]$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.96 : $(D^3 - 4D^2 + 13D)y = 1 + \cos 2x$

Solution : Step 1 : A.E. is $D^3 - 4D^2 + 13D = 0$

$$D(D^2 - 4D + 13) = 0$$

$$D = 0, D = 2 \pm 3i$$

Step 2 : The C.F. is $y_c = C_1 + e^{2x}(C_2 \cos 3x + C_3 \sin 3x)$

Step 3 : The P.I. is $y_p = \frac{1}{D^3 - 4D^2 + 13D} (1 + \cos 2x)$

$$= \frac{1}{(D^2 - 4D + 13)} \cdot \frac{1}{D} (1) + \frac{1}{-4D - 4(-4) + 13D} \cos 2x$$

$$= \frac{1}{13} \frac{1}{1 + \left(\frac{D^2 - 4D}{13}\right)} (x) + \frac{1}{16 + 9D} \cos 2x$$

$$= \frac{1}{13} \left[x - \frac{4}{13}\right] + \frac{16 - 9D}{256 - 81(-4)} \cos 2x$$

$$y_p = \frac{1}{13} \left[x - \frac{4}{13}\right] + \frac{1}{580} [16 \cos 2x + 9 \times 2 \sin 2x]$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.97 : $(D^5 - D)y = 12a^x + 8\sin x - 2$

Solution : Step 1 : A.E. is $D(D^4 - 1) = 0$

$D = 0,\ D = \pm 1,\ \pm i$

Step 2 : The C.F. is

$$y = C_1 + C_2 e^x + C_3 e^{-x} + C_4 \cos x + C_5 \sin x$$

Step 3 : The P.I. is

$$y_p = \frac{1}{(D^2-1)(D^2+1)(D)}[12a^x + 8\sin x - 2]$$

$$= \frac{12a^x}{(\log a)^5 - \log a} + 8x\frac{1}{5D^4 - 1}\sin x - \frac{1}{(0-1)^2}\frac{1}{D}(2)$$

$$y_p = \frac{12\,a^x}{(\log a)^5 - \log a} + 2x\sin x - 2x$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.98 : $(D^2 + 3D + 2)y = e^x + \cos e^{e^x}$

Solution : Step 1 : A.E. is $D^2 + 3D + 2 = 0$

$(D+2)(D+1) = 0$

$D = -2, -1$

Step 2 : The C.F. is $y_c = C_1 e^{-2x} + C_2 e^{-x}$

Step 3 : The P.I. is

$$y_p = \frac{1}{(D+1)(D+2)}[e^{e^x} + \cos e^x]$$

$$= \frac{1}{D+2}e^{-x}\int e^x (e^{e^x} + \cos e^x)\,dx$$

$$= \frac{1}{D+2}e^{-x}\left(e^{e^x} + \sin e^x\right)$$

$$= e^{-2x}\int e^{2x} e^{-x}\left(e^{e^x} + \sin e^x\right)dx$$

$$= e^{-2x}\int e^x\left(e^{e^x} + \sin e^x\right)dx$$

$$y_p = e^{-2x}\left[e^{e^x} - \cos e^x\right]$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.99 : $(D^2 + 3D + 2)y = \sin e^x$

Solution : Step 1 : A.E. is $D^3 + 3D + 2 = 0$

$D = -1, -2$

Step 2 : The C.F. is $y_c = C_1 e^{-x} + C_2 e^{-2x}$

Step 3 : The P.I. is $y_p = \frac{1}{(D+2)(D+1)}\sin e^x$

$$= \frac{1}{D+2}e^{-x}\int e^x \sin e^x\,dx$$

$$= \frac{1}{D+2}e^{-x}[-\cos e^x]$$

$$= -e^{-2x}\int e^{2x}e^{-x}\cos e^x\,dx$$

$$= -e^{-2x}\int e^x \cos e^x dx$$

$$y_p = -e^{-2x}\sin e^x$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.100 : $(D^2 + D)y = \frac{1}{1+e^x}$

Solution : Step 1 : A.E. is $D(D+1) = 0$

$D = 0, -1$

Step 2 : The C.F. is $y_c = C_1 + C_2 e^{-x}$

Step 3 : The P.I. is $y_p = \frac{1}{D(D+1)}\frac{1}{1+e^x}$

$$= \left(\frac{1}{D} - \frac{1}{D+1}\right)\frac{1}{1+e^x}$$

$$= \int \frac{1}{1+e^x}dx - e^{-x}\int e^x \frac{1}{1+e^x}dx$$

$$= \int \frac{e^x}{e^x(1+e^x)}dx - e^{-x}\log(1+e^x)$$

Put $1 + e^x = t,\ e^x dx = dt$

$$= \int \frac{dt}{t(t-1)} - e^{-x}\log(1+e^x)$$

$$= \int\left(\frac{1}{t-1} - \frac{1}{t}\right)dt - e^{-x}\log(1+e^x)$$

$$= \log(t-1) - \log(t) - e^{-x}\log(1+e^x)$$

$$= \log(e^x) - \log(1+e^x) - e^{-x}\log(1+e^x)$$

$$y_p = x - \log(1+e^x) - e^x\log(1+e^x)$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.101 : $(D^2 - 1)y = \frac{2}{1+e^x}$

Solution : Step 1 : A.E. is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

Step 2 : The C.F. is $y_c = C_1 e^x + C_2 e^x$

Step 3 : The P.I. is $y_p = \frac{1}{D^2-1}\frac{2}{1+e^x}$

$$= \left(\frac{\frac{1}{2}}{D-1} - \frac{\frac{1}{2}}{D+1}\right)\frac{2}{1+e^x}$$

$$= \left(\frac{1}{D-1} - \frac{1}{D+1}\right)\frac{1}{1+e^x}$$

$$= e^x \int e^x \frac{1}{1+e^x} dx - e^{-x} \int e^x \frac{1}{1+e^x} dx$$

Put $e^x = t$ in 1^{st} integral $\therefore e^x dx = dt$

$$= e^x \int \frac{1}{t^2(1+t)} dt - e^x \log(1+e^x)$$

$$= e^x \int \left[\frac{1}{t+1} + \frac{(-t+1)}{t^2}\right] dt - e^{-x} \log(1+e^x)$$

$$= e^x \left[\log(t+1) - \int \frac{1}{t} dt + \int t^{-2} dt\right] - e^{-x} \log(1+e^x)$$

$$= e^x \left[\log(t+1) - \log t + \frac{t^{-1}}{-1}\right] - e^{-x} \log(1+e^x)$$

$$y_p = e^x[\log(1+e^x) - 1 - e^{-x}] - e^{-x} \log(1+e^x)$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.102 : $(D^2+9)y = \text{cosec } 3x$

Solution : Step 1 : A.E. is $D^2 + 9 = 0 \Rightarrow D = \pm 3i$

Step 2 : The C.F. is $y_c = C_1 \cos 3x + C_2 \sin 3x$

Step 3 : The P.I. is $y_p = \frac{1}{D^2+9} \text{cosec } 3x$

$$= \frac{1}{(D+3i)(D+3i)} \text{cosec } 3x$$

$$y_p = \frac{1}{6i}\left[\frac{1}{D-3i} - \frac{1}{D+3i}\right] \text{cosec } 3x$$

$$= \frac{1}{6i}\left[\frac{1}{D-3i} \text{cosec } 3x - \frac{1}{D+3i} \text{cosec } 3x\right]$$

$$= \frac{1}{6i}[I_1 + I_2]$$

Consider $I_1 = \frac{1}{D-3i} \text{cosec } 3x$

$$= e^{3ix} \int e^{-3ix} \text{cosec } 3x \, dx$$

$$= e^{3ix} \int [\cos 3x - i \sin 3x] \text{cosec } 3x \, dx$$

$$= e^{3ix} \int (\cot 3x - i) \, dx$$

$$= e^{3ix}\left(\frac{\log \sin 3x}{3} - ix\right)$$

Similarly $I_2 = e^{-3ix}\left(\frac{\log \sin 3x}{3} + ix\right)$

$$\therefore y_p = \frac{1}{6i}\left[e^{3ix}\left(\frac{\log \sin 3x}{3} - ix\right) + e^{-3ix}\left(\frac{\log \sin 3x}{3} + ix\right)\right]$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.103 : $(D^2 - 3D + 2)y = \cos e^{-x}$

Solution : Step 1 : A.E. is $D^2 - 3D + 2 = 0$

$$\Rightarrow (D-2)(D-1) = 0$$

$$D = 1, 2$$

Step 2 : The C.F. is $y_c = C_1 e^x + C_2 e^{2x}$

Step 3 : The P.I. is $y_p = \frac{1}{(D-2)(D-1)} \cos e^{-x}$

$$= \frac{1}{D-2} e^x \int e^{-x} \cos e^{-x} \, dx$$

$$= -\frac{1}{D-2} e^x \sin e^{-x}$$

$$= -e^{2x} \int e^{-2x} e^x \sin e^{-x} \, dx$$

$$= -e^{2x} \int e^{-x} \sin e^{-x} \, dx$$

$$= +e^{2x} \cos e^{-x}$$

$$= e^{2x} \cos e^{-x}$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

Example 1.104 : $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = \cos(\log x) + x \sin(\log x)$

Solution : Step 1 : Given D.E. is Caucky's D.E., so we use substitution $x = e^z \Rightarrow z = \log x$ and $D \equiv \frac{d}{dz}$.

$\therefore$ We get $x\frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$ given DE becomes,

$$D(D-1)y - Dy + 4y = \cos z + e^z \sin z$$

$$(D^2 - 2D + 4)y = \cos z + e^z \sin z \quad \ldots (1)$$

which is linear D.E. with constant coefficients.

Step 2 : A.E. of equation (1) is $D^2 - 2D + 4 = 0$

$$\Rightarrow \quad D = 1 \pm i\sqrt{3}$$

Step 3 : The C.F. of equation (1) is

$$y_c = e^z[C_1 \cos\sqrt{3} z + C_2 \sin\sqrt{3} z]$$

Step 4 : The P.I. of equation (1) is

$$y_p = \frac{1}{D^2 - 2D + 4}(\cos z + e^z \sin z)$$

$$= \frac{1}{D^2 - 2D + 4}\cos z + \frac{1}{D^2 - 2D + 4} e^z \sin z$$

$$y_p = \frac{1}{3 - 2D}\cos z + e^z \frac{1}{D^2 + 3}\sin z$$

$$= \frac{2D + 3}{9 - 4D^2}\cos z + e^z \frac{1}{-1 + 3}\sin z$$

$$= \frac{1}{13}(-2\sin z + 3\cos z) + \frac{e^z}{2}\sin z$$

$$y_p = \frac{1}{13}[-2\sin(\log x) + 3\cos(\log x)] + \frac{x}{2}\sin(\log x)$$

Step 5 : The complete solution is

$$y = y_c + y_p$$

Example 1.105 : $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

Solution : Step 1 : Given D.E. is Cauchy's D.E., so put

$x = e^z \Rightarrow z = \log x$ and $D \equiv \frac{d}{dz}$.

$x^2 \frac{d^2y}{dx^2} = D(D-1)\,y$ and $x^3 \frac{d^3y}{dx^3} y = D(D-1)(D-2)y$

$\therefore$ Given D.E. becomes,

$[D(D-1)(D-2) + 2D(D-1) + 2]\,y = 10[e^z + e^{-z}]$ ... (1)

which is L.D.E with constant coefficients.

Step 2 : The A. E. of equation (1) is

$D^3 - D^2 + 2 = 0 \Rightarrow D = -1,\ 1 \pm i$

Step 3 : The C.F. is

$$y_c = C_1 e^{-z} + e^z[C_2 \cos z + C_3 \sin z]$$

Step 4 : The P.I. is

$$y_p = \frac{1}{D^3 - D^2 + 2} 10\,[e^z + e^{-z}]$$

$$= 10\left[\frac{1}{1 - 1 + 2} e^z + \frac{1}{3D^2 - 2D} e^{-z}\right]$$

$$= 5e^z + z\frac{10}{3 + 2} e^{-z}$$

$$= 5e^z + 2ze^{-z}$$

$$y_p = 5x + \frac{2}{x}\log x$$

Step 5 : The complete solution is

$$y = y_c + y_p$$

Example 1.106 : $x^2 \frac{d^2y}{dx} - 4x\frac{dy}{dx} + 6y = x^5$

SPPU : Dec.-17, May-17

Solution : Step 1 : Given D.E. is Cauchy's D.E. so, put

$x = e^z$, $z = \log x$, $\frac{d}{dz} \equiv D$

and $\quad x\frac{dy}{dx} = Dy,\ x^2 \frac{d^2y}{dx^2} = D(D-1)y$

Given D.E. becomes,

$$[D(D-1) - 4D + 6]y = e^{5z}$$

i.e. $$(D^2 - 5D + 6)y = e^{5z}$$

which is linear D.E. with constant coefficients

Step 2 : A.E. is $D^2 - 5D + 6 = 0 \Rightarrow$

$$(D-2)(D-3) = 0$$

$$D = 2, 3$$

Step 3 : The C.F. is $y_c = C_1 e^{2z} + C_2 e^{3z}$

Step 4 : The P.I. is $y_p = \frac{1}{(D-2)(D-3)} e^{5z}$

$$= \frac{1}{(5-2)(5-3)} e^{5z} = \frac{e^{5z}}{6}$$

Step 5 : The complete solution is

$$y = y_c + y_p = C_1 e^{2z} + C_2 e^{3z} + \frac{e^{5z}}{6}$$

$$y = C_1 x^2 + C_2 x^3 + \frac{x^5}{6}$$

Example 1.107 : $x^2 \frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 \log x$

Solution : Step 1 : We have

$$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^3 \log x$$

which is Cauchy's D.E.

Put, $x = e^z$, $z = \log x$, $D \equiv \frac{d}{dz}$

$\therefore$ We get

$$(D(D-1)(D-2) + 3D(D-1) + D)y = e^{3z} z$$

$$[D^3 - 3D^2 + 2D + 3D^2 - 3D + D]y = e^{3z} z$$

$$D^3 y = ze^{3z}$$

which is linear D.E. with constant coefficients.

Step 2 : A.E. is $D^3 = 0 \Rightarrow D = 0, 0, 0$

Step 3 : The C.F. is $y_c = C_1 + C_2 z + C_3 z^2$

Step 4 : The P.I. is

$$y_p = \frac{1}{D^3} z e^{3z}$$

$$y_p = \frac{1}{D^2}\int z e^{3z}\, dz$$

$$= \frac{1}{D^2}\left[z\frac{e^{3z}}{3} - (1)\frac{e^{3z}}{9}\right]$$

$$= \frac{1}{D}\int\left(\frac{ze^{3z}}{3} - \frac{e^{3z}}{9}\right)dz$$

$$= \frac{1}{D}\left[z\frac{e^{3z}}{9} - (1)\frac{e^{3z}}{27} - \frac{e^{3z}}{27}\right]$$

$$= z\frac{e^{3z}}{27} - \frac{e^{3z}}{81} - \frac{z\,e^{3z}}{81}$$

$$= z\frac{e^{3z}}{27} - \frac{e^{3z}}{27} = \frac{2^{3z}}{27}(z-1)$$

Step 4 : The complete solution is

$$y = y_c + y_p$$

$$= C_1 + C_2 z + C_3 z^2 + \frac{e^{3z}}{27}(z-1)$$

$$= C_1 + C_2 \log x + C_3 (\log x)^2 + \frac{x^3}{27}(\log x - 1)$$

Example 1.108 : $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = -x^4 \sin x$

Solution : Step 1 : Given D.E. is Cauchy's D.E. so put $x = e^z$

$$z = \log x \text{ and } D \equiv \frac{d}{dz}$$

$\therefore$ Given D.E. becomes,

$$(D^2 - 5D + 6)y = -e^{4z} \sin e^z$$

which is L.D.E. with constant coefficients

Step 2 : A.E. is $D^2 - 5D + 6 = 0$

$$(D-2)(D-3) = 0$$

$$\Rightarrow \quad D = 2, 3$$

Step 3 : The C.F. is $y_c = C_1 e^{2z} + C_2 e^{3z}$

Step 4 : The P.I. is $y_p = \frac{-1}{(D-2)(D-3)} e^{4z} \sin e^z$

$$= -e^{4z}\frac{1}{(D+2)(D+1)}\sin e^z$$

$$= -e^{4z}\frac{1}{D+2} e^{-z}\int e^z \sin e^z\, dz$$

$$= -e^{4z}\frac{1}{D+2} e^{-z}(-\cos e^z) \quad (\because \text{Put } e^z = t)$$

$$= +e^{4z} e^{-2z}\int e^{2z} e^{-z} \cos e^z\, dz$$

$$= e^{2z}\int e^z \cos e^z\, dz = e^{2z} \sin e^z$$

Step 5 : The complete solution is $y = y_c + y_p$

$$y = C_1 x^2 + C_2 x^3 + x^2 \sin x$$

Example 1.109 : $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

Solution : Step 1 : Given D.E. is Cauchy's D.E so put

$1 + x = e^z \Rightarrow z = \log(1+x)$ and $\frac{d}{dz} \equiv D$

$$\therefore \quad (1+x)^2\frac{d^2y}{dx^2} = D(D-1)y \text{ and } (1+x)\frac{dy}{dx} = Dy$$

$\therefore$ Given D.E. becomes

$$(D^2 + 1)y = 2 \sin z$$

which is L.D.E. with constant coefficients

Step 2 : A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

Step 3 : The C.F. is $y_c = C_1 \cos z + C_2 \sin z$

Step 4 : The P.I is $y_p = \frac{1}{D^2+1} 2 \sin z = 2z\frac{1}{2D}\sin z$

$$y_p = -z \cos z$$

Step 5 : The complete solution is $y = y_c + y_p$

$$y = C_1 \cos\log(1+x) + C_2 \sin\log(1+x) - \log(1+x)\cos\log(1+x)$$

Example 1.110 : $(D^2+4)y = \tan 2x$ *[by variation of parameters]* **SPPU : Dec.-18**

Solution : Step 1 : A.E. is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$$\therefore \quad y_c = C_1 \cos 2x + C_2 \sin 2x$$

Step 2 : Comparing with $y_c = C_1 y_1 + C_2 y_2$

$$\therefore \quad y_1 = \cos 2x,\ y_2 = \sin 2x$$

$$y_1' = -2\sin 2x,\ y_2' = 2\cos 2x$$

Step 3 : Assume P.I. $= uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv where

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$\Delta = 2\cos 2x \cos 2x + 2 \sin 2x \sin 2x = 2$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = -xy_2 = -\tan 2x \sin 2x$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = xy_1 = \tan 2x \cos 2x = \sin 2x$$

Step 5 : $u = \int \frac{\Delta u}{\Delta} dx = -\int \frac{\sin 2x \tan 2x}{2} dx$

$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx = -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x} dx$

$= -\frac{1}{2}\int (\sec 2x - \cos 2x) dx$

$u = -\frac{1}{2}\frac{\log(\sec 2x + \tan 2x)}{2} + \frac{1}{2}\frac{\sin 2x}{2}$

and $v = \int \frac{\Delta v}{\Delta} dx = \int \frac{\sin 2x}{2} dx$

$= \frac{-1}{2}\frac{\cos 2x}{2} = -\frac{1}{4}\cos 2x$

Step 6 : ∴ The P.I. is

$y_p = -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$

$+\frac{1}{4}\sin 2x \cos 2x - \frac{1}{4}\sin 2x \cos 2x$

$y_p = -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$

Step 7 : The complete solution is

$y = y_c + y_p$

$y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$

Example 1.111 : $(D^2 + a^2)y = \sec ax$ *(by variation of parameters)* **SPPU : Dec.-17**

Solution : Step 1 : A.E. is $D^2 + a^2 = 0$

$\Rightarrow$ $D = \pm ai$

∴ $y_c = C_1 \cos ax + C_2 \sin ax$

Step 2 : Comparing with $y_c = C_1 y_1 + C_2 y_2$

∴ $y_1 = \cos ax, \; y_2 = \sin ax$

$y_1' = -a \sin ax, \; y_2' = a \cos ax$

Step 3 : Let assume that $y_p = uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = a$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = -x y_2 = -\sec ax \sin ax$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = xy_1 = \sec ax \cos ax = 1$$

Step 5 : $u = \int \frac{\Delta u}{\Delta} dx = -\int \frac{\sec ax \sin ax}{a} dx$

$= -\frac{1}{a}\int \tan ax \, dx = -\frac{1}{a^2}\log \sec ax$

$v = \int \frac{\Delta v}{\Delta} dx = \frac{1}{a}\int dx = \frac{x}{a}$

Step 6 : The P.I. is

$y_p = -\frac{1}{a^2}\cos ax \log \sec ax + \frac{x}{a}\sin ax$

Step 7 : The complete solution is

$y = y_c + y_p$

$= C_1 \cos ax + C_2 \sin ax - \frac{1}{a^2}\cos ax \log \sec ax$

$+\frac{x}{a}\sin ax$

Example 1.112 : $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$ *(by variation of parameters)*

Solution : Step 1 : A.E. is $D^2 - 6D + 9 = 0 \Rightarrow D = 3, 3$

$y_c = (C_1 x + C_2)e^{3x} = C_1 x e^{3x} + C_2 e^{3x}$

Step 2 : Comparing y_c with $y_c = C_1 y_1 + C_2 y_2$ we get

$y_1 = xe^{3x}, \; y_2 = e^{3x}$

$y_1' = 3x e^{3x} + e^{3x}, \; y_2' = 3e^{3x}$

Step 3 : Assume that $y_p = uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = -e^{6x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = -xy_2 = -\frac{e^{3x}}{x^2}e^{3x} = \frac{-e^{6x}}{x^2}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = xy_1 = \frac{e^{3x}}{x^2}xe^{3x} = \frac{e^{6x}}{x}$$

Step 5 : Find u, v

$u = \int \frac{\Delta u}{\Delta} dx = \int -\frac{e^{6x}}{x^2}\frac{1}{-e^{6x}} dx = \int \frac{1}{x^2} = -\frac{1}{x}$

$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{e^{6x}}{x}\frac{1}{-e^{6x}} dx = -\int \frac{1}{x} dx$

$v = -\log x$

Step 6 : The P.I. is

$y_p = -\frac{1}{x} x e^{3x} - (\log x)(e^{3x})$

$= -e^{3x}(1 + \log x)$

Step 7 : The complete solution is

$y = y_c + y_p$

$= C_1 x e^{3x} + C_2 e^{3x} - e^{3x}(1 + \log x)$

Example 1.113 : $(D^2-1)y = \frac{1}{(1+e^{-x})^2}$

(by variation of parameters)

Solution : Step 1 : A.E. is $D^2-1=0 \Rightarrow D=\pm 1$

$$y_c = C_1 e^x + C_2 e^{-x}$$

Step 2 : Comparing y_c with $y_c = C_1 y_1 + C_2 y_2$

We get $y_1 = e^x$, $y_2 = e^{-x}$

$$y_1' = e^x, \; y_2' = -e^{-x}$$

Step 3 : Assume P.I. $= uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -1-1=-2$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = -e^{-x}(1+e^{-x})^{-2}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = e^x(1+e^{-x})^{-2}$$

Step 5 : Find u and v

$$u = \int \frac{\Delta u}{\Delta} dx = \frac{1}{2}\int \frac{e^{-x}}{(1+e^{-x})^2} dx$$

$$= \frac{1}{2}\int \frac{1}{e^x(1+e^{-x})^2} dx = \frac{1}{2}\int \frac{e^x}{(e^x+1)^2} dx$$

Put $e^x = t$, $e^x \; dx = dt$

$$u = \frac{1}{2}\int \frac{du}{(u+1)^2} = \frac{1}{2}\left(-\frac{1}{u+1}\right) = -\frac{1}{2(1+e^x)}$$

And

$$v = -\frac{1}{2}\int \frac{e^x dx}{(1+e^{-x})^2} = -\frac{1}{2}\int \frac{e^x}{(1+e^{-x})^2} dx$$

Put $e^x = t$, $e^x dx = dt$

$$\therefore \quad v = -\frac{1}{2}\int \frac{dt}{\left(1+\frac{1}{t}\right)^2} = -\frac{1}{2}\int t^2\left(\frac{1}{(t+1)^2}\right) dt$$

$$v = -\frac{1}{2}\left[t^2\left(-\frac{1}{t+1}\right) + (2t)\log(t+1)\right.$$

$$\left. -2(1+t)\log(t+1)+2t\right]$$

$$v = -\frac{1}{2}\left[-\frac{t^2}{t+1} - 2\log(t+1) + 2t\right]$$

$$v = \frac{1}{2}\frac{e^{2x}}{e^x+1} + \log(e^x+1) - 1e^x$$

Step 6 : The P.I. is

$$y = uy_1 + vy_2$$

$$= e^x\left(\frac{-1}{2(1+e^x)}\right) + e^{-x}\left[\frac{1}{2}\frac{e^{2x}}{e^x+1} + \log(e^x+1) - e^x\right]$$

$$= -1 + e^{-x}\log(e^x-1)$$

Step 7 : The complete solution is

$$y = y_c + y_p$$

$$= C_1 e^x + C_2 e^{-x} - 1 + e^{-x}\log(e^x+1)$$

Example 1.114 : $(D^2+3D+2)y = e^{e^x}$ *[by variation of parameters]* **SPPU : May-18**

Solution : Step 1 : A.E. is $(D+2)(D+1)=0$

$$D = -1, -2$$

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

Step 2 : Comparing y_c with $y_c = C_1 y_1 + C_2 y_2$

We have $y_1 = e^{-x}$, $y_2 = e^{-2x}$

$$y_1' = -e^{-x}, \; y_2' = -2e^{-2x}$$

Step 3 : Assume P.I. $= y_p = uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -e^{-3x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = -xy_2 = -e^{-2x}e^{e^x}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = xy_1 = e^{e^x}e^{-x}$$

Step 5 : Find u and v

$$u = \int \frac{\Delta u}{\Delta} dx = +\int \frac{e^{e^x}e^{-2x}}{e^{-3x}} dx = \int e^{e^x}e^x \; dx$$

$$\therefore \quad u = e^{e^x}$$

And similarly $v = -e^x e^{e^x} + e^{e^x}$

Step 6 : $y_p = e^{-x}e^{e^x} + e^{-2x}(-e^x e^{e^x}) + e^{-2x}e^{e^x}$

$$y_p = e^{-2x}e^{e^x}$$

Step 7 : The complete solution is

$$y = y_c + y_p$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x}e^{e^x}$$

Example 1.115 : $(D^2+6D+9) = y = 5^x - \log^2$

Solution : Step 1 : Auxiliary equation is

$$D^2+6D+9 = 0$$

$$\Rightarrow \quad (D+3)^2 = 0$$

$$D = -3, -3$$

$D = -3$ is a real and repeated twice.

Step 2 : The complementary function is

$$y_c = (C_1 + C_2 x) e^{-3x} \quad \dots (1)$$

Step 3 : The particular integral is

$$y_p = \frac{1}{f(D)} X = \frac{1}{(D+3)^2} [5^x - \log 2]$$

$$= \frac{1}{(D+3)^2} (5^x) - \frac{1}{(D+3)^2} (\log 2)$$

$$= \frac{1}{(D+3)^2} (e^{x \log 5}) - (\log 2) \frac{1}{(D+3)^2} (e^{0x})$$

$$= \frac{1}{(\log 5 + 3)^2} (5^x) - \frac{\log 2}{(0+3)^2}$$

$$y_p = \frac{5^x}{(\log 5 + 3)^2} - \frac{\log 2}{9}$$

Step 4 : The solution of given D.E. is

$$y = y_c + y_p$$

$$y = (C_1 + C_2 x) e^{-3x} + \frac{5^x}{(\log 5 + 3)^2} - \frac{\log 2}{9}$$

Example 1.116 : $(D-1)^2 (D^2+1) y = e^x + \sin^2(x/2)$

Solution : Step 1 : Auxilliary equation is

$$(D-1)^2 (D^2+1) = 0$$

$$(D-1)^2 = 0 \quad \text{or} \quad D^2 + 1 = 0$$

$$D = 1, 1 \quad \text{or} \quad D = \pm i$$

Two real roots and two complex roots.

Step 2 : The complementary function is

$$y_c = (C_1 + C_2 x) e^x + C_3 \cos x + C_4 \sin x \quad \dots (1)$$

Step 3 : $y_p = \frac{1}{f(D)} X$

$$= \frac{1}{(D-1)^2 (D^2+1)} [e^x + \sin^2(x/2)]$$

$$y_p = \frac{1}{(D-1)^2 (D^2+1)} e^x + \frac{1}{(D-1)^2 (D^2+1)} \left(\frac{1-\cos x}{2}\right)$$

$$= \frac{1}{(D-1)^2 (1+1)} e^x + \frac{1}{2} \frac{1}{(0-1)^2 (0+1)}$$

$$- \frac{1}{2} \frac{1}{(D-1)^2 (D^2+1)} \cos x$$

$$= \frac{1}{2} \frac{x^2}{2!} e^x + \frac{1}{2} - \frac{1}{2} \frac{1}{2(D^2+1)} \frac{1}{(-1-1)^2} \cos x$$

$$= \frac{x^2 e^x}{4} + \frac{1}{2} - \frac{1}{8} \left[\frac{1}{D^2+1} \cos x\right]$$

$$= \frac{x^2 e^x}{4} + \frac{1}{2} - \frac{1}{8} \left[\frac{x}{2(1)} \sin x\right]$$

$$y_p = \frac{x^2 e^x}{4} - \frac{x}{16} \sin x + \frac{1}{2}$$

Step 4 : Hence the complete solution is

$$y = y_c + y_p$$

$$y = (C_1 + C_2 x) e^x + C_3 \cos x + C_4 \sin x + \frac{x^2 e^x}{4} - \frac{x \sin x}{16} + \frac{1}{2}$$

Example 1.117 : $(D^3 - 4D) y = 2 \cosh^2(2x)$

Solution : Step 1 : A.E. is $D^3 - 4D = 0$

$$D(D^2 - 4) = 0$$

$$D(D-2)(D+2) = 0$$

$D = 0, 2, -2$; All roots are real and distinct.

Step 2 : $y_c = C_1 + C_2 e^{2x} + C_3 e^{-2x}$

Step 3 : $y_p = \frac{1}{D(D^2-4)} (2 \cosh^2 2x)$

$$= \frac{1}{D(D^2-4)} [1 + \cosh 2x]$$

$$= \frac{1}{D(D^2-4)} (1) + \frac{1}{D(D^2-4)} \cosh 2x$$

$$= \frac{1}{-4} \frac{1}{D} e^{0x} \frac{1}{D^2-4} \frac{\sin 2x}{2}$$

$$y_p = -\frac{1}{4} x e^{0x} + \frac{1}{2} \frac{x D}{2D^2} \sinh 2x$$

Step 4 : Hence the complete solution is

$$y = y_c + y_p$$

$$= -\frac{1}{4} x + \frac{x}{4(2)^2} D \sinh 2x$$

$$= -\frac{1}{4} x + \frac{2x}{16} \cosh 2x$$

$$y_p = -\frac{1}{4} x + \frac{x}{8} \cosh 2x$$

Example 1.118 : *Solve the following differential equations :*

i) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{-3x} \cos 4x + 6e^{2x}$

ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 16y = x^2 + 2^{\log x} + 4\cosh(\log x)$

iii) $\frac{d^2y}{dx^2}+y$ = cosec x, (by using method of variation of parameters) **SPPU : May-19, Marks 8**

Solution : i) A.E. is

$$D^2+6D+9 = 0$$
$$(D+3)(D+3) = 0$$
$$D = -3, -3$$

$\therefore$ $$y_c = (C_1+C_2x)e^{-3x}$$

and $$y_p = \frac{1}{(D+3)^2}[e^{-3x}\cos 4x+6e^{2x}]$$
$$= e^{-3x}\frac{1}{(D-3+3)^2}\cos 4x+\frac{1}{(D+3)^2}6e^{2x}$$
$$= e^{-3x}\frac{1}{D^2}\cos 4x+6\frac{1}{(2+3)^2}e^{2x}$$
$$y_p = e^{-3x}\left(-\frac{1}{16}\right)\cos 4x+\frac{6}{25}e^{2x}$$

$\therefore$ The complete solution is $y = y_c+y_p$

ii) Put $x = e^z \Rightarrow z = \log x$

$\therefore$ $$x\frac{dy}{dx} = D\,y \text{ where } D = \frac{d}{dz}$$
$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$ Given equation becomes,

$$D(D-1)y + Dy + 16y = e^{2z}+z^z+4\cos h\, z$$
$$(D^2+16)y = e^{2z}+z^z+4\cos h\, z \quad ...(1)$$

$\therefore$ A.E. is $D^2+16 = 0$

$$D^2 = -16 = 16\, i^z$$
$$D = \pm 4\, i$$

$\therefore$ $$y_c = C_1\cos 4z+C_2\sin 4z$$
$$y_p = \frac{1}{D^2+16}[e^{2z}+2^z+4\cos h\, z]$$
$$= \frac{1}{4+16}e^{2z}+\frac{1}{(\log 2)^2+16}2^z+4\frac{1}{1+16}\cos h\, z$$
$$y_p = \frac{1}{20}e^{2z}+\frac{2^z}{(\log 2)^2+16}+\frac{4}{17}\cos h\, z$$

$\therefore$ The complete solution is $y = y_c+y_p$.

iii) **Step 1 :** C.F. = $C_1\cos x+C_2\sin x$

Step 2 : Let P.I. = $u\,y_1+v\,y_2$

Step 3 : Find derivatives of y_1, y_2

$$y_1 = \cos x, \quad y_2 = \sin x$$
$$y_1' = -\sin x \quad y_2' = \cos x$$

Step 4 : Find Δ, Δu, Δv

$$\Delta = y_1y_2'-y_1'y_2 = \cos^2x+\sin^2x = 1$$
$$\Delta u = -x\,y_2 = -\text{cosec } x\sin x = -1$$
$$\Delta v = x\,y_1 = \text{cosec } x\cos x = \cot x$$

Step 5 : Find u, v

$$u = \int\frac{\Delta u}{\Delta}dx \qquad v = \int\frac{\Delta v}{\Delta}dx$$
$$= \int -1\,dx, \qquad = \int\cot x\,dx$$
$$= -x, \qquad = \log\sin x$$

Step 6 : $$\text{P.I.} = u\,y_1+v\,y_2$$
$$= -x\cos x+\sin x\log\sin x$$

Step 7 : $$y = \text{C.F.} + \text{P.I.}$$

$y = C_1\cos x+C_2\sin x-x\cos x+\sin x\log\sin x$ is the complete solution.

Example 1.119 : *Solve any two of the following differential equations :*

i) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y = x^2+x+1$

ii) $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+3y = \sin(\sqrt{3}\log x)+x^3$

iii) $\frac{d^2y}{dx^2}+4y$ = *tan 2x, by using the method of variation of parameters.*

SPPU : Dec.-18, Marks 8

Solution : i) A.E. is $D^2+D+1 = 0$

$$D = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{-3}}{2}$$
$$D = -\frac{1}{2}+i\frac{\sqrt{3}}{2}, -\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$\therefore$ $$\text{C.F.} = y_c = e^{-\frac{1}{2}x}\left[C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right]$$
$$\text{P.I.} = y_p = \frac{1}{1+(D+D^2)}(x^2+x+1)$$
$$= [1+(D+D^2)+(D+D^2)^2+...]\,(x^2+x+1)$$
$$= (x^2+x+1)+(2x+1)+2+2$$
$$y_p = x^2+3x+6$$

Thus, the complete solution is $y = y_c+y_p$

ii) Put $x = e^z \Rightarrow z = \log x$

$$x\frac{dy}{dx} = Dy \text{ where } D \equiv \frac{d}{dz}$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$ Given DE becomes

$$[D(D-1) - 3D + 3]y = \sin(\sqrt{3}z) + e^{3z}$$

$$[D^2 - 4D + 3]y = \sin(\sqrt{3}z) + e^{3z} \quad \ldots(1)$$

A.E. of (1) is $D^2 - 4D + 3 = 0$

$$(D-1)(D-3) = 0$$

$$D = 1, 3$$

$$\therefore \quad y_c = C_1 e^z + C_2 e^{3z}$$

$$y_p = \frac{1}{D^2 - 4D + 3}\sin(\sqrt{3}z) + \frac{1}{(D-1)(D-3)}e^{3z}$$

$$= \frac{1}{-3-4D+3}\sin(\sqrt{3}z) + \frac{1}{(3-1)}\frac{z}{1!}e^{3z}$$

$$= -\frac{1}{4}\left(-\frac{\cos(\sqrt{3}z)}{\sqrt{3}}\right) + \frac{1}{2}ze^{3z}$$

$$= \frac{1}{4\sqrt{3}}\cos(\sqrt{3}z) + \frac{1}{2}ze^{3z}$$

Thus the complete solution is $y = y_c + y_p$

iii) **Step 1 :** A.E. is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$$\therefore \quad y_c = C_1 \cos 2x + C_2 \sin 2x$$

Step 2 : Comparing with $y_c = C_1 y_1 + C_2 y_2$

$$\therefore \quad y_1 = \cos 2x, \; y_2 = \sin 2x$$

$$y_1' = -2\sin 2x, \; y_2' = 2\cos 2x$$

Step 3 : Assume P.I. $= uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv where

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$\Delta = 2\cos 2x \cos 2x + 2\sin 2x \sin 2x = 2$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ \phi(x) & y_2' \end{vmatrix} = -xy_2 = -\tan 2x \sin 2x$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & \phi(x) \end{vmatrix} = xy_1 = \tan 2x \cos 2x = \sin 2x$$

Step 5 :

$$u = \int \frac{\Delta u}{\Delta} dx$$

$$= -\int \frac{\sin 2x \tan 2x}{2} dx$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx = -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x) dx$$

$$u = -\frac{1}{2}\frac{\log(\sec 2x + \tan 2x)}{2} + \frac{1}{2}\frac{\sin 2x}{2}$$

and

$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{\sin 2x}{2} dx$$

$$= \frac{-1}{2}\frac{\cos 2x}{2} = -\frac{1}{4}\cos 2x$$

Step 6 : $\therefore$ The P.I. is

$$y_p = -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

$$+ \frac{1}{4}\sin 2x \cos 2x - \frac{1}{4}\sin 2x \cos 2x$$

$$y_p = -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Step 7 : The complete solution is

$$y = y_c + y_p$$

$$y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Example 1.120 : *Solve any two of the following :*

i) $(D^2 + 2D + 1)y = xe^{-x}\cos x$

ii) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{e^x}$

(using method of variation of parameter)

iii) $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

SPPU : May-18, Marks 8

Solution : i) A.E is

$$D^2 + 2D + 1 = 0$$

$$(D+1)^2 = 0$$

$$D = -1, -1$$

$$\therefore \quad y_c = (c_1 + c_2 x)e^{-x}$$

$$y_p = \frac{1}{(D+1)^2} xe^{-x}\cos x$$

$$= e^{-x}\frac{1}{(D-1+1)^2} x\cos x$$

$$= e^{-x}\frac{1}{D^2} x\cos x$$

$$= e^{-x}\left[x - \frac{2D}{D^2}\right]\frac{1}{D^2}\cos x$$

$$= e^{-x}\left[x - \frac{2D}{D^2}\right](-\cos x)$$

$$= e^{-x}\left[x(-\cos x) - 2D \cdot \frac{1}{D^2}(-\cos x)\right]$$

$$= -e^{-x}[x\cos x - 2D(-\cos x)]$$

$$y_p = -e^{-x}[x\cos x - 2\sin x]$$

$\therefore$ The complete solution is $y = y_c + y_p$

ii) **Step 1 :** A.E. is $(D+2)(D+1) = 0$

$$D = -1, -2$$

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

Step 2 : Comparing y_c with $y_c = C_1 y_1 + C_2 y_2$

We have $y_1 = e^{-x}$, $y_2 = e^{-2x}$

$$y_1' = -e^{-x},\ y_2' = -2e^{-2x}$$

Step 3 : Assume P.I. $= y_p = uy_1 + vy_2$

Step 4 : Find Δ, Δu, Δv

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -e^{-3x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ x & y_2' \end{vmatrix} = -xy_2 = -e^{-2x}e^{e^x}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & x \end{vmatrix} = xy_1 = e^{e^x}e^{-x}$$

Step 5 : Find u and v

$$u = \int \frac{\Delta u}{\Delta}dx = +\int \frac{e^{e^x}e^{-2x}}{e^{-3x}}dx = \int e^{e^x}e^x\,dx$$

$$\therefore \qquad u = e^{e^x}$$

And similarly $v = -e^x e^{e^x} + e^{e^x}$

Step 6 : $y_p = e^{-x}e^{e^x} + e^{-2x}(-e^x e^{e^x}) + e^{-2x}e^{e^x}$

$$y_p = e^{-2x}e^{e^x}$$

Step 7 : The complete solution is

$$y = y_c + y_p$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x}e^{e^x}$$

iii) Step 1 : Legendre's differential equation with $a = 3$, $b = 2$.

Put $3x + 2 = e^z$ i.e. $z = \log(3x+2)$

$$(3x+2)\frac{dy}{dx} = 3\,D\,y$$

$$(3x+2)^2\frac{d^2y}{dx^2} = 9\,D\,(D-1)\,y$$

Thus the equation becomes

$$9D(D-1)y + 3\cdot 3\,Dy - 36y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

$$9[D^2 - D + D - 4]\,y = 3\frac{(e^{2z}-4e^z+4)}{9} + \frac{4(e^z-2)}{3} + \frac{3}{3}$$

$$= \frac{e^{2z}-4e^z+4+4e^z-8+3}{3}$$

$$9(D^2-4)\,y = \frac{e^{2z}-1}{3}$$

$$(D^2-4)\,y = \frac{1}{27}(e^{2z}-1) \qquad \ldots(1)$$

Which is linear differential equation with constant coefficient.

Step 2 : A.E. $D^2 - 4 = 0$

$$(D-2)(D+2) = 0$$

$$D = 2, -2$$

$$\text{C.F.} = C_1 e^{2z} + C_2 e^{-2z}$$

Step 3 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{f(D)}Z$$

$$\text{P.I.} = \frac{1}{(D-2)(D+2)}\frac{1}{27}(e^{2z}-1)$$

Separate PI_1 and PI_2.

$$= \frac{1}{27}\left\{\frac{1}{(D-2)(D+2)}e^{2z} - \frac{1}{(D-2)(D+2)}1\right\}$$

$$= \frac{1}{27}\left\{\frac{1}{(D-2)}\cdot\frac{1}{(2+2)}e^{2z} - \frac{1}{(0-2)(0+2)}\right\}$$

$$= \frac{1}{27}\left\{\frac{1}{4}\cdot\frac{1}{D-2}e^{2z} + \frac{1}{4}\right\} = \frac{1}{108}[ze^{2z}+1]$$

Step 4 : The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{108}\{z\,e^{2z}+1\}$$

Put $e^z = 3x+2$ $y = C_1(3x+2)^2 + C_2(3x+2)^{-2} + \frac{1}{108}\{(3x+2)^2\log(3x+2)+1\}$

Example 1.121 : *Solve of the following :*
i) $(D^2 + 13D + 36)y = e^{-4x} + \sinh x$.

SPPU : May-17, Marks 8

Solution : i) $(D^2 + 13D + 36)y = e^{-4x} + \sinh x$

Step 1 : A.E. is $D^2 + 13D + 36 = 0$

$$(D+9)(D+4) = 0$$

$$D = -9, -4$$

Step 2 : The C.F. is $y_c = C_1 e^{-4x} + C_2 e^{-9x}$

Step 3 : The P.I. is $y_p = \frac{1}{D^2+13D+36}(e^{-4x} + \sinh x)$

$$= \frac{1}{(D+9)(D+4)} e^{-4x} + \frac{1}{D^2+13D+36} \sinh x$$

$$= x \frac{1}{2D+13} e^{-4x} + \frac{1}{13D+37} \sinh x$$

$$= x \frac{1}{-8+13} e^{-4x} + \frac{37-13D}{(37)^2 - 169D^2} \sinh x$$

$$y_p = \frac{xe^{-4x}}{5} + \frac{1}{1200}(37 \sinh x - 13 \cosh x)$$

Step 4 : The complete solution is $y = y_c + y_p$.

Example 1.122 : $(D^2+4)\,y = \cos 3x \cdot \cos x$

SPPU : May-14

Solution : Step 1 : $(D^2+4)\,y = \cos 3x \cdot \cos x$

A.E. is $D^2 + 4 = 0 \quad D^2 = -4 \Rightarrow D = \pm 2i$

$\therefore$ C.F. $= C_1 \cos 2x + C_2 \sin 2x,$

Step 2 : Now P.I. $= \frac{1}{D^2+4} (\cos 3x \cos x)$

$$\text{P.I.} = \frac{1}{2} \frac{1}{D^2+4} 2 [\cos 3x \cos x]$$

$$= \frac{1}{2} \frac{1}{D^2+4} (\cos (4x) + \cos (2x))$$

$$= \frac{1}{2} \frac{1}{D^2+4} \cos 4x + \frac{1}{2} \frac{1}{D^2+4} \cos 2x$$

$$= \frac{1}{2} \frac{1}{-16+4} \cos 4x + \frac{1}{2} \frac{1}{-4+4} \cos 2x$$

$$= \frac{1}{2} \frac{1}{(-12)} \cos 4x + \frac{1}{2} \text{ fail case}$$

$$= -\frac{1}{24} \cos 4x + \frac{1}{2} \frac{x}{2D} \cos 2x$$

$$= -\frac{1}{24} \cos 4x + \frac{1}{2} x \frac{\sin 2x}{2}$$

$$= -\frac{\cos 4x}{24} + \frac{x \sin 2x}{8}$$

Step 3 : The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 \cos 2x + C_2 \sin 2x - \frac{\cos 4x}{24} + \frac{x \sin 2x}{8}$$

Example 1.123 : $(D-1)^3 y = e^x + 5^x - 1$

Solution :

Step 1 : A.E. is $(D-1)^3 = 0$

$\Rightarrow \quad D = 1, 1, 1$

The complementary function is

$$y_c = (C_1 + C_2 x + C_3 x^2)\, e^x$$

Step 2 : The particular integral is

$$\text{P.I.} = y_p = \frac{1}{(D-1)^3} [e^x + 5^x - 1]$$

$$= \frac{1}{(D-1)^3} e^x + \frac{1}{(D-1)^3} 5^x + \frac{1}{(D-1)^3}(-1)$$

$$= \frac{x^3}{3!} e^x + \frac{1}{(\log 5 - 1)^3} 5^x + \frac{1}{(-1)^3}(-1)$$

$$y_p = \frac{x^3}{6} e^x + \frac{1}{(\log 5 - 1)^3} 5^x + 1$$

Step 3 : The complete solution is

$$y = y_c + y_p = (C_1 + C_2 x + C_3 x^2)\, e^x + \frac{x^3}{6} e^x + \frac{1}{(\log 5 - 1)^3} 5^x + 1$$

Example 1.124 : $\frac{d^3y}{dx^3} + \frac{dy}{dx} = \cos x$

SPPU : May-16

Solution : Step 1 : A.E. is $D^3 + D = 0$

$$D(D^2+1) = 0$$

$$D = 0, \quad D^2 = -1 \Rightarrow D = 0, \pm i$$

$\therefore$ The C.F. is $\quad y_c = C_1 + C_2 \cos x + C_3 \sin x \quad \ldots (1)$

Step 2 :

The P.I. is $\quad y_p = \frac{1}{D(D^2+1)} \cos x$

$$y_p = \frac{1}{D^2+1} \frac{1}{D} \cos x$$

$$= \frac{1}{D^2+1} \int \cos x\,dx = \frac{1}{D^2+1} \sin x$$

$$y_p = -\frac{x}{2} \cos x$$

Step 3 : The complete solution is $y = y_c + y_p$

$\therefore \quad y = C_1 + C_2 \cos x + C_3 \sin x - \frac{x}{2} \cos x$

Example 1.125 : $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + \sin x$

Solution : Step 1 : A.E. is

$$D^2 - 3D + 2 = 0$$

$$(D-1)(D-2) = 0$$

$\Rightarrow \quad D = 1, 2$

$\therefore$ The C.F. is $\quad y_c = C_1 e^x + C_2 e^{2x}$

Step 2 : The P.I. is

$$y_p = \frac{1}{(D-1)(D-2)}(x^2+\sin x)$$

$$= \left(\frac{-1}{D-1}+\frac{1}{D-2}\right)(x^2+\sin x)$$

$$= \left(\frac{1}{D-2}-\frac{1}{D-1}\right)x^2+\left(\frac{1}{D^2-3D+2}\right)\sin x$$

$$= \frac{1}{-2\left(1-\frac{D}{2}\right)}x^2+\frac{1}{1-D}x^2+\frac{1}{-1-3D+2}\sin x$$

$$= -\frac{1}{2}\left\{1+\frac{D}{2}+\frac{D^2}{4}+\ldots\right\}(x^2)+\{1+D+D^2+\ldots\}(x^2)$$

$$+\frac{1}{1-3D}\sin x$$

$$= -\frac{1}{2}\left\{x^2+x+\frac{1}{2}\right\}+(x^2+2x+2\}+\frac{1+3D}{1-9D^2}\sin x$$

$$= -\frac{1}{2}x^2+\frac{3}{2}x+\frac{7}{4}+\frac{1}{1-9(-1)}(1+3D)\sin x$$

$$= \frac{1}{2}\left(x^2+3x+\frac{7}{2}\right)+\frac{1}{10}(\sin x+3\cos x)$$

$$y_p = \frac{1}{2}\left(x^2+3x+\frac{7}{2}\right)+\frac{1}{10}(\sin x+3\cos x)$$

Step 3 : The complete solution is

$$y = y_c+y_p$$

Example 1.126 : *Solve* $(D^3+8)y = x^4+2x+1$

Solution : Step 1 : A.E. is $D^3+8=0$

By synthetic division method, we get,

$$(D+2)(D^2-2D+4) = 0$$

$$\therefore \quad D = -2,\ D = 1\pm i\sqrt{3}$$

The C.F. is

$$y_c = C_1e^{-2x}+e^x[C_2\cos\sqrt{3}\,x+C_3\sin\sqrt{3}\,x]$$

Step 2 : The P.I. is

$$y_p = \frac{1}{D^3+8}[x^4+2x+1]$$

$$= \frac{1}{8}\frac{1}{\left(1+\frac{D^3}{8}\right)}[x^4+2x+1]$$

$$= \frac{1}{8}\left[1-\frac{D^3}{8}+\left(\frac{D^3}{8}\right)^2\cdots\right][x^4+2x+1]$$

$$= \frac{1}{8}\left[x^4+2x+1-\frac{1}{8}(24\,x)+0\right]$$

$$= \frac{1}{8}[x^4-x+1]$$

Step 3 : The complete solution is

$$y = y_c+y_p$$

Example 1.127 : *Solve* $(D^2+6D+9)\,y = x^{-3}\,e^{-3x}$

SPPU : Dec.-14, Marks 4

Solution :

Step 1 : $D^2+6D+9=0$ $\therefore$ A.E. is $(D+3)^2=0$

$\Rightarrow D = -3,-3$

$$\therefore \quad \text{C.F.} = (C_1\,x+C_2)\,e^{-3x}$$

Step 2 : Now $\quad \text{P.I.} = \dfrac{1}{(D+3)^2}\,x^{-3}\,e^{-3x}$

$$= e^{-3x}\frac{1}{[D-3+3]^2}\,x^{-3}$$

$$= e^{-3x}\frac{1}{D^2}\,x^{-3} = e^{-3x}\frac{1}{D}\left(\frac{1}{-2x^2}\right)$$

$$\text{P.I.} = e^{-3x}\frac{1}{2x} = \frac{1}{2x\,e^{3x}}$$

Step 3 : The complete solution is

$$y = \text{C.F.}+\text{P.I.} = (C_1\,x+C_2)\,e^{-3x}+\frac{1}{2\,x\,e^{3x}}$$

Example 1.128 : $(D^2+9)y = x\sin 2x$

SPPU : May-15

Solution : Step 1 : A.E. is

$$D^2+9 = 0 \Rightarrow D^2 = -9$$

$$D = \pm 3i$$

$\therefore$ The C.F. is

$$y_c = C_1\cos 3x+C_2\sin 3x$$

Step 2 : The P.I. is

$$y_p = \frac{1}{D^2+9}(x\sin 2x)$$

$$= \left\{x-\frac{2D}{D^2+9}\right\}\frac{1}{D^2+9}\sin 2x$$

$$= \left\{x-\frac{2D}{D^2+9}\right\}\frac{1}{-4+9}\sin 2x$$

$$= \left\{x-\frac{2D}{D^2+9}\right\}\frac{1}{5}\sin 2x$$

$$= \frac{1}{5} x\sin 2x - 2D\frac{1}{25}\sin 2x$$

$$= \frac{1}{5} x\sin 2x - \frac{2}{25} D(\sin 2x)$$

$$y_p = \frac{1}{5} x \sin 2x - \frac{2}{25} (\cos 2x)(2)$$

$$= \frac{1}{5} x \sin 2x - \frac{4}{25} \cos 2x$$

Step 3 : The complete solution is $y = y_c + y_p$

$$y = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{5} x\sin 2x - \frac{4}{25}\cos 2x$$

Example 1.129 : $(D^2 - 6D + 13)y = e^{3x} \sin 4x + 3^x$

SPPU : May-15

Solution : Step 1 : A.E. is

$$D^2 - 6D + 13 = 0$$

$$D = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm 4i}{2} = 3 \pm 2i$$

$\therefore$ The C.F. is

$$y_c = e^{3x}[C_1 \cos 2x + C_2 \sin 3x]$$

Step 2 : The P.I. is

$$y_p = \frac{1}{D^2 + 6D + 13} e^{3x} \sin 4x + \frac{1}{D^2 - 6D + 13} 3^x$$

Let $$y_1 = \frac{1}{D^2 - 6D + 13} e^{3x} \sin 4x$$

$$= e^{3x} \frac{1}{(D+3)^2 - 6(D+3) + 13} \sin 4x$$

$$y_1 = e^{3x} \frac{1}{D^2 + 4} \sin 4x = e^{3x} \frac{1}{-16+4} \sin 4x$$

$$= -\frac{1}{12} e^{3x} \sin 4x$$

$$y_2 = \frac{1}{D^2 - 6D + 13} 3^x = \frac{1}{D^2 - 6D + 13} e^{x \log 3}$$

$$= \frac{1}{(\log 3)^2 - 6(\log 3) + 13} 3^x$$

$$\therefore \quad y_p = y_1 + y_2$$

$$= -\frac{1}{12} e^{3x} \sin 4x + \frac{1}{(\log 3)^2 - 6(\log 3) + 13} 3^x$$

Step 3 : The complete solution is

$$y = y_c + y_p$$

Example 1.130 : $(D^2 - 4D + 4)y = e^x \cos^2 x$

SPPU : May-14, Dec.-14

Solution : Step 1 : A.E. is $D^2 - 4D + 4 = 0 \Rightarrow (D-2)^2 = 0 \Rightarrow D = 2, 2$

$\therefore$ The C.F. is $y_c = (C_1 + C_2 x) e^{2x}$

Step 2 : The P.I. is

$$y_p = \frac{1}{(D-2)^2} e^x \cos^2 x$$

$$= e^x \frac{1}{(D+1-2)^2} \cos^2 x$$

$$e^x \frac{1}{(D-1)^2} \cos^2 x = e^x \frac{1}{D^2 - 2D + 1} \left[\frac{1 + \cos 2x}{2}\right]$$

$$= e^x \left\{\frac{1}{0-0+1}\left(\frac{1}{2}\right) + \frac{1}{2} \frac{1}{-4-2D+1} \cos 2x\right\}$$

$$= \frac{1}{2} e^x + \frac{(-1)}{2} \left\{\frac{(2D-3)}{(2D+3)(2D-3)} \cos 2x\right\}$$

$$= \frac{1}{2} e^x - \frac{1}{2} \frac{(2D-3)}{4D^2 - 9} \cos 2x$$

$$= \frac{1}{2} e^x - \frac{1}{2} \frac{1}{4(-4)-9} (2D-3) \cos 2x$$

$$= \frac{1}{2} e^x - \frac{1}{2}\left(\frac{1}{-25}\right)(-4 \sin 2x - 3\cos 2x)$$

$$y_p = \frac{1}{2} e^x - \frac{1}{50} (4\sin 2x + 3\cos 2x)$$

Step 3 : The complete solution is $y = y_c + y_p$

Example 1.131 : *Solve* $(D^2 + 5D + 6)y = e^{-2x} \sec^2 x(1 + 2 \tan x)$

SPPU : Dec.-14

Solution : Step 1 : A.E. is $D^2 + 5D + 6 = 0$

$$(D+2)(D+3) = 0 \Rightarrow D = -2, -3$$

$\therefore$ The C.F. is

$$y_c = C_1 e^{-2x} + C_2 e^{-3x}$$

Step 2 : The P.I. is

$$y_p = e^{-2x} \frac{1}{D^2 + 5D + 6} e^{-2x} \sec^2 x (1 + 2\tan x)$$

$$y_p = e^{-2x} \frac{1}{(D-2)^2 + 5(D-2) + 6} \sec^2 x (1 + 2\tan x)$$

$$= e^{-2x} \frac{1}{D^2 + D} \sec^2 x (1 + 2\tan x)$$

$$= e^{-2x} \left[\frac{1}{D} - \frac{1}{D+1}\right] \sec^2 x (1 + 2\tan x)$$

Let $$PI_1 = \frac{1}{D}\sec^2 x(1+2\tan x) = \int \sec^2 x(1+2\tan x)dx$$

Put $\tan x = t,\ \sec^2 x\, dx = dt$

$\therefore$ $$PI_1 = \int (1+2t)dt = \left[t+\frac{2t^2}{2}\right] = t+t^2 = \tan x+\tan^2 x$$

Now $$PI_2 = \frac{1}{D+1}\sec^2 x(1+2\tan x) = e^{-x}\int e^{x}(\sec^2 x+2\tan x\sec^2 x)dx$$

$$= e^{-x}(e^{x}\sec^2 x)\ (\because \int e^{x}[f(x)+f'(x)] = e^{x}f(x)) = \sec^2 x$$

$\therefore$ $$y_p = e^{-2x}[\tan x+\tan^2 x-\sec^2 x] = e^{-2x}(\tan x-1)$$

Step 3 : The complete solution is

$$y = y_c+y_p = C_1e^{-2x}+C_2e^{-3x}+e^{-2x}(\tan x-1)$$

Example 1.132 : $(D^2-2D+1)y = x\,e^{x}\sin x$

Solution : Step 1 : The A.E. is $D^2-2D+1 = 0$

$(D-1)^2 = 0 \Rightarrow D = 1, 1$

The C.F. is $y_c = (C_1+C_2x)\,e^{x}$

Step 2 : The P.I. is $y_p = \dfrac{1}{(D-1)^2}\, x\, e^{x}\sin x$

$$= e^{x}\frac{1}{[D+1-1]^2}\, x\sin x$$

$$y_p = e^{x}\frac{1}{D^2}\, x\sin x = e^{x}\left[x-\frac{2D}{D^2}\right]\frac{1}{D^2}\sin x$$

$$= e^{x}\left[x-\frac{2D}{D^2}\right](-\sin x) = -e^{x}\left[x-\frac{2D}{D^2}\right]\sin x$$

$$y_p = -e^{x}[x\sin x+2\cos x]$$

Step 3 : The complete solution is

$$y = y_c+y_p$$

Example 1.133 : $(3x+2)^2\dfrac{d^2y}{dx^2}+3(3x+2)\dfrac{dy}{dx}-36y = 3x^2+4x+1$

SPPU : May-14

Solution : Step 1 : Legendre's differential equation with a = 3, b = 2.

Put $3x+2 = e^z$ i.e. $z = \log(3x+2)$

$$(3x+2)\frac{dy}{dx} = 3\,D\,y$$

$$(3x+2)^2\frac{d^2y}{dx^2} = 9\,D\,(D-1)\,y$$

Thus the equation becomes

$$9D\,(D-1)\,y+3\cdot 3\,Dy-36y = 3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1$$

$$9\,[D^2 - D + D - 4]\,y = 3\frac{(e^{2z} - 4e^z + 4)}{9} + \frac{4\,(e^z - 2)}{3} + \frac{3}{3} = \frac{e^{2z} - 4e^z + 4 + 4e^z - 8 + 3}{3}$$

$$9\,(D^2 - 4)\,y = \frac{e^{2z} - 1}{3}$$

$$(D^2 - 4)\,y = \frac{1}{27}\,(e^{2z} - 1) \qquad \ldots(1)$$

Which is linear differential equation with constant coefficient.

Step 2 : A.E. $D^2 - 4 = 0$

$$(D - 2)\,(D + 2) = 0$$

$$D = 2, -2$$

$$\text{C.F.} = C_1 e^{2z} + C_2 e^{-2z}$$

Step 3 : Use P.I. formula.

$$\text{P.I.} = \frac{1}{f\,(D)}\,Z$$

$$\text{P.I.} = \frac{1}{(D-2)\,(D+2)}\,\frac{1}{27}\,(e^{2z} - 1)$$

Separate PI_1 and PI_2.

$$= \frac{1}{27}\left\{\frac{1}{(D-2)\,(D+2)}\,e^{2z} - \frac{1}{(D-2)\,(D+2)}\,1\right\}$$

$$= \frac{1}{27}\left\{\frac{1}{(D-2)}\cdot\frac{1}{(2+2)}\,e^{2z} - \frac{1}{(0-2)\,(0+2)}\right\}$$

$$= \frac{1}{27}\left\{\frac{1}{4}\cdot\frac{1}{D-2}\,e^{2z} + \frac{1}{4}\right\} = \frac{1}{108}\,[ze^{2z} + 1]$$

Step 4 : The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{108}\,\{z\,e^{2z} + 1\}$$

Put $e^z = 3x + 2$ $y = C_1 (3x+2)^2 + C_2 (3x+2)^{-2} + \frac{1}{108}\,\{(3x+2)^2 \log(3x+2) + 1\}$

Example 1.134 : $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2 \sin\,[\log\,(1+x)]$ **SPPU : May-16**

Solution : **Step 1 :** Put $x + 1 = e^z \Rightarrow z = \log\,(1+x)$ and $\dfrac{d}{dz} \equiv D$

$$\therefore \quad (1+x)^2 \frac{d^2y}{dx^2} = D(D-1)y, \; (1+x)\frac{dy}{dx} = Dy$$

$\therefore$ Given DE becomes

$$D(D-1)y + Dy + y = 2 \sin z$$

$$(D^2 + 1)y = 2 \sin z$$

Which is LDE with constant coefficients

Step 2 : A.E. is $D^2 + 1 = 0 \Rightarrow D^2 = -1$

$$\therefore \quad D = \pm i$$

$\therefore$ The C.F. is $\quad y_c = C_1 \cos z + C_2 \sin z$

Step 3 : The P.I. is $\quad y_p = \dfrac{1}{D^2+1} 2 \sin z = 2\left(-\dfrac{z}{2}\right)\cos z = -z \cos z$

Step 4 : The complete solution is

$$y = y_c + y_p$$

$$y = C_1 \cos \log(1+x) + C_2 \sin \log(1+x) - \log(1+x) \cos \log(1+x)$$

Example 1.135 : *Solve* $(2x+1)^2 \dfrac{d^2y}{dx^2} - 6(2x-1)\dfrac{dy}{dx} + 16y = 8(2x+1)^2$ **SPPU : May-15**

Solution : Step 1 : Put $2x + 1 = e^z \quad \therefore z = \log(2x+1)$

$(2x+1)\dfrac{dy}{dx} = 2Dy \quad$ where $\dfrac{d}{dz}$

$(2x+1)^2 \dfrac{d^2y}{dx^2} = 2^2 D(D-1)y = 4(D^2 - D)y$

$\therefore$ Given D.E. becomes

$$4(D^2 - D)y - 6(2Dy) + 16y = 8e^{2z}$$

$$(4D^2 - 16D + 16)y = 8e^{2z}$$

$$(D^2 - 4D + 4)y = 2e^{2z} \quad \ldots(1)$$

Step 2 : A.E. is $D^2 = 4D + 4 = 0 \quad \Rightarrow (D-2)^2 = 0$

$$D = 2, 2$$

$$\therefore \quad y_c = (C_1 + C_2 z)e^{2z}$$

Step 3 : Now $y_p = \dfrac{1}{D^2 - 4D + 4}(2e^{2z})$

$$y_p = 2\frac{1}{(D-2)^2}e^{2z} = 2\frac{z^2}{2!}e^{2z}$$

$$= z^2 e^{2z}$$

Step 4 : The complete solution is

$$y = y_c + y_p = (C_1 + C_2 z)e^{2z} + z^2 e^{2z}$$

$$\therefore \quad y = [C_1 + C_2 \log(2x+1)](2x+1)^2 + [\log(2x+1)]^2 (2x+1)^2$$

Example 1.136 : $(x^2D^2 - xD + 1)y = x \log x$ **SPPU : Dec.-15**

Solution : Step 1 : Given D.E. is Cauchy's D.E. $\therefore$ Put $x = e^z$

$$\therefore \quad z = \log x \text{ and } D \equiv \frac{d}{dz}$$

$$\therefore \quad x\frac{dy}{dx} = Dy \text{ and } x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$ Given D.E. becomes

$$D(D-1)y - Dy + y = e^z + z$$

$$(D^2 - 2D + 1)y = z + e^z$$

Which is LDE with constant coefficients

Step 2 : A.E. is

$$D^2 - 2D + 1 = 0$$

$$(D-1)^2 = 0 \Rightarrow D = 1, 1$$

$\therefore$ The C.F. is $\quad y_c = (C_1 + C_2 z)e^z$

Step 3 : The P.I. is $y_p = \dfrac{1}{(D-1)^2}(z + e^z)$

$$y_p = \frac{1}{(1-D)^2}(z) + \frac{1}{(1-D)^2}e^z$$

$$= (1-(-2)D+....)(z) + \frac{z^2}{2}e^z$$

$$= (1+2D+....)(z) + \frac{z^2}{2}e^z$$

$$y_p = z + 2 + \frac{z^2}{2}e^z = \frac{z^2}{2}e^z + z + 2$$

Step 4 : $\therefore$ The complete solution is $y = y_c + y_p$

$$y = (C_1 + C_2 z)e^z + \frac{z^2}{2}e^z + z + 2$$

$$y = (C_1 + C_2 \log x) + \frac{(\log x)^2}{2}x + \log x + 2$$

Example 1.137 : *Solve any two differential equations :*

i) $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} - 2y = e^{4x} \cosh 2x$

ii) $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 3y = x^2 \sin(\log x)$

iii) $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 16y = \dfrac{e^{4x}}{x^6}$, *by using the method of variation of parameters.*

SPPU : Dec.-19, Marks 8

Solution : i) A.E is $D^2 + 7D - 2 = 0$

$$D = \frac{-7 \pm \sqrt{49 - 4(1)(-2)}}{2} = \frac{-7 \pm \sqrt{57}}{2}$$

$$D = -\frac{7}{2} \pm \frac{\sqrt{57}}{2}$$

$$\therefore \quad \alpha_1 = -\frac{7}{2} + \frac{\sqrt{57}}{2} \text{ and } \alpha_2 = -\frac{7}{2} - \frac{\sqrt{57}}{2}$$

$$\therefore \quad y_c = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} \qquad ...(1)$$

$$y_p = \frac{1}{D^2 + 7D - 2}e^{4x}\cosh 2x$$

$$= e^{4x}\frac{1}{(D+4)^2 + 7(D+2) - 2}\cosh 2x$$

$$= e^{4x}\frac{1}{D^2 + 15D - 28}\cosh 2x \quad (\because D^2 \to 2^2)$$

$$= e^{4x}\frac{1}{4 + 15D - 28}\cosh 2x$$

$$= e^{4x}\frac{1}{(15D - 24)}\cosh 2x$$

$$= \frac{e^{4x}}{5}\frac{(3D+8)}{(3D-8)(3D+8)}\cosh 2x$$

$$= \frac{e^{4x}}{5}\frac{(3D+8)}{9D^2 - 64}\cosh 2x$$

$$= \frac{e^{4x}}{5}\frac{1}{36 - 64}(3D+8)\cosh 2x$$

$$= \frac{e^{4x}}{140}[3(2)\sinh 2x + 8\cosh 2x]$$

$$y_p = \frac{e^{4x}}{70}[3\sinh 2x + 4\cosh 2x]$$

$\therefore$ The complete solution is $y = y_c + y_p$

ii) Put $x = e^z \Rightarrow z = \log x$ and $x\dfrac{dy}{dx} = Dy$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$ We get

$$D(D-1)y - 3Dy + 3y = e^{2z} \sin z$$

$$(D^2 - 4D + 3)y = e^{2z}\sin z \qquad ...(2)$$

$\therefore \quad$ A.E. is $D^2 - 4D + 3 = 0$

$$(D-1)(D-3) = 0$$

$$D = 1 \text{ or } D = 3$$

$$\therefore \quad y_c = C_1 e^2 + C_2 e^{3z}$$

$$y_p = \frac{1}{(D-1)(D-3)}e^{2z}\sin z$$

$$= e^{2z}\frac{1}{(D+2-1)(D+2-3)}\sin z$$

$$= e^{2z}\frac{1}{(D+1)(D-1)}\sin z$$

$$y_p = e^{2z}\frac{1}{D^2-1}\sin z = e^{2z}\frac{1}{-1-1}\sin z = \frac{e^{2z}\sin z}{-2}$$

$$\therefore \quad y = y_c + y_p$$

iii) A.E is $D^2 - 8D + 16 = 0$

$$(D-4)(D-4) = 0$$

$$D = 4 \text{ or } 4$$

$$\therefore \quad y_c = (C_1 + C_2 x)e^{4x}$$

Let $\quad y_1 = e^{4x},\ y_2 = e^{4x}x$

$$y_1' = 4e^{4x},\ y_2' = e^{4x} + 4xe^{4x}$$

Assume that $y = uy_1 + vy_2$ is a particular solution of given DE.

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$\Delta = e^{4x}(e^{4x} + 4xe^{4x}) - xe^{4x}(4e^{4x}) = e^{8x}$$

$$\Delta u = \begin{vmatrix} 0 & y_2 \\ X & y_2' \end{vmatrix} = -y_2 X = xe^{4x}\frac{e^{4x}}{x^6} = \frac{1}{x^5}e^{8x}$$

$$\Delta v = \begin{vmatrix} y_1 & 0 \\ y_1' & X \end{vmatrix} = X y_1 = e^{4x}\frac{e^{4x}}{x^6} = \frac{e^{8x}}{x^6}$$

$$\therefore \quad u = \int \frac{\Delta u}{\Delta} dx = \int \frac{1}{x^5} dx = \frac{x^{-6}}{-6} = -\frac{1}{6x^6}$$

$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{1}{x^6} dx = \frac{x^{-7}}{-7} = -\frac{1}{7x^7}$$

$$\therefore \quad y_P = uy_1 + vy_2 = -\frac{1}{6x^6}e^{4x} - \frac{1}{7x^7}e^{4x}x$$

$$y_P = \frac{-e^{4x}}{x^6}\left[+\frac{1}{6}+\frac{1}{7}\right] = \frac{-13}{42}\frac{e^{4x}}{x^6}$$

Hence, $\quad y = y_C + y_P$

Example 1.138 : *Solve differential equations :*

i) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = e^{2x}\sin 4x + 2^{3x} + 6$

ii) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^4 + 3x + 1$

SPPU : May-19, Marks 8

Solution : i) A.E. is $D^2 - 4D + 4 = 0$

$$(D-2)^2 = 0$$

$$D = 2, 2$$

$$\therefore \quad y_c = (C_1 + C_2 x)e^{2x}$$

$$y_p = \frac{1}{(D-2)^2}\{e^{2x}\sin 4x + 2^{3x} + 6\}$$

$$= e^{2x}\frac{1}{(D+2-2)^2}\sin 4x + \frac{2^{3x}}{(\log 2^3 - 2)^2} + \frac{6}{(D-2)^2}$$

$$y_p = e^{2x}\frac{1}{-16}\sin 4x + \frac{2^{3x}}{(3\log 2 - 2)^2} + \frac{3}{2}$$

$\therefore$ $y = y_c + y_p$ is the complete solution.

ii) Put $x = e^z \Rightarrow z = \log x$ and $Dy = x\frac{dy}{dx}$

$$D(D-1)y = x^2\frac{d^2y}{dx^2}$$

$\therefore$ Given DE becomes

$$D(D-1)y + Dy - y = e^{4z} + 3e^z + 1$$

$$(D^2-1)y = e^{4z} + 3e^z + 1 \quad \ldots(1)$$

$\therefore$ A.E. is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$$y_c = C_1e^z + C_2e^{-z}$$

$$y_p = \frac{1}{(D+1)(D-1)}\{e^{4z} + 3e^z + 1\}$$

$$= \frac{1}{(4+1)(4-1)}e^{4z} + \frac{3z\,e^z}{(1+1)} + \frac{1}{(1)(-1)}$$

$$y_p = \frac{1}{15}e^{4z} + \frac{3}{2}ze^z - \frac{1}{2}$$

$\therefore$ The complete solution is

$$y = y_c + y_p$$

$$y = C_1e^z + C_2e^{-z} + \frac{1}{15}e^{4z} + \frac{3}{2}ze^z - \frac{1}{2}$$

$$y = C_1x + C_2\frac{1}{x} + \frac{1}{15}x^4 + \frac{3}{2}x\log x - \frac{1}{2}$$

Example 1.139 : *Solve differential equation :*

$$(2x+1)^2\frac{d^2y}{dx^2} + 2(2x+1)\frac{dy}{dx} + 4y = 4\sin[2\log(2x+1)]$$

SPPU : Dec.-18, Marks 8

Solution : Put $2x + 1 = e^z$ i.e. $z = \log(2x+1)$ in the given DE

and $$x = \frac{1}{2}(e^z - 1)$$

$$(2x+1)\frac{dy}{dx} = 2Dy$$

where $$D \equiv \frac{d}{dz}$$

$$(2x+1)^2 \frac{d^2y}{dx^2} = 4D(D-1)\,y$$

$\therefore$ Given DE becomes

$$[4D(D-1)+2(2D)+4]\,y = 4\sin 2z$$

$$[4D^2+4]\,y = 4\sin 2z$$

$$[D^2+1]\,y = \sin 2z$$

$\therefore$ A.E. is $D^2+1 = 0 \Rightarrow D = \pm i$

$$y_c = C_1\cos z + C_2 \sin z$$

and $$y_p = \frac{1}{D^2+1}\sin 2z = \frac{1}{-4+1}\sin 2z = -\frac{1}{3}\sin 2z$$

$$\therefore \quad y = y_c + y_p = C_1 \cos z + C_2 \sin z - \frac{1}{3}\sin 2z$$

$$\therefore \quad y = C_1 \cos\log(2x+1) + C_2 \sin\log(2x+1) - \frac{1}{3}\sin 2\log(2x+1)$$

Example 1.140 : *Solve* $(D^2+4)y = e^x + x^2$ **SPPU : May-18, Marks 8**

Solution :

A.E is $D^2+4 = 0$

$\Rightarrow$ $D^2 = -4 \Rightarrow D \pm 2i$

$\therefore$ $y_c = C_1\cos 2x + C_2 \sin 2x$

and $$y_p = \frac{1}{D^2+4}(e^x+x^2) = \frac{1}{D^2+4}e^x + \frac{1}{D^2+4}x^2$$

$$= \frac{1}{1+4}e^x + \frac{1}{4\left(1+\frac{D^2}{4}\right)}x^2$$

$$= \frac{1}{5}e^x + \frac{1}{4}\left[1 - \frac{D^2}{4} + \frac{D^4}{16} - \cdots\right]x^2$$

$$y_p = \frac{1}{5}e^x + \frac{1}{4}\left[x^2 - \frac{1}{2}\right]$$

$\therefore$ The complete solution is $y = y_C + y_P$

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{5}e^x + \frac{1}{4}\left[x^2 - \frac{1}{2}\right]$$

Example 1.141 : *Solve :*

i) $(D^2+4D+4)y = x^{-3}e^{-2x}$

ii) $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2 \sin(\log x)$

SPPU : Dec.-17, Marks 8

Solution : i) $(D^2+4D+4)\,y = x^{-3}e^{-2x}$

A.E. is $D^2+4D+4 = 0 \Rightarrow (D+2)^2 = 0$

$D = -2, -2$

$$\therefore \quad y_c = (C_1 + C_2\, x)\,e^{-2x},$$

$$y_1 = e^{-2x}, \quad y_2 = e^{-2x}\,x$$

$$y_1' = -2e^{-2x}, \quad y_2' = e^{-2x} - 2x\,e^{-2x}$$

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$= e^{-2x}e^{-2x}(1-2x) - (-2e^{-2x})(x\,e^{-2x})$$

$$\Delta = e^{-4x}$$

$$\Delta u = -X y_2 = -x^{-3}e^{-2x}(xe^{-2x}) = -x^{-2}e^{-4x}$$

$$\Delta v = X y_1 = x^{-3}e^{-2x}(e^{-2x}) = x^{-3}e^{-4x}$$

$$\therefore \quad u = \int \frac{\Delta u}{\Delta}dx = \int -x^{-2}dx = \frac{1}{x}$$

$$v = \int \frac{\Delta v}{\Delta}dx = \int x^{-3}dx = -\frac{1}{2x^2}$$

$$\therefore \quad y_p = u y_1 + v y_2 = \frac{1}{x}e^{-2x} - \frac{1}{2x^2}x\,e^{-2x}$$

$$= \frac{1}{x}e^{-2x}\left(1 - \frac{1}{2}\right)$$

$$y_p = \frac{1}{2}\frac{1}{x}e^{-2x}$$

$\therefore$ The complete solution is $y = y_c + y_p$

ii) put $x = e^z$, $z = \log x$, $x\dfrac{dy}{dx} = Dy$,

$$x^2\frac{d^2y}{dx^2} = D(D-1)\,y$$

$\therefore$ We get $D(D-1)\,y - 3Dy + 5y = e^{2z}\sin z$

$$(D^2 - 4D + 5)\,y = e^{2z}\sin z \qquad \ldots(1)$$

A.E. is $D^2 - 4D + 5 = 0$

$$D = \frac{4 \pm \sqrt{16 - 4(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

$$\therefore \quad y_c = e^{2z}(C_1 \cos z + C_2 \sin z)$$

$$y_p = \frac{1}{D^2 - 4D + 5}e^{2z}\sin z$$

$$= e^{2z}\frac{1}{(D+2)^2 - 4(D+2) + 5}\sin z$$

$$y_p = e^{2z}\frac{1}{D^2+1}\sin z = e^{2z}\left(-\frac{z}{2}\cos z\right)$$

$$y_p = \frac{-x^2}{2}\log x \cos\log x$$

$$\therefore \quad y = y_c + y_p$$

$$= x^2(C_1\cos\log x + C_2 \sin\log x) - \frac{x^2 \log x \cos\log x}{2}$$

Example 1.142 : *Solve of the following*

i) $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ *(use method of variation of parameters)*

ii) $(D^2 - 4)y = e^{4x} + 2x^3$

SPPU : May-17, Marks 8

Solution : i) A.E. is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$\therefore \quad y_c = c_1 e^x + c_2 e^{-x}$ where $y_1 = e^x,\ y_2 = e^{-x}$

Let $\quad y_p = u y_1 + v y_2$

$$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1-1 = -2$$

$$\Delta u = -xy_2 = \left(\frac{2}{1+e^x}\right)(e^{-x}) = -\frac{2e^{-x}}{1+e^x}$$

$$\Delta v = xy_1 = \frac{2e^x}{1+e^x}$$

$$u = \int \frac{\Delta u}{\Delta} dx = \int \frac{e^{-x}}{1+e^x} dx \quad \text{put } e^{-x} = t$$

$$-e^x dx = dt$$

$$u = -\int \frac{dt}{1+\frac{1}{t}} = -\int \frac{t}{1+t} dt$$

$$= -\int \frac{(t+1)-1}{t+1} dt = -\int \left[1 - \frac{1}{t+1}\right] dt$$

$$u = -\{t - \log(t+1)\} = \log(e^{-x}+1) - e^{-x}$$

$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{2e^x}{-2(1+e^x)} dx = -\log(1+e^x)$$

$$\therefore \quad y_p = e^x[\log(e^{-x}+1) - e^{-x}] - e^{-x}\log(1+e^{-x})$$

The complete solution is $y = y_c + y_p$

ii) A.E. is $D^2 - 4 = 0 \quad \Rightarrow \quad D = \pm 2$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_p = \frac{1}{D^2-4}(e^{4x} + 2x^3)$$

$$= \frac{1}{D^2-4} e^{4x} - \frac{1}{4\left(1 - \frac{D^2}{4}\right)}(2x^3)$$

$$= \frac{1}{16-4} e^{4x} - \frac{1}{4}\left[1 + \frac{D^2}{4} + \frac{D^4}{16} + \ldots\right](2x^3)$$

$$= \frac{1}{12} e^{4x} - \frac{1}{4}\left[2x^3 + \frac{12x}{4} + 0\right]$$

$$y_p = \frac{1}{12} e^{4x} - \frac{1}{2}x^3 - \frac{3}{4}x$$

$$\therefore \quad y = y_c + y_p$$

Example 1.143 : *Solve :*

$(D^2 - 1)y = \cos x \cosh x + 3^x$

SPPU : Dec.-16, Marks 8

Solution :

A.E. is $\quad D^2 - 1 = 0 \Rightarrow D^2 = 1$

$$D = \pm 1$$

$$\therefore \quad y_c = c_1 e^x + c_2 e^{-x}$$

$$\therefore \quad y_P = \frac{1}{D^2-1}[\cos x \cosh x + 3^x]$$

$$= \frac{1}{D^2-1}\left[\left(\frac{e^x - e^{-x}}{2}\right)\cos x + 3^x\right]$$

$$= \frac{1}{2}\frac{1}{D^2-1} e^x \cos x - \frac{1}{2}\frac{1}{D^2-1} e^{-x}\cos x + \frac{1}{D^2-1} 3^x$$

$$= \frac{1}{2} e^x \frac{1}{(D+1)^2 - 1}\cos x - \frac{1}{2}\frac{e^{-x}}{(D-1)^2-1}\cos x + \frac{1}{(\log 3)^2 - 1} 3^x$$

$$= \frac{1}{2} e^x \frac{1}{D^2+2D}\cos x - \frac{1}{2} e^{-x}\frac{1}{D^2-2D}\cos x + \frac{1}{(\log 3)^2-1} 3^x$$

$$= \frac{1}{2} e^x \frac{1}{-1+2D}\cos x - \frac{1}{2} e^{-x}\frac{1}{-1-2D}\cos x + \frac{1}{(\log 3)^2-1} 3^x$$

$$= \frac{1}{2} e^x \frac{(2D+1)}{(2D-1)(2D+1)}\cos x + \frac{1}{2} e^{-x}\frac{2D-1}{(2D+1)(2D-1)}\cos x + \frac{1}{(\log 3)^2-1} 3^x$$

$$= \frac{1}{2} e^x \frac{(2D-1)}{4D^2-1}\cos x + \frac{1}{2} e^{-x}\frac{(2D-1)}{4D^2-1}\cos x + \frac{3^x}{(\log 3)^2-1}$$

$$= \frac{1}{2} e^x \frac{1}{-4-1}(2D-1)\cos x + \frac{1}{2} e^{-x}\frac{(2D-1)}{-4-1}\cos x + \frac{3^x}{(\log 3)^2-1}$$

$$y_p = -\frac{1}{10} e^x[-2\sin x - \cos x] - \frac{1}{10} e^{-x}(-2\sin x - \cos x) + \frac{3^x}{(\log 3)^2-1}$$

$$y_p = -[-2\sin x - \cos x]\frac{1}{10}(e^x + e^{-x}) + \frac{3^x}{(\log 3)^2-1}$$

$\therefore$ The complete solution is $y = y_c + y_p$

Example 1.144 : *Solve :*
i) $(D^2 - 2D)y = e^x \sin x$
ii) $(D^2 + 9)y = x^2 + 2x + \cos 3x$

SPPU : May-16, Marks 8

Solution :

i) A.E. is $D^2 - 2D = 0$

$$D(D-2) = 0$$

$$\Rightarrow \quad D = 0,\ D = 2$$

$$\therefore \quad y_C = C_1 + C_2 e^{2x}$$

$$\text{P.I.} = y_p = \frac{1}{D(D-2)} e^x \sin x$$

$$= \frac{e^x}{(D+1)(D+1-2)} \sin x$$

$$= \frac{e^x}{(D+1)(D-1)} \sin x$$

$$= \frac{e^x}{D^2 - 1} \sin x = e^x \frac{1}{-1-1} \sin x$$

$$y_p = -\frac{e^x}{2} \sin x$$

$\therefore$ The complete solution is $y = y_c + y_p$

$$y = C_1 + C_2 e^{2x} - \frac{e^x}{2} \sin x$$

ii) A.E. is $D^2 + 9 = 0 \Rightarrow D = \pm 3i$

$$\therefore \quad y_c = C_1 \cos 3x + C_2 \sin 3x$$

$$\text{P.I.} = y_p = \frac{1}{D^2+9}[x^2 + 2x + \cos 3x]$$

$$= \frac{1}{9\left[1 + \frac{D^2}{9}\right]}[x^2 + 2x] + \frac{1}{D^2+9} \cos 3x$$

$$= \frac{1}{9}\left[1 - \frac{D^2}{9} + \frac{D^4}{81} \cdots\right](x^2 + 2x) + \frac{x}{2 \times 3} \sin 3x$$

$$y_p = \frac{1}{9}\left[x^2 + 2x - \frac{2}{9}\right] + \frac{x}{6} \sin 3x$$

$\therefore$ The complete solution is $y = y_c + y_p$

Example 1.145 : $(D^2 + 16)y = x \sin 3x + 2^{2x} + 16$

SPPU : Dec.-19

Solution : A.E. is $D^2 + 16 = 0 \Rightarrow D^2 = -16$

$$D = \pm 4i$$

$$y_c = C_1 \cos 4x + C_2 \sin 4x$$

$$y_p = \frac{1}{D^2+16}[x \sin 3x + 4^x + 16]$$

$$= \frac{1}{D^2+16}(x \sin 3x) + \frac{4^x}{(\log 4)^2 + 16} + \frac{16}{0+16}$$

Let

$$I = \frac{1}{D^2+16}(x \sin 3x)$$

$$= \left\{x - \frac{2D}{D^2+16}\right\} \frac{1}{D^2+16}(\sin 3x)$$

$$= \left\{x - \frac{2D}{D^2+16}\right\} \frac{1}{-9+16} \sin 3x$$

$$= \frac{x \sin 3x}{7} - 2D\left(\frac{1}{7}\right) \sin 3x$$

$$= \frac{1}{7}[x \sin 3x] - \frac{2}{7}(3 \cos 3x)$$

$\therefore$ The complete solution is $y = y_c + y_p$

Example 1.146 : *Solve* $\dfrac{d^2y}{dx^2} + 25y = \cot 5x$ *by variation of parameters.*

SPPU : Dec.-19

Solution : A.E. is $D^2 + 25 = 0$

$$D = \pm 5i$$

$$y_c = C_1 \cos 5x + C_2 \sin 5x$$

$$y_1 = \cos 5x,\ y_2 = \sin 5x$$

Assume that $\quad y_p = uy_1 + vy_2$

Now, $$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$= 5 \cos 5x \cos 5x - (-5) 5x \sin 5x = 5$$

$$\Delta u = -\phi(x) y_2 = -\cot 5x \sin 5x = -\cos 5x$$

$$\Delta v = \phi(x) y_1 = \cot 5x \cos 5x$$

$$\therefore \quad u = \int \frac{\Delta u}{\Delta} dx$$

$$u = \int \frac{-\cos 5x}{5} dx = -\frac{1}{5} \frac{\sin 5x}{5}$$

$$= \frac{-1}{25} \sin 5x$$

$$v = \int \frac{\Delta v}{\Delta} dx = \int \frac{\cot 5x \cos 5x \, dx}{5}$$

$$= \frac{1}{5} \int \frac{\cos^2 5x}{\sin 5x} dx = \frac{1}{5} \int \frac{(1 - \sin^2 5x)}{\sin 5x} dx$$

$$= \frac{1}{5} \int [\operatorname{cosec} 5x - \sin 5x] dx$$

$$v = \frac{1}{5} \log(\operatorname{cosec} 5x - \sin 5x) + \frac{1}{25} \cos 5x$$

$$\therefore \quad y_p = uy_1 + vy_2$$

and the complete solution is

$$y = y_c + y_p$$

1.8 University Questions

Dec. - 2016

Q.1 *Solve*

i) $(D^3 - 7D - 6)y = e^{2x}(1+x)$

ii) $(D^2+1)y = 3x - 8 \cot x$ *(by variation of parameter method).*

iii) $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 12x.$ **[8]**

May - 2017

Q.2 *Solve the following :*

i) $(D^2 + 13D + 36)y = e^{-4x} + \sin hx$

ii) $(D^2 - 2D + 2)y = e^x + \tan x$

(using method of variation of parameter)

iii) $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$ **[8]**

Dec. - 2017

Q.3 *Solve the following :*

i) $(D^2 - 4D + 3) = x^3 e^2 x$

ii) $(D^2+4)y = \sec 2x$

(using method of variation of parameter)

iii) $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$ **[8]**

May - 2018

Q.4 *Solve the following :* **[8]**

i) $(D^2 + 2D + 1)y = xe^{-x} \cos x$

ii) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

(using method of variation of parameter)

iii) $(3x+2)^2 + \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y$

$= 3x^2 + 4x + 1.$

Dec. - 2018

Q.5 *Solve the following differential equations :* **[8]**

i) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2 + x + 1$

ii) $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = \sin(\sqrt{3}\log x) + x^3$

iii) $\frac{d^2y}{dx^2} + 4y = \tan 2x$, *by using the method of variation of parameters.*

May - 2019

Q.6 *Solve the following differential equations :* **[8]**

i) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{-3x}\cos 4x + 6e^{2x}$

ii) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 16y$

$= x^2 + 2^{\log x} + 4 \cosh(\log x)$

iii) $\frac{d^2y}{dx^2} + y = \text{cosec } x,$

(by using method of variation of parameters)

Dec. - 2019

Q.7 *Solve the following differential equations :*

i) $\frac{d^2y}{dx^2} + 16y = x \sin 3x + 2^{2x} + 16$

ii) $(2x+1)^2 \frac{d^2y}{dx^2} - 4(2x+1)\frac{dy}{dx} + 8y$

$= \cos[\sqrt{2}\log(2x+1)]$

iii) $\frac{d^2y}{dx^2} + 25y = \cot 5x,$

by using the method of variation of parameters. **[8]**

❑❑❑

UNIT - I

2 Simultaneous Linear Differential Equations and Applications

2.1 Introduction

In this chapter we solve differential equations in which there is one independent variable and two or more dependent variables. For example,

$$\frac{dx}{dt} + y = t$$

$$\frac{dy}{dt} - x = e^t$$

Here t is independent variable and x, y are dependent variables. Such equations are known as simultaneous equations. Methods of solving these equations are similar to those of solving algebraic equations.

Illustrations on Simultaneous Differential Equation

Example 2.1 *The currents x and y in coupled circuits are given by*

$$\mathbf{L\frac{dx}{dt} + Rx + R(x-y) = E}$$

$$\mathbf{L\frac{dy}{dt} + Ry - R(x-y) = 0}$$

where L, R, E are constants. Find x and y in terms of t given x = 0, y = 0, when t = 0.

SPPU : Dec.-14

Solution : Step 1 : Use $D = \frac{d}{dt}$

$$L\,Dx + Rx + Rx - Ry = E$$

$$L\,Dy + Ry - Rx + Ry = 0$$

Collect the terms of x and y.

$$(LD + 2R)\,x - Ry = E \quad \text{... (1)}$$

$$-Rx + (LD + 2R)\,y = 0 \quad \text{... (2)}$$

Step 2 : Solving for x using Cramer's rule.

$$\begin{vmatrix} LD+2R & -R \\ -R & LD+2R \end{vmatrix} x = \begin{vmatrix} E & -R \\ 0 & LD+2R \end{vmatrix}$$

$$(L^2D^2 + 4RLD + 4R^2 - R^2)\,x = (LD+2R)\,E$$

$$(LD+R)(LD+3R)\,x = 2RE \text{ As } DE = \frac{d}{dt}E = 0$$

A.E. is

$$(LD+R)(LD+3R) = 0$$

$$D = \frac{-R}{L},\; D = \frac{-3R}{L}$$

$$x_c = \text{C.F} = C_1e^{-Rt/L} + C_2e^{-3Rt/L}$$

Find P.I for x.

$$x_p = \text{P.I.} = \frac{1}{(LD+R)(LD+3R)}2RE$$

$$x_p = \text{P.I.} = \frac{2RE}{3R^2} = \frac{2E}{3R}$$

Write $x = x_c + x_p$

$$x = C_1e^{-Rt/L} + C_2e^{-3Rt/L} + \frac{2E}{3R} \quad \text{... (3)}$$

Step 3 : Use the equation where the coefficient of y is simple i.e. equation (1).

$$\therefore \quad Ry = (LD + 2R)\,x - E$$

$$Ry = L\frac{dx}{dt} + 2Rx - E$$

Substitute x and $\frac{dx}{dt}$ to find y.

$$Ry = L\left[\frac{-R}{L}C_1e^{-Rt/L} - \frac{3RC_2}{L}e^{-3Rt/L}\right] + 2R\left[C_1e^{-Rt/L} + C_2e^{-3Rt/L} + \frac{2E}{3R}\right] - E$$

$$\therefore \quad Ry = C_1 Re^{-Rt/L} - RC_2 e^{-3Rt/L} + \frac{1}{3}E$$

$$\therefore \quad y = C_1 e^{-Rt/L} - C_2 e^{-3Rt/L} + \frac{E}{3R} \quad \dots (4)$$

Step 4 : Given at t = 0, x = 0 and y = 0

$\therefore$ To find C_1 and C_2 put t = 0, x = 0 in equation (3) and t = 0,

y = 0 equation (4).

$$0 = C_1 + C_2 + \frac{2E}{3R}$$

$$0 = C_1 - C_2 + \frac{E}{3R}$$

Find C_1 and C_2. Adding we get,

$$0 = 2C_1 + \frac{E}{R} \Rightarrow C_1 = \frac{-E}{2R}$$

Substituting we get $C_2 = -\frac{E}{6R}$

Substitute C_1 and C_2 in equations (3) and (4).

$$\therefore \quad x = \frac{-E}{2R} e^{-Rt/L} - \frac{E}{6R} e^{-3Rt/L} + \frac{2E}{3R}$$

$$y = \frac{-E}{2R} e^{-Rt/L} + \frac{E}{6R} e^{-3Rt/L} + \frac{E}{3R}$$

Example 2.2 *Solve* $\frac{dx}{dt} + 2x - 3y = t$

$\frac{dy}{dt} - 3x + 2y = e^{2t}$

Solution : Step 1 :

Use $D = \frac{d}{dt}$ hence equations becomes,

$$Dx + 2x - 3y = t$$

$$Dy - 3x + 2y = e^{2t}$$

Collect the terms of x and y.

$$(D+2)x - 3y = t \quad \dots (1)$$

$$(D+2)y - 3x = e^{2t} \quad \dots (2)$$

Step 2 : Solving for x

$$\begin{vmatrix} D+2 & -3 \\ -3 & D+2 \end{vmatrix} x = \begin{vmatrix} t & -3 \\ e^{2t} & D+2 \end{vmatrix}$$

$$[(D+2)^2 - 3^2]\,x = (D+2)\,t + 3\,e^{2t}$$

$$= 1 + 2t + 3e^{2t}$$

$$(D^2 + 4D - 5)x = 1 + 2t + 3e^{2t} \quad \dots (3)$$

A.E. is $D^2 + 4D - 5 = 0$ gives $D = -5, 1$ hence

$$x_c = C_1 e^{-5t} + C_2 e^t \quad \dots (4)$$

Now

$$x_p = \text{P.I.}$$

$$= \frac{1}{D^2 + 4D - 5}(1+2t) + \frac{3e^{2t}}{D^2 + 4D - 5}$$

$$= -\frac{1}{5}\left[1 - \frac{4D + D^2}{5}\right](1+2t) + \frac{3e^{2t}}{4+8-5}$$

$$= -\frac{1}{5}\left(1 + \frac{4D}{5}\right)(1+2t) + \frac{3}{7}e^{2t}$$

$$= -\frac{1}{5}\left(\frac{13}{5} + 2t\right) + \frac{3e^{2t}}{7} \quad \dots (5)$$

Hence G.S. is given by,

$$x = \text{C.F.} + \text{P.I.} = C_1 e^{-5t} + C_2 e^t - \frac{13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7} \quad \dots (6)$$

Now

$$\frac{dx}{dt} = -5C_1 e^{-5t} + C_2 e^t - \frac{2}{5} + \frac{6}{7}e^{2t}$$

Step 3 : Use the equation where coefficient of y is simple.

i.e. $(D+2)x - 3y = t$

$$3y = (D+2)x - t$$

Putting values of x and $\frac{dx}{dt}$ to find y.

$$y = \frac{1}{3}\left[\frac{dx}{dt} + 2x - t\right]$$

$$= \frac{1}{3}\left[-5C_1 e^{-5t} + C_2 e^t - \frac{2}{5} + \frac{6}{7}e^{2t} + 2C_1 e^{-5t} + 2C_2 e^t - \frac{26}{25} - \frac{4t}{5} + \frac{6e^{2t}}{7} - t\right]$$

Simplifying we get,

$$y = -C_1 e^{-5t} + C_2 e^{2t} - \frac{12}{25} - \frac{3t}{5} + \frac{4e^{2t}}{7} \quad \dots (7)$$

Example 2.3 $\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$,

$\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t$

Solution : Step 1 :

Use $D = \frac{d}{dt}$ hence system can be written as,

$$D^2x + Dy + 3x = e^{-t}$$

$$D^2y - 4Dx + 3y = \sin 2t$$

Collect the terms of x and y.

$$(D^2+3)\,x + D\,y = e^{-t}$$

$$-4\,Dx + (D^2+3)\,y = \sin 2t$$

Step 2 : Solving for x by Cramer's rule.

$$\begin{vmatrix} D^2+3 & D \\ -4D & D^2+3 \end{vmatrix} x = \begin{vmatrix} e^{-t} & D \\ \sin 2t & D^2+3 \end{vmatrix}$$

$$\left[(D^2+3)^2 + 4D^2\right] x = 4e^{-t} - 2\cos 2t$$

$$(D^2+1)(D^2+9)\,x = 4e^{-t} - 2\cos 2t$$

$(D^2+1)(D^2+9) = 0$ gives $\therefore$ $D = \pm i, \ \pm 3i$

$\therefore$ C.F. is $x_c = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t$

$$x_p = \text{P.I.}$$

$$= \frac{1}{(D^2+1)(D^2+9)} 4 \cdot e^{-t} - \frac{1}{(D^2+1)(D^2+9)} (2\cos 2t)$$

$$= \frac{1}{5} e^{-t} + \frac{2}{15} \cos 2t$$

$\therefore$ General solution for x = C.F. + P.I.

$\therefore$
$$x = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{1}{5} e^{-t} + \frac{2}{15} \cos 2t$$

Step 3 : And similarly solving for y we get,

$$(D^2+1)(D^2+9)\,y = -\sin 2t - 4e^{-t}$$

Auxiliary equation for y is same,

$(D^2+1)(D^2+9) = 0$ $\quad\therefore D = \pm i, \ \pm 3i$

$\therefore$ C.F. is $y_C = C_5 \cos t + C_6 \sin t + C_7 \cos 3t + C_8 \sin 3t$

P.I. for $y_p = \dfrac{1}{(D^2+1)(D^2+9)} (-\sin 2t - 4e^{-t})$

$$= (-1) \frac{1}{(D^2+1)(D^2+9)} \sin 2t - 4 \frac{1}{(D^2+1)(D^2+9)} e^{-t} = +\frac{1}{15} \sin 2t - \frac{1}{5} e^{-t}$$

$\therefore$ General solution for y = C.F. + P.I.

$\therefore$
$$y = C_5 \cos t + C_6 \sin t + C_7 \cos 3t + C_8 \sin 3t + \frac{1}{15} \sin 2t - \frac{1}{5} e^{-t}$$

Substituting these values of x and y in any one of given equations we get,

$C_5 - 2C_2,\ C_6 = -2C_1,\ C_7 = -2C_4,\ C_8 = 2C_3$

By comparing the coefficients of functions of t.

Step 4 : $\therefore$ Required solution for the given system is,

$$x = C_1 \cos t + C_2 \sin t + C_3 \cos 3t + C_4 \sin 3t + \frac{1}{5} e^{-t} + \frac{2}{15} \cos 2t$$

and
$$y = 2C_2 \cos t - 2C_1 \sin t - 2C_4 \cos 3t + 2C_3 \sin 3t - \frac{1}{5} e^{-t} + \frac{1}{15} \sin 2t$$

Example 2.4 *Solve* $\frac{dx}{dt} - \omega y = a \cos pt,\ \frac{dy}{dt} + \omega x = a \sin pt\ (\omega \neq p)$

Solution : Step 1 : Let $D = \frac{d}{dt}$

$\therefore$ Given system becomes,

$$Dx - \omega y = a \cos pt \quad \ldots (1)$$

$$\omega x + Dy = a \sin pt \quad \ldots (2)$$

Step 2 : Solving for x by Cramer's rule, we get,

$$\begin{vmatrix} D & -\omega \\ \omega & D \end{vmatrix} x = \begin{vmatrix} a \cos pt & -\omega \\ a \sin pt & D \end{vmatrix}$$

$$(D^2 + \omega^2)\, x = -ap \sin pt + a\,\omega \sin pt = a(\omega - p) \sin pt \quad \ldots (3)$$

A.E. of equation (3) is $D^2 + \omega^2 = 0$

$$D = \pm i\,\omega$$

The C.F. of equation (3) is $x_c = C_1 \cos \omega t + C_2 \sin \omega t$

The P.I. of equation (3) is

$$x_p = \frac{1}{D^2 + \omega^2} a(\omega - p) \sin pt = a(\omega - p) \frac{1}{-p^2 + \omega^2} \sin pt = \frac{a}{\omega + p} \sin pt$$

The complete solution of equation (3) is

$$x = C_1 \cos \omega t + C_2 \sin \omega t + \frac{a}{\omega + p} \sin pt \quad \ldots (4)$$

Step 3 : Substituting x in equation (1) we get,

$$\omega y = Dx - a \cos pt = -C_1\, \omega \sin \omega t + C_2\, \omega \cos \omega t + \frac{ap}{\omega + p} \cos pt - a \cos pt$$

$$\therefore \quad y = -C_1 \sin \omega t + C_2 \cos \omega t + \frac{ap}{\omega(\omega + p)} \cos pt - \frac{a}{\omega} \cos pt \quad \ldots (5)$$

Step 4 : Equations (4) and (5) together gives the complete solution of given system.

Exercise 2.1

1. Solve $\dfrac{du}{dx} + v = \sin x$

$\dfrac{dv}{dx} + u = \cos x$ given that when $x = 0,\ u = 1,\ v = 0.$

[Ans. : $u = \cosh x,\ v = \sin x - \sinh x$]

2. $\dfrac{dx}{dt} + 5x - 2y = t$

$\dfrac{dy}{dt} + 2x + y = 0$ given that $x = 0,\ y = 0$ at $t = 0.$

[Ans. : $x = -\dfrac{1}{27}(1+6t)\,e^{-3t} + \dfrac{1}{27}(1+3t)$

$y = -\dfrac{2}{27}(2+3t)\,e^{-3t} + \dfrac{2}{27}(2-3t)$]

3. $\dfrac{dx}{dt} + y = e^t$

$-\dfrac{dy}{dt} + x = e^{-t}$

[Ans. : $x = C_1 \cos t + C_2 \sin t + \dfrac{1}{2}\left(e^t - e^{-t}\right)$

$C_1 \sin t - C_2 \cos t + \dfrac{1}{2}\left(e^t - e^{-t}\right)$]

4. The equations of motion of a particle are given by

$\dfrac{dx}{dt} + \omega y = 0,\quad \dfrac{dy}{dt} - \omega x = 0.$

Find the path of the particle.

[Ans. : $x = A\cos\omega t + B\sin\omega t$

$y = A\sin\omega t - B\cos\omega t$]

5. Solve the simultaneous equations for r and θ

$\dfrac{dr}{dt} - 2r - \theta = 0$

$\dfrac{d\theta}{dt} + r - 4\theta = 0$

Given that $\theta(0) = 0$ and $r'(0) = 6$

[Ans. : $r = 3\left(e^{3t} - t\,e^{3t}\right)$

$\theta = -3t\ e^{3t}$]

6. A mechanical system with two degrees 0 freedom satisfies the equations,

$2\dfrac{d^2x}{dt^2} + 3\dfrac{dy}{dt} = 4$ and $2\dfrac{d^2y}{dt^2} - 3\dfrac{dx}{dt} = 0.$

Obtain expressions for x and y in terms of t given that x, y $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ all vanish at $t = 0.$

[Ans. : $x = -\dfrac{8}{9}\cos\dfrac{3}{2}t + \dfrac{8}{9}$

$y = \dfrac{4}{3}t - \dfrac{8}{9}\sin\dfrac{3t}{2}$]

7. $2\dfrac{dx}{dt} - x + 3y = \sin t$

$2\dfrac{dy}{dt} + 3y - y = \cos t$

Given $x = \dfrac{1}{4},\ y = \dfrac{-1}{20}$ at $t = 0.$

[Ans. : $x = \dfrac{1}{10}\left(e^{2t} + e^{-t}\right) + \dfrac{1}{20}(\cos t + 2\sin t)$

$y = \dfrac{-1}{10}e^{2t} + \dfrac{e^{-t}}{10} + \dfrac{2}{5}\sin t - \dfrac{1}{20}\cos t$]

8. The small oscillations of a certain system are given by

$D^2x + 3x - 2y = 0$

$D^2x + D^2y - 3x + 5y = 0$

Given $x = 0,\ y = 0,\ \dfrac{dx}{dt} = 3,\ \dfrac{dy}{dt} = 2$ at $t = 0$. Find x and y.

[Ans. : $x = \dfrac{11}{4}\sin(t) + \dfrac{1}{12}\sin(3t)$

$y = \dfrac{11}{4}\sin(t) - \dfrac{1}{4}\sin(3t)$]

2.2 Symmetrical form of Simultaneous Differential Equations

General form $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

where P, Q, R are functions of x, y, z are said to be symmetrical simultaneous differential equations. The solution of such a system of equations is given by a pair of relations $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$ which are independent of each other. We can solve such a system by following methods.

a) Method of combinations or grouping :

In this method we select two groups such that the third variable gets eliminated i.e. we get the variable separable form, then integration of this gives one part of the solution.

Similarly find second solution which is independent of first solution.

Illustrations on Type 1

Example 2.5 : $\dfrac{dx}{xy} = \dfrac{dy}{x^2} = \dfrac{dz}{xyz}$

Solution : By combinations

$$\frac{dx}{xy} = \frac{dy}{x^2}$$

$$\frac{x^2\,dx}{x} = y\,dy$$

$$x\,dx = y\,dy$$

which is variable separable integrating we get,

$$\frac{x^2}{2} = \frac{y^2}{2}+C$$

i.e. $$x^2 - y^2 = C_1 \quad \text{... (1)}$$

Again by combinations

$$\frac{dx}{xy} = \frac{dz}{xyz}$$

$$\frac{dx}{1} = \frac{dz}{z}$$

which is variable separable.

Integrating we get,

$$x = \log z + C_2$$

$$x - \log z = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.6 : $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{x^2y^2z^2}$

Solution : By combinations

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating we get,

$$-\frac{1}{x} = -\frac{1}{y}+C$$

i.e. $$\frac{1}{y} - \frac{1}{x} = C_1 \quad \text{... (1)}$$

Again by combinations,

$$\frac{dx}{y^2} = \frac{dz}{x^2y^2z^2}$$

$$\therefore \quad x^2\,dx = \frac{dz}{z^2}$$

Integrating we get

$$\frac{x^3}{3} = -\frac{1}{z}+C$$

i.e. $$\frac{x^3}{3}+\frac{1}{z} = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.7 : $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{nxy}$

Solution : By combinations,

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get,

$$\log x = \log y + \log C_1$$

$$\log\frac{x}{y} = \log C_1$$

$$\frac{x}{y} = C_1 \quad \text{... (1)}$$

Again by combinations,

$$\frac{dy}{y} = \frac{dz}{nxy}$$

$$x\,dy = \frac{dz}{n}$$

Now from equation (1) $x = y\,C_1$

$$\therefore \quad y\,C_1\,dy = \frac{dz}{n}$$

Integrating we get,

$$\frac{C_1}{2}y^2 = \frac{z}{n}+C_2$$

Again substitute $C_1 = \frac{x}{y}$

$$\therefore \quad \frac{x}{2y}\cdot y^2 - \frac{z}{n} = C_2$$

$$\frac{xy}{2} - \frac{z}{n} = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.8 : $\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{4xy^2-z}$

Solution : By combinations,

$$\frac{dx}{2x} = \frac{dy}{-y}$$

Integrating we get,

$$\frac{1}{2}\log x + \log y = \log C_1$$

$$\log y\sqrt{x} = \log C_1$$

$$y\sqrt{x} = C_1$$

i.e. $$xy^2 = C_1 \quad \text{... (1)}$$

Again by combinations,

$$\frac{dx}{2x} = \frac{dz}{4xy^2 - z}$$

From equation (1) $xy^2 = C_1$

$$\therefore \quad \frac{dx}{2x} = \frac{dz}{4C_1 - z}$$

Integrating we get,

$$\frac{1}{2}\log x = -\log(4C_1 - z) + \log C_2$$

$$\therefore \quad \log\sqrt{x}(4C_1 - z) = \log C_2$$

$$\sqrt{x}(4C_1 - z) = C_2$$

Again $\quad C_1 = xy^2$

$$\therefore \quad \sqrt{x}\left(4xy^2 - z\right) = C_2 \qquad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.9 : $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}$

Solution : By combinations,

$$\frac{dx}{1} = \frac{dy}{3}$$

$\therefore$ Integrating we get,

$$x = \frac{y}{3} + C$$

$$y - 3x = C_1 \qquad \text{... (1)}$$

Again by combinations,

$$\frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}$$

i.e.

$$\frac{dy}{3} = \frac{dz}{5z + \tan C_1}$$

Integrating we get,

$$\frac{y}{3} = \frac{1}{5}\log[5z + \tan C_1] + C$$

$$5y - 3\log[5z + \tan(y - 3x)] = C_2 \qquad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.10 : $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{\left(x^2 - 3y^2\right)y}$

Solution : By combinations,

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$x\,dx + y\,dy = 0$$

Integrating we get,

$$\frac{x^2}{2} + \frac{y^2}{2} = C$$

$$x^2 + y^2 = C_1 \qquad \text{... (1)}$$

Again by combinations,

$$\frac{dx}{y} = \frac{dz}{\left(x^2 - 3y^2\right)y}$$

Using equation (1) $\quad y^2 = C_1 - x^2$

$$\frac{dx}{1} = \frac{dz}{[x^2 - 3(C_1 - x^2)]}$$

$$\left(x^2 - 3C_1 + 3x^2\right)dx = dz$$

$$\left(4x^2 - 3C_1\right)dx = dz$$

Integrating we get,

$$4\frac{x^3}{3} - 3C_1 x = z + C_2$$

$$\frac{4}{3}x^3 - 3\left(x^2 + y^2\right)x - z = C_2 \qquad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.11 : $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$

Solution : By combinations,

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get,

$$\log x - \log y = \log C_1$$

$$\frac{x}{y} = C_1 \qquad \text{... (1)}$$

Again by combinations,

$$\frac{dx}{xy} = \frac{dz}{xyz - 2x^2}$$

From equation (1) $y = \frac{x}{C_1}$

$$\therefore \quad \frac{dx}{x \cdot \frac{x}{C_1}} = \frac{dz}{x \cdot \frac{x}{C_1} \cdot z - 2x^2}$$

$$C_1\,dx = \frac{dz}{\left(\frac{z}{C_1} - 2\right)}$$

$$C_1\, dx = \frac{C_1\, dz}{z-2C_1}$$

$$\frac{dx}{1} = \frac{dz}{z-2C_1}$$

Integrating we get,

$$x = \log(z-2C_1)+C_2$$

$$x-\log\left(z-2\frac{x}{y}\right) = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.12 : $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{x e^{x^2+y^2}}$

Solution : By combinations,

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$x\, dx + y\, dy = 0$$

Integrating,

$$\frac{x^2}{2}+\frac{y^2}{2} = C$$

$$x^2+y^2 = C_1 \quad \text{... (1)}$$

Again by combination,

$$\frac{dy}{-x} = \frac{dz}{x e^{x^2+y^2}}$$

$$\frac{dy}{-1} = \frac{dz}{e^{C_1}}$$

$$e^{C_1} \cdot dy = -dz$$

Integrating we get,

$$y e^{C_1} = -z+C_2$$

$$y e^{x^2+y^2}+z = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.13 : $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-a\sqrt{x^2+y^2+z^2}}$

Solution : By combinations,

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get,

$$\log x = \log y+\log C_1$$

$$x = y C_1 \quad \text{... (1)}$$

Again by combinations,

$$\frac{dy}{y} = \frac{dz}{z-a\sqrt{x^2+y^2+z^2}}$$

Using equation (1) $x = y\, C_1$

$$\frac{dy}{y} = \frac{dz}{z-a\sqrt{y^2 C_1^2+y^2+z^2}}$$

$$\frac{dz}{dy} = \frac{z-a\sqrt{(1+C_1^2)\,y^2+z^2}}{y}$$

Which is homogeneous $\therefore$ put $z = uy$

$$\therefore \quad \frac{dz}{dy} = u+y\frac{du}{dy}$$

$\therefore$ The equation becomes,

$$u+y\frac{du}{dy} = \frac{uy-a\sqrt{(1+C_1^2)\,y^2+u^2y^2}}{y}$$

$$= u-a\sqrt{(1+C_1^2)+u^2}$$

$$\frac{du}{\sqrt{(1+C_1^2)+u^2}} = \frac{a\,dy}{-y}$$

which is variable separable.

$\therefore$ Integrating we get,

$$\log\left[u+\sqrt{(1+C_1^2)+u^2}\right] = -a\log y+\log C_2$$

$$u+\sqrt{(1+C_1^2)+u^2} = \frac{C_2}{y^a}$$

Put $u = \frac{z}{y}$ and $C_1 = \frac{x}{y}$

$$\frac{z}{y}+\sqrt{1+\left(\frac{x}{y}\right)^2+\left(\frac{z}{y}\right)^2} = \frac{C_2}{y^a}$$

$$\frac{z+\sqrt{x^2+y^2+z^2}}{y} = \frac{C_2}{y^a}$$

$$z+\sqrt{x^2+y^2+z^2} = C_2\, y^{1-a}$$

b) Use of Multipliers

We know the property of ratio and proportion

$$\text{if } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx+m dy+n dz}{l P+m Q+n R}$$

$$= \frac{l_1 dx+m_1 dy+n_1 dz}{l_1 P+m_1 Q+n_1 R}$$

where l, m, n and l_1, m_1, n_1 are the multipliers constants or variables.

We choose l, m, n and l_1, m_1, n_1 in such a way that $lP + mQ + nR = 0$ and $l_1P + m_1Q + n_1R = 0$ then we have,

$l dx + m dy + n dz = 0$ and $l_1 dx + m_1 dy + n_1 dz = 0$

Integrating we get,

$$u(x, y, z) = C_1 \quad \dots (1)$$

and

$$v(x, y, z) = C_2 \quad \dots (2)$$

Equations (1) and (2) together constitute the solution of the system.

Note : Standard sets of multipliers (l, m, n), (lx, my, nz), (x, y, z), $\left(\frac{l}{x}, \frac{m}{y}, \frac{n}{z}\right)$, (lx^2, my^2, nz^2), $\left(\frac{l}{x^2}, \frac{m}{y^2}, \frac{n}{z^2}\right)$, (lx^3, my^3, nz^3) and so on where l, m, n are any constants.

Example 2.14 : $\dfrac{dx}{mz - ny} = \dfrac{dy}{nx - lz} = \dfrac{dz}{ly - mx}$

Solution : Use multipliers l, m, n

$$\therefore \text{Each ratio} = \frac{l dx + m dy + n dz}{lmz - lny + mnx - lmz + lny - mnx} = \frac{l dx + m dy + n dz}{0}$$

$$\Rightarrow \quad l dx + m dy + n dz = 0$$

Integrating we get,

$$lx + my + nz = C_1 \quad \dots (1)$$

Again use multipliers x, y, z

$$\therefore \text{Each ratio} = \frac{x dx + y dy + z dz}{mzx - nxy + nxy + lzy - lyz - mxz} = \frac{x dx + y dy + z dz}{0}$$

$$\Rightarrow \quad x dx + y dy + z dz = 0$$

Integrating we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

$$x^2 + y^2 + z^2 = C_2 \quad \dots (2)$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.15 : $\dfrac{dx}{x(2y^4 - z^4)} = \dfrac{dy}{y(z^4 - zx^4)} = \dfrac{dz}{z(x^4 - y^4)}$

Solution :

Use multipliers $\frac{1}{x}, \frac{1}{y}, \frac{2}{z}$

$$\therefore \quad \text{Each ratio} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{2}{z}dz}{2y^4 - z^4 + z^4 - 2x^4 + 2x^4 - 2y^4} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{2dz}{z}}{0}$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} + \frac{2}{z}dz = 0$$

Integrating we get,

$$\log x + \log y + 2\log z = \log C_1$$

$$\log xyz^2 = \log C_1$$

$$xyz^2 = C_1 \quad \dots (1)$$

Again use multipliers x^3, y^3, z^3

$$\therefore \text{Each ratio} = \frac{x^3 dx + y^3 dy + z^3 dz}{x^4(2y^4 - z^4) + y^4(z^4 - 2x^4) + z^4(x^4 - y^4)} = \frac{x^3 dx + y^3 dy + z^3 dz}{0}$$

$$\Rightarrow \quad x^3 dx + y^3 dy + z^3 dz = 0$$

Integrating we get,

$$\frac{x^4}{4} + \frac{y^4}{4} + \frac{z^4}{4} = C$$

$$x^4 + y^4 + z^4 = C_2 \quad \dots (2)$$

Equations (1) and (2) constitute the solution of the system.

Example 2.16 : $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{-y(z^2 + x^2)} = \dfrac{dz}{z(x^2 + y^2)}$

Solution :

Use multipliers $\frac{-1}{x}, \frac{1}{y}, \frac{1}{z}$

$$\therefore \quad \text{Each ratio} = \frac{-\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{-y^2 + z^2 - z^2 - x^2 + x^2 + y^2} = \frac{-\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \quad \frac{-dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating we get,

$$-\log x + \log y + \log z = \log C_1$$

$$\frac{yz}{x} = C_1 \quad \text{... (1)}$$

Again use multipliers x, y, z

$$\therefore \quad \text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

Integrating we get,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

$$x^2 + y^2 + z^2 = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.17 : $\dfrac{x\,dx}{z^2 - 2yz - y^2} = \dfrac{dy}{y+z} = \dfrac{dz}{y-z}$

Solution : By combinations,

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

$$y\,dy - z\,dy = y\,dz + z\,dz$$

$$y\,dz + z\,dy = y\,dy - z\,dz$$

$$d(yz) = y\,dy - z\,dz$$

Integrating,

$$yz = \frac{y^2}{2} - \frac{z^2}{2} + C$$

$$2yz - y^2 + z^2 = C_1 \quad \text{... (1)}$$

Use multipliers (1, y, z)

$$\text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\Rightarrow xdx + ydy + zdz = 0$

Integrating we get,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

$$x^2 + y^2 + z^2 = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Use of property of ratio and proportion :

Example 2.18 : $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{(1 + 2xy + 3x^2y^2)(x+y)z}$

Solution : By combination,

$$\frac{dx}{1} = \frac{dy}{1}$$

Integrating we get,

$$x - y = C_1 \quad \text{... (1)}$$

Also from first two groups

$$\text{Each ratio} = \frac{y\,dx + x\,dy}{y + x}$$

Equating with the third group,

$$\frac{y\,dx + x\,dy}{x+y} = \frac{dz}{(1 + 2xy + 3x^2y^2)(x+y)\,z}$$

$$(1 + 2xy + 3x^2y^2)\,d(xy) = \frac{dz}{z}$$

Put xy = u

$$(1 + 2u + 3u^2)\,du = \frac{dz}{z}$$

Integrating we get,

$$u + u^2 + u^3 = \log z + C_2$$

$$xy + x^2y^2 + x^3y^3 - \log z = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.19 : $\dfrac{dx}{y+z} = \dfrac{dy}{x+z} = \dfrac{dz}{x+y}$

Solution :

$$\text{Each ratio} = \frac{dx + dy + dz}{2(x+y+z)} \quad \text{... (A)}$$

Also $\quad \text{Each ratio} = \dfrac{dx - dy}{y - x}$ i.e. $\dfrac{dx - dy}{-(x-y)}$ ... (B)

Again $\quad \text{Each ratio} = \dfrac{dy - dz}{-(y-z)}$ ... (C)

$$\text{Each ratio} = \frac{dx - dz}{-(x-z)} \quad \text{... (D)}$$

Equation (A) = (B) gives,

$$\frac{dx + dy + dz}{2(x+y+z)} = \frac{dx - dy}{-(x-y)}$$

Integrating we get,

$$\frac{1}{2}\log(x+y+z) + \log(x-y) = \log C_1$$

$$(x-y)\sqrt{x+y+z} = C_1 \quad \text{... (1)}$$

Equation (C) = (D) gives,

$$\frac{dy - dz}{y - z} = \frac{dx - dz}{x - z}$$

Integrating we get,

$$\log(y - z) = \log(x - z) + \log C_2$$

$$\frac{y-z}{x-z} = C_2 \qquad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.20 : $\dfrac{dx}{x^2-yz} = \dfrac{dy}{y^2-zx} = \dfrac{dz}{z^2-xy}$

Solution :

$$\text{Each ratio} = \frac{dx-dy}{(x+y+z)(x-y)} \qquad \text{... (A)}$$

$$= \frac{dy-dz}{(x+y+z)(y-z)} \qquad \text{... (B)}$$

$$= \frac{dx-dz}{(x+y+z)(x-z)} \qquad \text{... (C)}$$

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} \qquad \text{... (D)}$$

$$= \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz} \qquad \text{... (E)}$$

Equation (A) = (B) gives,

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

Integrating we get,

$$\log(x-y)-\log(y-z) = \log C_1$$

$$\frac{x-y}{y-z} = C_1 \qquad \text{... (1)}$$

Equation (D) = (E) gives,

$$\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$= \frac{x\,dx+y\,dy+z\,dz}{\left(x^2+y^2+z^2-xy-yz-zx\right)(x+y+z)}$$

As $x^3+y^3+z^3-3xyz$

$= (x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)$

Thus $(x+y+z)\cdot[dx+dy+dz] = xdx + ydy + zdz$

Integrating we get,

$$\frac{(x+y+z)^2}{2} - \frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = C$$

i.e. $(x+y+z)^2-x^2-y^2-z^2 = C_2$

i.e. $x^2+y^2+z^2+2xy+2yz+2zx-x^2-y^2-z^2 = C_2$

$$2(xy+yz+zx) = C_2 \qquad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.21 : $\dfrac{dx}{x^2+y^2-yz} = \dfrac{dy}{xz-x^2-y^2} = \dfrac{dz}{xz-yz}$

Solution : Using multipliers 1, 1, – 1

$$\text{Each ratio} = \frac{dx+dy-dz}{0}$$

$\Rightarrow$ $dx + dy - dz = 0$

Integrating we get,

$$x + y - z = C_1 \qquad \text{... (1)}$$

$$\text{Again each ratio} = \frac{x\,dx+y\,dy}{x^3+xy^2-x^2y-y^3}$$

$$= \frac{x\,dx+y\,dy}{x\left(x^2+y^2\right)-y\left(x^2+y^2\right)} = \frac{x\,dx+y\,dy}{(x-y)\left(x^2+y^2\right)}$$

Equating with the third,

$$\frac{dz}{z(x-y)} = \frac{x\,dx+y\,dy}{(x-y)\left(x^2+y^2\right)}$$

Integrating we get,

$$\log z = \frac{1}{2}\log\left(x^2+y^2\right)+\log C_2$$

$$\log\frac{z}{\sqrt{x^2+y^2}} = \log C_2$$

$$\frac{z}{\sqrt{x^2+y^2}} = C_2 \qquad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.22 : $\dfrac{dx}{y^3x-2x^4} = \dfrac{dy}{2y^4-x^3y} = \dfrac{dz}{9z\left(x^3-y^3\right)}$

Solution :

$$\frac{dx}{x\left(y^3-2x^3\right)} = \frac{dy}{y\left(2y^3-x^3\right)} = \frac{dz}{9z\left(x^3-y^3\right)}$$

Use multipliers $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{3z}$

$$\text{Each ratio} = \frac{\dfrac{dx}{x}+\dfrac{dy}{y}+\dfrac{dz}{3z}}{0}$$

$$\Rightarrow \quad \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{3z} = 0$$

Integrating we get,

$$\log x+\log y+\frac{1}{3}\log z = \log C$$

$$\log x \cdot y \cdot z^{1/3} = \log C$$

$$x y z^{1/3} = C$$

$$x^3 y^3 z = C_1 \quad \text{... (1)}$$

$$\text{Each ratio} = \frac{x^2 dx + y^2 dy}{-2x^6 + 2y^6}$$

Equate with the third group

$$\frac{dz}{9z(x^3 - y^3)} = \frac{x^2 dx + y^2 dy}{-2(x^3 - y^3)(x^3 + y^3)}$$

$$\frac{-2}{3}\frac{dz}{z} = \frac{3x^2 dx + 3y^2 dy}{x^3 + y^3}$$

Integrating we get,

$$-\frac{2}{3}\log z = \log(x^3 + y^3) + \log C$$

$$\log(x^3 + y^3) + \frac{2}{3}\log z = -\log C$$

$$3\log(x^3 + y^3) + 2\log z = \log C_2$$

$$\log(x^3 + y^3)^3 \cdot z^2 = \log C_2$$

$$(x^3 + y^3)^3 \cdot z^2 = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.23 : $\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{dz}{z(x+y)^3}$

Solution :

$$\text{Each ratio} = \frac{dx + dy}{x^2 + y^2 + 2xy}$$

$$= \frac{dx + dy}{(x+y)^2} \quad \text{... (A)}$$

$$\text{Also each ratio} = \frac{dx - dy}{(x-y)^2} \quad \text{... (B)}$$

From equations (A) and (B)

$$\frac{dx + dy}{(x+y)^2} = \frac{dx - dy}{(x-y)^2}$$

Let $x + y = u$, $x - y = v$

$dx + dy = du$, $dx - dy = dv$

$$\therefore \quad \frac{du}{u^2} = \frac{dv}{v^2}$$

Integrating,

$$\frac{-1}{u} = \frac{-1}{v} + C$$

i.e.

$$\frac{-1}{x+y} + \frac{1}{x-y} = C_1$$

Also

$$\frac{dz}{z(x+y)^3} = \frac{dx + dy}{(x+y)^2}$$

i.e.

$$\frac{dz}{z u^3} = \frac{du}{u^2}$$

$$\frac{dz}{z} = u\, du$$

Integrating

$$\log z = \frac{u^2}{2} + C_2$$

$$\log z - \frac{(x+y)^2}{2} = C_2 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.24 : *Solve* $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$

Solution : Consider $\frac{dy}{2xy} = \frac{dz}{2xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$

Integrating, we get, $\log y = \log z + \log C_1$

$$y = C_1 z \quad \text{... (1)}$$

Let x, y, z be the set of multipliers for given equations then

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{x^3 + xy^2 + xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$$

Consider

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} = \frac{dy}{2xy}$$

$$\frac{2(x\,dx + y\,dy + z\,dz)}{x^2 + y^2 + z^2} = \frac{dy}{y}$$

Integrating we get,

$$\log(x^2 + y^2 + z^2) = \log y + \log C_2$$

$$x^2 + y^2 + z^2 - C_2 y = 0 \quad \text{... (2)}$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.25 : *Solve* $\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y}$

Solution : Choosing the multipliers 1, 1, 1

$$\text{Each ratio} = \frac{dx + dy + dz}{y + z - x - z + x - y}$$

$$\Rightarrow \quad dx + dy + dz = 0$$

Integrating, we get

$$x + y + z = C_1 \quad \dots (1)$$

Choosing another set of multipliers x, y, – z

$$\text{Each ratio} = \frac{x\,dx + y\,dy - z\,dz}{xy + xz - yx - yz - zx + yz}$$

$$\Rightarrow \quad x\,dx + y\,dy - z\,dz = 0$$

$$\frac{x^2}{z} + \frac{y^2}{z} - \frac{z^2}{z} = C_2 \quad \dots (2)$$

Equations (1) and (2) together constitute the solution of the system.

Example 2.26 : $\dfrac{dx}{3z-4y} = \dfrac{dy}{4x-2z} = \dfrac{dz}{2y-3x}.$

Solution : Using multipliers x, y, z, we get,

$$\text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0$$

Integrating we get,

$$x^2 + y^2 + z^2 = C_1 \quad \dots (1)$$

Now, using another set of multipliers 2, 3, 4, we get,

$$\text{Each ratio} = \frac{2\,dx + 3\,dy + 4\,dz}{0}$$

$$\Rightarrow \quad 2\,dx + 3\,dy + 4\,dz = 0$$

Integrating,

$$2x + 3y + 4z = C_2 \quad \dots (2)$$

∴ The equations (1) and (2) together constitutes the complete solution of given equations.

Example 2.27 : *Solve* $\dfrac{dx}{yz} = \dfrac{dy}{zx} = \dfrac{dz}{(x+y)(z^2+1)}$

Solution : Considering first and second ratio,

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow x\,dx = y\,dy$$

Integrating we get,

$$x^2 - y^2 = C_1 \quad \dots (1)$$

Adding first and second equating with third ratio, we get,

$$\frac{dx + dy}{z(x+y)} = \frac{dz}{(x+y)(z^2+1)}$$

$$dx + dy = \frac{z}{z^2+1}\,dz$$

Integrating, we get,

$$x + y - \frac{1}{2}\log(z^2+1) = C_2 \quad \dots (2)$$

The equations (1) and (2) together constitutes the complete solution of given equation.

Example 2.28 : $\dfrac{dx}{y-z} = \dfrac{dy}{z-x} = \dfrac{dz}{x-y}$

Solution : Using multipliers 1, 1, 1 we get,

$$\text{Each ratio} = \frac{dx + dy + dz}{y - z + z - x + x - y} = \frac{dx + dy + dz}{0}$$

$$\Rightarrow dx + dy + dz = 0$$

Integrating we get,

$$x + y + z = C_1 \quad \dots (1)$$

Now, again using multipliers x, y, z we get

$$\text{Each ratio} = \frac{x\,dx + y\,dy + z\,dz}{xy - zx + zy - xy + xz - yz} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$x\,dx + y\,dy + z\,dz = 0$$

Integrating, we get,

$$x^2 + y^2 + z^2 = C_2 \quad \dots (2)$$

Equations (1) and (2) together constitutes the complete solution of given equations.

Exercise 2.2

Solve the following

1. $\dfrac{dx}{yz^2} = \dfrac{dy}{xz^2} = \dfrac{dz}{xy^2}$ **[Ans. : $x^2 - y^2 = C_1, y^3 - z^3 = C_2$]**

2. $\dfrac{dx}{y^2z} = \dfrac{dy}{zx^2} = \dfrac{dz}{xy^2}$ **[Ans. : $x^3 - y^3 = C_1, x^2 - z^2 = C_2$]**

3. $\dfrac{dx}{xz} = \dfrac{dy}{yz} = \dfrac{2dz}{(x+y)^2}$ **[Ans. : $y = C_1x, (x+y)^2 - 2z^2 = C_2$]**

4. $\dfrac{dx}{y} = \dfrac{dy}{-x} = \dfrac{dz}{2x-3y}$ **[Ans. : $x^2 + y^2 = C_1, 3x + 2y + z = C_2$]**

5. $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$ **[Ans. : $x^2 - y^2 - 2xy = C_1, x^2 - y^2 - z^2 = C_2$]**

6. $\dfrac{x^2dx}{y^3} = \dfrac{y^2dy}{x^3} = \dfrac{dz}{z}$ **[Ans. : $x^6 - y^6 = C_1, x^3 + y^3 = C_2z^3$]**

7. $\dfrac{a\,dx}{(b-c)yz} = \dfrac{b\,dy}{(c-a)xz} = \dfrac{c\,dz}{(a-b)xy}$

[Ans. : $ax^2 + by^2 + cz^2 = C_1, a^2x^2 + b^2y^2 + c^2z^2 = C_2$]

8. $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$

[Ans. : $y = C_1z, x^2 + y^2 + z^2 = y\,C_2$]

9. $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{(x+y)\left[e^{xy} + \sin xy + x^2y^2\right]}$

[Ans. : $x - y = C_1, 3e^{xy} - 3\cos xy + (xy)^3 - 3z = C_2$]

10. $\dfrac{dx}{y-z} = \dfrac{dy}{z-x} = \dfrac{dz}{x-y}$

[Ans. : $x + y + z = C_1, x^2 + y^2 + z^2 = C_2$]

11. $\dfrac{dx}{3z-4y} = \dfrac{dy}{4x-2z} = \dfrac{dz}{2y-3x}$

[Ans. : $2x + 3y + 4z = C_1, x^2 + y^2 + z^2 = C_2$]

12. $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$

[Ans. : $x + y + z = C_1, xyz = C_2$]

13. $\dfrac{dx}{yz} = \dfrac{dy}{xz} = \dfrac{dz}{\left(x^2+y^2\right)\sqrt{1-x^2y^2}}$

[Ans. : $x^2 - y^2 = C_1, z^2 + \sin^{-1}xy + xy\sqrt{1-x^2y^2} = C_2$]

14. $\dfrac{dx}{x^3 + 3xy^2} = \dfrac{dy}{y^3 + 3x^2y} = \dfrac{dz}{2\left(x^2+y^2\right)z}$

[Ans. : $z^2 = C_1y, z = C_2\left(x^2 - y^2\right)$]

15. $\dfrac{dx}{x^2+y^2} = \dfrac{dy}{2xy} = \dfrac{dz}{z^2(x+y)^2}$

[Ans. : $\dfrac{1}{x+y} - \dfrac{1}{x-y} = C_1, x + y + \dfrac{1}{z} = C_2$]

16. $\dfrac{dx}{yz} = \dfrac{dy}{xz} = \dfrac{dz}{(x+y)\left(z^2+1\right)}$

[Ans. : $x^2 - y^2 = C_1, 2(x+y) = \log\left(z^2+1\right) + C_2$]

17. $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$

[Ans. : $y^2 - 2xy - x^2 = C_1, x^2 - y^2 - z^2 = C_2$]

18. $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{xyz^2\left(x^2-y^2\right)}$

[Ans. : $x^2 - y^2 = C_1, \left(x^2 - y^2\right)x^2 + \dfrac{2}{z} = C_2$]

2.3 Electro-Mechanical Analogy

The differential equations play a dominant role in differential theories of electrical and mechanical system. While making an electrical equivalent of mechanical system, the following correspondence between the electrical and mechanical quantities should be taken into account while formulating mathematical model :

Mechanical system	Electrical circuit (Series connection)
Mass (M)	Inductance (C)
Damping constant	Resistance (R)
Spring stiffness (k)	Reciprocal of capacitance $\left(\frac{1}{C}\right)$
Driving force ϕ cos nt	Electromotive force (EMF) E
Displacement x(t)	Current i (t) or charge 'q'

2.4 L-C-R Circuit

Consider the electrical circuit given in Fig. 2.1

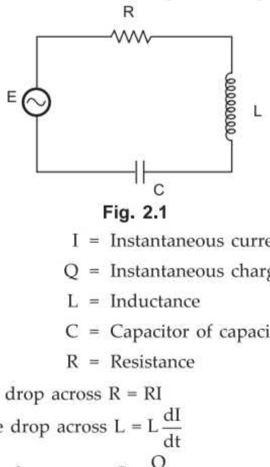

Fig. 2.1

Let I = Instantaneous current

Q = Instantaneous charge

L = Inductance

C = Capacitor of capacity

R = Resistance

$\therefore$ Voltage drop across R = RI

Voltage drop across L = $L\dfrac{dI}{dt}$

Voltage drop across C = $\dfrac{Q}{C}$

We know that $I = \dfrac{dQ}{dt}$ $\therefore$ $Q = \int I\,dt$

Kirchhoff's law : The algebraic sum of all the voltage drops in an electric circuit is zero. We consider the following cases.

Case 1 : The differential equation of electrical circuit consists of inductance L, capacitance C with emf E is $L\dfrac{dI}{dt} + \dfrac{Q}{C} = E$ $\therefore$ $L\dfrac{d^2Q}{dt^2} + \dfrac{Q}{C} = E$

Case 2 : The differential equation of electrical circuit consists of L - C without emf is

$$L\frac{d^2Q}{dt^2}+\frac{Q}{C} = 0 \quad \therefore \quad \frac{d^2Q}{dt^2}+\frac{Q}{LC} = 0$$

Case 3 : The differential equation of electrical circuit consists of inductance L, resistance R and capacitance C with emf E is $L\frac{dI}{dt}+RI+\frac{Q}{C} = E$

$$\text{or } L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C} = E \quad \therefore \frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q}{LC} = \frac{E}{L}$$

Case 4 : The differential equation of electrical circuit consists of L, R and C without E is

$$\frac{d^2Q}{dt^2}+\frac{R}{L}\frac{dQ}{dt}+\frac{Q}{LC} = 0$$

2.5 Illustrations

Example 2.29 : *In an L-C-R circuit, the charge q on the condenser is given by* $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E \sin \omega t$. *The circuit is tuned to resonate so that* $\omega^2 = \frac{1}{LC}$. *If initially the current and charge be zero, show that for small values of* $\frac{R}{L}$ *the current in the circuit at time t is given by* $\frac{Et}{2L} \sin \omega t$.

Solution : Step 1 : The given differential equation is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C} = E\sin \omega t$$

i.e. $\quad L D^2q + R D q + \frac{q}{C} = E \sin \omega t,$

where $\quad D = \frac{d}{dt}$

i.e. $\quad \left(D^2+\frac{R}{L}D+\frac{1}{LC}\right)q = \frac{E}{L} \sin \omega t$

i.e. $\quad \left(D^2 + \omega^2\right) q = \frac{E}{L} \sin \omega t \quad$ ... (1)

Step 2 : Note we can neglect $\left(\frac{RD}{L}\right)$ as $\frac{R}{L}$ is small and $\because \frac{1}{LC} = \omega^2$

Step 3 : To find C.F.

A.E. is $\quad D^2 + \omega^2 = 0$ i.e. $D^2 = -\omega^2 \quad D = \pm \omega i$

$\therefore \quad$ C.F. $= C_1 \cos \omega t + C_2 \sin \omega t$

Step 3 : To find P.I.

$$\text{P.I.} = \frac{E}{L} \cdot \frac{1}{\left(D^2 + \omega^2\right)} \sin \omega t = \frac{-Et}{2L\omega} \cos \omega t$$

Step 4 : $\therefore$ Complete solution is

$$q = C_1\cos \omega t + C_2 \sin \omega t - \frac{E}{2L\omega} t \cos \omega t \quad \text{... (2)}$$

Step 5 :

$$\therefore \quad i = \frac{dq}{dt} = -C_1\omega \sin \omega t + C_2\omega \cos \omega t$$

$$-\frac{E}{2L\omega}(-t\omega \sin \omega t + \cos \omega t) \quad \text{... (3)}$$

Step 6 : Now initially, at t = 0, q = 0 and i = 0.

$\therefore$ From equations (2) and (3) we get at t = 0

$$0 = C_1 + 0 + 0 \quad \therefore C_1 = 0 \text{ and}$$

$$0 = 0 + C_2\omega - \frac{E}{2L\omega}(0 + 1)$$

$$\therefore \quad C_2\omega = \frac{E}{2L\omega}$$

$\therefore$ Substituting in equation (3) we get

$$i = 0 + \frac{E}{2L\omega} \cos \omega t - \frac{E}{2L\omega}(-t\omega \sin \omega t + \cos \omega t)$$

$$= \left(\frac{E}{2L\omega} - \frac{E}{2L\omega}\right) \cos \omega t + \frac{Et}{2L} \sin \omega t = \frac{Et}{2L} \sin \omega t$$

Example 2.30 : *An uncharged condenser of capacity C is charged by applying an e.m.f of value* $E \sin \frac{t}{\sqrt{LC}}$ *through the leads of inductance L and negligible resistance. The charge Q on the plate of the condenser satisfies the differential equation* $\frac{d^2Q}{dt^2} + \frac{Q}{LC} = \frac{E}{L} \sin \frac{t}{\sqrt{LC}}$. *Prove that the charge at any time t is given by*

$$Q = \frac{EC}{2}\left[\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos\frac{t}{\sqrt{LC}}\right]$$

Solution : Step 1 : Let $p^2 = \frac{1}{LC}$ $\therefore$ The given differential equation becomes

$$\frac{d^2Q}{dt^2} + p^2Q = \frac{E}{L} \sin pt$$

i.e. $\quad \left(D^2 + p^2\right) Q = \frac{E}{L} \sin pt \quad$ ... (1)

Step 2 : To find C.F. A.E. is $D^2 + p^2 = 0$

i.e. $D^2 = -p^2 \therefore D = \pm p$

$\therefore$ C.F. $= C_1 \cos pt + C_2 \sin pt$

Step 3 : To find P.I.

$$\text{P.I.} = \frac{E}{L} \frac{1}{(D^2+p^2)} \sin pt$$

$$= \frac{E}{L} \cdot \frac{(-t)^1}{(2p)^1\, 1!} \sin\left(pt + 1\frac{\pi}{2}\right)$$

$$= -\frac{Et}{2\,pL} \cos pt$$

Step 4 : The complete solution is

$$\therefore Q = C_1 \cos pt + C_2 \sin pt - \frac{Et}{2\,pL} \cos pt \quad \text{... (2)}$$

Step 5 : At t = 0, Q = 0 and i = 0

$\therefore$ From equation (2) $0 = C_1 + 0 + 0 \therefore C_1 = 0$

$$\therefore Q = C_2 \sin pt - \frac{Et}{2pL} \cos pt \quad \text{... (3)}$$

$$\therefore i = \frac{dQ}{dt} = C_2\, p \cos pt - \frac{E}{2\,pL}(-tp \sin pt + \cos pt)$$

Step 6 : Now, at t = 0, i = 0

$$\therefore 0 = C_2 p\,(1) - \frac{E}{2pL}(0+1)$$

$$\therefore C_2 p = \frac{E}{2pL} \quad \therefore C_2 = \frac{E}{2p^2L}$$

Step 7 : Substituting in Q we get

$$Q = \frac{E}{2p^2L} \sin pt - \frac{Et}{2pL} \cos pt$$

$$= \frac{E}{2\frac{1}{LC}L} \sin pt - \frac{E}{2\frac{1}{\sqrt{LC}}L}\, t \cos pt \quad \text{...} \because p = \frac{1}{\sqrt{LC}}$$

$$= \frac{EC}{2} \sin pt - \frac{E}{2}\sqrt{\frac{C}{L}}\, t \cos pt$$

$$= \frac{EC}{2}\left(\sin pt - \frac{t}{\sqrt{LC}} \cos pt\right)$$

$$= \frac{EC}{2}\left(\begin{array}{l} \sin \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \\ \cos \frac{t}{\sqrt{LC}} \end{array}\right) \quad \text{...} \because p = \frac{1}{\sqrt{LC}}$$

❑❑❑

UNIT - II

3 Fourier Transforms

3.1 Introduction

Necessity is the origin of every research. In day to day life we deal with money. Suppose there are 10 coins of one rupee and one note of 10 rupees. i.e. there are two forms of money with same market value. If we need to do a telephone call then 10 ₹. note will not work. i.e. we need different forms of money for our individual use (i.e. application).

In electronics we deal with functions, naturally we need different forms of functions without changing their value.

We had studied the following transformations.

1) Matrices : For solving system of equations.
2) Logarithm : For solving the problems involving multiplication and division.
3) Fourier series : for the representation of periodic

   functions.

There are many non-periodic functions such as voltage, isolated pulse, decaying exponential unit step, which are non-periodic and suitable representation of these functions can be obtained by considering the limiting form of Fourier series of a periodic function, as the period becomes infinite.

Thus Fourier integral arises from Fourier series of a function with period ∞.

Thus Fourier integral is a transformation which transforms a non-periodic function of time t into a function of continuous frequency variable λ.

Fourier transforms are very useful in solving boundary value problems in partial differential equations.

3.2 Dirichlet's Conditions

i) A function f(x) is defined and single valued except at finite points in $(-L, L)$.

ii) f(x) is periodic in $(-L, L)$ with period 2 L.

iii) f(x) and $f'(x)$ are sectionally continuous.

iv) f(0x) is integrable in $(-L, L)$.

3.3 Complex Form of Fourier Series

If f(x) is a periodic function with period 2L defined in the interval $-L<x<L$ and satisfies Dirichlet's conditions then the Fourier series of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \quad \ldots (1)$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(u)\, du$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(u)\cos\left(\frac{n\pi u}{L}\right) du$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(u)\sin\left(\frac{n\pi u}{L}\right) du$$

Using $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

and rearranging the constants we can express f(x) in equation (1)

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{i\left(\frac{n\pi x}{L}\right)} \quad \ldots (2)}$$

where

$$\boxed{C_n = \frac{1}{2L}\int_{-L}^{L} f(u)\, e^{-i\left(\frac{n\pi u}{L}\right)} du \quad \ldots (3)}$$

which is called as complex exponential form of Fourier series.

3.4 Fourier Integral Theorem

If a function f(x) is such that

i) f(x) satisfies Dirichlet's conditions.

ii) f(x) is absolutely integrable i.e.

$\int_{-\infty}^{\infty} |f(x)|\, dx$ converges.

Then we can represent f(x) as a integral

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda} \quad \dots (4)$$

i.e. $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \left[\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du \right] d\lambda$

3.5 Equivalent Forms of Fourier Integral

From equation (4) (Fourier integral theorem)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda$$

We know that $e^{-i\theta} = \cos\theta - i \sin\theta$

$$\therefore\ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)$$

$$[\cos\lambda(u-x) - i \sin\lambda(u-x)]\, du\, d\lambda$$

We know that $\int_{-a}^{a} F(\lambda)\, d\lambda = 0$ if $F(\lambda)$ is an odd function of λ.

$= 2\int_{0}^{a} F(\lambda)\, d\lambda$ if $F(\lambda)$ is an even function of λ.

As $\sin\lambda(u-x)$ is an odd function of λ and $\cos\lambda(u-x)$ is an even function of λ.

$$\therefore\ \int_{-\infty}^{\infty} \sin\lambda(u-x)\, d\lambda = 0 \quad \text{and}$$

$$\int_{-\infty}^{\infty} \cos\lambda(u-x)\, d\lambda = 2\int_{0}^{\infty} \cos\lambda(u-x)\, d\lambda$$

$\therefore$ The integral becomes,

$$f(x) = \frac{2}{2\pi} \int_{\lambda=0}^{\infty} \int_{u=-\infty}^{\infty} f(u) \cos\lambda(u-x)\, du\, d\lambda$$

As $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\therefore\quad f(x) = \frac{1}{\pi} \int_{\lambda=0}^{\infty} \int_{u=-\infty}^{\infty} f(u)$$

$$[\cos\lambda u \cos\lambda x + \sin\lambda u \sin\lambda x]\, du\, d\lambda$$

Thus $$f(x) = \int_{0}^{\infty} [F_1(\lambda) \cos\lambda x + F_2(\lambda) \sin\lambda x]\, d\lambda \quad \dots (5)$$

is another equivalent form of Fourier integral.

where $$F_1(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos\lambda u\, du \quad \dots (6)$$

$$F_2(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin\lambda u\, du \quad \dots (7)$$

3.6 Fourier Sine Integral

If f(x) is an odd function of x then from equations (6) and (7)

$$F_1(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \cos\lambda u\, du = 0$$

and $$F_2(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(u) \sin\lambda u\, du$$

$$= \frac{2}{\pi} \int_{0}^{\infty} f(u) \sin\lambda u\, du$$

Substituting $F_1(\lambda)$ and $F_2(\lambda)$ in equation (5) we get

$$\boxed{f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u) \sin\lambda u \cdot \sin\lambda x\, du\, d\lambda} \quad \dots (8)$$

Equation (8) is called as Fourier sine integral of f(x). Similarly if f(x) is an even function of x then $F_1(\lambda) = \frac{2}{\pi} \int_{0}^{\infty} f(u) \cos\lambda u du$ and $F_2(\lambda) = 0$.

Substituting in equation (5) we get the Fourier cosine integral.

$$\boxed{f(x) = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(u) \cos\lambda u \cdot \cos\lambda x\, du\, d\lambda} \quad \dots (9)$$

3.7 Fourier Transforms

From Fourier integral theorem i.e. equation (4)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda(u-x)}\, du\, d\lambda$$

i.e. $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du \right] e^{i\lambda x}\, d\lambda$$

If
$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du \qquad \text{... (10)}$$

then
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda \qquad \text{... (11)}$$

The function $F(\lambda)$ is called as Fourier transform of f(x) (equation (10) gives the expression for Fourier Transform) while f(x) is called as inverse Fourier transform (equation (11) gives the expression for inverse).

3.8 Fourier Sine Transform

If a function defined in $-\infty < x < \infty$ is an odd function of x then from the expression of Fourier sine integral (equation (8)).

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(u) \sin \lambda u \cdot \sin \lambda x \, du\, d\lambda$$

i.e.
$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left[\int_0^{\infty} f(u) \sin \lambda u\, du \right] \sin \lambda x\, d\lambda$$

If we denote the integral as

$$F_s(\lambda) = \int_0^{\infty} f(u) \sin \lambda u\, du \qquad \text{... (12)}$$

then
$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\lambda) \sin \lambda x\, d\lambda \qquad \text{... (13)}$$

$F_s(\lambda)$ gives the Fourier sine transform of f(x) and f(x) gives the inverse Fourier sine transform of $F_s(\lambda)$.

3.9 Fourier Cosine Transform

If a function f(x) defined in $-\infty < x < \infty$ is an even function of x then from the expression of Fourier cosine integral i.e. equation no. (9) we have

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(u) \cos \lambda u \cdot \cos \lambda x \, du\, d\lambda$$

i.e.
$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left[\int_0^{\infty} f(u) \cos \lambda u\, du \right] \cos \lambda x\, d\lambda$$

We denote the integral as

$$F_c(\lambda) = \int_0^{\infty} f(u) \cos \lambda u\, du \qquad \text{... (14)}$$

then
$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\lambda) \cos \lambda x\, d\lambda \qquad \text{... (15)}$$

$F_c(\lambda)$ gives Fourier cosine transform of f(x) and f(x) gives inverse Fourier cosine transform of f(x).

Note

The expressions $f(x)$ and $F(\lambda)$ form a Fourier transform pair.

Important Results

1) Even and odd functions : If f(x) is defined in $-\infty < x < \infty$ and $f(-x) = f(x)$ then f (x) is said to be an even function of x.

If $f(-x) = -f(x)$ then it is said to be an odd function of x.

2) $\int_{-a}^{a} f(x)\, dx = 0$ if f(x) is an odd function.

$= 2\int_0^a f(x)\, dx$ if f(x) is an even function.

3) If f (x) is an even function of x then the graph of f (x) is symmetric about y axis. If f (x) is an odd function of x then the graph of f (x) is symmetric about opposite quadrants.

4) Arithmetics of odd and even functions are as follows.

Odd function	×	Odd function	= Even
Odd function	÷	Odd function	= Even
Odd	±	Odd	= Odd
Odd	×	Even	= Odd
Odd	÷	Even	= Odd
Even	±	Even	= Even
Even	±	Odd	= Neither odd nor even

5) $\int_0^{\infty} \frac{\sin ax}{x}\, dx = \frac{\pi}{2}$ if a is positive.

$= -\frac{\pi}{2}$ if a is negative.

6) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

7) $$\int_0^\infty e^{-au}\sin\lambda u\,du = \left\{\frac{e^{-au}}{a^2+\lambda^2}[-a\sin\lambda u - \lambda\cos\lambda u]\right\}_0^\infty$$

$$= \left\{0 - \frac{e^0}{a^2+\lambda^2}(0-\lambda)\right\}$$

$$= \frac{\lambda}{a^2+\lambda^2} \quad \left\{\begin{array}{l} \text{as } e^{-\infty}=0 \\ e^0 = 1 \\ \sin 0 = 0 \\ \cos 0 = 1 \end{array}\right.$$

8) $$\int_0^\infty e^{-au}\cos\lambda u\,du = \left\{\frac{e^{-au}}{a^2+\lambda^2}[-a\cos\lambda u + \lambda\sin\lambda u]\right\}_0^\infty$$

$$= 0 - \frac{e^{-0}}{a^2+\lambda^2}[-a+0]$$

$$= \frac{a}{a^2+\lambda^2}$$

9) DUIS rule

If $I(\alpha) = \int_a^b f(x,\alpha)\,dx$

then $$\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}$$

10) $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

11) $|x| \le a \Rightarrow -a \le x \le a$

and $|x| \ge a \Rightarrow x \ge a$ and $x \le -a$

12) Integration by parts

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

13) Generalized rule

$$\int uv\,dx = u\,v_1 - u'v_2 + u''v_3 \ldots$$

where dashes indicate derivatives and suffixes indicate integrals.

Generalized rule is useful for evaluating the integral of product of two functions where the first function is a function of positive powers of x like x^2, x^3+1, ... and the second function of the product must be sine, cosine or exponential function.

The following table gives the Fourier transform pairs for ready reference.

Sr. No.	Name of the transform	Expression for the transform $F(\lambda)$ =	Inverse transform f(x) =
1.	Fourier transform	$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\lambda u}du$	$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}d\lambda$
2.	Fourier cosine transform Fourier transform for even f(x)	$\sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos\lambda u\,du$	$\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$
3.	Fourier sine transform Fourier transform for odd f(x)	$\sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\sin\lambda u\,du$	$\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\lambda)\sin\lambda u\,d\lambda$

We may use the following table also (Note the constant multiples of the integrals)

Sr. No.	Name of the transform	Expression for the transform	Inverse transform
1.	Fourier transform	$F(\lambda) = \int_{-\infty}^{\infty} f(u)e^{-i\lambda u}du$	$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}d\lambda$
2.	Fourier cosine transform Fourier transform for even f(x)	$F(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$	$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$
3.	Fourier sine transform Fourier transform for odd f(x)	$F(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du$	$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\,d\lambda$

Note

1) For solving examples we use above formulae

2) $$\int_{-\infty}^{\infty} f(x)\,dx = 0 \quad \text{if } f(x) \text{ is odd.}$$

$$= 2\int_0^\infty f(x)\,dx \quad \text{if } f(x) \text{ is even.}$$

Example 3.1 : *Find Fourier cosine transform of*

$$f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x \geq a \end{cases}$$

SPPU : May-16

Solution : Consider F.C.T. formula

$$F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$$

Split the integral and put the value of f(u)

$$F_c(\lambda) = \frac{1}{2}\left\{\int_0^a 2\cos u \cdot \cos \lambda u \, du + \int_a^\infty 0\right\}$$

Use $2\cos \lambda u \cos u = \cos(\lambda-1)u + \cos(\lambda+1)u$

$$\therefore \quad F_c(\lambda) = \frac{1}{2}\int_0^a [\cos(\lambda-1)u + \cos(\lambda+1)u]du$$

$$\text{Integrate} = \frac{1}{2}\left[\frac{\sin(\lambda-1)u}{\lambda-1} + \frac{\sin(\lambda+1)u}{\lambda+1}\right]_0^a$$

Put the limits of u

$$F_c(\lambda) = \frac{1}{2}\left[\frac{\sin(\lambda-1)a}{\lambda-1} + \frac{\sin(\lambda+1)a}{\lambda+1}\right]$$

Example 3.2 : *Find Fourier sine transform of*

$$f(x) = \begin{cases} \sin x & 0 < x < a \\ 0 & x \geq a \end{cases}$$

Solution : Consider F.S.T. formula

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

Split the integral and put the value of f(u)

$$= \frac{1}{2}\left\{\int_0^a 2\sin u \cdot \sin \lambda u \, du + \int_a^\infty 0\right\}$$

Use $2\sin \lambda u \sin u = \cos(\lambda-1)u - \cos(\lambda+1)u$

$$\therefore \quad F_s(\lambda) = \frac{1}{2}\int_0^a [\cos(\lambda-1)u - \cos(\lambda+1)u]du$$

$$= \frac{1}{2}\left[\frac{\sin(\lambda-1)u}{\lambda-1} - \frac{\sin(\lambda+1)u}{\lambda+1}\right]_0^a$$

$$F_s(\lambda) = \frac{1}{2}\left(\frac{\sin(1-\lambda)a}{1-\lambda} - \frac{\sin(1+\lambda)a}{1+\lambda}\right)$$

Example 3.3 : *Find Fourier sine and cosine transform of $f(x) = e^{-x} + e^{-2x}$.*

SPPU : Dec.-16

Solution : Consider F.S.T. formula

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

$$= \left\{\int_0^\infty \left(e^{-u} + e^{-2u}\right) \cdot \sin \lambda u \, du\right\}$$

$$\therefore \quad F_s(\lambda) = \left[\frac{e^{-u}}{\lambda^2+1}(-\sin \lambda u - \lambda \cos \lambda u)\right]_0^\infty + \left[\frac{e^{-2u}}{\lambda^2+4}(-2\sin \lambda u - \lambda \cos \lambda u)\right]_0^\infty$$

$$= \left[\frac{\lambda}{\lambda^2+1} + \frac{\lambda}{\lambda^2+4}\right]$$

$$F_s(\lambda) = \frac{\lambda(2\lambda^2+5)}{\lambda^4+5\lambda^2+4}$$

For finding cosine transform

Consider F.C.T. formula

$$F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$$

$$F_c(\lambda) = \left\{\int_0^\infty \left(e^{-u} + e^{-2u}\right) \cos \lambda u \, du\right\}$$

$$\therefore \quad F_c(\lambda) = \left[\frac{e^{-u}}{\lambda^2+1}(-\cos \lambda u + \lambda \sin \lambda u)\right]_0^\infty + \left[\frac{e^{-2u}}{\lambda^2+4}(-2\cos \lambda u + \lambda \sin \lambda u)\right]_0^\infty$$

$$= \left[\frac{1}{\lambda^2+1} + \frac{2}{\lambda^2+4}\right]$$

$$F_c(\lambda) = \frac{3\lambda^2+6}{\lambda^4+5\lambda^2+4}$$

Example 3.4 : *Find Fourier sine and cosine transform of*

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \\ 0 & x > 2 \end{cases}$$

Solution : Consider F.S.T. formula

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

$$F_s(\lambda) = \int_0^1 u \sin \lambda u \, du + \int_1^2 (2-u) \sin \lambda u \, du$$

$$F_s(\lambda) = \left[u\frac{-\cos\lambda u}{\lambda}\right]_0^1 - \int_0^1 \frac{-\cos\lambda u}{\lambda}\,du$$

$$+\left[(2-u)\frac{-\cos\lambda u}{\lambda}\right]_1^2 - \int_1^2 -1\frac{-\cos\lambda u}{\lambda}\,du$$

$$= \left[\frac{-\cos\lambda}{\lambda} + \frac{\sin\lambda}{\lambda^2} + \frac{\cos\lambda}{\lambda} - \frac{\sin 2\lambda}{\lambda^2} + \frac{\sin\lambda}{\lambda^2}\right]$$

$$= \frac{2\sin\lambda}{\lambda^2} - \frac{\sin 2\lambda}{\lambda^2}$$

As $\sin 2\lambda = 2\sin\lambda\cos\lambda$ we get

$$\therefore F_s(\lambda) = \frac{2\sin\lambda(1-\cos\lambda)}{\lambda^2}$$

Now for finding cosine transform

Consider F.C.T. formula

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

$$F_c(\lambda) = \left[\int_0^1 u\cos\lambda u\,du + \int_1^2 (2-u)\cos\lambda u\,du\right]$$

$$F_c(\lambda) = \left[u\frac{\sin\lambda u}{\lambda}\right]_0^1 - \int_0^1 \frac{\sin\lambda u}{\lambda}\,du$$

$$+\left[(2-u)\frac{\sin\lambda u}{\lambda}\right]_1^2 - \int_1^2 -1\frac{\sin\lambda u}{\lambda}\,du$$

$$= \frac{\sin\lambda}{\lambda} + \frac{\cos\lambda}{\lambda^2} - \frac{1}{\lambda^2} - \frac{\sin\lambda}{\lambda} - \frac{\cos 2\lambda}{\lambda^2} + \frac{\cos\lambda}{\lambda^2}$$

Thus $\quad F_c(\lambda) = \dfrac{2\cos\lambda - \cos 2\lambda - 1}{\lambda^2}$

Example 3.5 : *Find the Fourier cosine transform of*

$$f(x) = x \quad \text{if } 0 < x < \frac{1}{2}$$
$$= 1-x \text{ if } \frac{1}{2} < x < 1$$
$$= 0 \quad \text{if } x > 1$$

Solution : Consider F.C.T. formula

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

Step 2 : $F_c(\lambda) = \displaystyle\int_0^{1/2} u\cos\lambda u\,du + \int_{1/2}^1 (1-u)\cos\lambda u\,du + 0$

Step 3 : Integrating by parts, we get

$$F_c(\lambda) = \left[u\left(\frac{\sin\lambda u}{\lambda}\right) - \left(\frac{-\cos\lambda u}{\lambda^2}\right)\right]_0^{1/2}$$

$$+\left[(1-u)\left(\frac{\sin\lambda u}{\lambda}\right) + \left(\frac{-\cos\lambda u}{\lambda^2}\right)\right]_{1/2}^1$$

$$= \left(\frac{1}{2}\frac{\sin\frac{1}{2}\lambda}{\lambda} + \frac{\cos\frac{1}{2}\lambda}{\lambda^2} - \frac{1}{\lambda^2}\right) + \left(-\frac{\cos\lambda}{\lambda^2} - \frac{1}{2}\frac{\sin\frac{1}{2}\lambda}{\lambda} + \frac{\cos\frac{1}{2}\lambda}{\lambda^2}\right)$$

Thus $F_c(\lambda)$ is

$$F_c(\lambda) = \frac{1}{\lambda^2}\left(2\cos\frac{1}{2}\lambda - \cos\lambda - 1\right)$$

Example 3.6 : *Find Fourier transform of*

$$f(x) = \begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases}$$

Solution : As f(x) is an even function of x

$$\because \quad f(-x) = \begin{cases} 1 & \text{if } |-x| \le a \\ 0 & \text{if } |-x| > a \end{cases}$$

$$= \begin{cases} 1 & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases} = f(x)$$

$\therefore$ We find F.C.T.

$$F(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

As $\quad f(x) = 1, \ -a < x < a$

$$\therefore \quad F_c(\lambda) = \int_{-a}^{a} 1\cdot\cos\lambda u\,du = 2\int_0^a \cos\lambda u\,du$$

$$F_c(\lambda) = 2\left[\frac{\sin\lambda u}{\lambda}\right]_0^a$$

Put the limits

$$F_c(\lambda) = 2\,\frac{\sin\lambda a}{\lambda}$$

Example 3.7 : *Find Fourier integral representation of*

$$f(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

Hence evaluate $\displaystyle\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$ *and* $\displaystyle\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda$

SPPU : Dec.-17

Solution : As f(x) is an even function of x we find F.C.T.

$$F(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

As $\quad f(x) = 1, \ -1 < x < 1$

$$\therefore \quad F(\lambda) = \int_0^1 1\cdot\cos\lambda u\,du$$

$$\therefore \quad F_c(\lambda) = \int_0^1 \cos \lambda u \, du = \left[\frac{\sin \lambda u}{\lambda}\right]_0^1$$

$$F_c(\lambda) = \frac{\sin \lambda}{\lambda}$$

Now to prove next part consider inverse Fourier cosine transform.

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda) \cos \lambda x \, d\lambda$$

Put the value of $F(\lambda)$

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin \lambda}{\lambda}\right) \cos \lambda x \, d\lambda$$

$$\frac{\pi}{2} f(x) = \int_0^\infty \frac{\sin \lambda}{\lambda} \cos \lambda x \, d\lambda$$

$\therefore$ The value of the integral $\int_0^\infty \frac{\sin \lambda}{\lambda} \cos \lambda x \, d\lambda$ is $\frac{\pi}{2} f(x)$

$$\therefore \int_0^\infty \frac{\sin \lambda}{\lambda} \cos \lambda x \, d\lambda = \frac{\pi}{2} f(x)$$

Again put x = 0

$$\therefore \quad \int_0^\infty \frac{\sin \lambda}{\lambda} d\lambda = \frac{\pi}{2} f(0)$$

Now from the definition of f(x), f(0) = 1

$$\therefore \quad \int_0^\infty \frac{\sin \lambda}{\lambda} d\lambda = \frac{\pi}{2}$$

Example 3.8 : *Find Fourier transform of*

$$f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \left(\frac{x \cos x - \sin x}{x^3}\right) \cos \frac{x}{2} \, dx$

SPPU : Dec.-16

Solution : As f(x) is an even function of x we find F.C.T. of f(x)

$$\text{F.C.T. } F(\lambda) = \int_0^\infty F(u) \cos \lambda u \, du$$

Split the integral into two parts as substitute values of f(u)

$$= \int_0^1 \left(1-u^2\right) \cos \lambda u \, du + \int_1^\infty 0$$

Use generalized formula for integration by parts

i.e. $\int uv \, dx = u v_1 - u' v_2 + u'' v_3 \ldots$

dashes indicates derivatives and suffixes indicates integrals.

$$= \int_0^1 \left(1-u^2\right) \cos \lambda u \, du$$

$$= \left[\left(1-u^2\right)\left(\frac{\sin \lambda u}{\lambda}\right) - (-2u)\left(\frac{-\cos \lambda u}{\lambda^2}\right) + (-2)\left(\frac{-\sin \lambda u}{\lambda^3}\right)\right]_0^1$$

$$= 0 - \frac{2 \cos \lambda}{\lambda^2} + \frac{2 \sin \lambda}{\lambda^3} - (0 - 0 - 0)$$

$$F(\lambda) = \frac{-2(\lambda \cos \lambda - \sin \lambda)}{\lambda^3}$$

As it is asked to find the integral therefore we get must find inverse.

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda) \cos \lambda x \, d\lambda$$

Substitute the value of $F(\lambda)$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{-2(\lambda \cos \lambda - \sin \lambda)}{\lambda^3} \cos \lambda x \, d\lambda$$

To get the required integral in terms of the parameter λ put x = 1/2

$$f\left(\frac{1}{2}\right) = \frac{-4}{\pi}\int_0^\infty \frac{\lambda \cos \lambda - \sin \lambda}{\lambda^3} \left(\cos \frac{\lambda}{2}\right) d\lambda$$

As $$f(x) = \begin{cases} 1-x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

$$\therefore \quad f\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} \quad \text{substituting we get}$$

$$\left(\frac{3}{4}\right)\left(\frac{-\pi}{4}\right) = \int_0^\infty \left(\frac{\lambda \cos \lambda - \sin \lambda}{\lambda^3}\right) \cos \frac{\lambda}{2} \, d\lambda$$

$\therefore$ As the variable is not important in definite integrals here replace λ by x

$$\therefore \quad \frac{-3\pi}{16} = \int_0^\infty \left(\frac{x \cos x - \sin x}{x^3}\right) \cos \frac{x}{2} \, dx$$

Example 3.9 : *Find Fourier sine transform of*

$$f(x) = \begin{cases} 1 & -2 \le x < 0 \\ -1 & 0 < x \le 2 \end{cases}$$

Hence show that

$$\int_0^\infty \frac{(\cos 2w - 1)\sin 2w}{w} dw = -\frac{\pi}{2}$$

Solution : Consider F.S.T. formula,

$$F(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

$$= \int_0^2 (-1) \sin \lambda u \, du + \int_2^\infty 0$$

$$= -\left[\frac{-\cos \lambda u}{\lambda}\right]_0^2$$

$$F(\lambda) = \frac{1}{\lambda}[\cos 2\lambda - 1]$$

Consider inverse Fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\lambda) \sin \lambda x \, d\lambda$$

$$\mathbf{f(x)} = \frac{2}{\pi}\int_0^\infty \frac{\cos 2\lambda - 1}{\lambda} \sin \lambda x \, d\lambda$$

Put x = 2

$$f(2) = \frac{2}{\pi}\int_0^\infty \frac{(\cos 2\lambda - 1)\sin 2\lambda}{\lambda} d\lambda$$

Put f(2) = – 1

$$(-1)\frac{\pi}{2} = \int_0^\infty \frac{(\cos 2\lambda - 1)\sin 2\lambda}{\lambda} d\lambda$$

As the variable is not important replacing λ by w we get

$$\int_0^\infty \frac{(\cos 2w - 1)\sin 2w}{w} dw = \frac{-\pi}{2}$$

Example 3.10 : *Find Fourier transform of* $e^{-|x|}$.

Solution : $e^{-|x|}$ is an even function. Hence we find Fourier cosine transform

Consider F.C.T. formula,

$$\therefore \quad F(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$$

$$\therefore \quad F(\lambda) = \int_0^\infty e^{-u} \cos \lambda u \, du$$

$$\therefore \quad F_c(\lambda) = \left[\frac{e^{-u}}{\lambda^2+1}(-\cos \lambda u + \lambda \sin \lambda u)\right]_0^\infty$$

$$\therefore \quad F_c(\lambda) = \frac{\mathbf{1}}{\lambda^2 + \mathbf{1}}$$

Example 3.11 : *Find Fourier sine transform of* $e^{-|x|}$. *Hence evaluate*

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx$$

Solution : Consider F.S.T. formula

$$F(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

$$= \left\{\int_0^\infty e^{-u} \sin \lambda u \, du\right\}$$

$$\therefore \quad F_s(\lambda) = \left[\frac{e^{-u}}{\lambda^2+1}(-\sin \lambda u + \lambda \cos \lambda u)\right]_0^\infty$$

$$F_s(\lambda) = \frac{\lambda}{\lambda^2 + \mathbf{1}}$$

Consider inverse Fourier transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

$$\therefore \quad e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

Put x = m and then λ = x we get

$$\therefore \quad e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{x \sin mx}{1+x^2} dx$$

$$\therefore \quad \int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$$

Example 3.12 : *Using Fourier integral representation show that* $\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$ *and*

$$\int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$$

Solution : As $\cos \lambda x$ is present in the equation we find Fourier cosine transform of $f(x) = \frac{\pi}{2} e^{-x}$

$$\therefore \quad F(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$$

$$= \int_0^\infty \frac{\pi}{2} e^{-u} \cos \lambda u \, du$$

$$= \frac{\pi}{2} \int_0^\infty e^{-u} \cos \lambda u \, du$$

$$= \frac{\pi}{2} \left\{ \frac{e^{-u}}{1+\lambda^2} [-\cos \lambda u + \lambda \sin \lambda u] \right\}_0^\infty$$

$$= \frac{\pi}{2} \frac{1}{1+\lambda^2}$$

Consider inverse Fourier cosine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F(\lambda) \cos \lambda u \, d\lambda$$

$$\therefore \quad \frac{\pi}{2} e^{-x} = \frac{2}{\pi} \cdot \int_0^\infty \frac{\pi}{2} \cdot \frac{1}{1+\lambda^2} \cos \lambda x \, d\lambda$$

Thus $\displaystyle\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$

To prove next part

$$\int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$$

As $\sin \lambda x$ is present in the equation we find Fourier sine transform of $f(x) = \frac{\pi}{2} e^{-x}$.

$$\therefore \quad F(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

$$= \int_0^\infty \frac{\pi}{2} e^{-u} \sin \lambda u \, du$$

$$= \frac{\pi}{2} \int_0^\infty e^{-u} \sin \lambda u \, du$$

$$= \frac{\pi}{2} \left\{ \frac{e^{-u}}{1+\lambda^2} [-\sin \lambda u - \lambda \cos \lambda u] \right\}_0^\infty$$

$$= \frac{\pi}{2} \left(\frac{\lambda}{1+\lambda^2} \right)$$

Consider inverse Fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F(\lambda) \sin \lambda u \, du$$

$$\therefore \quad \frac{\pi}{2} e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{2}{\pi} \left(\frac{\lambda}{1+\lambda^2} \right) \sin \lambda x \, d\lambda$$

Thus $\displaystyle\int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2} e^{-x}$

Example 3.13 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} d\lambda = \begin{cases} 0 & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

Solution : To prove the result consider the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

This function f(x) is defined in $-\infty < x < \infty$ and both $\sin \lambda x$ and $\cos \lambda x$ are present in the integrand so we use the general Fourier transform.

$\therefore$ We should find the F.T. of f(x).

We have

$$F(\lambda) = \int_{-\infty}^\infty f(x) e^{-i\lambda x} dx$$

$$F(\lambda) = \int_{-\infty}^0 f(x) e^{-i\lambda x} dx + \int_0^\infty f(x) e^{-i\lambda x} dx$$

$$F(\lambda) = 0 + \int_0^\infty \pi e^{-x} e^{-i\lambda x} dx$$

$$\therefore \quad F(\lambda) = \pi \int_0^\infty e^{-x(1+i\lambda)} dx$$

$$= \pi \left[\frac{1}{1+i\lambda} \right] = \pi \frac{(1-i\lambda)}{1+\lambda^2}$$

Consider the inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty F(\lambda) e^{i\lambda x} d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \pi \frac{(1-i\lambda)}{1+\lambda^2} [\cos \lambda x + i \sin \lambda x] \, d\lambda$$

$$= \frac{1}{2} \int_{-\infty}^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} d\lambda + \frac{1}{2} i \int_{-\infty}^\infty \frac{-\lambda \cos \lambda x + \sin \lambda x}{1+\lambda^2} d\lambda$$

$$= \frac{1}{2} 2 \int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} d\lambda + 0$$

$$= \int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} d\lambda$$

Thus

$$\int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} d\lambda = f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases} \quad \dots (1)$$

Putting x = 0 we get

$$f(0) = \int_0^\infty \frac{1}{1+\lambda^2} d\lambda = \frac{\pi}{2}$$

$\therefore$ We have

$$\int_0^\infty \frac{\cos \lambda x + \lambda \sin \lambda x}{1+\lambda^2} d\lambda = \begin{cases} 0 & \text{if } x<0 \\ \frac{\pi}{2} & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

Example 3.14 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{2}{1+\lambda^2} \cos \lambda x \, d\lambda = \begin{cases} 0 & x<0 \\ \pi & x=0 \\ \pi e^{-x} & x>0 \end{cases}$$

Solution : As $\cos \lambda x$ is present in the integral we find the cosine transform.

$$\therefore \quad F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$$

$$\therefore \quad F_c(\lambda) = \int_0^\infty \pi e^{-u} \cos \lambda u \, du$$

$$= \pi \left[\frac{e^{-u}}{1+\lambda^2}(-\cos \lambda u + \lambda \sin \lambda u)\right]_0^\infty = \frac{\pi}{1+\lambda^2}$$

Consider inverse Fourier cosine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \pi \cdot \frac{1}{1+\lambda^2} \cos \lambda x \, d\lambda$$

$$f(x) = \int_0^\infty \frac{2}{1+\lambda^2} \cos \lambda x \, d\lambda$$

$$= \pi e^{-x} \quad \text{... (1)}$$

Substituting x = 0 in equation (1) we get

$$f(0) = \int_0^\infty \frac{2}{1+\lambda^2} d\lambda$$

$$= [2 \tan^{-1}(\lambda)]_0^\infty$$

$$= 2[\tan^{-1} \infty - \tan^{-1} 0]$$

$$= 2\left[\frac{\pi}{2} - 0\right]$$

$$f(0) = \pi$$

And

$$\text{Thus } \int_0^\infty \frac{2}{1+\lambda^2} \cos \lambda x \, d\lambda = \begin{cases} 0 & \text{if } x<0 \\ \pi & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

Example 3.15 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{2\lambda}{1+\lambda^2} \sin \lambda x \, d\lambda = \begin{cases} 0 & x<0 \\ \pi/2 & x=0 \\ \pi e^{-x} & x>0 \end{cases}$$

Solution : As $\sin \lambda x$ is present in the integral, we should find Fourier sine transformation of

$$f(x) = \begin{cases} 0 & \text{if } x<0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

$\therefore$ Consider F.S.T.

$$F_s(\lambda) = \int_0^\infty f(u) \sin \lambda u \, du$$

$$= \int_0^\infty \pi e^{-u} \cdot \sin \lambda u \, du$$

$$\therefore \quad F_s(\lambda) = \left[\frac{\pi e^{-u}}{\lambda^2+1}(-\sin \lambda u - \lambda \cos \lambda u)\right]_0^\infty$$

$$F_s(\lambda) = \frac{\pi\lambda}{\lambda^2+1} = \frac{\pi\lambda}{1+\lambda^2}$$

Consider inverse Fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda$$

$$f(x) = 2\int_0^\infty \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda \quad \text{... (1)}$$

Substituting $x = \frac{\pi}{2}$

$$\text{Thus } 2\int_0^\infty \frac{\lambda}{\lambda^2+1} \sin \lambda x \, d\lambda = \begin{cases} 0 & x<0 \\ \pi/2 & x=0 \\ \pi e^{-x} & x>0 \end{cases}$$

Example 3.16 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{\cos \lambda \pi/2}{1-\lambda^2} \cos \lambda x \, d\lambda = \begin{cases} \frac{\pi}{2} \cos x & |x| \le \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases}$$

Solution : As $\cos \lambda x$ is present in the integral

$\therefore$ We should find cosine transform of

$$f(x) = \begin{cases} \frac{\pi}{2} \cos x, & |x| \le \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$$

Consider F.C.T. formula

$$F_c(\lambda) = \int_0^{\infty} f(u)\cdot \cos\lambda u \cdot du$$

$$= \int_0^{\pi/2} f(u)\cdot \cos\lambda u \cdot du + \int_{\pi/2}^{\infty} f(u)\cos\lambda u \; du$$

$$= \int_0^{\pi/2} \frac{\pi}{2} \cos u \cdot \cos\lambda u \cdot du + 0$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\int_0^{\pi/2} (2\cos\lambda u \cdot \cos u)\, du$$

$$2\cos\lambda u \cdot \cos u = \cos(\lambda+1)\, u + \cos(\lambda-1)\, u$$

$$\therefore \quad F_c(\lambda) = \frac{\pi}{4}\int_0^{\pi/2}\left[\cos(\lambda+1)\, u + \cos(\lambda-1)\, u\right] du$$

$$= \frac{\pi}{4}\left[\frac{\sin(\lambda+1)\, u}{\lambda+1} + \frac{\sin(\lambda-1)\, u}{\lambda-1}\right]_0^{\pi/2}$$

(Use $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$ and $\sin\left(\frac{\pi}{2}-\theta\right) = +\cos\theta$)

$$\left\{\sin(\lambda+1)\frac{\pi}{2} = \cos\frac{\pi\lambda}{2},\; \sin(\lambda-1)\frac{\pi}{2} = -\cos\frac{\pi\lambda}{2}\right\}$$

$$\therefore \quad F_c(\lambda) = \frac{\pi}{4}\left[\frac{\cos\frac{\pi\lambda}{2}}{\lambda+1} - \frac{\cos\frac{\pi\lambda}{2}}{\lambda-1}\right]$$

Take L.C.M

$$= \frac{\pi}{4}\left[\frac{-2}{\lambda^2-1}\right]\cos\frac{\pi\lambda}{2}$$

$$F_c(\lambda) = \frac{\pi}{2}\,\frac{1}{(1-\lambda^2)}\cos\frac{\pi\lambda}{2}$$

Consider inverse cosine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\lambda)\cdot\cos\lambda x\cdot d\lambda$$

Put the value of $F(\lambda)$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\pi}{2}\,\frac{1}{(1-\lambda^2)}\cos\frac{\pi\lambda}{2}\cdot\cos\lambda x\cdot d\lambda$$

$$\therefore \quad f(x) = \int_0^{\infty}\frac{\cos\frac{\pi\lambda}{2}\cos\lambda x}{1-\lambda^2}\, d\lambda$$

Example 3.17 : *Using Fourier integral representation show that*

$$\int_0^{\infty}\frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\, d\lambda = \begin{cases}\pi/2 & 0<x<\pi \\ 0 & x>\pi\end{cases}$$

Solution : As $\sin\lambda x$ is present in the integral

$\therefore$ We should find sine transform of

$$f(x) = \begin{cases}\frac{\pi}{2}, & 0<x<\pi \\ 0, & x>\pi\end{cases}$$

Consider F.S.T. formula

$$F_s(\lambda) = \int_0^{\infty} f(u)\sin\lambda u\, du$$

$$F_s(\lambda) = \int_0^{\pi} f(u)\sin\lambda u\, du + \int_{\pi}^{\infty} f(u)\sin\lambda u\, du$$

$$F_s(\lambda) = \int_0^{\pi}\frac{\pi}{2}\sin\lambda u\, du + \int_{\pi}^{\infty} 0\sin\lambda u\, du$$

$$= \frac{\pi}{2}\int_0^{\pi}\sin\lambda u\, du + 0 = \frac{\pi}{2}\left[\frac{-\cos\lambda u}{\lambda}\right]_0^{\pi}$$

$$= \frac{\pi}{2}\left[\frac{-\cos\lambda\pi-(-1)}{\lambda}\right]$$

$$= \frac{\pi}{2}\,\frac{(1-\cos\lambda\pi)}{\lambda}$$

Consider inverse sine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda)\sin\lambda x\, d\lambda$$

$$\therefore \quad f(x) = \frac{\pi}{2}\int_0^{\infty}\frac{2}{\pi}\,\frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\, d\lambda$$

$$f(x) = \int_0^{\infty}\frac{1-\cos\lambda\pi}{\lambda}\sin\lambda x\, d\lambda$$

Example 3.18 : *Using Fourier integral representation show that*

$$\int_0^{\infty}\frac{\lambda\sin\lambda\pi}{1-\lambda^2}\cos\lambda x\, d\lambda = \begin{cases}\frac{\pi}{2}\cos x & |x|\le\pi \\ 0 & |x|>\pi\end{cases}$$

Solution : As $\cos\lambda x$ is present in the integral

$\therefore$ We should find cosine transform of

$$f(x) = \begin{cases}\frac{\pi}{2}\cos x, & |x|\le\pi \\ 0, & |x|>\pi\end{cases}$$

Consider F.C.T. formula

$$F_c(\lambda) = \int_0^\infty f(u)\cos\lambda u\,du$$

$$= \int_0^\pi f(u)\cos\lambda u\,du + \int_\pi^\infty f(u)\cos\lambda u$$

$$= \frac{1}{2}\int_0^\pi \frac{\pi}{2}\,2\cos u\cos\lambda u\,du + 0$$

(Use $2\cos\lambda u\cos u = \cos(\lambda+1)u + \cos(\lambda-1)u$)

$$\therefore \quad F_c(\lambda) = \frac{\pi}{4}\int_0^\pi \left[\cos(\lambda+1) + \cos(\lambda-1)u\right]du$$

$$= \frac{\pi}{4}\left[\frac{\sin(\lambda+1)u}{\lambda+1} + \frac{\sin(\lambda-1)u}{\lambda-1}\right]_0^\pi$$

(Use $\sin(\pi+\theta) = -\sin\theta$ and $\sin(\pi-\theta) = \sin\theta$
$\{\sin(\lambda+1)\pi = -\sin\pi\lambda,\ \sin(\lambda-1)\pi = -\sin\pi\lambda\}$)

$$\therefore \quad F_c(\lambda) = \frac{\pi}{4}\left[\frac{-\sin\pi\lambda}{\lambda+1} - \frac{\sin\pi\lambda}{\lambda-1}\right]$$

$$= \frac{\pi}{4}\left[\frac{-2\lambda}{\lambda^2-1}\right]\sin\pi\lambda$$

$$F(\lambda) = \frac{\pi}{2}\frac{\lambda}{(1-\lambda^2)}\sin\pi\lambda$$

Consider inverse cosine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x\,d\lambda$$

Put the value of $F(\lambda)$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\frac{\lambda}{(1-\lambda^2)}\sin\pi\lambda\cos\lambda x\,d\lambda$$

$$\therefore \quad f(x) = \int_0^\infty \frac{\lambda\sin\pi\lambda}{(1-\lambda^2)}\cos\lambda x\,d\lambda$$

Example 3.19 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{\lambda\cos\lambda\pi/2}{1-\lambda^2}\sin\lambda x\,d\lambda = \begin{cases}\frac{\pi}{2}\sin x & 0 \le x \le \frac{\pi}{2}\\ 0 & x > \frac{\pi}{2}\end{cases}$$

Solution : As $\sin\lambda x$ is present in the integral
$\therefore$ We should find sine transform of

$$f(x) = \begin{cases}\frac{\pi}{2}\sin x, & 0 \le x \le \frac{\pi}{2}\\ 0 & ,\ x > \frac{\pi}{2}\end{cases}$$

Consider F.S.T. formula

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du$$

$$F_s(\lambda) = \int_0^{\pi/2} f(u)\sin\lambda u\,du + \int_{\pi/2}^\infty f(u)\sin\lambda u\,du$$

$$F_s(\lambda) = \int_0^{\pi/2} \frac{\pi}{2}\sin u\sin\lambda u\,du + \int_{\pi/2}^\infty 0\sin\lambda u\,du$$

(Use $2\sin\lambda u\cdot\sin u = \cos(\lambda-1)u - \cos(\lambda+1)u$)

$$\therefore \quad F_s(\lambda) = \frac{\pi}{4}\int_0^{\pi/2}\left[\cos(\lambda-1)u - \cos(\lambda+1)u\right]du$$

Integrate

$$= \frac{\pi}{4}\left[\frac{\sin(\lambda-1)u}{\lambda-1} - \frac{\sin(\lambda+1)u}{\lambda+1}\right]_0^{\pi/2}$$

(Use $\sin\left(\frac{\pi}{2}+\theta\right) = +\cos\theta$ and $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$)

$$= \frac{\pi}{4}\left[\frac{-\cos(\lambda\pi/2)}{\lambda-1} - \frac{\cos(\lambda\pi/2)}{\lambda+1}\right]$$

$$= \frac{\pi}{4}\left[\frac{-2\lambda}{\lambda^2-1}\right]\cos(\pi\lambda/2)$$

$$= \frac{\pi}{2}\frac{\lambda\cos(\lambda\pi/2)}{1-\lambda^2}$$

Consider inverse sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\,d\lambda$$

Put the value of $F(\lambda)$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\frac{\lambda\cos\left(\frac{\lambda\pi}{2}\right)}{1-\lambda^2}\,d\lambda$$

$$\therefore \quad f(x) = \int_0^\infty \frac{\lambda\cos(\lambda\pi/2)}{1-\lambda^2}\sin\lambda x\,d\lambda$$

Example 3.20 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{\sin\lambda\pi}{1-\lambda^2}\sin\lambda x\,d\lambda = \begin{cases}\frac{\pi}{2}\sin x & 0\le x\le\pi\\ 0 & x>\pi\end{cases}$$

Solution : As $\sin\lambda x$ is present in the integral

$\therefore$ We should find sine transform of

$$f(x) = \begin{cases}\frac{\pi}{2}\sin x, & 0\le x\le\pi\\ 0, & x>\pi\end{cases}$$

Consider F.S.T. formula

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\,du$$

$$F_s(\lambda) = \int_0^\pi f(u)\sin\lambda u\,du + \int_\pi^\infty f(u)\sin\lambda u\,du$$

Put the values of f(u)

$$F_s(\lambda) = \frac{1}{2}\int_0^\pi \frac{\pi}{2}\,2\sin u\sin\lambda u\,du + \int_\pi^\infty 0\sin\lambda u\,du$$

(Use $2\sin\lambda u\cdot\sin u = \cos(\lambda-1)u - \cos(\lambda+1)u$)

$$\therefore\quad F_s(\lambda) = \frac{\pi}{4}\int_0^\pi\left[\cos(\lambda-1)-\cos(\lambda+1)u\right]du$$

Integrating, we get

$$= \frac{\pi}{4}\left[\frac{\sin(\lambda-1)u}{\lambda-1} - \frac{\sin(\lambda+1)u}{\lambda+1}\right]_0^\pi$$

(Use $\sin(\pi+\theta) = -\sin\theta$ and $\sin(\pi-\theta)=\sin\theta$)

$\{\sin(\lambda+1)\pi = -\sin\pi\lambda,\ \sin(\lambda-1)\pi = -\sin\pi\lambda\}$

Put the limits of u.

$$F_s(\lambda) = -\frac{\pi}{4}\left[\frac{\sin\lambda\pi}{\lambda-1} + \frac{+\sin\lambda\pi}{\lambda+1}\right]$$

$$F_s(\lambda) = -\frac{\pi}{4}\left[\frac{2\lambda}{\lambda^2-1}\right]\sin\pi\lambda$$

$$F_s(\lambda) = \frac{\pi}{2}\frac{\lambda\sin\pi\lambda}{(1-\lambda^2)}$$

Consider inverse sine transform.

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\,d\lambda$$

Put the value of $F(\lambda)$

$$\therefore\quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\frac{\lambda\sin\pi\lambda}{(1-\lambda^2)}\sin\lambda x\,d\lambda = \frac{\pi}{2}\sin x$$

Example 3.21 : *Using Fourier integral representation show that*

$$\int_0^\infty\left[\frac{1-\lambda\sin\frac{\lambda\pi}{2}}{1-\lambda^2}\right]\cos\lambda x\,d\lambda = \begin{cases}\frac{\pi}{2}\sin x & 0\le x\le\frac{\pi}{2}\\ 0 & x>\frac{\pi}{2}\end{cases}$$

Solution : As $\cos\lambda x$ is present in the integral

$\therefore$ We should find cosine transform of

$$f(x) = \begin{cases}\frac{\pi}{2}\sin x & 0\le x\le\frac{\pi}{2}\\ 0 & x>\frac{\pi}{2}\end{cases}$$

Consider the F.C.T. formula

$$F_c(\lambda) = \int_0^\infty f(u)\cdot\cos\lambda u\,du$$

$$= \int_0^{\pi/2} f(u)\cdot\cos\lambda u\cdot du + \int_{\pi/2}^\infty f(u)\cos\lambda u\,du$$

$$= \frac{1}{2}\int_0^\pi \frac{\pi}{2}\,2\sin u\cdot\cos\lambda u\cdot du + 0$$

(Use $2\cos\lambda u\cdot\sin u = \sin(\lambda+1)u - \sin(\lambda-1)u$)

$$\therefore\quad F_c(\lambda) = \frac{\pi}{4}\int_0^{\pi/2}\left[\sin(\lambda+1)u - \sin(\lambda-1)u\right]du$$

Integrate

$$= \frac{\pi}{4}\left[\frac{-\cos(\lambda+1)u}{\lambda+1} + \frac{\cos(\lambda-1)u}{\lambda-1}\right]_0^{\pi/2}$$

(Use $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$ and $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$)

$\left\{\cos(\lambda+1)\frac{\pi}{2} = -\sin\frac{\pi\lambda}{2},\ \cos(\lambda-1)\frac{\pi}{2} = \sin\frac{\pi\lambda}{2}\right\}$

$$\therefore F_c(\lambda) = \frac{\pi}{4}\left[\frac{\sin(\lambda\pi/2)}{\lambda+1} + \frac{\sin(\lambda\pi/2)}{\lambda-1} + \frac{1}{\lambda+1} - \frac{1}{\lambda-1}\right]$$

$$= \frac{\pi}{4}\left[\frac{1+\sin\left(\frac{\lambda\pi}{2}\right)}{\lambda+1} + \frac{1-\sin\left(\frac{\lambda\pi}{2}\right)}{\lambda-1}\right]$$

$$= \frac{\pi}{4}\left[\frac{2-2\lambda\sin\left(\frac{\lambda\pi}{2}\right)}{1-\lambda^2}\right] = \frac{\pi}{2}\left[\frac{1-\lambda\sin\left(\frac{\lambda\pi}{2}\right)}{1-\lambda^2}\right]$$

Consider inverse F.C.T.

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\left[\frac{1-\lambda \sin\left(\frac{\lambda\pi}{2}\right)}{1-\lambda^2}\right]\cos\lambda x \, d\lambda$$

$$\therefore \quad f(x) = \int_0^\infty \left[\frac{1-\lambda \sin\frac{\lambda\pi}{2}}{1-\lambda^2}\right]\cos\lambda x \, d\lambda$$

Example 3.22 : *Using Fourier integral representation show that*

$$\int_0^\infty \frac{\lambda^3 \sin\lambda x}{\lambda^4+4}\, d\lambda = \frac{\pi}{2} e^{-x} \cos x$$

SPPU : May-17

Solution : As '$\sin\lambda x$' is present in the expression

$\therefore$ We find Fourier sine transform of

$$f(x) = \frac{\pi}{2}\cdot e^{-x} \cos x$$

Consider F.S.T. formula

$$\therefore \quad F_s(\lambda) = \int_0^\infty f(u) \sin\lambda u \, du$$

$$F_s(\lambda) = \int_0^\infty \frac{\pi}{2} e^{-u} \cos u \sin \lambda u \, du$$

Take $\frac{\pi}{2}$ outside the integral.

$$= \frac{\pi}{2}\int_0^\infty e^{-u} \cos u \sin \lambda u \, du$$

(Use $\{2 \sin A \cos B = \sin(A+B) + \sin(A-B)\}$)

$$F_s(\lambda) = \frac{\pi}{2}\int_0^\infty \frac{e^{-u}}{2}\left[\sin(\lambda+1)\,u + \sin(\lambda-1)\,u\right] du$$

$$= \frac{\pi}{4}\left[\int_0^\infty e^{-u} \sin(\lambda+1)\,u\,du + \int_0^\infty e^{-u} \sin(\lambda-1)\,u\,du\right] = \frac{\pi}{4}[(1)+(2)]$$

We know that $\int_0^\infty e^{-u} \sin\lambda u \, du = \frac{\lambda}{1+\lambda^2}$ thus, replacing λ by $(\lambda+1)$ we get,

$$(1) = \int_0^\infty e^{-u} \sin(\lambda+1)\,u\,du = \frac{(\lambda+1)}{1+(\lambda+1)^2}$$

Replacing $\lambda+1$ by $\lambda-1$ we get

$$(2) = \int_0^\infty e^{-u} \sin(\lambda-1)\,u\,du = \frac{\lambda-1}{1+(\lambda-1)^2}$$

Substitute in $F_s(\lambda)$, we get

$$F_s(\lambda) = \frac{\pi}{4}\left[\frac{\lambda+1}{1+(\lambda+1)^2} + \frac{\lambda-1}{1+(\lambda-1)^2}\right]$$

$$F_s(\lambda) = \frac{\pi}{4}\left[\frac{2\lambda^3}{\lambda^4+4}\right] = \frac{\pi}{2}\left[\frac{\lambda^3}{\lambda^4+4}\right]$$

Consider inverse Fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x\, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}\left(\frac{\lambda^3}{\lambda^4+4}\right)\sin\lambda x\cdot d\lambda$$

Put the value of f(x)

$$\therefore \int_0^\infty \frac{\lambda^3 \sin\lambda x}{\lambda^4+4}\, d\lambda = \frac{\pi}{2} e^{-x} \cos x$$

Example 3.23 : *Using Fourier integral representation show that* $\int_0^\infty \frac{2\lambda \sin\lambda x}{\lambda^4+4}\, d\lambda = \frac{\pi}{2} e^{-x} \sin x$

Solution : As $\sin\lambda x$ is present in the expression we find Fourier sine transform

$$\therefore \quad F(\lambda) = \frac{1}{2}\int_0^\infty \frac{\pi}{2} e^{-u}\, 2\sin u \sin\lambda u\, du$$

(Use $2\sin\lambda u \sin u = \cos(\lambda-1)u - \cos(\lambda+1)u$)

$$\therefore \quad F(\lambda) = \frac{\pi}{4}\int_0^\infty e^{-u}\left[\cos(\lambda-1)u - \cos(\lambda+1)u\right] du$$

$$= \frac{\pi}{4}\left\{\left[\frac{e^{-u}}{1+(\lambda-1)^2}\left(-\cos(\lambda-1)+(\lambda-1)\sin(\lambda-1)\right)\right]_0^\infty - \left[\frac{e^{-u}}{1+(\lambda+1)^2}\left(-\cos(\lambda+1)+(\lambda+1)\sin(\lambda+1)\right)\right]_0^\infty\right\}$$

$$= \frac{\pi}{4}\left[\frac{1}{1+(\lambda-1)^2} - \frac{1}{1+(\lambda+1)^2}\right] = \frac{\pi}{4}\left[\frac{\lambda^2+2\lambda+2-\lambda^2+2\lambda-2}{\lambda^4+4}\right] = \frac{\pi}{4}\left[\frac{4\lambda}{\lambda^4+4}\right]$$

Consider inverse fourier sine transform

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{4}\frac{4\lambda}{\lambda^4+4}\sin\lambda x\, d\lambda$$

$$\therefore \quad \frac{\pi}{2} e^{-x} \sin x = \int_0^\infty \frac{2\lambda}{\lambda^4+4}\sin\lambda x\, d\lambda$$

Example 3.24 : *Solve the integral equation.*

$$\int_0^{\infty} f(x)\cos\lambda x\,dx = \begin{cases}1-\lambda & 0\le\lambda\le 1\\ 0 & \lambda>1\end{cases}$$ *Hence show that* $\int_0^{\infty}\frac{\sin^2 x}{x^2}\,dx=\frac{\pi}{2}$

SPPU : May-19

Solution : As $\cos\lambda x$ is present in the integrand, we use F.C.T.

$$F_c(\lambda) = \int_0^{\infty} f(x)\cos\lambda x\,dx$$

$$= \begin{cases}1-\lambda, & 0\le\lambda\le 1\\ 0 & \lambda>1\end{cases}$$

To find f(x) consider inverse Fourier cosine transforms

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\lambda)\cos\lambda x\,d\lambda$$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 (1-\lambda)\cos\lambda x\,d\lambda+\int_1^{\infty} 0\,d\lambda\right]$$

$$f(x) = \frac{2}{\pi}\left[(1-\lambda)\left(\frac{\sin\lambda x}{x}\right)-(-1)\left(\frac{-\cos\lambda x}{x^2}\right)\right]_0^1$$

$$f(x) = \frac{2}{\pi}\left[\left(0-\frac{\cos x}{x^2}\right)-\left(0-\frac{1}{x^2}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)$$

(Use $(1-\cos x) = 2\sin^2(x/2)$)

$$= \frac{2}{\pi}\,\frac{2\sin^2\frac{x}{2}}{x^2}$$

$$f(x) = \frac{1}{\pi}\,\frac{\sin^2\frac{x}{2}}{\left(\frac{x^2}{4}\right)} = \frac{1}{\pi}\,\frac{\sin^2(x/2)}{(x/2)^2}$$

Substituting in the given equation, we get

$$\int_0^{\infty}\frac{1}{\pi}\,\frac{\sin^2(x/2)}{(x/2)^2}\cos\lambda x\,dx = \begin{cases}1-\lambda & 0\le\lambda\le 1\\ 0 & \lambda>1\end{cases}$$

Put $\lambda = 0$

$$\frac{1}{\pi}\int_0^{\infty}\frac{\sin^2(x/2)}{(x/2)^2}\,1\,dx = 1$$

Put $x/2 = u,\ x = 2u,\ dx = 2\,du$

$$\frac{1}{\pi}\int_0^{\infty}\frac{\sin^2 u}{u^2}\,2\,du = 1$$

$$\int_0^{\infty}\frac{\sin^2 u}{u^2}\,du = \frac{\pi}{2}$$

Example 3.25 : *Solve the integral equation*

$$\int_0^{\infty} f(x)\sin\lambda x\,dx = \begin{cases}1-\lambda & 0\le\lambda\le 1\\ 0 & \lambda>1\end{cases}$$

SPPU : Dec.-19, May-17, 18

Solution : As $\sin\lambda x$ is present in the integrand, we use F.S.T.

$$\therefore\quad F_s(T) = \int_0^{\infty} f(x)\sin\lambda x\,dx$$

$$F_s(\lambda) = \begin{cases}1-\lambda, & 0\le\lambda\le 1\\ 0 & \text{if } \lambda>1\end{cases}$$

To find f(x) consider inverse Fourier sine transform

$$\therefore\quad f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda)\sin\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\left[\int_0^1 (1-\lambda)\sin\lambda x\,d\lambda+\int_1^{\infty} 0\right]$$

$$= \frac{2}{\pi}\left[(1-\lambda)\left(-\frac{\cos\lambda x}{x}\right)-(-1)\left(\frac{-\sin\lambda x}{x^2}\right)\right]_0^1$$

$$= \frac{2}{\pi}\left[\frac{1}{x}-\frac{\sin x}{x^2}\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{x-\sin x}{x^2}\right]$$

Example 3.26 : *Solve the integral equation* $\int_0^{\infty} f(x)\cos\lambda x\,dx = e^{-\lambda},\ \lambda>0$

SPPU : May-19

Solution : Given

$$\therefore\ \int_0^{\infty} f(x)\cos\lambda x\,dx = e^{-\lambda},\ \lambda>0$$

As $\cos\lambda x$ is present in the integrand we use F.C.T.

$$\therefore\quad F_c(\lambda) = e^{-\lambda},\ \lambda>0$$

To find f(x) consider inverse Fourier cosine transform.

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F(\lambda)\cos\lambda x\,d\lambda$$

Put the value of $F(\lambda)$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^{\infty} e^{-\lambda}\cos\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\cos\lambda u + x\sin\lambda x)\right]_0^{\infty}$$

$$f(x) = \frac{2}{\pi}\left(\frac{1}{1+x^2}\right)$$

Example 3.27 : *Solve the integral equation* $\int_0^{\infty} f(x)\sin\lambda x\, dx = e^{-\lambda},\ \lambda > 0$

SPPU : Dec.-18, Marks 4

Solution : Given

$$\int_0^{\infty} f(x)\sin\lambda x\, dx = e^{-\lambda},\ \lambda > 0$$

As $\sin\lambda x$ is present in the integrand we use F.S.T.

$$\therefore \quad F_s(\lambda) = e^{-\lambda}$$

To find f(x) we use inverse Fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F(\lambda)\sin\lambda x\, d\lambda$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} e^{-\lambda}\sin\lambda x\, d\lambda$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\sin\lambda x - x\cos\lambda x)\right]_0^{\infty}$$

$$f(x) = \frac{2}{\pi}\frac{x}{1+x^2}$$

Example 3.28 : *Solve the integral equation*

$$\int_0^{\infty} f(x)\sin\lambda x\, dx = \begin{cases} 1 & 0<\lambda<1 \\ 2 & 1<\lambda<2 \\ 0 & \lambda>2 \end{cases}$$

Solution : As $\sin\lambda x$ is present in the integrand we use F.S.T.,

$$F_s(\lambda) = \begin{cases} 1, & 0<\lambda<1 \\ 2, & 1<\lambda<2 \\ 0, & \lambda>2 \end{cases}$$

Consider inverse sine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\lambda)\sin\lambda x\, d\lambda$$

Split the integral and substitute the value of $F(\lambda)$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 1\sin\lambda x\, d\lambda + \int_1^2 2\sin\lambda x\, d\lambda\right]$$

$$f(x) = \frac{2}{\pi}\left[\left(\frac{-\cos\lambda x}{x}\right)_0^1 - 2\left(\frac{\cos\lambda x}{x}\right)_1^2\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{-\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{1+\cos x - 2\cos 2x}{x}\right]$$

Example 3.29 : $\int_0^{\infty} f(x)\cos\lambda x\, dx = \begin{cases} 1 & 0<\lambda<1 \\ 2 & 1<\lambda<2 \\ 0 & \lambda>2 \end{cases}$

Solution : As $\cos\lambda x$ is present in the integrand, we use F.C.T.

$$\therefore \quad F_c(\lambda) = \begin{cases} 1 & 0<\lambda<1 \\ 2 & 1<\lambda<2 \\ 0 & \lambda>2 \end{cases}$$

Consider the inverse cosine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty} f(\lambda)\cos\lambda x\, d\lambda$$

Split the integral and substitute the value of $F(\lambda)$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 1\cos\lambda x\, d\lambda + \int_1^2 2\cos\lambda x\, d\lambda\right]$$

$$f(x) = \frac{2}{\pi}\left[\left(\frac{\sin\lambda x}{x}\right)_0^1 + 2\left(\frac{\sin\lambda x}{x}\right)_1^2\right]$$

$$f(x) = \frac{2}{\pi}\left[\frac{\sin x}{x} + \frac{2\sin 2x}{x} - \frac{2\sin x}{x}\right]$$

$$\therefore \quad f(x) = \frac{2}{\pi}\left[\frac{2\sin 2x - \sin x}{x}\right]$$

Example 3.30 : *Find the Fourier integral representation for the function*

$$f(x) = \begin{cases} 0 & x<0 \\ e^{-x} & x>0 \\ 1/2 & x=0 \end{cases}$$

Solution : Note that the given function f(x) is neither an even function nor an odd function.

Consider F.T. formula

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)e^{-i\lambda u}\, du$$

$$= \int_{-\infty}^{0} f(u)e^{-i\lambda u}\, du + \int_0^{\infty} f(u)e^{-i\lambda u}\, du$$

$$= \int_{-\infty}^{0} (0)\, e^{-i\lambda u}\, du + \int_{0}^{\infty} e^{-u}\, e^{-i\lambda u}\, du$$

$$= \left\{ \int_{0}^{\infty} e^{-(1+i\lambda)u}\, du \right\}$$

$$= \left[\frac{e^{-(1+i\lambda)u}}{-(1+i\lambda)} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\lambda}$$

Multiply and divide by $1-i\lambda$

$$= \frac{1}{\sqrt{2\pi}} \frac{1-i\lambda}{(1+i\lambda)(1-i\lambda)}$$

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \frac{1-i\lambda}{\left(1+\lambda^2\right)} \quad \text{... (1)}$$

By using inverse transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1-i\lambda}{1+\lambda^2} \right] e^{i\lambda x}\, d\lambda$$

(Use $e^{i\lambda x} = \cos\lambda x + i \sin\lambda x$)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1-i\lambda}{1+\lambda^2} \right] (\cos\lambda x + i \sin\lambda x)\, d\lambda$$

(Use $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{\cos\lambda x + \lambda \sin\lambda x}{1+\lambda^2} + i \frac{-\lambda\cos\lambda x + \sin\lambda x}{1+\lambda^2} \right] d\lambda$$

Separate the two integrals

$$= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \left[\frac{\cos\lambda x + \lambda \sin\lambda x}{1+\lambda^2} \right] d\lambda + i \int_{-\infty}^{\infty} \left[\frac{-\lambda\cos\lambda x + \sin\lambda x}{1+\lambda^2} \right] d\lambda \right\}$$

($\uparrow$ is an even function of λ) ($\uparrow$ is an odd function of λ)

(Use $\int_{-\infty}^{\infty} F(\lambda)\, d\lambda = 2\int_{0}^{\infty} F(\lambda)\, d\lambda$. If $F(\lambda)$ is an even function of λ and

$\int_{-\infty}^{\infty} F(\lambda)\, d\lambda = 0$ If $F(\lambda)$ is an odd function of λ.)

Thus

$$f(x) = \frac{2}{2\pi} \int_{0}^{\infty} \left[\frac{\cos\lambda x + \lambda \sin\lambda x}{1+\lambda^2} \right] d\lambda + 0$$

Simplify

$$\therefore \quad f(x) = \frac{1}{\pi} \int_{0}^{\infty} \left[\frac{\cos\lambda x + \lambda \sin\lambda x}{1+\lambda^2} \right] d\lambda$$

which is the Fourier integral representation of f(x).

Example 3.31 : *If* $f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & x < 0,\ x > \pi \end{cases}$

then prove that

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos\lambda x + \cos[\lambda(\pi - x)]}{1-\lambda^2}$$

Hence deduce that

$$\int_{0}^{\infty} \frac{\cos \lambda\pi/2}{1-\lambda^2}\, d\lambda = \frac{\pi}{2}$$

Solution : Here f(x) is defined over the interval $-\infty < x < \infty$, and f(x) is neither an even function nor an odd function.

Consider F.T. formula

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

$$= \int_{0}^{\pi} \sin u\ e^{-i\lambda u}\, du$$

$$= \left[\frac{e^{-i\lambda u}}{(-i\lambda)^2 + 1} (-i\lambda \sin u - \cos u) \right]_{0}^{\pi}$$

$$= \left[\frac{e^{-i\lambda\pi}}{-\lambda^2+1}(-\cos\pi) - \frac{1}{-\lambda^2+1}(-\cos 0) \right]$$

$$= \frac{e^{-i\lambda\pi}+1}{1-\lambda^2}$$

Consider inverse transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1+e^{-i\lambda\pi}}{1-\lambda^2} \right) e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{e^{i\lambda x} + e^{-i\lambda(\pi - x)}}{1-\lambda^2} \right) d\lambda$$

(Use $e^{ix} = \cos x + i \sin x$)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos\lambda x + i \sin\lambda x + \cos\lambda(\pi - x) + i \sin\lambda(\pi - x)}{\left(1-\lambda^2\right)} d\lambda$$

Consider F.T. formula

$$F(\lambda) = \int_{-\infty}^{\infty} f(u)\, e^{-i\lambda u}\, du$$

$$= \int_{0}^{\pi} \sin u \;\; e^{-i\lambda u}\, du$$

$$= \left[\frac{e^{-i\lambda u}}{(-i\lambda)^2+1} (-i\lambda \sin u - \cos u) \right]_0^{\pi}$$

$$= \left[\frac{e^{-i\lambda\pi}}{-\lambda^2+1}(-\cos\pi) - \frac{1}{-\lambda^2+1}(-\cos 0) \right]$$

$$= \frac{e^{-i\lambda\pi}+1}{1-\lambda^2}$$

Consider inverse transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)\, e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1+e^{-i\lambda\pi}}{1-\lambda^2} \right) e^{i\lambda x}\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{e^{i\lambda x}+e^{-i\lambda(\pi-x)}}{1-\lambda^2} \right) d\lambda$$

Use $\quad e^{ix} = \cos x + i \sin x$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos\lambda x + i\sin\lambda x + \cos\lambda(\pi-x) + i\sin\lambda(\pi-x)}{(1-\lambda^2)}\, d\lambda$$

Separate the real and imaginary integrals

$$= \int_{-\infty}^{\infty} \frac{\cos\lambda x + \cos\lambda(\pi-x)}{2\pi(1-\lambda^2)}\, d\lambda + \int_{-\infty}^{\infty} \frac{i\sin\lambda x + i\sin\lambda(\pi-x)}{2\pi(1-\lambda^2)}\, d\lambda$$

[↑ even function of λ] [↑ odd function of λ]

Use $\int_{-\infty}^{\infty} F(\lambda)\, d\lambda = 2\int_{0}^{\infty} F(\lambda)\, d\lambda$ If $F(\lambda)$ is an even function of λ and

$\int_{-\infty}^{\infty} F(\lambda)\, d\lambda = 0$ If $F(\lambda)$ is an odd function of λ.

Thus $f(x) = \dfrac{1}{2\pi}\, 2\int_{0}^{\infty} \left(\dfrac{\cos\lambda x + \cos\lambda(\pi-x)}{1-\lambda^2} \right) d\lambda + 0$

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos\lambda x + \cos[\lambda(\pi-x)]}{1-\lambda^2}\, d\lambda$$

Put $\quad x = \dfrac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos\lambda\pi/2 + \cos[\lambda(\pi-\pi/2)]}{1-\lambda^2}\, d\lambda$$

$\because \quad \left[f\left(\dfrac{\pi}{2}\right) = \sin\pi/2 = 1 \right]$

$$\int_{0}^{\infty} \frac{\cos\lambda\pi/2}{1-\lambda^2}\, d\lambda = \frac{\pi}{2}$$

This is the required deduction.

Example 3.32 : *a) Find Fourier transform of* $e^{-x^2/2}$.

b) Show that Fourier transform of $e^{-x^2/2}$ *is* $e^{-\lambda^2/2}$.

Solution : a) The Fourier transform of $f(x) = e^{-x^2/2}$ is given by

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{-i\lambda x}\, dx$$

$$\therefore \quad F(\lambda) = \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2}+i\lambda x\right)}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2i\lambda x)}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x+i\lambda)^2 - i^2\lambda^2]}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\lambda)^2}\, e^{-\frac{\lambda^2}{2}}\, dx$$

$$= e^{-\frac{\lambda^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\lambda)^2}\, dx$$

The above integrand is even function

$$\therefore \quad F(\lambda) = e^{-\lambda^2/2} \cdot 2\int_{0}^{\infty} e^{-\frac{1}{2}(x+i\lambda)^2}\, dx$$

Substitute $\dfrac{1}{2}(x+i\lambda)^2 = u$

$\therefore \quad \dfrac{1}{2}\, 2(x+i\lambda)\, dx = du$

$$\therefore \quad dx = \frac{du}{x+i\lambda} = \frac{du}{\sqrt{2u}}$$

∴ We get

$$F(\lambda) = e^{-\lambda^2/2}\, \frac{2}{\sqrt{2}} \int_{0}^{\infty} e^{-u}\, u^{-\frac{1}{2}}\, du$$

As $\overline{|n} = \int_0^\infty e^{-u}\, u^{n-1}\, du$

$$\therefore \quad F(\lambda) = e^{-\lambda^2/2}\sqrt{2}\,\overline{\left|\frac{1}{2}\right.} = \sqrt{2}\cdot\sqrt{\pi}\cdot e^{-\lambda^2/2}$$

$$F(\lambda) = \sqrt{2\pi}\; e^{-\lambda^2/2}$$

b) In this example use the formula

$$F(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{2\pi}\cdot e^{-\lambda^2/2} \qquad \text{(by part (a))}$$

$$F(\lambda) = e^{-\lambda^2/2}$$

Example 3.33 : *a) Find Fourier transform of* e^{-x^2}

b) Show that the Fourier transform of e^{-x^2} *is* $\frac{1}{\sqrt{2}}e^{-\lambda^2/4}$.

Solution : **a) Step 1 :** The F.T. $f(x) = e^{-x^2}$ is

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-x^2}\cdot e^{-i\lambda x}\, dx$$

Step 2 : $\therefore\; F(\lambda) = \int_{-\infty}^{\infty} e^{-\left(x^2+i\lambda x\right)}\, dx$

$$= \int_{-\infty}^{\infty} e^{-\left(x+\frac{i\lambda}{2}\right)^2-\frac{\lambda^2}{4}}\, dx$$

$$\left(\because x^2+i\lambda x = x^2+i\lambda x+\frac{i^2\lambda^2}{4}+\frac{\lambda^2}{4}\right)$$

$$= e^{-\frac{\lambda^2}{4}}\int_{-\infty}^{\infty} e^{-\left(x+\frac{i\lambda}{2}\right)^2}\, dx$$

Step 3 : Above integrand is an even function

$\therefore$ We have

$$F(\lambda) = 2\,e^{-\lambda^2/4}\int_0^\infty e^{-\left(x+i\frac{\lambda}{2}\right)^2}\, dx$$

Step 4 : Substitute $\left(x+i\frac{\lambda}{2}\right)^2 = u$

$$\therefore\; 2\left(x+i\frac{\lambda}{2}\right)dx = du$$

$$\therefore \quad dx = \frac{du}{2\sqrt{u}}$$

$$\therefore \quad F(\lambda) = 2\,e^{-\lambda^2/4}\int_0^\infty e^{-u}\,\frac{u^{-\frac{1}{2}}}{2}\, du$$

$$= e^{-\lambda^2/4}\,\overline{\left|\frac{1}{2}\right.} \qquad \left(\because \overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}\right)$$

$$F(\lambda) = \sqrt{\pi}\; e^{-\lambda^2/4}$$

b) In this example use the formula

$$F(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\sqrt{\pi}\; e^{-\lambda^2/4} \qquad \text{(by part (a))}$$

$$= \frac{1}{\sqrt{2}}\, e^{-\lambda^2/4}$$

Example 3.34 : *Find Fourier sine transform of* $\frac{1}{x}$.

Solution : Consider F. S. T. formula

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du$$

$$= \int_0^\infty \frac{1}{u}\sin\lambda u\, du$$

Put $\lambda u = t$, $\therefore\; du = \frac{dt}{\lambda}$

$$= \int_0^\infty \frac{\sin t}{t/\lambda}\,\frac{dt}{\lambda}$$

(Use standard integral $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$)

$$= \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

Example 3.35 : *Find Fourier sine transform of* $\frac{e^{-ax}}{x}$ *and hence evaluate* $\int_0^\infty \tan^{-1}\frac{\lambda}{a}\sin\lambda x\, d\lambda$.

SPPU : Dec.-17

Solution : Consider F.S.T. formula

$$F_s(\lambda) = \int_0^\infty f(u)\sin\lambda u\, du$$

$$= \int_0^\infty \frac{e^{-au}}{u}\sin\lambda u\, du$$

Use D.U.I.S

$$\therefore \quad \frac{d}{d\lambda}F_s(\lambda) = \int_0^\infty \frac{\partial}{\partial\lambda}\left(\frac{e^{-au}}{u}\sin\lambda u\right)du$$

$$= \int_0^\infty e^{-au} \cos \lambda u \, du$$

$$\frac{d}{d\lambda} F_s(\lambda) = \frac{a}{\lambda^2 + a^2}$$

Integrating, with respect to λ.

$$F_s(\lambda) = \tan^{-1}\left(\frac{\lambda}{a}\right) + A$$

To find constant A, we put $\lambda = 0$

$\therefore$ $F_s(0) = 0 + A$

$\therefore$ $0 = 0 + A$

Put value of A in step 6

$\therefore$ $F_s(\lambda) = \tan^{-1} \frac{\lambda}{a}$

Consider inverse sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \tan^{-1} \frac{\lambda}{a} \sin \lambda x \, d\lambda$$

$$\therefore \quad \frac{\pi}{2} \frac{e^{-ax}}{x} = \int_0^\infty \tan^{-1} \frac{\lambda}{a} \sin \lambda x \, d\lambda$$

Example 3.36 : *Find the Fourier sine and cosine transforms of the function $f(x) = x^{m-1}$.*

Solution : F.C.T. formula

$$F_c(\lambda) = \int_0^\infty u^{m-1} \cos \lambda u \, du \quad \text{... (1)}$$

F.S.T. formula

$$F_s(\lambda) = \int_0^\infty u^{m-1} \sin \lambda u \, du \quad \text{... (2)}$$

By definition of Gamma function

We know $\quad \overline{|m} = \int_0^\infty e^{-x} x^{m-1} \, dx \quad \text{... (3)}$

Put $x = i\lambda u$, thus $dx = i\lambda du$, $\therefore$ equation (3) becomes

$$\overline{|m} = \int_0^\infty e^{-i\lambda u} (i\lambda u)^{m-1} \, i\lambda \, du$$

$$\overline{|m} = i^m \lambda^m \int_0^\infty e^{-i\lambda u} u^{m-1} du$$

$(\because i = e^{i\pi/2})$ substituting we get

$$\overline{|m} = \left(e^{i\pi/2}\right)^m \lambda^m \int_0^\infty e^{-i\lambda u} u^{m-1} \, du$$

$$\frac{\overline{|m}}{\lambda^m} e^{-im\pi/2} = \int_0^\infty e^{-i\lambda u} u^{m-1} du$$

Use $e^{-ix} = \cos x - i \sin x$

$$\therefore \frac{\overline{|m}}{\lambda^m}\left(\cos \frac{m\pi}{2} - i \sin \frac{m\pi}{2}\right) = \int_0^\infty (\cos \lambda u - i \sin \lambda u) \, u^{m-1} \, du$$

Equating real and imaginary parts on both sides, we get

$$\int_0^\infty u^{m-1} \cos \lambda u \, du = \frac{\overline{|m}}{\lambda^m} \cos \frac{m\pi}{2}$$

$$\text{and} \int_0^\infty u^{m-1} \sin \lambda u \, du = \frac{\overline{|m}}{\lambda^m} \sin \frac{m\pi}{2}$$

Substituting in (1) and (2) we get,

$$F_c(\lambda) = \frac{\overline{|m}}{\lambda^m} \cos \frac{m\pi}{2}$$

$$F_s(\lambda) = \frac{\overline{|m}}{\lambda^m} \sin \frac{m\pi}{2}$$

Example 3.37 : *Find Fourier cosine transform of $\frac{1}{1+x^2}$. Hence find sine transform of $\frac{x}{1+x^2}$.*

Solution : F.C.T. formula

$$F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du$$

$$= \int_0^1 \frac{1}{1+u^2} \cos \lambda u \, du$$

Use D.U.I.S

$$F_c'(\lambda) = \int_0^\infty \frac{\partial}{\partial \lambda} \frac{1}{1+u^2} \cos \lambda u \, du$$

Differentiating w.r.t. λ

$$F_c'(\lambda) = \int_0^\infty -\frac{u}{1+u^2} \sin \lambda u \, du$$

Multiply and divide by u

$$= \int_0^\infty -\frac{u^2}{u(1+u^2)} \sin \lambda u \, du$$

Add and subtract 1

$$= \int_0^\infty \frac{-(u^2+1-1)}{u(1+u^2)} \sin \lambda u \, du$$

$$= \int_0^\infty \left(\frac{1}{u(1+u^2)} - \frac{1}{u} \right) \sin \lambda u \, du$$

$$= \left\{ \int_0^\infty \frac{\sin \lambda u}{u(1+u^2)} du - \int_0^\infty \frac{\sin \lambda u}{u} du \right\}$$

Putting $\lambda u = t$ in the second integral

$$= \left\{ \int_0^\infty \frac{\sin \lambda u}{u(1+u^2)} - \int_0^\infty \frac{\sin t}{t} dt \right\}$$

Use standard integral $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

$$= \left\{ \int_0^\infty \frac{\sin \lambda u}{u(1+u^2)} du - \frac{\pi}{2} \right\}$$

Again, using the D.U.I.S., we get

$$F_c''(\lambda) = \int_0^\infty \frac{\partial}{\partial \lambda} \frac{\sin \lambda u}{u(1+u^2)} du$$

Differentiating w.r.t. λ, we get

$$F_c''(\lambda) = \int_0^\infty \frac{\cos \lambda u}{1+u^2} du = F(\lambda)$$

$$\therefore \quad F_c''(\lambda) - F_c(\lambda) = 0$$

The general solution of this differential equation is given by

$$F_c(\lambda) = A e^{\lambda} + B e^{-\lambda} \quad \text{... (i)}$$

Differentiating w.r.t. λ, we get

$$F_c'(\lambda) = A e^{\lambda} - B e^{-\lambda} \quad \text{... (ii)}$$

Put $\lambda = 0$ in (i) and (ii)

$$F_c(0) = \left[\int_0^\infty \frac{\cos \lambda u}{1+u^2} du \right]_{\lambda=0} = A + B$$

$$F_c'(0) = \left[\int_0^\infty \frac{\sin \lambda u}{u(1+u^2)} du - \frac{\pi}{2} \right]_{\lambda=0} = A - B$$

$$\therefore \quad \int_0^\infty \frac{1}{1+u^2} du = A + B, \left\{ -\frac{\pi}{2} \right\} = A - B$$

Integrate first

$$\therefore \quad [\tan^{-1} u]_0^\infty = A + B$$

i.e. $$\left\{ \frac{\pi}{2} \right\} = A + B \text{ and } \left\{ -\frac{\pi}{2} \right\} = A - B$$

Solving we get, $A = 0$, $B = \frac{\pi}{2}$

Substituting A and B in equation (i) and (ii), we get

$$F_c(\lambda) = \frac{\pi}{2} e^{-\lambda}$$

and $$F_c'(\lambda) = -\frac{\pi}{2} e^{-\lambda}$$

Substituting f (λ) and f'(λ) in step 2 and step 4 we get

$$\therefore \quad \frac{\pi}{2} e^{-\lambda} = \int_0^\infty \frac{1}{1+u^2} \cos \lambda u \, du \text{ and}$$

$$-\frac{\pi}{2} e^{-\lambda} = \int_0^\infty \frac{-u}{1+u^2} \sin \lambda u \, du$$

Thus we can say that the Fourier cosine transform of $\frac{1}{1+x^2}$ is $\frac{\pi}{2} e^{-\lambda}$

Also we can say that the Fourier sine transform of $\frac{x}{1+x^2}$ is $\frac{\pi}{2} e^{-\lambda}$

Example 3.38 : *Using inverse sine transform, find f(x) if $F_s(\lambda) = \frac{1}{\lambda} e^{-a\lambda}$.*

Solution : Inverse sine transform of $F_s(\lambda)$ is given by (Note the formula for inverse)

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$$

Put the value of F (λ)

$$\therefore \quad f(x) = \frac{2}{\pi} \int_0^\infty \frac{e^{-a\lambda}}{\lambda} \sin \lambda x \, d\lambda$$

[Use the rule of D.U.I.S.]

$$\therefore \quad f'(x) = \frac{2}{\pi} \int_0^\infty \frac{\partial}{\partial x} \left(\frac{e^{-a\lambda}}{\lambda} \sin \lambda x \right) d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty e^{-a\lambda} \cos x\lambda \, d\lambda$$

$$= \frac{2}{\pi} \left[\frac{e^{-a\lambda}}{a^2+x^2} (-a \cos \lambda x + x \sin \lambda x) \right]_0^\infty$$

$$f'(x) = \frac{2}{\pi} \frac{a}{a^2+x^2} \quad \text{As } e^{-\infty} = 0, e^0 = 1$$

$$f(x) = \frac{2}{\pi} \tan^{-1} \frac{x}{a} + A$$

Put $x = 0$,

$$f(0) = 0 + A$$

To find f(0), put x = 0 in step 2.

$$f(x) = \left[\frac{2}{\pi}\int_0^\infty \frac{e^{-a\lambda}}{\lambda}\sin\lambda x \, d\lambda\right]_{x=0} = A \qquad \therefore A = 0$$

Put A = 0 in step 7

$$\therefore \qquad f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a}$$

Example 3.39 : *What is the function f(x), whose Fourier cosine tranform is* $\frac{\sin a\lambda}{\lambda}$ *?*

Solution :

Given that $F_c(\lambda) = \frac{\sin a\lambda}{\lambda}$. To find f(x) consider inverse cosine tranform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\lambda)\cos\lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin a\lambda}{\lambda}\cos\lambda x \, d\lambda$$

(Use 2sin A cos B = sin (A + B) + sin (A – B))

$$= \frac{1}{2}\frac{2}{\pi}\int_0^\infty \frac{\sin(a+x)\lambda + \sin(a-x)\lambda}{\lambda} dx$$

$$= \frac{1}{\pi}\left[\int_0^\infty \frac{\sin(a+x)\lambda}{\lambda} d\lambda + \int_0^\infty \frac{\sin(a-x)\lambda}{\lambda} d\lambda\right]$$

$$\text{Use } \because \int_0^\infty \frac{\sin ax}{x} dx = \begin{cases} \pi/2 & a > 0 \\ -\pi/2 & a < 0 \end{cases}$$

$$= \frac{1}{\pi}\begin{cases} \left[\frac{\pi}{2}+\frac{\pi}{2}\right] & a+x>0 \text{ and } a-x>0 \\ \left[\frac{\pi}{2}-\frac{\pi}{2}\right] & a+x>0 \text{ and } a-x<0 \end{cases}$$

$$f(x) = \frac{1}{\pi}\begin{cases} \pi & x>-a \text{ and } x<a \\ 0 & x>-a \text{ and } x>a \end{cases}$$

$$f(x) = \begin{cases} 1 & |x|<a \\ 0 & |x|>a \end{cases}$$

Example 3.40 : *Find Fourier transform of* $e^{-|x|}$ *hence show that* $\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1+\lambda^2} d\lambda = \pi e^{-|x|}$.

Solution : The Fourier transform of given function is

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x} dx$$

$$= \int_{-\infty}^{\infty} e^{-|x|}(\cos\lambda x - i\sin\lambda x)\, dx$$

As $e^{-|x|}$ is an even function, hence we get,

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-|x|}\cos\lambda x \, dx$$

$$= 2\int_0^\infty e^{-x}\cos\lambda x \, dx$$

$$F_c(\lambda) = 2\left[\frac{e^{-x}}{1+\lambda^2}(-\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty$$

$$F_c(\lambda) = \left[\frac{2}{1+\lambda^2}\right]$$

To find f(x) taking inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{-i\lambda x} d\lambda$$

Put the value of F(λ)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{1}{1+\lambda^2} e^{i\lambda x} d\lambda$$

$$\pi f(x) = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1+\lambda^2} d\lambda$$

$$\therefore \qquad \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1+\lambda^2} d\lambda = \pi e^{-|x|}$$

Exercise 3.1

1. *Find the Fourier integral representation of*

$$i)\ f(x) = \begin{cases} 1 & |x|<1 \\ 1/2 & |x|=1 \\ 0 & |x|>1 \end{cases}$$

[Ans. : $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda$**]**

$$ii)\ f(x) = \begin{cases} 0 & x<-a \\ 1 & -a \le x \le a \\ 0 & x>a \end{cases}$$

[Ans. : $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin a\lambda\cos\lambda x}{\lambda} d\lambda$**]**

$$iii)\ f(x) = \begin{cases} \frac{\pi}{2}\sin x & |x|<\pi \\ 0 & |x|>\pi \end{cases}$$

[Ans. : $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\pi\sin\lambda x}{1-\lambda^2} d\lambda$**]**

iv) $f(x)=\begin{cases}\frac{\pi}{2}\cos x & |x|<\pi \\ 0 & |x|>\pi\end{cases}$

[Ans. : $f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\lambda \sin\lambda\pi \cos\lambda x}{1-\lambda^2}\,d\lambda$]

v) $f(u)=e^{-|u|}\quad -\infty<u<\infty$

[Ans. : $f(u)=\frac{2}{\pi}\int_0^{\infty}\frac{\cos\lambda u}{1+\lambda^2}\,d\lambda$]

vi) $f(u)=e^{-u^2/2}\quad -\infty<u<\infty$

[Ans. : $f(u)=\frac{2}{\pi}\int_0^{\infty}e^{-\lambda^2/2}\cos\lambda u\,d\lambda$]

vii) $f(x)=\begin{cases}5 & |x|<a \\ 0 & |x|>a\end{cases}$

[Ans. : $f(x)=\frac{10}{\pi}\int_0^{\infty}\frac{\sin a\lambda}{\lambda}\cos\lambda x\,d\lambda$]

2. *Find Fourier Sine Transform of the following*

i) $f(x)=\begin{cases}0 & 0\le x\le 1 \\ x & 1<x\le 2 \\ 0 & x>2\end{cases}$

[Ans. : $\sqrt{\frac{2}{\pi}}\left\{\frac{\cos\lambda-\cos 2\lambda}{\lambda}+\frac{\sin 2\lambda-\sin\lambda}{\lambda^2}\right\}$]

ii) $f(x)=\begin{cases}x^2 & 0<x<1 \\ 0 & x>1\end{cases}$

[Ans. : $F_s(\lambda)=\sqrt{\frac{2}{\pi}}\left\{\frac{2\sin\lambda}{\lambda^2}-\frac{\cos\lambda}{\lambda}+\frac{2\cos\lambda}{\lambda^3}-\frac{2}{\lambda^3}\right\}$]

iii) $f(x)=\begin{cases}\pi & 0\le x\le\pi \\ 0 & x>\pi\end{cases}$

[Ans. : $F_s(\lambda)=\sqrt{\frac{2}{\pi}}\left\{\frac{\pi(1-\cos\lambda)}{\lambda}\right\}$]

iv) $f(x)=e^{-2x}+e^{-3x}\quad x>0$

[Ans. : $F_s(\lambda)=\sqrt{\frac{2}{\pi}}\left\{\frac{\lambda}{\lambda^2+4}+\frac{\lambda}{\lambda^2+9}\right\}$]

v) $f(x)=\begin{cases}\sin x & 0<x<m \\ 0 & x>m\end{cases}$

[Ans. : $F_s(\lambda)=\sqrt{\frac{2}{\pi}}\left\{\frac{\sin(\lambda-1)m}{\lambda-1}+\frac{\sin(\lambda+1)m}{\lambda+1}\right\}$]

vi) $f(x)=\begin{cases}0 & 0<x<a \\ x & a\le x\le b \\ 0 & x>b\end{cases}$

[Ans. : $\sqrt{\frac{2}{\pi}}\left\{\left(\frac{a\cos a\lambda-b\cos b\lambda}{\lambda}\right)+\left(\frac{\sin b\lambda-\sin a\lambda}{\lambda^2}\right)\right\}$]

3. *Find Fourier cosine transforms of the following*

i) $f(x)=\begin{cases}\pi & 0\le x\le\pi \\ 0 & x>\pi\end{cases}$

[Ans. : $F_c(\lambda)=\sqrt{\frac{2}{\pi}}\left(\frac{\pi\sin\lambda}{\lambda}\right)$]

ii) $f(x)=\begin{cases}x^2 & 0<x<1 \\ 0 & x>1\end{cases}$

[Ans. : $F_c(\lambda)=\sqrt{\frac{2}{\pi}}\left\{\frac{2\cos\lambda}{\lambda^2}+\frac{\sin\lambda}{\lambda}-\frac{2\sin\lambda}{\lambda^3}\right\}$]

iii) $f(x)=\begin{cases}x & 0\le x\le a \\ 0 & x>a\end{cases}$

[Ans. : $F_c(\lambda)=\sqrt{\frac{2}{\pi}}\left\{\frac{\sin a\lambda}{\lambda}+\frac{\cos a\lambda-1}{\lambda^2}\right\}$]

iv) $f(x)=\begin{cases}x & 0<x<1/2 \\ 1-x & 1/2<x<1 \\ 0 & x>1\end{cases}$

[Ans. : $F_c(\lambda)=\frac{2\cos\frac{\lambda}{2}-\cos\lambda-1}{\lambda^2}$]

v) $f(x)=2e^{-5x}+5e^{-2x}$

[Ans. : $F_c(\lambda)=\sqrt{\frac{2}{\pi}}\left\{\frac{10}{\lambda^2+25}+\frac{10}{\lambda^2+4}\right\}$]

vi) $f(x)=e^{-2x}+4e^{-3x}$

[Ans. : $F_c(\lambda)=2\sqrt{\frac{2}{\pi}}\left[\frac{1}{\lambda^2+4}+\frac{6}{\lambda^2+9}\right]$]

4. *Find Fourier transforms of the following*

i) $f(x)=\begin{cases}x^2 & |x|\le a \\ 0 & |x|>a\end{cases}$

[Ans. : $F(\lambda)=\frac{2}{\sqrt{2\pi}}\left\{\frac{a^2\sin a\lambda}{\lambda}+\frac{2a\cos a\lambda}{\lambda^2}-\frac{2\sin a\lambda}{\lambda^3}\right\}$]

ii) $f(x)=\begin{cases}\cos x+\sin x & |x|<\pi \\ 0 & |x|>\pi\end{cases}$

[Ans. : $F(\lambda)=\sqrt{\frac{2}{\pi}}\left(\frac{\lambda+i}{1-\lambda^2}\right)\sin\lambda\pi$]

iii) $f(x)=\begin{cases}\frac{\pi}{2}\cos x & |x|<\pi \\ 0 & |x|>\pi\end{cases}$

[Ans. : $F(\lambda)=\sqrt{\frac{2}{\pi}}\left(\frac{\pi\cdot\lambda\cdot\sin\lambda\pi}{1-\lambda^2}\right)$]

iv) $f(x)=\begin{cases}x & |x|\le a \\ 0 & |x|>a\end{cases}$

[Ans. : $F(\lambda)=\sqrt{\frac{2}{\pi}}\left[\frac{\sin a\lambda}{\lambda^2}-\frac{a\cos a\lambda}{\lambda}\right]$]

5. *Using Fourier integral representation show that*

i) $\frac{2}{\pi}\int_0^\infty \frac{k\cos\lambda x}{\lambda^2+k^2}d\lambda = e^{-kx}, \quad x>0$

ii) $\frac{2}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{\lambda^2+k^2}d\lambda = e^{-kx}; \quad x>0$

iii) $\frac{2}{\pi}\int_0^\infty \frac{(\lambda^2+2)\cos\lambda x}{\lambda^4+4}d\lambda = e^{-x}\cos x, \quad x>0$

iv) $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda = \begin{cases} \frac{\pi}{2} & 0\le x\le 1 \\ \frac{\pi}{4} & x=1 \\ 0 & x>1 \end{cases}$

v) $\frac{6}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+1)(\lambda^2+4)}d\lambda = e^{-x}-e^{-2x}$

vi) $\frac{12}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+4)(\lambda^2+16)}d\lambda = e^{-3x}\sinh x$

vii) $\frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\pi\sin\lambda x}{1-\lambda^2}d\lambda = \begin{cases} \sin x & |x|<\pi \\ 0 & |x|>\pi \end{cases}$

viii) $\int_0^\infty \left(\frac{1-\cos\lambda\pi}{\lambda}\right)\sin\lambda x\, d\lambda = \begin{cases} \frac{\pi}{2} & 0<x<\pi \\ 0 & x>\pi \end{cases}$

ix) $\frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos k\lambda}{\lambda}\right)\sin\lambda x\, d\lambda = \begin{cases} 1 & 0<x<k \\ \frac{1}{2} & x=k \\ 0 & x>k \end{cases}$

6. *Find the Fourier cosine integral representation for the following functions*

i) $f(x) = \begin{cases} x & 0\le x\le a \\ 0 & x>a \end{cases}$

[Ans. : $f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{\lambda a\sin a\lambda+\cos a\lambda-1}{\lambda^2}\right)\cos\lambda x\, d\lambda$**]**

ii) $f(x) = \begin{cases} x^2 & 0\le x\le 1 \\ 0 & x>1 \end{cases}$

[Ans. : $f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{(\lambda^2-2)\sin\lambda+2\lambda\cos\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda$**]**

iii) $f(x) = e^{-x}+e^{-2x}, \quad x\ge 0$

[Ans. : $f(x) = \frac{6}{\pi}\int_0^\infty \left(\frac{\lambda^2+2}{\lambda^4+5\lambda^2+4}\right)\cos\lambda x\, d\lambda$**]**

iv) $f(x) = \frac{1}{1+x^2}, \quad x\ge 0$

[Ans. : $f(x) = \int_0^\infty e^{-\lambda}\cos\lambda x\, d\lambda$**]**

7. *Find Fourier transform of* $f(x) = xe^{-x}, \ 0\le x\le\infty$

Hint : As $0\le x\le\infty$ *we can find F.C.T. or F.S.T. Let us find F.S.T. consider F.C.T of* e^{-x} *find it and then use D.U.I.S.*

[Ans. : $\sqrt{\frac{2}{\pi}}\left[\frac{2\lambda}{(1+\lambda^2)^2}\right]$**]**

8. *Find Fourier sine transform of* $f(x) = \begin{cases} 1 & 0\le x\le 1 \\ 0 & x>1 \end{cases}$ *hence evaluate* $\int_0^\infty \frac{\sin^3 x}{x^3}dx$

9. *Find Fourier cosine transform of* $f(x) = \begin{cases} \cos x & 0<x<a \\ 0 & x>a \end{cases}$ *and write integral Fourier representation for* $f(x)$ $f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin(\lambda+1)a}{\lambda+1}+\frac{\sin(\lambda-1)a}{\lambda-1}\right]\cos\lambda x\, d\lambda.$

10. *If* $f(x) = \begin{cases} \sin x & 0<x<\pi \\ 0 & x>\pi, x<0 \end{cases}$ *then prove that* $f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\cos\lambda x+\cos\lambda(\pi-x)}{1-\lambda^2}\right]d\lambda.$ *Hence deduce* $\int_0^\infty \frac{\cos\lambda\frac{\pi}{2}}{1-\lambda^2}d\lambda = \frac{\pi}{2}.$

11. *If* $f(x) = \begin{cases} \cos x & 0<x<\pi \\ 0 & x<0, \ x>\pi \end{cases}$ *then prove that* $f(x) = \frac{1}{\pi}\int_0^\infty \frac{\lambda[\sin\lambda x+\sin\lambda(\pi-x)]}{1-\lambda^2}d\lambda.$ *Hence deduce* $\int_0^\infty \frac{\lambda\sin\lambda\pi}{1-\lambda^2}d\lambda = \frac{-\pi}{2}.$

12. *Find Fourier cosine transform of* $f(x) = e^{-x}$ *and hence deduce* $\int_0^\infty \frac{\cos 2x}{1+x^2}dx = \frac{\pi}{2e^2}.$

13. *Find complex Fourier integral of* e^{-x^2}.

[Ans. : $f(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\lambda^2/4}\cos\lambda x\, d\lambda$**]**

14. Solve the following integral equations

i) $\int_0^\infty f(x)\cos\lambda x\,dx = \frac{\pi}{2}e^{-\lambda^2/2}\quad \lambda > 0$

$\left[\textbf{Ans. : } f(x) = \sqrt{\frac{\pi}{2}}\,e^{-x^2/2}\right]$

ii) $\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$

$\left[\textbf{Ans. : } f(x) = \frac{2}{\pi x^2}(x - \sin x)\right]$

iii) $\int_0^\infty f(x)\sin\lambda x\,dx = \frac{\lambda}{\lambda^2+k^2}$

$\left[\textbf{Ans. : } f(x) = e^{-kx}\right]$

iv) $\int_0^\infty f(x)\cos\lambda x\,dx = \frac{k}{\lambda^2+k^2}$

$\left[\textbf{Ans. : } f(x) = e^{-kx}\right]$

15. Use inverse Fourier Sine Transform to find f(x) if $F_s(\lambda) = \frac{\lambda}{a^2+\lambda^2}$

$\left[\textbf{Ans. : } e^{-ax}\right]$

16. Using inverse Fourier Cosine Transform find f(x) if

$F_c(\lambda) = \begin{cases} \sqrt{\frac{2}{\pi}}\left(a - \frac{\lambda}{2}\right) & \lambda \le 2a \\ 0 & \lambda > 2a \end{cases}$

$\left[\textbf{Ans. : } \frac{2\sin^2 ax}{\pi x^2}\right]$

3.11 Properties and Theorems of Fourier Transforms

If $F(\lambda) = F[f(x)]$ and $G(\lambda) = F[g(x)]$ are the complex Fourier transforms of f(x) and g(x) then :

1) Linearity Property

$$F[k_1 f(x) + k_2 g(x)] = k_1\,F[f(x)] + k_2 F[g(x)] = k_1\,F(\lambda) + k_2 G(\lambda)$$

2) Change of Scale Property

$$F[f(ax)] = \frac{1}{a}F\left(\frac{\lambda}{a}\right),\ a \ne 0$$

3) Shifting Property

$$F[f(x-a)] = e^{-i\lambda a}\,F(\lambda)$$

4) Modulation Theorem

$$F[f(x)\cos ax] = \frac{1}{2}[F(\lambda+a) + F(\lambda-a)]$$

Also,

a) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(\lambda+a) + F_c(\lambda-a)]$

b) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(\lambda+a) - F_s(\lambda-a)]$

c) $F_c[f(x)\cos ax] = \frac{1}{2}[F_s(\lambda+a) + F_s(\lambda-a)]$

d) $F_c[f(x)\sin ax] = \frac{1}{2}[F_c(\lambda-a) - F_c(\lambda+a)]$

This theorem is of great importance in the theory of communication.

5) Convolution Theorem

The convolution of the functions f (x) and g (x) over the interval $-\infty < x < \infty$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du \text{ and then}$$

$$F[f(x) * g(x)] = F[f(x)]\,F[g(x)] = F(\lambda)\,G(\lambda)$$

3.12 Finite Fourier Transforms

For a function f(x) is defined in the finite interval $(0 < x < L)$ and satisfying the Dirichlet's conditions, we can obtain its half range cosine or sine series. Using this representation, we define the Finite Cosine or Sine Transforms of f(x) as follows :

1) Finite Fourier Cosine Transform

For a function f (x) defined in $0 < x < L$, then half range cosine series of f (x) is given by :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

(where n is an integer) ... (1)

where $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$

and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

Now, the **Finite Fourier Cosine Transform** of f(x) is defined as :

$$\boxed{F_c[n] \text{ or } c(n) = \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx}$$ where n is an integer

So that $a_0 = \frac{2}{L}\int_0^L f(x)(1)\,dx = \frac{2}{L}F_c(0)$

and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}F_c[f(n)]$

and hence, using (1), the **Inverse Finite Fourier Cosine Transform** of $F_c[f(n)]$ is given by :

$$\boxed{f(x) = \frac{1}{L}F_c(0) + \frac{2}{L}\sum_{n=1}^{\infty} F_c[n]\cos\frac{n\pi x}{L}}$$

2) Finite Fourier Sine Transform

For a function f(x) is defined in $0 < x < L$, the half range sine series of f (x) is given by :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{... (2)}$$

(where n in an integer)

where, $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Now, the **Finite Fourier Sine Transform** of f(x) is defined as

$$\boxed{F_s[n] \text{ or } S(n) = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$$ where n is an integer

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}F_s[n]$$

and hence, using (2), the **Inverse Finite Fourier Sine Transform** of $F_s[n]$ is given by

$$\boxed{f(x) = \frac{2}{L}\sum_{n=1}^{\infty} F_s[n]\sin\frac{n\pi x}{L}}$$

3.13 Illustrations

Example 3.41 : *Find Finite Fourier sine transform of*

$$f(x) = \begin{cases} kx & 0 \le x \le \pi/2 \\ k(\pi - x) & \pi/2 \le x \le \pi \end{cases}$$

Solution : Here the interval (0, π) is finite. Hence, we get the Finite Fourier Transform.

Step 1 : By the definition the formula for Finite Fourier sine transform in (0, L) is

$$F_s[n] = \int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Put L = π $$= \int_0^{\pi} f(x)\sin n\pi x\,dx$$

Step 2 : Split the integral and substitute f(x).

$$= \int_0^{\pi/2} kx\sin nx\,dx + \int_{\pi/2}^{\pi} k(\pi - x)\sin nx\,dx$$

Step 3 : Integrate by parts and substitute limits of x

$$= k\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + k\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= k\left[\left(-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right) - (0+0)\right] + \left[(0-0) - \left(-\frac{\pi}{2n}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= k\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{1}{n^2}\sin\frac{n\pi}{2}+\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$\therefore \quad F_s[n] = k\frac{2}{n^2}\sin\frac{n\pi}{2}$$

Example 3.42 : *Find Fourier cosine transform of* $f(x) = lx - x^2$ *in* $0 \le x \le l$.

Solution : Step 1 : Here, the intervals of x are $[0, l]$ is finite. Hence, we get the Finite Fourier Transforms. By definition the Finite Fourier Cosine Transform of

$$F_c(n) = \int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

Step 2 : Substitute $f\underset{l}{(x)} = lx - x^2$

$$F_c(n) = \int_0^l \left(lx - x^2\right)\cos\left(\frac{n\pi x}{l}\right) dx$$

Step 3 : Integrate and substitute the limits of x.

$$F_c[n] = \int_0^L f(x)\cos\frac{n\pi x}{L}dx = \int_0^l \left(lx - x^2\right)\cos\frac{n\pi x}{l}dx \quad \text{... (1) using f(x)}$$

$$= \left[\left(lx-x^2\right)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)-(l-2x)\left(\frac{-l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)+(-2)\left(\frac{-l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

... integrating by parts (short cut)

$$= \left[\left(0-\frac{l^3}{n^2\pi^2}\cos n\pi+0\right)-\left(0+\frac{l^3}{n^2\pi^2}(1)+0\right)\right] \quad \text{... } \because \sin n\pi = 0$$

$$\therefore \quad F_c[n] = -\frac{l^3}{n^2\pi^2}(\cos n\pi + 1)$$

Step 4 : To find $F_c(0)$, $F_c(0) = \int_0^L f(x)\,dx$

$$\therefore \quad F_c(0) = \int_0^l \left(lx-x^2\right)(1)dx = \left[\frac{lx}{2}-\frac{x^3}{3}\right]_0^l = \frac{l^3}{2}-\frac{l^3}{3}-0 = \frac{l^3}{6}$$

Example 3.43 : *Solve the integral equation :* $\int_0^\infty f(x)\sin\lambda x\,dx = 4e^{-6\lambda}$, $\lambda > 0$.

SPPU : Dec.-18, Marks 4

Solution : Given that $\int_0^\infty f(x)\sin\lambda x dx = 4e^{-\sigma\lambda}$; $\lambda > 0$

As sin λ x is present in the integral and we use Fourier sine transform.

$$F_s(\lambda) = 4e^{-\sigma\lambda}$$

To find f(x), we use the inverse F.S.T.

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty F_s(\lambda)\sin\lambda x dx = \frac{2}{\pi}\int_0^\infty 4e^{-\sigma\lambda}\sin\lambda x d\lambda = \frac{8}{11}\left[\frac{e^{-\sigma\lambda}}{36+x^2}(-6\sin\lambda x - x\cos\lambda x)\right]_0^\infty$$

$$= \frac{8}{\pi}\left[0-\frac{1}{36+x^2}(-x)\right] = \frac{8}{\pi}\frac{x}{36+x^2}$$

3.14 University Questions

Dec. - 2016

Q.1 *Find the Fourier cosine transform of :* $f(x) = e^{-x} + e^{-2x},\ x > 0.$ **[4]**

May - 2017

Q.2 *Using Fourier integral representation show that :*

$$\int_0^{\infty} \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4} d\lambda = \frac{\pi}{2} e^{-x} \cos x,\ x > 0$$ **[4]**

Dec. - 2017

Q.3 *Find fourier sine transform of :* $\frac{e^{-ax}}{x}$ *where* $x > 0.$ **[4]**

May - 2018

Q.4 *Using suitable Fourier transform, solve the following equation :* **[4]**

$$\int_0^{\infty} f(x) \sin \lambda x \, dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda \ge 1 \end{cases}$$

Dec. - 2018

Q.5 *Solve the integral equation :* **[4]**

$$\int_0^{\infty} f(x) \sin \lambda x \, dx = 4e^{-6\lambda},\ \lambda > 0.$$

May - 2019

Q.6 *Solve the integral equation :*

$$\int_0^{\infty} f(x) \cos \lambda x \, dx = e^{-2\lambda},\ \lambda > 0$$ **[4]**

Q.7 *Solve the differential equation by Laplace transform method :* $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = te^t.$

where $y(0) = 0,\ y(0) = 3.$

Dec. - 2019

Q.8 *Solve the integral equation :* **[4]**

$$\int_0^{\infty} f(x) \sin \lambda x \, dx = \begin{cases} 4-\lambda, & 0 \le \lambda \le 4 \\ 0, & \lambda > 0 \end{cases}$$

❑❑❑

Notes

UNIT - II

4 Z-Transform

4.1 Introduction

The Z-transform is useful in solving difference equations which represent a discrete system. In the area of signal processing in electrical and electronics engineering, we encounter sequences of discrete signals. If we represent a sequence in signal processing as {f(n)}. Therefore the elements of sequence {f(n)} are functions of discrete non-negative integer valued arguments. The operation of such discrete systems is governed by a difference equation which are solved by using Z-transforms.

Thus Z-transform is the discrete analogue of Laplace transform therefore for almost every operational rule and application of Laplace transform there corresponds an operational rule and application of Z-transform.

The basic idea now known as the Z-transform was known to Laplace and it was reintroduced in 1947 by W. Hurewicz and others as a way to treat sampled data control systems used with radar. In 1952, it was later named "The Z-transform" by Ragazzini and Zadeh in the sample data control group.

4.2 Sequences

Definition : An ordered set of real or complex numbers is called as a sequence. It is denoted by {f(k)} or $\{f_k\}$ or (f_k) where k is an interger.

A sequence {f(k)} is represented in two ways.

I) Arrow representation : In this system, we use vertical arrow to denote the value corresponding to the position k = 0. i.e. $f(k) = \{..... f(-3), f(-2), f(-1), \underset{\uparrow}{f(0)}, f(1), f(2) ......\}$

e.g. $$f(k) = \{1, 5, 3, \underset{\uparrow}{2}, 8, 9, 11\}$$

where vertical arrow gives f(0) = 3.

$$\therefore \quad f(-1) = 5, f(-2) = 1, f(1) = 2, f(2)$$

Note

1. When the vertical arrow is not given then the starting term of a sequence denotes f(0).
2. A sequence may have finite or infinite number of terms. Every finite sequence can be written in the form of an infinite sequence by considering the terms on either sides as zeros. e.g. A sequence f(k) = {5, 8, 1, 3, 2} can be written as {f(k)} = {........ 0, 0, 5, 8, 1, 3, 2, 0, 0, ...} which is an infinite sequence.
3. Sequences $\{5, 8, \underset{\uparrow}{1}, 3, 2\}$ and $\{5, \underset{\uparrow}{8}, 1, 3, 2\}$ are not same as in first sequence f(0) = 1 and in second sequence f(0) = 8.

II) General representation :

In this system the general term of a sequence is given, by using this we can find the remaining required terms.

e.g. $$f(k) = 2^k, \forall\ k \in \mathbf{Z}$$

$$\therefore \quad \{f(k)\} = \{... 2^{-2}, 2^{-1}, 2^0 = 1, 2, 2^2, ...\}$$

Causal sequence

A sequence in which f(k) = 0 ∀ k < 0 is called as causal sequence.

e.g. $\{... 0, 0, \underset{\uparrow}{f(0)}, f(1), f(2), ...\} = \{f(0), f(1), f(2), ...\}$

Important Results :

1) $a + ar + ar^2 + ... = \dfrac{a}{1-r}$ if $|r| < 1$

Here a = First term of series

$$r = \text{Common ratio} = \frac{\text{Second term}}{\text{First term}}$$

2) $$\frac{1}{1-x} = 1 + x + x^2 + \ldots, \ |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots, \quad \text{if } |x| < 1$$

$$(1-x)^{-2} = \frac{1}{(1-x)^2}$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots \ \text{if } |x| < 1$$

3) If $z = x + iy$, then $|z| = +\sqrt{x^2 + y^2}$

Hence $|z| = a \Rightarrow x^2 + y^2 = a^2$ represents the points on the circle.

$|z| > a$ represents the points outside the circle $x^2 + y^2 = a^2$.

And $|z| < a$ represents the points inside the circle $x^2 + y^2 = a^2$

4) Unit Step Function or Sequence

$$u(k) = \begin{cases} 0 & \text{if } k < 0 \\ 1 & \text{if } k \geq 0 \end{cases}$$

And $$u(-k) = \begin{cases} 0 & \text{if } k > 0 \\ 1 & \text{if } k \leq 0 \end{cases}$$

5) Unit Impulse Function of Sequence.

$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

4.3 Z-transform

Definition :

1) The Z-transform of sequence $\{f(k)\}$ is defined as

$$Z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = F(z)$$

2) For causal sequence $\{f(k)\}$, where $0 \leq k < \infty$. The Z-transform is defined by

$$Z[f(k)] = \sum_{k=0}^{\infty} f(k)\, z^{-k} = F(z)$$

where $z = x + iy$ is a complex number,

z is the operator of Z-transform and F(z) is the Z-transform of sequence $\{f(k)\}$.

3) For a finite sequence $\{f(k)\}$ where $m \leq k \leq n$.

It's Z-transform is

$$Z[f(k)] = \sum_{k=m}^{n} f(k)\, z^{-k}$$

Note

1) The Z-transform of a sequence $\{f(k)\}$ exists if the series $\sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$ is convergent. i.e. The series tends to a finite value for some values of z. These values of z for which the series is convergent lie within a region known as Region of convergence (ROC) in the z-plane.

2) The Z-transform of a sequence is not unique.

3) For causal sequence, there is no need to specify the ROC.

4) In Z-transform ROC and the region of absolute convergence are same.

4.4 Z-transforms of Some Standard Sequences

1) Discrete unit step function :

$$u(k) = \begin{cases} 0 & \text{if } k < 0 \\ 1 & \text{if } k \geq 0 \end{cases}$$

$$Z[u(k)] = \sum_{k=-\infty}^{\infty} u(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} u(k)\, z^{-k} + \sum_{k=0}^{\infty} u(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0\, z^{-k} + \sum_{k=0}^{\infty} 1 z^{-k}$$

$$= 0 + \left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)$$

$$= \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

$$\therefore \quad \boxed{Z[u(k)] = \frac{z}{z-1}}$$

which is convergent if $\left|\frac{1}{z}\right| < 1$ i.e. $|z| > 1$

Hence $Z[u(k)] = \frac{z}{z-1}$ if $|z| > 1$

2) Unit impulse function :

$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$\therefore \quad Z[\delta(k)] = \sum_{k=-\infty}^{\infty} \delta(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} (0+...+0+1+0+...+0)z^{-k}$$

$$Z[\delta(k)] = 1 \cdot z^0 = 1$$

$$\therefore \quad \boxed{Z[\delta(k)] = 1}$$

3) $\{f(k)\} = \{a^k\}, \ k \geq 0$

$$Z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} a^k\, z^{-k}$$

$$= \sum_{k=0}^{\infty} (a\, z^{-1})^k$$

$$= 1 + a\,z^{-1} + (a\,z^{-1})^2 + ...$$

$$= \frac{1}{1-az^{-1}} \quad \text{if } \left|az^{-1}\right| < 1$$

$$= \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}, \text{ if } \left|\frac{a}{z}\right| < 1 \text{ i.e. } |z| > |a|$$

$$\therefore \quad \boxed{Z\,[a^k] = \frac{z}{z-a}; \ |z| > |a| \text{ and } k \geq 0}$$

Hence the region of convergence is the exterior of circle $x^2 + y^2 = a^2$

4) $\{f(k)\} = \{a^k\} \ : \ k < 0$

By definition

$$Z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^k\, z^{-k} + 0$$

Put $k = -r \ \therefore -k = r$

and $-\infty < k \leq -1 \Rightarrow \infty > -k \geq 1$ i.e. $1 \leq r < \infty$

$$\therefore \quad Z[f(k)] = \sum_{r=1}^{\infty} a^{-r} z^r$$

$$= \sum_{r=1}^{\infty} \left(\frac{z}{a}\right)^r$$

$$= \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + ...$$

$$= \frac{z/a}{1-z/a} \quad \text{provided } \left|\frac{z}{a}\right| < 1$$

$$= \frac{z}{a-z}, \quad |z| < |a|$$

Thus $\boxed{Z[f(k)] = z\,[a^k] = \frac{z}{a-z}, \ k < 0 \text{ and for } |z| < |a|}$

and hence region of convergence is the interior of the circle $x^2 + y^2 = a^2$.

5) $\{f(k)\} = \{a^{|k|}\}, \ \forall \ k \in \mathbb{Z}$

By definition,

$$Z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k)\, z^{-k} + \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{|k|}\, z^{-k} + \sum_{k=0}^{\infty} a^{|k|} z^{-k} \qquad ... (4.1)$$

we have $\quad |k| = k$ if $k \geq 0$

$= -k$ if $k < 0$

Thus equation (4.1) becomes,

$$Z[f(k)] = \sum_{k=-\infty}^{-1} a^{-k}\, z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= \sum_{k=-\infty}^{-1} (az)^{-k} + \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$

$$= [az + (az)^2 + (az)^3 + ...]$$

$$+ \left[1 + \frac{a}{z} - \left(\frac{a}{z}\right)^2 + ...\right]$$

$$= \frac{az}{1-az} + \frac{1}{1-\frac{a}{z}}$$

provided $|az| < 1$ and $\left|\frac{a}{z}\right| < 1$

$$= \frac{az}{1-az} + \frac{z}{z-a} \ : \ |z| < \frac{1}{|a|}$$

and $|z| > |a|$

Thus $\boxed{Z\,[a^{|k|}] = \frac{az}{1-az} + \frac{z}{z-a}, \ \forall \ k}$ and

$$|a| < |z| < \frac{1}{|a|}$$

Hence ROC is the annulus between the circles

$$x^2 + y^2 = a^2 \text{ and } x^2 + y^2 = \frac{1}{a^2}$$

6) $\{f(k)\} = \{e^{ak}\}$ for $k \geq 0$

$$z[e^{ak}] = \sum_{k=-\infty}^{\infty} e^{ak} z^{-k}$$

$$= \sum_{k=0}^{\infty} e^{+ak} z^{-k} = \sum_{k=0}^{\infty} \left(\frac{e^a}{z}\right)^k$$

$$= \frac{1}{1-e^a/z} \text{ for } \left|\frac{e^a}{z}\right| < 1$$

$$= \frac{z}{z-e^a} \text{ for } |z| > e^a$$

Thus $\boxed{Z[e^{ak}] = \frac{z}{z-e^a} \text{ for } |z| > e^a}$

7) $\{f(k)\} = \sin ak, \quad k \geq 0$

$$Z[\sin ak] = z\left[\frac{e^{iak} - e^{-iak}}{2i}\right]$$

$$= \frac{1}{2i}\{z[e^{iak}] - z[e^{-iak}]\}$$

$$= \frac{1}{2i}\left\{\frac{z}{z-e^{ia}} - \frac{z}{z-e^{-ia}}\right\}$$

for $|z| > \left|e^{\pm ia}\right| = 1$

$$= \frac{1}{2i}\left\{\frac{z^2 - ze^{-ia} - z^2 + ze^{ia}}{z^2 - (e^{ia} + e^{-ia})z + e^{ia}e^{-ia}}\right\}$$

$$= \frac{z\left(\frac{e^{ia} - e^{-ia}}{2i}\right)}{z^2 - 2z\cos a + 1}$$

$$\boxed{Z[\sin ak] = \frac{z \sin a}{z^2 - 2z\cos a + 1} \text{ for } |z| > 1}$$

8) $\{f(k)\} = \cos ak, \quad k \geq 0$

$$Z[\cos ak] = z\left[\frac{e^{iak} + e^{-iak}}{2}\right]$$

$$= \frac{1}{2}\{z[e^{iak}] + z[e^{-iak}]\}$$

$$= \frac{1}{2}\left\{\frac{z}{z-e^{ia}} + \frac{z}{z-e^{-ia}}\right\} \text{ for } |z| > \left|e^{\pm ia}\right|$$

$$= \frac{z}{2}\left\{\frac{z - e^{-ia} + z - e^{ia}}{z^2 - (e^{ia} + e^{-ia})z + e^{ia}e^{-ia}}\right\}$$

for $|z| > 1$

$$= \frac{z\left[z - \left(\frac{e^{ia} + e^{-ia}}{2}\right)\right]}{z^2 - 2z\cos a + 1}$$

$$\boxed{Z[\cos ak] = \frac{z[z - \cos a]}{z^2 - 2z\cos a + 1} \text{ for } |z| > 1}$$

9) $\{f(k)\} = \sinh ak, \quad k \geq 0$

$$Z[\sinh ak] = z\left[\frac{1}{2}(e^{ak} - e^{-ak})\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^a} - \frac{z}{z-e^{-a}}\right]$$

for $|z| > |e^a|$ and $|z| > |e^{-a}|$

$$= \frac{z}{2}\left[\frac{z - e^a - z + e^a}{z^2 - (e^a + e^{-a})z + e^a e^{-a}}\right]$$

for $|z| >$ maximum of $|e^a|$ and $|e^{-a}|$

$$= \frac{z\left(\frac{e^a - e^{-a}}{2}\right)}{z^2 - 2z\cosh a + 1}$$

$$\boxed{Z[\sinh ak] = \frac{z \sinh a}{z^2 - 2z\cosh a + 1}}$$

for $|z| > \max(|e^a|, |e^{-a}|)$

10) $\{f(k)\} = \cosh ak, \quad k \geq 0$

$$Z[\cosh ak] = z\left[\frac{e^{ak} + e^{-ak}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^a} + \frac{z}{z-e^{-a}}\right]$$

for $|z| > |e^a|$ and $|z| > |e^{-a}|$

$$= \frac{z}{2}\left[\frac{z - e^{-a} + z - e^a}{z^2 - (e^a + e^{-a})z + 1}\right]$$

$$= \frac{z\left[z - \left(\frac{e^a + e^{-a}}{2}\right)\right]}{z^2 - 2z\cosh a + 1}$$

$$\boxed{Z[\cosh ak] = \frac{z[z - \cosh a]}{z^2 - 2z\cosh a + 1}}$$

for $|z| > \max$ of $|e^a|$ and $|e^{-a}|$

11) a) $\{f(k)\} = {}^nC_k$ for $0 \leq k \leq n$

$$Z[{}^nC_k] = \sum_{k=-\infty}^{\infty} {}^nC_k \, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 + \sum_{k=0}^{n} {}^nC_k\, z^{-k} + \sum_{k=n+1}^{\infty} 0$$

$$= \sum_{k=0}^{n} {}^nC_k\, z^{-k}$$

$$= \sum_{k=0}^{n} {}^nC_k\, (z^{-1})^k$$

$$= (1+z^{-1})^n \qquad \text{for } |z^{-1}| < 1$$

$$= \left(1+\frac{1}{z}\right)^n \qquad \text{for } \left|\frac{1}{z}\right| < 1$$

$$\boxed{Z[{}^nC_k] = \left(\frac{z+1}{z}\right)^n ; \qquad \text{for } |z| > 1}$$

b) $\{f(k)\} = {}^kC_n$ for $k \geq n \geq 0$

$$Z[{}^kC_n] = \sum_{k=-\infty}^{\infty} {}^kC_n\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 + \sum_{k=0}^{n-1} 0 + \sum_{k=n}^{\infty} {}^kC_n\, z^{-k}$$

$$= \sum_{r=0}^{\infty} {}^{(n+r)}C_n\, z^{-(n+r)} \quad \text{put } k-n = r$$

$$= \sum_{r=0}^{\infty} {}^{n+r}C_r\, z^{-n}\, z^{-r}$$

$$(\because {}^nC_r = {}^nC_{n-r})$$

$$= z^{-n} \sum_{r=0}^{\infty} {}^{n+r}C_r\, z^{-r}$$

$$= z^{-n}(1-z^{-1})^{-(n+1)}$$

$$= z^{-n}\left(\frac{z}{z-1}\right)^{n+1} \qquad \text{for } k \geq n \geq 0$$

and $|z| < 1$

$$\boxed{Z[{}^kC_n] = z^{-n}\left(\frac{z}{z-1}\right)^{n+1} \qquad \text{for } |z| > 1, \quad k \geq n \geq 0}$$

c) $\{f(k)\} = {}^{k+n}C_n$

$$Z[{}^{k+n}C_n] = \sum_{k=0}^{\infty} {}^{k+n}C_n\, z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{(k+n)(k+n-1)\ldots(k+1)}{n!} z^{-k}$$

$$= 1 + (n+1)\left(\frac{1}{z}\right) + \frac{(n+1)(n+2)}{2!}\left(\frac{1}{z}\right)^2 + \ldots$$

$$= \left[1-\frac{1}{z}\right]^{-(n+1)} \qquad \text{(by above example)}$$

for $|z| > 1$

$$\boxed{Z[{}^{k+n}C_n] = \left[\frac{z}{z-1}\right]^{n+1}} \qquad \text{for } |z| > 1$$

Examples on Evaluation of Z-transform

Example 4.1 : *Find the Z-transform and its ROC i)* $3^k,\ k \geq 0$, *ii)* $\left(\frac{1}{4}\right)^k,\ k < 0$

Solution : i) We have,

$$f(k) = 3^k,\ k \geq 0$$

$\therefore$ By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} 3^k\, z^{-k}$$

$$= \sum_{0}^{\infty} (3z^{-1})^k \qquad \ldots \because k \geq 0$$

$$= 1 + (3z^{-1}) + (3z^{-1})^2 + (3z^{-1})^3 + \ldots$$

which is an infinite G.P.

$$= \frac{1}{1-3z^{-1}},\ \text{if } |3z^{-1}| < 1$$

$$\ldots \because S_\infty = \frac{a}{1-r},\ |r| < 1$$

$$= \frac{1}{1-\frac{3}{z}} = \frac{z}{z-3},\ \left|\frac{3}{z}\right| < 1$$

i.e. $|z| > |3|$

ii) We have, $f(k) = \left(\frac{1}{4}\right)^k,\ k < 0$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{4}\right)^k z^{-k}$$

$$\ldots \because k < 0 \text{ i.e. } -\infty \leq k \leq -1$$

Put $k = -r$ i.e. $-k = r$, $\therefore -\infty \le k \le -1 \Rightarrow -\infty \le -r \le -1$ i.e. $1 \le r \le \infty$

$$\therefore \quad Z\{f(k)\} = \sum_{r=\infty}^{1} \left(\frac{1}{4}\right)^{-r} z^r$$

$$= \sum_{1}^{\infty} \left[\left(\frac{1}{4}\right)^{-1} z\right]^r$$

$$= \sum_{1}^{\infty} (4z)^r$$

$$= 4z + (4z)^2 + (4z)^3 + \ldots$$

infinite G.P.

$$= \frac{4z}{1-4z}, |4z| < 1 \quad \text{... sum of G.P.}$$

$$= \frac{4z}{1-4z}, |z| < \frac{1}{4}$$

Example 4.2 : *Find the Z-transform of the following sequence.*

$$f(k) = \begin{cases} 2^k, & k < 0 \\ \left(\frac{1}{3}\right)^k, & k \ge 0 \end{cases}$$

SPPU : Dec.-17

Solution : We have,

$$f(k) = \begin{cases} 2^k, & k < 0 \\ \left(\frac{1}{3}\right)^k, & k \ge 0 \end{cases}$$

$\therefore$ By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 2^k z^{-k} + \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k z^{-k}$$

... using f (k)

$$= \sum_{r=\infty}^{1} 2^{-r} z^r + \sum_{k=0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^k$$

... Putting $k = -r$ in first summation

$$= \sum_{r=1}^{\infty} \left(\frac{z}{2}\right)^r + \sum_{k=0}^{\infty} \left(\frac{1}{3z}\right)^k$$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\frac{1}{3z}}, \left|\frac{z}{2}\right| < 1, \left|\frac{1}{3z}\right| < 1$$

Sum of infinite G.P.s $S_\infty = \frac{a}{1-r}, |r| < 1$

$$= \frac{2z}{2-z} + \frac{3z}{3z-1}, |z| < 2,$$

$$|z| > \frac{1}{3} \text{ i.e. } \frac{1}{3} < |z| < 2$$

Example 4.3 : *Find Z-transform of $f(k) = 3(2^k) + 4(-1)^k;\ k \ge 0$*

Solution : We have $f(k) = 3(2^k) + 4(-1)^k$

$$Z\{f(k)\} = Z\{3(2^k) + 4(-1)^k\};\ k \ge 0$$

$$= \frac{3z}{z-2} + \frac{4z}{z+1};\ |z| > 2,\ |z| > 1$$

$$= z\left\{\frac{3}{z-2} + \frac{4}{z+1}\right\};\ |z| > 2$$

$$= z\left\{\frac{3(z+1) + 4(z-2)}{(z+1)(z-2)}\right\};\ |z| > 2$$

$$Z\{f(k)\} = \frac{2(7z-5)}{(z+1)(z-2)};\ |z| > 2$$

$\therefore$ The region of convergence is exterior of the circle $x^2 + y^2 = 2^2$.

Example 4.4 : *Find the Z-transform and region of convergence.*

$$f(k) = \begin{cases} 2^k, & k \ge 0 \\ \left(\frac{1}{3}\right)^k, & k < 0 \end{cases}$$

Solution : We have,

$$f(k) = \begin{cases} 2^k, & k \ge 0 \\ \left(\frac{1}{3}\right)^k, & k < 0 \end{cases}$$

$\therefore$ By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left(\frac{1}{3}\right)^k z^{-k} + \sum_{k=0}^{\infty} 2^k z^{-k}$$

... using f(k)

$$= \sum_{r=\infty}^{1} \left(\frac{1}{3}\right)^{-r} z^r + \sum_{k=0}^{\infty} \left(2z^{-1}\right)^k$$

... putting $k = -r$ in first

$$= \sum_{r=1}^{\infty} (3z)^r + \sum_{k=0}^{\infty} \left(\frac{2}{z}\right)^k$$

$$= \frac{3z}{1-3z} + \frac{1}{1-\frac{2}{z}}, \quad |3z| < 1, \left|\frac{2}{z}\right| < 1$$

... sum of infinite G.P.

$$= \frac{3z}{1-3z} + \frac{z}{z-2}, \quad |z| < \frac{1}{3}, \quad |z| > 2$$

Now, for ROC : We see that $|z| < \frac{1}{3}$, $|z| > 2$.

Hence, there is no common ROC. Hence, we have,

$$Z\{f(k)\} = \frac{3z}{1-3z}, \quad k < 0,$$

for ROC : $|z| < \frac{1}{3}$ i.e. interior of the circle $x^2 + y^2 = \left(\frac{1}{3}\right)^2$

and $$Z\{f(k)\} = \frac{z}{z-2}, \quad k \geq 0$$

for ROC : $|z| > 2$ i.e. exterior of the circle $x^2 + y^2 = 2^2$

Example 4.5 : *Find the Z-transform of the following by showing the region of convergence*

$$f(k) = \begin{cases} -\left(-\frac{1}{3}\right)^k, & k < 0 \\ \left(-\frac{1}{4}\right)^k, & k \geq 0 \end{cases}$$

Solution : We have,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left[-\left(-\frac{1}{3}\right)^k\right] z^{-k}$$

$$+ \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k z^{-k} \quad \text{... using f (k)}$$

$$= -\sum_{k=-\infty}^{-1} \left(-\frac{1}{3}\right)^k z^{-k} + \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k z^{-k}$$

$$= -\frac{z}{\left(-\frac{1}{3} - z\right)} + \frac{z}{z - \left(-\frac{1}{4}\right)},$$

$$|z| < \left|-\frac{1}{3}\right|, |z| > \left|-\frac{1}{4}\right|$$

$$\because \quad Z\{a^k\} = \frac{z}{a-z}, k < 0, |z| < |a|$$

$$= \frac{z}{z-a}, k \geq 0, |z| > |a|$$

$$= +\frac{3z}{(1+3z)} + \frac{4z}{4z+1}, \frac{1}{4} < |z| < \frac{1}{3}$$

Alternately, we can use the sum of series actually, by putting $r = -k$ in first summation.

Example 4.6 : *Find* $Z\{f(k)\}$ *where*

$$f(k) = \begin{cases} -\left(-\frac{1}{4}\right)^k, & k < 0 \\ \left(-\frac{1}{5}\right)^k, & k \geq 0 \end{cases}$$

Solution : We have,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k) z^{-k} + \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} \left[-\left(-\frac{1}{4}\right)^k\right] z^{-k}$$

$$+ \sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k z^{-k}$$

$$\text{... } \because f(k) = -\left(-\frac{1}{4}\right)^k, \; k < 0$$

i.e. $-\infty \leq k \leq -1$

$$= \left(-\frac{1}{5}\right)^k, \; k \geq 0 \text{ i.e. } 0 \leq k \leq \infty$$

$$= -\sum_{k=-\infty}^{-1} \left(-\frac{1}{4}\right)^k z^{-k} + \sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k z^{-k}$$

$$= -\sum_{r=\infty}^{1} \left(-\frac{1}{4}\right)^{-r} z^r + \sum_{k=0}^{\infty} \left(-\frac{1}{5} z^{-1}\right)^k$$

... putting $k = -r$ in the first summation

$$= -\sum_{r=1}^{\infty} \left[\left(-\frac{1}{4}\right)^{-1} z\right]^r + \sum_{k=0}^{\infty} \left(-\frac{1}{5z}\right)^k$$

$$= -\sum_{r=1}^{\infty} (-4z)^r + \sum_{k=0}^{\infty} \left(-\frac{1}{5z}\right)^k$$

$$= -[(-4z) + (-4z)^2 + (-4z)^3 + \ldots]$$

$$+ \left[1 + \left(-\frac{1}{5z}\right) + \left(-\frac{1}{5z}\right)^2 + \ldots\right]$$

... infinite G.P.s

$$= -\left[\frac{-4z}{1-(-4z)}\right] + \left[\frac{1}{1 - \left(\frac{-1}{5z}\right)}\right], |-4z| < 1, \left|-\frac{1}{5z}\right| < 1$$

$$\text{... } S_\infty = \frac{a}{1-r}, |r| < 1$$

$$= \frac{4z}{1+4z} + \frac{1}{1+\frac{1}{5z}}, |z| < \frac{1}{|-4|}, |z| > \left|-\frac{1}{5}\right|$$

$$= \frac{4z}{1+4z} + \frac{5z}{5z+1}, |z| < \frac{1}{4}, |z| > \frac{1}{5}$$

i.e. ROC : $\frac{1}{5} < |z| < \frac{1}{4}$

Example 4.7 : *Find the Z-transform of the following by showing the region of convergence*

$$f(k) = 3^k, k < 0$$
$$= \left(\frac{1}{3}\right)^k, k = 0, 2, 4, 6, \ldots$$
$$= \left(\frac{1}{4}\right)^k, k = 1, 3, 5, 7, \ldots$$

Solution : We have,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 3^k z^{-k} + \sum_{\substack{k=0 \\ k\text{ - even}}}^{\infty} \left(\frac{1}{3}\right)^k z^{-k} + \sum_{\substack{k=0 \\ k\text{ - odd}}}^{\infty} \left(\frac{1}{4}\right)^k z^{-k}$$

$$\ldots \because f(k) = 3^k, k < 0$$
$$= \left(\frac{1}{3}\right)^k, k = 0, 2, 4, 6, \ldots$$
$$= \left(\frac{1}{4}\right)^k, k = 1, 3, 5, 7, \ldots$$

$$= \sum_{r=\infty}^{1} 3^{-r} z^{r} + \sum_{\substack{k=0 \\ k\text{ - even}}}^{\infty} \left(\frac{1}{3z}\right)^k + \sum_{\substack{k=0 \\ k\text{ - odd}}}^{\infty} \left(\frac{1}{4z}\right)^k$$

$$= \sum_{r=1}^{\infty} \left(\frac{z}{3}\right)^r + \left[1 + \left(\frac{1}{3z}\right)^2 + \left(\frac{1}{3z}\right)^4 + \ldots\right] + \left[\frac{1}{4z} + \left(\frac{1}{4z}\right)^3 + \left(\frac{1}{4z}\right)^5 + \ldots\right]$$

$$= \frac{\frac{z}{3}}{1-\frac{z}{3}} + \frac{1}{1-\left(\frac{1}{3z}\right)^2} + \frac{\frac{1}{4z}}{1-\left(\frac{1}{4z}\right)^2},$$

$$\left|\frac{z}{3}\right| < 1, \left|\left(\frac{1}{3z}\right)^2\right| < 1, \left|\left(\frac{1}{4z}\right)^2\right| < 1$$

$$\ldots \text{ sum of infinite G.P.s, } S_\infty = \frac{a}{1-r}, |r| < 1$$

$$= \frac{z}{3-z} + \frac{9z^2}{9z^2-1} + \frac{4z}{16z^2-1},$$

$|z| < 3$, $|z^2| > \frac{1}{9}$ i.e. $|z| > \frac{1}{3}$, $|z^2| > \frac{1}{16}$ i.e. $|z| > \frac{1}{4}$

i.e. ROC : $\frac{1}{3} < |z| < 3$

Example 4.8 : *Find the Z-transform of the following :*

$$f(k) = 3^k, k < 0$$
$$= \left(\frac{1}{3}\right)^k, k = 0, 2, 4, 6, \ldots$$
$$= \left(\frac{1}{2}\right)^k, k = 1, 3, 5, \ldots$$

Solution : We have,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} f(k) z^{-k} + \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 3^k z^{-k} + \sum_{\substack{k=0 \\ (k\text{ - even})}}^{\infty} f(k) z^{-k} + \sum_{\substack{k=0 \\ (k\text{ - odd})}}^{\infty} f(k) z^{-k}$$

$$\ldots \because f(k) = 3^k, k < 0$$

$$= \sum_{r=\infty}^{1} 3^{-r} z^{r} + \sum_{\substack{k=0 \\ (k\text{ - even})}}^{\infty} \left(\frac{1}{3}\right)^k z^{-k} + \sum_{\substack{k=0 \\ (k\text{ - odd})}}^{\infty} \left(\frac{1}{2}\right)^k z^{-k}$$

... putting $k = -r$ in first term and

$$\because f(k) = \left(\frac{1}{3}\right)^k, \text{ even } k$$
$$= \left(\frac{1}{2}\right)^k, \text{ odd } k$$

$$= \sum_{r=1}^{\infty} \left(3^{-1} z\right)^r + \sum_{\substack{k=0 \\ (k\text{ - even})}}^{\infty} \left(\frac{1}{3} z^{-1}\right)^k + \sum_{\substack{k=0 \\ (k\text{ - odd})}}^{\infty} \left(\frac{1}{2} z^{-1}\right)^k$$

$$= \left[\frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \ldots\right]$$
$$+ \left[1 + \left(\frac{1}{3z}\right)^2 + \left(\frac{1}{3z}\right)^4 + \ldots\right]$$
$$+ \left[\left(\frac{1}{2z}\right) + \left(\frac{1}{2z}\right)^3 + \left(\frac{1}{2z}\right)^5 + \ldots\right]$$

... $\because$ k = even i.e. 0, 2, 4 etc. in seond G.P.

and k = odd i.e. 1, 3, 5 etc. in the third G.P.

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\left(\frac{1}{3z}\right)^2} + \frac{\frac{1}{2z}}{1-\left(\frac{1}{2z}\right)^2},$$

provided $\left|\frac{z}{3}\right| < 1, \left|\left(\frac{1}{3z}\right)^2\right| < 1, \left|\left(\frac{1}{2z}\right)^2\right| < 1$

$$\ldots \because S_\infty = \frac{a}{1-r}, |r| < 1$$

$$= \frac{z}{3-z} + \frac{1}{1-\frac{1}{9z^2}} + \frac{\frac{1}{2z}}{1-\frac{1}{4z^2}}, |z| < 3, \left|\frac{1}{9z^2}\right| < 1,$$

$$\left|\frac{1}{4z^2}\right| < 1$$

$$\therefore \; Z\{f(k)\} = \frac{z}{3-z} + \frac{9z^2}{9z^2-1} + \frac{2z}{4z^2-1},$$

$|z| < 3, |z^2| > \left|\frac{1}{9}\right|$ i.e. $|z^2| > \frac{1}{9}$ i.e. $|z| > \frac{1}{3}$,

$|z^2| > \left|\frac{1}{4}\right|$ i.e. $|z|^2 > \frac{1}{4}$ i.e. $|z| > \frac{1}{2}$

$\therefore$ common ROC : $\frac{1}{2} < |z| < 3$ i.e. annulus

Example 4.9 : *Find the Z-transform of the following. Specify the ROC.*

$$f(k) = 4^k + 5^k, k \geq 0$$

Solution : We have,

$$f(k) = 4^k + 5^k, k \geq 0$$
$$\therefore \; Z\{f(k)\} = Z\{4^k + 5^k\}$$
$$= Z\{4^k\} + Z\{5^k\}, k \geq 0$$
$$= \frac{z}{z-4} + \frac{z}{z-5}, |z| > 4, |z| > 5$$
$$\ldots \because Z\{a^k\} = \frac{z}{z-a}, |z| > |a|$$
$$= z\left[\frac{(z-5)+(z-4)}{(z-4)(z-5)}\right]$$
$$= z\left(\frac{2z-9}{z^2-9z+20}\right), |z| > 5$$

and $\therefore$ the ROC is exterior of the circle, $x^2+y^2=5^2$

Example 4.10 : *Find the Z-transform of the following by showing the region of convergence.*

i) $f(k) = 2^k + 3^{-k}, k \geq 0$

ii) $\left(-\frac{1}{2}\right)^{k+1} + 3\left(\frac{1}{2}\right)^{k+1}, k \geq 0$

Solution : i) We have,

$$f(k) = 2^k + 3^{-k}, k \geq 0$$
$$= 2^k + \frac{1}{3^k} = 2^k + \left(\frac{1}{3}\right)^k, k \geq 0$$
$$\therefore \; Z\{f(k)\} = Z\left\{2^k + \left(\frac{1}{3}\right)^k\right\}$$
$$= Z\{2^k\} + Z\left\{\left(\frac{1}{3}\right)^k\right\}, k \geq 0$$

... using Linearity Property

$$= \frac{z}{z-2} + \frac{z}{z-\frac{1}{3}}, |z| > |2|, |z| > \left|\frac{1}{3}\right|$$

... using standard result :

$$Z\{a^k\} = \frac{z}{z-a}, k \geq 0, |z| > |a|$$
$$= \frac{z}{z-2} + \frac{3z}{3z-1}, |z| > 2$$

Thus, the region of convergence is the exterior of the circle, $x^2 + y^2 = 2^2$.

Example 4.11 : *Find the Z-transform of the following by showing the region of convergence.*

$$\left(-\frac{1}{2}\right)^{k+1} + 3\left(\frac{1}{2}\right)^{k+1}, k \geq 0$$

Solution : We have,

$$f(k) = \left(-\frac{1}{2}\right)^{k+1} + 3\left(\frac{1}{2}\right)^{k+1}, k \geq 0$$
$$= \left(-\frac{1}{2}\right)^k \left(-\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)$$
$$= -\frac{1}{2}\left(-\frac{1}{2}\right)^k + \frac{3}{2}\left(\frac{1}{2}\right)^k, k \geq 0$$

$$\therefore \quad Z\{f(k)\} = Z\left\{-\frac{1}{2}\left(-\frac{1}{2}\right)^k + \frac{3}{2}\left(\frac{1}{2}\right)^k\right\}$$

$$= -\frac{1}{2}Z\left\{\left(-\frac{1}{2}\right)^k\right\} + \frac{3}{2}Z\left\{\left(\frac{1}{2}\right)^k\right\},\ k \geq 0$$

$$= \left(-\frac{1}{2}\right)\frac{z}{z-\left(-\frac{1}{2}\right)} + \frac{3}{2}\cdot\frac{z}{z-\frac{1}{2}},$$

$$|z| > \left|-\frac{1}{2}\right|, |z| > \left|\frac{1}{2}\right|$$

$$\ldots \because Z\{a^k\} = \frac{z}{z-a},\ k \geq 0,$$

$$|z| > |a|$$

$$= -\frac{1}{2}\cdot\frac{2z}{2z+1} + \frac{3}{2}\cdot\frac{2z}{2z-1}, |z| > \frac{1}{2}$$

$$= -\frac{z}{2z+1} + \frac{3z}{2z-1}, |z| > \frac{1}{2}$$

and the ROC is the exterior of the circle, $x^2 + y^2 = \left(\frac{1}{2}\right)^2$.

Example 4.12 : *Find the Z-transforms of the following*

$$f(k) = 3\left(\frac{1}{4}\right)^{k+1} + 4\left(\frac{1}{5}\right)^{k+1},\ k \geq 0$$

Solution : We have,

$$f(k) = 3\left(\frac{1}{4}\right)^{k+1} + 4\left(\frac{1}{5}\right)^{k+1},\ k \geq 0$$

$$= 3\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^k + 4\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)^k,\ k \geq 0$$

$$\therefore \quad Z\{f(k)\} = \frac{3}{4}Z\left\{\left(\frac{1}{4}\right)^k\right\} + \frac{4}{5}Z\left\{\left(\frac{1}{5}\right)^k\right\},\ k \geq 0$$

$$= \frac{3}{4}\frac{z}{\left(z-\frac{1}{4}\right)} + \frac{4}{5}\frac{z}{\left(z-\frac{1}{5}\right)},$$

$$|z| > \frac{1}{4},\ |z| > \frac{1}{5}$$

$$= \frac{3}{4}\frac{4z}{(4z-1)} + \frac{4}{5}\frac{5z}{(5z-1)}, |z| > \frac{1}{4}$$

$$= \frac{3z}{4z-1} + \frac{4z}{5z-1}$$

$$= \frac{15z^2 - 3z + 16z^2 - 4z}{(4z-1)(5z-1)}$$

$$= \frac{31z^2 - 7z}{(4z-1)(5z-1)},\quad |z| > \frac{1}{4}$$

Example 4.13 : *Find the Z-transform and region of convergence*

$$f(k) = \left(\frac{1}{2}\right)^{|k|} \text{ for all } k.$$

Solution : We have,

$$f(k) = \left(\frac{1}{2}\right)^{|k|},\quad \forall\, k$$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{\infty}\left(\frac{1}{2}\right)^{|k|} z^{-k}$$

$$= \sum_{k=-\infty}^{-1}\left(\frac{1}{2}\right)^{|k|} z^{-k} + \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{|k|} z^{-k}$$

$$= \sum_{r=\infty}^{1}\left(\frac{1}{2}\right)^{|-r|} z^{r} + \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{|k|} z^{-k}$$

... putting k = – r in first

$$= \sum_{r=1}^{\infty}\left(\frac{1}{2}\right)^{r} z^{r} + \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k} z^{-k}$$

... ∵ k and r are positive

$$= \sum_{r=1}^{\infty}\left(\frac{z}{2}\right)^{r} + \sum_{k=0}^{\infty}\left(\frac{1}{2z}\right)^{k}$$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\frac{1}{2z}},\quad \left|\frac{z}{2}\right| < 1,\ \left|\frac{1}{2z}\right| < 1$$

... sum of infinite G.P.

$$= \frac{z}{2-z} + \frac{2z}{2z-1},\quad |z| < 2,\ |z| > \frac{1}{2}$$

$$\text{i.e. } \frac{1}{2} < |z| < 2$$

i.e. the ROC is the annulus between the circles $x^2 + y^2 = \left(\frac{1}{2}\right)^2$ and $x^2 + y^2 = 2^2$ in the z-plane.

Example 4.14 : *Find $Z\{f(k)\}$ if,*

$$f(k) = \frac{a^k}{k},\ k \geq 1$$

Solution : We have

$$f(k) = \frac{a^k}{k},\ k \geq 1$$

Since, not specified, we assume f(k) = 0, k ≤ 0

$$\therefore \quad Z\{f(k)\} = \sum_{k=1}^{\infty} \frac{a^k}{k} z^{-k}$$

$$= \sum_{k=1}^{\infty} \frac{(az^{-1})^k}{k}$$

$$= \frac{(az^{-1})}{1} + \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} + \ldots$$

$$= -\left[-(az^{-1}) - \frac{(az^{-1})^2}{2} - \frac{(az^{-1})^3}{3} \ldots\right]$$

... Note this step

$$= -\log(1 - a z^{-1})$$

... $\because$ Maclaurin's series :

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$$

$$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \ldots$$

This is a convergent series if $|az^{-1}| < 1$ i.e. $\left|\frac{a}{z}\right| < 1$

i.e. $|a| < |z|$ i.e. $|z| > |a|$ which is the ROC.

Alternately, we can use the property of 'Division by k' to find the Z-transform.

Example 4.15 : *Find $Z\{f(k)\}$ if, $f(k) = e^{-ak}$, $k \geq 0$*

Solution : We have, $f(k) = e^{-ak}$, $k \geq 0$

$$\therefore \quad Z\{f(k)\} = \sum_{k=0}^{\infty} e^{-ak} z^{-k}$$

$$= \sum_{0}^{\infty} \left(e^{-a} z^{-1}\right)^k$$

which is an infinite G.P. with first term, $a = (e^{-a}z^{-1})^0 = 1$ and common ratio, $r = e^{-a} z^{-1}$

$$\therefore \quad Z\{f(k)\} = \frac{1}{1 - e^{-a} z^{-1}}, \left|e^{-a} z^{-1}\right| < 1$$

... $\because$ sum of G.P., $S_\infty = \frac{a}{1 - r}$, $|r| < 1$

$$= \frac{1}{1 - \frac{e^{-a}}{z}}, \frac{|e^{-a}|}{|z|} < 1$$

$$= \frac{z}{z - e^{-a}}, |z| > |e^{-a}|$$

Example 4.16 : *Find the Z-transform of*

$$\cos\frac{2\pi k}{3} + \frac{1}{\sqrt{3}} \sin\frac{2\pi k}{3}$$

Solution : We have,

$$f(k) = \cos\frac{2\pi}{3}k + \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3}k$$

$$\therefore Z\{f(k)\} = Z\left\{\cos\frac{2\pi}{3}k\right\} + \frac{1}{\sqrt{3}} Z\left\{\sin\frac{2\pi}{3}k\right\}$$

$$= \frac{z\left(z - \cos\frac{2\pi}{3}\right)}{\left(z^2 - 2z\cos\frac{2\pi}{3} + 1\right)} + \frac{1}{\sqrt{3}} \frac{z\sin\frac{2\pi}{3}}{\left(z^2 - 2z\cos\frac{2\pi}{3} + 1\right)},$$

$|z| > 1$

$$= \frac{z}{\left(z^2 - 2z\cos\frac{2\pi}{3} + 1\right)} \left[\left(z + \frac{1}{2}\right) + \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{2}\right]$$

$$\because \cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

$$\ldots \sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$= \frac{z(z+1)}{z^2 - 2z\left(-\frac{1}{2}\right) + 1} = \frac{z(z+1)}{z^2 + z + 1}, |z| > 1$$

4.5 Properties of Z-transforms

1) Linearity : Let {f(k)} and {g(k)} be two sequences and a and b are real numbers. Then,

$$\mathbf{z\,[a\,f(k) + b\,g(k)] = a\,z\,[f(k)] + b\,z\,[g(k)]}$$

Proof : We have by definition of Z-transform

$$z[a\,f(k) + b\,g(k)] = \sum_{k=-\infty}^{\infty} [a\,f(k) + b\,g(k)]z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} [a\,f(k)\,z^{-k} + b\,g(k)\,z^{-k}]$$

$$= \sum_{k=-\infty}^{\infty} a\,f(k)\,z^{-k} + \sum_{k=-\infty}^{\infty} b\,g(k)\,z^{-k}$$

$$= a\sum_{k=-\infty}^{\infty} f(k)\,z^{-k} + b\sum_{k=-\infty}^{\infty} g(k)\,z^{-k}$$

$$= a\,z\,[f(k)] + b\,z\,[g(k)]$$

2) Change of scale : If Z[f(k)] = F(z) then

$$\mathbf{Z[a^k f(k)] = F\left(\frac{z}{a}\right)}$$

Proof : We have, $Z[f(k)] = F(z) = \sum_{k=-\infty}^{\infty} f(k)\,z^{-k}$

Replacing z by $\frac{z}{a}$, we get

$$\therefore \quad F\left(\frac{z}{a}\right) = \sum_{k=-\infty}^{\infty} f(k)\left(\frac{z}{a}\right)^{-k} = \sum_{k=-\infty}^{\infty} a^k f(k)\, z^{-k}$$

$$F\left(\frac{z}{a}\right) = Z\,\{a^k f(k)\}$$

$$\therefore \quad \boxed{Z\,[a^k f(k)] = F\left(\frac{z}{a}\right)}$$

3) If Z[f(k)] = F(z) then $\boxed{Z\,[e^{-ak}\, f(k)] = F\,(e^a z)}$

Proof : We have

$$Z[f(k)] = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$\therefore \quad Z\,[e^{-ak} f(k)] = \sum_{-\infty}^{\infty} e^{-ak} \cdot f(k)\, z^{-k}$$

$$= \sum_{-\infty}^{\infty} f(k)(e^a z)^{-k}$$

$$= F\,(e^a z)$$

4) Shifting property :

a) If Z [f(k)] = F(z) then $Z\,[f(k+n)] = z^n F(z)$ **and** $Z\,[f\,(k-n)] = z^{-n} F(z)$

Proof : We have,

$$Z\,[f(k)] = F\,(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$\therefore \quad Z\,[f(k+n)] = \sum_{k=-\infty}^{\infty} f(k+n)\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} f(k+n)\, z^{-(k+n)} \cdot z^n$$

$$= z^n \sum_{k=-\infty}^{\infty} f\,(k+n)\, z^{-(k+n)}$$

Put k + n = r if $k = -\infty \Rightarrow r = -\infty$

$$k = \infty \quad \Rightarrow r = \infty$$

$$\therefore \quad Z[f\,(k+n)] = z^n \sum_{r=-\infty}^{\infty} f(r)\, z^{-r} = z^n\, F\,(z)$$

$$\therefore \quad Z[f\,(k+n)] = z^n\, F\,(z)$$

Similarly,

$$Z\,[f(k-n)] = \sum_{k=-\infty}^{\infty} f\,(k-n)\, z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} f\,(k-n)\, z^{-(k-n)} \cdot z^{-n}$$

$$= z^{-n} \sum_{k=-\infty}^{\infty} f(k-n)\, z^{-(k-n)}$$

Put k – n = r; if $k = -\infty \Rightarrow r = -\infty$ and

$$k = \infty \;\Rightarrow r = \infty$$

$$= z^{-n} \sum_{r=-\infty}^{\infty} f\,(r)\, z^{-r} = z^{-n} F\,(z)$$

$$\therefore \quad Z[f(k-n)] = z^{-n} F(z)$$

b) For one sided Z-transform defined as

$$\therefore \quad \boxed{Z\,[f(k)] = \sum_{k=0}^{\infty} f(k)\, z^{-k}}$$

(i.e. Z-transform for $k \geq 0$), we have,

$$Z\,[f(k+n)] = z^n\, F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

$$\text{and} \quad Z\,[f(k-n)] = z^{-n}\, F(z) + \sum_{r=-n}^{-1} f\,(r)\, z^{-(n+r)}$$

Proof : We have for $k \geq 0$,

$$Z\,[f(k)] = \sum_{k=\infty}^{\infty} f(k)\, z^{-k}$$

$$Z\,[f(k+n)] = \sum_{k=0}^{\infty} f(k+n)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k+n)^{-(k+n)} \cdot z^n$$

Put k + n = r, when $k = 0 \Rightarrow r = n$

$$k = \infty \Rightarrow r = \infty$$

$$\therefore \quad Z\,[f(k+n)] = z^n \sum_{r=n}^{\infty} f(r)\, z^{-r}$$

$$\text{We have} \quad \sum_{r=n}^{\infty} = \sum_{r=0}^{\infty} - \sum_{r=0}^{n-1}$$

$$\therefore \quad Z\,[f(k+n)] = z^n \sum_{r=0}^{\infty} f(r)\, z^{-r} - z^n \sum_{r=0}^{n-1} f(r)\, z^{-r}$$

$$Z\,[f(k+n)] = z^n\, F(z) - z^n \sum_{r=0}^{n-1} f(r)\, z^{-r}$$

$$= z^n\, F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

Now,

$$Z\,[f(k-n)] = \sum_{k=0}^{\infty} f(k-n)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k-n)\, z^{-(k-n)} \cdot z^{-n}$$

$$= z^{-n} \sum_{k=0}^{\infty} f(k-n) z^{-(k-n)}$$

Put $k - n = r$, when $k = 0 \Rightarrow r = -n$

$$k = \infty \Rightarrow r = \infty$$

$$\therefore \quad Z[f(k+n)] = z^{-n} \sum_{r=-n}^{\infty} f(r)\, z^{-r}$$

We have $\displaystyle\sum_{r=-n}^{\infty} = \sum_{r=-n}^{-1} + \sum_{r=0}^{\infty}$

$$\therefore Z[f(k-n)] = z^{-n} \sum_{r=-n}^{-1} f(r)\, z^{-r} + z^{-n} \sum_{r=0}^{\infty} f(r)\, z^{-r}$$

$$\therefore Z[f(k-n)] = z^{-n} F(z) + \sum_{r=-n}^{-1} f(r)\, z^{-(n+r)}$$

Important Results :

1) If $\{f(k)\}$ is causal sequence then

$$Z[f(k-n)] = z^{-n} F(z)$$

because $f(-1)\, f(-2)\, f(-3) \ldots f(-n)$ are all zero.

2) $\quad Z[f(k-n)] = z^{-n} F(z)$

For $n = 1$,

$$Z[f(k-1)] = z^{-1} F(z),\ f(-1) = 0$$

$$Z[f(k-2)] = z^{-2} F(z),\ f(-1) = 0,\ f(-2) = 0$$

$$Z[f(k+1)] = z F(z) - z f(0)$$

$$Z[f(k+2)] = z^2 F(z) - z^2 f(0) - z f(1)$$

Shifting properties are very useful in Z-transforming linear difference equations, from which the solution is obtained by inverse transforming.

5. Multiplication by k :

If $Z[f(k)] = F(z)$ then $Z[k\, f(k)] = -z \dfrac{d}{dz} F(z)$

$\therefore$ **In general** $\quad \boxed{Z[k^n f(k)] = \left(-z\dfrac{d}{dz}\right)^n F(z)}$

Proof : We have,

$$Z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = F(z)$$

$$\therefore Z[k\, f(k)] = \sum_{k=-\infty}^{\infty} k\, f(k)\, z^{-k}$$

Multiply and divide by $(-z)$ on R.H.S.

$$= \sum_{k=-\infty}^{\infty} -k\, f(k)\, a^{-k-1}(-z)$$

$$= -z \sum_{k=-\infty}^{\infty} f(k)\, \{-k\, z^{-k-1}\}$$

$$= -z \sum_{k=-\infty}^{\infty} f(k) \left(\frac{d}{dz} z^{-k}\right)$$

$$= -z \frac{d}{dz} \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$\therefore Z[k\, f(k)] = -z \frac{d}{dz} F(z)$$

$$\therefore Z[k^2 f(k)] = Z[k \cdot k\, f(k)] = \left(-z\frac{d}{dz}\right) Z[k\, f(k)]$$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz} \cdot F(z)\right)$$

$$= \left(-z\frac{d}{dz}\right)^2 F(z)$$

$\therefore$ In general we get,

$$Z[k^n f(k)] = \left(-z\frac{d}{dz}\right)^n F(z)$$

Note :

1) $\left(-z\dfrac{d}{dz}\right)^2 \neq z^2 \dfrac{d^2}{dz^2}$ but it is a repeated operator $\left(-z\dfrac{d}{dz}\right)^2 = \left(-z\dfrac{d}{dz}\right)\left(-z\dfrac{d}{dz}\right)$.

2) Let $k \geq 0$ and let $f(k) = 1$.

$$Z[f(k)] = F(z) = \sum_{k=-\infty}^{\infty} 1 \cdot z^{-k}$$

$$Z[1] = 1 + z^{-1} + z^{-2} + \ldots$$

$$= \frac{1}{1-z^{-1}} \quad \text{if } |z^{-1}| < 1$$

$$Z[1] = (1-z^{-1})^{-1}, \quad |z| > 1.$$

$$\therefore \quad Z[k] = Z(k \cdot 1) = \left(-z\frac{d}{dz}\right) F(z)$$

$$= \left(-z\frac{d}{dz}\right) [(1-z^{-1}-1)]$$

$$= -z\left\{-1 \cdot (1-z^{-1})^{-2}\right\}$$

$$Z[k] = z(1-z^{-1})^{-2}, \quad |z| > 1$$

Similarly, $Z[k^n] = \left(-z\dfrac{d}{dz}\right)^n (1-z^{-1})^{-1}$, $|z|>1$, $k \geq 0$

6. Division by k :

If Z [f(k)] = F(z) then $\boxed{Z\left[\frac{f(k)}{k}\right] = \int_z^{\infty} \frac{F(z)}{z}\, dz}$

Proof : We have,

$$Z\,[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$Z\left[\frac{f(k)}{k}\right] = \sum_{k=-\infty}^{\infty} \frac{f(k)}{k} z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} f(k)\left(\frac{z^{-k}}{k}\right)$$

$$= \sum_{k=-\infty}^{\infty} f(k) \int_z^{\infty} z^{-k-1} dz$$

$$= \sum_{k=-\infty}^{\infty} f(k) \int_z^{\infty} \frac{z^{-k}}{z} dz$$

$$= \int_z^{\infty} \frac{1}{z} \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} dz$$

$$= \int_z^{\infty} \frac{1}{z} F(z)\, dz$$

7. Initial Value Theorem (One sided sequence) :

If Z {f(k)} = F(z) then $\boxed{f(0) = \lim_{z\to\infty} F(z)}$

Proof : We have,

$$F(z) = Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k}$$

$$\therefore \quad F(z) = f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3} + \ldots$$

$$\therefore \quad \lim_{z\to\infty} F(z) = \lim_{z\to\infty} [f(0) + f(1)\, z^{-1} + f(2) z^{-2} + f(3)\, z^{-3} + \ldots]$$

As $\lim_{z\to\infty} z^{-n} = 0$

$\therefore$ R.H.S. = f(0) + all vanishing terms

$$\therefore \quad f(0) = \lim_{z\to\infty} F(z)$$

8. Final Value Theorem (One sided sequence) :

$\lim_{k\to\infty} \{f(k)\} = \lim_{z\to 1} (z-1)\, F(z)$, **if limit exists.**

Proof : We have,

$$Z[f(k+1)] - f[(k)] = \sum_{k=0}^{\infty} [f(k+1) - f(k)] z^{-k}$$

$$\therefore\ Z[f(k+1)] - Z[f(k)] = \sum_{k=0}^{\infty} [f(k+1) - f(k)] z^{-k} \quad \ldots (A)$$

For causal sequence, we have,

$$Z[f(k+n)] = z^n\, F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

$\therefore$ For n = 1, Z {f(k + 1)} = z F(z) – f(0)

$\therefore$ From equation (A),

$$z\,F(z) - f(0) - F(z) = \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1) - f(k)] z^{-k}$$

$$\therefore \quad \lim_{z\to 1} (z-1) F(z) = f(0) + \lim_{z\to 1} \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1) - f(k)] z^{-k}$$

$$= f(0) + \lim_{n\to\infty} \sum_{k=0}^{n} f[(k+1) - f(k)] \lim_{z\to 1} z^{-k}$$

$$= f(0) + \lim_{n\to\infty} [f(1) - f(0) + f(2) - f(1) + f(3) - f(2) \ldots + f(n+1) - f(n)]$$

$$= \lim_{n\to\infty} [f(0) + f(1) - f(0) + f(2) - f(1) + f(3) - f(2) + \ldots + f(n+1) - f(n)]$$

$$= \lim_{n\to\infty} f(n+1) = \lim_{k\to\infty} f(k)$$

For k = n + 1, when $n\to\infty$, $k\to\infty$

$\therefore\ \lim_{z\to 1} (z-1) F(z) = \lim_{k\to\infty} f(k).$

9. Partial Sum :

If $\quad Z\{f(k)\} = F(z)$

then $\quad Z\left[\left\{\sum_{m=-\infty}^{k} f(m)\right\}\right] = \frac{F(z)}{1-z^{-1}}$

Proof : Form {g(k)} such that $g(k) = \sum_{m=-\infty}^{k} f(m).$

Hence we have to obtain Z[{g(k)}].

We have,

$$g(k) - g(k-1) = \sum_{m=-\infty}^{k} f(m) - \sum_{m=-\infty}^{k-1} f(m) = f(k)$$

$\therefore\ Z[\{g(k)\} - g(k-1)\}] = Z[\{f(k)\}] = F(z)$

$\therefore\ Z[\{g(k)\}] - Z[\{g(k-1)\}] = F(z)$

$\therefore\ G(z) - z^{-1}G(z) = F(z) \Rightarrow (1-z^{-1})\,G(z)F(z)$

$$\therefore \quad \sum_{m=-\infty}^{k} f(m) = G(z) = \frac{F(z)}{1-z^{-1}}$$

Alternative :

$$Z\left[\left\{\sum_{m=-\infty}^{k} f(m)\right\}\right] = \sum_{k=-\infty}^{\infty}\left[\sum_{k=-\infty}^{k} f(m)\right] z^{-k}$$

$$= \sum_{k=-\infty}^{\infty}[\ldots + f(k-3)z^{-k} + f(k-2)z^{-k} + f(k-1)z^{-k} + f(k)z^{-k}]$$

$$= \sum_{k=-\infty}^{\infty}[\ldots + f(k-3)z^{-(k-3)}z^{-3} + f(k-2)z^{-(k-2)}\cdot z^{-2} + f(k-1)z^{-(k-1)}z^{-1} + f(k)z^{-k}]$$

$$= \sum_{k=-\infty}^{\infty}\sum_{r=0}^{\infty} f(k-r)z^{-(k-r)}z^{-r}$$

$$= \sum_{r=0}^{\infty} z^{-r}\sum_{k=-\infty}^{\infty} f(k-r)z^{-(k-r)}, \qquad (\text{let } k-r=p)$$

$$= \sum_{r=0}^{\infty} z^{-r}\sum_{p=-\infty}^{\infty} f(p)z^{-p}$$

$$= \sum_{r=0}^{\infty} F(z)z^{-r}$$

$$= F(z)\sum_{r=0}^{\infty} z^{-r}$$

$$= F(z)(1+z^{-1}+z^{-2}+\ldots)$$

$$= F(z)\frac{1}{1-z^{-1}},\ |z^{-1}| < 1$$

$$Z\left[\left\{\sum_{m=-\infty}^{\infty} f(m)\right\}\right] = \frac{F(z)}{1-z^{-1}},\ |z| > 1.$$

Remark : $\lim\limits_{k\to\infty} g(k) = \lim\limits_{k\to\infty}\sum\limits_{m=-\infty}^{k} f(m) = \sum\limits_{m=-\infty}^{k} f(m)$

By final value theorem,

$$\lim_{k\to\infty} g(k) = \lim_{k\to\infty}(z-1)\left(\frac{F(z)}{1-z^{-1}}\right) \text{ (by using property 10).}$$

$$= \lim_{z\to 1}(z-1)\frac{F(z)}{z-1}\cdot z = F(1)$$

$$\boxed{\sum_{m=-\infty}^{\infty} f(m) = F(1)}$$

10. Convolution :

a) General case : Convolution of two sequences {f(k)} and {g(k)} denoted as {f(k)} ∗ {g(k)}, is defined as :

$$\{h(k)\} = \{f(k)\} * \{g(k)\}$$

Where $$h(k) = \sum_{m=-\infty}^{\infty} f(m)\,g(k-m)$$

(Replacing dummy index m by k – m)

$$= \sum_{m=-\infty}^{\infty} g(m)\,f(k-m)$$

$$= \{g(k)\} * \{f(k)\}$$

Taking Z-transform of both sides, we get

$$Z[\{h(k)\}] = \sum_{k=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} f(m)\,g(k-m)\right]z^{-k}$$

Since the power series converges absolutely, it converges uniformly also within the ROC, this allows us to interchange the order of summation, we get

$$Z[\{h(k)\}] = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f(m)\,g(k-m)\,z^{-k}$$

$$= \sum_{m=-\infty}^{\infty} f(m)z^{-m}\sum_{k=-\infty}^{\infty} g(k-m)z^{-(k-m)}$$

$$= \left[\sum_{m=-\infty}^{\infty} f(m)z^{-m}\right]G(z)$$

$$H(z) = F(z)\,G(z)$$

ROC of H(z) is common region of convergence of F(z) and G(z).

We have $\{f(k)\} * \{g(k)\} \leftrightarrow F(z)\,G(z)$.

b) Convolution of Causal Sequences,

In this case, f(k) and g(k) are zero for negative values of k, due to this

$$h(k) = \sum_{m=-\infty}^{\infty} f(m)\,g(k-m) \text{ becomes}$$

$$= \sum_{m=0}^{k} f(m)\,g(k-m)$$

Because for negative values of m, f (m) is zero and for values of m > k, g (k – m) becomes zero.

The Z-transform of $\{h(k)\} = Z[\{f(k)\} * \{g(k)\}]$

$$= F(z) \cdot G(z)$$

remains unchanged.

Table of Properties of Z-transforms

1) Definition

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k)\, z^{-k},$$

for causal sequence $(k \geq 0)$

2) Linearity

$$Z\{a f(k) + b g(k)\} = a Z\{f(k)\} + b Z\{g(k)\}$$

3) Change of Scale

$$Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$$

4) Corollary of Change of Scale

$$Z\{e^{-ak} f(k)\} = F(e^a z)$$

5) Shifting Property

a) For both sided sequence :

$$Z\{f(k+n)\} = z^n F(z) \text{ ... Left shifting}$$

$$Z\{f(k-n)\} = z^{-n} F(z) \text{ ... Right shifting}$$

b) For causal sequence $(k \geq 0)$:

$$Z\{f(k+n)\} = z^n F(z) - \sum_{r=0}^{n-1} f(r)\, z^{n-r}$$

$$\therefore \quad Z\{f(k+1)\} = z F(z) - z f(0)$$

$$Z\{f(k+2)\} = z^2 F(z) - z^2 f(0) - z f(1) \text{ etc.}$$

$$Z\{f(k-n)\} = z^{-n} F(z)$$

$$\therefore \quad Z\{f(k-1)\} = z^{-1} F(z)$$

$$Z\{f(k-2)\} = z^{-2} F(z) \text{ etc.}$$

6) Multiplication by k

$$Z\{k f(k)\} = -z \frac{d}{dz} F(z)$$

In general,

$$Z\{k^n f(k)\} = \left(-z \frac{d}{dz}\right)^n F(z)$$

7) Division by k

$$Z\left\{\frac{f(k)}{k}\right\} = \int_z^{\infty} \frac{F(z)}{z}\, dz$$

8) Initial Value Theorem (for one sided sequence e.g. k ≥ 0)

$$f(0) = \lim_{z \to \infty} F(z)$$

9) Final Value Theorem (for one sided sequence e.g. k ≥ 0)

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1) F(z)$$

10) Convolution Theorem

$$Z[\{f(k)\} * \{g(k)\}] = F(z) \cdot G(z)$$

where, convolution of $\{f(k)\}$ and $\{g(k)\}$ is

$$\{f(k)\} * \{g(k)\} = \sum_{m=-\infty}^{\infty} f(m)\, g(k-m)$$

and for casual sequence $(k \geq 0)$

$$\{f(k)\} * \{g(k)\} = \sum_{m=0}^{k} f(m)\, g(k-m)$$

Table 4.1 : Properties of Z-transforms

Examples on Properties of Z-transform

Example 4.17 : *Find the Z-transform of the following. Also, find the ROC :*
$f(k) = k^3,\ k \geq 0$

Solution :

We have, $\quad f(k) = k^3,\ k \geq 0$

Now, $\quad Z\{1\} = \dfrac{z}{z-1},\ |z| > 1$

$$\therefore \quad Z\{k^3\} = Z\{k^3 \cdot 1\}$$

$$= \left(-z \frac{d}{dz}\right)^3 Z\{1\}$$

$$= \left(-z \frac{d}{dz}\right)^2 \left(-z \frac{d}{dz}\right) \frac{z}{z-1}$$

$$= \left(-z \frac{d}{dz}\right)^2 \left[(-z) \cdot \frac{(z-1)-z}{(z-1)^2}\right]$$

$$= \left(-z \frac{d}{dz}\right)\left(-z \frac{d}{dz}\right) \frac{z}{(z-1)^2}$$

$$= \left(-z \frac{d}{dz}\right)\left[(-z)\, \frac{(z-1)^2 - z \cdot 2(z-1)}{(z-1)^4}\right]$$

$$= \left(-z \frac{d}{dz}\right)\left[(-z) \cdot \frac{(z-1) - 2z}{(z-1)^3}\right]$$

$$= -z \frac{d}{dz} \frac{z(z+1)}{(z-1)^3} = -z \frac{d}{dz} \frac{z^2+z}{(z-1)^3}$$

$$= -z\left[\frac{(z-1)^3 (2z+1) - (z^2+z) \cdot 3(z-1)^2}{(z-1)^6}\right]$$

$$= -z\left[\frac{(z-1)(2z+1)-3\left(z^2+z\right)}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2-4z-1}{(z-1)^4}\right]$$

$$= \frac{z\left(z^2+4z+1\right)}{(z-1)^4}, |z|>1$$

Example 4.18 : *Find* $Z\{f(k)\}$ *if*

i) $f(k)=2^k \cosh\alpha k,\ k\ge 0$

ii) $f(k)=3^k\cos(4k+5),\ k\ge 0$

iii) $2^{-t}\sin at,\ t\ge 0$

Solution : i) We have

$$f(k) = 2^k \cosh\alpha k,\ k\ge 0$$

[Here f(k) has the form : $\{a^k\ g(k)\}$. Hence use change of scale property]

Let, $g(k) = \cosh\alpha k,\ k\ge 0$

$\because$ $G(z) = Z\{g(k)\} = Z\{\cosh\alpha k\}$

$$= \frac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1},$$

$|z|>\max\left(\left|e^{\alpha}\right| \text{or} \left|e^{-\alpha}\right|\right)$ ... (1)

... Standard result

$\therefore$ $Z\{f(k)\} = Z\{2^k \cosh\alpha k\}$

$= Z\{2^k\ g(k)\}$

$= G\left(\frac{z}{2}\right)$ ... by change of scale property, $Z\{a^k g(k)\} = G\left(\frac{z}{a}\right)$

where, $G(z) = Z\{g(k)\}$

$$= \frac{\frac{z}{2}\left(\frac{z}{2}-\cosh\alpha\right)}{\left(\frac{z}{2}\right)^2-2\left(\frac{z}{2}\right)\cosh\alpha+1},$$

$\left|\frac{z}{2}\right|>\max\left(\left|e^{\alpha}\right| \text{or} \left|e^{-\alpha}\right|\right)$

... Replacing z by $\frac{z}{2}$ in (1)

$$= \frac{\frac{z}{4}(z-2\cosh\alpha)}{\frac{1}{4}(z^2-4z\cosh\alpha+4)},$$

$|z|>\max\left(2\left|e^{\alpha}\right| \text{or}\ 2\left|e^{-\alpha}\right|\right)$

$\therefore$ $$Z\{f(k)\} = \frac{z(z-2\cosh\alpha)}{z^2-4z\cosh\alpha+4},$$

$|z|>\max\left(\left|2e^{\alpha}\right| \text{or} \left|2e^{-\alpha}\right|\right)$

ii) We have, $f(k) = 3^k\cos(4k+5),\ k\ge 0$

[Form : $Z\{a^k\ f(k)\}$. Hence, use change of scale property]

Now,

$$\cos(4k+5) = \cos 4k\cos 5-\sin 4k\sin 5$$

$\therefore Z\{\cos(4k+5)\} = Z\{\cos 5\cdot\cos 4k - \sin 5\sin 4k\}$

$= \cos 5\cdot Z\{\cos 4k\} - \sin 5\ Z\{\sin 4k\}$

... Linearity Property

$$= \cos 5\left[\frac{z(z-\cos 4)}{z^2-2z\cos 4+1}\right]-\sin 5\left[\frac{z\sin 4}{z^2-2z\cos 4+1}\right]$$

... Standard Result

= say G(z) ... (1)

$\therefore$ By change of scale property

i.e. $Z\{a^k\ f(k)\} = F\left(\frac{z}{a}\right)$ we have,

$$Z\{3^k\cos(4k+5)\} = G\left(\frac{z}{3}\right)$$

$$= \cos 5\left[\frac{\frac{z}{3}\left(\frac{z}{3}-\cos 4\right)}{\left(\frac{z}{3}\right)^2-2\left(\frac{z}{3}\right)\cos 4+1}\right] - \sin 5\left[\frac{\frac{z}{3}\sin 4}{\left(\frac{z}{3}\right)^2-2\left(\frac{z}{3}\right)\cos 4+1}\right]$$

... using G(z) from (1)

$$= \cos 5\left[\frac{\frac{z}{9}(z-3\cos 4)}{\frac{z^2-6z\cos 4+9}{9}}\right] - \sin 5\left[\frac{\frac{z}{9}(3\sin 4)}{\frac{z^2-6z\cos 4+9}{9}}\right]$$

$$= \frac{(\cos 5)\cdot z\,(z-3\cos 4)-(\sin 5)\cdot 3z\sin 4}{z^2-6z\cos 4+9}$$

iii) We have, $f(t) = 2^{-t}\sin at,\ t\ge 0$

$$= \left(\frac{1}{2}\right)^t \sin at,\ t\ge 0$$

$\therefore$ Its Z-transform is

$$Z\{f(t)\} = Z\left\{\left(\frac{1}{2}\right)^t \sin at\right\}, t \geq 0$$

$$= \frac{\left(\frac{1}{2}\right) z \sin a}{z^2 - 2\left(\frac{1}{2}\right) z \cos a + \left(\frac{1}{2}\right)^2},$$

$|z| > \left|\frac{1}{2}\right|$ $\because$ Standard Result :

$$Z\{c^k \sin \alpha k\} = \frac{cz \sin \alpha}{z^2 - 2cz\cos\alpha + c^2}, \ |z| > |c|$$

$$= \frac{1}{2} \frac{z \sin a}{\left(z^2 - z \cos a + \frac{1}{4}\right)} = \frac{1}{2} \frac{z \sin a}{\frac{1}{4}\left(4z^2 - 4z \cos a + 1\right)}$$

$$= \frac{2z \sin a}{4z^2 - 4z \cos a + 1}, |z| > \frac{1}{2}$$

Alternately we can write $Z\{\sin at\}$ first, then use the change of scale property to find $Z\left\{\left(\frac{1}{2}\right)^t \sin at\right\}$.

Example 4.19 : *Find* $Z\{f(k)\}$ *if*

i) $f(k) = e^{-ak} \cos bk, \ k \geq 0$

ii) $f(k) = e^{-3k} \sinh 4k, \ k \geq 0$

Solution : [Form : $Z\{e^{-ak} f(k)\}$. Hence use, the property (case of change of scale, $a \to e^{-a}$).]

i) We have,

$$Z\{\cos bk\} = \frac{z(z - \cos b)}{z^2 - 2z \cos b + 1},$$

$|z| > 1$ ... (1) Standard Result

Now, $\because Z\{e^{-ak} f(k)\} = F(e^a z)$ ... Property where, $Z\{f(k)\} = F(z)$

$$\therefore \quad Z\{e^{-ak} \cos bk\} = \frac{e^a z\left(e^a z - \cos b\right)}{\left(e^a z\right)^2 - 2\left(e^a z\right) \cos b + 1},$$

$|e^a z| > 1$

... Replacing z by $(e^a z)$ in (1)

$$= \frac{e^a \cdot e^a z\left(z - e^{-a} \cos b\right)}{e^{2a} \cdot z^2 - 2 e^a z \cos b + 1}$$

$$= \frac{e^{2a} \cdot z\left(z - e^{-a} \cos b\right)}{e^{2a}\left(z^2 - 2 e^{-a} z \cos b + e^{-2a}\right)}$$

$$= \frac{z\left(z - e^{-a} \cos b\right)}{z^2 - 2 e^{-a} z \cos b + e^{-2a}}, \ |z| > \frac{1}{e^a}$$

ii) We have, $f(k) = e^{-3k} \sinh 4k, \ k \geq 0$

$$Z\{\sinh 4k\} = \frac{z \sinh 4}{z^2 - 2z \cosh 4 + 1}, \ k \geq 0$$

$$\ldots \because Z\{\sinh \alpha k\} = \frac{z \sinh \alpha}{z^2 - 2z \cosh \alpha + 1}$$

$= \text{say } G(z)$ ... (1)

$$\therefore Z\left\{e^{-3k} \sinh 4k\right\} = G\left(e^3 z\right)$$

$$\ldots \because \text{Property} Z\left\{e^{-ak} f(k)\right\} = F\left(e^a z\right)$$

$$= \frac{e^3 z \sinh 4}{\left(e^3 z\right)^2 - 2\left(e^3 z\right) \cosh 4 + 1}$$

... using G(z) from (1)

$$= \frac{e^3 z \sinh 4}{e^6 z^2 - 2e^3 z \cosh 4 + 1}$$

$$= \frac{e^3 z \sinh 4}{e^3\left(e^3 z^2 - 2z \cosh 4 + e^{-3}\right)}$$

$$= \frac{z \sinh 4}{e^3 z^2 - 2z \cosh 4 + e^{-3}}$$

Example 4.20 : *Find the Z-transform of the following. Specify the ROC.*

$$f(k) = e^{-3k} \sin 4k, \ k \geq 0$$

Solution : We have,

$$f(k) = e^{-3k} \sin 4k, \ k \geq 0$$

Now, $$Z\{\sin 4k\} = \frac{z \sin 4}{z^2 - 2z \cos 4 + 1}, |z| > 1$$

$$\therefore \quad Z\left\{e^{-3k} \sin 4k\right\} = \frac{\left(e^3 z\right) \sin 4}{\left(e^3 z\right)^2 - 2\left(e^3 z\right) \cos 4 + 1},$$

$|e^3 z| > 1$

$$\ldots \because Z\left\{e^{-ak} f(k)\right\} = F\left(e^a z\right)$$

$$= \frac{e^3 z \sin 4}{e^6 z^2 - 2e^3 z \cos 4 + 1}$$

$$= \frac{e^3 z \sin 4}{e^3\left(e^3 z^2 - 2z \cos 4 + e^{-3}\right)}$$

$$= \frac{z \sin 4}{e^3 z^2 - 2z \cos 4 + e^{-3}}, \quad |z| > \frac{1}{e^3}$$

and $\therefore$ the ROC is the exterior of the circle : $x^2+y^2 = \left(\frac{1}{e^3}\right)^2$.

Example 4.21 : *Find z-transform of*

$f(x) = e^{-3k} \sin 2k \cos 2k, \quad k \geq 0$

Solution : We have

$$f(k) = e^{-3k} \sin 2k \cos 2k$$

$$f(k) = \frac{1}{2} e^{-3k} \sin 4k$$

We know that

$$Z\{\sin 4k\} = \frac{z \sin 4}{z^2 - 2z \cos 4 + 1}; \ |z| > 1$$

$$\therefore \quad Z\left\{\frac{1}{2} e^{-3k} \sin 4k\right\} = \frac{1}{2} \frac{(e^3 \sin 4)(z)}{(e^3 z)^2 - 2(e^3 z) \cos 4 + 1}$$

$$Z\{f(k)\} = \frac{(e^3 \sin 4) z}{2[e^6 z^2 - 2e^3 z \cos 4 + 1]}$$

$$; |z| > e^{-3}$$

Example 4.22 : *Find the Z-transform of the following. Also, find the ROC :*
$f(k) = \alpha^k \cos(\beta k + \gamma), \ k \geq 0$

Solution : We have,

$$f(k) = \alpha^k \cos(\beta k + \gamma), \ k \geq 0$$

$$= \alpha^k (\cos \beta k \cos \gamma - \sin \beta k \sin \gamma)$$

$$= \cos \gamma \left(\alpha^k \cos \beta k\right) - \sin \gamma \left(\alpha^k \sin \beta k\right)$$

$$\therefore Z\{f(k)\} = \cos \gamma \, Z\left\{\alpha^k \cos \beta k\right\} - \sin \gamma \, Z\left\{\alpha^k \sin \beta k\right\}$$

$$= \cos \gamma \left[\frac{z(z - \alpha \cos \beta)}{z^2 - 2\alpha z \cos \beta + \alpha^2}\right] - \sin \gamma \left[\frac{\alpha z \sin \beta}{z^2 - 2\alpha z \cos \beta + \alpha^2}\right] |z| > |\alpha|$$

$$\ldots \because Z\left\{c^k \cos ak\right\} = \frac{z(z - c \cos a)}{z^2 - 2cz \cos a + c^2}$$

$$\text{and } Z\left\{c^k \sin ak\right\} = \frac{cz \sin a}{z^2 - 2cz \cos a + c^2}, \ |z| > |c|$$

$$= \frac{z^2 \cos \gamma - \alpha z \cos \beta \cos \gamma - \alpha z \sin \beta \sin \gamma}{z^2 - 2\alpha z \cos \beta + \alpha^2}$$

$$= \frac{z^2 \cos \gamma - \alpha z \cos(\beta - \gamma)}{z^2 - 2\alpha z \cos \beta + \alpha^2}, |z| > |\alpha|$$

Example 4.23 : *Find the Z-transforms of the following*
i) $f(k) = 3^k \sin(2k+3), \ k \geq 0$
ii) $f(k) = k e^{-k} \sin ak, \ k \geq 0$

Solution : i) We have,

$$f(k) = 3^k \sin(2k+3)$$

Now,

$$Z\{\sin(2k+3)\} = Z\{\sin 2k \cos 3 + \cos 2k \sin 3\}$$

$$= (\cos 3) Z\{\sin 2k\} + (\sin 3) Z\{\cos 2k\}$$

$$= \cos 3 \frac{z \sin 2}{z^2 - 2z \cos 2 + 1} + \sin 3 \frac{z(z - \cos 2)}{z^2 - 2z \cos 2 + 1}, \ |z| > 1$$

$$= \text{say } G(z), |z| > 1 \qquad \ldots (1)$$

$$\therefore \quad Z\{f(k)\} = Z\left\{3^k \sin(2k+3)\right\}$$

$$= G\left(\frac{z}{3}\right) \qquad \ldots \text{change of scale : } Z\left\{a^k f(k)\right\} = F\left(\frac{z}{a}\right)$$

$$= \cos 3 \frac{\frac{z}{3} \sin 2}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right) \cos 2 + 1} + \sin 3 \cdot \frac{\frac{z}{3}\left(\frac{z}{3} - \cos 2\right)}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right) \cos 2 + 1}, \left|\frac{z}{3}\right| > 1 \ \ldots \text{using (1)}$$

$$= \frac{\frac{z}{3} \cos 3 \sin 2}{\frac{1}{9}\left(z^2 - 6z \cos 2 + 9\right)} + \frac{\frac{z}{3} \sin 3\left(\frac{z}{3} - \cos 2\right)}{\frac{1}{9}\left(z^2 - 6z \cos 2 + 9\right)}$$

$$= \frac{3z}{\left(z^2 - 6z \cos 2 + 9\right)}\left[\cos 3 \sin 2 + \frac{z}{3} \sin 3 - \sin 3 \cos 2\right]$$

$$= \frac{3z}{(z^2 - 6z \cos 2 + 9)}\left[\frac{z}{3} \sin 3 - (\sin 3 \cos 2 - \cos 3 \sin 2)\right]$$

$$= \frac{3z}{\left(z^2 - 6z \cos 2 + 9\right)}\left[\frac{z}{3} \sin 3 - \sin 1\right], |z| > 3$$

Alternately, we can use

$$Z\left\{c^k \sin\alpha k\right\} = \frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}, |z| > |c| \text{ etc,}$$

directly

ii) We have, $f(k) = k\,e^{-k}\sin ak,\ k \geq 0$

Now,

$$Z\{\sin ak\} = \frac{z\sin a}{z^2 - 2z\cos a + 1}, |z| > 1$$

$$\therefore \quad Z\{k\sin ak\} = \left(-z\frac{d}{dz}\right)\frac{z\sin a}{z^2 - 2z\cos a + 1}$$

... multiplication by k

$$= -z(\sin a)\left[\frac{\left(z^2 - 2z\cos a + 1\right) - z(2z - 2\cos a)}{\left(z^2 - 2z\cos a + 1\right)^2}\right]$$

$$= -z(\sin a)\left[\frac{-z^2 + 1}{\left(z^2 - 2z\cos a + 1\right)^2}\right]$$

$$= z(\sin a)\left[\frac{z^2 - 1}{\left(z^2 - 2z\cos a + 1\right)^2}\right]$$

$= \text{say } G(z),\ |z| > 1 \quad \dots (1)$

$$\therefore\ Z\left\{e^{-k}\cdot k\sin ak\right\} = G\left(e^1 z\right), |ez| > 1$$

$$\dots \because Z\left\{e^{-ak} f(k)\right\} = F\left(e^a z\right)$$

$$= (ez)(\sin a)\cdot\left[\frac{(ez)^2 - 1}{\left[(ez)^2 - 2(ez)\cos a + 1\right]^2}\right], |z| > \frac{1}{e}$$

... using (1)

Example 4.24 : *Find the Z-transform of the following. Also, find the ROC :*

$$f(k) = k\,5^k$$

Solution : We have,

$$f(k) = k\,5^k, \text{ Assume } k \geq 0$$

Now,

$$Z\left\{5^k\right\} = \frac{z}{z-5}, |z| > 5$$

$$\therefore \quad Z\left\{k\,5^k\right\} = -z\frac{d}{dz}\left(\frac{z}{z-5}\right)$$

... Multiplication by k

$$= -z\left[\frac{(z-5) - z}{(z-5)^2}\right]$$

$$= \frac{5z}{(z-5)^2}, |z| > 5$$

Example 4.25 : *Find Z{f(k)} if* $f(k) = 4^k e^{-5k} k;\ K \geq 0$ **SPPU : May-19**

Solution : Let $g(k) = k$

$$Z\{g(k)\} = Z\{k\} = \left(-z\frac{d}{dz}\right)z\{1\}$$

$$= \left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right);\ |z| > 1$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= (-z)\left[\frac{(z-1) - z(1)}{(z-1)^2}\right];\ |z| > 1$$

$$= (-z)\frac{-1}{(z-1)^2}$$

$$Z\{g(k)\} = \frac{z}{(z-1)^2};\ |z| > 1$$

$$\therefore \quad Z\{4^k k\} = \frac{\frac{z}{4}}{\left(\frac{z}{4} - 1\right)^2};\ |z| > 1$$

Now

$$Z\{4^k k\,e^{-5k}\} = \frac{\frac{ze^5}{4}}{\left(\frac{ze^5}{4} - 1\right)^2};\ |z| > 1$$

$$= \frac{4e^5 z}{(ze^5 - 4)^2};\ |z| > 1$$

Example 4.26 : *Find the Z-transform of the following sequence.*

$$k^2 a^k,\ k \geq 0$$

Solution : $f(k) = k^2 a^k,\ k \geq 0$

Now, $$Z\left\{a^k\right\} = \frac{z}{z-a}, |z| > |a|$$

$$\therefore \quad Z\left\{k^2 a^k\right\} = \left(-z\frac{d}{dz}\right)^2 Z\left\{a^k\right\}$$

... Multiplication by k^2

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-a}\right)$$

$$= \left(-z\frac{d}{dz}\right)\left[(-z)\frac{(z-a)-z}{(z-a)^2}\right]$$

$$= -z\frac{d}{dz}\frac{az}{(z-a)^2}$$

$$= -za\left[\frac{(z-a)^2 - z\cdot 2(z-a)}{(z-a)^4}\right]$$

$$= -za\left[\frac{(z-a) - 2z}{(z-a)^3}\right]$$

$$= \frac{az(z+a)}{(z-a)^3}, \quad |z| > |a|$$

Example 4.27 : *Find* $Z\{f(k)\}$ *if*

i) $f(k) = (k+1)\,2^k$

ii) $f(k) = k^2 e^{-ak}$, $k \geq 0$

SPPU : Dec.-19, Marks 4 (a = 1)

Solution : [Here we have the forms $Z\{k\,f(k)\}$. Hence use the property of multiplication by k.]

i) We have, $f(k) = (k+1)\,2^k$

$$= k\,2^k + 2^k, \; k \geq 0$$

(Assumption)

$$\therefore \quad Z\{f(k)\} = Z\{k\,2^k + 2^k\}$$

$$= Z\{k\,2^k\} + Z\{2^k\}$$

... Linearity Property

$$= \left(-z\frac{d}{dz}\right)[Z\{2^k\}] + \frac{z}{z-2}$$

$$\ldots \because Z\{k\,f(k)\} = -z\frac{d}{dz}F(z)$$

and $$Z\{a^k\} = \frac{z}{z-a} = -z\frac{d}{dz}\left(\frac{z}{z-2}\right) + \frac{z}{z-2}$$

$$= -z\left[\frac{(z-2)1 - z(1)}{(z-2)^2}\right] + \frac{z}{z-2}$$

$$= \frac{2z}{(z-2)^2} + \frac{z}{z-2} = \frac{2z + z(z-2)}{(z-2)^2}$$

$$= \frac{z^2}{(z-2)^2}$$

ii) We have, $f(k) = k^2 e^{-ak}$, $k \geq 0$

Now, $$Z\{e^{-ak}\} = Z\{(e^{-a})^k\}$$

$$= \frac{z}{z - e^{-a}}$$

$$\ldots \because Z\{a^k\} = \frac{z}{z-a}$$

$$\therefore \quad Z\{k\,e^{-ak}\} = -z\frac{d}{dz}Z\{e^{-ak}\}$$

$$\ldots \because Z\{k\,f(k)\} = -z\frac{d}{dz}F(z)$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-e^{-a}}\right)$$

$$= -z\left[\frac{(z-e^{-a})1 - z(1)}{(z-e^{-a})^2}\right]$$

$$= \frac{z\,e^{-a}}{(z-e^{-a})^2} \quad \ldots (1)$$

Now, using $Z\{k\,f(k)\} = -z\dfrac{d}{dz}F(z)$ again, we get

$$Z\{k^2 e^{-ak}\} = Z\{k(ke^{-ak})\}$$

$$= -z\frac{d}{dz}Z\{ke^{-ak}\}$$

$$= -z\frac{d}{dz}\left[\frac{z\,e^{-a}}{(z-e^{-a})^2}\right] \quad \ldots \text{using (1)}$$

$$= -ze^{-a}\frac{d}{dz}\frac{z}{(z-e^{-a})^2}$$

$$= -ze^{-a}\left[\frac{(z-e^{-a})^2\,1 - z\;2(z-e^{-a})1}{(z-e^{-a})^4}\right]$$

$$= -ze^{-a}\left[\frac{(z-e^{-a}) - 2z}{(z-e^{-a})^3}\right]$$

$$= -ze^{-a}\left[\frac{-z-e^{-a}}{(z-e^{-a})^3}\right]$$

$$\therefore \quad Z\{k^2 e^{-ak}\} = \frac{ze^{-a}(z+e^{-a})}{(z-e^{-a})^3}$$

OR, we can directly write

$$Z\{k^2 e^{-ak}\} = \left(-z\frac{d}{dz}\right)^2 Z\{e^{-ak}\}$$

... multiplication by k^2

Example 4.28 : *Find $Z\{f(k)\}$ if $f(k) = (k+1)(k+2)a^k,\ k \geq 0$*

SPPU : May-17

Solution : We have,

$f(k) = (k+1)(k+2)a^k = (k^2 + 3k + 2)\,a^k,\ k \geq 0$

$$\therefore \quad Z\{f(k)\} = Z\left\{(k^2 + 3k + 2)a^k\right\}$$

$$= Z\left\{k^2a^k\right\} + 3Z\left\{ka^k\right\} + 2Z\left\{a^k\right\} \quad \text{... (1)}$$

... using Linearity Property

Now, $$Z\left\{a^k\right\} = \frac{z}{z-a} \quad \text{... (2)}$$

... Standard Result

$$\therefore \quad Z\left\{ka^k\right\} = -z\frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$\text{... } \because Z\{k\,f(k)\} = -z\frac{d}{dz}F(z)$$

i.e. Property of multiplication by k

$$= -z\left[\frac{(z-a)\,1 - z(1)}{(z-a)^2}\right]$$

$$= \frac{az}{(z-a)^2} \quad \text{... (3)}$$

Using the property of multiplication by k again, we get,

$$Z\left\{k^2a^k\right\} = Z\left\{k\left(ka^k\right)\right\}$$

$$= -z\frac{d}{dz}Z\left\{ka^k\right\}$$

$$= -z\frac{d}{dz}\left[\frac{az}{(z-a)^2}\right] \quad \text{... using (3)}$$

$$= -az\left[\frac{(z-a)^2\,1 - z\cdot 2(z-a)}{(z-a)^4}\right]$$

$$= -az\left[\frac{(z-a) - 2z}{(z-a)^3}\right]$$

$$\therefore \quad Z\left\{k^2a^k\right\} = \frac{az(a+z)}{(z-a)^3} \quad \text{... (4)}$$

$\therefore$ Substituting from (2), (3) and (4) in (1) we get

$$Z\{f(k)\} = \frac{az(a+z)}{(z-a)^3} + 3\frac{az}{(z-a)^2} + 2\frac{z}{z-a}$$

$$= \frac{(a^2z + az^2) + 3az(z-a) + 2z(z-a)^2}{(z-a)^3}$$

$$= \frac{a^2z + az^2 + 3az^2 - 3a^2z + 2z(z^2 - 2az + a^2)}{(z-a)^3}$$

$$= \frac{2z^3}{(z-a)^3}$$

To check : By direct Standard Result,

$$Z\left\{\frac{(k+1)(k+2)a^k}{2!}\right\} = \frac{z^3}{(z-a)^3}$$

$$\therefore Z\{(k+1)(k+2)a^k\} = 2!\frac{z^3}{(z-a)^3} = \frac{2z^3}{(z-a)^3}$$

Example 4.29 : *Find $Z\{f(k)\}$ for the following sequences :*

$$f(k) = k^2C_k$$

Solution : $f(k) = k^2C_k$

and we must have, $0 \leq k \leq 2$ for ${}^2C_k$ to exist.

Now, we have,

$$Z\{{}^2C_k\} = (1 + z^{-1})^2,\ |z| > 1$$

$$\text{... } \because Z\{{}^nC_k\} = (1 + z^{-1})^n,\quad 0 \leq k \leq n,\ |z| > 1$$

$$\therefore \quad Z\{k^2C_k\} = \left(-z\frac{d}{dz}\right)(1 + z^{-1})^2$$

... Multiplication by k

$$= -z\frac{d}{dz}\left(1 + \frac{1}{z}\right)^2$$

$$= (-z)\left[2\left(1 + \frac{1}{z}\right)\cdot\left(\frac{-1}{z^2}\right)\right]$$

$$= 2z\left(\frac{z+1}{z}\right)\cdot\frac{1}{z^2} = \frac{2}{z}\left(\frac{z+1}{z}\right)$$

$$\therefore \quad Z\{k^2C_k\} = \frac{2z+2}{z^2}$$

Example 4.30 : *Find $Z\{f(k)\}$ if $f(k) = k\,e^{-2k}\cos 3k,\ k \geq 0$*

Solution : We have,

$$f(k) = k\,e^{-2k}\cos 3k,\ k \geq 0$$

Now, $$Z\{\cos 3k\} = \frac{z(z - \cos 3)}{z^2 - 2z\cos 3 + 1},\ |z| > 1$$

... (1) Standard Result

$$\therefore \quad Z\{e^{-2k}\cos 3k\} = \frac{(e^2z)(e^2z - \cos 3)}{(e^2z)^2 - 2(e^2z)\cos 3 + 1}$$

$$\text{... } \because Z\{e^{-ak}f(k)\} = F(e^az)$$

$$= \frac{e^2 z\,(e^2 z - \cos 3)}{e^4 z^2 - 2\,e^2 z \cos 3 + 1}$$

$$= \frac{e^2 z\,(e^2 z - \cos 3)}{e^2 (e^2 z^2 - 2z \cos 3 + e^{-2})}$$

$$\therefore \quad Z\,\{e^{-2k} \cos 3k\} = \frac{z\,(e^2 z - \cos 3)}{e^2 z^2 - 2z \cos 3 + e^{-2}}$$

$$\therefore \quad Z\,\{k \cdot e^{-2k} \cos 3k\} = \left(-z \frac{d}{dz}\right) Z\,\{e^{-2k} \cos 3k\}$$

... Multiplication by k

$$= -z \frac{d}{dz}\left[\frac{e^2 z^2 - z \cos 3}{e^2 z^2 - 2z \cos 3 + e^{-2}}\right]$$

$$= -z \left[\frac{(e^2 z^2 - 2z \cos 3 + e^{-2})(2e^2 z - \cos 3) - (e^2 z^2 - z \cos 3)(2e^2 z - 2 \cos 3)}{(e^2 z^2 - 2z \cos 3 + e^{-2})^2}\right]$$

$$= \frac{-z\left[\begin{array}{l}\left(2e^4 z^3 - e^2 z^2 \cos 3 - 4\,e^2 z^2 \cos 3 + 2z(\cos 3)^2 \right. \\ \left. + 2z - e^{-2} \cos 3\right) - \left(2e^4 z^3 - 2e^2 z^2 \cos 3 \right. \\ \left. - 2e^2 z^2 \cos 3 + 2z\,(\cos 3)^2\right)\end{array}\right]}{(e^2 z^2 - 2z \cos 3 + e^{-2})^2}$$

$$= \frac{-z[-e^2 z^2 \cos 3 + 2z - e^{-2} \cos 3]}{(e^2 z^2 - 2z \cos 3 + e^{-2})^2}$$

$$= \frac{z\,[(e^2 z^2 + e^{-2}) \cos 3 - 2z]}{(e^2 z^2 - 2z \cos 3 + e^{-2})^2}$$

Note : Here we may find $Z\{k \cos 3k\}$ first and then find $Z\,\{e^{-2k} (k \cos 3k)\}$, which is more convenient.

Example 4.31 : *Find the Z-transforms of the following*
$f(k) = k\,a^{k-1}\,U(k-1),\ k \geq 0$

Solution : We have,

$$f(k) = k\,a^{k-1}\,U(k-1),\ k \geq 0$$

Now,

$$Z\,\{U(k)\} = \frac{z}{z-1},\ |z| > 1$$

... standard result

$$\therefore \quad Z\{a^k\,U(k)\} = \frac{\frac{z}{a}}{\frac{z}{a} - 1},\ \left|\frac{z}{a}\right| > 1$$

... change of scale

$$= \frac{z}{z-a},\ |z| > |a|$$

$\therefore$ By shifting property,

$$Z\,\{a^{(k-1)}\,U(k-1)\} = z^{-1}\left(\frac{z}{z-a}\right)$$

$$\ldots \because Z\,\{f(k-n)\} = z^{-n}\,F(z),\ k \geq 0$$

$$= \frac{1}{z-a}$$

$$\therefore \quad Z\{k a^{k-1} U(k-1)\} = -z \frac{d}{dz}\left(\frac{1}{z-a}\right)$$

... Multiplication by k

$$= -z\left[-\frac{1}{(z-a)^2}\right]$$

$$= \frac{z}{(z-a)^2}$$

Example 4.32 : *Find* $Z\,\{f(k)\}$ *if*
$k^2\,a^{k-1}\,U(k-1),\ k \geq 0$

Solution : We have,

$$f(k) = k^2 a^{k-1}\,U(k-1),\ k \geq 0$$

Now, we have,

$$Z\,\{U(k)\} = \frac{z}{z-1},\ |z| > 1 \qquad \text{... Standard result}$$

Let, $\quad g(k) = a^k\,U(k)$

$$\therefore \quad Z\{g(k)\} = Z\,\{a^k\,U(k)\} = \frac{\frac{z}{a}}{\frac{z}{a} - 1}$$

... Change of scale i.e. $Z\,\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

where, $F\,(z) = Z\,\{f(k)\}$

i.e. $\quad G\,(z) = \dfrac{z}{z-a},\ \left|\dfrac{z}{a}\right| > 1$ i.e. $|z| > |a|$ ... (1)

Now, by the shifting property we have,
$Z\,\{f(k-n)\} = z^{-n}\,F(z)$ i.e. $z^{-n}\,Z\{f(k)\}$.

Hence,

$$Z\,\{a^{(k-1)} U\,(k-1)\} = Z\,\{g(k-1)\}$$

$$\ldots \because\ g(k) = a^k\,U(k)$$

$$= z^{-1}\, G(z) \quad \text{... using (1) ... (2)}$$

$$= z^{-1} \cdot \frac{z}{z-a} = \frac{1}{z-a}$$

$$\therefore Z\{k^2 a^{k-1} U(k-1)\} = \left(-z\frac{d}{dz}\right)^2 Z\{a^{k-1}\ U(k-1)\}$$

... multiplication by k^2

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\frac{1}{(z-a)}$$

... using (2)

$$= -z\frac{d}{dz}\left[(-z)\frac{-1}{(z-a)^2}\right]$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-a)^2}\right]$$

$$= -z\left[\frac{(z-a)^2\, 1 - z\cdot 2(z-a)}{(z-a)^4}\right]$$

$$= -z\left[\frac{(z-a)-2z}{(z-a)^3}\right]$$

$$\therefore Z\{k^2 a^{k-1} U(k-1)\} = \frac{z(z+a)}{(z-a)^3},\ |z| > |a|$$

Example 4.33 : *Find the Z-transform of the following*

i) $f(k) = \dfrac{\sin\alpha\, k}{k},\ k > 0$

SPPU : May-15

ii) $f(k) = \dfrac{5^k}{k},\ k \geq 1$

Solution : [Form is $Z\left\{\dfrac{f(k)}{k}\right\}$. Hence, use the property of division by k.]

i) We have, $f(k) = \dfrac{\sin\alpha k}{k},\ k > 0$ i.e. $k \geq 1$

Now, $$Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1},\ k \geq 0$$

... (1) ... Standard result

which is same for the case $k > 0$ i.e. $k \geq 1$ also.

Now, by property of division by k i.e.

$$Z\left\{\frac{f(k)}{k}\right\} = \int_z^\infty \frac{F(z)}{z}\, dz$$

where $F(z) = Z\{f(k)\}$

$$\therefore \quad Z\left\{\frac{\sin\alpha k}{k}\right\} = \int_z^\infty \frac{Z\{\sin\alpha k\}}{z}\, dz$$

$$= \int_z^\infty \frac{1}{z}\frac{z\sin\alpha}{\left(z^2 - 2z\cos\alpha + 1\right)}\, dz$$

... using (1)

$$= \sin\alpha \int_z^\infty \frac{1}{z^2 - 2z\cos\alpha + \left(\cos^2\alpha + \sin^2\alpha\right)}\, dz$$

... **Note this step**

$$= \sin\alpha \int_z^\infty \frac{1}{(z-\cos\alpha)^2 + \sin^2\alpha}\, dz$$

$$= \sin\alpha\left[\frac{1}{\sin\alpha}\tan^{-1}\left(\frac{z-\cos\alpha}{\sin\alpha}\right)\right]_z^\infty$$

$$\ldots \because \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{z-\cos\alpha}{\sin\alpha}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{z-\cos\alpha}{\sin\alpha}\right)$$

$$\therefore \quad Z\left\{\frac{\sin\alpha k}{k}\right\} = \cot^{-1}\left(\frac{z-\cos\alpha}{\sin\alpha}\right)$$

ii) We have, $f(k) = \dfrac{5^k}{k},\ k > 0$ i.e. $k \geq 1$

[Note that here we cannot use the Standard result for $Z\left\{a^k\right\}$, $k \geq 0$ as here $k \geq 1$ and $k \neq 0$]

$\therefore$ By definition,

$$Z\left\{5^k\right\} = \sum_{k=1}^{\infty} 5^k z^{-k}$$

$$= \sum_{k=1}^{\infty}\left(5z^{-1}\right)^k$$

... infinite G.P. with $a = \left(5z^{-1}\right) = 5z^{-1}$ and $r = 5z^{-1}$

$$= \frac{5z^{-1}}{1-5z^{-1}},\ \left|5z^{-1}\right| < 1 \text{ i.e. } \left|\frac{5}{z}\right| < 1$$

$$= \frac{\frac{5}{z}}{1-\frac{5}{z}} = \frac{5}{z-5},\ |z| > 5$$

... [Note : for $k \geq 0$, $Z\left\{5^k\right\} = \dfrac{z}{z-5}$]

$$\therefore \quad Z\left\{\frac{5^k}{k}\right\} = \int_z^\infty \frac{1}{z}\cdot\frac{5}{z-5}\, dz$$

$$\ldots \because Z\left\{\frac{f(k)}{k}\right\} = \int_z^\infty \frac{F(z)}{z}\, dz$$

$$= \int_z^\infty \frac{5}{z(z-5)}\, dz$$

$$= \int_z^\infty \left(\frac{1}{z-5} - \frac{1}{z}\right) dz$$

... Partial fractions

$$= [\log(z-5) - \log z]_z^\infty$$

$$= \left[\log \frac{z-5}{z}\right]_z^\infty$$

$$= \left[\log\left(1 - \frac{5}{z}\right)\right]_z^\infty$$

$$= \log(1-0) - \log\left(1 - \frac{5}{z}\right)$$

$$= 0 - \log\left(\frac{z-5}{z}\right)$$

$$= \log \frac{z}{z-5} \; |z| > 5$$

Example 4.34 : *Find the Z-transform of the following. Also, find the ROC :*

$$f(k) = \frac{2^k}{k}, \; k \geq 1$$

Solution : We have

$$f(k) = \frac{2^k}{k}, \; k \geq 1$$

Now, $Z\{2^k\} = \sum_{k=1}^{\infty} 2^k z^{-k}$ ... by definition

and note that $k \neq 0$

$$= \sum_1^\infty \left(\frac{2}{z}\right)^k = \frac{\frac{2}{z}}{1 - \frac{2}{z}}, \left|\frac{2}{z}\right| < 1$$

... sum of GP

$$= \frac{2}{z-2}, |z| > 2$$

$$\therefore \quad Z\left\{\frac{2^k}{k}\right\} = \int_z^\infty \frac{1}{z} Z\{2^k\}\, dz \quad \text{... Division by k}$$

$$= \int_z^\infty \frac{1}{z} \cdot \frac{2}{(z-2)}\, dz = \int_z^\infty \left(\frac{1}{z-2} - \frac{1}{z}\right) dz$$

$$= [\log(z-2) - \log z]_z^\infty = \left[\log \frac{z-2}{z}\right]_z^\infty$$

$$= \left[\log\left(1 - \frac{2}{z}\right)\right]_z^\infty$$

$$= \log(1-0) - \log\left(1 - \frac{2}{z}\right)$$

$$= -\log \frac{z-2}{z} = \log \frac{z}{z-2}, |z| > 2$$

Example 4.35 : *Find $Z\{f(k)\}$ if,*

i) $f(k) = \frac{a^k}{k}, \; k \geq 1$

ii) $f(k) = e^{-ak}, \; k \geq 0$

Solution : i) We have

$$f(k) = \frac{a^k}{k}, \; k \geq 1$$

Since, not specified, we assume $f(k) = 0, \; k \leq 0$

$$\therefore \quad Z\{f(k)\} = \sum_{k=1}^{\infty} \frac{a^k}{k} z^{-k}$$

$$= \sum_{k=1}^{\infty} \frac{(az^{-1})^k}{k}$$

$$= \frac{(az^{-1})}{1} + \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} + \ldots$$

$$= -\left[-(az^{-1}) - \frac{(az^{-1})^2}{2} - \frac{(az^{-1})^3}{3} \ldots\right]$$

... Note this step

$$= -\log(1 - a z^{-1})$$

... $\because$ Maclaurin's series :

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \ldots$$

This is a convergent series if $|az^{-1}| < 1$ i.e. $\left|\frac{a}{z}\right| < 1$

i.e. $|a| < |z|$ i.e. $|z| > |a|$ which is the ROC.

Alternately, we can use the property of 'Division by k' to find the Z-transform.

ii) We have, $f(k) = e^{-ak}, \; k \geq 0$

$$\therefore \quad Z\{f(k)\} = \sum_{k=0}^{\infty} e^{-ak} z^{-k}$$

$$= \sum_0^\infty (e^{-a} z^{-1})^k$$

which is an infinite G.P. with first term,

$$a = \left(e^{-a} z^{-1}\right)^0 = 1$$

and common ratio, $r = e^{-a} z^{-1}$

$$\therefore \quad Z\{f(k)\} = \frac{1}{1 - e^{-a} z^{-1}}, \left|e^{-a} z^{-1}\right| < 1$$

$$\ldots \because \text{sum of G.P., } S_\infty = \frac{a}{1-r}, |r| < 1$$

$$= \frac{1}{1 - \frac{e^{-a}}{z}}, \frac{\left|e^{-a}\right|}{|z|} < 1$$

$$= \frac{z}{z - e^{-a}}, |z| > \left|e^{-a}\right|$$

Example 4.36 : *Find the Z-transform of*

$$\frac{1 - \cos \alpha k}{k}, \; k \geq 0$$

Solution : We have

$$f(k) = \frac{1 - \cos \alpha k}{k}, \; k \geq 0$$

Now,

$$Z\{1 - \cos \alpha k\} = Z\{1\} - Z\{\cos \alpha k\}, \; k \geq 0$$

$$= \frac{z}{z-1} - \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1},$$

$$|z| > 1$$

$$\therefore \quad Z\left\{\frac{1 - \cos \alpha k}{k}\right\} = \int_z^\infty \frac{1}{z} \cdot \left[\frac{z}{z-1} - \frac{z(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1}\right] dz$$

... division by k

$$= \int_z^\infty \left[\frac{1}{z-1} - \frac{(z - \cos \alpha)}{z^2 - 2z \cos \alpha + 1}\right] dz$$

$$= \int_z^\infty \left[\frac{1}{z-1} - \frac{1}{2} \cdot \frac{(2z - 2\cos \alpha)}{(z^2 - 2z \cos \alpha + 1)}\right] dz$$

... Note this step

$$= \left[\log(z-1) - \frac{1}{2} \log(z^2 - 2z \cos \alpha + 1)\right]_z^\infty$$

$$= \left[\log \frac{z-1}{\sqrt{z^2 - 2z \cos \alpha + 1}}\right]_z^\infty$$

$$= \left[\log \frac{1 - \frac{1}{z}}{\sqrt{\frac{z^2 - 2z \cos \alpha + 1}{z^2}}}\right]_z^\infty$$

$$= \left[\log \frac{1 - \frac{1}{z}}{\sqrt{1 - \frac{2}{z} \cos \alpha + \frac{1}{z^2}}}\right]_z^\infty$$

$$= \log \frac{1-0}{\sqrt{1 - 0 + 0}} - \log \frac{1 - \frac{1}{z}}{\sqrt{1 - \frac{2}{z} \cos \alpha + \frac{1}{z^2}}}$$

$$= 0 - \log \frac{z-1}{\sqrt{z^2 - 2z \cos \alpha + 1}}$$

$$= \log \left[\frac{\sqrt{z^2 - 2z \cos \alpha + 1}}{z-1}\right], \quad |z| > 1$$

Example 4.37 : *Find the Z-transform of the following :*

i) $f(k) = 2^k e^{-4k}$

ii) $f(k) = \dfrac{e^{-k} - e^{-2k}}{k}, \quad k \geq 0$

Solution :

i) We have $Z\{e^{-4k}\} = \dfrac{z}{z - e^{-4}}, \quad |z| > 2e^{-4}$

and $\quad Z\{2^k e^{-4k}\} = \dfrac{z}{z/2 - e^4}$

$$= \frac{z}{z - 2e^{-4}} + |z| > 2e^{-4}$$

ii) We have

$$Z\{e^{-4k}\} = \frac{z}{z - e^{-1}} \text{ and } z\{e^{-2k}\} \frac{z}{z - e^{-2}}$$

$$\therefore \quad Z\{e^{-k} - e^{-2k}\} = \frac{z}{z - e^{-1}} - \frac{z}{z - e^{-2}}$$

Therefore,

$$Z\left\{\frac{e^{-k} - e^{-2k}}{k}\right\} = \int_z^\infty \frac{1}{z}\left\{\frac{z}{z - e^{-1}} - \frac{z}{z - e^{-2}}\right\} dz$$

$$= \int_z^\infty \left\{\frac{z}{z - e^{-1}} - \frac{z}{z - e^{-2}}\right\} dz$$

$$= \{\log(z - e^{-1}) - \log(z - e^{-2})\}_z^\infty$$

$$= \log(z - e^{-2}) - \log(z - e^{-1})$$

$$= \log\left(\frac{z - e^{-2}}{z - e^{-1}}\right)$$

Example 4.38 : *Show that* $Z\left\{\frac{1}{k!}\right\} = e^{1/z}$.

Hence, evaluate $Z\left\{\frac{1}{(k+1)!}\right\}$

Solution : Let

$$f(k) = \frac{1}{k!}$$

$\therefore$ we have by definition,

$$F(z) = Z\{f(k)\}$$

$$= Z\left\{\frac{1}{k!}\right\} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k}$$

... $k \geq 0$ (for k! to exist)

$$= 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \ldots$$

$$= e^{(z^{-1})} \quad \ldots \because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\therefore \quad F(z) = Z\left\{\frac{1}{k!}\right\} = e^{1/z}$$

Now, we have the shifting property (for a causal sequence, $k \geq 0$)

$$Z\{f(k+n)\} = z^n F(z) - \sum_{r=0}^{n-1} f(r) z^{n-r}$$

In particular, if n = 2, we get

$$Z\{f(k+2)\} = z^2 F(z) - \sum_{r=0}^{1} f(r) z^{2-r}$$

$$= z^2 F(z) - z^2 f(0) - z f(1) \quad \ldots (1)$$

Now, $\because \quad f(k) = \frac{1}{k!}$

$\therefore \quad f(k+2) = \frac{1}{(k+2)!}$ and $F(z) = e^{1/z}$

$\therefore$ from (1)

$$Z\left\{\frac{1}{(k+2)!}\right\} = z^2 e^{1/z} - z^2\left(\frac{1}{0!}\right) - z\left(\frac{1}{1!}\right)$$

$$= z^2 e^{1/z} - z^2 - z$$

Example 4.39 : *Solve the following :*

i) Find $Z\left\{\cos\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\}, k \leq 0$ **May-18, Marks 4**

Solution : We have

$$Z\left\{\cos\left(\frac{k\pi}{2} + \frac{\pi}{4}\right)\right\} = z\left\{\cos\frac{k\pi}{2}\cos\frac{\pi}{4} - \sin\frac{k\pi}{2}\sin\frac{\pi}{4}\right\}$$

$$= z\left\{\frac{1}{\sqrt{2}}\left(\cos\frac{k\pi}{2} - \sin\frac{k\pi}{2}\right)\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{z\left(z - \cos\frac{\pi}{2}\right)}{z^2 - 2z\cos\frac{\pi}{2} + 1} - \frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1}\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right\} = \frac{1}{\sqrt{2}}\frac{z(z-1)}{z^2+1}$$

Example 4.40 : *Find* $Z\{f(k)\}$ *for the following sequences :*

i) $f(k) = \delta(k+1)$

ii) $f(k) = \delta(k-2)$

Solution : i) We have,

$$f(k) = \delta(k+1)$$

Now, $\quad Z\{\delta(k)\} = 1 \quad$ ... Standard result

$\therefore$ By shifting (Right) property for both sided sequence i.e.

$$Z\{f(k+n)\} = z^n F(z) \text{ we have,}$$

$$Z\{\delta(k+1)\} = z^1 Z\{\delta(k)\} = z(1) = z$$

ii) Also, by the shifting (left) property, for both sided sequence,

i.e. $\quad Z\{f(k-n)\} = z^{-n} F(z)$, we get

$$Z\{\delta(k-2)\} = z^{-2} Z\{\delta(k)\}$$

$$= \frac{1}{z^2} \cdot (1) = \frac{1}{z^2}$$

Example 4.41 : *If* $F(z) = \frac{z}{(z-1)^2}$, *evaluate* $f(0)$ *and* $\lim_{k \to \infty} f(k)$.

Solution : We have,

$$F(z) = \frac{z}{(z-1)^2}$$

$\therefore$ By initial value theorem we have,

$$f(0) = \lim_{z \to \infty} F(z)$$

$$= \lim_{z \to \infty} \frac{z}{(z-1)^2}$$

$$= \lim_{z \to \infty} \frac{\frac{z}{z^2}}{\frac{(z-1)^2}{z^2}}$$

$$= \lim_{z \to \infty} \frac{\frac{1}{z}}{\left(1 - \frac{1}{z}\right)^2} = \frac{0}{(1-0)^2}$$

$\therefore \quad f(0) = 0$

Now, by final value theorem we have,

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1)\, F(z)$$

$$= \lim_{z \to 1} (z-1) \frac{z}{(z-1)^2}$$

$$= \lim_{z \to 1} \frac{z}{(z-1)}$$

$$= \frac{1}{(0)} = \infty$$

Example 4.42 : *Verify Convolution Theorem for $f_1(k) = k$ and $f_2(k) = k$.*

Solution : We have, $f_1(k) = k$, $f_2(k) = k$ and we assume, $k \geq 0$.

Now, we have to verify the Convolution Theorem of Z-transforms

i.e. $Z\left[\{f_1(k)\} * \{f_2(k)\}\right] = F_1(z) \cdot F_2(z)$ ... (1)

where, $F_1(z) = Z\{f_1(k)\}$, $F_2(z) = Z\{f_2(k)\}$

Now, $\because$ $Z\{f_1(k)\} = Z\{k\}$

$$= Z\{k \cdot 1\} = -z \frac{d}{dz} Z\{1\}$$

... Multiplication by k

$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

... $\because Z\{1\} = Z\{1^k\} = \dfrac{z}{z-1}$

$$= -z \left[\frac{(z-1)1 - z(1)}{(z-1)^2}\right]$$

$$= \frac{z}{(z-1)^2}$$

$\therefore \quad F_1(z) = Z\{f_1(k)\} = \dfrac{z}{(z-1)^2}$

Similarly, $F_2(z) = Z\{f_2(k)\}$

$$= Z\{k\} = \frac{z}{(z-1)^2}$$

$\therefore \quad \text{RHS} = F_1(z) \cdot F_2(z)$

$$= \frac{z}{(z-1)^2} \cdot \frac{z}{(z-1)^2}$$

$$= \frac{z^2}{(z-1)^4} \quad \text{... (2)}$$

Now, by definition of convolution,

$$\{f_1(k)\} * \{f_2(k)\} = \sum_{m=0}^{\infty} f_1(m)\, f_2(k-m)$$

... $\because k \geq 0$

$$= \sum_{m=0}^{\infty} m(k-m)$$

... $\because f_1(k) = k,\ f_2(k) = k$

$$= k \sum_{m=0}^{\infty} m - \sum_{m=0}^{\infty} m^2$$

... $\because$ k and m are independent integers

$$= k \sum_{k=0}^{\infty} k - \sum_{k=0}^{\infty} k^2$$

... $\because$ Replacing m by k in the summation, both being integers such that, $m, k \geq 0$

$$= k\,\frac{k(k+1)}{2} - \frac{k(k+1)(2k+1)}{6}$$

... $\because$ Standard summations

$$= \frac{k(k+1)}{6}\left[3k - (2k+1)\right]$$

$$= \frac{k}{6}(k+1)(k-1)$$

$$\therefore \quad \{f_1(k)\} * \{f_2(k)\} = \frac{k}{6}(k^2 - 1)$$

$\therefore$ Taking the Z-transform, we get

$$\text{LHS} = Z\left[\{f_1(k)\} * \{f_2(k)\}\right]$$

$$= Z\left\{\frac{k}{6}(k^2-1)\right\}$$

$$= \frac{1}{6}\left[Z\{k^3\} - Z\{k\}\right]$$

$$= \frac{1}{6}\left[Z\{k^2 \cdot k\} - Z\{k\}\right]$$

$$= \frac{1}{6}\left[\left(-z\frac{d}{dz}\right)^2 Z\{k\} - Z\{k\}\right]$$

... Multiplication by k^2

$$= \frac{1}{6}\left[\left(-z\frac{d}{dz}\right)^2 \frac{z}{(z-1)^2} - \frac{z}{(z-1)^2}\right]$$

... (3) $\because Z\{k\} = \dfrac{z}{(z-1)^2}$

Now,

$$\left(-z\frac{d}{dz}\right)^2 \frac{z}{(z-1)^2} = \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\frac{z}{(z-1)^2}$$

$$= \left(-z\frac{d}{dz}\right)\left[(-z)\frac{(z-1)^2\cdot 1 - z\cdot 2(z-1)}{(z-1)^4}\right]$$

$$= \left(-z\frac{d}{dz}\right)\left[(-z)\frac{(z-1)-2z}{(z-1)^3}\right] = -z\frac{d}{dz}\left[\frac{z(z+1)}{(z-1)^3}\right]$$

$$= -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$= -z\left[\frac{(z-1)^3(2z+1)-(z^2+z)\cdot 3(z-1)^2}{(z-1)^6}\right]$$

$$= -z\left[\frac{(z-1)(2z+1)-3(z^2+z)}{(z-1)^4}\right]$$

$$= -z\left[\frac{2z^2-z-1-3z^2-3z}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2-4z-1}{(z-1)^4}\right] = \frac{z(z^2+4z+1)}{(z-1)^4}$$

$\therefore$ Substituting in (3) we get,

$$\text{LHS} = \frac{1}{6}\left[\frac{z(z^2+4z+1)}{(z-1)^4} - \frac{z}{(z-1)^2}\right]$$

$$= \frac{1}{6}\cdot\frac{z}{(z-1)^2}\left[\frac{z^2+4z+1}{(z-1)^2} - 1\right]$$

$$= \frac{z}{6(z-1)^2}\left[\frac{(z^2+4z+1)-(z-1)^2}{(z-1)^2}\right]$$

$$= \frac{z}{6(z-1)^2}\left[\frac{(z^2+4z+1)-(z^2-2z+1)}{(z-1)^2}\right]$$

$$= \frac{z}{6(z-1)^2}\left[\frac{6z}{(z-1)^2}\right]$$

$$\therefore \quad \text{L.H.S.} = \frac{z^2}{(z-1)^4} \quad \dots (4)$$

$\therefore$ Comparing (2) and (4) we have

L.H.S. = R.H.S. i.e. the Convolution Theorem is verified.

Example 4.43 : *Find the Z-transform of $\{x_k\}$ if,*

$$x_k = \frac{1}{1^k} * \frac{1}{2^k} * \frac{1}{3^k},\ k \geq 0$$

Solution : We have,

$$x_k = \frac{1}{1^k} * \frac{1}{2^k} * \frac{1}{3^k},\ k \geq 0$$

Let $\quad f(k) = \frac{1}{1^k},\ g(k) = \frac{1}{2^k},\ h(k) = \frac{1}{3^k}$

$\therefore \quad x_k = f(k) * g(k) * h(k)$

$\therefore$ By Convolution Theorem,

$$Z\{x_k\} = Z\{f(k) * g(k) * h(k)\}$$

$$= Z\{f(k)\}\cdot Z\{g(k)\}\cdot Z\{h(k)\} \quad \dots (1)$$

Now,

$$Z\{f(k)\} = Z\left\{\frac{1}{1^k}\right\} = Z\left\{\left(\frac{1}{1}\right)^k\right\}$$

$$= Z\left\{(1)^k\right\} = \frac{z}{z-1},\ |z| > 1$$

$$\dots \because Z\left\{a^k\right\} = \frac{z}{z-a}$$

$$Z\{g(k)\} = Z\left\{\frac{1}{2^k}\right\} = Z\left\{\left(\frac{1}{2}\right)^k\right\}$$

$$= \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1},\ |z| > \frac{1}{2}$$

and

$$Z\{h(k)\} = Z\left\{\frac{1}{3^k}\right\} = Z\left\{\left(\frac{1}{3}\right)^k\right\}$$

$$= \frac{z}{z-\frac{1}{3}} = \frac{3z}{3z-1},\ |z| > \frac{1}{3}$$

$\therefore$ Substituting in (1) we get

$$Z\{x_k\} = \left(\frac{z}{z-1}\right)\cdot\left(\frac{2z}{2z-1}\right)\cdot\left(\frac{3z}{3z-1}\right)$$

Common ROC : $|z| > 1$

Example 4.44 : *Find the Z-transform of the following by showing the region of convergence* $x_k = \frac{1}{3^k}\cdot\frac{1}{2^k}\cdot\frac{1}{4^k},\ k \geq 0.$

Solution : We have,

$$x_k = \frac{1}{3^k}\cdot\frac{1}{2^k}\cdot\frac{1}{4^k},\ k \geq 0$$

$$= \left(\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{4}\right)^k = \left(\frac{1}{24}\right)^k,\ k \geq 0$$

$$\therefore \quad Z\{x_k\} = Z\left\{\left(\frac{1}{24}\right)^k\right\},\ k \geq 0$$

$$= \frac{z}{z-\frac{1}{24}},\ |z| > \left|\frac{1}{24}\right|$$

$$= \frac{24z}{24z-1},\ |z| > \frac{1}{24}$$

Example 4.45 : *Find the Z-transform of*

$$\sum_{m=0}^{k} \frac{a^m}{(m-2)!}$$

Solution : We have to find the Z-transform of the Partial sum given by $\sum_{m=-\infty}^{k} f(m)$. Hence, we use another property of Z-transforms given as,

$$Z\left\{\sum_{m=-\infty}^{k} f(m)\right\} = \frac{F(z)}{1-z^{-1}},$$

where $F(z) = Z\{f(k)\}$

By using the final value theorem we also get the result,

$$\sum_{m=-\infty}^{\infty} f(m) = F(1)$$

If, $\{f(k)\}$ is causal so that $k \geq 0$, then

$$\left.\begin{aligned} Z\left\{\sum_{m=0}^{k} f(m)\right\} &= \frac{F(z)}{1-z^{-1}} \\ \text{and} \quad \sum_{m=0}^{\infty} f(m) &= F(1) \end{aligned}\right]$$

Now here we have,

$$f(m) = \frac{a^m}{(m-2)!}$$

$$\therefore \quad f(k) = \frac{a^k}{(k-2)!}$$

Now, we have,

$$Z\left\{\frac{a^k}{k!}\right\} = e^{a/z},\ k \geq 0 \qquad \text{... Standard result}$$

$\therefore$ By shifting property

$$Z\left\{\frac{a^{k-2}}{(k-2)!}\right\} = z^{-2}\, Z\left\{\frac{a^k}{k!}\right\}$$

$$\text{... } \because Z\{f(k-n)\} = z^{-n} F(z)$$

$$\therefore \quad Z\left\{\frac{a^k \cdot a^{-2}}{(k-2)!}\right\} = \frac{1}{z^2} e^{a/z}$$

i.e.

$$\frac{1}{a^2} Z\left\{\frac{a^k}{(k-2)!}\right\} = \frac{1}{z^2} e^{a/z}$$

$$\therefore \quad Z\left\{\frac{a^k}{(k-2)!}\right\} = \frac{a^2}{z^2} e^{a/z}$$

$$\therefore \quad F(z) = Z\{f(k)\} = \frac{a^2}{z^2} e^{a/z}$$

$$\text{... } \because f(k) = \frac{a^k}{(k-2)!}$$

$\therefore$ By the property of Partial sum we have

$$\therefore \quad Z\left\{\sum_{m=0}^{k} f(m)\right\} = \frac{F(z)}{1-z^{-1}}$$

$$\text{i.e.} \quad Z\left\{\sum_{m=0}^{k} \frac{a^m}{(m-2)!}\right\} = \frac{1}{(1-z^{-1})}\frac{a^2}{z^2} e^{a/z}$$

$$= \frac{z}{z-1}\cdot\frac{a^2}{z^2} e^{a/z} = \frac{a^2}{z(z-1)} e^{a/z}$$

4.6 Inverse Z-transform

If $Z[f(k)] = F(z)$ then

$Z^{-1}F(z) = [f(k)]$ determines the sequence $\{f(k)\}$ which generates the given Z-transform is known as inverse Z-transform.

For example $\quad Z\{a^k\}\ k \geq 0 = \dfrac{z}{z-a}$ for $|z| > |a|$

$$\therefore \quad z^{-1}\left\{\frac{z}{z-a}\right\}_{|z|>|a|} = a^k \text{ for } k \geq 0.$$

Using standard results of Z-transform we can form the following table of inverse Z-transforms.

Table of Inverse Z-transform

No.	F (z)	f (k) = Z^{-1} F (z) ($\lvert z\rvert > \lvert a\rvert$, $k \geq 0$)	f (k) = Z^{-1} F (z) ($\lvert z\rvert < \lvert a\rvert$, $k < 0$)
1.	$\frac{z}{z-a}$	a^k	$-a^k$
2.	$\frac{z}{(z-a)^2}$	ka^{k-1}	$-ka^{k-1}$

3.	$\frac{z^2}{(z-a)^2}$	$(k+1)a^k$	$-(k+1)a^k$
4.	$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(k+1)(k+2)a^k$	$-\frac{1}{2!}(k+1)(k+2)a^k$
5.	In general, $\frac{z^n}{(z-a)^n}$	$\frac{1}{(n-1)!}\begin{bmatrix}(k+1)(k+2)\\ \dots(k+(n-1))\end{bmatrix}a^k$	$-\frac{1}{(n-1)!}\begin{bmatrix}(k+1)(k+2)\\ \dots(k+(n-1))\end{bmatrix}a^k$
6.	$\frac{1}{z-a}$	$a^{k-1}U(k-1)$	$-a^{k-1}U(-k)$
7.	$\frac{1}{(z-a)^2}$	$(k-1)a^{k-2}U(k-2)$	$-(k-1)a^{k-2}U(-k+1)$
8.	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(k-2)(k-1)a^{k-3}U(k-3)$	$-\frac{1}{2}(k-2)(k-1)a^{k-3}U(-k+2)$
9.	$\frac{z}{z-1}$	U (k)	
10.	$\frac{z\sin\alpha}{z^2-2z\cos\alpha+1}$ $\lvert z\rvert > 1$	$\sin\alpha k$	
11.	$\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$ $\lvert z\rvert > 1$	$\cos\alpha k$	

Methods for finding inverse Z-transform :

Type 1 : Partial fraction method.

Type 1 (a) : If linear factorisation of denominator is possible (real factors only).

Step 1) Find $\frac{F(z)}{z}$

Step 2) Find its partial fractions in usual manner.

Step 3) Rewrite F(z) by multiplying throughout by z.

Step 4) Take inverse Z-transform of each term to get {f(k)}.

Note :

1) $\frac{F(z)}{z}$ here must be a proper fraction (i.e. the degree of the denominator must be greater than that of the numerator). If not so, carry out actual division to get

$$\frac{F(z)}{z} = Q(z) + \frac{R(z)}{P(z)},$$

where $\frac{R(z)}{P(z)}$ is a proper fraction, and then proceed.

2) If the factors of the denominator of the expression $\frac{F(z)}{z}$ are linear and not-repeated we get

$$\frac{F(z)}{z} = \frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c} + \dots \text{ etc}$$

$$\therefore \quad F(z) = A\left(\frac{z}{z-a}\right) + B\left(\frac{z}{z-b}\right) + C\left(\frac{z}{z-c}\right) + \dots \text{ etc. so that}$$

$$\{f(k)\} = Z^{-1}F(z) = A\,Z^{-1}\left(\frac{z}{z-a}\right) + B\,Z^{-1}\left(\frac{z}{z-b}\right) + C\,Z^{-1}\left(\frac{z}{z-c}\right) + \dots$$

$$\therefore \quad \{f(k)\} = A\{a^k\} + B\{b^k\} + C\{c^k\} + \dots,\ k \geq 0$$

if $|z| > |a|, |z| > |b|, |z| > |c|$ ... etc.

And,

$$\{f(k)\} = A\{-a^k\} + B\{-b^k\} + C\{-c^k\} + \dots,\ k < 0$$

if $|z| < |a|, |z| < |b|, |z| < |c|$ ... etc.

3) For Linear and repeated factors, we find the partial fractions as usual (as in case of Inverse Laplace Transforms) and proceed.

Illustrations :

Example 4.46 : *Show that*

$$Z^{-1}\left[\frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}\right] = \{x_k\} \text{ for } |z| > \frac{1}{2} \text{ where}$$

$$x_k = 6\left[\left(\frac{1}{2}\right)^{k-1} - \left(\frac{1}{3}\right)^{k-1}\right], \ k \geq 1$$

Solution : We have

$$X(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \ |z| > \frac{1}{2}$$

$$= \frac{6}{z-\frac{1}{2}} - \frac{6}{z-\frac{1}{3}}$$

$$\text{... Partial Fractions} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{3}}$$

where $A = \dfrac{1}{\frac{1}{2}-\frac{1}{3}} = 6$, $B = \dfrac{1}{\frac{1}{3}-\frac{1}{2}} = -6$

$$= \frac{6}{z\left(1-\frac{1}{2z}\right)} - \frac{6}{z\left(1-\frac{1}{3z}\right)} \qquad \text{...} \because |z| > \frac{1}{2} \therefore |z| > \frac{1}{3}$$

$$\therefore \left|\frac{1}{2z}\right| < 1 \text{ and } \left|\frac{1}{3z}\right| < 1$$

$$= \frac{6}{z}\left(1-\frac{1}{2z}\right)^{-1} - \frac{6}{z}\left(1-\frac{1}{3z}\right)^{-1}$$

$$= \frac{6}{z}\left[1+\left(\frac{1}{2z}\right)+\left(\frac{1}{2z}\right)^2+\ldots\right] - \frac{6}{z}\left[1+\left(\frac{1}{3z}\right)+\left(\frac{1}{3z}\right)^2+\ldots\right]$$

$$= 6\left[\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{2^2z^3}+\ldots\right] - 6\left[\frac{1}{z}+\frac{1}{3z^2}+\frac{1}{3^2z^3}+\ldots\right]$$

$$= 6\sum_{k=1}^{\infty}\frac{1}{2^{k-1}z^k} - 6\sum_{k=1}^{\infty}\frac{1}{3^{k-1}z^k} \quad \text{... General terms}$$

$$= 6\sum_{k=1}^{\infty}\left[\left(\frac{1}{2}\right)^{k-1}z^{-k} - \left(\frac{1}{3}\right)^{k-1}z^{-k}\right]$$

$$= \sum_{k=1}^{\infty} 6\left[\left(\frac{1}{2}\right)^{k-1} - \left(\frac{1}{3}\right)^{k-1}\right]z^{-k} \qquad \text{... (1)}$$

$$\therefore \quad Z^{-1}X(z) = \{x_k\} \text{ where}$$

$$\{x_k\} = \text{coeff. of } z^{-k} \text{ in (1)}$$

$$= 6\left[\left(\frac{1}{2}\right)^{k-1} - \left(\frac{1}{3}\right)^{k-1}\right]$$

$$k \geq 1 \text{ for } |z| > \frac{1}{2}$$

Example 4.47 : *Find*

$$Z^{-1}\left[\frac{1}{(z-1)(z-2)}\right], \ 1 < |z| < 2$$

Solution : We have,

$$F(z) = \frac{1}{(z-1)(z-2)}$$

$$= -\left[\frac{1}{z-1} - \frac{1}{z-2}\right]$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

... Direct partial fractions

$$\therefore \quad F(z) = \frac{1}{z-2} - \frac{1}{z-1}, \ 1 < |z| < 2$$

$$= \frac{1}{2\left(\frac{z}{2}-1\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} \quad \because 1 < |z| \therefore \left|\frac{1}{z}\right| < 1$$

$$\text{and } \because |z| < 2 \therefore \left|\frac{z}{2}\right| < 1$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots\right]$$

$$-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\ldots\right]$$

$$= -\left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\ldots\right]$$

$$-\left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right]$$

$$F(z) = -\sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}} - \sum_{k=1}^{\infty}\frac{1}{z^k}$$

$\therefore$ coeff. of z^k in 1^{st} series $= \dfrac{1}{2^{k+1}}$, $k \geq 0$

$\therefore f_1(k) =$ coeff. of z^{-k} in 1^{st} series $= \dfrac{1}{2^{-k+1}} = 2^{k-1}$,

$-k \geq 0$ i.e. $k \leq 0$

and $f_2(k)$ = coeff. of z^{-k} i.e. $\frac{1}{z^k}$ in 2^{nd} series = 1,

$k \geq 1$

$\therefore$ Combining we get

$$f(k) = -f_1(k) - f_2(k)$$
$$= -2^{k-1} - 1$$
$$(k \leq 0) \quad (k \geq 1)$$

$$\therefore Z^{-1}\left[\frac{1}{(z-1)(z-2)}\right] = \{f(k)\} \text{ where,}$$

$$f(k) = -2^{k-1}, k \leq 0$$
$$= -1, k \geq 1$$

Alternately, we can use the Standard result,

$$Z^{-1}\left(\frac{1}{z-a}\right) = -a^{k-1} U(-k), \text{ for } |z| < |a|$$
$$= a^{k-1} U(k-1), \text{ for } |z| > |a|$$

Thus, we have,

$$\because \quad F(z) = \frac{1}{z-2} - \frac{1}{z-1}, 1 < |z| < 2$$

$$\therefore Z^{-1} F(z) = Z^{-1}\left[\frac{1}{z-2}\right] - Z^{-1}\left[\frac{1}{z-1}\right], \quad |z| < |2|,$$

$$|z| > |1|$$
$$= -2^{k-1} U(-k) - 1^{k-1} U(k-1)$$
$$= \text{say } f_1(k) + f_2(k) \quad \text{... (1)}$$

Now, by definition of unit step function

$$U(t) = 0, \; t < 0$$
$$= 1, \; t \geq 0$$

$$\therefore \quad U(-k) = 0, \; -k < 0 \quad \text{i.e. } k > 0 \text{ i.e. } k \geq 1$$
$$= 1, \; -k \geq 0 \quad \text{i.e. } k \leq 0 \text{ and}$$
$$U(k-1) = 0, \; k-1 < 0 \text{ i.e. } k < 1 \text{ i.e. } k \leq 0$$
$$= 1, \; k-1 \geq 0 \text{ i.e. } k \geq 1$$

$$\therefore \quad f_1(k) = -2^{k-1} U(-k) = 0, \quad k \geq 1$$
$$= -2^{k-1}, \quad k \leq 0$$

$$\text{and} \quad f_2(k) = -1^{k-1} U(k-1) = 0, \quad k \leq 0$$
$$= -1^{k-1}, \quad k \geq 1$$

$\therefore$ from (1) we have,

$$Z^{-1} F(z) = \{f(k)\}$$

where $$f(k) = -2^{k-1} + 0 = -2^{k-1}, \; k \leq 0$$
$$= 0 - 1 = -1, \; k \geq 1$$

Example 4.48 : *Find inverse Z-transform of* $F(z) = \frac{1}{z^2 - z - 6}; \; |z| > 3$

Solution : By partial fractions

$$F(z) = \frac{\frac{1}{5}}{z-3} - \frac{\frac{1}{5}}{z+2}; \; |z| > 3$$
$$= \frac{1}{5} \frac{1}{z\left(1-\frac{3}{z}\right)} - \frac{1}{5} \frac{1}{z\left(1+\frac{2}{z}\right)},$$

$$1 > \left|\frac{3}{z}\right| \text{ and } 1 > \left|\frac{2}{z}\right|$$

$$= \frac{1}{5z}\left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \ldots\right]$$
$$- \frac{1}{5z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \ldots\right]$$

$$F(z) = \frac{1}{5}\left[\frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \ldots\right]$$
$$- \frac{1}{5}\left[\frac{1}{z} - \frac{2}{z^2} + \frac{2^2}{z^3} + \ldots\right]$$

$\therefore$ The coefficient of z^{-k} in first series is $= \frac{1}{5} 3^{k-1}$

and the coefficient of z^{-k} in second series is $= -\frac{1}{5}(-1)^{k+1} 2^{k-1}$

$$f(k) = \frac{1}{5}\left(3^{k-1} - (-1)^{k+1} \; 2^{k-1}\right)$$

Example 4.49 : *Find the inverse Z-transform of*

$$F(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \; \frac{1}{3} < |z| < \frac{1}{2}$$

Solution : We have,

$$F(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}, \; \frac{1}{3} < |z| < \frac{1}{2}$$
$$= \frac{6}{\left(z-\frac{1}{2}\right)} + \frac{(-6)}{\left(z-\frac{1}{3}\right)}$$

... Partial fractions : $A = \frac{1}{\frac{1}{2} - \frac{1}{3}} = 6, \; B = \frac{1}{\frac{1}{3} - \frac{1}{2}} = -6$

$$= \frac{6}{\left(-\frac{1}{2}\right)(-2z+1)} - \frac{6}{z\left(1-\frac{1}{3z}\right)}, |z| > \frac{1}{3}$$

$\therefore \left|\frac{1}{3z}\right| < 1,\ |z| < \frac{1}{2}\ \therefore |2z| < 1$

$$= -12\,(1-2z)^{-1} - \frac{6}{z}\left(1-\frac{1}{3z}\right)^{-1},$$

$$|2z| < 1, \left|\frac{1}{3z}\right| < 1$$

$$= -(12)\left[1+2z+(2z)^2+(2z)^3+\dots\right]$$

$$-\frac{6}{z}\left[1+\left(\frac{1}{3z}\right)+\left(\frac{1}{3z}\right)^2+\dots\right]$$

$$= -12\sum_{k=0}^{\infty} 2^k z^k - \frac{6}{z}\sum_{k=0}^{\infty}\frac{1}{3^k z^k}$$

$$\therefore \quad F(z) = -12\sum_{k=0}^{\infty} 2^k z^k - 6\sum_{k=0}^{\infty}\frac{1}{3^k z^{k+1}} \quad \dots (1)$$

$\therefore$ coeff. of z^k in 1^{st} series $= 2^k,\ k \ge 0$

$\therefore$ $f_1(k)$ = coeff. of z^{-k} in 1^{st} series $= 2^{-k},\ -k \ge 0$
i.e. $k \le 0$ and coeff. of $z^{-(k+1)}$ i.e. $\frac{1}{z^{k+1}}$ in 2^{nd} series $= 3^{-k},\ k \ge 0$

Put $k + 1 = n \therefore k = n - 1$

$\therefore$ coeff. of z^{-n} in 2^{nd} series $= 3^{-(n-1)}$,
$n - 1 \ge 0$ i.e. $n \ge 1$

$\therefore$ $f_2(k)$ = coeff. of z^{-k} in 2^{nd} series $= 3^{1-k},\ k \ge 1$

... replacing n by k, both being integers

$\therefore$ Combining we have from (1)

$$\{f(k)\} = Z^{-1}\,F(z)$$

$$= -12\{f_1(k)\}_{k\le 0} - 6\,\{f_2(k)\}_{k\ge 1}$$

$$= -12\left\{2^{-k}\right\}_{k\le 0} - 6\left\{3^{1-k}\right\}_{k\ge 1}$$

i.e. $\quad f(k) = -12\cdot 2^{-k},\ k \le 0$

$$= -6\cdot 3^{1-k} = -18\cdot 3^{-k},\ k \ge 1$$

Alternately, we have,

$$F(z) = \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{6}{\left(z-\frac{1}{2}\right)} - \frac{6}{\left(z-\frac{1}{3}\right)}, \frac{1}{3} < |z| < \frac{1}{2}$$

$$\therefore \quad \{f(k)\} = 6\,Z^{-1}\left(\frac{1}{z-\frac{1}{2}}\right) - 6\,Z^{-1}\left(\frac{1}{z-\frac{1}{3}}\right),$$

$$|z| < \frac{1}{2}, |z| > \frac{1}{3}$$

$$= 6\left\{-\left(\frac{1}{2}\right)^{k-1} U(-k)\right\}$$

$$- 6\left\{\left(\frac{1}{3}\right)^{k-1} U(k-1)\right\}$$

$$\dots \because Z^{-1}\left(\frac{1}{z-a}\right) = a^{k-1}\,U(k-1), |z| > |a|$$

$$= -a^{k-1}\,U(-k), |z| < |a|$$

$$= \left\{-6\left(\frac{1}{2}\right)^{k-1} U(-k) - 6\left(\frac{1}{3}\right)^{k-1} U(k-1)\right\}$$

Now, $\left(\frac{1}{2}\right)^{k-1} U(-k) = 0,\ -k < 0$ i.e. $k > 0$ i.e. $k \ge 1$

$$= \left(\frac{1}{2}\right)^{k-1}(1),\ -k \ge 0 \text{ i.e. } k \le 0$$

... by using definition of unit step function.

and $\left(\frac{1}{3}\right)^{k-1} U(k-1) = 0,\ k-1 < 0$ i.e. $k < 1$

i.e. $k \le 0$

$$= \left(\frac{1}{3}\right)^{k-1}(1),\ k-1 \ge 0$$

i.e. $k \ge 1$

$$\therefore \quad f(k) = -6\left(\frac{1}{2}\right)^{k-1} + 0$$

$$= -12\,(2)^{-k},\ k \le 0$$

$$= 0 - 6\left(\frac{1}{3}\right)^{k-1} - 18(3)^{-k},\ k \ge 1$$

Example 4.50 : *Find the inverse Z-transform of* , $\frac{1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}, |z| < \frac{1}{5}$

Solution : We have,

$$F(z) = \frac{1}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}, |z| < \frac{1}{5}$$

[... Numerator is constant $\therefore$ Use Power series Method]

$$= \frac{20}{z-\frac{1}{4}} - \frac{20}{z-\frac{1}{5}},$$

$|z| < \frac{1}{5} \Rightarrow |z| < \frac{1}{4}$ and $\therefore |5z| < 1, |4z| < 1$

$$= \frac{20}{\frac{1}{4}(4z-1)} - \frac{20}{\frac{1}{5}(5z-1)}$$

$$= -\frac{80}{(1-4z)} + \frac{100}{(1-5z)}$$

$$= -80(1-4z)^{-1} + 100(1-5z)^{-1}$$

$$= -80[1+(4z)+(4z)^2+\ldots] + 100[1+5z+(5z)^2+\ldots]$$

$$\therefore \quad F(z) = -80\sum_{k=0}^{\infty} 4^k z^k + 100\sum_{k=0}^{\infty} 5^k z^k \quad \ldots (1)$$

$\therefore$ coeff. of z^k in 1st series $= 4^k$, $k \geq 0$

$\therefore$ $f_1(k)$ = coeff. of z^{-k} in 1st series $= 4^{-k}$, $-k \geq 0$ i.e. $k \leq 0$ coeff. of z^k in 2nd series $= 5^k$, $k \geq 0$

$\therefore$ $f_2(k)$ = coeff. of z^{-k} in 2nd series $= 5^{-k}$, $-k \geq 0$ i.e. $k \leq 0$

$\therefore$ combining from (1) we get,

$$\{f(k)\} = Z^{-1} F(z)$$

$$= -80\{f_1(k)\} + 100\{f_2(k)\}$$

$$= -80\{4^{-k}\} + 100\{5^{-k}\}, k \leq 0$$

$$= \{100(5)^{-k} - 80(4)^{-k}\}, k \leq 0$$

Example 4.51 : *Find* $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$

if i) $|z| \leq 1$ *ii)* $|z| > 2$.

SPPU : Dec.-19, Marks 4

Solution : i) $F(z) = \frac{z}{(z-1)(z-2)}, |z| \leq 1$

[Note that the numerator is not a pure constant and involves all terms in powers of z. Hence, we can conveniently use Partial Fraction Method]

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{(z-1)(z-2)} \quad \ldots \text{(Dividing by z)}$$

$$= -\left[\frac{1}{z-1} - \frac{1}{z-2}\right] = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

... Partial Fractions of $\frac{F(z)}{z}$

$$\therefore \quad F(z) = \frac{z}{z-2} - \frac{z}{z-1}, |z| \leq 1$$

$\therefore$ $|z| < 2$ ... Rewriting F(z)

$$\therefore Z^{-1}F(z) = Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z-1}\right),$$

$|z| < 2, |z| \leq 1$ ... Inverting

i.e. $\{f(k)\} = \{-2\}^k - \{-1^k\} = \{-2^k\} + \{1\}$

$(k < 0)\ (k \leq 0) \quad (k < 0)\ (k \leq 0)$

... Note : $|z|$ (< or =) 1 and

Thus,

$$Z^{-1}\frac{z}{(z-1)(z-2)} = \{f(k)\}$$

where, $f(k) = 1, \quad k = 0 = -2^k + 1, k < 0$

ii) Here we have $|z| > 2 \therefore |z| > 1$ and we get

$$Z^{-1}F(z) = Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z-1}\right)$$

$|z| > 2, |z| > 1$

$$= \{2^k\} - \{1^k\}, k \geq 0$$

$$\ldots \because Z^{-1}\left(\frac{z}{z-a}\right) = a^k, k \geq 0, \text{ for } |z| > |a|$$

$$= \{2^k - 1\}, \quad k \geq 0$$

Example 4.52 : *Find the inverse Z-transform of*

$$\frac{z}{(z-1)(z-2)}, 1 < |z| < 2$$

Solution : We have,

$$F(z) = \frac{z}{(z-1)(z-2)}, 1 < |z| < 2$$

... Numerator involves 'z'

$\therefore$ use Partial Fraction Method

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{(z-1)(z-2)}$$

$$= \left[\frac{1}{(z-2)} - \frac{1}{(z-1)}\right]$$

... Partial fractions

$$\therefore \quad F(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

$$\therefore \quad \{f(k)\} = Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z-1}\right)$$

$|z| > 1, |z| < 2$

$$= \{-2^k\} - \{1^k\} = \{-2^k\} - \{1\}$$

$(k < 0)\ (k \geq 0)\ (k < 0)\ (k \geq 0)$

$$\ldots \because Z^{-1}\left(\frac{z}{z-a}\right) = -a^k, k < 0, |z| < |a|$$

i.e. $f(k) = -2^k, k < 0$

$= -1, k \geq 0$

Example 4.53 : *Find* $Z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right]$ *if*

i) $|z| > \frac{1}{4}$ **SPPU : Dec.-14**

ii) $\frac{1}{5} < |z| < \frac{1}{4}$

iii) $|z| < \frac{1}{5}$

Solution : We have

$$F(z) = \frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$$

Let us use Partial Fractions Method which is more convenient here

$$\therefore \quad \frac{F(z)}{z} = \frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{5}}$$

Partial fractions $A = \dfrac{1/4}{1/4 - 1/5} = 5$ and

$$B = \frac{1/5}{1/5 - 1/4} = -4 = \frac{5}{z-\frac{1}{4}} - \frac{4}{z-\frac{1}{5}}$$

$$\therefore \quad F(z) = 5\frac{z}{z-\frac{1}{4}} - 4\frac{z}{z-\frac{1}{5}}$$

$\therefore$ Inverting we get

$$\{f(k)\} = Z^{-1} F(z)$$

$$= 5Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - 4Z^{-1}\left(\frac{z}{z-\frac{1}{5}}\right) \quad \dots (1)$$

Case I :

$|z| > \frac{1}{4} \Rightarrow |z| > \frac{1}{5}$

$$\therefore \quad \{f(k)\} = 5Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - 4Z^{-1}\left(\frac{z}{z-\frac{1}{5}}\right),$$

$$|z| > \frac{1}{4}, |z| > \frac{1}{5}$$

$$= 5\left\{\left(\frac{1}{4}\right)^k\right\} - 4\left\{\left(\frac{1}{5}\right)^k\right\}$$

$$(k \geq 0) \qquad (k \geq 0)$$

$$\dots \because Z^{-1}\left(\frac{z}{z-a}\right) = a^k, k \geq 0 \text{ for } |z| > |a|$$

$$= \left\{5\left(\frac{1}{4}\right)^k - 4\left(\frac{1}{5}\right)^k\right\}, k \geq 0$$

Case II :

$\frac{1}{5} < |z| < \frac{1}{4} \quad \therefore$ From (1) we have

$$\{f(k)\} = 5Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - 4Z^{-1}\left(\frac{z}{z-\frac{1}{5}}\right),$$

$$|z| < \frac{1}{4}, |z| > \frac{1}{5}$$

$$= 5\left\{-\left(\frac{1}{4}\right)^k\right\} - 4\left\{\left(\frac{1}{5}\right)^k\right\}$$

$$\dots \because Z^{-1}\left(\frac{z}{z-a}\right) = -a^k, k < 0$$

for $|z| < |a|$

$= (k < 0)\ (k \geq 0)\ a^k, k \geq 0$ for $|z| > |a|$

$$= \left\{-5\left(\frac{1}{4}\right)^k\right\} + \left\{-4\left(\frac{1}{5}\right)^k\right\}$$

$$(k < 0) \qquad (k \geq 0)$$

Thus, $Z^{-1}\left[\dfrac{z^{-1}}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right] = \{f(k)\}$

where, $f(k) = -5\left(\frac{1}{4}\right)^k, k < 0$

$= -4\left(\frac{1}{5}\right)^k, k \geq 0$

Case III :

$|z| < \frac{1}{5} \Rightarrow |z| < \frac{1}{4}$

$\therefore$ from (1) we have,

$$\{f(k)\} = 5\,Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) - 4\,Z^{-1}\left(\frac{z}{z-\frac{1}{5}}\right),$$

$$|z| < \frac{1}{4},\quad |z| < \frac{1}{5}$$

$$= 5\left\{-\left(\frac{1}{4}\right)^k\right\} - 4\left\{-\left(\frac{1}{5}\right)^k\right\}$$

$$(k < 0) \qquad (k < 0)$$

$$\ldots Z^{-1}\left(\frac{z}{z-a}\right) = -a^k,\ k < 0 \text{ for } |z| < |a|$$

$$= \left\{4\left(\frac{1}{5}\right)^k - 5\left(\frac{1}{4}\right)^k\right\},\quad k < 0$$

Example 4.54 : *Find ,*

$$Z^{-1}\left[\frac{2z^2-3z}{(z+2)(z-4)}\right], |z| < 2$$

Solution : We have,

$$F(z) = \frac{2z^2-3z}{(z+2)(z-4)}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{2z-3}{(z+2)(z-4)}$$

$$= \frac{\left(\frac{7}{6}\right)}{z+2} + \frac{\left(\frac{5}{6}\right)}{z-4} \quad \ldots A = \frac{2(-2)-3}{-2-4} = \frac{7}{6},$$

$$B = \frac{2(4)-3}{4+2} = \frac{5}{6}$$

$$\therefore \quad F(z) = \frac{7}{6}\cdot\frac{z}{z+2} + \frac{5}{6}\,\frac{z}{z-4}$$

$\therefore$ Inverting,

$$\{f(k)\} = Z^{-1}\,F(z)$$

$$= \frac{7}{6}Z^{-1}\left[\frac{z}{z-(-2)}\right] + \frac{5}{6}z^{-1}\left[\frac{z}{z-4}\right]$$

Now, $\because |z| < 2 \therefore |z| < |-2|$

and $|z| < |4|$

$$\therefore \quad \{f(k)\} = \frac{7}{6}\{-(-2)^k\} + \frac{5}{6}\{-4^k\},$$

$$k < 0 \ldots \because Z^{-1}\left(\frac{z}{z-a}\right) = -a^k,\ k < 0 \text{ for } |z| < |a|$$

$$= \left\{-\frac{7}{6}(-2)^k - \frac{5}{6}4^k\right\},\ k < 0$$

Example 4.55 : *Find Inverse Z-transform of the following* $\frac{3z^2+2z}{z^2-3z+2}$, $1 < |z| < 2$

May-15

Solution : We have,

$$F(z) = \frac{3z^2+2z}{z^2-3z+2},\ 1 < |z| < 2$$

$$\therefore \quad \frac{F(z)}{z} = \frac{3z+2}{z^2-3z+2} = \frac{3z+2}{(z-2)(z-1)}$$

$$= \frac{8}{z-2} + \frac{(-5)}{z-1}$$

Partial Fractions : $A = \frac{3(2)+2}{2-1} = 8$

$$B = \frac{3(1)+2}{1-2} = -5$$

$$\therefore \quad F(z) = 8\left(\frac{z}{z-2}\right) - 5\left(\frac{z}{z-1}\right)$$

$$\therefore \quad \{f(k)\} = 8\,Z^{-1}\left(\frac{z}{z-2}\right) - 5\,Z^{-1}\left(\frac{z}{z-1}\right)$$

$$\ldots |z| < 2,\ |z| > 1$$

$$= 8\{-2^k\} - 5\{1^k\}$$

$$(k < 0) \quad (k \ge 0)$$

$$= -8\{2^k\} - 5\{1\}$$

$$(k < 0)\ (k \ge 0)$$

i.e. $$f(k) = -8\cdot 2^k,\ k < 0$$

$$= -5,\ k \ge 0$$

Example 4.56 : *Find the Inverse Z-transform of ,*

$$\frac{3z^2+2z}{z^2+3z+2},\ 1 < |z| < 2$$

Solution : We have,

$$F(z) = \frac{3z^2+2z}{z^2+3z+2},\ 1 < |z| < 2$$

$$\therefore \quad \frac{F(z)}{z} = \frac{3z+2}{z^2+3z+2} = \frac{3z+2}{(z+1)(z+2)}$$

$$= \frac{(-1)}{z+1} + \frac{(4)}{z+2} \quad \ldots A = \frac{3(-1)+2}{-1+2} = -1$$

$$B = \frac{3(-2)+2}{-2+1} = 4$$

$$\therefore \quad F(z) = -\frac{z}{z+1} + 4\frac{z}{z+2}$$

$$\therefore \quad \{f(k)\} = Z^{-1} F(z)$$

$$= -Z^{-1}\left(\frac{z}{z+1}\right) + 4Z^{-1}\left(\frac{z}{z+2}\right)$$

$$= -Z^{-1}\left[\frac{1}{z-(-1)}\right] + 4Z^{-1}\left[\frac{z}{z-(-2)}\right]$$

$|z| > 1$ i.e. $|z| > |-1|$

and $|z| < 2$ i.e. $|z| < |-2|$

$$= -\underset{(k\ge 0)}{\{(-1)^k\}} + 4\underset{(k<0)}{\{-(-2)^k\}}$$

$$\ldots \because Z^{-1}\left(\frac{z}{z-a}\right) = a^k,\ k \ge 0,\ |z| > |a|$$

$$= -a^k,\ k < 0,\ |z| < |a|$$

$$= \underset{(k \ge 0)}{\{-1\}^{k+1}} - \underset{(k<0)}{4\{(-2)^k\}}$$

i.e. $\quad f(k) = -4(-2)^k,\ k < 0$

$$= (-1)^{k+1},\ k \ge 0$$

Example 4.57 : *Find* $Z^{-1}\left[\dfrac{z(z+1)}{(z-1)^2}\right], |z| > 1$

Solution : We have,

$$F(z) = \frac{z(z+1)}{(z-1)^2}, |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z+1}{(z-1)^2}$$

$$= \frac{(z-1)+2}{(z-1)^2}$$

$$= \frac{1}{(z-1)} + \frac{2}{(z-1)^2}$$

$$\therefore \quad F(z) = \frac{z}{z-1} + 2\frac{z}{(z-1)^2}$$

$\therefore$ Inverting,

$$\{f(k)\} = Z^{-1} F(z)$$

$$= Z^{-1}\left(\frac{z}{z-1}\right) + 2Z^{-1}\left[\frac{z}{(z-1)^2}\right],$$

$|z| > 1$

$$= \{1^k\} + 2\{k(1)^{k-1}\},\ k \ge 0$$

$$\ldots \because Z^{-1}\frac{z}{(z-a)^2} = ka^{k-1},$$

$k \ge 0$ for $|z| > |a|$

$$\therefore \quad \{f(k)\} = \{1 + 2k\},\ k \ge 0$$

Example 4.58 : *Find the inverse Z-transform of*

$$\frac{z+1}{(z-1)^2}, |z| > 1.$$

Solution : We have,

$$F(z) = \frac{z+1}{(z-1)^2}, |z| > 1 \therefore \left|\frac{1}{z}\right| < 1$$

$$= \frac{(z+1)}{z^2\left(1-\frac{1}{z}\right)^2}$$

$$= (z+1)\frac{1}{z^2}\left(1-\frac{1}{z}\right)^{-2}, \left|\frac{1}{z}\right| < 1$$

$$= (z+1)\frac{1}{z^2}\left[1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+\ldots\right]$$

$$\ldots \because (1-x)^{-2} = 1+2x+3x^2+\ldots$$

$$= (z+1)\left[\frac{1}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}+\ldots\right]$$

$$= \left[\frac{1}{z}+\frac{2}{z^2}+\frac{3}{z^3}+\ldots\right] + \left[\frac{1}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}+\ldots\right]$$

$$= \sum_{k=1}^{\infty}\frac{k}{z^k} + \sum_{k=1}^{\infty}\frac{k}{z^{k+1}}$$

$$= \sum_{k=1}^{\infty} kz^{-k} + \sum_{k=1}^{\infty} kz^{-(k+1)}$$

Now, coefficient of z^{-k} in 1^{st} series $= f_1(k) = k$, $k \ge 1$

coefficient of $z^{-(k+1)}$ in 2^{nd} series $= k$, $k \ge 1$

Put $k + 1 = n \therefore k = n - 1$

$\therefore$ coefficient of z^{-n} in 2^{nd} series $= n - 1$, $n - 1 \ge 1$

i.e. $n \ge 2$

$\therefore$ $f_2(k)$ = coefficient of z^{-k} in 2^{nd} series $= k - 1$, $k \ge 2$

... replacing n by k

$\therefore$ Combining we get

$$Z^{-1} F(z) = \{f(k)\} = \{f_1(k)\} + \{f_2(k)\}$$

$$= \{k\} + \{k-1\}$$

$$k \geq 1 \quad k \geq 2$$

i.e. $f(k) = 0, k < 1$

$= k, k = 1$

$= k + (k-1) = 2k - 1, k \geq 2$

i.e. $f(k) = 0, k < 1$

$= 2k - 1, k \geq 1$

... at $k = 1$ we get $[2(1) - 1 = 1 = k]$

Alternately, we have

$$F(z) = \frac{z+1}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2} + \frac{1}{(z-1)^2}, |z| > 1$$

$$\therefore \quad \{f(k)\} = Z^{-1} F(z)$$

$$= Z^{-1}\left[\frac{z}{(z-1)^2}\right] + Z^{-1}\left[\frac{1}{(z-1)^2}\right], \quad |z| > 1$$

$$= \{k(1)^{k-1}\} + \{(k-1)(1)^{k-2}\, U(k-2)\}$$

$$(k \geq 0) \qquad (k \geq 0)$$

$$\ldots \because Z^{-1} \frac{z}{(z-a)^2} = k a^{k-1},$$

$k \geq 0, |z|>|a|$ and

$$Z^{-1} \frac{1}{(z-a)^2} = (k-1)\, a^{k-2}\, U(k-2), |z|>|a|$$

$$= \{k\} + \{(k-1)\, U(k-2)\}, \quad k \geq 0$$

$\therefore$ Now,

$(k-1)\, U(k-2) = 0, \quad k - 2 < 0$ i.e. $k < 2$

$= (k-1)\cdot(1), k \geq 2$

$\therefore \quad f(k) = 0 + 0, k \leq 0$

$= 1 + 0, k = 1$

$= k + (k-1), k \geq 2$

i.e. $f(k) = 0, k < 1$

$= 2k - 1, k \geq 1$

Example 4.59 : *Find*

$$Z^{-1}\left[\frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}\right], |z| > 1$$

Solution : We have

$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}, |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2} \quad \ldots (1)$$

Now,

$$\text{let } \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2} = \frac{A}{(z-1)} + \frac{B}{\left(z-\frac{1}{2}\right)} + \frac{C}{\left(z-\frac{1}{2}\right)^2} \quad \ldots (2)$$

... Partial fractions

$\therefore$ Multiplying throughout by the denominator $(z-1)\left(z-\frac{1}{2}\right)^2$ we get

$$z^2 = A\left(z-\frac{1}{2}\right)^2 + B(z-1)\left(z-\frac{1}{2}\right) + C(z-1) \quad \ldots (3)$$

$\therefore$ Putting $z = 1$, $\quad 1 = A\left(\frac{1}{2}\right)^2 + 0 + 0$

$$\therefore \quad 1 = \frac{1}{4} A \quad \therefore A = 4$$

Putting $z = \frac{1}{2}$ in (3),

$$\left(\frac{1}{2}\right)^2 = 0 + 0 + C\left(\frac{1}{2} - 1\right)$$

$$\therefore \quad \frac{1}{4} = -\frac{1}{2} C$$

$$\therefore \quad C = -\frac{1}{2}$$

and putting $z = 0$ in (3),

$$0 = A\left(-\frac{1}{2}\right)^2 + B(-1)\left(-\frac{1}{2}\right) + C(-1)$$

$$= \frac{A}{4} + \frac{B}{2} - C$$

$$\therefore \quad \frac{B}{2} = C - \frac{A}{4} = -\frac{1}{2} - \frac{4}{4} = -\frac{3}{2}$$

$$\ldots \because C = -\frac{1}{2}, A = 4$$

$$\therefore \quad B = -3$$

$\therefore$ Substituting for A, B and C in (2) we get,

$$\frac{F(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$$= \frac{4}{z-1} - \frac{3}{z-\frac{1}{2}} - \frac{1}{2\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad F(z) = 4\left(\frac{z}{z-1}\right) - 3\left(\frac{z}{z-\frac{1}{2}}\right) - \frac{1}{2}\frac{z}{\left(z-\frac{1}{2}\right)^2},$$

$$|z| > 1 \Rightarrow |z| > \frac{1}{2}$$

$\therefore$ Inverting we get

$$\{f(k)\} = Z^{-1} F(z)$$

i.e. $$\{f(k)\} = 4Z^{-1}\left(\frac{z}{z-1}\right) - 3Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)$$

$$-\frac{1}{2}Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)^2}\right], |z| > 1, |z| > \frac{1}{2}$$

$$= 4\{1^k\} - 3\left\{\left(\frac{1}{2}\right)^k\right\} - \frac{1}{2}\left\{k\left(\frac{1}{2}\right)^{k-1}\right\}$$

$$(k \geq 0) \qquad (k \geq 0) \qquad (k \geq 0)$$

$$\ldots \because Z^{-1}\left(\frac{z}{z-a}\right) = \{a^k\},$$

$$k \geq 0 \text{ for } |z| > |a|$$

and $$Z^{-1}\left[\frac{z}{(z-a)^2}\right] = ka^{k-1},$$

$$k \geq 0 \text{ for } |z| > |a|$$

$$= 4\{1\} - 3\left\{\left(\frac{1}{2}\right)^k\right\} - \left\{k\left(\frac{1}{2}\right)^k\right\}, k \geq 0$$

$$= \left\{4 - 3\left(\frac{1}{2}\right)^k - k\left(\frac{1}{2}\right)^k\right\}, k \geq 0$$

$$\therefore \quad \{f(k)\} = \left\{4 - (k+3)\left(\frac{1}{2}\right)^k\right\}, k \geq 0$$

Example 4.60 : *Find the inverse Z-transform of*

$$\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2}, |z| > 1$$

Solution : We have,

$$F(z) = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)^2}, |z| > 1$$

... Numerator involves 'z'

$$\therefore \quad \frac{F(z)}{z} = \frac{z}{(z-1)\left(z-\frac{1}{2}\right)^2} \quad \ldots (1)$$

Now, let

$$\frac{z}{(z-1)\left(z-\frac{1}{2}\right)^2} = \frac{A}{(z-1)} + \frac{B}{\left(z-\frac{1}{2}\right)} + \frac{C}{\left(z-\frac{1}{2}\right)^2} \quad \ldots (2)$$

Partial fractions

$$\therefore z = A\left(z-\frac{1}{2}\right)^2 + B(z-1)\left(z-\frac{1}{2}\right) + C(z-1)$$

Put $z = 1 \quad \therefore \quad 1 = A\left(\frac{1}{4}\right) \quad \therefore A = 4$

Put $z = \frac{1}{2} \quad \therefore \quad \frac{1}{2} = C\left(-\frac{1}{2}\right) \quad \therefore C = -1$

Put $z = 0 \quad \therefore \quad 0 = A\left(\frac{1}{4}\right) + B(-1)\left(-\frac{1}{2}\right) + C(-1)$

$$= 1 + \frac{B}{2} + 1 \quad \because A = 4, C = -1$$

$$\therefore \quad B = -4$$

$\therefore$ From equation (2) we get,

$$\frac{F(z)}{z} = \frac{4}{z-1} - \frac{4}{\left(z-\frac{1}{2}\right)} - \frac{1}{\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad F(z) = 4\frac{z}{(z-1)} - 4\frac{z}{\left(z-\frac{1}{2}\right)} - \frac{z}{\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad \{f(k)\} = 4\,Z^{-1}\left(\frac{z}{z-1}\right) - 4\,Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)$$

$$- Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)^2}\right] \quad ... |z| > 1 \therefore |z| > \frac{1}{2}$$

$$= 4\{1^k\} - 4\left\{\left(\frac{1}{2}\right)^k\right\} - \left\{k\left(\frac{1}{2}\right)^{k-1}\right\},\ k \ge 0$$

$$... \because Z^{-1}\left(\frac{z}{z-a}\right) = a^k, k \ge 0,$$

$$\text{for } |z| > |a|,\ Z^{-1}\frac{z}{(z-a)^2}$$

$$= ka^{k-1},\ k \ge 0,\ |z| > |a|$$

$$= \left\{4 - 4\left(\frac{1}{2}\right)^k - k\left(\frac{1}{2}\right)^{k-1}\right\},\ k \ge 0$$

Example 4.61 : *Find inverse Z-transform of the following* $\frac{z^2}{z^2+a^2}$, $|z| > |a|$.

Solution : We have,

$$F(z) = \frac{z^2}{z^2+a^2},\ |z| > |a|$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z}{z^2+a^2} = \frac{z}{(z-ia)(z+ia)}$$

$$= \frac{1}{2}\left[\frac{1}{z-ia} + \frac{1}{z+ia}\right] \quad ... \text{Partial fractions}$$

$$\therefore \quad F(z) = \frac{1}{2}\frac{z}{(z-ia)} + \frac{1}{2}\frac{z}{(z+ia)}$$

$$\therefore \quad \{f(k)\} = Z^{-1}F(z)$$

$$= \frac{1}{2}Z^{-1}\left(\frac{z}{z-ia}\right) + \frac{1}{2}Z^{-1}\left[\frac{z}{z-(-ia)}\right]$$

now, $\because |z| > |a|$ and

$\because |ia| = |i|\,|a| = |a|$

and $|-ia| = |-i|\,|a| = |a|$

$\therefore |z| > |ia|,\ |z| > |-ia|$

$$\therefore \quad \{f(k)\} = \frac{1}{2}\{(ia)^k\} + \frac{1}{2}\{(-ia)^k\},\ k \ge 0$$

$$= \frac{1}{2}\{a^k[i^k + (-i)^k]\} \quad ... (1)$$

Now, $\quad i = 0 + 1\,i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

and $-i = e^{-i\pi/2}$

$$\therefore \quad (i)^k + (-i)^k = e^{ik\pi/2} + e^{-ik\pi/2}$$

$$= 2\cos\frac{k\pi}{2} \quad ... \because e^{i\theta} + e^{-i\theta}$$

$$= (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$

$$= 2\cos\theta$$

$\therefore$ From (1)

$$\{f(k)\} = \frac{1}{2}\left\{a^k\, 2\cos\frac{k\pi}{2}\right\}$$

$$= \left\{a^k \cos\frac{k\pi}{2}\right\},\ k \ge 0$$

Alternately, we have,

$$F(z) = \frac{z^2}{z^2+a^2} = \frac{z(z-0)}{z^2-(0)+a^2}$$

$$= \frac{z\left[z - a\cos\frac{\pi}{2}\right]}{z^2 - 2az\left(\cos\frac{\pi}{2}\right) + a^2},\ |z| > |a|$$

$$... \because \cos\frac{\pi}{2} = 0$$

$$\therefore Z^{-1}F(z) = Z^{-1}\left[\frac{z\left(z - a\cos\frac{\pi}{2}\right)}{z^2 - 2az\cos\frac{\pi}{2} + a^2}\right]$$

$$= a^k\cos\frac{k\pi}{2},\ k \ge 0$$

$$... \because Z\{c^k\cos\alpha k\} = \frac{z(z-c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}\ k \ge 0,\ |z| > |c|$$

$$\therefore Z^{-1}\left[\frac{z(z-\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}\right] = c^k\cos\alpha k,\ k \ge 0$$

Example 4.62 : *Find Inverse Z-transform of the following functions :*

$$\frac{z^2}{(z^2+1)},\ |z| > 1$$

Solution : We have,

$$F(z) = \frac{z^2}{z^2+1},\ |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z}{z^2+1} = \frac{z}{(z-i)(z+i)}$$

... Note this step

$$= \frac{1}{2}\left(\frac{1}{z-i}+\frac{1}{z+i}\right) \quad \text{... Partial fractions}$$

$$\therefore \quad F(z) = \frac{1}{2}\frac{z}{(z-i)} + \frac{1}{2}\frac{z}{[z-(-i)]}$$

$$\text{...} |i| = \sqrt{0+1} = 1, |-i| = 1,$$

$$\therefore |z| > 1 \Rightarrow |z| > |i|, |z| > |-i|$$

$\therefore$ Inverting we get,

$$\{f(k)\} = Z^{-1}\ F(z)$$

$$= \frac{1}{2}Z^{-1}\left(\frac{z}{z-i}\right) + \frac{1}{2}Z^{-1}\left(\frac{z}{z-(-i)}\right),$$

$$|z| > |i|, |z| > |-i|$$

$$= \frac{1}{2}\{i^k\} + \frac{1}{2}\{(-i)^k\},\ k \ge 0$$

$$= \frac{1}{2}\left\{i^k + (-i)^k\right\},\ k \ge 0$$

Now, $\because \quad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

and $\quad -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = e^{-i\pi/2}$

$$\therefore \quad \{f(k)\} = \frac{1}{2}\left\{e^{\frac{i\pi k}{2}} + e^{-\frac{i\pi k}{2}}\right\}$$

$$= \frac{1}{2}\left\{\left(2\cos\frac{\pi k}{2}\right)\right\} = \left\{\cos\frac{k\pi}{2}\right\},\ k \ge 0$$

Alternately, we can write

$$F(z) = \frac{z^2}{z^2+1} = \frac{z(z-0)}{z^2-2z(0)+1}$$

... Note this step

$$= \frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\left(\cos\frac{\pi}{2}\right)+1}, \quad \text{...} \because \cos\frac{\pi}{2} = 0$$

... Note this step

$$\therefore \quad \{f(k)\} = Z^{-1}\left[\frac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\cos\frac{\pi}{2}+1}\right], |z| > 1$$

$$= \cos\left(\frac{\pi}{2}k\right),\ k \ge 0$$

$$\text{...} \because Z^{-1}\left[\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}\right] = \cos\alpha k,\ k \ge 0 \text{ for } |z| > 1$$

Type 1 (b) If linear real factorisation of denominator is not possible.

Example 4.63 : *Evaluate*

$$Z^{-1}\left(\frac{2z^2+3z}{z^2+z+1}\right), |z| > 1$$

Solution : We have,

$$F(z) = \frac{2z^2+3z}{z^2+z+1}, |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{2z+3}{z^2+z+1}$$

Now consider

$$z^2+z+1 = 0$$

$$\therefore \quad z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore \ z^2+z+1 = \left[z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]\left[z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right]$$

i.e. complex factors

$$\therefore \quad \frac{F(z)}{z} = \frac{2z+3}{\left[z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]\left[z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right]}$$

$$= \frac{A}{z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)} + \frac{B}{z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}$$

$$\text{where} \quad A = \frac{2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)+3}{\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{2+i\sqrt{3}}{i\sqrt{3}}$$

$$B = \frac{2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)+3}{\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{2-i\sqrt{3}}{-i\sqrt{3}}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{(2+i\sqrt{3})}{i\sqrt{3}\left[z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]} - \frac{(2-i\sqrt{3})}{i\sqrt{3}\left[z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right]}$$

$$\therefore \quad F(z) = \frac{1}{i\sqrt{3}}\left[(2+i\sqrt{3})\left(\frac{z}{z-a}\right) - (2-i\sqrt{3})\left(\frac{z}{z-b}\right)\right]$$

where $a = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$ and $b = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

so that $|a| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ and $|b| = 1$

Now $\therefore |z| > 1 \therefore |z| > |a|$, $|z| > |b|$

$\therefore$ Inverting we get

$$\{f(k)\} = Z^{-1} F(z)$$
$$= \frac{1}{i\sqrt{3}}\left[(2+i\sqrt{3})Z^{-1}\left(\frac{z}{z-a}\right) -(2-i\sqrt{3})Z^{-1}\left(\frac{z}{z-b}\right)\right],$$
$$|z|>|a|, |z|>|b|$$
$$= \frac{1}{i\sqrt{3}}\left[(2+i\sqrt{3})\{a^k\} - (2-i\sqrt{3})\{b^k\}\right],$$
$$k \geq 0 \quad \ldots (1)$$

Now, $\because \quad a = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\therefore \quad r = |a| = 1$

and $\quad \theta = \tan^{-1}\frac{\sqrt{3}/2}{-1/2} = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$

$\left[\text{as } \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right.$ belongs to second quadrant in the Argand's Diagram (i.e. Complex Plane) $\Big]$

$\therefore \quad a = r e^{i\theta} = 1 \cdot e^{i\frac{2\pi}{3}}$

Similarly, $\quad b = 1 \cdot e^{-i\frac{2\pi}{3}}$

$\therefore$ Substituting in (1) we get

$$\{f(k)\} = \frac{1}{i\sqrt{3}}\left[(2+i\sqrt{3})\left\{e^{i\frac{2\pi k}{3}}\right\} - (2-i\sqrt{3})\left\{e^{-i\frac{2\pi k}{3}}\right\}\right]$$
$$= \frac{1}{i\sqrt{3}}\left\{2\left(e^{i\frac{2\pi k}{3}} - e^{-i\frac{2\pi k}{3}}\right) + i\sqrt{3}\left(e^{i\frac{2\pi k}{3}} + e^{-i\frac{2\pi k}{3}}\right)\right\}$$
$$= \frac{1}{i\sqrt{3}}\left\{2\left(2\,i \sin\frac{2\pi k}{3}\right) + i\sqrt{3}\left(2\cos\frac{2\pi k}{3}\right)\right\}$$

$\ldots \because \; e^{ix} - e^{-ix} = 2\,i \sin x,$

$e^{ix} + e^{-ix} = 2\cos x$

$$\therefore \quad Z^{-1}\left[\frac{2z^2+3z}{z^2+z+1}\right] = \left\{\frac{4}{\sqrt{3}}\sin\frac{2\pi k}{3} + 2\cos\frac{2\pi k}{3}\right\},$$
$$k \geq 0$$

> **Note :** Here we cannot express $\frac{2z^2+3z}{z^2+z+1}$ in the form $\frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$ where $-1 \leq \cos\alpha \leq 1$.

Example 4.64 : *Find the Inverse Z-transform of* $\frac{z(z+1)}{z^2+2z+2}$, $|z| > \sqrt{2}$

Solution : We have,

$$F(z) = \frac{z(z+1)}{z^2+2z+2}, \; |z| > \sqrt{2}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z+1}{z^2+2z+2} \quad \ldots (1)$$

Now, consider $z^2 + 2z + 2 = 0$

$$\therefore \quad z = \frac{-2 \pm \sqrt{4-8}}{2}$$
$$= \frac{-2 \pm 2i}{2} = -1 \pm i \quad \text{are its roots}$$

$$\therefore \quad z^2+2z+2 = [z-(-1+i)]\,[z-(-1-i)]$$

... factors

$$\therefore \text{ From (1)} \quad \frac{F(z)}{z} = \frac{z+1}{[z-(-1+i)][z-(-1-i)]}$$
$$= \frac{A}{z-(-1+i)} + \frac{B}{z-(-1-i)}$$

where $\quad A = \frac{(-1+i)+1}{(-1+i)-(-1-i)} = \frac{i}{2i} = \frac{1}{2}$

$$B = \frac{(-1-i)+1}{(-1-i)-(-1+i)} = \frac{-i}{-2i} = \frac{1}{2}$$
$$= \frac{1}{2}\cdot\frac{1}{[z-(-1+i)]} + \frac{1}{2}\cdot\frac{1}{[z-(-1-i)]}$$
$$= \frac{1}{2}\cdot\frac{1}{(z-a)} + \frac{1}{2}\cdot\frac{1}{(z-b)}$$

where, $a = -1+i$, $b = 1-i$

so that $|a| = \sqrt{1+1} = \sqrt{2}$, $|b| = \sqrt{1+1} = \sqrt{2}$

$$\therefore \quad F(z) = \frac{1}{2}\cdot\frac{z}{(z-a)} + \frac{1}{2}\cdot\frac{z}{(z-b)},$$

$|z| > \sqrt{2}$ i.e. $|z| > |a|$, $|z| > |b|$

$$\therefore \quad \{f(k)\} = \frac{1}{2} Z^{-1}\left(\frac{z}{z-a}\right) + \frac{1}{2} Z^{-1}\left(\frac{z}{z-b}\right),$$

$$|z| > |a|, |z| > |b|$$

$$= \frac{1}{2}\{a^k\} + \frac{1}{2}\{b^k\}, \ k \geq 0$$

$$= \frac{1}{2}\{a^k + b^k\}, \ k \geq 0 \qquad \ldots (2)$$

Now, $a = -1 + i \ \therefore r = \sqrt{2}, \ \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

... $a = -1 + i$ belongs to 2^{nd} quadrant

$$\therefore \quad a = r\,e^{i\theta} = \sqrt{2}\,e^{i\frac{3\pi}{4}}$$

and $b = -1 - i = r\,e^{-i\theta} = \sqrt{2}\,e^{-i\frac{3\pi}{4}}$

... $\because$ b belong to 3^{rd} quadrant

$$\therefore \quad a^k + b^k = \left(\sqrt{2}\right)^k e^{i\frac{3\pi k}{4}} + \left(\sqrt{2}\right)^k e^{-i\frac{3\pi k}{4}}$$

$$= \left(\sqrt{2}\right)^k \left[e^{i\frac{3\pi k}{4}} + e^{-i\frac{3\pi k}{4}}\right]$$

$$= \left(\sqrt{2}\right)^k 2\cos\frac{3\pi k}{4}$$

$\therefore$ From (2) we get

$$\{f(k)\} = \frac{1}{2}\left\{2\left(\sqrt{2}\right)^k \cos\frac{3\pi k}{4}\right\}, \ k \geq 0$$

$$= \left\{\left(\sqrt{2}\right)^k \cos\frac{3\pi k}{4}\right\}, \ k \geq 0$$

Note : We can write

$$F(z) = \frac{z(z+1)}{z^2+2z+2}, \ |z| > \sqrt{2}$$

$$= \frac{z\left[z-\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)\right]}{z^2 - 2(\sqrt{2})\,z\left(-\frac{1}{\sqrt{2}}\right) + (\sqrt{2})^2}$$

$$= \frac{z\left(z-\sqrt{2}\cos\frac{3\pi}{4}\right)}{z^2 - 2(\sqrt{2})\,z\cos\frac{3\pi}{4} + (\sqrt{2})^2},$$

$$|z| > \sqrt{2} \quad \ldots \because \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\therefore \{f(k)\} = Z^{-1}\left[\frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}\right],$$

$|z| > |c|$ where $c = \sqrt{2}, \ \alpha = \frac{3\pi}{4}$

$$= \{c^k \cos\alpha k\}, \ k \geq 0$$

$$= \left\{\left(\sqrt{2}\right)^k \cos\frac{3\pi k}{4}\right\}, \ k \geq 0$$

Example 4.65 : *Find* $Z^{-1}\left[\frac{z(z+1)}{(z-1)(z^2+z+1)}\right]$ $|z| > 1$

Solution : We have,

$$F(z) = \frac{z(z+1)}{(z-1)(z^2+z+1)}, \ |z| > 1$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z+1}{(z-1)(z^2+z+1)} \qquad \ldots (1)$$

Now,

$$\frac{(z+1)}{(z-1)\left(z^2+z+1\right)} = \frac{A}{(z-1)} + \frac{(Bz+C)}{\left(z^2+z+1\right)}$$

... $z^2 + z + 1$ being quadratic factor

$\therefore$ Multiplying throughout by the denominator,

$$(z+1) = A(z^2+z+1) + (Bz+C)(z-1)$$

$$= (Az^2 + Az + A) + (Bz^2 - Bz + Cz - C)$$

$$z + 1 = (A+B)z^2 + (A-B+C)z + (A-C)$$

$\therefore$ Comparing the coefficients of like powers on both the sides,

$$A + B = 0$$

$$A - B + C = 1$$

$$A - C = 1$$

$$\therefore \quad B = -A$$

$$C = A - 1$$

$$\therefore \quad A - B + C = 1 \Rightarrow A + A + A - 1 = 1$$

$$\therefore \quad A = \frac{2}{3}$$

$$\therefore \quad B = -A = -\frac{2}{3} \text{ and } C = A - 1$$

$$= \frac{2}{3} - 1 = -\frac{1}{3}$$

$\therefore$ Using A, B, C and (1) we get,

$$\frac{F(z)}{z} = \frac{2/3}{z-1} + \frac{\left(-\frac{2}{3}z - \frac{1}{3}\right)}{z^2+z+1}$$

$$\therefore \quad F(z) = \frac{2}{3}\left(\frac{z}{z-1}\right) - \frac{z\frac{2}{3}\left(z+\frac{1}{2}\right)}{z^2+z+1}$$

∴ Inverting,

$$\{f(k)\} = Z^{-1}\ F(z)$$

$$= \frac{2}{3} Z^{-1}\left(\frac{z}{z-1}\right) - \frac{2}{3} Z^{-1}\left[\frac{z\left(z+\frac{1}{2}\right)}{z^2+z+1}\right],$$

$$|z| > 1 \quad \text{... (2)}$$

Now, $Z^{-1}\left(\frac{z}{z-1}\right) = 1,\ k \geq 0$ for $|z| > 1$ and

$$\because \quad \frac{z\left(z+\frac{1}{2}\right)}{z^2+z+1} = \frac{z\left[z-\left(-\frac{1}{2}\right)\right]}{z^2-2z\left(-\frac{1}{2}\right)+1},\ |z| > 1$$

Now, $Z^{-1}\left(\frac{z}{z-1}\right) = 1,\ k \geq 0$ for $|z| > 1$ and

$$\because \quad \frac{z\left(z+\frac{1}{2}\right)}{z^2+z+1} = \frac{z\left[z-\left(-\frac{1}{2}\right)\right]}{z^2-2z\left(-\frac{1}{2}\right)+1},\ |z| > 1$$

i.e. of the form $\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$ where $\cos\alpha = -\frac{1}{2}$

$$\therefore \quad Z^{-1}\left[\frac{z\left(z+\frac{1}{2}\right)}{z^2+z+1}\right] = \cos \alpha k,\ k \geq 0$$

where $\cos\alpha = -\frac{1}{2}$

$$= \cos\frac{2\pi k}{3},\ k \geq 0$$

$$\text{...} \because \cos\alpha = -\frac{1}{2}\ \therefore -\cos\alpha = \frac{1}{2} \text{ i.e. } \cos(\pi-\alpha) = \frac{1}{2}$$

$$\therefore \pi - \alpha = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}\ \therefore \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

From (2) we get,

$$\{f(k)\} = \frac{2}{3}\{1\} - \frac{2}{3}\left\{\cos\frac{2\pi}{3}k\right\},\ k \geq 0$$

i.e. $$\{f(k)\} = \left\{\frac{2}{3}\left(1-\cos\frac{2\pi}{3}k\right)\right\},\ k \geq 0$$

Example 4.66 : *Solve any one of the following :*

Obtain $Z^{-1}\left[\frac{6z}{(z+2)(z-4)}\right],\ |z| > 4,\ K \geq 0.$

SPPU : Dec.-18, Marks 4

Solution :

Let $$F(z) = \frac{6z}{(z+2)(z-4)}$$

$$\frac{F(z)}{z} = \frac{1}{z-4} - \frac{1}{z+2}$$

$$F(z) = \frac{z}{z-4} - \frac{z}{z+2}$$

∴ Taking inverse transform, we get,

$$\{f(k)\} = z^{-1}\left[\frac{z}{z-4}\right] - z^{-1}\left[\frac{z}{z+2}\right]$$

$$\{f(k)\} = \{+4\}^k - \{-2\}^k = 4^k - (-2)^k$$

Example 4.67 : *Solve :*

Find : $Z^{-1}\left\{\frac{1}{(z-4)(z-5)}\right\}$ *by inversion integral method.*

SPPU : May-17, Marks 4

Solution :

i) Let $$F(z) = \frac{1}{(z-4)(z-5)}$$

$F(z) \to \infty$ at $z = 4,\ z = 5$

∴ $z = 4$ and $z = 5$ are simple poles of F(z)

Residue of $F(z)\cdot z^{k-1}$ at pole $z = 4$ is

$$= [(z-4)\,F(z)\,z^{k-1}]_{z=4}$$

$$= \left[\frac{z^{k-1}}{z-5}\right]_{z=4} = -(4)^{k-1}$$

Residue of $F(z)\cdot z^{k-1}$ at pole $z = 5$ is

$$= [(z-5)\,F(z)\cdot z^{k-1}]_{z=5}$$

$$= \left[\frac{z^{k-1}}{z-4}\right]_{z=5} = 5^{k-1}$$

Now, $$Z^{-1}[F(z)] = f(k) = \text{Res}(z=4) + \text{Res}(z=5)$$

$$= 5^{k-1} - 4^{k-1}\ ;\ k \geq 0$$

Example 4.68 : *Find the inverse z-transform* $F(z) = \frac{1}{(z-3)(z-4)},\ |z| < 3.$

SPPU : Dec.-16, Marks 4

Solution : We have

$$F(z) = \frac{1}{(z-3)\,(z-4)}$$

$$F(z) = \frac{-1}{z-3} + \frac{1}{z-4} = \frac{1}{3\left(\frac{z}{3}-1\right)} + \frac{1}{4\left(\frac{z}{4}-1\right)}$$

$$= -\frac{1}{3}\frac{1}{\left(1-\frac{z}{3}\right)} - \frac{1}{4}\frac{1}{1-\frac{z}{4}}$$

$$= -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\ldots\ldots\right)$$

$$-\frac{1}{4}\left(1+\frac{z}{4}+\frac{z^2}{4^2}+\cdots\right)$$

$$= -\frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{z}{3}\right)^{k-1} - \frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{z}{4}\right)^{k-1}$$

$$F(z) = \sum_{k=1}^{\infty}\left[-\frac{1}{3}\left(\frac{1}{3}\right)^{k-1} - \frac{1}{4}\left(\frac{1}{4}\right)^{k-1}\right] z^{k-1}$$

$$= \sum_{k=1}^{\infty}\left[-\left(\frac{1}{3}\right)^{k} - \left(\frac{1}{4}\right)^{k}\right] z^{k-1}$$

$$\therefore\ z^{-1}[F(z)] = \{x_k\} \Rightarrow$$

$$\{x_k\} = -\left(\frac{1}{3}\right)^{-k+1} - \left(\frac{1}{4}\right)^{-k+1}$$

$$= -(3)^{k-1} - (4)^{k-1};\ k \geq 1$$

Type 2) Power series method

1) By actual division :

If F (z) is a ratio of two polynomials which cannot be factorized we take the actual division and we have :

Case I :

Where the ROC is of the form $|z| < |C|$.

Step 1 : Carry out the division by beginning with the 'lowest power term of z in the denominator i.e. a_n'. [The Quotient gives a power series in z which converges in the same region as F (z)].

Step 2 : Note down the coefficient of the general term z^k. Then,

$$f(-k) = \text{coefficient of } z^k$$

Step 3 : Replacing (– k) by k throughout, we get f(k).

Case II :

Where the ROC is of the form $|z| > |C|$.

Step 1 : Carry out the division by beginning with the 'highest power term of z in the denominator i.e. $b_0 z^n$', to get the power series in z.

Step 2 : Note down the coefficient of the general term z^{-k}. Then,

$$f(k) = \text{coefficient of } z^{-k}$$

2) By use of binomial theorem

Step 1 : Take out a suitable term (depending upon the ROC) common from the denominator so as to get a term of the form $(1 \pm r)^p$, where $|r| < 1$, in the denominator.

> **Note :** If ROC is $|z| < |c|$ we have, $r = \frac{z}{c}$ so that $\left(|r| = \left|\frac{z}{c}\right|\right) < 1.$
>
> If ROC is $|z| > |c|$ we have, $r = \frac{c}{z}$ so that $\left(|r| = \left|\frac{c}{z}\right|\right) < 1$]

Step 2 : Expand $\frac{1}{(1 \pm r)^p} = (1 \pm r)^{-p}$, using Binomial Theorem and proceed to get a power series in z.

Step 3 : Note down the coefficient of z^{-k} in the series and then

$$f(k) = \text{coefficient of } z^{-k}$$

> **Note :** Useful Binomial Expansions :
>
> $$\frac{1}{(1+x)} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \ldots$$
>
> $$\frac{1}{(1-x)} = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 \ldots$$
>
> $$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$$
>
> $$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

Example 4.69 : *Find* $Z^{-1}\left(\frac{1}{z-a}\right)$ *when* *i)* $|z|<|a|$ *ii)* $|z|>|a|$.

Solution : We have

$$F(z) = \frac{1}{z-a}$$

Use Power Series Method to find Z^{-1}

Case I : $|z|<|a|$

$\therefore$ Divide by beginning with (– a) to get

$$F(z) = \frac{1}{z-a}$$

$$\text{i.e. } -a+z)\ 1\ \left(-\frac{1}{a}-\frac{z}{a^2}-\frac{z^2}{a^3}+\ldots\right.$$

$$\begin{array}{l} 1-\frac{z}{a} \\ \hline \frac{z}{a} \\ \frac{z}{a}-\frac{z^2}{a^2} \\ \hline \frac{z^2}{a^2} \\ \frac{z^2}{a^2}-\frac{z^3}{a^3} \\ \hline \vdots \end{array}$$

$$\therefore \quad F(z) = \frac{1}{(z-a)} = -\frac{1}{a}-\frac{z}{a^2}-\frac{z^2}{a^3}\ \ldots$$

$$= \sum_{k=0}^{\infty} -\frac{z^k}{a^{k+1}} \qquad \text{... general term}$$

$$\therefore \quad f(-k) = \text{coefficient of } z^k$$

$$= -\frac{1}{a^{k+1}},\ k \geq 0 \qquad \text{... (1)}$$

$$\therefore \quad f(k) = \text{coefficient of } z^{-k}$$

$$= -\frac{1}{a^{-k+1}} = -a^{k-1},\ -k \geq 0 \text{ i.e. } k \leq 0$$

... Replacing (– k) by k in (1)

$$\therefore Z^{-1}\left(\frac{1}{z-a}\right) = \{f(k)\} = \{-a^{k-1}\},\ k \leq 0, \text{ if } |z|<|a|$$

Case II : $|z|>|a|$

$\therefore$ Begin the division by 'z'.

$$\therefore \quad F(z) = \frac{1}{z-a}$$

$$z-a)\ 1\ \left(\frac{1}{z}+\frac{a}{z^2}+\frac{a^2}{z^3}+\ldots\right.$$

$$\begin{array}{l} 1-\frac{a}{z} \\ \hline \frac{a}{z} \\ \frac{a}{z}-\frac{a^2}{z^2} \\ \hline \frac{a^2}{z^2} \\ \frac{a^2}{z^2}-\frac{a^3}{z^3} \\ \hline \vdots \end{array}$$

$$\therefore \quad F(z) = \frac{1}{z-a} = \frac{1}{z}+\frac{a}{z^2}+\frac{a^2}{z^3}+\ldots$$

$$= \sum_{k=1}^{\infty} \frac{a^{k-1}}{z^k} = \sum_{k=1}^{\infty} a^{k-1} z^{-k}$$

... general term

$$\therefore \quad f(k) = \text{Coefficient of } z^{-k}$$

$$= a^{k-1},\ k \geq 1$$

$$\therefore Z^{-1}\left(\frac{1}{z-a}\right) = \{f(k)\}$$

$$= \{a^{k-1}\},\ k \geq 1, \text{ if } |z|>|a|$$

Note : Alternately, in both the above cases we can use Binomial Theorem conveniently.

Example 4.70 : *Find the Inverse Z-transform of* $\frac{z}{z-4}$ *if i)* $|z|>4$ *ii)* $|z|<4$.

Solution : We have

$$F(z) = \frac{z}{z-4}$$

Let's use power series method (Using Binomial Theorem) to find $Z^{-1}F(z)$

Case I :

$|z|>4 \quad \therefore \left|\frac{4}{z}\right|<1$

$\therefore$ Let , $\quad r = \frac{4}{z}$ so that $|r|<1$

$\therefore$ In order to get term (1 – r) we take out 'z' common from the denominator term.

$$\therefore \quad F(z) = \frac{z}{z-4} = \frac{z}{z\left(1-\frac{4}{z}\right)} \quad \ldots |z|>4 \text{ i.e. } \left|\frac{4}{z}\right|<1$$

$$= \left(1 - \frac{4}{z}\right)^{-1}$$

$$= 1 + \left(\frac{4}{z}\right) + \left(\frac{4}{z}\right)^2 + \left(\frac{4}{z}\right)^3 + \ldots$$

$$\because (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$= \frac{4^0}{z^0} + \frac{4}{z} + \frac{4^2}{z^2} + \frac{4^3}{z^3} + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{4^k}{z^k} = \sum_{k=0}^{\infty} 4^k z^{-k} \ldots \text{ general term}$$

$\therefore \quad f(k) = \text{coefficient of } z^{-k} = 4^k, \; k \geq 0$

$\therefore \; Z^{-1}\left(\frac{z}{z-4}\right) = \{f(k)\} = \{4^k\}, \; k \geq 0 \text{ if } |z| > 4$

Case II :

$|z| < 4 \quad \therefore \left|\frac{z}{4}\right| < 1 \Rightarrow r = \frac{z}{4}$ so that $|r| < 1$

$\therefore$ In order to get the term $(1 - r)$, we take out '4' common from the denominator.

$$\therefore \quad F(z) = \frac{z}{z-4} = \frac{z}{4\left(\frac{z}{4}-1\right)} \quad \ldots |z| < 4 \; \therefore \left|\frac{z}{4}\right| < 1$$

$$= \frac{z}{-4\left(1-\frac{z}{4}\right)} = -\frac{z}{4}\left(1-\frac{z}{4}\right)^{-1}$$

$$= -\frac{z}{4}\left[1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + \ldots\right]$$

$$= -\frac{z}{4}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \ldots\right]$$

$$= -\frac{z}{4} - \frac{z^2}{4^2} - \frac{z^3}{4^3} \ldots$$

i.e. $\quad F(z) = \sum_{k=1}^{\infty} \frac{-z^k}{4^k} \quad \ldots$ general term

$\therefore$ coefficient of $z^k = -\frac{1}{4^k}, \; k \geq 1$

$\therefore$ Replacing k by $(-k)$ we get,

coefficient of $z^{-k} = -\frac{1}{4^{-k}} = -4^k, \; -k \geq 1$

i.e. $k \leq -1$ i.e. $k < 0$

$\therefore \quad f(k) = \text{coefficient of } z^{-k} = -4^k, \; k < 0$

Thus,

$Z^{-1}\left(\frac{z}{z-4}\right) = \{f(k)\} = \{-4^k\},$

$k < 0$ if $|z| < 4$

Note : Here we can also write,

$$F(z) = \sum_{k=1}^{\infty} \frac{-z^k}{4^k}$$

$\ldots$ general term

$\therefore \quad$ coefficient of $z^k = -\frac{1}{4^k}, \; k \geq 1$

$\therefore$ Put $n = -k$ i.e. $k = -n$

$\therefore$ coefficient of $z^{-n} = -\frac{1}{4^{-n}} = -4^n, \; -n \geq 1$

i.e. $n \leq -1$ i.e. $n < 0$

$\therefore$ Replacing n by k (both being integers) we get

coefficient of $z^{-k} = -4^k, \; k < 0$

$\therefore \quad f(k) = -4^k, \; k < 0$

Example 4.71 : *Find the Inverse Z-transform of*

i) $\frac{1}{(z-a)^2}$ *if* $|z| < |a|$

ii) $\frac{1}{(z-5)^3}, \; |z| > 5$

Solution : i) We have

$$F(z) = \frac{1}{(z-a)^2}, \; |z| < |a|, \; \therefore \left|\frac{z}{a}\right| < 1$$

$$\therefore \quad F(z) = \frac{1}{\left[a\left(\frac{z}{a} - 1\right)\right]^2}$$

$\ldots$ Taking out 'a' common

$$= \frac{1}{\left[-a\left(1 - \frac{z}{a}\right)\right]^2}$$

$$= \frac{1}{a^2}\left(1 - \frac{z}{a}\right)^{-2} \qquad \ldots 1 - r = 1 - \frac{z}{a}$$

$$= \frac{1}{a^2}\left[1 + 2\left(\frac{z}{a}\right) + 3\left(\frac{z}{a}\right)^2 + 4\left(\frac{z}{a}\right)^3 + \ldots\right]$$

$\ldots \because (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

$$= \frac{1}{a^2}\left[1 + \frac{2}{a}z + \frac{3}{a^2}z^2 + \frac{4}{a^3}z^3 + \ldots\right]$$

$$= \frac{1}{a^2} + \frac{2}{a^3}z + \frac{3}{a^4}z^2 + \frac{4}{a^5}z^3 + \ldots$$

$\therefore \quad F(z) = \sum_{k=0}^{\infty} \frac{(k+1)}{a^{k+2}} z^k \quad \ldots$ general term

$\therefore$ coefficient of $z^k = \dfrac{k+1}{a^{k+2}}, \ k \geq 0$

$\therefore$ coefficient of $z^{-k} = \dfrac{-k+1}{a^{-k+2}}, \ -k \geq 0$ i.e. $k \leq 0$

... Replacing k by (– k)

$\therefore \quad f(k) =$ coefficient of $z^{-k} = \dfrac{1-k}{a^{2-k}}, \ k \leq 0$

$$\therefore \quad Z^{-1}\left[\frac{1}{(z-a)^2}\right] = \{f(k)\} = \left\{\frac{1-k}{a^{2-k}}\right\},$$

$k \leq 0$ if $|z| < |a|$

ii) We have, $\quad F(z) = \dfrac{1}{(z-5)^3}, |z| > 5 \therefore \left|\dfrac{5}{z}\right| < 1$

$$\therefore F(z) = \frac{1}{\left[z\left(1-\frac{5}{z}\right)\right]^3} = \frac{1}{z^3\left(1-\frac{5}{z}\right)^3} \quad \text{... } 1 - r = 1 - \frac{5}{z}$$

$$= \frac{1}{z^3}\left(1-\frac{5}{z}\right)^{-3}$$

$$= \frac{1}{z^3}\left[1-(-3)\left(\frac{5}{z}\right) + \frac{(-3)(-4)}{2!}\left(\frac{5}{z}\right)^2 - \frac{(-3)(-4)(-5)}{3!}\left(\frac{5}{z}\right)^3 + \ldots\right]$$

$\because$ Binomial Theorem :

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$= \frac{1}{z^3}\left[1 + 3\left(\frac{5}{z}\right) + 6\left(\frac{5^2}{z^2}\right) + 10\left(\frac{5^3}{z^3}\right) + \ldots\right]$$

$$= \frac{1}{z^3}\left[\frac{(1)(2)}{2}\frac{5^0}{z^0} + \frac{(2)(3)}{2}\frac{5}{z} + \frac{3(4)}{2}\frac{5^2}{z^2} + \frac{4(5)}{2}\frac{5^3}{z^3} + \ldots\right]$$

... Note this step

$$= \frac{1}{z^3}\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}5^k z^{-k}$$

... general term of series in bracket

$$= \sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}5^k z^{-k-3}$$

... general term

$$\therefore \quad F(z) = \sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}5^k z^{-(k+3)}$$

$\therefore$ coefficient of $z^{-(k+3)} = \dfrac{(k+1)(k+2)}{2}5^k, \ k \geq 0$

Put $k + 3 = n \therefore k = n - 3$ **... Note this step**

$\therefore$ coefficient of $z^{-n} = \dfrac{(n-3+1)(n-3+2)}{2}5^{n-3},$

$n - 3 \geq 0$

$= \dfrac{(n-2)(n-1)}{2}5^{n-3}, \ n \geq 3$

$\therefore$ coefficient of $z^{-k} = \dfrac{(k-1)(k-2)}{2}5^{k-3}, \ k \geq 3$

... Replacing n by k

$\therefore \quad f(k) =$ coefficient of z^{-k}

$= \dfrac{(k-1)(k-2)}{2}5^{k-3}, \ k \geq 3$

$$\text{Thus,} \quad Z^{-1}\frac{1}{(z-5)^3} = \{f(k)\}$$

$$= \left\{\frac{(k-1)(k-2)}{2}5^{k-3}\right\}, k \geq 3$$

Example 4.72 : *Show that*

$$Z^{-1}\left[\frac{z+2}{z^2-2z+1}\right] = \{x_k\} \textit{ for } |z| > 1 \textit{ where}$$

$$x_k = 0, k < 1$$
$$= 3k - 1, k \geq 1$$

SPPU : Dec.-15

Solution : We have,

$$X(z) = \frac{z+2}{z^2-2z+1}, |z| > 1$$

$$= \frac{(z-1)+3}{(z-1)^2}$$

$$= \frac{1}{(z-1)} + \frac{3}{(z-1)^2}, |z| > 1$$

$$= \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{3}{z^2\left(1-\frac{1}{z}\right)^2}$$

$\because |z| > 1 \therefore \left|\dfrac{1}{z}\right| < 1$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{3}{z^2}\left(1-\frac{1}{z}\right)^{-2}$$

$$= \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1^2}{z^2} + \frac{1^3}{z^3} + \ldots\right]$$

$$+ \frac{3}{z^2}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \ldots\right]$$

$$= \left[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right] + 3\left[\frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \ldots\right]$$

$\therefore \quad X(z) = \sum_{k=1}^{\infty} \frac{1}{z^k} + 3 \sum_{k=1}^{\infty} \frac{k}{z^{k+1}}$

$= \sum_{k=1}^{\infty} z^{-k} + \sum_{k=1}^{\infty} 3\,k\,z^{-(k+1)}$

$\therefore$ coefficient of z^{-k} in 1^{st} series = 1

$\therefore \quad f_1(k) = 1,\ k \geq 1$

coefficient of $z^{-(k+1)}$ in 2^{nd} series = 3k, $k \geq 1$

Put $(k + 1 = n)$ $\therefore k = n - 1$

$\therefore$ coefficient of z^{-n} in 2^{nd} series = 3 (n – 1), n – 1 ≥ 1

i.e. $n \geq 2$

$\therefore$ coefficient of z^{-k} in 2^{nd} series = 3 (k – 1), $k \geq 2$,

... Replacing n by k

$\therefore \quad f_2(k) = 3\,k - 3,\ k \geq 2$

$\therefore$ Combining we get

$Z^{-1}\,X(z) = \{x_k\} = \{f_1(k)\} + \{f_2(k)\}$

$= \{1\} + \{3k - 3\}$

$(k \geq 1) \quad (k \geq 2)$

$$\therefore \quad x_k = 0,\ k < 1$$

$$= 1,\ k = 1$$

$$= 1 + (3k - 3) = 3k - 2,\ k \geq 2$$

Thus, $\quad x_k = 0,\ k < 1$

$= 3k - 2,\ k \geq 1$ ... $\because$ at k = 1, 3 (k) – 2

$= 3 - 2 = 1$

Alternately, as we have

$$X(z) = \frac{1}{(z-1)} + \frac{3}{(z-1)^2},\ |z| > 1$$

$\therefore$ Inverting, $\quad \{x_k\} = Z^{-1}\,X(z)$

$$= z^{-1}\left(\frac{1}{z-1}\right) + 3\,z^{-1}\left[\frac{1}{(z-1)^2}\right],$$

$|z| > 1$

$$= \left\{1^{k-1}\,U(k-1)\right\}$$
$$+ 3\left\{(k-1)\,1^{k-2}\,U(k-2)\right\}$$

... Standard result :

$$Z^{-1}\,\frac{1}{(z-a)} = a^{k-1}\,U(k-1)$$

$$Z^{-1}\,\frac{1}{(z-a)^2} = (k-1)\,a^{k-2}\,U(k-2)$$

$\therefore \quad \{x_k\} = \{f_1(k)\} + \{f_2(k)\}$

where, $\quad f_1(k) = 1^{k-1}\,U(k-1)$

Now,

$\because \quad U(k-1) = 0,\ k - 1 < 0$ i.e. $k < 1$

$= 1,\ k \geq 1$

$\therefore \quad f_1(k) = 0,\ k < 1$

$= 1\ (1),\ k \geq 1$

Similarly, $\quad f_2(k) = 3(k-1)\,1^{k-2}\,U(k-2)$

$\because \quad U(k-2) = 0,\ k < 2$

$= 1,\ k \geq 2$

$\therefore \quad f_2(k) = 0,\ k < 2$

$= 3(k-1) \cdot 1,\ k \geq 2$

$\therefore$ Combining we get,

$$x_k = 0 + 0 = 0,\ k < 1$$

$$= 1 + 0 = 1,\ k = 1$$

$$= 1 + 3\,(k - 1) = 3\,k - 2,\ k \geq 2$$

i.e. $\quad x_k = 0,\ k < 1$

$= 3\,k - 2,\ k \geq 1$

Exercise 4.1

Find the inverse Z-transform of the following functions :

1) $\frac{1}{z+a},\ |z| > |a|,\ |z| < |a|$

[Ans. : $(-a)^{k-1},\ k \geq 1;\ (-1)^{-k}\,a^{k-1},\ k < 0$]

2) $\frac{1}{z-3},\ |z| < 3,\ |z| > 3$

[Ans. : $-3^{k-1},\ k \leq 0;\ 3^{k-1},\ k \geq 1$]

3) $\frac{1}{(z-a)^3},\ |z| > a,\ |z| < a$

[Ans. : $\frac{(k-1)\,(k-2)}{2}\,a^{k-3},\ k \geq 3;$ $\frac{-(-k+1)\,(-k+2)}{2}\,a^{k-3},\ k \leq 0$]

4) $\frac{1}{(z-2)\,(z-3)},\ |z| < 2,\ 2 < |z| < 3,\ |z| > 3$

[Ans. : $2^{k-1} + 3^{k-1},\ k \leq 0$; $f(k) = -3^{k-1}\ (k \leq 0)$ $= -2^{k-1}\ (k \geq 0)$ $f(k) = 3^{k-1} - 2^{k-1},\ k \geq 1$]

5) $\frac{z+3}{z^2 - 2z + 1},\ |z| > 1$ **[Ans. : $f(k) = 0,\ k < 1 = 4k - 3,\ k \geq 1$]**

6) $\frac{2\left(z^2 - 5z + 6.5\right)}{(z-2)\,(z-3)^2},\ 2 < |z| < 3$

[Ans. : $f(k) = 2^{k-1},\ k \geq 1 = -(k+2)\,3^{k-2},\ k \leq 0$]

7) $\dfrac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}$, $|z| < \frac{1}{5}$, $\frac{1}{5} < |z| < \frac{1}{4}$, $|z| > \frac{1}{4}$

[Ans. : i) $20\left[\left(\frac{1}{5}\right)^k - \left(\frac{1}{4}\right)^k\right], k < 0$

ii) $f(k) = -20\left(\frac{1}{5}\right)^k, k \geq 0 = -20\left(\frac{1}{4}\right)^k, k < 0$

iii) $f(k) = 20\left[\left(\frac{1}{4}\right)^k - \left(\frac{1}{5}\right)^k\right], k \geq 0$]

8) $\dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$, $|z| < \frac{1}{3}$, $\frac{1}{3} < |z| < \frac{1}{2}$

$|z| > \frac{1}{2}$

[Ans. : i) $2\left(\frac{1}{3}\right)^k - 3\left(\frac{1}{2}\right)^k, k < 0$

ii) $f(k) = -3\left(\frac{1}{2}\right)^k, k < 0 = -2\left(\frac{1}{3}\right)^k, k \geq 0$

iii) $3\left(\frac{1}{2}\right)^k - 2\left(\frac{1}{3}\right)^k, k \geq 0$]

9) $\dfrac{z^2+5z}{z^2-5z+6}$, $2<|z|<3$

[Ans. : $f(k) = -8\ (3)^k, k < 0$
$-7\ (2)^k, k \geq 0$]

10) $\dfrac{z^3}{(z-3)\ (z-2)^2}$, $|z| > 3$

[Ans. : $3^{k+2} - 2^{k+3} - k\ 2^{k+1}, k \geq 0$]

11) $\dfrac{z^2}{z^2+4}$, $|z| > 2$

[Ans. : $2^k \cos \frac{k\pi}{2}$]

12) $\dfrac{2z^2+3z}{z^2+z+\frac{1}{16}}$, $|z| > 2+\sqrt{3}$

[Ans. : $2\left\{\left(-\frac{1}{4}\right)^k \cosh \alpha k - \frac{8}{\sqrt{3}}\left(-\frac{1}{4}\right)^k \sinh \alpha k\right\}$,
$k \geq 0$, where $\alpha = \cosh^{-1} 2$]

13) $\dfrac{2z^2+3z}{(z+2)\ (z-4)}$, $|z| > 4$

[Ans. : $\frac{1}{6}(-2)^k + \frac{11}{6}(4)^k, k \geq 0$]

14) $\dfrac{z}{(z-a)^2}$, $|z|>|a|$, $|z|<|a|$

[Ans. : $ka^{k-1}, k \geq 0; -ka^{k-1}, k < 0$]

15) $\dfrac{z}{z^2-5z+6}$, $2 < |z| < 3$

[Ans. : $f(k) = -3^k, k < 0 = -2^k, k \geq 0$]

4.7 Zeroes, Singular Point, Pole and Residue

a) Zero :

A zero of an analytic function is that value of z for which $f(z) = 0$.

b) Singular Point :

As explained earlier, a singular point of a function f(z) is the point at which the function ceases to be analytic.

Now, if z = a is a singularity of f(z), such that f(z) has no other singularity in the small neighbourhood of 'a', then z = a is called an isolated singularity. In this case, f(z) can be expanded in the form of a Laurent's series around z = a, given as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots + a_{-n}(z-a)^{-n} + \ldots \quad \ldots (4.2)$$

where, $a_n = \dfrac{1}{2\pi i}\oint\limits_C \dfrac{f(z)}{(z-a)^{n+1}}\,dz$, $n = 0, \pm 1, \pm 2, \ldots$

and C is a circle surrounding point z = a.

c) Pole :

If in the Laurent's expansion, there are n terms having negative powers of (z – a) [i.e. upto $a_{-n}(z-a)^{-n}$], then the singular point z = a is called a pole of order n.

Thus, if there is only one such term [i.e. $a_{-1}(z-a)^{-1}$] then z = a is called a simple pole. If there are two such terms [i.e. $a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2}$] then z = a is a double pole or pole of order two etc.

d) Residue :

The coefficient (a_{-1}) of $(z-a)^{-1}$ in the Laurent's expansion of f(z) around an isolated singular point (z = a), is called as the Residue of f(z) at that point. It is given by

$$\text{Residue of } f(z) \text{ at } (z = a) = a_{-1} = \frac{1}{2\pi i}\oint\limits_C f(z)\,dz.$$

where C is circle surrounding point z = a.

4.8 Cauchy's Residue Theorem

If f(z) is analytic within and on a closed curve C, except at a finite number of isolated singular points within C, then

$$\boxed{\oint_C f(z)\,dz = 2\pi i \text{ (Sum of residues at these singular points)} = 2\pi i\,(r_1 + r_2 + r_3 + \ldots)}$$

where r_1, r_2, r_3 ... etc. are the residues of f(z) at the singular points $z = a_1, a_2, a_3$ ... etc. respectively.

[Here, $r_1 = \frac{1}{2\pi i}\int_{C_1} f(z)dz,\ r_2 = \frac{1}{2\pi i}\int_{C_2} f(z)dz$ etc.]

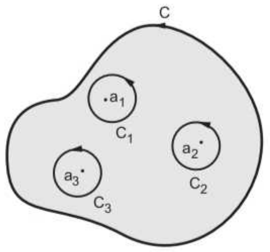

Fig. 4.1

4.9 Working Rule for Finding the Poles and Residues

a) To find the poles of the function : $f(z) = \frac{\phi(z)}{\psi(z)}$:

1. Consider the equation (from the denominator) $\psi(z) = 0$. Its solution, say, z = a, z = b ... etc. gives the poles of f(z).
2. If (z = a) is not a repeated root of $\psi(z) = 0$, then (z = a) is a simple pole of f(z).
3. If (z = b) is two times repeated root of $\psi(z) = 0$, then (z = b) is a double pole i.e. a pole of order (n = 2) of f(z). Similarly, if (z = b) is repeated thrice, it is a pole of order (n = 3).

b) To find residue of f(z) at simple pole (z = a) :

1. It is given by,

   Res f(a) = $\lim_{z \to a} [(z-a)f(z)]$, which is a non-zero finite value.

2. Sometimes we use the following result :

   If, $f(z) = \frac{\phi(z)}{\psi(z)}$ and $\psi(z) = (z-a)\,F(z)$ (say) where $F(a) \neq 0$, then

$$\boxed{\text{Res } f(a) = \frac{\phi(a)}{\psi'(a)} = \left[\frac{\phi(z)}{\frac{d}{dz}\psi(z)}\right]_{z=a}}$$

c) To find the residue of f(z) at the pole (z = b) of order n :

1. It is given by

$$\boxed{\text{Res } f(b) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-b)^n f(z)\right]\right\}_{z=b}}$$

Thus if (z = b) is a double pole i.e. n = 2 then,

$$\text{Res } f(b) = \frac{1}{(2-1)!}\left\{\frac{d^{2-1}}{dz^{2-1}}\left[(z-b)^2 f(z)\right]\right\}_{z=b} = \left\{\frac{d}{dz}\left[(z-b)^2 f(z)\right]\right\}_{z=b}$$

and here, $\lim_{z \to b} [(z-b)^2 f(z)]$ is non-zero and finite

and if (z = b) is a pole of order n = 3 then

$$\text{Res } f(b) = \frac{1}{2!}\left\{\frac{d^2}{dz^2}[(z-b)^3 f(z)]\right\}_{z=b}$$

and here $\lim_{z \to b} [(z-b)^3 f(z)]$ is non-zero and finite.

4.10 Inversion Integral Method by using Residues

The Inverse Z-transform of F(z) can be easily obtained by using the Inversion Integral of F(z), given by

$$f(k) = \frac{1}{2\pi i}\int_C F(z) \cdot z^{k-1}\,dz$$

where, C is the closed curve (contour) (drawn according to the given ROC) such that all the poles of F(z) (i.e. the values of z where F(z) becomes infinite) lie inside it.

More, conveniently we have,

$f(k) = \sum$ Residues of $[F(z) \cdot z^{k-1}]$ at the poles of F(z).

Note :

1) Pole of F(z) is the value of z for which F(z) is infinite

e.g. If $F(z) = \dfrac{z}{(z-a)(z-b)(z-c)^2}$

Then, $F(z) \to \infty$ for z = a, z = b, z = c (repeated twice)

Hence, z = a and z = b are the simple poles of F(z) while z = c is a double (i.e. multiple) pole of F(z) etc.

2) From the Theory of complex variables we have :

a) Residue at a simple pole (z = a) is

$$\left[(z-a)F(z)\cdot z^{k-1}\right]_{z=a}$$

b) Residue at r times repeated pole (i.e. multiple pole) at (z = a) is

$$\frac{1}{(r-1)!}\left[\frac{d^{r-1}}{dz^{r-1}}\left[(z-a)^r F(z)\cdot z^{k-1}\right]\right]_{z=a}$$

$\therefore$ If (r = a) is a double pole, then

Residue at

$$(z = a) = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left[(z-a)^2 F(z)\cdot z^{k-1}\right]\right]_{z=a}$$

3) Working Rule for using Inversion Integral Method

Step 1 : Find the poles of F(z).

Step 2 : Find the expression for $F(z)\cdot z^{k-1}$

Step 3 : Find the residues of $\left[F(z)z^{k-1}\right]$ at all the poles of F (z) using proper formulae.

Step 4 : Take algebraic sum of the residues to get f(k).

Solved Examples on Inversion Integral Method

Example 4.73 : *Use Inversion Integral Method to find* $Z^{-1}\left[\dfrac{10z}{(z-1)(z-2)}\right]$

SPPU : Dec.-14, May-18

Solution :

Step 1 : To find poles of F(z) :

We have, $$F(z) = \frac{10z}{(z-1)(z-2)} \quad \text{... (1)}$$

$\therefore F(z) \to \infty$ at z = 1, z = 2 $\therefore$ (z = 1) and (z = 2) are simple poles of F(z)

Step 2 : To find $F(z)\cdot z^{k-1}$

Now, using F(z) we have,

$$F(z)\cdot z^{k-1} = \frac{10z}{(z-1)(z-2)}z^{k-1}$$

$$= 10\frac{z^k}{(z-1)(z-2)}$$

Step 3 : Find Residues of $F(z)\cdot z^{k-1}$ at poles

$\because$ (z = 1) and (z = 2) are simple poles

$\therefore$ Residue of $\left[F(z)\cdot z^{k-1}\right]$ at (z = 1)

$$= \left[(z-1)F(z)z^{k-1}\right]_{z=1}$$

$$= \left[(z-1)\frac{10z^k}{(z-1)(z-2)}\right]_{z=1}$$

$$= 10\left[\frac{z^k}{(z-2)}\right]_{z=1} = 10\left(\frac{1}{1-2}\right)$$

$\therefore$ Residue at (z = 1) = – 10 ... (2)

Similarly,

$$\text{Residue at } (z = 2) = \left[(z-2)\frac{10z^k}{(z-1)(z-2)}\right]_{z=2}$$

$$= 10\left[\frac{z^k}{z-1}\right] = 10\cdot\frac{2^k}{(2-1)}$$

$$= 10\cdot 2^k \quad \text{... (3)}$$

Step 4 : To take algebraic sum of all residues.

$\therefore$ By Inversion integral method.

$$Z^{-1}F(z) = \{f(k)\} \text{ where}$$

$$f(k) = \sum \text{Residues of } F(z)\,z^{k-1}$$

$$= \text{Res }(z = 1) + \text{Res }(z = 2)$$

$$= -10 + 10\cdot 2^k,\ k \geq 0$$

... using (2) and (3)

$$= 10\left(2^k - 1\right),\ k \geq 0$$

Example 4.74 : *Use Inversion Integral Method to find Inverse Z-transform of* $\dfrac{1}{(z-a)^3}$.

Solution : We have

$$F(z) = \frac{1}{(z-a)^3}$$

$\therefore$ $F(z) \to \infty$ at z = 3 which is repeated (r = 3) times.

$\therefore$ (z = a) is a triple pole of F(z)

Now,

$$F(z)\cdot z^{k-1} = \frac{1}{(z-a)^3}\cdot z^{k-1} \quad \text{... (1)}$$

$\therefore$ Residue of $[F(z) \cdot z^{k-1}]$ at the triple (i.e. r = 3 times repeated) pole (z = a) of F(z) is

$$\text{Residue } (z = a) = \frac{1}{(3-1)!}\left[\frac{d^{3-1}}{dz^{3-1}}\left[\frac{(z-a)^3}{F(z)\cdot z^{k-1}}\right]\right]_{z=a}$$

$$= \frac{1}{2}\left[\frac{d^2}{dz^2}\left[(z-a)^3 \frac{z^{k-1}}{(z-a)^3}\right]\right]_{z=a} \quad \text{... using (1)}$$

$$= \frac{1}{2!}\left[\frac{d^2}{dz^2} z^{k-1}\right]_{z=a}$$

$$= \frac{1}{2!}\left[\frac{d}{dz}\left[(k-1) z^{k-2}\right]\right]_{z=a}$$

$$= \frac{1}{2!}\left[(k-1)(k-2) z^{k-3}\right]_{z=a}$$

$$= \frac{(k-1)(k-2) a^{k-3}}{2!}$$

$$\therefore \quad Z^{-1} F(z) = \{f(k)\} \text{ where}$$

$$f(k) = \sum \text{Residues}$$

$$= \frac{(k-1)(k-2) a^{k-3}}{2}, \quad k \geq 0$$

Example 4.75 : *Find Inverse Z-transform using Inversion Integral Method*

$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}, \quad |z| > 1$$

Solution : We have

$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$\therefore$ F(z) has simple pole at (z = 1) and a double pole at $\left(z = \frac{1}{2}\right)$

$$\boxed{\text{Now, } F(z) z^{k-1} = \frac{z^3 \cdot z^{k-1}}{(z-1)\left(z-\frac{1}{2}\right)^2}}$$

$$= \frac{z^{k+2}}{(z-1)\cdot\left(z-\frac{1}{2}\right)^2} \quad \text{... (1)}$$

$\therefore$ Residue of $[F(z) z^{k-1}]$ at simple pole (z = 1)

$$= \left[(z-1) F(z) z^{k-1}\right]_{z=1}$$

$$= \left[\frac{z^{k+2}}{\left(z-\frac{1}{2}\right)^2}\right]_{z=1} \quad \text{... using (1)}$$

$$= \frac{1^{k+2}}{\left(1-\frac{1}{2}\right)^2} = 4$$

and Residue at the double pole $\left(z = \frac{1}{2}\right)$

$$= \frac{1}{(2-1)!}\left[\frac{d}{dz}\left[\left(z-\frac{1}{2}\right)^2 F(z)\cdot z^{k-1}\right]\right]_{z=\frac{1}{2}}$$

$$= \left[\frac{d}{dz}\frac{z^{k+2}}{z-1}\right]_{z=\frac{1}{2}} \quad \text{... using (1)}$$

$$= \left[\frac{(z-1)(k+2) z^{k+1} - z^{k+2}}{(z-1)^2}\right]_{z=\frac{1}{2}}$$

$$= \frac{\left(-\frac{1}{2}\right)(k+2)\left(\frac{1}{2}\right)^{k+1} - \left(\frac{1}{2}\right)^{k+2}}{\left(\frac{1}{4}\right)}$$

$$= 4\left[-(k+2)\left(\frac{1}{2}\right)^{k+2} - \left(\frac{1}{2}\right)^{k+2}\right]$$

$$= -4(k+2+1)\left(\frac{1}{2}\right)^{k+2}$$

$$= -4(k+3)\left(\frac{1}{2}\right)^{k+2} = -(k+3)\left(\frac{1}{2}\right)^{k}$$

$$\therefore Z^{-1} F(z) = \{f(k)\}$$

$$\text{where } f(k) = \sum \text{Residues}$$

$$= 4 - (k+3)\left(\frac{1}{2}\right)^k, \; k \geq 0, |z| > 1$$

Example 4.76 : *Find inverse Z-transform for* $F(z) = \frac{z}{(z-2)(z+4)^2}$ *by inversion integral method.*

Solution : To find poles of F(z)

We have $F(z) = \dfrac{z}{(z-2)(z+4)^2}$

$\therefore$ $F(z) \to \infty$ at z = 2 and z = – 4

Hence z = 2 is a simple pole and z = – 4 is a double pole of F(z).

We have $F(z)z^{k-1} = \dfrac{z\, z^{k-1}}{(z-2)(z+4)^2} = \dfrac{z^k}{(z-2)(z+4)^2}$

Now, find residue of $F(z)\, z^{k-1}$ at poles.

So residue of $F(z)\, z^{k-1}$ at $z = 2$ is

$$= [(z-2)F(z)z^{k-1}]_{z=2} = \left[(z-2)\frac{z^k}{(z-2)(z+4)^2}\right]_{z=2}$$

$$= \frac{1}{36}2^k;\ k = 0.$$

Now, we have

Residue of $[F(z)z^{k-1}]$ at $z = -4$ is

$$= \frac{1}{(2-1)!}\left[\frac{d}{dz}(z+4)^2 F(z)z^{k-1}\right]_{z=-4}$$

$$= \left[\frac{d}{dz}\frac{z^k}{z-2}\right]_{z=-4} = \left[\frac{(z-2)\,k - z^{k-1} z^k}{(z-2)^2}\right]_{z=-4}$$

$$= \frac{1}{36}[-6k\,(-4)^k - (-4)^{k-1}]$$

Thus, $f(k) = \frac{1}{36}[2^k - 6k(-4)^{k-1} - (-4)^k]$

Example 4.77 : *Use Inversion Integral Method to find* $Z^{-1}\left[\frac{z^3}{(z-3)\,(z-2)^2}\right]$

Solution :

Step 1 : To find poles of F(z) :

We have, $$F(z) = \frac{z^3}{(z-3)\,(z-2)^2}$$

$\therefore$ $F(z) \to \infty$ at $z = 3$ and $z = 2$ which is repeated $(r = 2)$ times.

Hence, $z = 3$ is a simple pole of F(z) and $z = 2$ is a double pole of F(z).

Step 2 : To find expression for $F(z)\cdot z^{k-1}$:

Now, we have,

$$F(z)\cdot z^{k-1} = \frac{z^3}{(z-3)\,(z-2)^2}\cdot z^{k-1} = \frac{z^{k+2}}{(z-3)\,(z-2)^2} \quad \text{... (1)}$$

Step 3 : To find Residues of $F(z)\cdot z^{k-1}$ at the poles of F(z) :

Now, for the simple pole at $(z = 3)$ we have,

Residue of $\left[F(z)\cdot z^{k-1}\right]$ at $(z = 3)$

$$= \left[(z-3)\ F(z)\cdot z^{k-1}\right]_{z=3}$$

... $\because$ Res $(z = a)$

$$= \left[(z-a)\ F(z)\ z^{k-1}\right]_{z=a} \quad \text{for a simple pole}$$

$$= \left[(z-3)\frac{z^{k+2}}{(z-3)\,(z-2)^2}\right]_{z=3} \quad \text{... using (1)}$$

$$= \left[\frac{z^{k+2}}{(z-2)^2}\right]_{z=3} = \frac{3^{k+2}}{(3-2)^2} = 3^{k+2}$$

$\therefore$ Res $(z = a) = 3^{k+2},\ k \ge 0$ ... (2)

Also, for the double (i.e. $r = 2$) pole at $(z = 2)$ we have,

Residue of $\left[F(z)\cdot z^{k-1}\right]$ at $(z = 2)$

$$= \frac{1}{(2-1)!}\left[\frac{d^{2-1}}{dz^{2-1}}\left[(z-2)^2\ F(z)\cdot z^{k-1}\right]\right]_{z=2}$$

... $\because$ Res $(z = a)$

$$= \frac{1}{(r-1)!}\left[\frac{d^{r-1}}{dz^{r-1}}\left[(z-a)^r\ F(z)\cdot z^{k-1}\right]\right]_{z=a}$$

for r times repeated pole

$$= \frac{1}{1}\left[\frac{d}{dz}\left[(z-2)^2\ \frac{z^{k+2}}{(z-3)\,(z-2)^2}\right]\right]_{z=2}$$

... using (1)

$$= \left[\frac{d}{dz}\left(\frac{z^{k+2}}{z-3}\right)\right]_{z=2}$$

$$= \left[\frac{(z-3)\,(k+2)\,z^{k+1} - z^{k+2}\,(1)}{(z-3)^2}\right]_{z=2}$$

$$= \frac{(2-3)\,(k+2)\cdot 2^{k+1} - 2^{k+2}}{(2-3)^2} = -(k+2)\,2^{k+1} - 2^{k+2}$$

$$= -(k+2)\,2^{k+1} - 2\cdot 2^{k+1} = -[(k+2)+2]\,2^{k+1}$$

$\therefore$ Residue $(z = 2) = -(k+4)\,2^{k+1},\ k \ge 0$ ... (3)

Step 4 : Take algebraic sum of all the Residues.

Thus, using (2) and (3) we have, by inversion integral method

$$Z^{-1}[F(z)] = \{f(k)\}$$

where, $f(k) = \sum$ Residues of $\left[F(z)\,z^{k-1}\right]$ at poles of F (z)

$$= 3^{k+2} - (k+4)\,2^{k+1},\ k \ge 0$$

Note : Here we assume the ROC as $|z| > 3$. Hence, its contour is the circle, $C : |z| = 3$ and hence both the poles, $(z = 2)$ and $(z = 3)$ lie inside C.

Example 4.78 : *Find Inverse Z-transform using Inversion Integral Method,*

$$f(z) = \frac{z^2}{z^2+1}$$

SPPU : Dec.-16, 17

Solution : We have,

$$F(z) = \frac{z^2}{z^2+1} = \frac{z^2}{(z-i)(z+i)}$$

... Note this step

$\therefore$ F (z) has simple poles at (z = i) and (z = – i)

$$\text{Now,} \quad F(z)\, z^{k-1} = \frac{z^2 \cdot z^{k-1}}{(z-i)(z+i)}$$

$$= \frac{z^{k+1}}{(z-i)(z+i)} \qquad \text{... (1)}$$

$\therefore$ Residue of $[F(z)\, z^{k-1}]$ at (z = i)

$$= [(z-i)\, F(z)\, z^{k-1}]_{z=i}$$

$$= \left[\frac{z^{k+1}}{z+i}\right]_{z=i} \qquad \text{... using (1)}$$

$$= \frac{(i)^{k+1}}{i+i} = \frac{1}{2i}(i)^{k+1} = \frac{(i)^k}{2}$$

and Residue at (z = – i)

$$= [[z-(-i)]\, F(z)\, z^{k-1}]_{z=-i}$$

$$= \left[\frac{z^{k+1}}{z-i}\right]_{z=-i}$$

$$= \frac{(-i)^{k+1}}{-i-i} = \frac{(-i)^{k+1}}{2(-i)} = \frac{(-i)^k}{2}$$

$$\therefore \quad Z^{-1}\left(\frac{z^2}{z^2+1}\right) = \{f(k)\}$$

$$\text{where,} \quad f(k) = \sum \text{Residues}$$

$$= \frac{(i)^k}{2} + \frac{(-i)^k}{2}$$

$$= \frac{1}{2}[(i)^k + (-i)^k]$$

$$\text{Now,} \quad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$$

$$\text{and} \quad -i = e^{-i\pi/2}$$

$$\therefore \quad (i)^k + (-i)^k = e^{i\frac{k\pi}{2}} + e^{-i\frac{k\pi}{2}} = 2\cos\frac{k\pi}{2}$$

$$\text{...} \because e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\therefore \quad f(k) = \frac{1}{2}\left(2\cos\frac{k\pi}{2}\right) = \cos\frac{k\pi}{2},\ k \geq 0$$

Example 4.79 : *Find the inverse z transform :* $F(z) = \dfrac{1}{(z-2)(z-3)}$, *by using inversion integral method.*

SPPU : May-16, Marks 4

Solution :

$$F(z) = \frac{1}{(z-2)(z-3)}$$

$$F(z) \to \infty \text{ as } z \to 2 \text{ or } 3$$

$$\therefore \quad z = 2 \text{ and } 3 \text{ are poles of } F(z)$$

$$\text{Now } F(z)\, z^{k-1} = \frac{z^{k-1}}{(z-2)(z-3)}$$

Residue of $[F(z)\, z^{k-1}]$ at z = 2 is given by

$$R_1 = [(z-2)\, F(z)\, z^{k-1}]_{z=2}$$

$$= \left[\frac{z^{k-1}}{z-3}\right]_{z=2} = -(2)^{k-1}$$

Residue of $[F(z)\, z^{k-1}]$ at z = 3 is given by

$$R_2 = [(z-3)\, F(z)\, z^{k-1}]_{z=3} = \left[\frac{z^{k-1}}{z-2}\right]_{z=3}$$

$$= 3^{k-1}$$

$$\therefore \quad f(k) = Z^{-1}\,[F(z)] = R_1 + R_2$$

$$= 3^{k-1} - 2^{k-1} \quad k \geq 0$$

Exercise 4.2

Find Inverse Z-transform using Inversion Integral Method

1) $\dfrac{1}{(z-2)(z-3)}$ **[Ans. : $3^{k-1} - 2^{k-1}$]**

2) $\dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$ **[Ans. : $3\left(\frac{1}{2}\right)^k - 2\left(\frac{1}{3}\right)^k$]**

3) $\left(\dfrac{z}{z-a}\right)^2$ **[Ans. : $(k+1)\, a^k$]**

4) $\dfrac{z^2+z}{(z-1)(z^2+z+1)}$ **[Ans. : $\frac{2}{3}\left(1-\cos\frac{2\pi k}{3}\right)$]**

5) $\dfrac{2z}{(z-1)(z^2+1)}$ **[Ans. : $1 - \frac{(i)^n}{(1+i)} - \frac{(-i)^n}{(1-i)}$]**

4.11 Solutions of Simple Difference Equations (with constant coefficients) using Z-transforms

A linear equation between f(k) and f(k + 1), f(k + 2), f(k + 3) ... etc. is called a Linear Difference Equation and the expression, f(k) in terms of k, which satisfies this difference equation is called its solution.

e.g. $f(k+1)+\frac{1}{4}f(k)=\left(\frac{1}{2}\right)^k,\ k\geq 0$

is a difference equation and the function f(k) satisfying it is its solution.

The performance of discrete systems is expressed using suitable difference equations. The Z-transforms are very useful in solving these difference equations, similar to the Laplace Transforms as in case of Linear Differential Equations (representing continuous systems).

Working Rule for solving Linear Difference Equations with constant coefficients (using Z-transforms) :

Step 1 : Take the Z-transform of both sides of the given difference equation by using suitable standard formulae. Use given conditions, if any, properly.

Step 2 : Rearrange the resultant algebraic equation in z to express F(z) in terms of z.

Step 3 : Use standard methods to find the Inverse Z-transform of F(z), i.e. f(k).

> **Note**
>
> 1) {f(k)} is considered as a causal sequence, where $k \geq 0$
>
> 2) For a causal sequence $\{f(k)\}$, we have the standard results :

$$\begin{aligned} Z\{f(k)\} &= F(z) \\ Z\{f(k+1)\} &= z\,F(z)-z\,f(0) \\ Z\{f(k+2)\} &= z^2\,F(z)-z^2 f(0)-z\,f(1) \text{ etc.} \\ Z\{f(k-1)\} &= z^{-1}\,F(z) \\ Z\{f(k-2)\} &= z^{-2}\,F(z) \text{ etc.} \end{aligned}$$

Example 4.80 : *Obtain $f(k)$ given that $12f(k+2)-7f(k+1)+f(k)=0,\ k\geq 0,\ f(0)=0,\ f(1)=3$*

SPPU : Dec.-16

Solution : We have,

$$12\,f(k+2)-7f(k+1)+f(k)=0$$

$$\therefore\ 12Z\{f(k+2)\}-7Z\{f(k+1)\}+Z\{f(k)\}=0$$

i.e. $$12\left[z^2\,F(z)-z^2 f(0)-z\,f(1)\right] -7[z\,F(z)-z\,f(0)]+F(z)=0$$

i.e. $$12\left[z^2\,F(z)-0-3z\right]-7[z\,F(z)-0]+F(z)=0$$

$$\dots \because f(0)=0,\ f(1)=3$$

i.e. $$\left(12z^2-7z+1\right)F(z) = 36z$$

$$\therefore\ F(z) = \frac{36z}{12z^2-7z+1}$$

$$\therefore\ \frac{F(z)}{z} = \frac{36}{(4z-1)(3z-1)}$$

$$= 36\,\frac{1}{(4z-1)(3z-1)}$$

$$= 36\left[\frac{-4}{(4z-1)}+\frac{3}{(3z-1)}\right]$$

$$\dots \text{Partial fraction } A=\frac{1}{\frac{3}{4}-1}=-4$$

$$B=\frac{1}{\frac{4}{3}-1}=3$$

$$\therefore\ F(z) = 36\left[\frac{3z}{3z-1}-\frac{4z}{4z-1}\right]$$

$$= 36\left(\frac{z}{z-\frac{1}{3}}-\frac{z}{z-\frac{1}{4}}\right)$$

$$\therefore\ \{f(k)\} = Z^{-1}\,F(z)$$

$$= 36\,Z^{-1}\left(\frac{z}{z-\frac{1}{3}}\right)-36\,Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$= 36\left\{\left(\frac{1}{3}\right)^k\right\}-36\left\{\left(\frac{1}{4}\right)^k\right\},\ k\geq 0$$

$$\dots \text{assuming } |z|>\frac{1}{3} \text{ since here } k\geq 0$$

$$\therefore\ f(k) = 36\left[\left(\frac{1}{3}\right)^k-\left(\frac{1}{4}\right)^k\right],\ k\geq 0$$

Example 4.81 : *Use Z-transforms to solve* $f(k+1) - f(k) = 1,\ f(0) = 0$

SPPU : Dec.-14

Solution : We have the difference equation,

$$f(k+1) - f(k) = 1,\ f(0) = 0$$

$\therefore Z\{f(k+1)\} - Z\{f(k)\} = Z\{1\}$

i.e. $[z F(z) - z f(0)] - F(z) = \dfrac{z}{z-1},\ |z| > 1$

i.e. $[z F(z) - 0] - F(z) = \dfrac{z}{(z-1)}$ ... $\because f(0) = 0$

i.e. $(z-1) F(z) = \dfrac{z}{(z-1)}$

$\therefore \quad F(z) = \dfrac{z}{(z-1)^2}$

$\therefore \quad \{f(k)\} = Z^{-1} F(z) = Z^{-1}\left[\dfrac{z}{(z-1)^2}\right]$

$= \{k(1^{k-1})\},\ k \geq 0$

... $\because Z^{-1}\left[\dfrac{z}{(z-a)^2}\right] = ka^{k-1},\ k \geq 0$

$= \{k\},\ k \geq 0$

$\therefore \quad f(k) = k,\ k \geq 0$

Example 4.82 : *Solve, the following difference equation to find* $\{f(k)\}$ *:* $f(k+2) + 3f(k+1) + 2f(k) = 0,\ f(0) = 0,\ f(1) = 1.$

SPPU : May-15,16, Dec.-18

Solution : We have the difference equation

$$f(k+2) + 3f(k+1) + 2f(k) = 0$$

Step 1 : Take the Z-transform of both the sides

$\therefore\ Z\{f(k+2)\} + 3Z\{f(k+1)\} + 2\{f(k)\} = 0$

i.e. $\left[z^2 F(z) - z^2 f(0) - z f(1)\right]$
$+ 3[z F(z) - z f(0)] + 2 F(z) = 0$

where $F(z) = Z\{f(k)\}$ ...using Standard results

i.e.

$\left[z^2 F(z) - 0 - z(1)\right] + 3[z F(z) - 0] + 2 F(z) = 0$

... $\because f(0) = 0,\ f(1) = 1$ (given)

Step 2 : Rearrange this algebraic equation to find $F(z)$:

Thus, $\left(z^2 + 3z + 2\right) F(z) - z = 0$

$\therefore \quad F(z) = \dfrac{z}{z^2 + 3z + 2}$

Step 3 : Find $Z^{-1} F(z)$:

Now, $F(z) = \dfrac{z}{z^2 + 3z + 2}$

$\therefore \quad \dfrac{F(z)}{z} = \dfrac{1}{(z+1)(z+2)}$

$\therefore \quad \dfrac{F(z)}{z} = \dfrac{1}{z+1} - \dfrac{1}{z+2}$ ... Partial fractions

$\therefore \quad F(z) = \dfrac{z}{z+1} - \dfrac{z}{z+2}$

$\therefore$ Inverting we get,

$\{f(k)\} = Z^{-1} F(z)$

$= Z^{-1}\left[\dfrac{z}{z-(-1)}\right] - Z^{-1}\left[\dfrac{z}{z-(-2)}\right]$

$= \{(-1)^k\} - \{(-2)^k\},\ k \geq 0,$

... assuming $|z| > 2$ i.e. $|z| > |-2|$ and

$\therefore |z| > |-1|$ and $\because Z^{-1}\left(\dfrac{z}{z-a}\right) = a^k,\ k \geq 0$ for $|z| > |a|$

$= \{(-1)^k - (-2)^k\},\ k \geq 0$

$\therefore \quad f(k) = (-1)^k - (-2)^k,\ k \geq 0$

Example 4.83 : *Obtain* $f(k)$, *given that* $f(k+1) + \frac{1}{4} f(k) = \left(\frac{1}{4}\right)^k,\ k \geq 0,\ f(0) = 0.$

SPPU : Dec.-15

Solution : We have, the difference equation :

$$f(k+1) + \frac{1}{4} f(k) = \left(\frac{1}{4}\right)^k,\ k \geq 0$$

Step 1 : Taking Z-transform of both the sides, we get

$Z\{f(k+1)\} + \frac{1}{4} Z\{f(k)\} = Z\left\{\frac{1}{4}\right\}^k$

i.e. $[z F(z) - z f(0)] + \frac{1}{4} F(z) = \dfrac{z}{z - \frac{1}{4}},\ |z| > \frac{1}{4}$

... using Standard results

i.e. $[z F(z) - 0] + \frac{1}{4} F(z) = \dfrac{z}{z - \frac{1}{4}}$

... $\because f(0) = 0$ (given)

Step 2 : To find $F(z)$:

$\therefore \quad \left(z + \frac{1}{4}\right) F(z) = \dfrac{z}{z - \frac{1}{4}}$

$$\therefore \quad F(z) = \frac{z}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)} \qquad \ldots |z| > \frac{1}{4}$$

Step 3 : To find $Z^{-1}\ F(z)$

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)}$$

$$= \frac{\frac{1}{2}}{\left(z-\frac{1}{4}\right)} + \frac{\left(-\frac{1}{2}\right)}{\left(z+\frac{1}{4}\right)}$$

... Partial fractions

$$\therefore \quad F(z) = \frac{1}{2}\frac{z}{\left(z-\frac{1}{4}\right)} - \frac{1}{2}\frac{z}{\left(z+\frac{1}{4}\right)}$$

$\therefore$ Inverting,

$$Z^{-1}\ F(z) = \{f(k)\} = \frac{1}{2} Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right)$$

$$-\frac{1}{2} Z^{-1}\left[\frac{z}{z-\left(-\frac{1}{4}\right)}\right]$$

$$\ldots |z| > \frac{1}{4} \text{ i.e. } |z| > \left|-\frac{1}{4}\right|$$

$$\therefore \quad \{f(k)\} = \frac{1}{2}\left\{\left(\frac{1}{4}\right)^k\right\} - \frac{1}{2}\left\{\left(-\frac{1}{4}\right)^k\right\},\ k \ge 0$$

$$\therefore \quad f(k) = \frac{1}{2}\left[\left(\frac{1}{4}\right)^k - \left(-\frac{1}{4}\right)^k\right],\ k \ge 0$$

Example 4.84 : *Use Z-transforms to solve* $y_k - \frac{5}{6} y_{k-1} + \frac{1}{6} y_{k-2} = \left(\frac{1}{2}\right)^k,\ k \ge 0$

Solution : We have the difference equation,

$$y_k - \frac{5}{6} y_{k-1} + \frac{1}{6} y_{k-2} = \left(\frac{1}{2}\right)^k,\ k \ge 0$$

$$\therefore Z\{y_k\} - \frac{5}{6} Z\{y_{k-1}\} + \frac{1}{6} Z\{y_{k-2}\} = Z\left\{\left(\frac{1}{2}\right)^k\right\}$$

$$\text{i.e. } Y(z) - \frac{5}{6} z^{-1}\ Y(z) + \frac{1}{6} z^{-2}\ Y(z) = \frac{z}{z-\frac{1}{2}},\ |z| > \frac{1}{2}$$

$$\ldots \because Z\{f(k-1)\} = z^{-1}\ F(z)$$
$$\text{and } Z\{f(k-2)\} = z^{-2}\ F(z)$$

$$\text{i.e.} \quad \left(1-\frac{5}{6z}+\frac{1}{6z^2}\right) Y(z) = \frac{z}{\left(z-\frac{1}{2}\right)}$$

$$\therefore \quad \frac{1}{z^2}\left(z^2-\frac{5}{6}z+\frac{1}{6}\right) Y(z) = \frac{z}{\left(z-\frac{1}{2}\right)}$$

$$\therefore \quad Y(z) = \frac{z^3}{\left(z-\frac{1}{2}\right)\left(z^2-\frac{5}{6}z+\frac{1}{6}\right)}$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$= \frac{z^2}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)^2} \qquad \ldots (1)$$

Now, consider

$$\frac{z^2}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)^2} = \frac{A}{\left(z-\frac{1}{3}\right)} + \frac{B}{\left(z-\frac{1}{2}\right)} + \frac{C}{\left(z-\frac{1}{2}\right)^2}$$

... Partial fractions

$$\therefore \quad z^2 = A\left(z-\frac{1}{2}\right)^2 + B\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right) + C\left(z-\frac{1}{3}\right) \qquad \ldots (2)$$

Put $z = \frac{1}{2}$

$$\therefore \quad \frac{1}{4} = C\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{C}{6}$$

$$\therefore \quad C = \frac{6}{4} = \frac{3}{2}$$

Put $z = \frac{1}{3}$ in (2),

$$\therefore \quad \frac{1}{9} = A\left(\frac{1}{3}-\frac{1}{2}\right)^2 = A\left(-\frac{1}{6}\right)^2$$

$$\therefore \quad A = \frac{36}{9} = 4$$

Put $z = 0$ in (2),

$$\therefore \quad 0 = A\left(\frac{1}{4}\right) + B\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right) + C\left(-\frac{1}{3}\right)$$

$$= 4\left(\frac{1}{4}\right) + \frac{B}{6} + \frac{3}{2}\left(\frac{-1}{3}\right)$$

$$= 1 + \frac{B}{6} - \frac{1}{2} = \frac{1}{2} + \frac{B}{6}$$

$$\ldots \because A = 4,\ C = \frac{3}{2}$$

$\therefore \quad \frac{B}{6} = -\frac{1}{2}$

$\therefore \quad B = -3$

$\therefore$ Substituting for A, B and C and using (1) we get

$$\frac{Y(z)}{z} = \frac{4}{\left(z-\frac{1}{3}\right)} - \frac{3}{\left(z-\frac{1}{2}\right)} + \frac{3/2}{\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad Y(z) = 4\left(\frac{z}{z-\frac{1}{3}}\right) - 3\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{3}{2}\frac{z}{\left(z-\frac{1}{2}\right)^2}$$

$$\therefore \quad \{y_k\} = Z^{-1}\,Y(z) = 4\,Z^{-1}\left(\frac{z}{z-\frac{1}{3}}\right)$$

$$- 3\,Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{3}{2}Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)^2}\right]$$

$$|z| > \frac{1}{2} \Rightarrow |z| > \frac{1}{3}$$

$$= 4\left\{\left(\frac{1}{3}\right)^k\right\} - 3\left\{\left(\frac{1}{2}\right)^k\right\} + \frac{3}{2}\left\{k\left(\frac{1}{2}\right)^{k-1}\right\},$$

$k \geq 0$ ... standard results

$$= \left\{4\left(\frac{1}{3}\right)^k - 3\left(\frac{1}{2}\right)^k + 3\,k\left(\frac{1}{2}\right)^k\right\}, \; k \geq 0$$

$$\therefore \quad y_k = 4\left(\frac{1}{3}\right)^k - 3\left(\frac{1}{2}\right)^k + 3k\left(\frac{1}{2}\right)^k, \; k \geq 0$$

Example 4.85 : *Obtain output of the system where input U_k and the system is given by : $y_{k+2} - 3\,y_{k+1} + 2\,y_k = U_k$*

where $\quad U_k = \left(\frac{1}{3}\right)^k, \; k \geq 0$

$\quad = 0, \; k < 0$

Solution : Here y_k is the output of the system satisfying the difference equation :

$$y_{k+2} - 3\,y_{k+1} + 2\,y_k = U_k \qquad \text{... (1)}$$

Step 1 : Taking Z-transform of both the sides

$$Z\{y_{k+2}\} - 3\{y_{k+1}\} + 2\{y_k\} = Z\{U_k\}$$

i.e. $\left[z^2\,Y(z) - z^2 y_0 - z\,y_1\right]$

$$- 3\left[z\,Y(z) - z\,y_0\right] + 2\,Y(z) = Z\left\{\left(\frac{1}{3}\right)^k\right\},$$

$k \geq 0 \qquad$ ... (2)

where $Z\{y_k\} = Y(z)$

Now, $\because \quad U_k = \left(\frac{1}{3}\right)^k, \; k \geq 0$

$\quad = 0, \; k < 0$

$\therefore$ Putting $k = -2$ in (1)

$\therefore \; y_0 - 3y_{-1} + 2\,y_{-2} = U_{(-2)}$

$\therefore \quad y_0 = 0 \qquad$ ... $\because y_k$ is causal

$\therefore \quad y_{-1} = y_{-2} = 0$ and for $k < 0$,

$U_k = 0$

Also, putting $\quad k = -1$ in (1)

$y_1 - 3\,y_0 + 2\,y_{-1} = U_{-1}$

$\therefore \quad y_1 - 3\,y_0 = 0$

$\therefore \quad y_1 = 3y_0 = 0 \qquad \because y_0 = 0$

$\therefore \quad y_0 = 0$ and $y_1 = 0$

$\therefore$ From (2) we get

$$z^2\,Y(z) - 3\,zY(z) + 2\,Y(z) = Z\left\{\left(\frac{1}{3}\right)^k\right\} = \frac{z}{z-\frac{1}{3}}$$

i.e. $\quad (z^2 - 3z + 2)\,Y(z) = \frac{z}{z-\frac{1}{3}}$

$$\therefore \quad Y(z) = \frac{z}{\left(z-\frac{1}{3}\right)\left(z^2 - 3z + 2\right)}$$

$$= \frac{z}{\left(z-\frac{1}{3}\right)(z-1)\,(z-2)} \qquad \text{... (3)}$$

Let us use Inversion Integral method to find $\{y_k\}$.

Now from (3) we see that

Y(z) has simple poles at $z = \frac{1}{3}$, $z = 1$, $z = 2$.

Let $|z| > 2$ so that all these poles lie inside the contour $C : |z| = 2$

$$\text{Now,} \quad Y(z)\,z^{k-1} = \frac{z^k}{\left(z-\frac{1}{3}\right)(z-1)\,(z-2)}$$

... using (3) ... (4)

$\therefore$ Residue of $\left[Y(z)\,z^{k-1}\right]$ at pole $\left(z = \frac{1}{3}\right)$

$$= \left[\left(z-\frac{1}{3}\right) Y(z) z^{k-1}\right]_{z=\frac{1}{3}}$$

$$= \left[\frac{z^k}{(z-1)(z-2)}\right]_{z=\frac{1}{3}} \quad \text{... using (4)}$$

... using standard formula for residue

$$= \frac{\left(\frac{1}{3}\right)^k}{\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}$$

$$= \frac{\left(\frac{1}{3}\right)^k}{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)} = \frac{9}{10}\left(\frac{1}{3}\right)^k$$

Residue at

$$(z = 1) = \left[(z-1) Y(z) z^{k-1}\right]_{z=1}$$

$$= \left[\frac{z^k}{\left(z-\frac{1}{3}\right)(z-2)}\right]_{z=1}$$

... using (4)

$$= \frac{1^k}{\left(\frac{2}{3}\right)(-1)} = -\frac{3}{2} 1^k = -\frac{3}{2}$$

Residue at

$$(z = 2) = \left[(z-2) Y(z) z^{k-1}\right]_{z=2}$$

$$= \left[\frac{z^k}{\left(z-\frac{1}{3}\right)(z-1)}\right]_{z=2}$$

... using (4)

$$= \frac{2^k}{\frac{5}{3}(1)} = \frac{3}{5} 2^k$$

$\therefore \quad Z^{-1} Y(z) = \{y_k\}$

where, $\quad y_k = \sum$ Residues

$$= \frac{9}{10}\left(\frac{1}{3}\right)^k - \frac{3}{2} + \frac{3}{5} 2^k,$$

$k \geq 0, |z| > 2$

Example 4.86 : *Obtain the output of the system, where the input is U_k and the system is given by $y_k - 4y_{k-2} = U_k$*

where $\quad U_k = \left(\frac{1}{2}\right)^k, \; k \geq 0$

$\quad = 0, \; k < 0$

Solution : Here the output of the system is $\{y_k\}$ which is assumed to be causal.

Hence, $y(-1) = y(-2) = ... = 0$

Now, the difference equation of the system is

$$y_k - 4y_{k-2} = U_k$$

$\therefore$ Taking Z-transform of both the sides,

$Z\{y_k\} - 4Z\{y_{k-2}\} = Z\{U_k\}$

i.e. $Y(z) - 4\left[z^{-2} Y(z)\right] = Z\left\{\left(\frac{1}{2}\right)^k\right\}, \because k \geq 0$ where

$Y(z)=Z\{y_k\}$ ... using Standard Result for $Z\{f(k-2)\}$

and $\because U_k = \left(\frac{1}{2}\right)^k, \; k \geq 0$

i.e. $$\left(1-\frac{4}{z^2}\right) Y(z) = \frac{z}{z-\frac{1}{2}}, |z| > \frac{1}{2}$$

i.e. $$\left(\frac{z^2-4}{z^2}\right) Y(z) = \frac{z}{\left(z-\frac{1}{2}\right)}$$

$$\therefore \quad Y(z) = \frac{z^3}{\left(z-\frac{1}{2}\right)(z^2-4)}$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{z^2}{\left(z-\frac{1}{2}\right)(z-2)(z+2)}$$

$$= \frac{\left(-\frac{1}{15}\right)}{z-\frac{1}{2}} + \frac{\left(\frac{2}{3}\right)}{z-2} + \frac{\left(\frac{2}{5}\right)}{z+2}$$

... Partial fractions : $A = \dfrac{\frac{1}{4}}{\left(-\frac{3}{2}\right)\left(\frac{5}{2}\right)} = -\dfrac{1}{15}$

$$B = \frac{4}{\left(\frac{3}{2}\right)(4)} = \frac{2}{3},$$

$$C = \frac{4}{\left(-\frac{5}{2}\right)(-4)} = \frac{2}{5}$$

$\therefore$ $$Y(z) = -\frac{1}{15}\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{2}{3}\left(\frac{z}{z-2}\right) + \frac{2}{5}\left(\frac{z}{z+2}\right)$$

$\therefore$ Inverting

$$Z^{-1}Y(z) = \{y_k\} = -\frac{1}{15}Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)$$

$$+\frac{2}{3}Z^{-1}\left(\frac{z}{z-2}\right)+\frac{2}{5}Z^{-1}\left[\frac{z}{z-(-2)}\right]$$

and assume $|z| > 2 \therefore |z| > |-2|, |z| > \frac{1}{2}$

$\therefore$ $$\{y_k\} = -\frac{1}{15}\left\{\left(\frac{1}{2}\right)^k\right\} + \frac{2}{3}\{(2)^k\}$$

$$+\frac{2}{5}\{(-2)^k\},\ k \geq 0$$

$\therefore$ Output of the system is $\{y_k\}$ where

$$y_k = \frac{2}{3}(2^k)+\frac{2}{5}(-2)^k-\frac{1}{15}\left(\frac{1}{2}\right)^k, \qquad k \geq 0$$

Example 4.87 : *Solve the difference equation $f(k)+\frac{1}{4}f(k-1) = \delta(k)+\frac{1}{3}\delta(k-1)$ where $\delta(k)$ is a unit Impulse sequence.*

Solution : We have

$$f(k)+\frac{1}{4}f(k-1) = \delta(k)+\frac{1}{3}\delta(k-1)$$

$$\therefore Z\{f(k)\}+\frac{1}{4}Z\{f(k-1)\} = Z\{\delta(k)\}+\frac{1}{3}Z\{\delta(k-1)\}$$

$$\text{i.e.} F(z)+\frac{1}{4}z^{-1}F(z) = Z\{\delta(k)\} + \frac{1}{3}z^{-1}Z\{\delta(k)\}$$

$$\ldots \because Z\{f(k-1)\} = z^{-1}Z\{f(k)\}$$

$\therefore$ $$\left(1+\frac{1}{4z}\right)F(z) = 1+\frac{1}{3z}(1) \qquad .. \because Z\{\delta(k)\} = 1$$

$\therefore$ $$F(z) = \frac{1+\frac{1}{3z}}{1+\frac{1}{4z}}$$

$$= \left(1+\frac{1}{3z}\right)\left(1+\frac{1}{4z}\right)^{-1}$$

$$= \left(1+\frac{1}{3z}\right)\left[1-\frac{1}{4z}+\left(\frac{1}{4z}\right)^2-\left(\frac{1}{4z}\right)^3+\ldots\right]$$

... Assuming $\left|\frac{1}{4z}\right| < 1$ i.e. $|z| > \frac{1}{4}$ and

$\because (1+x)^{-1} = 1-x+x^2-x^3+\ldots,\ |x| < 1$

$$= \left[1-\frac{1}{4z}+\frac{1}{4^2z^2}-\frac{1}{4^3z^3}+\ldots\right]$$

$$+\frac{1}{3}\left[\frac{1}{z}-\frac{1}{4z^2}+\frac{1}{4^2z^3}\ldots\right]$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{4^k z^k}+\frac{1}{3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{4^{k-1}z^k}$$

$$= \sum_{k=0}^{\infty}\left(-\frac{1}{4}\right)^k z^{-k}+\frac{1}{3}\sum_{k=1}^{\infty}\left(-\frac{1}{4}\right)^{k-1}z^{-k}$$

$\therefore$ Coefficient of z^{-k} in 1st series

$$= f_1(k) = \left(-\frac{1}{4}\right)^k, \qquad k \geq 0$$

Coefficient of z^{-k} in 2nd series

$$= f_2(k) = \frac{1}{3}\left(-\frac{1}{4}\right)^{k-1}, \qquad k \geq 1$$

$\therefore$ $$Z^{-1}F(z) = \{f(k)\}$$

where $$f(k) = f_1(k)+f_2(k)$$

$$= \left(-\frac{1}{4}\right)^k + \frac{1}{3}\left(-\frac{1}{4}\right)^{k-1}$$

$$(k \geq 0) \qquad (k \geq 1)$$

$\therefore$ $$f(k) = 1 + 0 = 1,\ k = 0$$

$$= \left(-\frac{1}{4}\right)^k + \frac{1}{3}\left(-\frac{1}{4}\right)^{k-1}$$

$$= \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1}+\frac{1}{3}\left(-\frac{1}{4}\right)^{k-1},$$

$$k \geq 1$$

$\therefore$ $$f(k) = 1,\ k = 0$$

$$= \left(\frac{1}{3}-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1}$$

$$= \frac{1}{12}\left(-\frac{1}{4}\right)^{k-1},\ k \geq 1$$

Thus, $$f(k) = 1,\ k = 0$$

$$= \frac{1}{12}\left(-\frac{1}{4}\right)^{k-1},\ k \geq 1$$

Exercise 4.3

Solve the following difference equations to find f (k) :

1) $f(k+1)+\frac{1}{2}f(k)=\left(\frac{1}{2}\right)^k,\ k\geq 0,\ f(0)=0.$

$$\left[\text{Ans. : } f(k)=\left(\frac{1}{2}\right)^k-\left(-\frac{1}{2}\right)^k,\ k\geq 0\right]$$

2) $f(k+2)+4f(k+1)+3f(k)=3^k,\quad k\geq 0,\quad f(0)=0,$ $f(1)=1.$

$$\left[\text{Ans. : } f(k)=\frac{3}{8}(-1)^k+\frac{1}{24}3^k-\frac{5}{12}(-3)^k,\ k\geq 0\right]$$

3) $y_{k+2}+6y_{k+1}+9y_k=2^k,\ y_0=y_1=0$

$$\left[\text{Ans. : } f(k)=\frac{1}{25}\left[2^k-(-3)^k+\frac{5}{3}k(-3)^k\right],\ k\geq 0\right]$$

4) $4f(k)-f(k+2)=0,\ f(0)=0,\ f(1)=2$

$$\left[\text{Ans. : } f(k)=\frac{1}{2}\left[2^k-(-2)^k\right],\ k\geq 0\right]$$

5) $f(k+2)-3f(k+1)+2f(k)=U(k)$ where $k\geq 1$ and

$U(k)=0,\ k<0,\ k>0$

$=1,\ k=0$

$$\left[\text{Ans. : } f(k)=2^{k-1}-1,\ k>0,\ |z|>2\right]$$

6) $U_{n+2}+U_{n+1}+U_n=0,\ U_0=U_1=1$

$$\left[\text{Ans. : } U_n=\cos\frac{2n\pi}{3}+\sqrt{3}\sin\frac{2n\pi}{3},\ n\geq 0\right]$$

7) Obtain the output of the system, where input u_k is given by $y_k-3y_{k-1}+2y_{k-2}=u_k$ where

$u_k=\left(\frac{1}{2}\right)^k,\ k\geq 0=0,\ k<0$

$$\left[\text{Ans. : } y_k=\frac{1}{10}\left(\frac{1}{3}\right)^k+\frac{12}{5}2^k-\frac{3}{2},\ k\geq 0\right]$$

□□□

Notes

UNIT - III

5 Statistics

5.1 Introduction

The word Statistics has been derived from the word state. A state means a kingdom. Formerly kings ruling in different parts of the world were always keeping records of their land revenue, manpower, roads etc. These records were named as Statistics. But nowadays this word is used in two different senses. i) Singular ii) Plural sense. When this word is used in plural sense it means a collection of numbers relating some phenomenon, e.g. statistics of marks of students.

When it is used in singular sense it means the science of statistics in which statistical methods are developed or studied for analysing numerical data. The aim of this chapter is to introduce the simple aspects of collection, classification and different methods of processing numerical data which are necessary for developing modern statistical techniques.

Definition of Statistics :

It is the branch of science in which we study methods applied in collecting, analysing and interpreting quantitive data, affected by various causes in any department of enquiry.

5.2 Some Useful Definitions

1) Variable or Variate : A quantity which can vary from one individual to another is called variable or variate.

For e.g. weight, heights, marks.

Quantities which can take continuous values in certain range are known as continuous variables.

Quantities which can take discrete values are known as discrete variable e.g. no. of children.

2) Ungrouped data : A simple or raw collection of a certain experimental observations.

3) Grouped data (classified data) : When the observations are arranged in a particular order in number of classes or groups such a data is called classified or grouped data.

There are two methods of classifications.

i) Classification by class intervals or frequency distribution.

ii) Classification by attributes.

4) Frequency Distribution : When a data is classified into class intervals and frequency of each class interval is recorded in a tabular form, such a presentation of the data is called frequency distribution.

Class interval	Class mark	Frequency
0 - 10	5	2
10 - 20	15	3
20 - 30	25	5
30 - 40	35	6
40 - 50	45	4
Total		20

5) Class mark : Mid value of a class interval is called class mark.

6) Frequency : Number of observations in a particular class interval is called the frequency of that class.

Rules for tabulation

i) The class interval must be well defined for e.g. the value x = 20 must be put in the class interval 20 - 30 not in 10 - 20 while x = 50 must be put in 40 - 50.

ii) Not a single observation should escape from classification.

iii) Entire range of observations should be divided into well defined class intervals.
iv) The class intervals should be uniform.
v) The number of class intervals should be adequate.
vi) Large volume of statistical data should not be crowded in a single class interval.

5.3 Measures of Central Tendency

When measurements are taken on a phenomenon, it is observed that there is a number (say A) and most of the values **Cluster** somewhere in the neighbourhood of that particular number (A). That means observations have a tendency to cluster in some part of the range, usually the central part of the range. This tendency is known as central tendency. The particular number around which all the other observations cluster is known as a **measure** of central tendency.

There are five methods of finding a measure of central tendency.

i) Arithmetic mean ii) Median iii) Mode

iv) Geometric mean v) Harmonic mean.

These are known as measures of central tendency.

A) Arithmetic Mean

1) For ungrouped data :

If $x_1, x_2, \cdots x_n$ are n observations

$$\text{Mean} = \bar{x} = \frac{\sum x_i}{n} = \frac{\text{Sum of the observations}}{\text{Total number of observations}}$$

2) For grouped data : If x_i are class marks and f_i are their respective frequencies for $1 \le i \le n$ then

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Note : We can write $\sum f_i = N =$ Total frequency

3) Method of step deviation (to simplify the calculations) :

If $u_i = \dfrac{x_i - A}{h}$, where

A is the working mean or assumed mean of given data and h is the class width or gcd of all $x_i - A$

$$\bar{x} = A + h\bar{u}$$
$$= A + h\left(\frac{\sum f_i u_i}{\sum f_i}\right)$$

Example 5.1 : *Calculate the A.M. for the following data.*

Wages in ₹	25-30	30-35	35-40	40-45	45-50	50-55	55-60	60-65	65-70	70-75
No. of Workers	10	13	18	21	24	28	20	11	8	7

Solution : The frequency distribution table

Let $A = 47.5$

$$\therefore \quad u_i = \frac{x_i - A}{h} = \frac{x_i - 47.5}{5}$$

x_i = Mid values

h = Class width

Wages	x_i	f_i	$d_i = x_i - 47.5$	$u_i = \frac{x_i - 47.5}{5}$	$f_i u_i$
25-30	27.5	10	– 20	– 4	– 40
30-35	32.5	13	– 15	– 3	– 39
35-40	37.5	18	– 10	– 2	– 36
40-45	42.5	21	– 5	– 1	– 21
45-50	47.5	24	0	0	0
50-55	52.5	28	5	1	28
55-60	57.5	20	10	2	40
60-65	62.5	11	15	3	33
65-70	67.5	8	20	4	32
70-75	72.5	7	25	5	35
Total		160			32

Now $\bar{u} = \dfrac{\sum f_i u_i}{\sum f_i} = \dfrac{32}{160}$

$$\therefore \quad \bar{x} = A + h(\bar{u})$$
$$= 47.5 + 5\left(\frac{32}{160}\right)$$
$$= 48.5 \text{ ₹}$$

B) Median

1) For ungrouped data :

It divides total set of data into two equal points.

Median is value of **middle** most terms of series when arranged in **ascending** or **descending** order of magnitude.

If n is odd then median $= \left(\dfrac{n+1}{2}\right)^{th}$ observation

If n is even then there are two values in the middle so we take mean of these two values

$$\text{median} = \frac{\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}}{2} \text{ observation}$$

2) For group frequency distribution :

$$\boxed{\text{Median} = l + \frac{h\left(\frac{N}{2} - C\right)}{f}}$$

l = Lower limit of median class.

f = Frequency of median class.

h = Width of median class.

C = Coefficient of the class preceding the median class.

$N = \sum f_i$

Median class is the class where $\left(\frac{N}{2}\right)^{th}$ observation lies.

Example 5.2 : *Find the median for the following distribution.*

Marks obtained	*Number of students*
below 10	15
below 20	35
below 30	60
below 40	84
below 50	94
below 60	127
below 70	198
below 80	249

Solution : Cumulative frequency distribution table

Marks (x_i)	No. of students (f_i)	Cumulative frequency
0-10	15	15
10-20	20	35
20-30	25	60
30-40	24	84
40-50	10	94
50-60	33	127
60-70	71	198
70-80	51	249
	$\sum f_i = 249$	

Median class is the class where $\frac{N}{2} = \frac{249}{2} = 124.5$ lies

124.5 lies before 127

Thus 50-60 in the median class

$\therefore l = 50,\ h = 10,\ f = 33,\ c = 94$

$$\text{Median} = l + \frac{h\left(\frac{N}{2} - c\right)}{f}$$

$$= 50 + \frac{10(124.5 - 94)}{33}$$

$$= 59.24$$

C) Mode

1) For ungrouped data : Mode is the most repeated observation in the given set of observation. So mode is not unique. If given data has only one mode, then it is known as unimodal otherwise multimodal.

Ex. 1) Mode of 1, 2, 3, 4, 2, 3, 2 is 2.

2) Mode of 1, 2, 3 is 1, 2, 3.

2) For grouped frequency distribution.

$$\text{Mode} = l + \frac{h\,(f_1 - f_0)}{2 f_1 - f_0 - f_2}$$

l = Lower limit of modal class.

h = Width of the modal class.

f_1 = Frequency of the modal class.

f_0 = Frequency of the class preceding the modal class.

f_2 = Frequency of the class succeeding the modal class.

Modal class is the class with highest frequency.

Example 5.3 : *Find mean, mode, median for the following distribution.*

CI	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70
f	4	7	8	12	25	18	10

Solution :

CI	x_i Midvalues	f_i	$u_i = \frac{x_i - 35}{10}$	CF	$f_i u_i$
0 - 10	5	4	– 3	4	– 12
10 - 20	15	7	– 2	11	– 14
20 - 30	25	8	– 1	19	– 8
30 - 40	35	12	0	31	0

40 - 50	45	25	1	56	25
50 - 60	55	18	2	74	36
60 - 70	65	10	3	84	30
Total		**84**			**57**

a) Mean $= A + h\left(\frac{\sum f_i u_i}{\sum f_i}\right) = 35 + 10\left(\frac{57}{84}\right) = 41.785$

b) Median class is the class in which

$$N/2 = 84/2 = 42$$

$\therefore$ lies 40 - 50 is median class

$$l = 40 \quad h = 10 \quad f = 25 \quad c = 31 \quad N = 84$$

$$\text{Median} = l + \frac{h\left(\frac{N}{2} - c\right)}{f}$$

$$= 40 + \frac{10\left(\frac{84}{2} - 31\right)}{25}$$

$$= 44.44$$

c) Modal class is the class with highest frequency i.e. 25

$\therefore$ Modal class = 40 – 50

$l = 40 \quad h = 10 \quad f_1 = 25 \quad f_0 = 12 \quad f_2 = 18$

$$\text{Mode} = l + \frac{h(f_1 - f_0)}{2f_1 - f_0 - f_2}$$

$$= 40 + \frac{10\,(25-12)}{50-12-18}$$

$$= 46.5$$

D) Geometric Mean

i) For ungrouped data : Geometric mean or G.M. of n observations $x_1, x_2 \ldots x_n$ $(x_i \neq 0)$ is the n^{th} root of their product.

i.e. $\quad$ G.M. $= (x_1 . x_2 \ldots x_n)^{1/n}$

Take log on both sides

$$\log(GM) = \frac{1}{n}(\log x_1 + \log x_2 + \ldots + \log x_n)$$

$$\therefore \quad GM = \text{Antilog}\left[\frac{\sum \log x_i}{n}\right]$$

ii) For grouped data : For $x_1, x_2 \ldots x_n$ having corresponding frequencies $f_1, f_2 \ldots f_n$

$$\text{G.M.} = \left[(x_1)^{f_1} \cdot (x_2)^{f_2} \cdot (x_3)^{f_3} \ldots (x_n)^{f_n}\right]^{1/N}$$

Take log on both sides

$$\log \text{G.M.} = \frac{1}{N}\left[f_1 \log x_1 + f_2 \log x_2 \ldots + f_n \log x_n\right]$$

$$\text{G.M.} = \text{Antilog}\left[\frac{\sum f_i \log x_i}{N = \sum f_i}\right]$$

E) Harmonic Mean

Harmonic mean of a number of observations is the reciprocal of the arithmetic mean of the reciprocals of the given values.

i) For ungrouped data :

$$\text{H.M.} = \frac{n}{\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)}$$

i.e. $$\text{H.M.} = \frac{1}{\left(\frac{\sum\left(\frac{1}{x_i}\right)}{n}\right)}$$

ii) For grouped data :

For $x_1, x_2 \ldots x_n$ having corresponding frequencies $f_1\ f_2 \ldots f_n$

$$\text{H.M.} = \frac{N}{\left(\frac{f_1}{x_1} + \frac{f_2}{x_2} + \ldots + \frac{f_n}{x_n}\right)} = \frac{N}{\sum_{i=1}^{n}\left(\frac{f_i}{x_i}\right)}$$

where $N = \sum f_i$

5.4 Dispersion

Meaning of dispersion is scatteredness. To find whether measures of central tendencies are true representative of the data we calculate dispersion.

To measure the scatteredness of data from mean we use

a) Mean deviation

b) Standard deviation

a) Mean Deviation :

i) For ungrouped data :

For variates $x_1, x_2 \ldots x_n$

Deviation from average A = $d_i = x_i - A$

Deviation from mean = $d_i = x_i - \bar{x}$

$$\text{Mean deviation} = \frac{\sum |d_i|}{n}$$

ii) For grouped data :

For variate $x_1, x_2 ... x_n$ with corresponding frequencies $f_1, f_2 ... f_n$

$$\text{Mean deviation} = \frac{\sum f_i d_i}{\sum f_i} = \frac{\sum f_i |x_i - A|}{\sum f_i}$$

b) Standard Deviation (σ) :

i) For ungrouped data : $x_1, x_2, x_3 ... x_n$

$$\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$$

Simplifying we get

$$\sigma = \sqrt{\frac{\sum (x_i)^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{\sum (x_i)^2}{n} - (\bar{x})^2}$$

ii) For grouped data :

$$\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}}$$

Simplifying we get

$$\sigma = \sqrt{\frac{\sum f_i x_i^2}{N} - \left(\frac{\sum f_i x_i}{N}\right)^2}$$

where $\sum f_i = N$

i.e. $$\sigma = \sqrt{\frac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2}$$

If we use method of step deviation for simplification of our calculations

i.e. if $u_i = \dfrac{x_i - A}{h}$ then

i) For ungrouped data

$$\sigma_u = \sqrt{\frac{\sum u_i^2}{n} - \left(\frac{\sum u_i}{n}\right)^2}$$

$$\sigma_x = h\sigma_u$$

ii) For grouped data

$$\sigma_u = \sqrt{\frac{\sum f_i u_i^2}{\sum f_i} - \left(\frac{\sum f_i u_i}{\sum f_i}\right)^2}$$

$$\sigma_x = h\sigma_u$$

$$\bar{x} = A + h\bar{u}$$

Note :

1) The square of standard deviation is called variance given by σ^2.

2) The coefficient of variation is given by

$$\text{C.V.} = \frac{\sigma}{\text{A.M.}} \times 100$$

For comparing the variability of two series, we calculate the coefficient of variations for each series. The series having lesser C.V. is said to be more consistent.

Example 5.4 : *Goals scored by two teams A and B in a football season were as follows. Determine which team is more consistent.*

Number of goals scored	*Number of matches*	
	Team A	*Team B*
0	27	17
1	9	9
2	8	6
3	5	5
4	4	3

Solution : Frequency distribution table for team A.

No. of goals (x_i)	Matches f_i	$d_i = x_i - 2$	$f_i d_i$	$f_i d_i^2$
0	27	− 2	− 54	108
1	9	− 1	− 9	9
2	8	0	0	0
3	5	1	5	5
4	4	2	6	12
	53		− 50	138

Thus for team A

$$\bar{x} = A + \left(\frac{\sum f_i d_i}{\sum f_i}\right)$$

$$= 2 + \frac{-50}{53} = 1.06$$

$$\sigma_A = \sqrt{\frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2}$$

$$= \sqrt{\frac{138}{53} - \left(\frac{-50}{53}\right)^2}$$

$$= 1.31$$

$$\text{C.V.} = \frac{\sigma_A}{\bar{x}} \times 100$$

$$= \frac{1.31}{1.06} \times 100 = 123.6$$

Frequency distribution table for team B

No. of goals (x_i)	Matches f_i	$d_i = x_i - 2$	$f_i d_i$	$f_i d_i^2$
0	17	– 2	– 34	68
1	9	– 1	– 9	9
2	6	0	0	0
3	5	1	5	5
4	3	2	6	12
	40		– 32	94

Thus for team B

$$\bar{x} = A + \frac{\sum f_i d_i}{\sum f_i}$$

$$= 2 + \left(\frac{-32}{40}\right) = 1.2$$

$$\sigma_B = \sqrt{\frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2}$$

$$= \sqrt{\frac{94}{40} - \left(\frac{-32}{40}\right)^2} = 1.3$$

$$\text{C.V.} = \frac{\sigma_B}{\bar{x}} \times 100$$

$$= \frac{1.3}{1.2} \times 100 = 108.3$$

Since $(\text{C.V.})_B < (\text{C.V.})_A$

$\therefore$ Team B is more consistent.

Example 5.5 : *Fluctuations in the aggregate of marks obtained by two groups of students are given below. Find out which of the two shows greater variability.*

Group A : 518, 519, 530, 530, 544, 542, 518, 550, 527, 531, 550, 550, 529, 528, 527.

Group B : 825, 830, 830, 819, 814, 814, 844, 842, 842, 826, 832, 835, 835, 840, 840.

Solution : To solve this problem,

We have to determine coefficient of variation $\frac{\sigma}{\text{AM}} \times 100$ in each case. First we present the data in frequency distribution form.

Group A

x	f	d = x – 530	$f_i d_i$	$f_i d_i^2$
518	2	– 12	– 24	288
519	1	– 11	– 11	121
527	2	– 3	– 6	18
528	1	– 2	– 2	4
529	1	– 1	– 1	1
530	2	0	0	0
531	1	1	1	1
542	1	12	12	144
544	1	14	14	196
550	3	20	60	1200
Total	$\sum$ **f = 15**		$\sum$ **$f_i d_i$ = 43**	**1973**

$$\text{A.M.} = 530 + \frac{\sum f_i d_i}{\sum f_i}$$

$$= 530 + \frac{43}{15} = 532.866$$

$$\sigma = \sqrt{\frac{1}{N} \sum f_i d_i^2 - \left(\frac{\sum f_i d_i}{N}\right)^2}$$

$$= \sqrt{\frac{1973}{15} - \left(\frac{43}{15}\right)^2}$$

$$= \sqrt{131.533 - 8.218}$$

$$= 11.105$$

$$\text{Coefficient of variation} = \frac{\sigma}{\text{AM}} \times 100 = 2.0840$$

Frequency distribution table for group B

x	f	d = x – 830	$f_i d_i$	$f_i d_i^2$
814	2	–16	– 32	512
819	1	– 11	– 11	121
825	1	– 5	– 5	25

826	1	– 4	– 4	16
830	2	0	0	0
832	1	2	2	4
835	2	5	10	50
840	2	10	20	200
842	2	12	24	288
844	1	14	14	196
	$\sum f = 15$		$\sum f_i d_i = 18$	$\sum f_i d_i^2 = 1412$

$$\text{A.M.} = 830 + \frac{18}{15} = 831.2$$

$$\sigma = \sqrt{\frac{1412}{15} - \left(\frac{18}{15}\right)^2}$$

$$= \sqrt{94.133 - 1.44}$$

$$= 9.628$$

$$\text{Coefficient of variation} = \frac{9.628}{831.2} \times 100 = 1.158$$

Coefficient of variation of group A is greater than that of group B.

$\therefore$ Group A has greater variability or group B is more consistent.

5.5 Moments

i) Central moments : Moments about the mean are known as central moments. The arithmetic mean of various powers of the deviation $(x_i - \bar{x})$ is called central moment of the distribution and is denoted by μ_i.

Thus for ungrouped data

$$\mu_1 = \frac{\sum (x_i - \bar{x})}{n}$$

$$\mu_2 = \frac{\sum (x_i - \bar{x})^2}{n}$$

$$\mu_3 = \frac{\sum (x_i - \bar{x})^3}{n}$$

$$\vdots$$

$$\mu_r = \frac{\sum (x_i - \bar{x})^r}{n}$$

Also for grouped data

$$\mu_1 = \frac{\sum f_i (x_i - \bar{x})}{N} \qquad \text{where } N = \sum f_i$$

$$\mu_2 = \frac{\sum f_i (x_i - \bar{x})^2}{N}$$

$$\vdots$$

$$\mu_r = \frac{\sum f_i (x_i - \bar{x})^r}{N}$$

μ_r is the r^{th} moment about the mean of a distribution.

Note : Subsituting r = 0, 1, 2, we get

i) $r = 0 \Rightarrow \quad \mu_0 = \dfrac{\sum f_i}{N} = \dfrac{N}{N} = 1$

ii) $r = 1 \Rightarrow \quad \mu_1 = \dfrac{\sum f_i (x_i - \bar{x})}{N}$

$$= \frac{\sum f_i x_i}{N} - \left(\frac{\sum f_i}{N}\right)\bar{x}$$

$$= \bar{x} - \frac{N}{N}\cdot\bar{x}$$

$$= \bar{x} - \bar{x}$$

$$= 0$$

iii) $r = 2 \Rightarrow \quad \mu_2 = \dfrac{\sum f_i (x_i - \bar{x})^2}{N}$

which gives the variance σ^2 of the distribution.

Thus
$$\boxed{\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= \frac{\sum f_i (x_i - \bar{x})^2}{N} \\ \mu_3 &= \frac{\sum f_i (x_i - \bar{x})^3}{N} \\ \mu_4 &= \frac{\sum f_i (x_i - \bar{x})^4}{N} \end{aligned}}$$

are the first four moments of the distribution about mean.

ii) Raw moments : Moment about any point of observations different from mean is known as raw moments. The r^{th} moment about any number A is denoted by μ_r' and is given by

$$\mu_r' = \frac{\sum f_i (x_i - A)^r}{N}$$

Substituting r = 0, 1, 2, ... we get

$$\mu_0' = 1$$

$$\mu_1' = \frac{\sum f_i (x_i - A)}{N}$$

$$= \frac{\sum f_i x_i}{N} - \left(\frac{\sum f_i}{N}\right) A$$

$$= \bar{x} - A$$

$$\mu_2' = \frac{\sum f_i (x_i - A)^2}{N}$$

$= s^2$ = Mean square deviation

$$\mu_3' = \frac{\sum f_i (x_i - A)^3}{N}$$

and $$\mu_4' = \frac{\sum f_i (x_i - A)^4}{N}$$

Note : Proper choice of A can reduce that calculations of calculating μ_r' than that of μ_r.

Relations between μ_r' and μ_r :

By definition

$$\mu_r' = \frac{\sum f_i (x_i - A)^r}{N}$$

Let $d_i = x_i - A$

$$\therefore \quad \mu_r' = \frac{\sum f_i (d_i)^r}{N}$$

Also
$$\mu_r = \frac{\sum f_i (x_i - \bar{x})^r}{N}$$

$$= \frac{1}{N} \sum f_i (x_i - A + A - \bar{x})^r$$

$$= \frac{1}{N} \sum f_i (d_i - \mu_1')^r \quad \{\text{As } \mu_1' = \bar{x} - A\}$$

using binomial expansion

$$= \frac{1}{N} \sum f_i \begin{Bmatrix} (d_i)^r - r_{C_1} (d_i)^{r-1} (\mu_1') + r_{C_2} (d_i)^{r-2} (\mu_1')^2 \\ \ldots + (-1)^r (\mu_1')^r \end{Bmatrix}$$

As $\mu_r' = \dfrac{\sum f_i (d_i)^r}{N}$ we get

$$\mu_r = \mu_r' - r_{C_1} \mu_{r-1}' (\mu_1') + r_{C_2} \mu_{r-2}' (\mu_1')^2 \ldots + (-1)^r (\mu_1')^r$$

Substituting r = 2, 3, 4 we get

$$\mu_2 = \mu_2' - 2_{C_1} \mu_1' \mu_1' + (\mu_1')^2$$

$$= \mu_2' - 2(\mu_1')^2 + (\mu_1')^2$$

$$= \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3_{C_1} \mu_2' \mu_1' + 3_{C_2} \mu_1' (\mu_1')^2 - (\mu_1')^3$$

$$= \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4_{C_1} \mu_3' \mu_1' + 4_{C_2} \mu_2' (\mu_1')^2 - 4_{C_3} \mu_1' (\mu_1')^3 + 4_{C_4} (\mu_1')^4$$

$$= \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' (\mu_1')^2 - 4(\mu_1')^4 + (\mu_1')^4$$

$$= \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' (\mu_1')^2 - 3(\mu_1')^4$$

Thus
$$\boxed{\begin{aligned} \mu_0 &= 1 \\ \mu_1 &= 0 \\ \mu_2 &= \mu_2' - (\mu_1')^2 \\ \mu_3 &= \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3 \\ \mu_4 &= \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' (\mu_1')^2 - 3(\mu_1')^4 \end{aligned}}$$

5.6 Sheppard's Correction for Moments

If we use $u_i = \dfrac{x_i - A}{h}$ then also we get the same relations. This involves some error in calculations of moments.

$\therefore$ By W.F. Sheppard the corrected formulae are

$$\mu_2 \text{ (corrected)} = \mu_2 - \frac{h^2}{12}$$

(where h = Width of the interval)

$$\mu_3 \text{ (corrected)} = \mu_3$$

$$\mu_4 \text{ (corrected)} = \mu_4 - \frac{h^2}{2} \mu_1 + \frac{7h^4}{240}$$

5.7 Skewness

To get the idea about the shape of the curve we study skewness. Skewness signifies departure from symmetry.

a) Positive skewness :

If the mean lies to the right of mode then the frequency curve streches to the right then the distribution is right skewed or positively skewed.

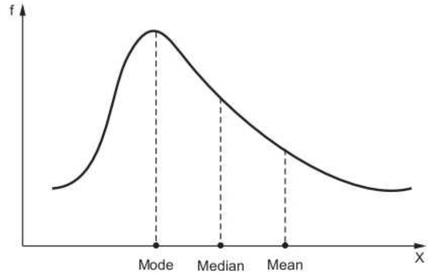

Fig. 5.1

b) Negative skewness :

If the mean lies to the left side of mode then the frequency curve stretches to the left then the distribution is left skewed or negatively skewed.

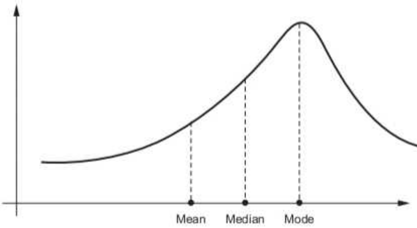

Fig. 5.2

The different measures of skewness are

i) Skewness $= \dfrac{3\,(\text{Mean} - \text{Median})}{\text{Standard deviation}}$

ii) Coefficient of skewness : $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$

and $\gamma_1 = +\sqrt{\beta_1}$

5.8 Kurtosis

If we know the measures of central tendency, dispersion and skewness, we still cannot have a complete idea about the distribution. Observe the Fig. 5.3 there are three curves $C_1C_2C_3$ which are symmetrical about mean and have the same range. Therefore we should know about the flatness or peakedness of the curve.

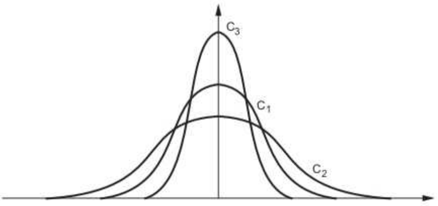

Fig. 5.3

Kurtosis (convexity of curve) is a measure which gives an idea about the flatness or peakedness of the curve. It is measured by the coefficient β_2.

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} \quad \text{or} \quad \gamma_2 = \beta_2 - 3$$

a) Mesokurtic curve : (Normal curve)

The curve C_1 which is neither flat nor peaked is called the normal curve or mesokurtic curve, for which $\beta_2 = 3$ or $\gamma_2 = 0$.

b) Platykurtic curve :

The curve (C_2) which is flatter than C_1 is platykurtic curve, for which $\beta_2 < 3$ or $\gamma_2 < 0$.

c) Leptokurtic curve :

The curve (C_3) which is more peaked than C_1 is Leptokurtic curve, for which $\beta_2 > 3$ or $\gamma_2 > 0$.

Example 5.6 : *The first four moments of a distribution about the value of 4 of the variable are – 1.5, 17, – 30 and 108. Find the moments about mean and β_1 and β_2.*

SPPU : May-16, 17, Dec.-18, Marks 4

Solution : $A = 4,\ \mu_1' = -1.5,\ \mu_2' = 17,$

$\mu_3' = -30,\ \mu_4' = 108$

$\therefore\ \mu_2 = \mu_2' - (\mu_1')^2 = 17 - (1.5)^2 = 14.75$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1)^3$

$= -30 - 3\,(17)\,(-1.5) + 2\,(-1.5)^3 = 39.75$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$= 108 - 4\,(-30)\,(-1.5) + 6\,(17)\,(-1.5)^2 - 3(-1.5)^4$

$= 108 - 180 + 229.5 - 15.1875 = 142.3125$

$$\therefore\ \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(39.75)^2}{(14.75)^3} = 0.4926$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{142.3125}{(14.75)^2} = 0.6543$$

Example 5.7 : *The first four moments of a distribution about the value 4 of the variable are – 1.5, 17, – 30 and 108. Find the moments about the origin and the point $x = 2$.*

Solution : (From above problem)

Given $A = 4,\ \mu_1' = -1.5,\ \mu_2' = 17,\ \mu_3' = -30,\ \mu_4' = 108$

We know that

$\bar{x} = A + \mu_1'$

$\therefore\quad \bar{x} = 4 + (-1.5)$

$= 2.5$

Also $\quad \bar{x} = A + \mu_1' = 4 + (-1.5) = 2.5$

where μ_1' is the first moment about $x = A$ taking $A = 0$, we get the first moment about the origin $\mu_1' = \bar{x} = 2.5$.

As $\quad \mu_2 = \mu_2' - (\mu_1')^2$

$\therefore\quad \mu_2' = \mu_2 + (\mu_1')^2 = 14.75 + (2.5)^2 = 21 \quad \ldots (1)$

As $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$\therefore$ $\mu_3' = \mu_3 + 3\mu_2'\mu_1' - 2(\mu_1')^3$

$= \mu_3 + 3\left[\mu_2 + (\mu_1')^2\right](\mu_1') - 2(\mu_1')^3$

$= \mu_3 + 3\mu_2\mu_1' + 3(\mu_1')^3 - 2(\mu_1')^3$

$\therefore$ $\mu_3' = \mu_3 + 3\mu_2\mu_1' + \mu_1'^3$ ... (2)

$= 39.75 + 3\ (14.75)\ (2.5) + (2.5)^3 = 166$

As $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$\therefore$ $\mu_4 = \mu_4' - 4\left[\mu_3 + 3\mu_2\mu_1' + (\mu_1')^3\right]$

$+ 6\left[\mu_2 + (\mu_1')^2\right] - 3(\mu_1')^4$

$\therefore$ $\mu_4 = \mu_4' - 6\mu_2\mu_1'^2 - 4\mu_3\mu_1' - (\mu_1')^4$

$\therefore$ $\mu_4' = \mu_4 + \mu_3\mu_1' + 6\mu_2(\mu_1')^2 + (\mu_1')^4$ ... (3)

$\therefore$ $\mu_4' = 142.312 + 4\ (39.75)\ (0.5)$

$+ 6\ (14.75)\ (0.5)^2 + (0.5)^4$

$= 244$

We have $\bar{x} = A + \mu_1'$

$\mu_1' = \bar{x} - 2 = 2.5 - 2 = 0.5$

$\therefore$ $\mu_2' = \mu_2 + \mu_1'^2 = 14.75 + (0.5)^2$

$= 15$

$\therefore$ $\mu_3' = \mu_3 + 3\mu_2\mu_1' + (\mu_1')^3$

$= 39.75 + 3\ (14.75)\ (0.5) + (0.5)^3$

$= 62$

$\mu_4' = \mu_4 + 4\mu_3\mu_1' + 6\mu_2\mu_1'^2 + (\mu_1')^4$ from equation (3)

$= 142.3125 + 4\ (39.75)\ (0.5)$

$+ 6\ (14.75)\ (0.5)^2 + (0.5)^4 = 244$

Example 5.8 : *Calculate the first four moments about the mean of the given distribution. Also find β_1 and β_2.*

x_i	2.0	2.5	3.0	3.5	4.0	4.5	5.0
f_i	4	36	60	90	70	40	10

Solution : Let A = 3.5 and $u_1' = \dfrac{x_i - 3.5}{0.5}$

Frequency distribution table.

x_i	f_i	$u_i = \dfrac{x_i - 3.5}{0.5}$	$f_i u_i$	$f_i u_i^2$	$f_i u_i^3$	$f_i u_i^4$
2.0	4	− 3	− 12	36	− 108	342
2.5	36	− 2	− 72	144	− 288	576
3.0	60	− 1	− 60	60	− 60	60
3.5	90	0	0	0	0	0
4.0	70	1	70	70	70	70
4.5	40	2	80	160	320	640
5.0	10	3	30	90	270	810
	310		$\sum fu = 36$	$\sum fu^2 = 560$	$\sum fu^3 = 204$	$\sum fu^4 = 2480$

We know $\mu_1' = \dfrac{\sum f_i u_i}{\sum f_i} = \dfrac{36}{310} = 0.166$

$\mu_2' = \dfrac{\sum f_i u_i^2}{\sum f_i} = \dfrac{560}{310} = 1.806$

$\mu_3' = \dfrac{\sum f_i u_i^3}{\sum f_i} = \dfrac{204}{310} = 0.658$

$\mu_4' = \dfrac{\sum f_i u_i^4}{\sum f_i} = \dfrac{2480}{310} = 8.0$

Now using relations

$\mu_1 = 0$

$\mu_2 = \mu_2' - \mu_1^2 = 1.806 - 0.013456 = 1.7925$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= 0.658 - 0.6258 + 0.003122 = 0.03262$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 8.0 - 0.3053 + 0.1458 - 0.000543 = 7.8399$

$\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{0.001064}{5.7594} = 0.000185$

$\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{7.8399}{3.213} = 2.44$

Example 5.9 : *Calculate the first four moments of the following distribution about the mean and find skewness and kurtosis.* **SPPU : Dec.-18**

x	0	1	2	3	4	5	6	7	8
f	1	8	28	56	70	56	28	8	1

Solution : We first calculate moments about x = 4 (Assumed mean)

x_i	f_i	$d_i = x_i - 4$	$f_i d_i$	$f_i d_i^2$	$f_i d_i^3$	$f_i d_i^4$
0	1	– 4	– 4	16	– 64	256
1	8	– 3	– 24	72	– 216	648
2	28	– 2	– 56	112	– 224	448
3	56	– 1	– 56	56	– 56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448
7	8	3	24	72	216	648
8	1	4	4	16	64	256
	$\sum f_i = 256$		$\sum f_i d_i = 0$	$\sum f_i d_i^2 = 512$	$\sum f_i d_i^3 = 0$	$\sum f_i d_i^4 = 2816$

We know that

$$\mu_r' = \frac{1}{N}\sum f_i (x_i - 4)^r = \frac{1}{N}\sum f_i\ d_i^r$$

$$\mu_1' = \frac{1}{N}\sum f_i\ d_i = 0$$

$$\mu_2' = \frac{1}{N}\sum f_i\ d_i^2 = \frac{512}{256} = 2$$

$$\mu_3' = \frac{1}{N}\sum f_i\ d_i^3 = 0$$

$$\mu_4' = \frac{1}{N}\sum f_i\ d_i^4 = \frac{2816}{256} = 11$$

using the relations between μ_r and μ_r'

$\therefore$ Moments about mean are

$$\mu_1 = 0 \text{ always}$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 2 - 0$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= 0 - 3\times 2\times 0 + 2\times 0 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 11 - 4(0)(0) + 6(2)(0) - 3\times 0 = 11$$

$$\therefore \quad \text{Skewness } \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0}{2^3} = 0$$

$$\text{Kurtosis } \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{11}{4} = 2.75$$

Example 5.10 : *The first four moments about the working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 400.25. Calculate the moments about the mean. Also evaluate β_1, β_2 and comment upon the skewness and kurtosis of the distribution.*

SPPU : Dec.-16, 17, Marks 4

Solution : Given the first four moments about the arbitrary origin 30.2 are

$$\mu_1' = 0.255,\ \mu_2' = 6.222,$$

$$\mu_3' = 30.211,\ \mu_4' = 400.25$$

We know that $\mu_1' = \bar{x} - A$

$$0.255 = \bar{x} - 30.2$$

$$\therefore \quad \bar{x} = 30.455$$

Now using the relations between μ_r and μ_r'

$$\mu_2 = \mu_2' - \mu_1'^2 = 6.222 - (0.255)^2 = 6.15698$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= 30.211 - 3\,(6.222)\,(0.255) + 2\,(0.255)^3$$

$$= 30.211 - 4.75983 + 0.03316275$$

$$\mu_3 = 25.48433$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= 440.25 - 4\,(30.211)\,(0.255) + 6\,(6.222)$$

$$(0.255) - 3\,(0.255)^4$$

$$\mu_4 = 378.9418$$

$$\therefore \quad \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(25.48433)^2}{(6.15698)^3} = 2.78255$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{378.9418}{(6.15698)^2}$$

$$\beta_2 = 9.99625$$

$$\therefore \quad \gamma_1 = \sqrt{\beta_1} = \sqrt{2.78255} = 1.6681$$

which indicates considerable skewness of the distribution.

$$\gamma_2 = \beta_2 - 3 = 9.99625 = 6.99625$$

which shows that the distribution is leptokurtic (As $\beta_2 > 3$ or $\gamma_2 > 0$)

Example 5.11 : *Calculate the first four moments about the mean of the given distribution. Also find skewness and kurtosis.*

Class intervals	1.75-2.25	2.25-2.75	2.75-3.25	3.25-3.75	3.75-4.25	4.25-4.75	4.75-5.25
Frequency	4	36	60	90	70	40	10

Solution : Here H = 0.5 (class width)

Let $$u_i = \frac{x_i - A}{h} = \frac{x_i - 3.5}{0.5}$$

Frequency distribution table

Class intervals	x_i Mid values	f_i	$u_i = \frac{x_i - A}{h}$	$f_i u_i$	$f_i u_i^2$	$f_i u_i^3$	$f_i u_i^4$
1.75 - 2.25	2.0	4	– 3	– 12	36	– 108	324
2.25 - 2.75	2.5	36	– 2	– 72	144	– 288	576
2.75 - 3.25	3.0	60	– 1	– 60	60	– 60	60
3.25 - 3.75	3.5	90	0	0	0	0	0
3.75 - 4.25	4.0	70	1	70	70	70	70
4.25 - 4.75	4.5	40	2	80	160	320	640
4.75 - 5.25	5.0	10	3	30	90	270	810
Total		$\sum f_i = 310$		$\sum f_i u_i = 36$	$\sum f_i u_i^2 = 560$	$\sum f_i u_i^3 = 204$	$\sum f_i u_i^4 = 2480$

$\therefore$ Moments about A = 3.5 are

$$\mu_1' = \frac{\sum f_i u_i}{N} = \frac{36}{310} = 0.116$$

$$\mu_2' = \frac{\sum f_i u_i^2}{N} = \frac{560}{310} = 1.806$$

$$\mu_3' = \frac{\sum f_i u_i^3}{N} = \frac{204}{310} = 0.658$$

$$\mu_4' = \frac{\sum f_i u_i^4}{N} = \frac{2480}{310} = 8.0$$

To find moments about mean use the relations between μ_r and μ_r'.

$$\mu_0 = 1, \ \mu_1 = 0 \text{ (always)}$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 1.806 = 0.013456$$

$$\mu_2 = 1.7925$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 0.658 - 0.6285 + 0.003122$$

$$\mu_3 = 0.03262$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 8.0 - 0.3053 + 0.1458 - 0.00543$$

$$\mu_4 = 7.8399$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0.001064}{5.7594} = 0.000185$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{7.8399}{3.213} = 2.44$$

Example 5.12 : *The first four moments of a distribution about the values 5 are 2, 20 , 40 and 50. From the given information obtain the first four central moment, mean, standard deviation and coefficient of skewness and kurtosis.*

SPPU : May-18

Solution : $A = 5, \mu_1' = 2, \mu_2' = 20, \mu_3' = 40$ and $\mu_4' = 50$.

We know that

$$\mu_1' = \bar{x} - A$$

$$\therefore \quad \bar{x} = A + \mu_1' = 5 + 2 = 7$$

To use the relations between μ_r and μ_r'.

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 20 - (2)^2 = 16$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 = 40 - 3(2)(20) + 2(2)^3 = 40 - 120 + 16 = -64$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4 = 50 - 4(2)(20) + 6(2)^2(2) - 3(2)^4 = 50 - 160 + 480 - 48 = 322$$

We know

$$\therefore \quad \text{Variance} = \mu_2 = 16$$

$\therefore$ Standard deviation $= \sqrt{\mu_2} = \sqrt{16} = 4$

Coefficient of skewness is given by,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(-64)^2}{(16)^3} = 1$$

Since $\beta_1 = 1$, the distribution is positively skewed. Coefficient of kurtosis is given by,

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{322}{(16)^2} = 1.26$$

Since the value of β_2 is less than 3, hence the distribution is platykurtic.

Example 5.13 : *For a distribution the mean is 10, variance is 16, $\gamma_1 = +1$ and $\beta_2 = 4$. Find first four moments about the origin. Comment upon the nature of distribution.*

Solution :

Here Mean $\bar{x} = 10, \mu_2 = \sigma^2 = 16, \gamma_1 = +1$ and $\beta_2 = 4$

First four moments about the origin $(\mu_1', \mu_2', \mu_3', \mu_4')$

$$\mu_1' = \text{Mean as } \bar{x} = A + \mu_1' \text{ and } A = 0$$

$$\therefore \quad \mu_1' = 10$$

For $\quad \mu_2 = \mu_2' - \mu_1'^2 \Rightarrow \mu_2' = \mu_2 + \mu_1'^2 = 16 + 10^2$

$$\therefore \quad \mu_2' = 116$$

$$\gamma_1 = +1 \quad \text{since } \gamma_1 = \sqrt{\beta_1} = +\sqrt{\frac{\mu_3^2}{\mu_2^3}}$$

$$\therefore \quad \gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} = 1 \Rightarrow \mu_3 = \mu_2^{3/2} = (16)^{3/2} = 64$$

$$\therefore \quad \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

We have $\mu_3' = \mu_3 + 3\mu_2'\mu_1' - 2\mu_1'^3$

$$= 64 + 3 \times 116 \times 10 - 2 \times 1000 = 1544$$

$$\therefore \quad \mu_3' = 1544$$

Also $\quad \beta_2 = \frac{\mu_4}{\mu_2^2} = 4 \Rightarrow \frac{\mu_4}{16^2} = 4$

$$\therefore \quad \mu_4 = 4 \times 256 = 1024$$

for $\quad \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$$\therefore \quad \mu_4' = \mu_4 + 4\mu_3'\mu_1' - 6\mu_2'\mu_1'^2 + 3\mu_1'^4$$

$$\therefore \quad \mu_4' = 1024 + 4 \times 1544 \times 10 - 6 \times 116 \times 100 + 3 \times 10000$$

$$\therefore \quad \mu_4' = 23184$$

As $\gamma_1 = +1$, the distribution is positively skewed. i.e. for the given distribution, tail will be longer towards right.

Also $\quad \beta_2 = 4 > 3$, Distribution is leptokurtic

Example 5.14 : *The first three moments about the value 2 of a distribution are 1, 16 and – 40. Find the mean, standard deviation and β_1 of the distribution.* **SPPU : Dec.-19, Marks 4**

Solution : We have $A = 2, \mu_1' = 1, \mu_2' = 16, \mu_3' = -40$ we know that $\bar{x} = A + \mu_1' \Rightarrow \bar{x} = 2 + 1 = 3$.

$\therefore$ Mean is 3

$\mu_1 = 0, \mu_2 = \mu_2' - (\mu_1')^2 = 16 - (1)^2 = 15$

$$\therefore \quad \mu_2 = 15 = \sigma^2$$

$$\Rightarrow \quad \sigma = \sqrt{15} = \text{Standard deviation}$$

Now $\quad \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$$= -40 - 3(16)(1) + 2(1)^3 = -40 - 48 + 2$$

$$\mu_3 = -86$$

$$\therefore \quad \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(-86)^2}{(15)^3} = \frac{7396}{3375}$$

Example 5.15 : *Calculate the first four moments about the mean of the following distribution.*

Marks	No. of students
0 - 10	06
10 - 20	26
20 - 30	47
30 - 40	15
40 - 50	06

Solution :

Class	Mid pt. x	Freq. f	$\mu = \frac{x-25}{10}$	fu	fu^2	fu^3	fu^4
0 - 10	5	6	− 2	− 12	24	− 48	96
10 - 20	15	26	− 1	− 26	26	− 26	26
20 - 30	25	47	0	0	0	0	0
30 - 40	35	15	1	15	15	15	15
40 - 50	45	6	2	12	24	48	96
Total	-	100	-	− 11	89	− 11	233

The moments about the arbitrary mean A = 25 are

$$\mu'_r = h^r \frac{\sum fu^r}{\sum f} \quad \text{for } r = 0, 1, 2, 3 \ldots$$

$$\mu'_1 = -1.1, \mu'_2 = 89, \mu'_3 = -110, \mu'_4 = 23300$$

∴ The central moments are

$\mu_1 = 0, \ \mu_2 = 87.79$

$\mu_3 = 181.038, \ \mu_4 = 23457.7477$

Example 5.16 : *Calculate the first four moments of the following distribution about the mean and find skewness and kurtosis.*

x	0	1	2	3	4	5	6	7	8
f	1	8	28	56	70	56	28	8	1

Solution : We first calculate moments about x = 4 (Assumed mean)

x_i	f_i	$d_i = x_i - 4$	$f_i d_i$	$f_i d_i^2$	$f_i d_i^3$	$f_i d_i^4$
0	1	− 4	− 4	16	− 64	256
1	8	− 3	− 24	72	− 216	648
2	28	− 2	− 56	112	− 224	448
3	56	− 1	− 56	56	− 56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448
7	8	3	24	72	216	648
8	1	4	4	16	64	256
	$\sum f_i = 256$		$\sum f_i d_i = 0$	$\sum f_i d_i^2 = 512$	$\sum f_i d_i^3 = 0$	$\sum f_i d_i^4 = 2816$

We know that

$$\mu'_r = \frac{1}{N} \sum f_i (x_i - 4)^r = \frac{1}{N} \sum f_i \ d_i^r$$

$$\mu'_1 = \frac{1}{N} \sum f_i \ d_i = 0$$

$$\mu'_2 = \frac{1}{N} \sum f_i \ d_i^2 = \frac{512}{256} = 2$$

$$\mu'_3 = \frac{1}{N} \sum f_i \ d_i^3 = 0$$

$$\mu'_4 = \frac{1}{N} \sum f_i \ d_i^4 = \frac{2816}{256} = 11$$

using the relations between μ_r and μ'_r

∴ Moments about mean are

$$\mu_1 = 0 \text{ always}$$

$$\mu_2 = \mu'_2 - \mu_1'^2 = 2 - 0$$

$$\mu_3 = \mu'_3 - 3\mu'_2 \mu'_1 + 2\mu_1'^3$$

$$= 0 - 3 \times 2 \times 0 + 2 \times 0 = 0$$

$$\mu_4 = \mu'_4 - 4\mu'_3 \mu'_1 + 6\mu'_2 \mu_1'^2 - 3\mu_1'^4$$

$$= 11 - 4(0)(0) + 6(2)(0) - 3 \times 0 = 11$$

∴ Skewness $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0}{2^3} = 0$

Kurtosis $\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{11}{4} = 2.75$

Example 5.17 : *Calculate the first four moments of the following distribution about the mean and find skewness and kurtosis.*

x	2	2.5	3	3.5	4	4.5	5
f	4	36	60	90	70	40	10

SPPU : May-15

Solution : Let, A = 3.5, $u_i = \frac{x_i - 3.5}{0.5}$ and h = 0.5

x_i	f_i	$u_i = \frac{x_i - 3.5}{0.5}$	$f_i u_i$	$f_i u_i^2$	$f_i u_i^3$	$f_i u_i^4$
2.0	4	− 3	− 12	36	− 108	342
2.5	36	− 2	− 72	144	− 288	576
3.0	60	− 1	− 60	60	− 60	60
3.5	90	0	0	0	0	0
4.0	70	1	70	70	70	70

4.5	40	2	80	160	320	640
5.0	10	3	30	90	270	810
Total	**310**		$\sum fu$ = **36**	$\sum fu^2$ = **560**	$\sum fu^3$ = **204**	$\sum fu^4$ = **2480**

We know $\mu_1' = \frac{\sum f_i u_i}{\sum f_i} = \frac{36}{310} = 0.166$

$$\mu_2' = \frac{\sum f_i u_i^2}{\sum f_i} = \frac{560}{310} = 1.806$$

$$\mu_3' = \frac{\sum f_i u_i^3}{\sum f_i} = \frac{204}{310} = 0.658$$

$$\mu_4' = \frac{\sum f_i u_i^4}{\sum f_i} = \frac{2480}{310} = 8.0$$

Now using relations

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 1.806 - 0.013456 = 1.7925$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= 0.658 - 0.6258 + 0.003122 = 0.03262$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_2'^2 - 3\mu_1'^4$$

$$= 8.0 - 0.3053 + 0.1458 - 0.000543 = 7.8399$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^2} = \frac{0.001064}{5.7594} = 0.000185$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{7.8399}{3.213} = 2.44$$

Example 5.18 : *The first four moments of a distribution about the values 5 are 2, 20 , 40 and 50. From the given information obtain the first four central moment, mean, standard deviation and coefficient of skewness and kurtosis.*

SPPU : May-15

Solution : A = 5, $\mu_1' = 2$, $\mu_2' = 20$, $\mu_3' = 40$ and $\mu_4' = 50$.

We know that

$$\mu_1' = \bar{x} - A$$

$$\therefore \quad \bar{x} = A + \mu_1' = 5 + 2 = 7$$

To use the relations between μ_r and μ_r'.

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 20 - (2)^2 = 16$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

$$= 40 - 3(2)(20) + 2(2)^3$$

$$= 40 - 120 + 16 = -64$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4$$

$$= 50 - 4(2)(20) + 6(2)^2(2) - 3(2)^4$$

$$= 50 - 160 + 480 - 48 = 322$$

We know

$\therefore$ Variance $= \mu_2 = 16$

$\therefore$ Standard deviation $= \sqrt{\mu_2} = \sqrt{16} = 4$

Coefficient of skewness is given by,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(-64)^2}{(16)^3} = 1$$

Since $\beta_1 = 1$, the distribution is positively skewed. Coefficient of kurtosis is given by,

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{322}{(16)^2} = 1.26$$

Since the value of β_2 is less than 3, hence the distribution is platykurtic.

Example 5.19 : *The first four moments about the working mean 3.5 of a distribution are 0.0375, 0.4546, 0.0609 and 0.5074. Calculate the first four moments about the mean. Also calculate the coefficient of skewness.* **SPPU : May-16, Marks 4**

Solution : Given that $\mu_1' = 0.0375$, $\mu_3' = 0.0609$, $\mu_4' = 0.5074$, A = 3.5

$\therefore$ The central moments are

$$\mu_1 = 0, \mu_2 = \mu_2' - \mu_1'^2 = 0.4546 - (0.0375)^2 = 0.0453$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^2 = 0.0600$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_1'^2\mu_2' - 3\mu_1'^4 = 3.2385$$

Now,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(0.06)^2}{(0.0453)^3} = 32.22$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3.2385}{(0.0453)^2} = 1578.21$$

Example 5.20 : *Calculate the first four central moments from the following data and hence find β_1 and β_2 :*

x	0	1	2	3	4	5	6
f	5	15	17	25	19	14	5

SPPU : Dec.-17, Marks 4

Solution : First we calculate moments about the assumed mean x = 3

x_i	f_i	$d_i = x_i - 3$	$f_i d_i$	$f_i d_i^2$	$f_i d_i^3$	$f_i d_i^4$
0	5	– 3	– 15	45	– 135	405
1	15	– 2	– 30	60	– 120	240
2	17	– 1	– 17	17	– 17	17
3	25	0	0	0	0	0
4	19	1	19	19	19	19
5	14	2	28	56	112	224
6	5	3	15	45	135	405
$\sum f_i$ = 100			$\sum f_i d_i = 0$	$\sum f_i d_i^2 = 242$	$\sum f_i d_i^3 = 6$	$\sum f_i d_i^4 = 1310$

$$\mu_r' = \frac{1}{N}\sum f_i (x_i - 3)^r = \frac{1}{N}\sum f_i d_i^r$$

$$\mu_1' = 0, \quad \mu_2' = 2.42, \quad \mu_3' = 0.06, \quad \mu_4' = 13.10$$

Moments about mean are $\mu_1 = 0$

$$\mu_2 = \mu_2' - \mu_1'^2 = 2.42$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 0.06$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = \mu_4' = 13.10$$

and $$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(0.06)^2}{(2.42)^3} = 0.000254$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 2.2368$$

Exercise 5.1

1. *Find the arithmetic mean, median, standard deviation first four moments about mean of the following distribution. Find the coefficient of skewness and kurtosis.*

x	1	2	3	4	5	6	7	8	9	10
f	6	15	23	42	62	60	40	24	13	5

[Ans. : $\mu_1 = 0, \mu_2 = 3.3778, \mu_3 = -0.0824, \mu_4 = 37.7721, \beta_1 = 0.002, \beta_2 = 3.3106$]

2. *The first four moments of a distribution about x = 2 are 1, 2.5, 5.5 and 16. Calculate first four moments about mean and about zero.* **[Ans. : $\mu_1 = 0, \mu_2 = 1.5, \mu_3 = 0, \mu_4 = 6$**

about zero : 3 10.5, 40.5, 168]

3. *Compute coefficient of skewness and kurtosis for the data*

x	4.5	14.5	24.5	34.5	44.5	54.5	64.5	74.5	84.5	94.5
f	1	5	12	22	17	9	4	3	1	1

[Ans. : $\mu_1 = 0, \mu_2 = 2.83, \mu_3 = 3.38, \mu_4 = 30.295 \therefore \beta_1 = 0.504, \beta_2 = 3.782$]

4. *Calculate first four moments about the mean for the distribution.*

x	1	2	3	4	5	6	7	8	9
f	1	6	13	25	30	22	9	5	2

[Ans. : $\mu_1 = 0, \mu_2 = 2.49, \mu_3 = 0.68, \mu_4 = 18.26$]

5. *First four moments of a distribution about the value 5 of the variable are 2, 20, 40 and 50. Find the moments about the mean.* **[Ans. : 0, 16, – 64, 162]**

6. *The scores of two golfers for 10 rounds each are*

A	58	59	60	54	65	66	52	75	69	52
B	84	56	92	65	86	78	44	54	78	68

Which may be regarded as more consistent player. **[Ans. : 'A']**

7. *A Collar manufacturer is considering production of a new type of Collar to attract youngmen. The following statistics of neck circumferences are available based upon the measurements of typical groups of college students.*

Mid value of students	12.5	13.0	13.5	14.0	14.5	15.0	15.5	16.0	16.5
No. of Students	4	19	30	63	66	29	18	1	1

Compute mean, standard deviation and variance. **[Ans. : 14.24, 0.72, 0.52]**

8. *Calculate mean and standard deviation for the data*

x	56	63	70	77	84	91	98
f	3	6	14	16	13	6	2

[Ans. : 76.53, 9.87]

9. *Compute Mean deviation from median for the data :*

Class interval	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
No. of students	5	10	20	5	10

[Ans. : 9]

5.9 Curve Fitting

Important Points to Remember

1) **Curve fitting :** The process of binding the curve of best fit which is most suitable for predicting the unknown values is known as curve fitting.

2) **Method of least squares :** The method of least squares is the most systematic procedure to fit a unique curve through the given points.

a) Normal equations for straight line.

Let $y = a + bx$ be the straight line to be fitted to the given data points $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$.

$$e_i^2 = (y_i - a - bx_i)^2$$

$$\therefore \quad S = \sum e_i^2 = \sum (y_i - a - bx_i)^2$$

For S to be minimum $\dfrac{\partial s}{\partial a} = 0$ and $\dfrac{\partial s}{\partial b} = 0$

$$\Rightarrow \sum (y - a - bx) = 0 \text{ and } \sum \left(xy - ax - bx^2\right) = 0$$

$$\Rightarrow \sum y = na + b\sum x$$

$\sum xy = a\sum x + b\sum x^2$ which are known as normal equations of straight line.

b) Table for the equation of curve and their normal equations

Sr. No.	Equation of curve	Normal equations
1)	$y = ax^b$ or $Y = A + Bx$	$\sum Y = nA + B\sum X$ and $\sum XY = A\sum X + B\sum X^2$
2)	$y = ab^x$ $\log y = \log a + (\log b)\, x$ $Y = A + Bx$	$\sum Y = nA + B\sum X$ and $\sum XY = A\sum X + B\sum X^2$
3)	$xy = b + ax$ $y = \dfrac{b}{x} + a$ $Y = bX + a$	$\sum Y = nA + b\sum X$ $\sum XY = a\sum X + b\sum X^2$
4)	$y = ax + bx^2$	$\sum xy = a\sum x^2 + b\sum x^3$ and $\sum x^2 y = a\sum x^3 + b\sum x^4$
5)	$y = ax + \dfrac{b}{x}$	$\sum xy = a\sum x^2 + nb$ and $\sum \dfrac{y}{x} = na + b\sum \dfrac{1}{x^2}$
6)	$y = a + \dfrac{b}{x} + \dfrac{c}{x^2}$	$\sum y = na + b\sum \dfrac{1}{x} + c\sum \dfrac{1}{x^2}$ $\sum \dfrac{y}{x} = a\sum \dfrac{1}{x} + b\sum \dfrac{1}{x^2} + c\sum \dfrac{1}{x^3}$ and $\sum \dfrac{y}{x^2} = a\sum \dfrac{1}{x^2} + b\sum \dfrac{1}{x^3} + c\sum \dfrac{1}{x^4}$
7)	$y = \dfrac{a}{x} + b\sqrt{x}$	$\sum \dfrac{y}{x} = a\sum \dfrac{1}{x^2} + b\sum \dfrac{1}{\sqrt{x}}$ and $\sum y\sqrt{x} = a\sum \dfrac{1}{\sqrt{x}} + bx$
8)	$2^x = ax^2 + bx + c$	$\sum 2^x x^2 = a\sum x^4 + b\sum x^3 + c\sum x^2$ $\sum 2^x x = a\sum x^3 + b\sum x^2 + c\sum x$ $\sum 2^x = a\sum x^2 + b\sum x + nc$
9)	$y = a\,e^{-3x} + b\,e^{-2x}$	$\sum y\,e^{-3x} = a\sum e^{-6x} + b\sum e^{-5x}$ $\sum y\,e^{-2x} = a\sum e^{-5x} + b\sum e^{-4x}$

Example 5.21 : *By the method of least squares, find the straight line that best fits the following data :*

x :	*1*	*2*	*3*	*4*	*5*
y :	*14*	*27*	*40*	*55*	*68*

Solution : Let the straight line of best fit be

$$y = a + bx \quad \ldots(1)$$

Normal equations are

$$\sum y = ma + b\sum x \quad \ldots(2)$$

and

$$\sum xy = a\sum x + b\sum x^2 \quad \ldots(3)$$

Here m = 5

$\therefore$

x	y	xy	x^2
1	14	14	1
2	27	54	4
3	40	120	9

4	55	220	16
5	68	340	25
$\sum x = 15$	$\sum y = 204$	$\sum xy = 748$	$\sum x^2 = 55$

Substituting in equation (2) and (3) we have,

$204 = 5a + 15b, \quad 748 = 15a + 55b$

Solving above equation a = 0, b = 13.6

Hence required straight line is y = 13.6 x .

Example 5.22 : *Using method of least squares, derive the normal equations to fit the curve $y = ax^2 + bx$. Hence fit this curve to the following data :*

x	1	2	3	4	5	6	7	8
y	1	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Solution : Given curve is $y = ax^2 + bx$

$$s = \sum\left(y - ax^2 - bx^2\right) \quad ...(1)$$

Differentiating equation (1) w.r.t. a partially

$$\frac{\partial s}{\partial a} = 2\sum\left(y - ax^2 - bx\right)\left(-x^2\right)$$

Differentiating equation (1) w.r.t. b partially

$$\frac{\partial s}{\partial b} = 2\sum\left(y - ax^2 - bx\right)(-x)$$

For minimization of error, $\frac{\partial s}{\partial a} = 0, \frac{\partial s}{\partial b} = 0$

$$\sum\left(yx^2 - ax^4 - bx^3\right) = 0$$

$$\sum x^2y = a\sum x^4 + b\sum x^3 \quad ... (2)$$

and $\sum xy - a\sum x^3 - b\sum x^2 = 0$

$$\sum xy = a\sum x^3 + b\sum x^2 = 0 \quad ... (3)$$

Equation (2) and (3) are the two normal equations for given curve.

x	y	x^2	x^3	x^4	x^2y	xy
1	1.0	1	1	1	1	1
2	1.2	4	8	16	4.8	2.4
3	1.8	9	27	81	16.2	5.4
4	2.5	16	64	256	40	10.0
5	3.6	25	125	625	90	18.0
6	4.7	36	216	1296	169.2	28.2
7	6.6	49	343	2401	323.4	46.2
8	9.1	64	512	4096	582.4	72.8

From above,
$$\sum x^4 = 8772$$
$$\sum x^3 = 1296$$
$$\sum x^2 = 204$$
$$\sum x^2y = 1227$$
$$\sum xy = 184$$

Putting all these values in equations (2) and (3) we have

$$1227 = 8772a + 1296b \quad ...(4)$$

$$184 = 1296a + 204b \quad ...(5)$$

On solving, equations (4) and (5)

a = 0.1078

b = 0. 2171

Thus the required curve is $y = 0.1078x^2 + 0.2171x$.

Example 5.23 : *Obtain a relation of the form $y = ab^x$ for the following data by the method of least squares :*

x	2	3	4	5	6
y	8.3	15.4	33.1	65.2	127.4

Solution : The curve to be fitted is $y = ab^x$

$$Y = A + Bx$$

where, $A = \log_{10} a$, $B = \log_{10} b$ and $Y = \log_{10} y$

∴ The normal equation are $\sum Y = 5A + B\sum x$

$$\sum xy = A\sum x + B\sum x^2.$$

x	y	$Y = \log_{10} y$	x^2	xY
2	8.3	0.9191	4	1.8382
3	15.4	1.1872	9	3.5616
4	33.1	1.5198	16	6.0792
5	65.2	1.8142	25	9.0710
6	127.4	2.1052	36	12.6312
$\sum x = 20$		$\sum Y = 7.5455$	$\sum x^2 = 90$	$\sum xY = 33.1812$

Substituting the above values, we get.

$7.5455 = 5A + 20B$ and $33.1812 = 20A + 90B$.

Solving equation A = 0.31 and B = 0.3

a = Antilog A = 2.04 and b = Antilog B = 1.995

Hence the required curve is $y = 2.04\,(1.995)^x$

Example 5.24 : *Obtain the least squares fit of the form.* $f(t) = a e^{-3t} + b e^{-2t}$ *for the data :*

t :	0.1	0.2	0.3	0.4
f(t) :	0.76	0.58	0.44	0.35

Solution : Normal equations to the curve $f(t) = a e^{-3t} + b e^{-2t}$

are
$$\sum f(t)\, e^{-3t} = a \sum e^{-6t} + b \sum e^{-5t} \quad ...(1)$$
$$\sum f(t)\, e^{-2t} = a \sum e^{-5t} + b \sum e^{-4t} \quad ...(2)$$

Table of values are

t	f(t)	e^{-4t}	e^{-5t}	e^{-6t}	$f(t)\, e^{-2t}$	$f(t)\, e^{-3t}$
0.1	0.76	0.6703	0.6065	0.5488	0.6222	0.5630
0.2	0.58	0.4493	0.3679	0.3012	0.3888	0.3183
0.3	0.44	0.3012	0.2231	0.1653	0.2415	0.1789
0.4	0.35	0.2019	0.1353	0.0907	0.1573	0.1054
		$\sum e^{-4t} = 1.6227$	$\sum e^{-5t} = 1.3328$	$\sum e^{-6t} = 1.106$	$\sum f(t)\, e^{-2t} = 1.4098$	$\sum f(t)\, e^{-3t} = 1.1656$

Substituting the values in equation (1) and equation (2) we have

$1.106\ a + 1.3328\ b = 1.1656$

$1.3328\ a + 1.6227\ b = 1.4098$

On solving we get $a = 0.6778$, $b = 0.3121$

Hence the least squares fit is $f(t) = 0.6778\, e^{-3t} + 0.3121\, e^{-2t}$

Example 5.25 : *Use the method of least squares to fit the curve* $y = \frac{C_0}{x} + C_1 \sqrt{x}$ *to the following table.*

x	0.1	0.2	0.4	0.5	1	2
y	21	11	7	6	5	6

Solution : Normal equations to the curve $y = \frac{C_0}{x} + C_1 \sqrt{x}$

are
$$\sum \frac{y}{x} = C_0 \sum \frac{1}{x^2} + C_1 \sum \frac{1}{\sqrt{x}} \quad ... (1)$$
$$\sum y \sqrt{x} = C_0 \sum \frac{1}{\sqrt{x}} + C_1 \sum x \quad ... (2)$$

∴ Table is

x	y	y/x	$y\sqrt{x}$	$1/\sqrt{x}$	$1/x^2$
0.1	21	210	6.64078	3.16228	100
0.2	11	55	4.91935	2.23607	25
0.4	7	17.5	4.42719	1.58114	6.25
0.5	6	12	4.24264	1.41421	4
1	5	5	5	1	1
2	6	3	8.48528	0.70711	0.25
$\sum x = 4.2$		$\sum (y/x) = 302.5$	$\sum y\sqrt{x} = 33.71524$	$\sum 1/\sqrt{x} = 10.10081$	$\sum 1/x^2 = 136.5$

From equation (1) and (2) we have

$$302.5 = 136.5\ C_0 + 10.10081\ C_1$$

$$33.71524 = 10.10081\ C_0 + 42\ C_1$$

Solving above equation $C_0 = 1.97327$ and $C_1 = 3.28182$

Hence the required equation of curve is

$$y = \frac{1.97327}{x} + 3.28182\sqrt{x}$$

Example 5.26 : *Fit a second degree parabola to the following data.*

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	8	13	5

Solution : We have n = 9 and $y = a + bx + cx^2$

Let $$X = \frac{x-5}{1} = x-5 \text{ and } Y = y-8$$

$\therefore$ New parabola is $Y = a + bx + cx^2$

Let us consider the following table :

x	y	X	Y	XY	X^2	$X^2 Y$	X^3	X^4
1	2	– 4	– 6	24	16	– 96	– 64	256
2	6	– 3	– 2	6	9	–18	– 27	81
3	7	– 2	– 1	2	4	– 4	– 8	16
4	8	– 1	0	0	1	0	– 1	1
5	10	0	2	0	0	0	0	0
6	11	1	3	3	1	3	1	1
7	8	2	0	0	4	0	8	16
8	13	3	5	15	9	45	27	81
9	5	4	–3	–12	16	– 48	64	256
		$\sum X = 0$	$\sum Y = -2$	$\sum XY = 38$	$\sum X^2 = 60$	$\sum X^2 Y = -118$	$\sum X^3 = 0$	$\sum X^4 = 708$

The normal equations are

$$\sum Y = na + b\sum X + c\sum X^2$$
$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3$$
$$\sum X^2 Y = a\sum X^2 + b\sum X^3 + c\sum X^4$$

Putting all values from table, we get

$$-2 = 9a + 0b + 60c \quad ...(1)$$
$$38 = 0a + 60b + 0c \quad ...(2)$$
$$-118 = 60a + 0b + 708c \quad ...(3)$$

Equation (2) $\Rightarrow$ $$b = \frac{38}{60} = \frac{19}{30}$$

Equation (1) $\times$ 60 $\Rightarrow$ $$-120 = 540a + 360c \quad ...(4)$$

Equation (3) $\times$ 9 $\Rightarrow$ $$-1062 = 540a + 6372c \quad ...(5)$$

Equation (5) – (4) $\Rightarrow$ $$-942 = 6012c$$

$\Rightarrow$ $$c = -\frac{942}{6012} = -\frac{471}{306}$$

$$\therefore \quad a = \frac{1}{9}[-2 + 60\,c]$$

$$= \frac{1}{9}\left[-2 + 60 \times \frac{471}{306}\right]$$

$$= \frac{1}{9}\left[-2 + \frac{471}{51}\right] = 41$$

Thus

$$Y = 41 + \frac{19}{30}X + \left(-\frac{471}{306}\right)X^2$$

$$y-8 = 41 + \frac{19}{30}(X-5) - \frac{471}{306}(X-5)^2$$

$$y = 7.35+(16.02)\,x-1.53\,x^2$$

5.10 Correlation

Bivariate distribution : The distribution for one variate x is known as univariate distribution. The distribution which involves more than one variable is known as bivariate distribution. If a change in one variable x affects the change in other variable y then the variables are said to be correlated.

If increase in x ⇒ increase in y ⇒ direct or
(decrease) (decrease) positive correlation

If increase in x ⇒ decrease in y ⇒ negative correlation
(decrease) (increase) (inverse correlation)

For variables x and y if the ratio $\frac{y}{x}$ = Constant then the correlation is linear, otherwise non-linear.

Karl Person's Coefficient of Correlation

To measure the intensity or degree of linear relationship between two variables, Karl Pearson developed a formula called correlation coefficient.

a) Correlation coefficient between two variables x and y is denoted by $r(x, y)$ and is defined as

$$\boxed{r\,(x, y) = \frac{\text{cov}\,(x, y)}{\sigma_x\,\sigma_y}}$$

where $\text{cov}\,(x, y)$ = Co-variance of (x, y)

$$= \frac{1}{n}\sum (x_i-\bar{x})(y_i-\bar{y})$$

where $\bar{x} = \frac{\sum x_i}{n} \qquad \bar{y} = \frac{\sum y_i}{n}$

$$\sigma_x = \sqrt{\frac{1}{n}\sum (x_i-\bar{x})^2}$$

$$\sigma_y = \sqrt{\frac{1}{n}\sum (y_i-\bar{y})^2}$$

Now $\text{cov}\,(x, y) = \frac{1}{n}\sum (x_i-\bar{x})(y_i-\bar{y})$

$$= \frac{1}{n}\sum (x_i\,y_i - \bar{y}\cdot x_i - \bar{x}\cdot y_i + \bar{x}\,\bar{y})$$

$$= \frac{1}{n}\left(\sum x_i y_i - \bar{y}\sum x_i - \bar{x}\sum y_i + \sum \bar{x}\bar{y}\right)$$

$$= \frac{1}{n}\sum x_i\,y_i - \bar{y}\left(\frac{\sum x_i}{n}\right) - \bar{x}\left(\frac{\sum y_i}{n}\right) + \frac{1}{n}\bar{x}\,\bar{y}\,n$$

$$= \frac{1}{n}\sum x_i\,y_i - \bar{y}\,\bar{x} - \bar{x}\cdot\bar{y} + \bar{x}\cdot\bar{y}$$

$$= \frac{1}{n}\sum x_i\,y_i - \bar{x}\cdot\bar{y}$$

i.e.

$$\boxed{\begin{aligned} \text{cov}\,(x, y) &= \frac{1}{n}\sum x_i\,y_i - (\bar{x})(\bar{y}) \\ \text{cov}\,(x, y) &= \frac{1}{n}\sum x_i\,y_i - \left(\frac{\sum x_i}{n}\right)\left(\frac{\sum y_i}{n}\right) \end{aligned}}$$

Also

$$\sigma_x^2 = \frac{1}{n}\sum (x_i-\bar{x})^2$$

$$= \frac{1}{n}\sum \left(x_i^2 - 2\,\bar{x}\,x_i + (\bar{x})^2\right)$$

$$= \frac{1}{n}\sum x_i^2 - 2\,\bar{x}\left(\frac{\sum x_i}{n}\right) + (\bar{x})^2\frac{1}{n}\,n$$

$$= \frac{1}{n}\sum x_i^2 - 2\,\bar{x}(\bar{x}) + (\bar{x})^2$$

$$= \frac{1}{n}\sum x_i^2 - (\bar{x})^2$$

i.e.

$$\boxed{\begin{aligned} \sigma_x^2 &= \frac{1}{n}\sum x_i^2 - (\bar{x})^2 \\ \sigma_x^2 &= \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2 \end{aligned}}$$

Similarly

$$\boxed{\sigma_y^2 = \frac{\sum y_i^2}{n} - \left(\frac{\sum y_i}{n}\right)^2}$$

b) Method of step deviation

If $u_i = \frac{x_i - A}{n}$ and $v_i = \frac{y_i - B}{k}$

$$\boxed{\begin{array}{l} \text{cov}(u, v) = \dfrac{\sum u_i v_i}{n} - \bar{u}\cdot\bar{v} \\ \sigma_u^2 = \dfrac{\sum u_i^2}{n} = \bar{u}, \quad \sigma_v^2 = \dfrac{\sum v_i^2}{n} - \bar{v} \\ \bar{u} = \dfrac{\sum u_i}{n}, \quad \bar{v} = \dfrac{\sum v_i}{n} \end{array}}$$

then

and $\quad r(u, v) = \dfrac{\text{cov}(u, v)}{\sigma_u \sigma_v}$

Note that $\quad r(x, y) = r(u, v)$

Note : 1) If $r = 0$ then there is lack of relationship between x and y.

2) If $r = \pm 1$ then the relationship between x and y is very strong.

c) Correlation coefficient for bivariate frequency distribution

When a data is presented in bivariate frequency distribution then also

$$r(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$$

where $\boxed{\text{cov}(x, y) = \dfrac{\sum f_i x_i y_i}{\sum f_i} - \bar{x}\,\bar{y}}$

$$= \frac{\sum f_i x_i y_i}{\sum f_i} - \left(\frac{\sum f_i x_i}{\sum f_i}\right)\left(\frac{\sum f_i y_i}{\sum f_i}\right)$$

i.e. $\text{cov}(x, y) = \dfrac{\sum f_i x_i y_i}{N} - \left(\dfrac{\sum f_i x_i}{N}\right)\left(\dfrac{\sum f_i y_i}{N}\right)$

Also $\boxed{\sigma_x^2 = \dfrac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2}$

and $\boxed{\sigma_y^2 = \dfrac{\sum f_i y_i^2}{\sum f_i} - (\bar{y})^2}$

d) Method of step deviation

If $\quad u_i = \dfrac{x_i - A}{n}, \quad v_i = \dfrac{y_i - B}{k}$

$$\boxed{\begin{array}{l} \text{cov}(u, v) = \dfrac{\sum f_i u_i v_i}{\sum f_i} - \bar{u}\cdot\bar{v} \\ \sigma_u^2 = \dfrac{\sum f_i u_i^2}{\sum f_i} - (\bar{u})^2 \\ \sigma_v^2 = \dfrac{\sum f_i v_i^2}{\sum f_i} - (\bar{v})^2 \end{array}}$$

where $\quad \bar{u} = \dfrac{\sum f_i u_i}{\sum f_i} \quad \bar{v} = \dfrac{\sum f_i v_i}{\sum f_i}$

and $r(x, y) = r(u, v) = \dfrac{\text{cov}(u, v)}{\sigma_u \cdot \sigma_v}$

Substituting all the above values we can write

$$r(x, y) = \frac{\dfrac{\sum f_i u_i v_i}{N} - \left(\dfrac{\sum f_i u_i}{N}\right)\left(\dfrac{\sum f_i v_i}{n}\right)}{\sqrt{\dfrac{\sum f_i u_i^2}{N} - \left(\dfrac{\sum f_i u_i}{N}\right)^2}\sqrt{\dfrac{\sum f_i v_i^2}{N} - \left(\dfrac{\sum f_i v_i}{N}\right)^2}}$$

i.e. $$\boxed{r(x, y) = \frac{N\cdot\sum f_i u_i v_i - \left(\sum f_i u_i\right)\left(\sum f_i v_i\right)}{\sqrt{N\sum f_i u_i^2 - \left(\sum f_i u_i\right)^2}\sqrt{N\sum f_i v_i^2 - \left(\sum f_i v_i\right)^2}}}$$

Example 5.27 : *Compute the coefficient of correlation for the following data.*

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Solution : Let $A = 22 \quad \therefore u_i = \dfrac{x_i - A}{h} = \dfrac{x_i - 22}{4}$

and $\quad B = 24 \quad \therefore v_i = \dfrac{y_i - B}{k} = \dfrac{x_i - 24}{6}$

Table

x_i	y_i	$u_i = \dfrac{x_i - 22}{4}$	$v_i = \dfrac{y_i - 24}{6}$	u_i^2	v_i^2	$u_i v_i$
10	18	− 3	− 1	9	1	3
14	22	− 2	− 2	4	4	4
18	24	− 1	0	1	0	0

22	6	0	– 3	0	9	0
26	30	1	1	1	1	1
30	36	2	2	4	4	4
Total		– 3	– 3	19	19	12

$$\bar{u} = \frac{\sum u_i}{n} = \frac{-3}{6} = \frac{-1}{2}$$

$$\bar{v} = \frac{\sum v_i}{n} = \frac{-3}{6} = \frac{-1}{2}$$

$$\text{cov}(u, v) = \frac{\sum u_i v_i}{n} - (\bar{u})(\bar{v})$$

$$= \frac{12}{6} - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$$

$$= 2.0 - 0.25$$

$$= 1.75$$

$$\sigma_u^2 = \frac{\sum u_i^2}{n} - (\bar{u})^2 = \frac{19}{6} - \left(\frac{-1}{2}\right)^2$$

$$= 3.1666 - 0.25$$

$$= 2.9166$$

$$\sigma_v^2 = \frac{\sum v_i^2}{n} - (\bar{v})^2 = \frac{19}{6} - \left(-\frac{1}{2}\right)^2$$

$$= 2.9166$$

$$\therefore \quad r(x, y) = r(u, v)$$

$$= \frac{\text{cov}(u, y)}{\sigma_u \sigma_v} = \frac{1.75}{\sqrt{(2.9166)(2.9166)}}$$

$$= \frac{1.75}{2.9166}$$

$$= 0.60$$

Example 5.28 : *Following are the marks of 10 students in Maths III and strength of materials calculate the coefficient of correlation.*

Roll No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
SOM	*78*	*36*	*98*	*25*	*75*	*82*	*90*	*62*	*65*	*39*
M. III	*84*	*51*	*91*	*60*	*68*	*62*	*86*	*58*	*53*	*47*

Solution : Let x, y represents the marks in two subjects.

Arrange x in increasing order and write corresponding y in front of x.

Let $u_i = x_i - 65, \quad v_i = y_i - 66$

x_i	y_i	$u_i = x_i - 65$	$v_i = y_i - 66$	u_i^2	v_i^2	$u_i v_i$
25	60	– 40	– 6	1600	36	240
36	51	– 29	– 15	841	225	435
39	47	– 26	– 19	676	361	494
62	58	– 3	– 8	9	64	24
65	53	0	– 13	0	169	0
75	68	10	2	100	4	20
78	84	13	18	169	324	234
82	62	17	– 4	289	16	– 68
90	86	25	20	625	400	500
98	91	33	25	1089	625	825
		0	0	5398	2224	2734

$$\bar{u} = \frac{\sum u_i}{n} = 0, \qquad \bar{v} = \frac{\sum v_i}{n} = 0$$

$$\text{cov}(u, v) = \frac{\sum u_i v_i}{n} - \bar{u} \cdot \bar{v} = \frac{2734}{10} = 273.4$$

$$\sigma_u^2 = \frac{\sum u_i^2}{n} - (\bar{u})^2 = \frac{5398}{10} - 0 = 539.8$$

$$\sigma_v^2 = \frac{\sum v_i^2}{n} - (\bar{v})^2 = \frac{2224}{10} - 0 = 222.4$$

$$r(x, y) = r(u, v) = \frac{\text{cov}(u, y)}{\sigma_u \sigma_v}$$

$$= \frac{273.4}{\sqrt{(529.8)(222.4)}} = 0.787$$

Example 5.29 : *The following data are ages of husband and wife for twenty couples calculate coefficient of correlation.*

x	*22*	*24*	*26*	*26*	*27*	*30*	*27*	*28*	*28*	*29*	*39*	*39*	*31*	*32*	*33*	*36*	*34*	*25*	*35*	*37*
y	*18*	*20*	*20*	*24*	*22*	*32*	*24*	*27*	*24*	*21*	*25*	*29*	*27*	*27*	*30*	*30*	*27*	*30*	*31*	*32*

Solution : Let $u_i = x_i - 30 \qquad v_i = y_i - 26$

x_i	y_i	$u_i = x_i - 30$	$v_i = y_i - 26$	u_i^2	v_i^2	$u_i v_i$
22	18	– 8	– 8	64	64	64
24	20	– 6	– 6	36	36	36

26	20	– 4	– 6	16	36	24
26	24	– 4	– 2	16	4	8
27	22	– 3	– 4	9	16	12
30	32	0	6	0	36	0
27	24	– 3	– 2	9	4	6
28	27	– 2	1	4	1	– 2
28	24	– 2	– 2	4	4	4
29	21	– 1	– 5	1	25	5
30	25	0	– 1	0	1	0
30	29	0	3	0	9	0
31	27	1	1	1	1	1
32	27	2	1	4	1	2
33	30	3	4	9	16	13
36	30	6	4	36	16	24
34	27	4	1	16	1	4
25	30	5	4	25	16	20
35	31	5	5	25	25	25
37	32	7	6	49	36	42
		0	0	324	348	287

Now $\bar{u} = \frac{\sum u_i}{n} = 0, \quad \bar{v} = \frac{\sum v_i}{n} = 0$

$$\text{cov}(u, v) = \frac{\sum u_i v_i}{n} - \bar{u}\,\bar{v}$$

$$= \frac{287}{20} - (0)(0) = 14.35$$

$$\sigma_u = \sqrt{\frac{\sum u_i^2}{n} - (\bar{u})^2} = \sqrt{\frac{324}{20}} = 4.05$$

$$\sigma_y = \sqrt{\frac{\sum v_i^2}{n} - (\bar{v})^2} = \sqrt{\frac{348}{20}} = 4.2$$

$$\therefore \quad r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \sigma_v} = 0.85$$

Example 5.30 : *From a group of 10 students marks obtained by each in papers of Mathematics and Applied Mechanics are given as.*

x - marks in maths	*23*	*28*	*42*	*17*	*26*	*35*	*29*	*37*	*16*	*46*
y - marks in ap mech	*25*	*22*	*38*	*21*	*27*	*39*	*24*	*32*	*18*	*44*

Calculate Karl Pearson's coefficient of correlation.

Solution :

x_i	y_i	$u_i = x_i - 35$	$v_i = y_i - 39$	u_i^2	v_i^2	$u_i v_i$
16	18	– 19	– 21	361	441	399
17	21	– 18	– 18	324	324	324
23	25	– 12	– 14	144	196	168
26	27	– 9	– 12	81	144	108
28	22	– 7	– 17	49	289	119
29	24	– 6	– 15	36	225	90
35	39	0	0	0	0	0
37	32	2	– 7	4	49	– 14
42	38	7	– 1	49	1	– 7
46	44	11	5	121	25	55
		$\sum u = -51$	$\sum v = -100$	$\sum y^2 = 1169$	1694	1242

$$\bar{u} = \frac{\sum u_i}{n}$$

$$\therefore \quad \bar{u} = \frac{-51}{10} = -5.1 \qquad \bar{u}^2 = 26.01$$

$$\therefore \quad \bar{v} = -10 \qquad \bar{v}^2 = 100$$

Now $\text{cov}(u, v) = \frac{1}{n}\sum u_i v_i - \bar{u}\,\bar{v}$

$$= \frac{1}{10}(1242) - 51 = 73.2$$

$$\sigma_u^2 = \frac{1}{n}\sum u_i^2 - (\bar{u})^2$$

$$= \frac{1169}{10} - 26.01 = 90.89$$

$$\sigma_u = \sqrt{90.89} = 9.5336$$

$$\sigma_v^2 = \frac{1}{n}\sum v_i^2 - \bar{v}^2$$

$$= \frac{1694}{10} - 100 = 69.4$$

$$\sigma_v = \sqrt{69.4} = 8.33$$

$$r(x, y) = r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \cdot \sigma_v}$$

$$= \frac{73.2}{9.534 \times 8.33}$$

$$= 0.9217$$

Example 5.31 : *Calculate the coefficient of correlation.*

x	5	9	15	19	24	28	32
y	7	9	14	21	23	29	30
f	6	9	13	20	16	11	7

Solution :

x_i	y_i	f_i	$u_i = x_i - 19$	$v_i = y_i - 21$	$f_i u_i$	$f_i v_i$	$f_i u_i^2$	$f_i v_i^2$	$f_i u_i v_i$
5	7	6	– 14	– 14	– 84	– 84	1176	1176	1176
9	9	9	– 10	– 12	– 90	– 108	900	1296	1080
15	14	13	– 4	– 7	– 52	– 91	208	637	364
19	21	20	0	0	0	0	0	0	0
24	23	16	5	2	80	32	400	64	160
28	29	11	9	8	99	88	891	704	792
32	30	7	13	9	91	63	1183	567	819
		82			**44**	**– 100**	**4758**	**4444**	**4391**

$$\bar{u} = \frac{\sum f_i u_i}{\sum f_i} = \frac{44}{82} = 0.5366 \qquad \bar{u}^2 = 0.288$$

$$\bar{v} = \frac{\sum f_i v_i}{\sum f_i} = \frac{-100}{82} = -1.2196 \qquad \bar{v}^2 = 1.4872$$

$$\text{cov}(u, v) = \frac{\sum f_i u_i v_i}{\sum f_i} - \bar{u} \cdot \bar{v} = 52.89$$

$$\sigma_u^2 = \frac{\sum f_i u_i^2}{\sum f_i} - \bar{u}^2 = \frac{4.758}{82} - 0.288 = 57.7364$$

$$\sigma_v^2 = \frac{\sum f_i v_i^2}{\sum f_i} - \bar{v}^2 = \frac{4444}{82} - 1.4872 = 52.708$$

$$\sigma_u = 7.598$$

$$\sigma_v = 7.26$$

$$r(u, v) = \frac{\cos(u, v)}{\sigma_u \cdot \sigma_v} = \frac{52.89}{55.16} = 0.9588$$

$\therefore$ Coefficient of correlation = r (x, y) = 0.9588

Example 5.32 : *The following marks have been obtained by a class of students in 2 papars of mathematics.*

Paper I	45	55	56	58	60	65	68	70	75	80	85
Paper II	56	50	48	60	62	64	65	70	74	82	90

Calculate the coefficient of correlation for the above data.

Solution : Let A = 68, B = 70

$$u_i = x_i - 68, \quad v_i = y_i - 70$$

Consider the following table

x_i	y_i	$u_i = x_i - 68$	$v_i = y_i - 70$	u_i^2	v_i^2	$u_i v_i$
45	56	– 23	– 14	529	196	322
55	50	– 13	– 20	169	400	260
56	48	– 12	– 22	144	484	264
58	60	– 10	– 10	100	100	100
60	62	– 8	– 8	64	64	64
65	64	– 3	– 6	9	36	18
68	65	0	– 5	0	25	0
70	70	2	0	4	0	0
75	74	7	4	49	16	28
80	82	12	12	144	144	144
85	90	17	20	289	400	340
		– 31	– 49	1501	1865	1540

$$\bar{u} = \frac{\sum u}{n} = \frac{-31}{11} = 2.818$$

$$\bar{v} = \frac{\sum v}{n} = \frac{-49}{11} = -4.455$$

$$\text{cov}(u, v) = \frac{\sum u_i v_i}{n} - \bar{u}\bar{v}$$

$$= 140 - 12.554$$

$$\text{cov}(u, v) = 127.446$$

$$\sigma_u^2 = \frac{\sum u_i^2}{n} - \bar{u}^2 = 128.512$$

$$\sigma_u = 11.336$$

$$\sigma_v^2 = \frac{\sum v_i^{-2}}{n} - \bar{v}^2 = 149.7, \quad \sigma_v = 12.2353$$

$$\therefore \quad r(x, y) = r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \sigma_v} = \mathbf{0.9188}$$

Example 5.33 : *Two examiners A and B award marks to seven students.*

Roll No.	1	2	3	4	5	6	7
Marks by A	40	44	28	30	44	38	31
Marks by B	32	39	26	30	38	34	28

Solution : $u_i = x_i - A$ and $v_i = y_i - B$

were $A = 38, \; B = 34$

Consider the table

x_i	y_i	u_i	v_i	u_i^2	v_i^2	$u_i v_i$
40	32	2	−2	4	4	−4
44	39	6	5	36	25	30
28	26	−10	−8	100	64	80
30	30	−8	−4	64	16	32
44	38	6	4	36	16	24
38	34	0	0	0	0	0
31	28	−7	−6	49	36	42
		$\sum u = -11$	$\sum v_i = -11$	$\sum u^2 = 289$	$\sum u_i^2 = 161$	204

$$\bar{u} = \frac{\sum u_i}{7} = -1.571, \; \bar{v} = \frac{\sum v_i}{n} = -1.571$$

$$\text{cov}(u, v) = \frac{\sum u_i v_i}{n} - \bar{u}\,\bar{v} = 26.673$$

$$\sigma_u^2 = \frac{\sum u_i^2}{n} - u^2 = 38.816, \; \sigma_u = 6.23$$

$$\sigma_v^2 = 20.530, \; \sigma_v = 4.531$$

$$r(x, y) = r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \sigma_v} = \mathbf{0.9448}$$

Problems on bivariate distribution

Here we can use the direct formula

$$r(x, y) = \frac{N \sum f_i u_i v_i - \left(\sum f_i u_i\right)\left(\sum f_i v_i\right)}{\sqrt{\left[N \sum f_i u_i^2 - \left(\sum f_i u_i\right)^2\right]\left[N \sum f_i v_i^2 - \left(\sum f_i v_i\right)^2\right]}}$$

where $N = \sum f_i \quad u_i = \frac{x_i - A}{n} \qquad v_i = \frac{y_i - B}{k}$

Note : 1) Multiply u_i, v_i and the respective frequency of each cell and write $(f_i \; u_i \; v_i)$ in the right hand upper corner of each cell.

i.e. for each cell $\longrightarrow$ $\boxed{f_i \; u_i \; v_i \\ f_i}$

2) Obtain sums of $f_i \; u_i \quad f_i \; u_i^2 \; f_i \; v_i \quad f_i \; v_i^2 \quad f_i \; u_i \; v_i$ and substitute in the above formula to find r.

Example 5.34 : *Calculate Karl Pearson's coefficient of correlation from the following data.*

y \ x	200 - 300	300 - 400	400 - 500	500 - 600	600 - 700
10 - 15	-	-	-	3	7
15 - 20	-	4	9	4	3
20 - 25	7	6	12	5	-
25 - 30	3	10	19	8	-

Solution : Prepare the frequency distribution table.

Let $\quad u_i = \frac{x_i - 450}{100} \qquad v_i = \frac{y_i - 17.5}{5}$

Class intervals			200-300	300-400	400-500	500-600	600-700				
Class intervals		x	250	350	450	550	650				
	y	v \ u	u = −2	−1	0	+1	+2	f	fv	fv^2	fuv
10-15	12.5	v = −1	-	-	-	−3 3	−14 7	10	−10	10	−17
15-20	17.5	0	-	0 4	0 9	0 4	0 3	20	0	0	0
20-25	22.5	1	−14 7	−6 6	0 12	5 5	-	30	30	30	−15
25-30	27.5	2	−12 3	−20 10	0 19	16 8	-	40	80	160	−16
		f	10	20	40	20	10	N = 100	Σ fv = 100	Σ fv^2 = 200	−48
		fu	−20	−20	0	20	20	Σ fu = 00			
		fu^2	40	20	0	20	40	Σ fu^2 = 120			
		fuv	−26	−26	0	18	−14	Σ fuv = −98			

Thus $N = 100, \; \sum f_i \; u_i = 0, \quad \sum f_i \; v_i = 100,$

$\sum f_i \; u_i^2 = 120, \; \sum f_i \; v_i^2 = 200,$

$\sum f_i \; u_i \; v_i = -48$

Substituting in we get

$$r(x, y) = \frac{N \sum f_i u_i v_i - \sum f_i u_i \sum f_i v_i}{\sqrt{N \sum f_i u_i^2 - \left(\sum f_i u_i\right)^2} \sqrt{N \sum f_i v_i^2 - \left(\sum f_i v_i\right)^2}}$$

$$r(x, y) = \frac{100(-48) - 0 \times 100}{\sqrt{100(120) - (00)^2}\ \sqrt{100(200) - (100)^2}}$$

$$= \frac{-4800}{\sqrt{12000}\ \sqrt{10000}} = \frac{-4800}{10954} = \mathbf{-0.438}$$

Example 5.35 : *The following data gives according to age the frequency of marks obtained by 100 students in an intelligence test.*

Age in yrs / Marks	*18*	*19*	*20*	*21*	*Total*
10 - 20	*4*	*2*	*2*	*-*	*8*
20 - 30	*5*	*4*	*6*	*4*	*19*
30 - 40	*6*	*8*	*10*	*11*	*35*
40 - 50	*4*	*4*	*6*	*8*	*22*
50 - 60	*-*	*2*	*4*	*4*	*10*
60 - 70	*-*	*2*	*3*	*1*	*6*
Total	*19*	*22*	*31*	*28*	*100*

Calculate the coefficient of correlation between age and intelligence.

Solution : We denote the age and intelligence by x and y respectively.

Also let, $u = \frac{y-45}{10}$, $v = x - 20$

Mid value	y \ x	18	19	20	21	f_i	u_i	fu	fu^2	fuv
15	10-20	24 4	6 2	0 2	0 –	8	– 3	– 24	72	30
25	20-30	20 5	8 4	0 6	– 8 4	19	– 2	– 38	76	20
35	30-40	12 6	8 8	0 10	– 11 11	35	– 1	– 35	35	09
45	40-50	00 4	0 4	0 6	0 8	22	0	00	00	00
55	50-60	0 –	– 2 2	0 4	4 4	10	1	10	10	02
65	60-70	0 –	– 4 2	0 3	2 1	6	2	12		– 2
	f	19	22	31	28	$\Sigma f_i = 100$		$\Sigma fu = -75$	$\Sigma fu^2 = 217$	$\Sigma fuv = 59$
	v	– 2	– 1	0	1					
	fv	– 38	– 22	0	28	$\Sigma fv = -32$				
	fv^2	76	22	0	28	$\Sigma fv^2 = 126$				
	fuv	56	16	0	– 13	$\Sigma fuv = 59$				

Example 5.36 : *The following table gives, according to age the frequency of marks obtained by 200 students in a test to determine talent in mathematics :*

Age in yrs / Marks	*20*	*21*	*22*	*23*	*24*	*Total*
0 - 10	*10*	*8*	*6*	*10*	*4*	*38*
10 - 20	*8*	*10*	*8*	*-*	*11*	*37*
20 - 30	*-*	*11*	*7*	*8*	*5*	*31*
30 - 40	*20*	*-*	*10*	*12*	*10*	*52*
40 - 50	*2*	*6*	*7*	*15*	*12*	*42*
Total	*40*	*35*	*38*	*45*	*42*	*200*

Solution :

	y_i		20	21	22	23	24	Total	$f_i u_i$	$f_i u_i^2$	$f_i u_i v_i$
CI	$v_i = y_i - 22$		– 2	– 1	0	1	2	–	–	–	–
	x_i	$u_i = \frac{x_i - 25}{10}$	40	16	0	– 20	– 16	38	– 76	152	20
0-10	5	– 2	10	8	6	10	4				
			16	10	0	0	– 22	37	– 37	37	4
10-20	15	– 1	8	10	8	0	11				
			0	0	0	0	0	31	0	0	0
20-30	25	0	0	11	7	8	5				
			– 40	0	0	12	20	52	52	52	– 8
30-40	35	1	20	0	10	12	10				
			– 8	– 12	0	30	48	42	84	168	58
40-50	45	2	2	6	7	15	12				
Total			40	35	38	45	42	200	23	409	74
$f_i v_i$			$40 \times (-2) = -80$	– 35	0	45	84	14			
$f_i v_i^2$			$40 \times 4 = 160$	35	0	45	168	408			
$f_i v_i u_i$			8	14	0	22	30	74			

Substituting in

$$r(x, y) = \frac{N \sum f_i u_i v_i - \left(\sum f_i u_i\right)\left(\sum f_i v_i\right)}{\sqrt{\left[N \sum f_i u_i^2 - \left(\sum f_i u_i\right)^2\right]\left[N \sum f_i v_i^2 - \left(\sum f_i v_i\right)^2\right]}}$$

$$r = \frac{(200)(74) - (14)(23)}{\sqrt{[(200)(408) - (14)^2][(200)(409) - (23)^2]}}$$

$$r = 0.178076$$

5.11 Regression

If x and y are correlated. If the points in scatter diagram lies on some curve then that curve is called curve of regression. If the curve is a straight line then it is called as a regression line in such a case the relation between the two variables is a linear relation.

Lines of regression are used for estimating the value of one variable for a given value of other variable.

The regression line is obtained using the method of least squares.

Consider the set of values of (x_i, y_i) i = 1, 2, ... n

Let the line of regression of y on x be

$$y = mx + c$$

$$\therefore \quad \sum y_i = m\sum x_i + c\sum i$$

$$\text{i.e.} \quad \sum y_i = nc + m\sum x_i \quad \text{... (I)}$$

$$\text{Also} \sum x_i y_i = m\sum x_i^2 + c\sum x_i \quad \text{... (II)}$$

Dividing equation (I) by n

$$\frac{1}{n}\sum y_i = c + m\frac{\sum x_i}{n}$$

$$\text{i.e.} \quad \bar{y} = c + m\bar{x} \quad \text{... (III)}$$

Which shows that the point $(\bar{x}, \bar{y})$ lies on the line of regression.

We know that

$$\text{cov}(x, y) = \frac{1}{n}\sum x_i y_i - \bar{x}\,\bar{y}$$

Let $\text{cov}(x, y) = \mu_{11}$

$$\text{Then } \mu_{11} + \bar{x}\,\bar{y} = \frac{1}{n}\sum x_i y_i \quad \text{... (IV)}$$

$$\text{Also} \quad \sigma_x^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

$$\therefore \quad \frac{1}{n}\sum x_i^2 = \sigma_x^2 + \bar{x}^2 \quad \text{... (V)}$$

Dividing equation (II) by N

$$\frac{1}{n}\sum x_i y_i = m\frac{\sum x_i^2}{N} + c\frac{\sum x_i}{N} \quad \text{... (VI)}$$

$$\therefore \quad \mu_{11} + \bar{x}\,\bar{y} = m\left(\sigma_x^2 + \bar{x}^2\right) + c\,\bar{x} \quad \text{... (VII)}$$

Multiplying equation (III) by $\bar{x}$ we have

$$\bar{x}\,\bar{y} = c\,\bar{x} + m\,\bar{x}^2$$

Subtract from equation (VIII)

$$\mu_{11} = m\,\sigma_x^2 \Rightarrow m = \frac{\mu_{11}}{\sigma_x^2}$$

$\therefore$ Equation of regression line which passes through $(\bar{x}, \bar{y})$ is

$$y - \bar{y} = m(x - \bar{x})$$

$$\text{or} \quad y - \bar{y} = \frac{\mu_{11}}{\sigma_x^2}(x - \bar{x})$$

This gives regression line of y on x we know that

$$r(x, y) = \frac{\text{cov}(x, y)}{\sigma_x\,\sigma_y} = \frac{\mu_{11}}{\sigma_x \cdot \sigma_y}$$

$$\therefore \quad \frac{\mu_{11}}{\sigma_x^2} = r(x, y)\cdot\frac{\sigma_y}{\sigma_x}$$

$$\therefore \quad \boxed{y - \bar{y} = r(x, y)\cdot\frac{\sigma_y}{\sigma_x}(x - \bar{x})}$$

is the line of regression of y on x.

Similarly if we start from x = my + c we can get the line of regression of x and y as

$$\boxed{x - \bar{x} = r(x, y)\cdot\frac{\sigma_x}{\sigma_y}(y - \bar{y})}$$

Thus the equation of line of regression of y on x is given by

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

where $\bar{x}, \bar{y}$ are means of distributions for x and y respectively. The equation of line of regression of x and y is given by

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$\text{where, } r\frac{\sigma_y}{\sigma_x} = \text{Regression coefficient of y on x} = b_{yx}$$

$$r\frac{\sigma_x}{\sigma_y} = \text{Regression coefficient of x on y} = b_{xy}$$

$$\text{Thus} \quad b_{yx}\cdot b_{xy} = r^2$$

Example 5.37 : *Find the lines of regression for the data.*

x	*10*	*14*	*19*	*26*	*30*	*34*	*39*
y	*12*	*16*	*18*	*26*	*29*	*35*	*38*

and estimate y for x = 14.5 and x for y = 29.5

Solution : Let A = 26, B = 26 $\quad u = x - A$,

$$v = y - B$$

$$\therefore \quad u = x - 26$$

$$v = y - 26$$

x	y	u = x − a x − 26 a	v = y − b y − 26 b	u^2	v^2	uv
10	12	− 16	− 14	256	196	224
14	16	− 12	− 10	144	100	120
19	18	− 7	− 8	49	64	56
26	26	0	0	0	0	0
30	29	4	3	16	9	12
34	35	8	9	64	81	72
39	38	13	12	169	144	156
		$\sum u = -10$	$\sum v = -8$	$\sum u^2 = 698$	$\sum v^2 = 594$	$\sum uv = 640$

$$\bar{u} = \frac{\sum u}{n} \quad \text{and} \quad \bar{v} = \frac{\sum v}{n}$$

here $n = 7\bar{u} = -\frac{10}{7} = -1.429$

$$\bar{v} = -\frac{8}{7} = 1.143$$

$$\bar{u}^2 = 2.042$$

$$\bar{v}^2 = 1.306$$

$$\text{cov}(u, v) = \frac{1}{n}\sum uv - \bar{u}\,\bar{v}$$

$$= \frac{1}{7}(640) - (1.429)(1.143)$$

$$= 89.795$$

$$\sigma_u^2 = \frac{1}{n}\sum u_i^2 - \bar{u}^2 = \frac{1}{7}(698) - 2.042$$

$$= 97.672$$

$$\sigma_u = 9.883$$

$$\sigma_v^2 = \frac{1}{n}\sum v_i^2 - \bar{v}^2 = \frac{1}{7}(594) - 1.306 = 83.55$$

$$\sigma_v = 9.14$$

$$r(x, y) = r(u, v) = \frac{\text{cov}(u\ v)}{\sigma_u \sigma_v} = \frac{89.795}{9.883 \times 9.14}$$

$$\therefore \quad r(x, y) = 0.9941$$

$$\therefore \quad b_{yx} = r \times \frac{\sigma_y}{\sigma_x} = r \times \frac{\sigma_v}{\sigma_u} = 0.9941 \times \frac{9.14}{9.883}$$

$$= 0.9194$$

$$b_{xy} = r \times \frac{\sigma_y}{\sigma_x} = 0.9941 \times \frac{9.883}{9.14} = 1.0749$$

$$\bar{x} = a + \bar{u} = 26 - 1.429 = 24.571$$

$$\bar{y} = b + \bar{v} = 26 - 1.143 = 24.857$$

$\therefore$ Regression line of y on x

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 24.857 = 0.9194\,(x - 24.571) \quad \text{... (I)}$$

Regression line of x and y

$$x - \bar{x} = \frac{r\sigma_y}{\sigma_x}(y - \bar{y})$$

$$x - 24.571 = 1.0749\,(y - 24.857) \quad \text{... (II)}$$

$\therefore$ put $x = 14.5$ in equation (I)

put $y = 29.5$ in equation (II)

$$\therefore \quad y = 15.5977$$

$$x = 29.56176$$

Example 5.38 : *Calculate the coefficient of correlation and obtain the regression line of y on x for the following data :*

x :	1	2	3	4	5	6	7	8	9
y :	9	8	10	12	11	13	14	16	15

Also find an estimate of y which correspond to x = 6.2.

Solution :

x	y	u = x − 5	v = y − 12	u^2	v^2	uv
1	9	− 4	− 3	16	9	12
2	8	− 3	− 4	9	16	12
3	10	− 2	− 2	4	4	4
4	12	− 1	0	1	0	0
5	11	0	− 1	0	1	0
6	13	1	1	1	1	1
7	14	2	2	4	4	4
8	16	3	4	9	16	12
9	15	4	3	16	9	12
		$\sum u = 0$	$\sum v = 0$	$\sum u^2 = 60$	$\sum v^2 = 60$	$\sum uv = 57$

$$\bar{u} = \frac{\sum u_i}{n} = 0, \qquad \bar{v} = \frac{\sum v_i}{n} = 0$$

$$\text{cov}(u, v) = \frac{\sum u_i v_i}{n} - \bar{u}\,\bar{v}$$

$$= \frac{57}{9} - 0\ 0$$

$$= 6.333$$

$$\sigma_u^2 = \frac{\sum u_i^2}{n} - (\bar{u})^2$$

$$= \frac{60}{9} - 0$$

$$= 6.666$$

$$\sigma_v^2 = \frac{\sum v_i^2}{n} - (\bar{v})^2$$

$$= \frac{60}{9} - 0$$

$$= 6.6666$$

$$\therefore \quad r(x, y) = r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \sigma_v}$$

$$= \frac{6.333}{\sqrt{(6.666)(6.666)}} = 0.95$$

As $\bar{x} = A + h\bar{u} = A + \bar{u}$

$$= 5 + 0 = 5$$

$\bar{y} = B + k\bar{v} = B + \bar{v}$ As $h = k = 1$

$$= 12 + 0$$

$$= 12$$

Now $\sigma_x = h\sigma_u$ and $\sigma_y = k\sigma_v$

$= \sqrt{6.666}$ $= \sqrt{6.666}$

$$\therefore \quad b_{yx} = r\frac{\sigma_y}{\sigma_x} = (0.95)(1) = 0.95$$

$\therefore$ The regression line of y on x is

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 12 = (0.95)(x - 5)$$

$$y = 0.95x + 7.25$$

put $x = 6.2$

$$y = (0.95)(6.2) + 7.25$$

$$= 13.14$$

Example 5.39 : *Obtain regression lines for the following data.*

SPPU : May-17, Dec.-16 Marks 4

x	6	2	10	4	8
y	9	11	5	8	7

Solution : To find regression coefficient b_{xy} and b_{yx} prepare the following table.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
6	9	36	81	54
2	11	4	121	22
10	5	100	25	50
4	8	16	64	32
8	7	64	49	56
$\sum x_i = 30$	$\sum y_i = 40$	$\sum x_i^2 = 220$	$\sum y_i^2 = 340$	$\sum x_i y_i = 214$

Here $n = 5$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{30}{5} = 6 \quad \text{and}$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{40}{5} = 8$$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 = \frac{220}{5} - (6)^2$$

$$= 44 - 36 = 8$$

$$\sigma_y^2 = \frac{\sum y_i^2}{n} - (\bar{y})^2 = \frac{340}{5} - (8)^2$$

$$= 68 - 64 = 4$$

$$\text{cov}(x, y) = \frac{\sum (x_i y_i)}{n} - \bar{x}\,\bar{y}$$

$$= \frac{214}{5} - 6 \times 8$$

$$\text{cov}(x, y) = 42.8 - 48 = -5.2$$

$$b_{yx} = \frac{\text{cov}(x, y)}{\sigma_x^2} = \frac{-5.2}{8} = -0.65$$

$$b_{xy} = \frac{\text{cov}(x, y)}{\sigma_y^2} = \frac{-5.2}{6} = -1.3$$

Regression line of Y on X is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 8 = -0.65(x - 6)$$

$$y = -0.65\ x + 3.9 + 8$$

$$y = -0.65\ x + 11.9$$

Regression line of X on Y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 6 = -1.3\ (y - 8)$$

$$x - 6 = -1.3\ y + 10.4$$

$$x = -1.3\ y + 10.4 + 6$$

$$x = -1.3\ y + 16.4$$

Example 5.40 : *The following are marks obtained by 10 students in Statistics and Economics.*

No.	1	2	3	4	5	6	7	8	9	10
Marks in economics	25	28	35	32	31	36	29	38	34	32
Marks in statistics	43	46	49	41	36	32	31	30	33	39

Marks are out of 50. Obtain regression equation to estimate marks in statistics if mark in economics are 30.

Solution : n = 10. Let marks in economics be x and marks in statistics be y.

Let u = x – 30 and v = y – 35.

x	y	u = x – 30	v = y – 35	u^2	v^2	uv
25	43	– 5	8	25	64	– 40
28	46	– 2	11	4	121	– 22
35	49	5	14	25	196	70
32	41	2	6	4	36	12
31	36	1	1	1	1	1
36	32	6	– 3	36	9	– 18
29	31	– 1	– 4	1	16	4
38	30	8	– 5	64	25	– 40
34	33	4	– 2	16	4	– 8
32	39	2	4	4	16	8
–	–	$\sum u = 2$	$\sum v = 30$	$\sum u^2 = 180$	$\sum v^2 = 488$	$\sum uv = -33$

$$\bar{u} = \frac{\sum u}{n} = \frac{20}{10} = 2 \quad \text{and}$$

$$\bar{v} = \frac{\sum v}{n} = \frac{30}{10} = 3$$

$u = x - 30 \quad \therefore \quad \bar{u} = \bar{x} - 30$

$\therefore \quad \bar{x} = \bar{u} - 30 = 2 + 30 = 32$

$v = y - 35 \quad \therefore \quad \bar{v} = \bar{y} - 35$

$\therefore \quad \bar{y} = \bar{v} + 35 = 3 + 35 = 38$

$$\sigma_u^2 = \frac{\sum u^2}{n} - (\bar{u})^2 = \frac{180}{10} - (2)^2$$

$$= 18 - 4 = 14$$

$$\sigma_v^2 = \frac{\sum v^2}{n} - (\bar{v})^2 = \frac{488}{10} - (3)^2$$

$$= 48.8 - 9 = 39.8$$

$\therefore \quad \sigma_u = 3.742$ and $\sigma_v = 6.309$

If $\quad u_i = \dfrac{x_i - A}{h} \qquad v_i = \dfrac{y_i - B}{k}$

then $\quad \sigma_x = h\,\sigma_u \qquad \sigma_y = k\,\sigma_v$

here $\quad h = k = 1$

$\therefore \quad \sigma_x = \sigma_u \qquad \sigma_y = \sigma_v$

$\therefore \quad \sigma_x = 3.743$ and $\sigma_y = 6.309$

$\therefore \quad \sigma_x^2 = 14$ and $\sigma_y^2 = 39.8$

$$\text{cov}(u, v) = \frac{\sum uv}{n} - \bar{u}\,\bar{v} = \frac{-33}{10} - 2\,(3)$$

$$= -3.3 - 6$$

$\therefore \quad \text{cov}(u, v) = -9.3$

Now $\quad r(x, y) = r(u, v) = \dfrac{\text{cov}(u, v)}{\sigma_u\,\sigma_v}$

$$= \frac{-9.3}{(3.742)(6.309)}$$

$$= -0.3943$$

$\therefore \quad b_{yx} = -r\,\dfrac{\sigma_y}{\sigma_x}$

$$= -0.3943\,\frac{6.309}{3.742}$$

$$= -0.6646$$

$\therefore$ The regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 38 = -0.664\ (x - 32)$$

$\therefore \quad y = -0.664\ x + 59.248$

Put x = 30 and find y

$$y = -0.664\,(30) + 59.248$$
$$= 39.328$$

Thus marks in economics are approximately 39.

Example 5.41 : *Find the coefficient of correlation for distribution in which S.D. of $x = 4$ and S.D. of $y = 1.8$. Coefficient of regression of y on x is 0.32.*

Solution : $\sigma_x = 4, \quad \sigma_y = 1.8 \quad$ and $\quad b_{yx} = 0.32$

We have, $b_{yx} = r\,\dfrac{\sigma_y}{\sigma_x}$

$$0.32 = r \times \frac{1.8}{4}$$

$$\therefore \quad r = \frac{0.32 \times 4}{1.8}$$

$$r = 0.711$$

Example 5.42 : *Given $n = 6$, $\sum (x-18.5) = -3$, $\sum (y-50) = 0$, $\sum (x-18.5)^2 = 19$, $\sum (y-50)^2 = 850$, $\sum (x-18.5)(y-50) = -120$. Calculate coefficient of correlation.*

Solution : Let $u = x - 18.5$ and $v = y - 50$

$$\therefore \quad \bar{u} = \frac{-3}{6} = -0.5$$

and
$$\bar{v} = \frac{20}{6} = 3.33$$

From the given data $\sum u = -3$, $\sum v = 20$, $\sum u^2 = 19$, $\sum v^2 = 850$ and $\sum uv = -120$.

Coefficient of correlation is given by

$$r = \frac{n\left(\sum uv\right) - \left(\sum u\right)\left(\sum v\right)}{\sqrt{\left[n\sum u^2 - \left(\sum u\right)^2\right]\left[n\sum v^2 - \left(\sum v\right)^2\right]}}$$

$$r = \frac{6\,(-120) - (-3)\,(20)}{\sqrt{[6\,(19) - (-3)^2]\;[6\,(850) - (20)^2]}}$$

$$r = \frac{-720 + 60}{\sqrt{[105]\;[4700]}} = -0.9395$$

Example 5.43 : *Given the following information*

	Variable x	*Variable y*
Arithmetic mean	*8.2*	*12.4*
Standard deviation	*6.2*	*20*

Coefficient of correlation between x and y is 0.9. Find the linear regression estimate of x given $y = 10$.

Solution : Given that $\bar{x} = 8.2$, $\bar{y} = 12.4$, $\sigma_x = 6.2$, $\sigma_y = 20$ and $r_{xy} = 0.9$. We want to find x for $y = 10$.

Line of regression of x on y is

$$x - \bar{x} = b_{xy}\,(y - \bar{y})$$

$$b_{xy} = r\,\frac{\sigma_x}{\sigma_y} = 0.9 \times \frac{6.2}{20}$$

$$= 0.279$$

Substituting value of $\bar{x}$, $\bar{y}$ and b_{xy} in above equation we get,

$$x - 8.2 = 0.279\,(y - 12.4)$$
$$x = 0.279\,y - 3.4596 + 8.2$$
$$x = 0.279\,y + 4.7404$$

Putting $y = 10$ in equation we get,

$$x = 0.279 \times 10 + 4.7404$$
$$x = 7.5304$$

Example 5.44 : *Given :*

	x series	*y series*
Mean	*18*	*100*
Standard deviation	*14*	*20*

and coefficient of correlation is 0.8. Find most probable values of y if $x = 70$ and most probable values of x if $y = 90$.

Solution : We are given $\bar{x} = 18 \quad \sigma_x = 14$
$\bar{y} = 100 \quad \sigma_y = 20$

The equation of line of regression of y on x is

$$y - \bar{y} = r\,\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 100 = \frac{0.8 \times 20}{14}(x - 18)$$

$$y = 1.413\,x + 79.421 \quad \ldots (1)$$

Probable value of y when $x = 70$ is given by (1)

$$y = 1.413\,(70) + 79.421$$
$$= 178.338$$

Equation of line of regression of x on y is

$$x - \bar{x} = r\,\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$x - 18 = (0.8)\,\frac{14}{20}(y - 100)$$

i.e.
$$x = 0.56\,y - 38 \quad \ldots (2)$$

Probable value of x when $y = 90$ is given by equation (2)

$$x = 0.56\,(90) - 38$$
$$= 12.4$$

Example 5.45 : *From record of analysis of correlation data the following results are available variance of x = 9 and lines of regression are given by*

8x – 10y + 66 = 0

40 x – 18y = 214

Find out a) Mean values for x and y services. b) Standard deviation of y services. c) Coefficient of correlation between x and y services.

SPPU : May-19, Dec.-17

Solution : Let $\bar{x}$, $\bar{y}$ be the means of x and y series then as $(\bar{x}, \bar{y})$ satisfy equations of lines of regression we have

$$8\bar{x} - 10\bar{y} + 66 = 0 \quad \text{... (1)}$$
$$40\bar{x} - 18\bar{y} - 214 = 0$$

Solving the above equation we get

$$\bar{x} = 13,\ \bar{y} = 17$$

Expressing equation (1) of lines of regression y in terms of x and x in terms of y, we have

$$8x - 10y + 66 = 0 \text{ as } \begin{cases} y = \frac{8}{10}x + \frac{66}{10} & \text{... (2)} \\ x = \frac{10}{8}y - \frac{66}{8} & \text{... (3)} \end{cases}$$

and

$$40x - 18y - 214 = 0 \text{ as } \begin{cases} y = \frac{40}{18}x - \frac{214}{18} & \text{... (4)} \\ x = \frac{18}{40}y + \frac{214}{40} & \text{... (5)} \end{cases}$$

Now we take combination of a line y on x with the other line x on y from equation (2), (3), (4) and (5) thus

$$y = \frac{8}{10}x + \frac{66}{10} \quad \text{and}$$
$$x = \frac{18}{40}y + \frac{214}{40} \quad \text{... (6)}$$
$$x = \frac{10}{8}y - \frac{66}{8} \quad \text{and}$$
$$y = \frac{40}{18}x - \frac{214}{18} \quad \text{... (7)}$$

For pair of lines given by equation (6), we have coefficient of regression as

$$b_{yx} = \frac{8}{10} \quad b_{xy} = \frac{18}{40}$$
$$r^2 = b_{yx}\ b_{xy} = \frac{8 \times 18}{10 \times 40} = 0.3600$$

Hence as r > 0, we consider the equation (6) as the lines of regression for x, y distribution

$$r^2 = 0.36$$
$$\therefore \quad r = \pm 0.6$$

For these lines equation (6) $b_{yx} = \frac{8}{10}$ and $b_{xy} = \frac{18}{40}$ are positive we take r as positive.

$\therefore r = 0.6 =$ Correlation coefficient between x and y.

Now $b_{yx} = r\frac{\sigma_y}{\sigma_x}$ As $\sigma_x^2 = 9 \quad \therefore \sigma_x = 3$

$$\therefore \quad \frac{8}{10} = 0.6\,\frac{\sigma_y}{3}$$
$$\therefore \quad \sigma_y = \frac{24}{6} = 4$$

Example 5.46 : *If the two lines of regression are $9x + y - \lambda = 0$ and $4x + y = \mu$ and the means of x and y are 2 and – 3 respectively, find the values of λ, μ and the coefficient of correlation between x and y.*

SPPU : Dec.-18, Marks 4

Solution : $\bar{x} = 2$ and $\bar{y} = -3$

The lines of regression are $9x + y = \lambda$ and $4x + y = \mu$.

The point of intersection of two regression lines is (x, y) i.e. $(\bar{x}, \bar{y})$ lies on both the regression lies.

$$9\bar{x} + \bar{y} = \lambda \quad \text{... (1)}$$
$$4\bar{x} + \bar{y} = \mu \quad \text{... (2)}$$

Substituting values of $\bar{x}$ and $\bar{y}$ we get,

$$9(2) + (-3) = \lambda$$
$$\lambda = 18 - 3 = 15$$

and
$$4(2) + (-3) = \mu$$
$$\therefore \quad \mu = 8 - 3 = 5$$

Thus, the regression lies are,

$$9x + y = 15 \quad \text{and} \quad 4x + y = 15$$

Let 9x + y = 15 be the regression line of x and y, so it can be written as

$$x = \frac{15}{9} - \frac{y}{9}$$
$$\therefore \quad b_{xy} = -\frac{1}{9} = -0.11$$

Let 4x + y = 5 be the regression line of y on x. So it can be written as y = 5 – 4x.

$$\therefore \quad b_{yx} = -4$$

Correlation coefficient between x and y is given as,

$$r = \sqrt{b_{yx}\ b_{xy}} = \sqrt{(-4) \times (-0.11)}$$

$$= \sqrt{0.44}$$
$$= 0.663$$

Alternate Method : Since the given equations (1) and (2) pass through $(\bar{x}, \bar{y})$

$\therefore$ we have

$$\bar{x} = 19.13 - 0.87\,\bar{y}$$

and $$\bar{y} = 11.64 - 0.50\,\bar{x}$$

Solving these we get

$$\bar{x} = 15.79$$
$$\bar{y} = 3.74$$

These are the required mean of x and y respectively.

Also from equation (1) and (2), we have

$$b_{xy} = -0.87$$

and $$b_{yx} = -0.50$$

$\therefore$ Coefficient of correlation

$$r = \pm\sqrt{(b_{xy})\times(b_{yx})}$$

$$\Rightarrow \quad r = \pm\sqrt{(0.87)(-0.50)}$$
$$= -0.66$$

We choose here negative sign, because b_{xy} and b_{yx} are both having negative signs.

Example 5.47 : *Obtain regression lines for the following data.*

x	*6*	*2*	*10*	*4*	*8*
y	*9*	*11*	*5*	*8*	*7*

Solution : To find regression coefficient b_{xy} and b_{yx} prepare the following table.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
6	9	36	81	54
2	11	4	121	22
10	5	100	25	50
4	8	16	64	32
8	7	64	49	56
$\sum x_i = 30$	$\sum y_i = 40$	$\sum x_i^2 = 220$	$\sum y_i^2 = 340$	$\sum x_i y_i = 214$

Here $$n = 5$$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{30}{5} = 6 \quad \text{and}$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{40}{5} = 8$$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 = \frac{220}{5} - (6)^2$$
$$= 44 - 36 = 8$$

$$\sigma_y^2 = \frac{\sum y_i^2}{n} - (\bar{y})^2 = \frac{340}{5} - (8)^2$$
$$= 68 - 64 = 4$$

$$\text{cov}(x, y) = \frac{\sum (x_i y_i)}{n} - \bar{x}\,\bar{y}$$
$$= \frac{214}{5} - 6\times 8$$

$$\text{cov}(x, y) = 42.8 - 48 = -5.2$$

$$b_{yx} = \frac{\text{cov}(x, y)}{\sigma_x^2} = \frac{-5.2}{8} = -0.65$$

$$b_{xy} = \frac{\text{cov}(x, y)}{\sigma_y^2} = \frac{-5.2}{6} = -1.3$$

Regression line of Y on X is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$
$$y - 8 = -0.65(x - 6)$$
$$y = -0.65x + 3.9 + 8$$
$$y = -0.65x + 11.9$$

Regression line of X on Y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$
$$x - 6 = -1.3(y - 8)$$
$$x - 6 = -1.3y + 10.4$$
$$x = -1.3y + 10.4 + 6$$
$$x = -1.3y + 16.4$$

Example 5.48 : *Find lines of regression for the following data.*

x	*2*	*3*	*5*	*7*	*9*	*10*	*12*	*15*
y	*2*	*5*	*8*	*10*	*12*	*14*	*15*	*16*

Find estimate of y when x = 6

SPPU : Dec.-15, May-16

Solution : To find regression lines, we require to calculate regression coefficients σ_{xy} and σ_{yx}. So consider the following table.

x_i	y_i	x_i^2	y_i^2	x_iy_i
2	2	4	4	4
3	5	9	25	15
5	8	25	64	40
7	10	49	100	70
9	12	81	144	108
10	14	100	196	140
12	15	144	225	180
15	16	225	256	240
Total = 63	**82**	**637**	**1014**	**797**

We have

$$n = 8$$

$$x = \frac{\sum x_i}{n} = \frac{63}{8} = 7.873$$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - \bar{x}^2 = \frac{637}{8} - (7.873)^2$$

$$= 17.6094$$

$$\bar{y} = \frac{\sum y}{n} = \frac{82}{8} = 10.25,$$

$$\sigma_y^2 = \frac{\sum y_i^2}{n} - \bar{y}^2 = \frac{1014}{8} - (10.25)^2$$

$$= 21.6875$$

$$\text{cov}(x, y) = \frac{\sum x_i y_i}{n} - \bar{x}\bar{y}$$

$$= \frac{797}{8} - (7.873)(10.25)$$

$$= 18.9063$$

$$b_{yx} = \frac{\text{cov}(x, y)}{\sigma_x^2} = \frac{18.9063}{17.6094} = 1.0736$$

$$b_{xy} = \frac{\text{cov}(x, y)}{\sigma_y^2} = \frac{18.9063}{21.6875} = 0.8718$$

i) Regression line of y on x :

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 10.25 = 1.0736(x - 7.875)$$

$$y = 1.0736x + 1.7954$$

$\therefore$ At $x = 6$, $y = 1.0736 \times 6 + 1.7954 = 8.237$

ii) Regression line of x on y :

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 7.875 = (0.8718)(y - 10.25)$$

$$x = 0.8718y - 1.06095$$

Example 5.49 : *Obtain the regression line of y on x for the following data :* **SPPU : Dec.-18, Marks 4**

x	*5*	*1*	*10*	*3*	*9*
y	*10*	*11*	*5*	*10*	*6*

Solution : Consider the following table

x_i	y_i	x_i^2	x_iy_i
5	10	25	50
1	11	01	11
10	5	100	50
3	10	09	30
9	6	81	54
$\sum x_i = 28$	$\sum y_i = 42$	$\sum x_i^2 = 216$	$\sum x_iy_i = 195$

Here $n = 5$, $\bar{x} = \frac{\sum x_i}{n} = \frac{28}{5} = 5.6$, $\bar{y} = \frac{\sum y_i}{5} = 8.4$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 = \frac{216}{5} - (5.6)^2 = 11.84$$

$$\text{cov}(x, y) = \frac{\sum x_i y_i}{n} - \bar{x}\bar{y} = \frac{195}{5} - (5.6)(8.4) = -8.04$$

$$\therefore \quad b_{yx} = \frac{\text{cov}(x, y)}{\sigma_x^2} = \frac{-8.04}{11.84} = -0.679$$

The regression Line of Y on X is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 8.4 = -0.679(x - 5.6)$$

$$y = 0.679x + 12.2024$$

Example 5.50 : *Obtain the regression line of y on x for the following data :* **SPPU : May-19, Marks 4**

x	*y*
1	*2*
2	*5*
3	*3*
4	*8*
5	*7*

Solution : Consider the following table

x	y	x^2	y^2	xy
1	2	1	4	2
2	5	4	25	10
3	3	9	09	9
4	8	16	64	32
5	7	25	49	35
$\sum x = 15$	$\sum y = 25$	$\sum x^2 = 55$	$\sum y^2 = 151$	$\sum xy = 88$

Here $n = 5, \bar{x} = \frac{\sum x}{n} = \frac{15}{5} = 3,$

$$\bar{y} = \frac{\sum y}{n} = \frac{25}{5} = 5$$

$$\text{cov}(x, y) = \frac{\sum x_i y_i}{n} - \bar{x}\bar{y}$$

$$= \frac{88}{5} - 15 = 17.6 - 15 = 2.6$$

$$\sigma_x^2 = \frac{\sum x^2}{n} - x^{-2} = \frac{55}{5} - 9 = 3$$

$$\sigma_y^2 = \frac{\sum y^2}{n} - y^{-2} = \frac{151}{5} - 25$$

$$= 30.2 - 25 = 5.2$$

$$r = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} = \frac{2.6}{\sqrt{3} \cdot \sqrt{5.2}} = 0.6582$$

The regression line of y on x is,

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 5 = (0.6582)\left(\frac{\sqrt{5.2}}{\sqrt{3}}\right)(x - 3)$$

$$y = 0.8665\,(x - 3) + 5$$

$$\therefore \quad y = 0.8665x + 2.40031$$

Example 5.51 : *Calculate the coefficient of correlation from the following information :* $n = 10$, $\Sigma x = 40$, $\Sigma x^2 = 190$, $\Sigma y^2 = 200$, $\Sigma xy = 150$, $\Sigma y = 40$. **SPPU : May-16, Marks 4**

Solution : Given that $n = 10$, $\Sigma x = 40$, $\Sigma x^2 = 190$, $\Sigma y^2 = 200$

$$\Sigma xy = 150, \Sigma y = 40, \bar{x} = 4 = \bar{y}$$

$$\text{COV}(x, y) = \frac{\Sigma xy}{n} - \bar{x}\,\bar{y}$$

$$= \frac{\Sigma xy}{n} - \left(\frac{\Sigma x}{n}\right)\left(\frac{\Sigma y}{n}\right)$$

$$= \frac{150}{10} - \left(\frac{40}{10}\right)\left(\frac{40}{10}\right) = 15 - 16 = -1$$

$$\sigma_x = \left[\frac{\Sigma x^2}{n} - \bar{x}^2\right]^{1/2} = \left[\frac{190}{10} - 16\right]^{\frac{1}{2}} = [19 - 16]^{1/2}$$

$$\sigma_x = \sqrt{3} = 1.7320$$

$$\sigma_y = \left[\frac{\Sigma y^2}{n} - \bar{y}^2\right]^{1/2} = \left[\frac{200}{10} - 16\right]^{1/2} = [4]^{\frac{1}{2}} = 2.$$

$$\therefore \ r = \frac{COV(x,y)}{\sigma_x \sigma_y} = \frac{-1}{2 \times (1.7320)} = -0.2886$$

Example 5.52 : *If* $\sum x_i = 30$, $\sum y_i = 40$, $\sum x_i^2 = 220$, $n = 5$, $\sum y_i^2 = 340$ *and* $\sum x_i y_i = 214$, *then obtain the regression lines for this data.* **SPPU : Dec.-16, Marks 4**

Solution :

Given : $\sum x_i = 30$, $\sum y_i = 40$, $\sum x_i^2 = 220$, $n = 5$,

$$\sum y_i^2 = 340, \sum x_i y_i = 214, \therefore \bar{x} = \frac{\sum x_i}{n} = \frac{30}{5} = 6$$

$$\bar{y} = \frac{40}{5} = 8$$

$$\text{cov}(x, y) = \frac{\sum xy}{n} - \bar{x}\bar{y} = \frac{214}{5} - 6 \times 8 = -5.2$$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - \bar{x}^2 = \frac{220}{5} - 36 = 44 - 36 = 8$$

$$\therefore \quad \sigma_x = \sqrt{8}$$

$$\sigma_y^2 = \frac{\sum y^2}{n} - \bar{y}^2 = \frac{340}{5} - 64 = 68 - 64 = 4$$

$$\therefore \quad \sigma_y = 2$$

$$\therefore \quad r = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} = \frac{-5.2}{\sqrt{8} \times 2} = -0.9192$$

Example 5.53 : *The two variables x and y have regression lines :*

$3x + 2y - 26 = 0$ and $6x + y - 31 = 0$

Find :

i) The mean values of x and y and

ii) Correlation coefficient between x and y. **SPPU : Dec.-17, Marks 4**

Solution : Let $(\bar{x}, \bar{y})$ be the mean of regression series of x and y.

$\therefore$ $(\bar{x}, \bar{y})$ satisfies equations of lines of regressions.

$$3\bar{x} + 2\bar{y} = 26 \quad \rightarrow (1)$$

$$6\bar{x} + \bar{y} = 31$$

$$12\bar{x} + 2\bar{y} = 62 \quad \rightarrow (2)$$

$$- \quad - \quad -$$

$$-9\bar{x} = -36$$

$$\bar{x} = 4$$

$\therefore$ $$\bar{y} = 31 - 6\bar{x} = 31 - 24 = 7$$

Now $$3x + 2y = 26 \Rightarrow x = \frac{26}{3} - \frac{2}{3}y$$

$\therefore$ $$b_{xy} = -\frac{2}{3}$$

$$6x + y = 31 \Rightarrow y = 31 - 6x$$

$\therefore$ $$b_{yx} = -6$$

Thus, $$r^2 = b_{xy}\, b_{yx} = \left(-\frac{2}{3}\right)(-6) = 4$$

which is not possible

$\therefore$ $$3x + 2y = 26 \Rightarrow y = \frac{26}{2} - \frac{3}{2}x \quad \therefore b_{yx} = -\frac{3}{2}$$

and $$6x + y = 31 \Rightarrow x = \frac{31}{6} - \frac{1}{6}y \Rightarrow b_{xy} = -\frac{1}{6}$$

$\therefore$ $$r^2 = \left(-\frac{3}{2}\right)\left(-\frac{1}{6}\right) = \frac{1}{4} = 0.25$$

$$r = 0.5$$

Example 5.54 : *Obtain the line of regression of y on x for the following data, Also, estimate the value of y for x = 10.* **SPPU : May-18, Marks 4**

x	2	4	5	6	8	11
y	18	12	10	8	7	5

Solution : Consider the following table

$$u = x - 6,\ v = y - 8$$

x	y	u = x – 6	v = y – 8	u^2	v^2	uv
2	18	– 4	10	16	100	– 40
4	12	– 2	4	4	16	– 8
5	10	– 1	2	1	04	– 2
6	8	0	0	0	00	0
8	7	2	– 1	4	1	– 2
11	5	5	– 3	25	9	– 15
		$\sum u = 0$	$\sum v = 12$	$\sum u^2 = 50$	$\sum v^2 = 130$	$\sum uv = -67$

Now $$\bar{u} = \frac{\sum u}{6} = 0,\ \bar{v} = \frac{\sum v}{6} = 2$$

$$\text{cov}(u, v) = \frac{\sum uv}{n} - \bar{u}\,\bar{v} = \frac{-67}{6} = -11.167$$

$$\sigma_u^2 = \frac{\sum u^2}{n} - \bar{u}^2 = \frac{50}{6} = 8.333$$

$$\sigma_v^2 = \frac{\sum v^2}{n} - \bar{v}^2 = \frac{130}{6} - 4 = \frac{106}{6} = 17.667$$

$$r(x, y) = r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u\, \sigma_v} = \frac{-11.167}{\sqrt{8.333}\sqrt{17.667}}$$

$$r(u, v) = -0.9203$$

$$\bar{x} = A + \bar{u} = 6 + 0 = 6$$

$$\bar{y} = B + \bar{v} = 8 + 2 = 10$$

$$\sigma_x = \sigma_u \text{ and } \sigma_y = \sigma_v$$

$\therefore$ $$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{(-0.9203)(4.203)}{(2.8867)} = 1.3399$$

$\therefore$ The regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 10 = 1.399(x - 6)$$

$\therefore$ $$y = 1.3399\,x - 1.9606$$

$\therefore$ At $$x = 10$$

$$y = 13.399 - 1.9606 = 11.4384$$

Example 5.55 : *Obtain the regression line of y on x for the following data :* **SPPU : Dec.-18, Marks 4**

x	5	1	10	3	9
y	10	11	5	10	6

Solution : Consider the following table

x_i	y_i	x_i^2	$x_i y_i$
5	10	25	50
1	11	01	11
10	5	100	50
3	10	09	30
9	6	81	54
$\sum x_i = 28$	$\sum y_i = 42$	$\sum x_i^2 = 216$	$\sum x_i y_i = 195$

Here $n = 5,\ \bar{x} = \frac{\sum x_i}{n} = \frac{28}{5} = 5.6,\ \bar{y} = \frac{\sum y_i}{5} = 8.4$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 = \frac{216}{5} - (5.6)^2 = 11.84$$

$$\text{cov}(x, y) = \frac{\sum x_i y_i}{n} - \bar{x}\bar{y} = \frac{195}{5} - (5.6)(8.4) = -8.04$$

$$\therefore \quad b_{yx} = \frac{\text{cov}(x,y)}{\sigma_x^2} = \frac{-8.04}{11.84} = -0.679$$

The regression Line of Y on X is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 8.4 = -0.679\ (x - 5.6)$$

$$y = 0.679\ x + 12.2024$$

Example 5.56 : *Obtain the regression line of y on x for the following data :*

SPPU : May-19, Marks 4

x	*y*
1	2
2	5
3	3
4	8
5	7

Solution : Consider the following table

x	y	x^2	y^2	xy
1	2	1	4	2
2	5	4	25	10
3	3	9	09	9
4	8	16	64	32
5	7	25	49	35
$\sum x = 15$	$\sum y = 25$	$\sum x^2 = 55$	$\sum y^2 = 151$	$\sum xy = 88$

Here $n = 5,\ \bar{x} = \frac{\sum x}{n} = \frac{15}{5} = 3,$

$$\bar{y} = \frac{\sum y}{n} = \frac{25}{5} = 5$$

$$\text{cov}(x, y) = \frac{\sum x_i y_i}{n} - \bar{x}\bar{y}$$

$$= \frac{88}{5} - 15 = 17.6 - 15 = 2.6$$

$$\sigma_x^2 = \frac{\sum x^2}{n} - \bar{x}^2 = \frac{55}{5} - 9 = 3$$

$$\sigma_y^2 = \frac{\sum y^2}{n} - \bar{y}^2 = \frac{151}{5} - 25 = 30.2 - 25 = 5.2$$

$$r = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} = \frac{2.6}{\sqrt{3} \cdot \sqrt{5.2}} = 0.6582$$

The regression line of y on x is,

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 5 = (0.6582)\left(\frac{\sqrt{5.2}}{\sqrt{3}}\right)(x - 3)$$

$$y = 0.8665\ (x - 3) + 5$$

$$\therefore \quad y = 0.8665x + 2.40031$$

Exercise 5.2

1. *Calculate the coefficient of correlation for the following distribution.*

y \ x	16 - 18	18 - 20	20 - 22	22 - 24
10 - 20	2	1	1	–
20 - 30	3	2	3	2
30 - 40	3	4	5	6
40 - 50	2	2	3	4
50 - 60	–	1	2	2
60 - 70	–	1	2	1

[Ans. : 0.28]

2. *Calculate the coefficient of correlation between the marks obtained by 8 students in Maths and statistics, from the following table. Find also lines of regression.*

Student	A	B	C	D	E	F	G	H
Maths (x)	25	30	32	35	37	40	42	45
Statistics (y)	8	10	15	17	20	22	24	25

3. *Calculate coefficient of correlation for the following ages of Husband and Wifes.*

Husband's age	x	23	27	28	28	29	30	31	33	35	36
Wife's age	y	18	20	22	27	21	29	27	29	28	29

[Ans. : 0.82]

4. *Calculate correlation coefficient for the following heights in inches of fathers and their sons :*

Father's height	x	65	66	67	67	68	69	70	72
Son's height	y	67	68	65	68	72	72	69	72

[Ans. : 0.603]

5. *Find the correlation coefficient and equations of regression lines for the values of x and y :*

x	1	2	3	4	5
y	2	5	3	8	7

[Ans. : r = 0.8, y = 1.3 x + 1.1, x = 0.5 y + 0.5]

6. *Obtain the regression of y on x and x on y from the table and estimate the blood pressure when the age is 45 years.*

Age in years	x	56	42	72	36	63	47	55	49	38	42	68	60
Blood pressure	y	147	125	160	118	149	128	150	145	115	140	152	155

[Ans. : y = 1.138 x + 80.778, y = 131.988 for x = 45]

7. *Following data refers to year of service in a factory of 7 persons in a specialised field and their monthly incomes.*

Year in Service	11	7	9	5	8	6	10
Income in Hundreds of rupees	7	5	3	2	6	4	8

Find the regression equation of income on years of service. Using it what initial start would you recommend for a person applying for a job after having served in another factory in a similar field for 12 years ? **[Ans. : 800]**

8. *Calculate the correlation coefficient for the following heights in inches of fathers (x) and their sons (y).*

x	65	66	67	67	68	69	70	72
y	67	68	65	68	72	72	69	71

Also estimate the height of son when the height of father is 71 cms. **[Ans. : r = 0.55, Height of son (y) = 70.8 cm]**

9. *The following regression equations were obtained from a correlation table :*

$y = 0.516\, x + 33.73$

$x = 0.512\, y + 32.52$

Find the value of (i) mean of x's and y's (ii) the correlation coefficient. **[Ans. : i) $\bar{x}$ = 67, $\bar{y}$ = 68.61, ii) r = 0.514]**

10. *Two examiners A and B independently award marks to seven students.*

R. No.	1	2	3	4	5	6	7
Marks by A	40	44	28	30	44	38	31
Marks by B	32	39	26	30	38	34	28

Obtain the equations of regression lines. If examiner A awards 36 marks to Roll No. 8, what would be the marks expected to be awarded by examiner B to the same candidate ? **[Ans. : y = 11.885 + 0.587x, 33.017]**

11. *Determine the equations of regression lines for the following data :*

x	1	2	3	4	5	6	7	8	9
y	9	8	10	12	11	13	14	16	15

and obtain an estimate of y for x = 4.5. **[Ans. : 0.95x + 7.25, x = 11.525]**

12. *Determine the reliability of estimates for the data :*

x	10	14	19	26	30	34	39
y	12	16	18	26	29	35	38

[Ans. : r^2 = 0.988 high]

13. *The following marks have been obtained by a group of students in Engineering Mathematics.*

Paper I	80	45	55	56	58	60	65	68	70	75	85
Paper II	82	56	50	48	60	62	64	65	70	74	90

Calculate the coefficient of correlation. **[Ans. 9277]**

14. *For the following tabulated data, find the coefficient of correlation.*

y \ x	18	18	20	21	Total
10 - 20	4	2	2	–	8
20 - 30	5	4	6	4	19
30 - 40	6	8	10	11	35
40 - 50	4	4	6	8	22
50 - 60	–	2	4	4	10
60 - 70	–	2	3	1	6
Total	19	22	31	28	100

[Ans. : 0.25]

15. The two regression equations of variables x and y are $x = 4y - 3$ and $9y = x + 13$ find :

i) mean of x and y and **[Ans. : $\bar{x} = 5, \bar{y} = 2$]**

ii) coefficient of correlation between x and y. **[Ans. : $r = \frac{2}{3}$]**

16. A panel of two judges 'A' and B graded dramatic performances by independently awarding marks as follows :

Performance No.	1	2	3	4	5	6	7	8
Marks by A	36	32	34	31	32	32	35	38
Marks by B	35	33	31	30	34	32	36	?

The eight performance, however, which judge B could not attend, got 38 marks by judge A. If judge B had also been present, how many marks would be expected to have been awarded by him to the eight performance ?

[Ans. : 35.9 ≅ 36 (Approximately)]

17. For a group of children, mean age is 10 years with standard deviation of 2.5 years. The average height of the group is 125 cms, with standard deviation of 13 cms, the coefficient of correlation between the age and height is 0.6, write the equations of two regression lines and explain their use.

[Ans. : Line of regression of y on x is, $y = 3.12x + 93.8$,

Line of regression of x on y is, $x = 0.115y - 4.43$]

18. Find the correlation coefficient and the equations of regression lines from the following data :

a)

x	1	2	3	4	5
y	2	5	3	8	7

[Ans. : $r = 0.81$, $x = 0.5y + 0.5$, $y = 1.3x + 1.1$]

b)

x	80	45	55	56	58	60	65	68	70	75	85
y	8	56	50	48	60	62	64	65	70	74	90

[Ans. : $r = 0.918$, $y - 65.45 = 0.989\ (x - 65.18)$, $x - 65.18 = 0.85\ (y - 65.45)$]

c)

x	2	4	5	6	8	11
y	18	12	10	8	7	5

[Ans. : $r = -0.92$, $y - 10 = -1.34\ (x - 6)$, $x - 6 = -0.632\ (y - 10)$]

5.12 University Questions

Dec.-2016

Q.1 *The first four moments of a distribution about the value 2 are – 2, 12, – 20 and 100. Find the first four central moments and B_1, B_2.* **[4]**

Q.2 *Find coefficient of correlation for the following data :* **[4]**

x	y
6	9
2	11
10	5
4	8
8	7

May-2017

Q.3 *If*

$\sum f = 27,\ \sum fx = 91,\ \sum fx^2 = 359,$

$\sum fx^3 = 1567,\ \sum fx^4 = 7343.$

Find the first four moments about origin. Also find μ_2, μ_3, μ_4. **[4]**

Q.4 *Obtain regression lines for the following data :* **[4]**

x	y
6	9
2	11
10	5
4	8
8	7

Dec.-2017

Q.5 *The first four central moments about the working mean 30.2 of a distribution are 0.255, 6.222, 30.211 and 400.25 Calculate the first four moments about the mean. Also evaluate β_1, β_2 and comment upon the skewness and kurtosis of the distribution.* **[4]**

Q.6 *The two regression equations of the variables x and y are $x = 19.13 - 0.87\ y$, $y = 11.64 - 0.50\ x$, find $\bar{x}, \bar{y}$ and coefficient of correlation between x and y.* **[4]**

May-2018

Q.7 *The first four moments of a distribution about the value 5 are 2, 20, 40 and 50. From the given information obtain the first four central moments and coefficient of skewness and kurtosis.* **[4]**

Q.8 *Find the regression line of y on x for the following data :* **[4]**

x	y
10	18
14	12
18	24
22	6
26	30
30	36

Dec.-2018

Q.9 *Calculate the first four moments about the mean of the following frequency distribution :* **[4]**

x	f
0	1
1	8
2	28
3	56
4	70
5	56
6	28
7	8
8	1

Q.10 *Obtain the regression line of y on x for the following data :* **[4]**

x	y
5	10
1	11
10	5
3	10
9	6

May-2019

Q.11 *The first four moments of a distribution about the value 2.5 are 1, 10, 20 and 25. Obtain first four central moments. Also calculate coefficient of skewness (β_1) and coefficient of kurotsis (β_2).* **[4]**

Q.12 *Obtain the regression line of y on x for the following data :* **[4]**

x	y
1	2
2	5
3	3
4	8
5	7

Dec.-2019

Q.13 *The first four moments of a distribution about 2 are 1, 2.5, 5.5 and 16. Calculate first four moments about the mean. Also obtain the coefficient of skewness (β_1) and coefficient of kutosis (β_2) .* **[4]**

Q.14 *Calculate the coefficient of correlation for the following data :* **[4]**

x	y
1	9
2	8
3	10
4	12
5	11
6	13
7	14
8	16
9	15

❑❑❑

Notes

UNIT - IV

6 Probability and Probability Distributions

6.1 Introduction

If an experiment is conducted and each outcome of the experiment has the same chance of appearing as any other then we call the outcomes as equally likely.

$$P(A) = \text{Probability of occurrence of any event A}$$
$$= \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$$

When an event succeeds in "S" ways and fails in "F" ways (n = F + S = Total number of ways) then

$$P(\text{Success}) = \frac{S}{S+F} \quad \text{and} \quad P(\text{Failure}) = \frac{F}{S+F}$$

Generally the probability of success is denoted by p and the probability of failure is denoted by q.
Obviously $p + q = 1$ i.e. $q = 1 - p$.

6.2 Theory of Probability

Some important definitions

1) Trial and event : Let an experiment be repeated under essentially the same conditions and let it result in any one of the several possible outcomes. Then, the experiment is called a trial and the possible outcomes are known as events or cases. For example, tossing of a coin is a trial and the turning up of head or tail is an event.

2) Independent events : Two events are said to be independent when the outcome of one does not affect and is not affected by the other.

e.g. coin is tossed twice, outcome of 2^{nd} throw is independent of the outcome of 1^{st}.

3) Dependent events : Dependent events are those in which the occurrence or non-occurrence of one event in any trial affects the probability of other event in other trials.

4) Mutually exclusive : Two events are said to be mutually exclusive or incompatible when both cannot happen simultaneously in a single trial or in other words the occurrence of any one of them precludes the occurrence of the other. If E_1 and E_2 are events then $E_1 \cap E_2 = \phi$.

e.g. coin is tossed, either it will be head or tail, both cannot be up at the same time.
Similarly person cannot be alive and dead simultaneously.

$\therefore$ Mutually exclusive events are either or type.

5) Simple and compound events : In case of simple events we consider the probability of the happening or not happening of single events.

e.g. we might be interested in probability of red ball from 2 bag of 10 white and 6 red balls.

On the other hand if a bag contains 10 white and 6 red balls and two successive draws of 3 balls each are made. Then finding probability of 3 white in first draw and 3 red in second is the case of joint occurrence. In this case we are dealing with compound event.

6) Favourable events : The cases which entail the happening of an event are said to be favourable to event. It is the total number of possible outcomes in which the specified event happens. In throwing of two dice the number of cases favourable to getting a sum '6' is five i.e. (1, 5), (5, 1), (2, 4), (4, 2), (3, 3).

7) Equally likely events : Events are said to be equally likely if there is no reason to expect any other. In throwing a die, all the six faces are equally likely to come.

8) Exhaustive events : Events are said to be exhaustive when their totality includes all the possible outcomes of a random experiment.

e.g. while tossing a die, the possible outcome are 1, 2, 3, 4, 5, 6.

$\therefore$ Exhaustive number of cases is '6'.

If two dices are thrown once (together) the possible outcomes are $6 \times 6 = 36$.

Similarly for three dices it will be $6 \times 6 \times 6 = 216$ as total number of cases.

Similarly black and red cards are examples of collectively exhaustive events each being 26 in number.

9) Complementary events : Two events are said to be complementary if they are mutually exclusive and exhaustive. If E_1 and E_2 are complementary events then $E_1 \cup E_2 = S$ and $E_1 \cap E_2 = \phi$.

for e.g. a dice is thrown, getting an even 2, 4, 6 and odd no. (1, 3, 5) are complementary events.

i.e. if A occurs B does not (Exclusive) and vice-versa.

6.3 Theorems on Probability

a) Addition theorem

1) If A and B are mutually exclusive events.

then Prob (A or B)

i.e. $$P(A \cup B) = P(A) + P(B)$$

If A, B, C are mutually exclusive then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

2) When the events are not mutually exclusive then probability that atleast one of the two events A and B will occur is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i.e. $$P(A \text{ or } B) = P(A) + P(B) - P(A \,\&\, B)$$

In case of three events

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) \\ &= -P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= P(A \cap B \cap C) \end{aligned}$$

b) Multiplication theorem

If A and B are two independent events, then the probability that both will occur is equal to the product of their individual probabilities.

$$P(A \,\&\, B) = P(A) \times P(B) = P(A \cap B)$$

Similarly $$P(A, B \,\&\, C) = P(A) \times P(B) \times P(C) = P(A \cap B \cap C)$$

c) Conditional probability

Multiplication theorem is not applicable when events are dependent.

e.g. when we are computing probability of a particular event A. If given information about occurrence of B. Such a probability is referred to as conditional probability.

$\therefore$ for two dependent events A and B probability of B, given A has occurred is denoted by

$$P(B|A) = \frac{P(A, B)}{P(A)} = \frac{P(A \cap B)}{A(A)}$$

Similarly, probability of A given B has occurred is

$$P(A/B) = \frac{P(A, B)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

$$\therefore \quad P(A \,\&\, B) = P(A) \times P(B/A)$$

$$P(A \,\&\, B) = P(B) \times P(A/B)$$

for three events A, B and C.

$$P(A, B \,\&\, C) = P(A) \times P(B/A) \times P(C/A, B)$$

Example 6.1 : *What is the probability that a leap year will contain 53 Mondays ?*

Solution : A leap year has 366 days.

This contains complete 52 weeks and two more days. These two days may take following combinations.

(i) Monday - Tuesday (ii) Tuesday - Wednesday (iii) Wednesday - Thursday (iv) Thursday - Friday (v) Friday - Saturday (vi) Saturday - Sunday and (vii) Sunday - Monday.

Out of these 7 combinations only 2 contain Monday.

$\therefore$ Required probability $= \frac{2}{7}$

Example 6.2 : *Prof. X and Madam Y appear for an interview for two posts. The probability of Prof. X's selection is $\frac{1}{7}$ and that of Madam Y's selection is $\frac{1}{5}$.*

Find the probability that only one of them is selected. What is probability that at least one of them is selected ?

Solution : Given that,

$$P(X) = \frac{1}{7} \qquad P(Y) = \frac{1}{5}$$

$$P(\overline{X}) = 1 - \frac{1}{7} = \frac{6}{7} \qquad P(\overline{Y}) = 1 - \frac{1}{5} = \frac{4}{5}$$

As only one of this is selected

$\Rightarrow$ If X is selected, Y is not selected (Case A)

$\Rightarrow$ If Y is selected, X is not selected (Case B)

$$P(A) = P(X) \times P(\overline{Y}) = \frac{1}{7} \times \frac{4}{5} = \frac{4}{35}$$

$$P(B) = P(\overline{X}) \times P(Y) = \frac{6}{7} \times \frac{1}{5} = \frac{6}{35}$$

$\therefore$ Required probability

$P(A \cup B) = P(A \text{ or } B) = P(A) + P(B)$

$$\frac{4}{35} + \frac{6}{35} = \frac{10}{35} = \frac{2}{7}$$

$P(\overline{X}) = \frac{6}{7}, \quad P(\overline{Y}) = \frac{4}{5}$

$\therefore$ Probability that none is selected $= P(\overline{X}) \times P(\overline{Y})$

$$= \frac{6}{7} \times \frac{4}{5}$$

$$= \frac{24}{35}$$

$\therefore$ P (at least one is selected) $= 1 - \frac{24}{35}$

$$= \frac{11}{35}$$

Example 6.3 : *A can hit the target 1 out of 4 times*

B can hit the target 2 out of 3 times

C can hit the target 3 out of 4 times

Find the probability that at least two hit the target.

Solution : Given that

$$P(A) = \frac{1}{4} \qquad P(\overline{A}) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(B) = \frac{2}{3} \qquad P(\overline{B}) = 1 - \frac{2}{3} = \frac{1}{3}$$

$$P(C) = \frac{3}{4} \qquad P(\overline{C}) = 1 - \frac{3}{4} = \frac{1}{4}$$

Required probability $= P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) + P(A \cap B \cap C)$

$$= P(A) \cdot P(B) \cdot P(\overline{C}) + P(A) \cdot P(\overline{B}) \cdot P(C) + P(\overline{A}) \cdot P(B) \cdot P(C) + P(A) \cdot P(B) \cdot P(C)$$

$$= \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{3}{4}$$

$= 29/48$

Example 6.4 : *There are two bags. One bag contains 4 white and 2 black balls. Second bag contains 5 white and 4 black balls. Two balls are transferred from first bag to second bag. Then 1 ball is drawn from second. Find the probability that it is white.*

Solution : Two balls transferred from first bag to second bag can be

I) Both white II) Both black III) One white and one black.

Now 2 balls out of 6 can be drawn in ${}^6C_2$ ways.

2 white out of 4 can be drawn in ${}^4C_2$ ways.

$\therefore$ If both balls transferred are white can be

$$\frac{{}^4C_2}{{}^6C_2} = \frac{\frac{4!}{2!\,2!}}{\frac{6!}{4!\,2!}} = \frac{\frac{4 \times 3}{2}}{\frac{6 \times 5}{2}} = \frac{2}{5}$$

In second bag there are 9 balls out of which 5 are white.

If two white balls are transferred into it,

then total no. $= 11$ white $= 7$

$\therefore$ Probability of 1 white $= \frac{7}{11}$

$\therefore$ Required probability $= \frac{2}{5} \times \frac{7}{11} = \frac{14}{55}$

$\therefore$ Probability that equipment will not fail

$$= P(\overline{A}) \times P(\overline{B}) \times P(\overline{C})$$

$$= \frac{5}{6} \times \frac{19}{20} \times \frac{9}{10}$$

$\therefore$ Required probability $= 1 - P(\overline{A}) \times P(\overline{B}) \times P(\overline{C})$

Example 6.5 : *Find the probability of drawing a Queen, King and Jack in that order from a pack of cards in three consecutive draws without replacement.*

Solution : P (Queen) $= \frac{4}{52}$

P (King when Queen has been drawn) $= \frac{4}{51}$

P (Jack when Q and K have been drawn) $= \frac{4}{50}$

$$P(A, B, C) = P(A) \times P(B/A) \times P(C/AB)$$

$$= \frac{4}{52} \times \frac{4}{51} \times \frac{4}{50} = \frac{64}{132600}$$

Example 6.6 : *If 9 persons take seats at random at a round table, find the probability that two specified persons are seated next to each other.*

Solution : Nine persons can be seated on a round table in (9 – 1) (i.e. 8 !) ways.

If two persons are to be seated together, then we may regard them to have been 'tied' up so that now the seating can take place in (8 – 1) ! (i.e. 7 !) ways. But these two persons can also seat themselves together in 2 ways.

$$p = \frac{\text{Favourable ways}}{\text{Total number of ways}} = \frac{7!\ 2}{8!} = \frac{1}{4}$$

Example 6.7 : *A problem of statistics is given to 5 students A, B, C, D and E. Their chances of solving the problem are 1/2, 1/3, 1/4, 1/5 and 1/6. What is the probability that problem is solved ?*

Solution : Probability that problem is solved.
Atleast one of the student solve it.

$$P(A) = \frac{1}{2} \Rightarrow P(\overline{A}) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(B) = \frac{1}{3} \Rightarrow P(\overline{B}) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(C) = \frac{1}{4} \Rightarrow P(\overline{C}) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(D) = \frac{1}{5} \Rightarrow P(\overline{D}) = 1 - \frac{1}{5} = \frac{4}{5}$$

$$P(E) = \frac{1}{6} \Rightarrow P(\overline{E}) = 1 - \frac{1}{6} = \frac{5}{6}$$

$\therefore$ Probability that problem is not solved.

$$= P(\overline{A}) \times P(\overline{B}) \times P(\overline{C}) \times P(\overline{D}) \times P(\overline{E})$$

$\therefore$ Probability that problem is solved.

$$= 1 - P(\overline{A}) \times P(\overline{B}) \times P(\overline{C}) \times P(\overline{D}) \times P(\overline{E})$$

$$= 1 - \frac{1}{6} = \frac{5}{6}$$

Example 6.8 : *An aircraft gun can take a minimum of four shots at an enemy plane moving away from it. The probability of hitting plane at 1^{st}, 2^{nd}, 3^{rd}, 4^{th} shot are 0.4, 0.3, 0.2, and 0.1 respectively. What is the probability that gun hits the plane ?*

Solution : Let probability of hitting at 1^{st}, 2^{nd}, 3^{rd} and 4^{th} shots be $P(S_1)$, $P(S_2)$, $P(S_3)$ and $P(S_4)$.

$$P(S_1) = 0.4 \Rightarrow P(\overline{S}_1) = 1 - 0.4 = 0.6$$

$$P(S_2) = 0.3 \Rightarrow P(\overline{S}_2) = 1 - 0.3 = 0.7$$

$$P(S_3) = 0.2 \Rightarrow P(\overline{S}_3) = 1 - 0.2 = 0.8$$

$$P(S_4) = 0.1 \Rightarrow P(\overline{S}_4) = 1 - 0.1 = 0.9$$

$\therefore$ Probability that no shot hits the plane.

$$P(\overline{S}_1, \overline{S}_2, \overline{S}_3, \overline{S}_4) = P(\overline{S}_1) \times P(\overline{S}_2) \times P(\overline{S}_3) \times P(\overline{S}_4)$$

$\therefore$ Probability that at least one shot hits the plane.

$$= 1 - P(\overline{S}_1, \overline{S}_2, \overline{S}_3, \overline{S}_4)$$

$$= 1 - P(\overline{S}_1) \times P(\overline{S}_2) \times P(\overline{S}_3) \times P(\overline{S}_4)$$

$$= 0{\cdot}6976$$

Example 6.9 : *A bag contains 3 red and 5 black balls and a 2^{nd} bag contains 6 red and 4 black balls. A ball is drawn from each bag. Find the probability that one is red and other is black.*

Solution : For 1^{st} bag

$$P(r_1) = \frac{3}{8} \quad \text{and } P(b_1) = \frac{5}{8}$$

For 2^{nd} bag.

$$P(r_2) = \frac{6}{10} \quad \text{and } P(b_2) = \frac{4}{10}$$

$\therefore$ A ball is drawn from each bag 1 red from 1^{st} and 1 black from 2^{nd} or 1 black from 1^{st} and 1 red from 2^{nd}.

From first case

$$P(r_1 \text{ and } b_2) = P(r_1) \times P(b_2) = \frac{3}{8} \times \frac{4}{10} = \frac{12}{80}$$

Second case $\quad P(b_1 \text{ and } r_2) = \frac{5}{8} \times \frac{6}{10} = \frac{30}{80}$

$$\therefore \quad \text{Required probability} = \frac{12}{80} + \frac{30}{80} = \frac{42}{80}$$

Mutually exclusive case.

Example 6.10 : *Four cards are drawn from a pack of cards find the probability that*
i) All are diamonds; ii) There is one card of each suit and; iii) There are two spades and two hearts.

Solution : 4 cards can be drawn from a pack of 52 cards in ${}^{52}C_4$ ways (i.e. exhaustive number of cases).

i) There are 13 diamonds in the pack and '4' can be drawn out of them in ${}^{13}C_4$.

$\therefore$ Favourable number of cases.

$$= {}^{13}C_4 = \frac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = 715$$

$$\therefore \quad P = \frac{{}^{13}C_4}{{}^{52}C_4} = \frac{715 \times 4 \times 3 \times 2 \times 1}{52 \times 51 \times 50 \times 49} = \frac{11}{4165}$$

$$= 0.002641$$

ii) There are 4 suits, each containing 13 cards.

$\therefore$ Favourable number of cases $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$

$\therefore$ Required probability

$$P = \frac{13 \times 13 \times 13 \times 13}{{}^{52}C_4}$$

$$= \frac{13 \times 13 \times 13 \times 13}{270725} = \frac{2197}{20825}$$

$$= 0.10549$$

iii) 2 spades out of 13 can be drawn in $^{13}C_2$ ways.
2 hearts out of 13 can be drawn in $^{13}C_2$ ways.

$\therefore$ Favourable number of cases = $^{13}C_2 \times {}^{13}C_2$

$= \dfrac{13\times12\times13\times12}{2\times1\times2\times1} = 6084$

$\therefore$ Required probability P = $\dfrac{6084}{270725}$ = 0.0.02247

Example 6.11 : *Find the probability of drawing i) a card of spades ii) a king, iii) a king or a queen or a knave from a pack of cards.*

Solution : i) There are 13 cards of spades in a pack of cards and drawing any shall be a success. Obviously the total number of ways is 52.

$$\therefore \quad P = \frac{13}{52} = 0.25$$

ii) There are 4 kings in a pack of cards and drawing any one of these shall be a success.

$$\therefore \quad P = \frac{4}{52} = \frac{1}{13}$$

iii) These are 4 kings, 4 queues, 4 jacks i.e. a total of 12 and drawing any one of these is a success.

$$\therefore \quad P = \frac{12}{52} = \frac{3}{13}$$

Example 6.12 : *From a pack of cards, four are drawn at random. What is the probability that there will be the four honours from the same suit ?*

Solution : Four cards can be selected out of 52 in $^{52}C_4$ ways and this gives us the total number of ways.
There are only four favourable ways, as there can be 4 honours from the same suit in spade, heart, diamond or clud.

Thus,
$$P = \frac{4\times4!}{^{52}C_4} = \frac{4\times4\times3\times2\times1}{52\times51\times50\times49}$$
$$= \frac{96}{6497400} = 0.0000147$$

Example 6.13 : *A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they will both be white.*

Solution : Total number of balls = 7 + 6 + 5 = 18.
Out of 18 balls, two can be drawn in $^{18}C_2$ ways.

$\therefore$ Exhaustive number of cases = $^{18}C_2 = \dfrac{18\times17}{2\times1} = 153$

out of 7 white balls 2 can be drawn in $^{7}C_2$ ways $= \dfrac{7\times6}{2\times1} = 21$

$\therefore$ Favourable number of cases = 21

$$\therefore \quad P = \frac{21}{153} = \frac{7}{51}$$

Example 6.14 : *A bag contains 6 red and green balls. If a draw of three is made, what is chance that the three balls drawn are red ?*

Solution : Total number of balls in bag are $6+4=10$.
Three can be drawn in $^{10}C_3$ ways.
The event shall be a success if all the balls are red i.e. the balls drawn are from the 6 red balls. Thus the number of favourable ways is $^{6}C_3$.

$$\therefore \quad P = \frac{^{6}C_3}{^{10}C_3} = \frac{6\times5\times4}{10\times9\times8} = \frac{1}{6}$$

Example 6.15 : *A bag contains 10 white and 15 black balls. Two balls are drawn in succession, what is the probability that*
i) One of the ball is black and other white
ii) Both of them are black.

Solution : Total number of balls in bag are $10+15=25$.
i) It will contain one white and one black in $^{10}C_1 \times {}^{10}C_1$ ways.

$$\therefore \quad P = \frac{^{10}C_1 \times {}^{15}C_1}{^{25}C_2} = \frac{10\times15\times2}{25\times24} = \frac{1}{2}$$

ii) If both balls are black then favourable ways $= {}^{15}C_2$

$$\therefore \quad P = \frac{^{15}C_2}{^{25}C_2} = \frac{15\times14}{25\times24} = \frac{7}{20}$$

Example 6.16 : *A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that both balls drawn are black.*

Solution : There are 5 + 3 = 8 balls.
Probability of drawing black P (A) = $\dfrac{3}{8}$

Probability of drawing 2^{nd} black given that 1^{st} ball drawn is black

$$P\,(B/A) = \frac{2}{7}$$

$\therefore$ Probability that both balls drawn are black is given by

$$P\,(A\ \&\ B) = P(A)\times P(B/A)$$
$$= \frac{3}{8}\times\frac{2}{7} = \frac{3}{28}$$

Example 6.17 : *An urn contains 10 black and 10 white balls. Find the probability of drawing two balls of the same colour.*

Solution : Probability of drawing two black balls $= \frac{^{10}C_2}{^{20}C_2}$

Probability of drawing two red balls $= \frac{^{10}C_2}{^{20}C_2}$

$\therefore$ Probability of two balls of the same colour

$$= \frac{^{10}C_2}{^{20}C_2} + \frac{^{10}C_2}{^{20}C_2} = 2\,\frac{10\times 9}{20\times 19} = \frac{9}{19}$$

Example 6.18 : *Among five books, there are two books of Mathematics. These books are arranged in a random order on a shelf. Find the probability that the two books of Mathematics are always together.*

Solution : Five books can be arranged in 5 ! ways. Two books can be kept together by 2! ways and for remaining three places 3 ! ways.

$$\therefore \text{ Required probability } = \frac{2!\cdot 3!}{5!}$$

$$= \frac{2\times 6}{120} = \frac{1}{10} = 0.1$$

Example 6.19 : *If 3 out of 20 tubes are defective and 4 of them are randomly chosen for inspection then what is the probability that only one of the defective tubes will be selected.*

Solution : Out of 3 defective 1 can be chosen in $^3C_1$ ways and out of 17 non-defective 3 can be chosen in $^{17}C_3$ ways and out of total 20 tubes 4 can be chosen in $^{20}C_4$ ways

$$\therefore \text{ Required probability } = \frac{^3C_1 \times {^{17}C_3}}{^{20}C_4}$$

$$= \frac{2040}{4845} = 0.42$$

Example 6.20 : *Three coins are tossed once. Find the probability of getting exactly 2 heads. Exactly two heads are possible in how many ways. HTH, HHT, THH.*

Solution : Probability (H) $= \frac{1}{2}$, Probability (T) $= \frac{1}{2}$

$$\therefore \text{Probability (HTH)} = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$

$$\text{Probability (HHT)} = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$

$$\text{Probability (THH)} = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$$

There are mutually exclusive events.

$$\therefore \text{ P (A or B or C)} = \frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{3}{8}$$

Example 6.21 : *A, B play a game of alternate tossing a coin one who gets head first wins the game. Find the probability that B wins the game if A has a start.*

Solution : Following are the cases where B wins the game

1) TH 2) TTTH 3) TTTTTH ....

We know $P(T) = \frac{1}{2}$ $\quad P(H) = \frac{1}{2}$

$$\therefore \text{ P ((1))} = P(T)\cdot P(H) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^2}$$

$$\text{P ((2))} = P(T)\,P(T)\,P(T)\,P(T)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^4}$$

$$\text{P ((3))} = P(T)\cdot P(T)\cdot P(T)\cdot P(T)\cdot P(T)\cdot P(H)$$

$$= \frac{1}{2^6}$$

$$\therefore \text{ Required probability } = \frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}\ \ldots$$

which is a geometric series

$$a + a_r + a_r^2 \ \ldots\ldots + \frac{a}{1-r} \quad \text{with } a = \frac{1}{2^2} \text{ and } r = \frac{1}{2^2}$$

$$\therefore \text{ Required probability } = \frac{1/4}{1-1/4} = \frac{1}{3}$$

Example 6.22 : *A six faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. What is the probability that the sum of the two numbers thrown is even ?*

Solution : Let probability of an odd number be P, so probability of an even number appearing is 2P. Now there are 6 outcomes (1, 2, 3, 4, 5, 6)

$$\therefore \quad P(1) = P(3) = P(5) = P$$

$$P(2) = P(4) = P(6) = 2P$$

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

$$\therefore\ P = 1 \qquad \therefore\ P = \frac{1}{9}$$

When a die is thrown twice, the sample space is described as,

(1, 1),	(1, 2),	(1, 3),	(1, 4),	(1, 5),	(1, 6)
(2, 1),	(2, 2),	(2, 3),	(2, 4),	(2, 5),	(2, 6)
(3, 1),	(3, 2),	(3, 3),	(3, 4),	(3, 5),	(3, 6)
(4, 1),	(4, 2),	(4, 3),	(4, 4),	(4, 5),	(4, 6)
(5, 1),	(5, 2),	(5, 3),	(5, 4),	(5, 5),	(5, 6)
(6, 1),	(6, 2),	(6, 3),	(6, 4),	(6, 5),	(6, 6)

We want to find the probability of event of sum of the two numbers even.

In first row, cases (1, 1), (1, 3), (1, 5) are favourable to the event with probabilities

$$P(1, 1) = \frac{1}{9} \times \frac{1}{9} = \frac{1}{81}, \quad P(1, 3) = \frac{1}{81},$$

$$P(1, 5) = \frac{1}{81}$$

In second row, cases (2, 2), (2, 4), (2, 6) are favourable to the event where

$$P(2, 2) = P(2, 4) + P(2, 6) = \frac{2}{9} \times \frac{2}{9} = \frac{4}{81}$$

Similarly 3^{rd} and 5^{th} row have cases (3, 1), (3, 3), (3, 5) and (5, 1), (5, 3), (5, 5) and 4^{th} and 6^{th} row have cases (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6).

Required probability P is given by,

$$\begin{aligned} P = {} & P(1, 1) + P(1, 3) + P(1, 5) \\ & + P(2, 2) + P(2, 4) + P(2, 6) \\ & + P(3, 1) + P(3, 3) + P(3, 5) \\ & + P(4, 2) + P(4, 4) + P(4, 6) \\ & + P(5, 1) + P(5, 3) + P(5, 5) \\ & + P(6, 2) + P(6, 4) + P(6, 6) \\ = {} & \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{4}{81} + \frac{4}{81} + \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{4}{81} + \frac{4}{81} + \frac{4}{81} \\ & + \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{4}{81} + \frac{4}{81} + \frac{4}{81} + \frac{4}{81} = \frac{45}{81} = \frac{5}{9} \end{aligned}$$

Example 6.23 : *A is one of the eight horses entered for a race and is to be ridden by one of the two jockeys B and C. It is 2 to 1 that B rides A, in which case all the horses are equally likely to win, with rider C, A's chance is doubled.*

1) Find the probability that A wins.

2) What are odds against A's winning ?

Solution : 1) A can win in the following two mutually exclusive cases.

i) B rides A and A wins.

ii) C rides A and A wins.

$$P(i) = \frac{2}{3} \times \frac{1}{8} = \frac{1}{12}$$

$$P(ii) = \frac{1}{3} \times \frac{2}{8} = \frac{1}{12}$$

Probability of A winning = P (i) + P (ii)

$$= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

2) Probability of A's losing $= 1 - \frac{1}{6} = \frac{5}{6}$

Hence odds against A's winning are $\frac{5}{6} ; \frac{1}{6}$, i.e. 5 : 1.

Example 6.24 : *In a single throw of two dice, determine the probability of obtaining a total of 7 or 9.*

Solution : Total of 7 can be obtained as (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)

Total of 9 (4, 5), (3, 6), (6, 3), (5, 4)

When two dices are thrown

Exhaustive no. $= 6 \times 6 = 36$

$$P(7) = \frac{6}{36}, \quad P(9) = \frac{4}{36}$$

$$\therefore \quad P(7 \text{ or } 9) = \frac{6}{36} + \frac{4}{36} = \frac{10}{36}$$

Example 6.25 : *A student takes his examination in four subjects $\alpha, \beta, \gamma, \delta$. He estimates his chances of passing α as $\frac{4}{5}$, in β as $\frac{3}{4}$ in γ as $\frac{5}{6}$ in δ as $\frac{2}{3}$. To qualify, he must pass in α and at least two other subjects. What is the probability that he qualifies.*

Solution : Here

$$P(\alpha) = \frac{4}{5}, \quad P(\beta) = \frac{3}{4},$$

$$P(\gamma) = \frac{5}{6}, \text{ and } P(\delta) = \frac{2}{3}$$

$$P(\bar{\alpha}) = 1 - \frac{4}{5} = \frac{1}{5}, \quad P(\bar{\beta}) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P(\gamma) = 1 - \frac{5}{6} = \frac{1}{6}, \quad P(\bar{\delta}) = 1 - \frac{2}{3} = \frac{1}{3}$$

There are four possibilities of passing at least two subjects,

I) Passing in β, γ and failing in δ

$$= P(\beta) \times P(\gamma) \times P(\bar{\delta})$$

$$= \frac{3}{4} \times \frac{5}{6} \times \frac{1}{3} = \frac{5}{24}$$

II) Passing in γ,δ and failing in β

$$= P(\gamma)\times P(\delta)\times P(\bar{\beta})$$
$$= \frac{5}{6}\times\frac{2}{3}\times\frac{1}{4} = \frac{5}{36}$$

III) Passing in δ,β and failing in γ

$$= P(\delta)\times P(\beta)\times P(\bar{\gamma})$$
$$= \frac{2}{3}\times\frac{3}{4}\times\frac{1}{6} = \frac{1}{12}$$

IV)Passing in β,γ,δ $= P(\beta)\times P(\gamma)\times P(\delta)$

$$= \frac{3}{4}\times\frac{5}{6}\times\frac{2}{3} = \frac{5}{12}$$

$\therefore$ Probability of passing in at least two other subjects

$$= \frac{5}{24}+\frac{5}{36}+\frac{1}{12}+\frac{5}{12} = \frac{61}{72}$$

$\therefore$ Probability of passing α and at least two other subjects

$$= \frac{4}{5}\times\frac{61}{72} = \frac{61}{90}$$

Example 6.26 : *Find the probability of throwing '6' in the first only of two successive throws with an ordinary dice.*

Solution : We required a throw of '6' in the first throw and any number other than '6' in the second throw. Thus this is a problem of multiplication of the probabilities.

The probability of the first event (of throwing '6' in a single throw) $= \frac{1}{6}$

The probability of the second event (of throwing a number other than 6, in the second throw) $= \frac{5}{6}$.

Hence the probability of happening of both the events.

$$= \frac{1}{6}\times\frac{5}{6} = \frac{5}{36}$$

Example 6.27 : *A box contains 3 black, 4 white and 5 red balls, what is the probability of drawing 2 black balls in succession if the first ball is replaced after drawing ?*

Solution : Obviously one ball is drawn, in each draw.

The probability of drawing a black ball in one draw.

$$= \frac{3}{3+4+5} = \frac{1}{4}$$

When this ball is replaced, and another drawn, the probability of black is again 1/4.

The compound probability, that the ball is black in both the draws is, therefore.

$$= \frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$$

Example 6.28 : *A can hit the target 1 out of 4 times B can hit the target 2 out of 3 times, C can hit the target 3 out of 4 times. Find the probability of at least two hit the target.*

Solution : Let A be the event of A hitting the target, B be the event of B hitting th target and C be the event of C hitting the target.

$$\therefore \quad P(A) = \frac{1}{4}, \quad P(B) = \frac{2}{3} \text{ and } P(C) = \frac{3}{4}$$

$$\therefore \quad P(\bar{A}) = \frac{3}{4}, \quad P(\bar{B}) = \frac{1}{3} \text{ and } P(\bar{C}) = \frac{1}{4}$$

$\therefore$ The probability of at least two hitting the target

= P (A & B hit but C not)

+ P (A & C hit but B not)

+ P (B & C hit but A not)

+ P (A, B, C hit)

$$= P(A\cap B\cap\bar{C}) + P(A\cap C\cap\bar{B}) + P(\bar{A}\cap B\cap C) + P(A\cap B\cap C)$$
$$= \frac{1}{4}\times\frac{2}{3}\times\frac{1}{4}+\frac{1}{4}\times\frac{3}{4}\times\frac{1}{3}+\frac{3}{4}\times\frac{2}{3}\times\frac{3}{4}+\frac{1}{4}\times\frac{2}{3}\times\frac{3}{4} = \frac{29}{48}$$

Example 6.29 : *An envelope contains 6 tickets with numbers 1, 2, 3, 5, 6, 7. Another envelope contains 4 tickets with numbers 1, 3, 5, 7. An envelope is chosen at random and ticket is drawn from it. Find the probability that the ticket bears the numbers i) 2 or 5 ii) 2.*

Solution : i) Required event can happen in the following two mutually exclusive ways.

1) First envelope is chosen and then ticket is drawn.

2) Second envelope is chosen and then thicket is drawn.

The probability of choosing an envelope is $\frac{1}{2}$.

$\therefore$ The required probability = P(1) + P(2)

$$= \frac{1}{2}\times\frac{2}{6}+\frac{1}{2}\times\frac{1}{4} = \frac{7}{24}$$

ii) Similarly, the required probability of ticket bearing number 2 is = P(1) + P (2)

$$= \frac{1}{2}\times\frac{1}{6}+\frac{1}{2}\times 0 = \frac{1}{12}$$

Example 6.30 : *A riddle is given to three students whose probabilities of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that the riddle is solved.*

Solution : Let P(A), P(B) and P(C) be probability that A, B and C can solve the problem respectively.

$$\therefore \quad P(A) = \frac{1}{2}, \quad P(\bar{A}) = \frac{1}{2}$$

$$P(A) = \frac{1}{3}, \quad P(\bar{B}) = 1-\frac{1}{3} = \frac{2}{3}$$

$$P(C) = \frac{1}{4}, \quad P(\bar{C}) = 1-\frac{1}{4} = \frac{3}{4}$$

The probability that the riddle (problem) is solved means probability that at least one student solved the riddle.

$\therefore$ Required Probability

$$= 1 - \text{Probability that riddle is not solved}$$

$$= 1 - P(\bar{A})P(\bar{B})P(\bar{C})$$

$$= 1-\frac{1}{2}\left(\frac{2}{3}\right)\left(\frac{3}{4}\right) = 1-\frac{1}{4} = \frac{3}{4}$$

Exercise 6.1

1. *A box contains 6 red balls 4 white balls 5 blue balls. Three balls are drawn successively from the box. Find the probability that they are drawn in the order red, white and blue if each ball is not replaced.* **[Ans. : $\frac{6}{15}, \frac{4}{14}, \frac{5}{13} = \frac{4}{91}$]**

2. *An urn contains 6 white and 8 red balls. Second urn contains 9 white and 10 red balls. One ball is drawn at random from the first urn and put into the second urn without noticing its colour. A ball is then drawn at random from the second urn. What is the probability that it is red.* **[Ans. : $\frac{3}{7}\cdot\frac{1}{2}+\frac{4}{7}\cdot\frac{11}{20} = \frac{259}{490}$]**

3. *Supposing that out of 12 test matches played between India and Shrilanka during last 3 years, 6 are won by India, 4 are won by Shrilanka and 2 are drawn. If they agree to play a test series consisting of three matches. Find the probability that India wins the test series on the basis of past performance.* **[Ans. : $\frac{19}{72}$]**

4. *A throw is made with two dice. Find the probability that (i) The sum is 7 or less, (ii) The sum is a perfect square.* **[Ans. : (i) $\frac{7}{12}$, (ii) $\frac{7}{36}$]**

5. *Three coins are tossed simultaneously. Find the probability of getting at least 2 Heads.* **[Ans. : $\frac{1}{2}$]**

6. *A bag contains 6 white and a 10 black balls. What is the probability that just 3 will be white out of 8 drawn ?* **[Ans. : $\frac{56}{143}$]**

7. *Assuming that the ratio of male children is $\frac{1}{2}$, find the probability that in a family of 6 children :*

   i) All children will be of same sex. **[Ans. : $\frac{1}{32}$]**

   ii) The four eldest children will be boys. **[Ans. : $\frac{1}{64}$]**

   iii) Exactly three children will be boys. **[Ans. : $\frac{1}{1280}$]**

8. *What is the chance of throwing '3' in a single die ?* **[Ans. : p = $\frac{1}{6}$]**

9. *What is the probability of throwing a number greater than 3 in a single throw of one dice ?* **[Ans. : $\frac{1}{2}$]**

10. *Find the probability of throwing '9' with two dice.* **[Ans. : $\frac{1}{9}$]**

11. *A problem in mathematics is given to three students A, B and C whose chances solving it are $\frac{1}{2}, \frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved ?* **[Ans. : $\frac{29}{32}$]**

12. *From a deck of 52 cards, two cards are drawn at random. Find the probability that (i) Both are hearts, (ii) Both the cards are of different suits.* **[Ans. : (i) $\frac{39}{613}$, (ii) $\frac{13}{17}$]**

13. *There are six married couples in a room. If two persons are chosen at random, find the probability that (i) They are of different sex, (ii) They are married to each other.* **[Ans. : (i) $\frac{6}{11}$; (ii) $\frac{1}{11}$]**

14. *A committee consists of 9 students two of which are from first year, three from second year and four from third year. Three students are to be removed at random. What is the chance that :*

   i) The three students belong to different classes. **[Ans. : $\frac{2}{7}$]**

ii) Two belong to the same class and third to the different class. **[Ans. : $\frac{55}{84}$]**

iii) The three belong to the same class. **[Ans. : $\frac{5}{84}$]**

15. *If 15 persons take seats at random at a round table, find the probability that two specified, persons are seated next to each other.* **[Ans. : $\frac{1}{7}$]**

16. *Urn I contains 6 white and 4 black balls and urn II contains 4 white and 5 black balls. From urn I, two balls are transferred to urn II without noticing the colour. Sample of size 2 is then drawn without replacement from urn II. What is the probability that the sample contains exactly 1 white ball ?* **[Ans. : $\frac{4}{5}$]**

17. *Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Show that the probability that exactly two of them will be children is $\frac{10}{21}$.*

18. *A bag contains 5 green and 7 red balls. Two balls are drawn. What is the probability that one is green and the other red ?* **[Ans. : $\frac{35}{66}$]**

19. *One shot is fired from each of the three guns. E_1, E_2, E_3 denote the events that the target is hit by the first, second and third guns respectively. If $P(E_1) = 0.5$, $P(E_2) = 0.6$, $P(E_3) = 0.7$ and E_1, E_2, E_3 are independent events, then find the probability that at least two hits are registered.* **[Ans. : 0.65]**

20. *A problem on computer Mathematics is given to the three students A, B and C whose chances of solving it are $\frac{1}{2}, \frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved ?* **[Ans. : $\frac{29}{32}$]**

21. *What is the probability of getting a total of 5 or 8 in a single throw with two dice ?* **[Ans. : $\frac{1}{4}$]**

22. *Find the probability of drawing a king, a queen and a knave in that order from a pack of cards in three consecutive draws, the cards not being replaced.* **[Ans. : $\frac{8}{13 \times 51 \times 25}$]**

23. *Find the probability that six people selected at random will have six different birth dates.* **[Ans. : 61]**

24. *A five figure number is formed by the digits 0, 1, 3, 4, (without repetition). Find the probability that the number formed is divisible by 4.* **[Ans. : $\frac{3}{10}$]**

25. *A, B, C throw the coin alternatively in that order. One who gets tail first wins the game. Find the probability of B winning the game if C has a start.* **[Ans. : $\frac{1}{7}$]**

26. *A box contains 5 red and 4 white marbles. Two marbles are drawn successively from the box without replacement and it is noted that the second one is white. What is the probability that the first is also white ?* **[Ans. : $\frac{1}{6}$]**

27. *Box A contains 3 red and 2 blue marbles. The box B contains 2 red and 8 blue marbles. A fair coin is tossed. If the coin shows head, a marble is chosen from box A, if it shows Tail, a marble is chosen from box B. Find the probability that a red marble is chosen.* **[Ans. : $\frac{2}{5}$]**

6.4 Baye's Theorem

If $E_1, E_2, \ldots E_n$ are mutually exclusive and exhaustive events with $P(E_i) \neq 0$ $(i = 1, 2, \ldots n)$ of a random experiment then for any arbitrary event A of the sample space with $P(A) > 0$.

We have,

$$P(E_i / A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i) P(A/E_i)}$$

Proof : Let S be the sample space of the random experiment.

The events $E_1, E_2, \ldots E_n$ being exhaustive

$$S = E_1 \cup E_2 \cup \ldots \cup E_n$$

$$\therefore \quad A = A \cap S = A \cap (E_1 \cup E_2 \cup \ldots \cup E_n)$$

$$= (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)$$

$$\Rightarrow P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n)$$

$$= P(E_1) P(A/E_1) + P(E_2) P(A/E_2) + \ldots + P(E_n) P(A/E_n)$$

$$= \sum_{i=1}^{n} P(E_i)\, P(A/E_i) \qquad \ldots (i)$$

Now, $P(A \cap E_i) = P(A) P(E_i / A)$

$$\Rightarrow \qquad P(E_i / A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)} \qquad \text{...(Using equation (i))}$$

Note : The significance of Baye's theorem may be understood in the following manner :

$P(E_i)$ Is the probability of occurrence of E_i. The experiment is performed and we are told that the event A has occurred. With this information, the probability $P(E_i)$ is changed to $P(E_i / A)$. Baye's theorem enables us to evaluate $P(E_i / A)$ if all the $P(E_i)$ and the conditional probabilities $P(A/E_i)$ are known.

Example 6.31 : *In a bolt factory, machine A, B and C manufacture respectively 25 %, 35 % and 40 % of the total of their output 5, 4 and 2 percent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactures by machine B ?*

Solution : Let, E_1, E_2 and E_3 denote the events that a bolt selected at random is manufacture by the machines A, B, and C respectively and let H denote the event of its being defective.

Then, $P(E_1) = 0.25$, $P(E_2) = 0.35$, $P(E_3) = 0.40$

The probability of drawing a defective bolt manufactured by machine A is $P(H/E_1) = 0.05$

Similarly, $P(H/E_2) = 0.04$

And, $P(H/E_3) = 0.02$

By Baye's theorem, we have

$$P(E_2 / H) = \frac{P(E_2)P(H/E_2)}{P(E_1)\, P(H/E_1) + P(E_2)P(H/E_2) + P(E_3)P(H/E_3)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = \frac{0.0140}{0.0345} = \mathbf{0.41}$$

Example 6.32 : *The contents of urns I, II and III are as follows :*
1 white, 2 black and 3 red balls, 2 white, 1 black and 1 red balls and
4 white, 5 black and 3 red balls.
One urn is chosen at random and two balls drawn.They happen to be white and red. What is the probability that they come from urns I,II or III ?

Solution : Let,

E_1 : urn I is chosen,

E_2 : urn II is chosen

E_3 : urn III is chosen

and A : the two balls are white and red.

We have to find $P(E_1 / A)$, $P(E_2 / A)$, $P(E_3 / A)$

Now, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$$P(A/E_1) = P(\text{a white and a red ball are drawn from urn I}) = \frac{{}^{1}C_1 \times {}^{3}C_1}{{}^{6}C_2} = \frac{1}{5}$$

$$P(A/E_2) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{1}{3}$$

$$P(A/E_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{2}{11}$$

By Baye's theorem, we have,

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)} = \frac{\frac{1}{3}\times\frac{1}{5}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{11}} = \frac{33}{118}$$

Similarly, $P(E_2/A) = \frac{55}{118}$

$$P(E_3/A) = \frac{15}{59}$$

Example 6.33 : *A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.*

Solution : Let, E_1 : The ball is drawn from bag X, E_2 : The ball is drawn from bag Y

A : The ball is red

We have to find $P(E_2/A)$.

By Baye's theorem,

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)} \quad \ldots(1)$$

Since the two bags are equally likely to be selected

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Also, $P(A/E_1) = P(\text{a red ball is drawn from bag X}) = \frac{3}{5}$

$P(A/E_2) = P(\text{a red ball is drawn from bag Y}) = \frac{5}{9}$

From equation (1), we have

$$P(E_2/A) = \frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}} = \frac{25}{52}$$

6.5 Expected Values and Variance

Let a random variable X assume values $x_1, x_2, \ldots x_n$ with probabilities $P_1, P_2, P_3 \ldots P_n$ respectively,

Where, $P(X = x_i) = P_i \geq 0$ for each x_i

And, $P_1 + P_2 + P_3 \ldots + P_n = \sum_{i=1}^{n} P_i = 1$

$$X : x_1, x_2, x_3, \ldots x_n, \quad P(X) : P_1, P_2, P_3, \ldots P_n$$

is called the **discrete probability distribution** for X and spells out how a total probability of 1 is distributed over several values of the random variable.

We denote the mean by μ and define $\mu = \frac{\sum P_i x_i}{\sum P_i} = \sum P_i x_i$

Other names for the mean are average or expected value E(X) we denote the variance by σ^2 and define $\sigma^2 = \sum P_i (x_i - \mu)^2$.

If μ is not a whole number, then $\sigma^2 = \sum P_i x_i^2 - \mu^2$

Standard deviation $\sigma = +\sqrt{\text{variance}}$

Example 6.34 : *A random variable X has the following probability function :*

Values of x, x	: 0	1	2	3	4	5	6	7
P(x)	: 0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

i) Find K, ii) Evaluate $P(X<6)$, $P(X\geq 6)$, $P(3<x\leq 6)$, *iii) Find the minimum value of x so that* $P(X\leq x) > \frac{1}{2}$.

Solution : i) Since $\sum_{x=0}^{7} P(x) = 1$

We have, $0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1 \Rightarrow 10k^2+9K-1 \Rightarrow (10K-1)(K+1) = 0$

$$\therefore \quad K = \frac{1}{10}$$

ii)

$$P(X<6) = P(X=0)+P(X=1)+...+P(X=5) = 0+K+2K+3K+K^2 = 8K+K^2 = \frac{8}{10}+\frac{1}{100} = \frac{81}{100}$$

$$P(X\geq 6) = P(X=6)+P(X=7) = 2K^2+7K^2+K = \frac{9}{100}+\frac{1}{10} = \frac{19}{100}$$

$$P(3<x\leq 6) = P(X=4)+P(X=5)+P(X=6) = 3K+K^2+2K^2 = \frac{3}{10}+\frac{3}{100} = \frac{33}{100}$$

iii)

$$P(X\leq 1) = K = \frac{1}{10} < \frac{1}{2}$$

$$P(X\leq 2) = K+2K = \frac{3}{10} < \frac{1}{2}$$

$$P(X\leq 3) = K+2K+2K = \frac{5}{10} = \frac{1}{2}$$

$$P(X\leq 4) = K+2K+2K+3K = \frac{8}{10} > \frac{1}{2}$$

$\therefore$ The maximum value of x so that $P(X\leq x) > \frac{1}{2}$ is 4.

Example 6.35 : *A die is tossed thrice. A success is getting 1 or 6' on a toss. Find the mean and the variance of the number of successes.*

Solution : Let X denote the number of success. Clearly X can take the value 0, 1, 2 or 3.

Probability of success $= \frac{2}{6} = \frac{1}{3}$, Probability of failure $= 1-\frac{1}{3} = \frac{2}{3}$

$P(X=0) = P$ (no success) $= P$(all 3 failures) $= \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3} = \frac{8}{27}$

$P(X=1) = P$(one success and 2 failures) $= 3_{C_1} \times\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3} = \frac{12}{27}$

$P(X=2) = P(\text{two success and 1 failures}) = 3_{C_2} \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{6}{27}$

$P(X=3) = P(\text{all 3 successes}) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

$\therefore$ The probability distribution of the random variable X is :

X :	0	1	2	3
P(X)	$8/27$	$12/27$	$6/27$	$1/27$

To find the mean and variance

x_i	P_i	$P_i\, x_i$	$P_i\, x_i^2$
0	$8/27$	0	0
1	$12/27$	$12/27$	$12/27$
2	$6/27$	$12/27$	$24/27$
3	$1/27$	$3/27$	$9/27$
		1	$5/3$

Mean $\mu = \sum P_i x_i = 1$

Variance $\sigma^2 = \sum P_i x_i^2 - \mu^2 = \frac{5}{3} - 1 = \frac{2}{3}$

6.6 Probability Distribution

We know that frequency distribution gives tabulated values of x and corresponding frequencies. Probability distribution gives tabulated values of x and corresponding probabilities.

Continuous and discrete random variables :

Continuous random variable is a variable which can take all continuous values within the interval. A discrete random variable is a variable which takes only isolated values.

6.7 Discrete Probability Distribution

If a random variable X assumes values $x_1, x_2, x_3, \ldots x_n$ with corresponding probabilities $P_1, P_2, P_3, \ldots.. P_n$ respectively, where

$$\sum_{i=1}^{n} P_i = 1, \quad \text{then}$$

The distribution :

X	x_1	x_2	....	x_n
P (X)	p_1	p_2	....	p_n

is called discrete probability distribution for X and it spells out how a total probability of 1 is distributed over several values of the random variable.

6.8 Mean and Variance of Random Variables

For the probability distribution

X	x_1	x_2	x_3	....	x_n
P (X)	p_1	p_2	p_3	....	p_n

Mean is denoted by μ and is given by

$$\mu = \frac{\sum p_i x_i}{\sum p_i}$$

As $\sum p_i = 1$, mean $\mu = \sum p_i x_i$

Mean is also called Mathematical expectation denoted by E (X).

Variance σ^2 is defined as

$$\sigma^2 = \frac{\sum p_i (x_i - \mu)^2}{\sum p_i}$$

As $\sum p_i = 1 \quad \therefore \sigma^2 = \sum p_i (x_i - \mu)^2$

Simplifying we get

$$\sigma^2 = \sum p_i x_i^2 - \mu^2$$

i.e. $$\sigma^2 = \sum p_i x_i^2 - \left(\sum p_i x_i\right)^2$$

Standard deviation $= \sqrt{\text{Variance}}$

6.9 Illustrations

Example 6.36 : *Find the probability distribution for number of sixes in three tosses of a dice.*

Solution : Let X denote the random variable which is the number of sixes obtained in 3 tosses.

$\therefore$ X can take values 0, 1, 2, 3.

Probability of getting a six $= \frac{1}{6}$ i.e. $P = \frac{1}{6}$

$\therefore$ Probability of not getting a six $= 1 - \frac{1}{6} = \frac{5}{6}$

i.e. $q = \frac{5}{6}$

$$\therefore P(X = 0) = (q \times q \times q) = \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) = \frac{125}{216}$$

$$P(X = 1) = (p \times q \times q) + (q \times p \times q) + (q \times q \times p)$$
$$= \left(\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$$
$$= 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) = \frac{75}{216}$$

$$P(x = 2) = (p \times p \times q) + (p \times q \times p) + (q \times p \times p)$$
$$= 3\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{15}{216}$$

$$P(X = 3) = (p \times p \times p) = \left(\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{216}$$

$\therefore$ Probability distribution

X	0	1	2	3
P (X)	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Example 6.37 : *A dice is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and variance of the number of success.*

Solution : Let

X be the random variable = getting 1 or 6

$\therefore$ X can take values 0, 1, 2, 3

Probability of getting (1 or 6) $= \frac{2}{6} = \frac{1}{3} \Rightarrow p = \frac{1}{3}$

$\therefore$ Probability of not getting (1 or 6) $= 1 - \frac{1}{3} = \frac{2}{3}$

$$\therefore \quad P(X = 0) = q \times q \times q = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$$

$$P(X = 1) = 3(p \times q \times q) = 3 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{12}{27}$$

$$P(X = 2) = 3(p \times p \times q) = 3 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{6}{27}$$

$$P(X = 3) = p \times p \times p = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$\therefore$ Probability distribution

X	0	1	2	3
P (X)	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

$$\therefore \text{ Mean } = \sum p_i x_i = 0 \times \frac{8}{27} + 1 \times \frac{12}{27} + 2 \times \frac{6}{27} + 3 \times \frac{1}{27}$$

$$\therefore \quad \mu = \frac{27}{27} = 1$$

Variance $\sigma^2 = \sum p_i (x_i - \mu)^2$

x_i	p_i	$x_i - \mu$	$p_i(x_i - \mu)^2$
0	8/27	0 – 1 = – 1	$\frac{8}{27}$
1	12/27	1 – 1 = 0	0
2	6/27	2 – 1 = 1	$\frac{6}{27}$
3	1/27	3 – 1 = 2	$\frac{4}{27}$
			$\sum p_i (x_i - \mu)^2 = \frac{18}{27}$

$$\therefore \quad \sigma^2 = \frac{18}{27}$$

Example 6.38 : *Compute variance of the probability distribution of the number of doubles in four throws of a pair of dice.*

Solution : Let X be random variable representing number of doublets.

Clearly X can take values 0, 1, 2, 3, 4.

Pair of dice is thrown.

Total number of outcomes = $6 \times 6 = 36$

doubles are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)

$\therefore$ Probability of doublets $= \frac{6}{36} = \frac{1}{6}$ i.e. $P = \frac{1}{6}$

Probability of not a doublet $= 1 - \frac{1}{6} = \frac{5}{6}$ i.e. $q = \frac{5}{6}$

$$P(X = 0) = q \times q \times q \times q = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$$

$$P(X = 1) = 4(p \times q \times q \times q) = 4 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^3 = \frac{500}{1296}$$

$$P(X = 2) = 6(p \times p \times q \times q) = 6 \times \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 = \frac{150}{1296}$$

$$P(X = 3) = 4(p \times p \times p \times q) = 4\left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right) = \frac{20}{1296}$$

$$P(X = 4) = \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$$

X	P (X)	$p_i x_i$	x_i^2	$p_i x_i^2$
0	$\frac{625}{1296}$	0	0	0
1	$\frac{500}{1296}$	$\frac{500}{1296}$	1	$\frac{500}{1296}$
2	$\frac{150}{1296}$	$\frac{300}{1296}$	4	$\frac{600}{1296}$
3	$\frac{20}{1296}$	$\frac{60}{1296}$	9	$\frac{180}{1296}$
4	$\frac{1}{1296}$	$\frac{4}{1296}$	16	$\frac{16}{1296}$
	$\sum p_i = 1$	$\sum p_i x_i = \frac{864}{1296}$		$\sum p_i x_i^2 = 1$

$$\therefore \quad \mu = \sum p_i x_i = \frac{864}{1296}$$

$$\therefore \sigma^2 = \sum p_i x_i^2 - \mu^2$$

$$= 1 - \left(\frac{864}{1296}\right)^2 = 0.55$$

6.10 Theoretical Distributions

Frequency distributions can be classified as

1) Observed frequency distributions.

2) Theoretical or expected frequency distributions.

Observed frequency distributions are based on actual observations and experimentation.

Theoretical distributions are derived mathematically with some assumptions. There are three types of theoretical distributions.

1) Binomial distribution

2) Poisson's distribution

3) Normal distribution

6.11 Binomial Distribution

Consider the experiment in which we perform a series of n independent trials. Each trial has only two outcomes or two mutually exclusive possibilities, a success or a failure.

Let p = Probability of getting a success

q = Probability of getting a failure

Since there are only two outcomes p + q = 1

$\therefore$ P (r successes in n trials)

$$= P\left[(S \cdot S \ldots S)_{r \text{ times}} (F \cdot F \ldots F)_{n-r \text{ times}}\right]$$

$$= [P(S) \cdot P(S) \ldots P(S)]_{r \text{ times}} [P(F) \cdot P(F) \ldots P(F)]_{(n-r) \text{ times}}$$

$$= [P \cdot P \cdot P \ldots P]_{r \text{ times}} [q \cdot q \ldots q]_{(n-r) \text{ times}}$$

$$= p^r \cdot q^{n-r}$$

As r successes and $(n-r)$ failures can occur in ${}^nC_r$ mutually exclusive cases

$\therefore$ P [r successive in n trials] $= {}^nC_r \cdot p^r \cdot q^{n-r}$

Substituting r = 0, 1, 2, 3 ..... n we get the following table.

r	0	1	2	3	......	n
p (r)	${}^nC_0 p^0 q^n$	${}^nC_1 p^1 q^{n-1}$	${}^nC_2 p^2 q^{n-2}$	${}^nC_3 p^3 q^{n-3}$	......	${}^nC_n p^n q^{n-n}$

$${}^nC_0 = 1, \quad {}^nC_n = 1$$

Consider now the Binomial expansion of

$$(q+p)^n = q^n + {}^nC_1 q^{n-1} p + {}^nC_2 q^{n-2} p^2 + \ldots + p^n$$

Terms of R.H.S. of this expansion give probability of r = 0, 1, 2, ..., n success. This is the reason for above probability distribution to be called Binomial probability distribution. It is denoted by B (n, p, r).

Thus $\quad B(n, p, r) = {}^nC_r \, p^r \, q^{n-r}$

6.12 Mean and Variance of the Binomial Distribution

1) Mean for binomial distribution

$P(r) = {}^nC_r\, p^r\, q^{n-r}$

$$\text{Mean } \mu = \sum_{r=0}^{n} r\cdot P(r) = \sum_{r=0}^{n} r\cdot {}^nC_r\, p^r\, q^{n-r}$$

$$= 0+1\cdot {}^nC_1 p\cdot q^{n-1} + 2\cdot {}^nC_2 p^2\cdot q^{n-2} + 3\cdot {}^nC_3 p^3\cdot q^{n-3} + \ldots\ldots + n\cdot {}^nC_n p^n q^0$$

$$= n\cdot p\cdot q^{n-1} + 2\cdot\frac{n(n-1)}{2}\cdot p^2 q^{n-2} + 3\cdot\frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}\ldots p^3\cdot q^{n-3} + \ldots\ldots n\cdot p^n$$

$$= np\left[q^{n-1} + (n-1)q^{n-2}\cdot p + \frac{(n-1)(n-2)}{2\cdot 1} q^{n-3}\cdot p^2 + \ldots + (n-1)_{C_{(n-1)}}\, p^{n-1}\right]$$

$$= np\,(q+p)^{n-1}$$

$\therefore\ \mu = np$ as $p+q=1$

$\therefore$ Mean of binomial distribution is np.

2) Variance

$$\sigma^2 = \sum_{r=0}^{n} r^2\, P(r) - \mu^2$$

$$= \sum_{r=0}^{n}\left(r + r^2 - r\right)\cdot p(r) - \mu^2$$

$$= \sum_{r=0}^{n}\{r + r(r-1)\}\, p(r) - \mu^2$$

$$= \sum_{r=0}^{n} r\cdot p(r) + \sum_{r=0}^{n} r(r-1)\, p(r) - \mu^2$$

$$= \mu + \sum_{r=2}^{n} r(r-1)\, {}^nC_r\, p^r\, q^{n-r} - \mu^2$$

$$= m + \left[2\cdot 1\cdot {}^nC_2 p^2\cdot q^{n-2} + 3\cdot 2\, {}^nC_3 p^3\cdot q^{n-3} + \ldots\ldots + n(n-1)\cdot {}^nC_n\, p^n\, q^0\right] - \mu^2$$

$$= \mu + \left[2.1\frac{n(n-1)}{2\cdot 1} p^2\cdot q^{n-2} + 3\cdot 2\cdot\frac{n(n-1)(n-2)}{3\cdot 2\cdot 1} p^3\cdot q^{n-3} + \ldots + n(n-1)\, p^n\right]$$

$$= \mu + n(n-1)p^2\left[q^{n-2} + (n-2)q^{n-3}\cdot p + \ldots + p^{n-2}\right] - \mu^2$$

$$= \mu + n(n-1)p^2\left[{}^{n-2}C_0\cdot q^{n-2} + {}^{n-2}C_1\, q^{n-3}\cdot p + \ldots + (n-2)_{C_{(n-2)}}\cdot p^{n-2}\right] - \mu^2$$

$$= \mu + n(n-1)\,p^2\,[q+p]^{n-2} - \mu^2$$

as $p+q = 1$ and $\mu = np$

$$\text{Variance } \sigma^2 = np + n(n-1)\,p^2 - n^2 p^2$$

$$= np[1 + (n-1)p - np]$$

$$= np\,[1 + np - p - np] = np(1-p)$$

$\therefore$ Variance, $\sigma^2 = npq$

$\therefore$ Standard deviation of binomial distribution is

$$\sigma = +\sqrt{npq}$$

6.13 Recurrence Formula for Binomial Distribution

We know

$$B(n, p, r) = {}^nC_r\, p^r\, q^{n-r}$$

i.e. $$p(r) = {}^nC_r\, p^r\, q^{n-r}$$

$\therefore$ $$P(r+1) = {}^nC_{r+1}\, p^{r+1}\, q^{n-(r+1)}$$

$\therefore$ $$\frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}\cdot p^{r+1}\cdot q^{n-(r+1)}}{{}^nC_r\cdot p^r\cdot q^{n-r}}$$

$$= \frac{\left[\dfrac{n!}{(r+1)!(n-r-1)!}\right]}{\left[\dfrac{n!}{r!(n-r)!}\right]} \times \frac{p}{q}$$

$$= \frac{(n-r)!}{(n-r-1)!}\ \frac{r!}{(r+1)!}\ \frac{p}{q}$$

$$= \frac{(n-r)\times\cancel{(n-r-1)!}}{\cancel{(n-r-1)!}}\ \frac{r!}{(r+1)\cdot r!}\ \frac{p}{q}$$

$$\frac{p(r+1)}{p(r)} = \frac{(n-r)}{(n+1)}\frac{p}{q}$$

$\therefore$ $$p(r+1) = \left(\frac{n-r}{r+1}\right)\cdot\frac{p}{q}\cdot p(r)$$

which is the required recurrence formula.

Thus if P (0) is known, we can find P (1), P (2), P (3) .....

Thus for Binomial Distribution :

If p = Probability of success.

q = Probability of failure.

Then, $B(n, p, r) = {}^nC_r\, p^r\, q^{n-r}$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

where $B(n, p, r)$ = Probability r successes in n trials

6.14 Illustrations

Example 6.39 : *An unbiased coin is thrown 10 times. Find the probability of getting exactly 6 heads, at least 6 heads.* **SPPU : May-17**

Solution : Here $p = q = \frac{1}{2}$ $n = 10$

Occurrence of head is treated as successes.

Probability of getting exactly 6 Heads is

$$P(6) = {}^{10}C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^4 = {}^{10}C_6 \left(\frac{1}{2}\right)^{10}$$

Event at least six heads occurs when coin shows up Head 6, 7, 8, 9 or 10 times the probability for these events are

$$P(7) = {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 = {}^{10}C_7 \left(\frac{1}{2}\right)^{10}$$

$$P(8) = {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 = {}^{10}C_8 \left(\frac{1}{2}\right)^{10}$$

$$P(9) = {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 = {}^{10}C_9 \left(\frac{1}{2}\right)^{10}$$

$$P(10) = {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0 = \left(\frac{1}{2}\right)^{10}$$

$$P(\text{at least 6 heads}) = P(6)+P(7)+P(8)+P(9)+P(10)$$

$$= \left(\frac{1}{2}\right)^{10} \left[{}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right]$$

$$= \left(\frac{1}{2}\right)^{10} \left[\frac{10!}{6!\,4!} + \frac{10!}{7!\,3!} + \frac{10!}{8!\,2!} + \frac{10!}{9!\,1!} + \frac{10!}{10!\,0!}\right]$$

$$= \left(\frac{1}{2}\right)^{10} \left[\frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} + \frac{10 \cdot 9}{2 \cdot 1} + \frac{10}{1} + 1\right]$$

$$= \left(\frac{1}{2}\right)^{10} [210+120+45+10+1]$$

$$= \frac{386}{2^{10}} = \frac{386}{1024} = 0.3769$$

Example 6.40 : *A pair of dice is thrown 10 times. If getting a doublet is considered a success, find the probability of (i) 4 successes, (ii) No success.*

Solution : Here $n = 10$

$$p = \text{Probability of getting a doublet}$$
$$= \frac{3}{36} = \frac{1}{6}$$

$$q = \text{Probability of not getting a doublet}$$
$$= 1 - \frac{1}{6} = \frac{5}{6}$$

$$\therefore \quad p(r) = {}^nC_r\, p^r\, q^{n-r}$$

$$p(4) = {}^{10}C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^{10-4}$$

$$= \frac{10!}{6!\,4!} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = \frac{(5)^6}{(6)^{10}}$$

Probability of no success i.e.

$$p(0) = {}^nC_0\, p^0\, q^{n-0}$$

$$= {}^{10}C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{10} = \left(\frac{5}{6}\right)^{10}$$

Example 6.41 : *Probability of man aged 60 years will live for 70 year is 1/10. Find the probability of 5 men selected at random 2 will live for 70 years.*

Solution : Here $p = \frac{1}{10}$ $q = \frac{9}{10}$ $r = 2$ $n = 5$

$$p(\text{2 men live for 70 years}) = {}^5C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^3$$

$$\text{i.e. } p(2) = \frac{5.4}{1.2}\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)\left(\frac{9}{10}\right)\left(\frac{9}{10}\right) = 0 \cdot 054675$$

Example 6.42 : *Twelve coins are thrown and the number of heads recorded. If the experiment is repeated 4096 times. Find the theoretical frequencies of different number of heads.*

Solution : Here $N = 4096$, $N = 12$, $p = \frac{1}{2}$, $q = \frac{1}{2}$

Thus the theoretical frequencies are given by

$$N(p+q)^n = 4096\left(\frac{1}{2}+\frac{1}{2}\right)^{12}$$

$$= 4096\left[\left(\frac{1}{2}\right)^{12} + {}^{12}C_1\left(\frac{1}{2}\right)^{11}\left(\frac{1}{2}\right) + {}^{12}C_2\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^2 + {}^{12}C_3\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^3\right.$$
$$+ {}^{12}C_4\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^4 + {}^{12}C_5\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^5 + {}^{12}C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^6$$
$$+ {}^{12}C_7\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^7 + {}^{12}C_8\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^8 + {}^{12}C_9\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^9$$
$$\left. + {}^{12}C_{10}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{10} + {}^{12}C_{11}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{11} + {}^{12}C_{12}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{12}\right]$$

$$= 4096 \cdot \left(\frac{1}{2}\right)^{12} [1+12+66+220+495+792+924+792$$
$$+495+220+66+12+1]$$

$$= 4096\left[\frac{1}{4096} + \frac{12}{4096} + \frac{66}{4096} + \frac{220}{4096} + \frac{495}{4096} + \frac{792}{4096} + \frac{924}{4096}\right.$$

$$+\frac{792}{4096}+\frac{495}{4096}+\frac{220}{4096}+\frac{66}{4096}+\frac{12}{4096}+\frac{1}{4096}\Bigg]$$

Thus we have the following theoretical frequencies,

No. of Heads	0	1	2	3	4	5	6	7	8	9	10	11	12
Frequency	1	12	66	220	495	792	924	792	495	220	66	12	1

The sum of frequencies = 4096

Example 6.43 : *Four coins are tossed simultaneously. What is the probability of getting (i) Two heads and two tails (ii) At least two heads (iii) At least one head.*

Solution : In single toss

$$P(H) = \frac{1}{2} \quad \text{i.e.} \quad P = \frac{1}{2}$$

$$\therefore \quad P(T) = P(\overline{H}) = 1-\frac{1}{2} = \frac{1}{2} \quad \text{i.e.} \quad q = \frac{1}{2}$$

$$p(r) = {}^{n}C_r\, p^r\, q^{n-r}$$

$$\therefore \quad p(2) = {}^{4}C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2} = \frac{4!}{2!\,2!}\left(\frac{1}{2}\right)^4 = 6\left(\frac{1}{2}\right)^4$$

This is prob. of two heads and two tails.

i) We know that

$$\therefore p(x \geq 2) = p(2) + p(3) + p(4)$$

$$= {}^{4}C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2} + {}^{4}C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{4-3} + {}^{4}C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6$$

$$= 6\left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{11}{16}$$

ii) Probability of at least one head = 1 – Probability of no head

$$= 1-{}^{4}C_0\, p^0\, q^{4-0}$$

$$= 1-\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 = 1-\frac{1}{16} = \frac{15}{16}$$

Example 6.44 : *Ten percent of articles from a certain machine are defective. What is the probability that there shall be 6 defectives in a sample of 25 ?*

Solution : Here $n = 25,\ p = \frac{10}{100} = 0.1,\ q = 1 - p = 0.9$

$$\therefore \quad p(r) = {}^{n}C_r\, p^r\, q^{n-r}$$

$$p(r) = {}^{25}C_r\,(0.1)^r\,(0.9)^{25-r}$$

$$\therefore \quad p(6) = {}^{25}C_6\,(0.1)^6\,(0.9)^{19}$$

$$= \frac{25\times24\times23\times22\times21\times20}{6\times5\times4\times3\times2\times1}\times\frac{(9)^{19}}{(10)^{25}} = 0.024$$

Example 6.45 : *A department has 10 machines which may need adjustment from time to time during the day. Three of them are old, each having a probability of 1/10 of needing adjustment during the day and 7 are new, having corresponding probability 1/20. Assuming that no machine needs adjustment twice on the same day, determine the probability that on a particular day*

i) Just two old and no new machine need adjustment.

ii) Just two machines need adjustment which are of the same type.

Solution : Out of 3 old machines, if 2 need adjustment then combination of 2 needing adjustment and one cannot occur in ${}^{3}C_2$ ways.

p_1 = Probability of old machine needing adjustment

$$= 1 = \frac{1}{10}$$

q_1 = Probability of old machine needing adjustment

= 9/10

p_2 = Probability of new machine needing adjustment

= 1/20

q_2 = Probability of new machine not needing an adjustment

= 19/20

Consider an event A where two old machines needs adjustments and zero new machines needs adjustments.

i.e. p (Two old + Zero new)

$$= \left[{}^{3}C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{3-2}\right]\left[{}^{7}C_0\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^{7-0}\right]$$

$$\text{Hence } P(A) = {}^{3}C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)\left(\frac{19}{20}\right)^7$$

$$= \frac{3\times9\times(19)^7}{10^3\,(20)^7} = 0.0188$$

Also consider an event B in which case zero old machines needs adjustment and two new machines needs adjustment.

$\therefore$ p [Zero old + Two new]

$$= \left[{}^{3}C_0\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{3-0}\right]\left[{}^{7}C_2\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^{7-2}\right]$$

$$\text{Hence} \quad P(B) = \left(\frac{9}{10}\right)^3\,{}^{7}C_2\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^5$$

$$= \frac{7 \cdot 6}{1 \cdot 2} \times \left(\frac{9}{10}\right)^3 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^5 = 0.0296$$

Event, two machines needing an adjustment which are of the same type A + B.

$\therefore P(A+B) = P(A) + P(B)$ [A,B are mutually exclusive]

$= 0.0188 + 0.0296$

Required probability $= 0.0484$

Example 6.46 : *A dice is thrown 5 times. If getting an odd number is a success, what is the probability of (i) 4 successes (ii) At least 4 success.*

SPPU : May-19

Solution : Odd numbers are 1, 3, 5

$\therefore$ Probability of an odd $= \frac{3}{6} = \frac{1}{2}$ i.e. $p = \frac{1}{2}$

$\therefore$ Probability of not an odd $= 1 - \frac{1}{2} = \frac{1}{2}$ i.e. $q = \frac{1}{2}$

$$\therefore \quad P(r) = {}^nC_r\, p^r\, q^{n-r}$$

$$P(4) = {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4} = 5\left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

Probability of at least four, i.e.

$$P(x \geq 4) = p(4) + p(5)$$

$$= {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4} + {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{5-5}$$

$$= \frac{5}{32} + \frac{1}{32} = \frac{6}{32}$$

Example 6.47 : *An average box containing 10 articles is likely to have 2 defectives. If we consider a consignment of 100 boxes, how many of them are expected to have three or less defectives.*

SPPU : Dec.-19

Solution : Let p = Probability of box containing defective articles

$= 2/10 = 1/5$

q = Probability on non defective item = 4/5

Probability of box containing three or less defective articles.

$= P(r \leq 3) = P(r = 0) + P(r = 1) + P(r = 2) + P(r = 3)$

r = Number of defective items

$$P(r = 0) = {}^{10}C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10} = 0.10738$$

$$P(r = 1) = {}^{10}C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^9 = 0.2684$$

$$P(r = 2) = {}^{10}C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^8 = 0.302$$

$$P(r = 3) = {}^{10}C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^7 = 0.2013$$

$$P(r \leq 3) = 0.10738 + 0.2684 + 0.302 + 0.2013 = 0.8791$$

$$100 \times 0.8791 = 87.91$$

88 boxes are expected to contain three or less defectives.

Example 6.48 : *Probability of a man hitting a target is 1/4. If he fires seven times, what is the probability of his hitting the target at least twice.*

Solution : Here $p = \frac{1}{4}$ $q = \frac{3}{4}$ $n = 7$

$$P(r) = {}^nC_r\, (p)^r\, (q)^{n-r}$$

$$p \text{ (at least twice)} = p(x \geq 2)$$

$$= P(2) + P(3) + P(4) + P(5) + P(6) + P(7)$$

As the probability = 1

i.e. $P(0) + P(1) + \ldots + P(7) = 1$

$\therefore$ $P(2) + P(3) + P(4) + P(5) + P(6) + P(7)$

$$= 1 - P(0) - P(1)$$

$$= 1 - \left[{}^7C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^7\right] - \left[{}^7C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^6\right]$$

$$= \frac{9094}{16384} = 0.555$$

Example 6.49 : *Assume that on average telephone number out of 15 called between 2 pm to 3 pm on week days is busy. What is the probability that is 6 randomly selected telephone numbers called.*

i) Not more that 3 ii) At least 3 of them is busy.

Solution : The probability that the telephone no, called between 2 pm and 3 pm, is busy is

$$p = \frac{1}{15} \quad q = 1 - \frac{1}{15} = \frac{14}{15}$$

Hence probability that r nos. called out of 6 called are.

$$P(r) = {}^6C_r\, p^r\, q^{6-r} = {}^6C_r \left(\frac{1}{15}\right)^r \left(\frac{14}{15}\right)^{6-r} \quad \ldots (1)$$

i) For not more than 3 calls are busy

$$P(r \leq 3) = P(0) + P(1) + P(2) + P(3)$$

$$= {}^6C_0 \left(\frac{14}{15}\right)^6 + {}^6C_1 \left(\frac{1}{15}\right)\left(\frac{14}{15}\right)^5 + {}^6C_2 \left(\frac{1}{15}\right)^2 \left(\frac{14}{15}\right)^4 + {}^6C_3 \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3$$

$$= \frac{(14)^3}{(15)^6}\left[(14)^3 + 6(14)^2 + \frac{6.5}{1.2}(14) + \frac{6\cdot5\cdot4}{1\cdot2\cdot3}\right] = 0.9997$$

ii) For at least 3 calls to be busy of 6 calls we have the probability as.

$$P(r \geq 3) = 1 - P(r < 3)$$

$$= 1 - \left[{}^6C_0\left(\frac{14}{15}\right)^6 + {}^6C_1\left(\frac{1}{15}\right)\left(\frac{14}{15}\right)^5 + {}^6C_2\left(\frac{1}{15}\right)^2\left(\frac{14}{15}\right)^4\right]$$

$$= 0.0051$$

Example 6.50 : *20 % of bolts produced by a machine are defective. Determine the probability that out of 4 bolts chosen at random. a) 1 is defective b) Zero is defective c) At most 2 bolts are defective.*

Solution : The probability of defective bolt is.

$$p = \frac{20}{100} = 0.2$$

$$\therefore\ q = 0.8 \quad [p + q = 1]$$

a) The probability of having 1 defective bolt out of 4 is

$$p(r = 1) = {}^4C_1\,(0.2)\,(0.8)^3$$
$$= 4\,(0.2)\,(0.8)^3 = 0.4096$$

b) The probability of having zero bolts defective

$$p(r = 0) = {}^4C_0\,(0.2)^0\,(0.8)^4 = 0.4096$$

c) The probability of having at most 2 bolts is

$$P(r \leq 2) = P(0) + P(1) + P(2)$$

$$= {}^4C_0(0.2)^0(0.8)^4 + {}^4C_1(0.2)(0.8)^3 + {}^4C_2(0.2)^2(0.8)^2$$

$$= 0.9728$$

Example 6.51 : *Out of 2000 families with 4 children each, how many would you expect to have a) At least a boy b) Two boys c) 1 or 2 girls d) No girls.*

Solution : p = Probabilities of having a boy $= \frac{1}{2}$

q = Probabilities of having a girl $= 1 - \frac{1}{2} = \frac{1}{2}$

Hence binomial distribution for 2000 families

a) P (At lest a boy) $= P(r \geq 1) = 1 - P(r = 0)$

$$= 1 - {}^4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4$$

$$= 1 - \frac{1}{16} = \frac{15}{16} = 0.9375$$

Hence expected no. of families having at least boy

$$= 2000\,P(r \geq 1) = 2000 \times \frac{15}{16} = 1875$$

b) Expected no. of families having 2 boys

$$= 2000\,{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 750$$

c) Expected no. of families having 1 or 2 girls. (i.e. having 3 boys or 2 boys)

$$= 2000\,[P(3) + P(2)]$$

$$= 2000\left[{}^4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2\right] = 1250$$

d) Expected no. of families having no girls (having 4 boys)

$$= 2000\,{}^4C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = 125$$

Example 6.52 : *Out of 800 families with 4 children each, how many families would be expected to have (I) 2 boys and 2 girls (II) At least one boy (III) No girl (IV) At most two girls ? Assume equal probability for boys and girls.*

Solution : Here probability of boy and girl is equal.

$$\therefore\ P = \frac{1}{2},\quad q = \frac{1}{2}$$

$$n = 4,\quad N = 800$$

$\therefore$ Binomial distribution is $800\left(\frac{1}{2} + \frac{1}{2}\right)^4$

$\therefore$ I) P (2 boys and 2 girls)

$$= 800 \times {}^4C_2\,p^2q^{4-2} = 800 \times \frac{4!}{2!2!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 300$$

II) P (at least 1 boy) $= P(1) + P(2) + P(3) + P(4)$

$$= 800 \times [1 - p\,(\text{no boy})] = 800 \times [1 - p(0)]$$

$$= 800 \times \left[1 - {}^4C_n\,p^0q^{4-0}\right] = 800 \times \left[1 - 1\cdot\left(\frac{1}{2}\right)^4\right] = 750$$

III) P (no girls) $= P$ (all 4 boys) $= P(4)$

$$= 800 \times {}^4C_4\,(P)^4\,(q)^{4-0} = 800 \times \left(\frac{1}{2}\right)^4 = 50$$

IV) P (at most two girls) = P (at least two boys)

$$= 800 \times [P(2) + P(3) + P(4)]$$

$$= 800\left[{}^4C_2\,p^2\,q^{4-2} + {}^4C_3\,p^3\,q^{4-3} + {}^4C_4\,p^4\,q^{4-0}\right]$$

$$= 800 \times \left[\frac{4!}{2!2!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + 1\cdot\left(\frac{1}{2}\right)^4\right] = 550$$

Example 6.53 : *An Urn contain 25 balls, of which 10 balls bear a mark X and remaining 15 bear a mark Y. A ball is drawn at random from the Urn. Its mark is recorded and it is replaced. If 6 balls are drawn in this way, find the probability that (I) All will bear X mark (II) Not more than two bear X mark.*

Solution : Here probability of getting a ball with mark X = $\frac{10}{25} = \frac{2}{5}$ i.e. $p = \frac{2}{5}$

$\therefore$ Probability of getting a ball with mark Y = $1 - \frac{2}{5} = \frac{3}{5}$ i.e. $q = \frac{3}{5}$

Here, n = 6 $\quad P(r) = {}^nC_r\, p^r\, q^{n-r}$

$$P(6) = {}^6C_6\,(p)^6\,(q)^{6-6}$$

$$= 1 \cdot \left(\frac{2}{5}\right)^6 \times \left(\frac{3}{5}\right)^0 = \frac{64}{15625}$$

II) Probability of not more than 2 X marks

$= P(0) + P(1) + P(2)$

$= {}^6C_0\,(P)^0\,q^{6-0} + {}^6C_1\,p^1\,q^{6-1} + {}^6C_2\,p^2\,q^{6-2}$

$= \left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^6 + 6 \times \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^5 + \frac{6!}{4!2!}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^4$

$= \left(\frac{3}{5}\right)^6 + 4 \times \left(\frac{3}{5}\right)^6 + 20\frac{(3)^5}{5^6}$

$= \frac{3^5}{5^6}[1 + 12 + 20] = \frac{729 \times 33}{15625}$

III) Probability of at least 1 ball with X marks

$= 1 -$ Probability of none

$= 1 - P(0) = 1 - {}^6C_0\,(P)^0\,q^{6-0}$

$= 1 - \left(\frac{3}{5}\right)^6 = \frac{15625 - 729}{15625}$

$= \frac{14896}{15625} = 0.953344$

Note : Mean of binomial distribution is np and variance = npq.

Example 6.54 : *Point out fallacy of the statement. The mean of binomial distribution is 3 and variance 5.*

Solution : Consider $= \frac{\text{Variance}}{\text{Mean}}$

$$= \frac{npq}{np} = q.$$

$$npq = 5, \; np = 3$$

$$3 \times q = 5$$

$$q = 5/3$$

which is not possible

Example 6.55 : *Mean and variance of binomial distribution are 6 and 2 respectively. Find $P(r \geq 1)$.*

Solution : Mean = np = 6

Variance = npq = 2

$6 \times q = 2$

$q = 1/3$

$P = 1 - \frac{1}{3} = \frac{2}{3}$

As $np = 6 \Rightarrow n \cdot \frac{2}{3} = 6 \Rightarrow n = 9$

$$\therefore P(r \geq 1) = 1 - \left\{{}^nC_0\, p^r \cdot q^{n-r}\right\}$$

$$= 1 - \left[{}^9C_0\left(\frac{2}{3}\right)^0 \times \left(\frac{1}{3}\right)^9\right] = 0.9999$$

Example 6.56 : *A group of 20 a aeroplane are sent on an operational flight. The chances that the aeroplane fails to return from the flight is 5 percent. Determine the probability that*

i) No plane returns.

ii) At most 3 planes do not return.

Solution : Let P be the probability that aeroplane fails to return from the flight.

$\therefore \quad P = \frac{5}{100} = \frac{1}{20}$ and $q = 1 - \frac{1}{20} = \frac{19}{20}$

and $\quad n = 20$

$$P(r) = {}^nC_r p^r q^{n-r}, \quad r = 0, 1, 2, \ldots h$$

i) The probability that no plane returns

$= P(r = 0) = {}^{20}C_0\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^{20} = 0.35848$

ii) The probability that at most 3 planes do not return

$= P(r \leq 3)$

$= P(r = 0) + P(r = 1) + P(r = 2) + P(r = 3)$

$= 0.35848 + {}^{20}C_1\left(\frac{1}{20}\right)^1\left(\frac{19}{20}\right)^{19} + {}^{20}C_2\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^{18} + {}^{20}C_3\left(\frac{1}{20}\right)^3\left(\frac{19}{20}\right)^{17}$

$= 0.98408$

Example 6.57 : *In 100 sets of 10 tosses of a coin, in how many ways do you expect.*

i) 7 heads and 3 tails ii) at least 7 heads.

Solution : Let P be the probability of gettting head

$\therefore \quad P = \frac{1}{2}, \quad q = \frac{1}{2}, \; n = 10$

i) The probability of getting 7 heads and 3 tails

$$= P(r = 7) = {}^{10}C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 = \frac{12}{100} = 0.12$$

$\therefore$ The expected number of ways to get 7 heads and 3 tails is $100 \times 0.12 = 12$

ii) $P(r \geq 7) = P(7) + P(8) + P(9) + P(10)$

$$= {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$= 0.17$

$\therefore$ The expected no. of ways to get at least 7 heads $= 100 \times 0.17 = 17$

Example 6.58 : *Team A has a probability of $\frac{2}{3}$ of winning whenever the team plays a particular game. If team A plays 4 games, find the probability that the team wins : i) Exactly two games and ii) At least two games.* **SPPU : May-17, Marks 4**

Solution : Let P be the probability of winning of team A $= \frac{2}{3}$

$\therefore \quad q = 1 - p = \frac{1}{3} \; ; \quad n = 4$

By Binomial distribution

$$P(r) = {}^nC_r\, p^r q^{n-r}$$

i) The probability that team A wins exactly two games

$$= P(2) = {}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = 0.2962$$

ii) The probability that the team A wins at least two games

$$= P(r \geq 2) = 1 - P(r < 2) = 1 - P(0) - P(1)$$

$$= 1 - {}^4C_0\, p^0 q^4 - {}^4C_1\, p^1 q^3$$

$$= 1 - \left(\frac{1}{3}\right)^4 - 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3$$

$$= 1 - \frac{1}{81} - 0.09876 = 0.8889$$

Example 6.59 : *An insurance agent accepts policies of 5 men of identical age and in good health. The probability that a man of this age will be alive 30 years hence is 2/3. Find the probability that in 30 years :*
i) all five men and
ii) at least one man will be alive.
SPPU : Dec.-17, Marks 4

Solution :

Let $\quad p$ = probability that man alive $= \frac{2}{3}$

$$q = 1 - p = \frac{1}{3}, \quad n = 5$$

By Binomial distribution

$$P(r) = {}^nC_r\, p^r q^{n-r} = {}^5C_r\, p^r q^{5-r}$$

i) The probability that all five men alive is

$$= P(5) = {}^5C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0$$

$$= \left(\frac{2}{3}\right)^5 = 0.1316$$

ii) The probability that at least one man will be alive is

$$= P(r \geq 1) = 1 - P(r < 1)$$

$$= 1 - P(0)$$

$$= 1 - {}^5C_0\, p^0 q^5 = 1 - 0.004115$$

$$= 0.99588$$

Example 6.60 : *A series of five one day matches is to be played between India and Sri Lanka. Assuming that the probability of India's win in each match as 0.6 and results of all the five matches independent of each other, find the probability that India wins the series.* **SPPU : Dec.-18, Marks 4**

Solution : Let p be the probability that India wins the match.

$\therefore \quad p = 0.6$ and $q = 1 - p = 0.4$, $n = 5$

The probability that India wins the series

$$= p(r = 3) + p(r = 4) + p(r = 5)$$

$$= {}^nC_3\, p^3 q^2 + {}^nC_4\, p^4 q + {}^nC_5\, p^5 q^0$$

$$= {}^5C_3 (0.6)^3 (0.4)^2 + {}^5C_4 (0.6)^4 (0.4) + {}^5C_5 (0.6)^5$$

$$= 10(0.6)^3 (0.4)^2 + 5(0.6)^4 (0.4) + (0.6)^5$$

$$= 0.68256$$

6.15 Poisson Distributions

When 'p' the probability of success, is very small and 'n', the number of trials, is very large and np is finite, we get another distribution called 'Poisson distribution.' It is considered as limiting case of Binomial distribution with $n \to \infty$, $p \to 0$ and np remaining finite.

Consider the Binomial distribution

$$B(n, p, r) = {}^nC_r\, p^r q^{n-r}$$

$$= \frac{n(n-1)(n-2)\,.....\,(n-r+1)}{r!}\, p^r (1-p)^{n-r}$$

Let $z = np \quad \therefore \; p = \frac{z}{n}$

$$\therefore B(n, p, r) = \frac{np(np-p)(np-2p).....(np-(r-1)p)}{r!} \times \frac{\left(1-\frac{z}{n}\right)^n}{\left(1-\frac{z}{n}\right)^r}$$

Taking the limit as $n \to \infty$ and $np = z$

$$p \to 0$$

$$\text{Lim } B(n, p, r) = \frac{z^r \cdot e^{-z}}{r!} \qquad \left[\lim_{n \to \infty} \left(1-\frac{z}{n}\right)^n = e^{-z} \right]$$

The distribution with frequencies given by, $\sum_{r=0}^{\infty} \frac{e^{-z} \cdot z^r}{r!}$ corresponding to 0, 1, 2, ... r, success is called Poisson distribution.

6.16 Mean and Variance of the Poisson's Distribution

We have $P(r) = \frac{e^{-z} z^r}{r!}$

$$\text{Mean} \quad \mu = \sum_{r=0}^{\infty} r \cdot P(r) = \sum_{r=0}^{\infty} r \cdot \frac{e^{-z} \cdot z^r}{r!}$$

$$= e^{z} \sum_{r=1} \frac{z^r}{(r-1)!}$$

$$= e^{-z}\left[z + \frac{z^2}{1!} + \frac{z^3}{2!} + ...\right]$$

$$= z \cdot e^{-z}\left[1 + \frac{z}{1!} + \frac{z^2}{2!} + ...\right]$$

$$= z \cdot e^{-z} \cdot e^{z}$$

$$\text{as} \quad e^z = 1 + z + \frac{z^2}{2!} + ...$$

$$= z = np$$

$$\therefore \text{ Mean} \quad \mu = np$$

$$\text{Variance} \quad \sigma^2 = \sum_{r=0}^{\infty} r^2 P(r) - \mu^2$$

$$= \sum_{r=0}^{\infty} r^2 \cdot \frac{e^{-x} z^r}{r!} - z^2$$

$$= e^{-z}\left[\frac{1^2 \cdot z}{1!} + \frac{2^2 \cdot z^2}{2!} + \frac{3^2 z^3}{3!} + ...\right] - z^2$$

$$= z \cdot e^{-z}\left[1 + \frac{2z}{1!} + 3\frac{z}{2!} + ...\right] - z^2$$

$$= z \cdot e^{-z}\left[1 + (1+1)\frac{z}{1!} + \frac{(1+2)z^2}{2!} + \frac{(1+3)z^3}{3!} + ...\right] - z^2$$

$$= z \cdot e^{-z}\left[\left\{1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^2}{3!} + ...\right\} + \left\{\frac{z}{1!} + \frac{2z^2}{2!} + \frac{3z^3}{3!} + ...\right\}\right] - z^2$$

$$= z \cdot e^{-z} \cdot \left[e^z + z\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + ...\right)\right] - z^2$$

$$= z \cdot e^{-z} \cdot \left[e^z + z \cdot e^z\right] - z^2$$

$$= z(1+z) - z^2 = z$$

$\therefore$ Variance $\quad \sigma^2 = \lambda = nP$

$\therefore$ Mean and Variance of Poisson's distribution = nP

6.17 Recurrence Formula for Poisson's Distribution

We have $P(r) = \frac{e^{-\lambda} \lambda^r}{r!}$

$$\therefore \quad P(r+1) = \frac{e^{-\lambda} \lambda^{r+1}}{(r+1)!}$$

$$\therefore \quad \frac{P(r+1)}{P(r)} = \frac{\frac{e^{-\lambda} \cdot \lambda^{r+1}}{(r+1)!}}{\frac{e^{-\lambda} \lambda^r}{r!}} = \frac{\lambda \cdot r!}{(r+1)!}$$

$$\therefore \quad P(r+1) = \frac{\lambda}{r+1} P(r)$$

$$r = 0, 1, 2, 3 ....$$

Thus for Poisson Distribution

When P is very small and n is very large and np is finite, the binomial distribution reduce to Poisson's Distribution.

$$P(r) = \frac{z^r \cdot e^{-z}}{r!} \quad \text{where } z = np$$

Mean np = z and variance npq = z.

As probability of success (p) is very small, q will be almost 1.

$\therefore$ Variance = np

6.18 Illustrations

Example 6.61 : *A manufacturer of cotter pins knows that 2 % of his products is defective. If he sells cotter pins in a box of 100 pins and guarantees that not more than 5 pins are defective in a box. Find the approximate probability that a box will fail to meet the guaranteed quantity.*

Solution : $p = 0.02 \quad q = 1 - 0.02$

$$n = 100 \quad z = np = 0.02 \times 100 = 2$$

$$\therefore \quad P(r) = \frac{z^r \cdot e^{-z}}{r!}$$

$$P(r > 5) = 1 - P[r \leq 5]$$

$$= 1 - [P(0) + P(1) + P(2) + P(3) + P(4) + P(5)]$$

$$= 1 - \left\{ \begin{array}{l} \frac{2^0 e^{-2}}{0!} + \frac{2! e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} \\ + \frac{2^3 e^{-2}}{3!} + \frac{2^4 e^{-2}}{4!} + \frac{2^5 e^{-2}}{5!} \end{array} \right\} = 0.0165$$

Example 6.62 : *In a certain factory turning out razor blades there is a small chance of 1/500 for any blade to be defective. The blades are supplied in a packet of 10. Calculate approximate number of packets containing no defective and 2 defective blades in a consignment of 10,000 packets.*

SPPU : May-18, Dec.-18

Solution : $P = \left(\frac{1}{500}\right)$ defective

$$n = 10 \quad z = 10/500 = 1/50$$

Defective = 2, No defective = 0

$$P(2) = \frac{e^{-0.02} (0.02)^2}{2!}$$

No. of packets = $10000 \times P(2) = 9802$

$$P(0) = \frac{e^{-0.02} (0.02)^0}{0!}$$

Number of packets = $10000 \times P(0) = 2$

Example 6.63 : *Find probability that almost 5 defective fuses will be found in a box of 200 fuses if 2 % of such fuses are defective.*

Solution : $n = 200$, $p = 0.02$

$z = np = 200 \times 0.02 = 4$

$\therefore$ Required probability

$= P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$$= \frac{4^0 (e)^{-4}}{0!} + \frac{4^1 (e)^{-e}}{1!} + \frac{4^2 e^{-4}}{2!} + \frac{4^3 e^{-4}}{3!} + \frac{4^4 e^{-4}}{4!} + \frac{4^5 e^{-4}}{5!} = 0.735$$

Example 6.64 : *A telephone switch board handle 600 calls on the average during hour. The board can make a maximum 20 connections/minute. Use Poisson distribution to estimate the probability that board will be overtaxed during any given minute.*

Solution :

Consider $z = np = \frac{600}{60} = 10$

$$\therefore \quad P(r) = \frac{10^r e^{-10}}{r!}$$

$$P(r > 20) = 1 - P(r \leq 20)$$

$$= 1 - \left[\begin{array}{l} \frac{10^{20} e^{-10}}{20!} + \frac{10^{19} e^{-10}}{19!} \\ + \dots + \frac{10^0 e^{-10}}{0!} \end{array} \right]$$

$$= 1 - \sum_{r=0}^{20} \frac{(10)^r e^{-10}}{r!}$$

Example 6.65 : *In a Poisson's distribution.*

P(1) = 2P(2), find P(3).

Solution : $P(3) = \frac{z^3 e^{-z}}{3!}$

$$P(1) = 2P(2)$$

$$\frac{z \cdot e^{-z}}{1!} = \frac{2 \cdot z^2 e^{-z}}{2!}$$

$$z^2 = z$$

$$z = 1$$

$$\therefore \quad P(3) = \frac{z^3 e^{-z}}{3!} = \frac{1 \cdot e^{-1}}{3!} = 0.0613$$

Example 6.66 : *The accidents per shift in factory are given by,*

Acc / Shift	0	1	2	3	4	5
Frequency	142	158	67	27	5	1

Find a Poisson distribution.

Solution :

x_i	f_i	$f_i x_i$	
0	142	0	
1	158	158	Mean $= \frac{\sum f_i x_i}{\sum f_i}$
2	67	134	$= \frac{398}{400}$
3	27	81	
4	5	20	
5	1	5	z = 0.995
	400	398	

x_i	f_i	$f_i x_i$	$P(r) = \frac{z^r e^{-z}}{r!}$	Exp ≠ 400 × P (r)
0	142	0	0.3697	148
1	158	158	0.3678	147
2	67	134	0.183	73
3	27	81	0.607	24
4	5	20	0.0157	6.0
5	1	5	0.003	1

Example 6.67 : *S.T. in a Poisson's distribution with unit mean, mean deviation about mean is 2/e times the standard deviation.*

Solution : Mean z = 1

$$\text{Standard deviation} = \sqrt{\text{Variance}} = \sqrt{z} = 1$$

$$P(r) = \frac{z^r e^{-z}}{r!} = \frac{1^r e^{-1}}{r!}$$

Deviations : $di = |r-1|$ These are deviation about mean = 1

$$\text{Mean deviation} = \frac{\sum Pi\,di}{\sum Pi} = \frac{\sum Pi\,di}{1}$$

$$= \frac{\sum e^{-1}}{r!}|r-1|$$

Mean deviation about mean (1) is

$$\sum_{r=0}^{\infty} |r-1| \cdot P(r) = \sum_{0}^{\infty} |r-1| \cdot \frac{e^{-1}}{r!}$$

$$= e^{-1} \sum_{0}^{\infty} \frac{|r-1|}{r!}$$

$$\text{M.D.} = e^{-1}\left[1 + \frac{1}{2!} + \frac{1}{3!} + \frac{3}{4!} \cdots\right]$$

We know $\frac{n}{(n+1)!} = \frac{n+1-1}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!}$

$$= \frac{1}{n!} - \frac{1}{(n+1)!}$$

Thus

$$\text{M.D.} = e^{-1}\left[1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) \cdots\right]$$

$$= e^{-1}[1+1] = \frac{2}{e}$$

$$= \frac{2}{e}(1) = \frac{2}{e} \text{ (Standard deviation)}$$

Example 6.68 : *A car hire firm has 2 cars which it hires out day by day. The no. of demands for the car on each day is distributed as Poisson's distribution with parameter 1.5. Calculate the probability of days on which neither car is used and for days on which demand is refused.*

Solution : Let r be demands for each day, where r takes the values 0, 1, 2. The Poisson probability distribution for demand of r cars on each day is given by

$$P(r) = \frac{e^{-z} z^r}{r!}$$

$$= \frac{e^{-1.5} (1.5)^r}{r!} \qquad [z = 1.5] \ldots (1)$$

a) From (1), the probability of the days on which neither car is used is

$$P(r = 0) = \frac{e^{-1.5} (1.5)^0}{0!} = 0.22$$

b) When r > 2, the demand will be refused by the firm. Hence the probability of days when the demands is refused is.

$$P(r > 2) = 1 - P(r \le 2)$$

$$= 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - e^{-1.5}\left[1 + \frac{(1.5)}{1!} + \frac{(1.5)^2}{2!}\right] = 0.2025$$

Example 6.69 : *One percent of articles from a certain machine are defective. What is the probability of i) No defective ii) One defective iii) Two defectives iv) Two or more defective in a sample of 100 ?*

Solution : Here $p = \frac{1}{100}$, $n = 100$, $np = z = 1$

Thus, the Poisson's distribution takes the form $\sum_{r=0}^{\infty} e^{-1} \cdot \frac{1}{r!}$

Thus, i) The probability of no defective

$= \frac{e^{-1}}{0!} = e^{-1} = 0.3679$

ii) The probability of one defective

$= \frac{e^{-1}}{1!} = e^{-1} = 0.3679$

iii) The probability of two defective

$= \frac{e^{-1}}{2!} = \frac{0.3679}{2} = 0.1840$

iv) The probability of two or more defectives

$= \frac{e^{-1}}{2!} + \frac{e^{-1}}{3!} + \frac{e^{-1}}{4!} + \ldots = e^{-1}\left\{\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\right\}$

$= e^{-1}\left[\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) - 1 - 1\right]$

$= e^{-1}\{e - 2\} = 0.3679 \times (2.7183 - 2)$

$= 0.3679 \times 0.7183 = 0.2642$

Example 6.70 : *In a certain factory producing cycle tyres there is a small chance of 1 in 500 tyres to be defective. The tyres are supplied in lots of 10. Using Poisson distribution, calculate the approximate number of lots containing*
i) No defective ii) One defective
iii) Two defective tyres, respectively, in consignment of 1000 lots.

Solution : Here $p = \frac{1}{500}$, $n = 10$,

$$z = np = 10 \times \frac{1}{500} = \frac{1}{50} = 0.02$$

$$\therefore \quad p(r) = \frac{e^{-z} \cdot z^r}{r!}$$

i) Probability of no defective i.e.

$$p(0) = \frac{e^{-0.002}\ (0.02)}{0!} = e^{-0.02} = 0.9802$$

$\therefore$ Number of lots of containing no defectives

$= 10000 \times 0.9802 = 9802$ lots.

ii) Probability of one defective i.e.

$$p(1) = \frac{e^{-0.02}\ (0.02)^1}{1!}$$

$$= 0.9802 \times 0.02 = 0.019604$$

$\therefore$ Number of lots of containing no defectives

$= 10000 \times 0.019604 = 196$ lots

iii) Probability of two defectives i.e.

$$p(1) = \frac{e^{-0.02} \cdot (0.02)^2}{2!}$$

$$= 0.9802 \times 0.02 = 0.00019604$$

$\therefore$ Number of lots of containing no defective $= 10000 \times 0.00019604 = 1.96 \cong 2$ lots

Example 6.71 : *Assuming that the probability of an individual coal miner being killed in a mine accident during a year is $\frac{1}{2400}$. Use Poisson's distribution to calculate the Probability that in a mine employing 200 miners there will be at least one fatal accident in a year.*

Solution : Here $P = \frac{1}{2400}$ $n = 200$

$$\therefore \quad \lambda = np = \frac{1}{2400} \times 200 = 0.083$$

$$P(r) = \frac{e^{-\lambda} \cdot \lambda}{r!} = \frac{e^{-.083}\ (0.083)^r}{r!}$$

P(at least one fatal accident) = 1 – Probability (none)

$$= 1 - P(0)$$

$$= 1 - \frac{e^{-0.083} \times (0.083)^0}{0!}$$

$$= 1 - 0.92 = 0.08$$

Example 6.72 : *A manufacturer knows that the razor blades he makes contain on the average 0.5 % of defectives. He packs them in packets of 5. What is the probability that a packet picked at random will contain 3 or more faulty blades ?*

Solution : Here $p = 0.5\ \% = \frac{0.5}{100} = 0.005$

$n = 5$, $z = np = 5 \times 0.005 = 0.025$

Then $\quad P(r) = \frac{e^{-z} \cdot z^r}{r!}$

gives $\quad P(r \geq 3) = P(3) + P(4) + P(5)$

$$= \frac{e^{-0.025}\ (0.025)^3}{3!} + \frac{e^{-0.025}\ (0.025)^4}{4!} + \frac{e^{-0.025}(0.025)^5}{5!}$$

$$= \frac{e^{-0.025}\ (0.025)^3}{5!}\left[20 + 5\,(0.025) + (0.025)^2\right]$$

$$= \frac{0.975 \times 0.000015625 \times 20.125625}{120} = 0.000002555$$

Example 6.73 : *If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the Probability out of 2000 individuals. I) Exactly 3 II) More than 2 will suffer a bad reaction.* **SPPU : May-17**

Solution : Here $P = 0.001$

$n = 2000$

$\therefore \quad \lambda = nP = 0.001 \times 2000 = 2$

$$\therefore \quad P(r) = \frac{e^{-\lambda}\lambda^r}{r!} = \frac{e^{-2}\,2^r}{r!}$$

$$\therefore \text{I)} \quad P(3) = \frac{e^{-2}\,(2)^3}{3!} = 0.136 \times \frac{8}{6}$$

II) Prob (more than 2) = P (3) + P (4) .... + P (2000)

$$= 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - \left[\frac{e^{-2} \times 2^0}{0!} + \frac{e^{-2} \times 2^1}{1!} + \frac{e^{-2} \times 2^2}{2!}\right]$$

$$= 1 - e^{-2}\,[1+2+2]$$

$$= 1 - 0.136 \times 5$$

$$= 0.32$$

Example 6.74 : *Following is the data of men killed by the kick of a horse in a certain army corps in 20 years. Calculate the theoretical frequencies using Poisson's distribution.*

Number of deaths	0	1	2	3	4
Frequency	109	65	22	3	1

Solution : Frequency distribution

Number of deaths x_i	Frequency f_i	$f_i x_i$
0	109	100
1	65	65
2	22	44
3	3	9
4	1	4
	$\sum f_i = 200$	$\sum f_i x_i = 122$

$$\therefore \quad \text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{122}{200} = 0.61$$

$\therefore \quad \lambda = 0.61$

$$\therefore \quad P(r) = \frac{e^{-0.61} \times (0.61)^r}{r!}$$

$$\therefore \quad P(0) = \frac{e^{-0.61} \times (0.61)^0}{0!} = 0.54335$$

$$P(1) = \frac{e^{-0.61} \times (0.61)^1}{1!}$$

$$= 0.54335 \times 0.61 = 0.3314435$$

$$P(2) = \frac{e^{-0.61} \times (0.61)^2}{2!}$$

$$= \frac{0.54335 \times 0.3721}{2} = 0.101090$$

$$P(3) = \frac{e^{-0.61} \times (0.61)^3}{3!}$$

$$= \frac{0.54335 \times 0.226981}{6} = 0.020490$$

$$P(4) = \frac{e^{-0.61} \times (0.61)^4}{4!}$$

$$= \frac{0.54335 \times 0.138458}{24} = 0.0031248$$

$\therefore$ Theoretical frequencies are

$P(0) \times 200 = 0.54335 \times 200 = 108.67 = 109$

$P(1) \times 200 = 0.3314435 \times 200 = 66.29 = 66$

$P(2) \times 200 = 0.101090 \times 200 = 20.22 = 20$

$P(3) \times 200 = 0.20490 \times 200 = 4.10 = 4$

$P(4) \times 200 = 0.0031248 \times 200 = 0.63 = 1$

Example 6.75 : *In a town, 10 accidents took place in a span of 50 days. Assuming that the number of accidents per day follows Poisson distribution, find the probability that there will be 3 or more accidents in a day ?* **SPPU : Dec.-19**

Solution : Here $n = 50$, p = Probability of accident in a day $= \frac{10}{50}$

$p = 0.2$

$z = np = 50 \times 0.2 = 10$

$\therefore$ By Poisson distribution $P(r) = \frac{e^{-z} z^r}{r!}$

$\therefore$ The required probability $= P(r \geq 3) = 1 - P(r < 3)$

$$= 1 - P(r = 0) - P(r = 1) - P(r = 2)$$

$$= 1 - \frac{e^{-10}10^0}{0!} - \frac{e^{-10}10^1}{1!} - \frac{e^{-10}10^2}{2!}$$

$$= 1 - (0.002745) = 0.997255$$

Example 6.76 : *In a telephone exchange, the probability that any one call is wrongly connected is 0.02. What is the minimum number of calls required to ensure a probability 0.1 that at least one call is wrongly connected.*

Solution : We have $p = 0.02$, $z = np$

$\therefore$ The probability that at least one call is wrongly connected = 0.1.

$$= P(r \geq 1) = 1 - P(r < 1)$$

$$0.1 = 1 - P(0) = 1 - e^{-z}$$

$$\Rightarrow \quad e^{-z} = 1 - 0.1 = 0.9$$

$$\therefore \quad -z = \log_e 0.9 = -0.105$$

$$\therefore \quad z = 0.105$$

Now $\quad z = np \Rightarrow n = \dfrac{z}{p} = \dfrac{0.105}{0.02} = 5.26$

$$\Rightarrow \quad n = 5.26 \simeq 6 \text{ calls approximately}$$

Example 6.77 : *Suppose 300 misprints are distributed randomly throughout a book of 500 pages. Find the probability that the given page contains.*

i) Exactly two misprints.

ii) Two or more misprints.

Solution : Given that 300 misprints are distributed randomly throughout the book of 500 pages.

$\therefore$ z = Mean = probability of getting misprint in a page.

$$z = \frac{300}{500} = 0.6$$

i) The probability of getting exactly two misprints.

$$= P(r = 2) = \frac{e^{-z} z^r}{r!} = \frac{e^{-0.6}(0.6)^2}{2!} = 0.0988 \sim 0.1$$

ii) The probability of getting exactly two or more misprints =

$$P(r \geq 2) = 1 - P(r < 2) = 1 - P(r = 0) - P(r = 1)$$

$$= 1 - (0.549) - (0.329) = 0.122$$

Example 6.78 : *Fit Poisson's distribution to following data and calculate theoretical frequencies.*

X	0	1	2	3	4
f	122	60	15	2	1

Solution : Let z = Mean of the distribution

$$= \frac{\sum x_i f_i}{\sum f_i} = \frac{100}{200} = 0.5$$

$$P(r) = \frac{e^{-z} z^r}{r!} \text{ when } r = 0, 1, 2, 3, 4$$

$$\therefore \quad P(0) = e^{-0.5} = 0.6065$$

$$P(1) = 0.30325$$

$$P(2) = 0.07581$$

$$P(3) = 0.01263$$

$$P(4) = 0.001579$$

$\therefore$ Theoretical frequencies are

$$P(0) \times 200 = 121$$

$$P(1) \times 200 = 61$$

$$P(2) \times 200 = 15$$

$$P(3) \times 200 = 3$$

$$P(4) \times 200 = 1$$

Example 6.79 : *If 10 % of the rivet's produced by the machine are defective, find the probability that out of 5 rivets chosen at random.*

i) None will be defective.

ii) One will be defective.

iii) At least two will be defective.

Solution : Let p = Probabillity of getting defective rivet

$$p = \frac{10}{100} = 0.1$$

$$n = 5$$

$$z = np = 0.1 \times 5 = 0.5$$

By Poisson distribution

$$P(r) = \frac{e^{-z} z^r}{r!}, \ r = 0, 1, 2, \ldots$$

i) Probability that none will be defective

$$= \frac{e^{-0.5}(0.5)^0}{0!} = e^{-0.5} = 0.6065$$

ii) Probability that one rivet will be defective

$$= \frac{e^{-0.5}(0.5)^1}{1!} = 0.3032$$

iii) Probability at least 2 rivet will be defective

$$= P(r \geq 2) = 1 - P(r = 0) - P(r = 1)$$

$$= 1 - 0.6065 - 0.3032 = 0.09023$$

Example 6.80 : *The number of breakdowns of a computer in a week is a Poisson variable with $\lambda = np = 0.3$. What is the probability that the computer will operate :*

i) With no breakdown and

ii) At the most one breakdown in a week.

SPPU : Dec.-17, Marks 4

Solution : Given that

$$\lambda = np = 0.3$$

$$P(r) = \frac{e^{-\lambda}\lambda^r}{r!}, \quad r = 0, 1, 2, \ldots$$

i) The probability that the computer has no breakdown = P (0)

$$= \frac{e^{-\lambda}\lambda^0}{0i} = e^{-\lambda} = e^{-0.3} = 0.7408$$

ii) The probability that there is at most one breakdown = P (0) + P (1)

$$= 0.7408 + \frac{e^{-\lambda}\lambda}{1}$$

$$= 0.7408 + 0.2222 = 0.96304$$

Example 6.81 : *During working hours, on an average 3 phone calls are coming into a company within an hour. Using Poisson distribution, find the probability that during a particular working hour, there will be at the most one phone call.*

SPPU : May-19, Marks 4

Solution : The probability that phone call coming into a company is

$$= \frac{3}{60} = 0.05$$

$$\therefore \quad z = 0.05$$

The probability of getting at most one phone call is

$$= P(r \geq 1) = P(r = 0) + P(r = 1)$$

$$= \frac{e^{-z}z^0}{0!} + \frac{e^{-z}z^1}{1!} = e^{-z}[1+z] = 0.9987$$

Example 6.82 : *On an average, there are 2 printing mistakes on a page of a book. Using Poisson distribution, find the probability that a randomly selected page from the book has at least one printing mistake.* **SPPU : Dec.-19, Marks 4**

Solution : Given that, z = mean = 2

The probability that a page from the book has at least one printing mistake

$$P(r \geq 1) = 1 - P(r < 1)$$

$$= 1 - P(r = 0)$$

$$= 1 - \frac{z^0 e^{-z}}{0!} = 1 - 2e^{-2}$$

$$= 0.7293$$

6.19 Normal Distribution

The normal distribution is a continuous distribution. It can be derived from the Binomial distribution in the limiting case when n, the number of trials is very large and p, the probability of a success, is close to $\frac{1}{2}$. The general equation of the normal distribution is given by

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where the variable 'x' assumes all values from $-\infty$ to ∞ and σ, called the parameters of the distribution respectively are known as mean and standard deviation of the distribution and $-\infty < \mu < \infty$, $\sigma > 0$ x is called the normal variate and f (x) is probability density function of the normal distribution.

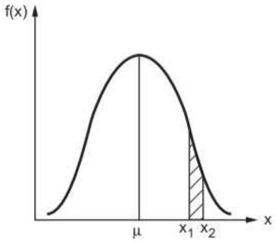

Fig. 6.1

The graph of the normal distribution is called the normal curve (some times known as normal probability curve or normal curve of errors). It is bell - shaped and symmetrical about the mean 'μ' as shown in the figure. The two tails of the curve extend to $+\infty$ and $-\infty$ towards the positive and negative directions of the X-axis respectively and gradually approach the X-axis without ever meeting it. The line $x = \mu$ divides the area under the normal curve above, X-axis into two equals parts. The area under the normal curve between any two given ordinates $x = x_1$ and $x = x_2$ represents the probability of values failing into the given interval. The total area under the normal curve above the x-axis is '1' i.e.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Thus $P(x_1 < x < x_2) = \int_{x_1}^{x_2} f(x)\, dx$

$$= \int_{x_1}^{x_2} \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

6.20 Standard Form of the Normal Distribution

If 'X' is a normal random variable with mean 'μ' and standard deviation σ, then the random variable $z = \frac{X-\mu}{\sigma}$ has the normal distribution with mean '0' and standard deviation 1. The random variable z is called the Standard normal random variable.

Thus probability density function or the normal distribution in standard form is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad (-\infty < z < \infty)$$

It is free from any parameter. This is useful to compute areas under the normal probability curve by making use of standard tables.

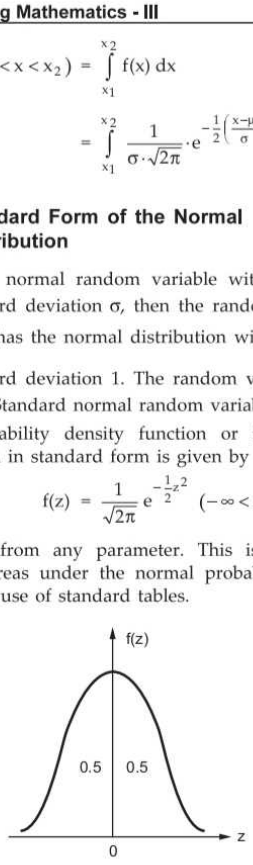

Fig. 6.2

Area under the Normal Curve :

The area under the normal curve is divided into two equal parts by z = 0. Left hand side area and right hand side area to z = 0 is 0.5.

6.21 Area Property (Normal Probability Integral)

The probability that random value x will be between $x = \mu$ and $x = x_1$ is given by

Table of Area

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5259
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9494	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9653	0.9664	0.9871	0.9678	0.9686	0.9693	0.9699	0.9708
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9883	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9634	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9809	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9988	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9998	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9898

Table 6.1
In each row and each colomn 0.5 to be substracted

Thus for Normal distribution note the following

Normal distribution (Continuous distribution)

Normal distribution is obtained as limiting form of Binomial distribution when n is very large and neither P nor q is very small.

The normal distribution curve is given by.

$$y = \frac{1}{\sigma\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \right]$$

where μ = mean σ = standard deviation shape of the curve is like a bell.

1) The total area under the curve

= Sum of the probabilities = 1 i.e. $\int_{-\infty}^{\infty} y\,dx$

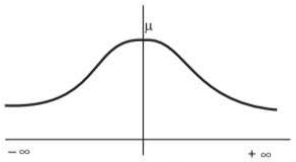

Fig. 6.3

2) $P(x_1 < x < x_2) = \int_{x_1}^{x_2} y\,dx$

= Area under the curve from x_1 to x_2

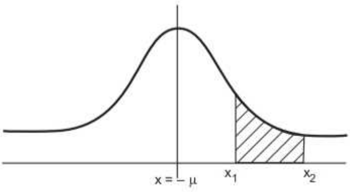

Fig. 6.4

3) $P(\mu < x < x_1) = \int_{\mu}^{x_1} y\,dx$

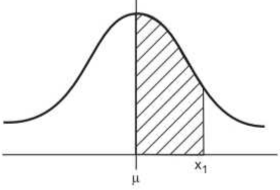

Fig. 6.5

Put $\frac{x-\mu}{\sigma} = z, \quad \frac{dx}{\sigma} = dz$

$$P(0 < z < z_1) = \int_{0}^{z_1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

$$= \int_{0}^{z_1} (z)\,dz$$

is known as normal integral gives the area under the standard normal curve between $z = 0$ and $z = z_1$.

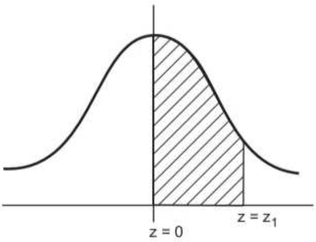

Fig. 6.6

4) $P(z_1 < z < z_2) = A(z_2) - A(z_1)$

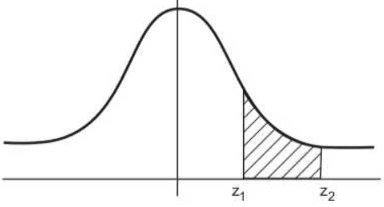

Fig. 6.7

5) $P(z > z_1) = 0.5 - A(z_1)$

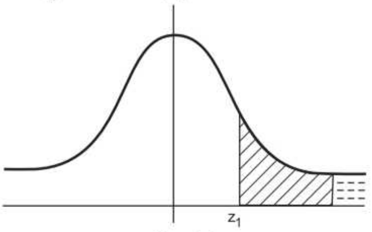

Fig. 6.8

6) $P(z < -z_1) = 0.5 - A(z_1)$

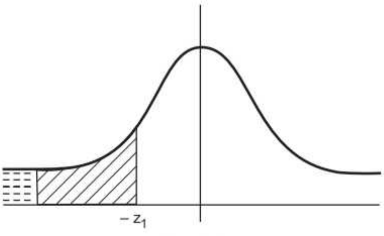

Fig. 6.9

7) $P(-z_1 < z < -z_2) = A(z_1) - A(z_2)$

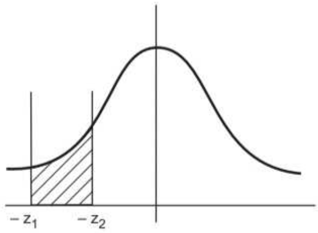

Fig. 6.10

8) $P(-z_1 < z < -z_2) = A(z_1) + A(z_2)$

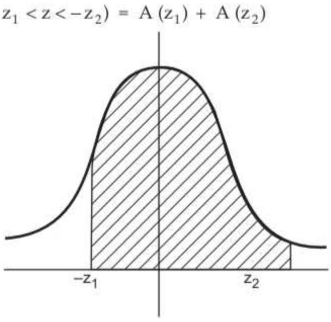

Fig. 6.11

6.22 Illustrations

Example 6.83 : *Assuming that the diagram of 1000 brass plugs taken consecutively from machine from a normal distribution with mean 0.7515 cm and standard deviation 0.0020 cm how many of the plugs are likely to be approved if the acceptable diagram is 0.752 ± 0.004 cm.*

SPPU : May-16

Solution : $\sigma = 0.0020, \ \mu = 0.7515$

$$x_1 = 0.752 + 0.004 = 0.756$$

$$x_2 = 0.752 - 0.004 = 0.748$$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{0.756 - 0.7515}{0.0020} = 2.25$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{0.748 - 0.7515}{0.0020} = -1.75$$

A_1 corresponding to $z_1 = 2.25 = 0.4878$

A_2 corresponding to $z_2 = 1.75 = 0.4599$

$P(0.748 < x < 0.756) = 0.4878 + 0.4599 = 0.9477$

Number of plugs likely to be approved

$$= 1000 \times 0.9477 = 948 \text{ approximately}$$

Example 6.84 : *The mean weight of 500 students is 63 kg and the standard deviation is 8 kgs. Assuming that the weights are normally distributed find how many student weight 52 kgs. The weights are recorded to the nearest kg.*

Solution : The frequency curve for the given distribution is since the weights are recorded to the nearest kg., the students weighing 52 kgs. have their actual weights between x = 51.5 and 52.5 kg. So the area under the curve (1) from $x_1 = 51.5$ to $x_2 = 52.5$ is to be obtained. As $z = \frac{x-\mu}{\sigma}$

$$\therefore \quad z_1 = \frac{x_1 - 63}{8} = z_1 = \frac{51.5 - 63}{8} = -1.4375$$

$$\text{and } z_2 = \frac{x_2 - 63}{8} = z_2 = \frac{52.5 - 63}{8} = -1.3125$$

The number of student weighing 52 kg.

$$= 500(A_1 - A_2)$$

where A_1 is area under z = 1.4315

A_2 is area under z = 1.3125

$\therefore$ Number of students = 500 (0.4326 – 0.4049)

= 9 students approximately

Example 6.85 : *In a certain examination test 200 students appeared in subject of statistics. Average marks obtained were 50 % with standard deviation 5 %. How many students do you expect to obtain more than 60 % of marks, supposing that marks are distributed normally.*

SPPU : Dec.-17,18,19

Solution : $\mu = 0.5, \ \sigma = 0.05, \ x_1 = 0.6$

$$z_1 = \frac{0.6 - 0.5}{0.05} = 2$$

A corresponding to z = 2 is 0.4772

$$P(x \geq 6) = 0.5 - 0.4772 = 0.0228$$

Number of students expected to get more than 60 % marks

$$= 0.0228 \times 200$$

$$= 46 \text{ students approximately.}$$

Example 6.86 : *In a distribution exactly normal 7 % of the items are under 35 and 89 % are under 63. Find the mean and standard deviation of the distribution.*

Solution : From Fig. 6.12 it is clear that 7 % of items are under 35 means area under 35 is 0.07. Similarly area for $x \geq 63$ is 0.11.

$P(x < 35) = 0.07, \ P(x > 63) = 0.11$

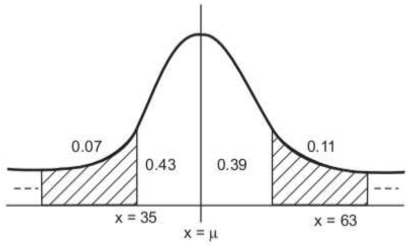

Fig. 6.12

$x = 35$, $x = 63$ are located as shown in Fig. 6.12

When $\quad x = 35,\ z = \dfrac{35-\mu}{\sigma} = -z_1$ (say)

(Negative sign because $x = 35$ is to the left of $x = \mu$)

When $x = 63,\ z = \dfrac{63-\mu}{\sigma} = z_2$ (say)

Area A_2 for $P(0 < z < z_2) = 0.39$

$\therefore$ By Table 6.1, $z_2 = 1.23$

Area A_1 for $P(0 < z < z_1) = 0.43$

Corresponding $z_1 = 1.48$

Thus we get two simultaneous equation

$$\frac{35-\mu}{\sigma} = -1.48 \qquad \text{... (1)}$$

$$\frac{63-\mu}{\sigma} = 1.23 \qquad \text{... (2)}$$

Solving these two equations we get.

$\therefore \quad \sigma = 10.33 \quad \mu = 50.3$ (approx)

Example 6.87 : *In a normal distribution 31 % of the items are under 45 and 8 % are over 64. Find mean and standard deviation of the distribution.*

Solution : Let μ and σ be the mean and standard deviation given, for $x = 45$, $z_1 = 31\ \% = 0.31$

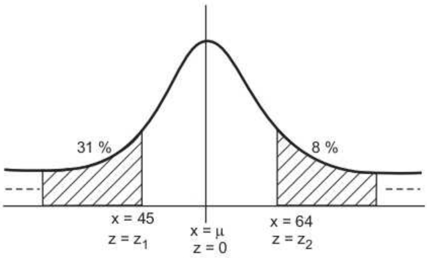

Fig. 6.13

i.e. area to the left of ordinate $x = 45$ is 0.31

$\therefore \quad P(z_1 < z < 0) = 0.5 - 0.31 = 0.19$

The value of z corresponding to this area is 0.5

$$z_1 = -0.5\ (z < 0)$$

for $\quad x = 64,\ z_2 = 8\ \% = 0.08$

$\therefore \quad P(0 < z < z_2) = 0.5 - 0.08 = 0.452$

Value of z corresponding to this is 1.4

$$-0.5 = \frac{45-\mu}{\sigma}$$

$$1.4 = \frac{64-\mu}{\sigma}$$

Solving we get $\quad \mu = 50,\ \sigma = 10$

Example 6.88 : *In a sample of 1000 cases, the mean of a certain test is '14' and standard deviation is 2.5. Assuming the distribution to be normal find ;*
i) How many students score between 12 and 15 ?
ii) How many score above 18 ?
iii) How many score below 8 ?
iv) How may score 16 ? **SPPU : May-19**

Solution : Here $\quad n = 1000,\ \mu$ (or $\bar{x}$) $= 14,\ \sigma = 2.5$

$$\therefore \quad z = \left(\text{or } \frac{x}{\sigma}\right) = \frac{x-\mu}{\sigma} = \frac{x-14}{2.5}$$

i) When $x = 12$ then $z_1 = \dfrac{12-14}{2.5} = -0.8$

and where $x = 15$

$$z_2 = \frac{15-14}{2.5} = 0.4$$

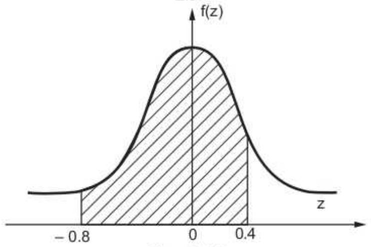

Fig. 6.14

$\therefore A_1 \rightarrow$ Area corresponding to

$(z_1 = 0.8) = 0.2881$ (Refer Table 6.1 for $z = \dfrac{x}{\sigma}$)

$A_2 \rightarrow$ Area corresponding to

$(z_2 = 0.4) = 0.1554$

Required probability $p\ (12 < x < 15) = A_1 + A_2$

$= 0.2881 + 0.1555 = 0.4435$

∴ Required number of students = 1000×0.4435

$\cong 443$

ii) When x = 18, $z = \frac{18 - 14}{2.5} = 1.6$

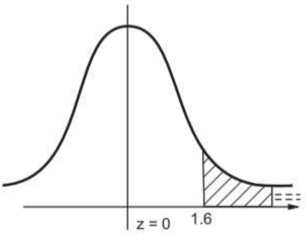

Fig. 6.15

∴ Probability P (x > 18) = Area right to z = 1.6 (i.e. $z \geq 1.6$)

$= P(0 < z < \infty) - P(0 < z(1.6))$

$= 0.5 - 0.4452 = 0.0548$

[∵ The total area from the mean to the right is 0.5]

∴ Required number of students = $1000 \times 0.0548 = 54.8$

$\cong 55$

iii) When x = 8, $z = \frac{8 - 14}{2.5} = -2.4$

∴ Probability P (x < 8) = Area left to

$z = -2.4$ (i.e. $z \leq -2.4$)

$= P(0 < z < -\infty) - P(0 < z < 2.4)$

$= 0.5 - 0.4918 = 0.0082$

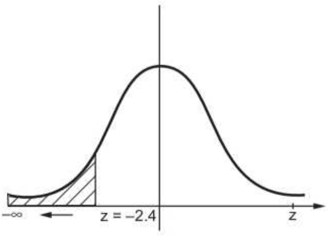

Fig. 6.16

(∵ The total area from the mean to the left is 0.5)

∴ Required number of students = $1000 \times 0.0082 = 8.2 \cong 8$

iv) 16 means between 15.5 and 16.5

∴ for x = 15.5, $z_1 = \frac{15.5 - 14}{2.5} = 0.6$

for x = 16.5, $z_2 = \frac{16.5 - 14}{2.5} = 1$

Required probability P (15.5 < x 16.5)

$= P(0 < z_2 < 1) - P(0 < z_1 < 0.6)$

$= 0.3413 - 0.2257 = 0.1156$

Required number of students = 1000×0.1156

$= 115.6 \cong 116$

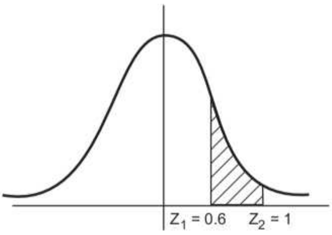

Fig. 6.17

Example 6.89 : *A sample of 100 dry battery cells tested to find the length of life produced the following results.*

Mean $\mu = 12$ hours, Standard deviation $\sigma = 3$ hours.

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life ?

i) More than 15 hours ii) Less than 6 hours

iii) Between 10 and 14 hours ?

Solution : Here X denotes length of life of dry battery cells.

Also $z = \frac{x - \mu}{\sigma} = \frac{x - 12}{3}$

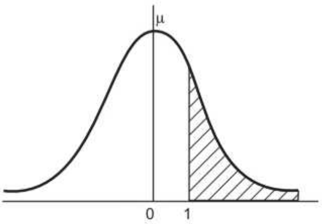

Fig. 6.18

i) Here x = 15,

∴ $Z = \frac{15 - 12}{3} = 1$

$\therefore \quad P(x > 15) = P(z > 1)$

$= P(0 < z < \infty) - P(0 < z < 1)$

$= 0.5 - 0.3413 = 0.1587 = 15.87\ \%$

ii) For $x = 6$, $z = \dfrac{6 - 12}{3} = -2$

$\therefore \quad (x < 6) = (z < -2) = P(z > 2)$

$= P(0 < z < \infty) - P(0 < z < z)$

$= 0.5 - 0.4772 = 0.0228 = 2.28\ \%$

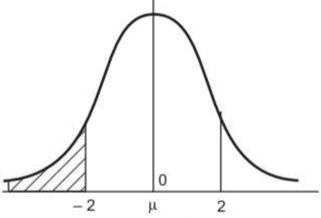

Fig. 6.19

iii) For $x = 10$, $z_1 = \dfrac{10 - 12}{3} = -\dfrac{2}{3} = -0.67$

For $x = 14$, $z_1 = \dfrac{14 - 12}{3} = +\dfrac{2}{3} = 0.67$

$\therefore \ P(10 < x < 14) = P(z_1 < z < z_2)$

$= P(-0.67 < z < 0.67)$

$= 2\,P(0 < z < 0.67)$

$= 2 \times 0.2487 = 0.4974 = 49.74\ \%$

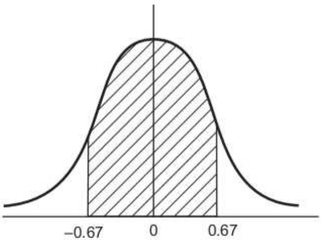

Fig. 6.20

Example 6.90 : *A normal distribution has a mean 15.73 and a standard deviation of 2.08. Find the percentage of cases that fall between 17.81 and 13.65. Find also the % of cases lying above 18.85.*

Solution : Here

$\mu(\text{or } \bar{x}) = 15.73, \quad \sigma = 2.08$

then in first case, when

$x_1 = 13.65$

$z_1 = \dfrac{x - \bar{x}}{\sigma} = \dfrac{13.65 - 15.73}{2.08} = -\dfrac{2.08}{2.08} = -1$

and when $x_2 = 17.81$

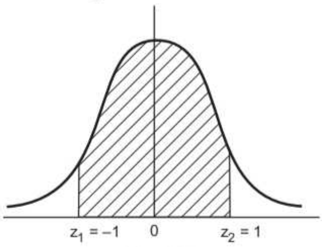

Fig. 6.21

$z_2 = \dfrac{17.81 - 15.73}{2.08} = \dfrac{2.08}{2.08} = 1$

$\therefore$ Required probability $P(13.65 < x < 17.81)$

$= P(0 < z_1 < -1) + P(0 < z_2 < 1)$

$= 0.34134 + 0.34134 = 0.68268$

$\therefore$ Thus required % $= 100 \times 0.68268 = 68.268$

For next part, when $x = 18.85$

$z = \dfrac{18.85 - 15.73}{2.08} = \dfrac{3.12}{2.08} = 1.5$

$\therefore$ Probability $P(z > 1.5)$ = Area of right to

$z = 1.5\ (z \geq 1.5)$

$= P(0 < z < \infty) - P(0 < z < 1.5)$

$= 0.5 - 0.43349 = 0.06651$

[$\because$ For $z = 1.5$, the probability (from Table 8.1) is 0.43349]

[$\because$ The total area from the mean to the right $P(0 < z < \infty)$ is 0.5]

$\therefore$ Percentage of cases above

$18.85 = 100 \times 0.0665 = 6.651$

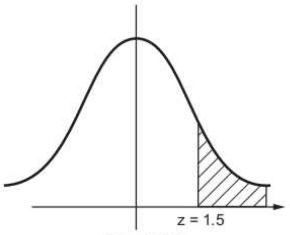

Fig. 6.22

Example 6.91 : *Five thousand students appeared in an examination carrying a maximum of 100 marks. It was found that the marks were normally distributed with mean = 39.5 and standard deviation = 12.5. Determine approximately the number of candidates who secured a first class for which a minimum of 60 marks is necessary. The following table gives deviation from mean and proportion of whole area of normal curve lying to the left of the ordinate at the deviation* $\frac{x}{\sigma}$.

$\frac{x}{\sigma}$	1.5	1.6	1.7	1.8
A	0.93319	0.94520	0.95543	0.96407

Solution : Here $\mu = 39.5$

$\sigma = 12.5$

For x = 60 $\quad z = \dfrac{60 - 39.5}{12.5} = 1.64$

$\therefore$ Required area is for z = 1.64

From table

area for z = 1.6 is 0.94520

area for z = 1.7 is 0.99543

$\therefore$ Difference for 0.1 is 0.01023

$\therefore$ Difference for 0.04 = 0.004092

$\therefore$ Area for z = 1.64 = 0.94520 + 0.004092

= 0.949292

This is total area lying to left of z = 1.64

$\therefore$ Area right to z = 1.64 = 1 – 0.949292

= 0.050708

(As total area under normal curve is 1)

$\therefore$ Required Number of students = 5000 × 0.050708

= 253.540

= 253 students approximately

Example 6.92 : *Of the 1000 students of a college, the mean height is 68 inches, with a standard deviation of 5 inches. Between what limits will the middle 60 % of the heights lie ? Assume the distribution to be normal.*

Solution : The middle 60 % of the height means 30 % on either side of the mean. Now the probability 0.30 will lie between the values 0.8 and 0.9 from $z\left(=\frac{x}{\sigma}\right)$ tables. By interpolation or from accurate tables, we have $z = \left(\frac{x}{\sigma}\right) = 0.8418$ for a probability of 0.30.

$\therefore \quad x = \sigma \times 0.8418 = 5 \times 0.8418 = 4.2090$ inches

And the limits shall be 68 + 4.2090 and 68 – 4.2090 i.e. 72.2090 inches and 63.7910 inches.

Note : Here the given number 1000 has not been used.

Example 6.93 : *For a normal distribution when mean* $\bar{x} = 1$, *S.D. = 3. Find the probabilities for the intervals :*

i) $3.43 \le x \le 6.19$ *ii)* $-1.43 \le x \le 6.19$

Solution : We have $z\left(=\frac{x}{\sigma}\right) = \frac{x - \bar{x}}{\sigma} = \frac{x-1}{3}$

i) When x = 3.43,

$$z_1 = \frac{-1.43 - 1}{3} = -0.81$$

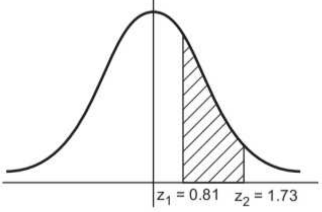

Fig. 6.23

When x = 6.19 $\quad z_2 = \dfrac{6.19 - 1}{3} = 1.73$

Required probability P (– 3.43 ≤ x ≤ 6.19)

$= P(0 < z_1 < -1.73) - P(0 < z_2 < 0.81)$

= 0.4582 + 0.2910 (see table) = 0.1672

ii) When x = – 1.43

$$z_1 = \frac{1.43 - 1}{3} = -0.81$$

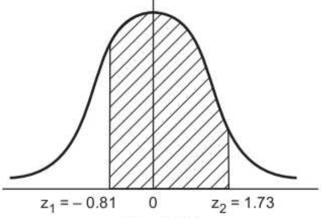

Fig. 6.24

When x = 6.19, $\quad z_2 = \dfrac{6.19 - 1}{3} = 1.73$

Required probability P (– 1.43 ≤ x ≤ 6.19)

$= P(0 < z_1 < -0.81) - P(0 < z_2 < 1.73)$

= 0.2910 + 0.4582 = 0.7492

Example 6.94 : *The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 cm and standard deviation is 0.005 cm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 0.496 to 0.508 cm otherwise washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed.*

Solution : Here $\mu = 0.502, \ \sigma = 0.005$

For x = 0.496 $\quad z_1 = \dfrac{0.496 - 0.502}{0.005} = -1.2$

For x = 0.508 $\quad z_2 = \dfrac{0.508 - 0.502}{0.005} = 1.2$

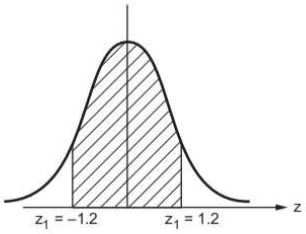

Fig. 6.25

$$\therefore \ P(0.496 < x < 0.508) = P(z_1 < z < z_2)$$
$$= P(-1.2 < z < 1.2)$$
$$= 2\,P\,(0 < z < 1.2) = 2 \times 0.3849$$
$$= 0.7698 = 75.98\ \%$$

∴ Percentage of defective washers

$$= 100 - 76.98 = 23.02\ \%$$

Example 6.95 : *Local authorities in a certain city install 2000 electric bulbs on the streets. If the bulbs have average life of 1000 burning hours with standard deviation of 200 hours. Assuming normality find*

i) What number of bulbs might be expected to fail in first 700 hours.

ii) After what period of burning hours would it be expected that 10 % of bulbs would have failed.

Solution : Here $\mu = 1000, \ \sigma = 200$

For x = 700, $\quad z = \dfrac{700 - 1000}{200} = -1.5$

$$\therefore \quad P\,(x < 700) = P\,(z < -1.5)$$
$$= P\,(z > 1.5)$$

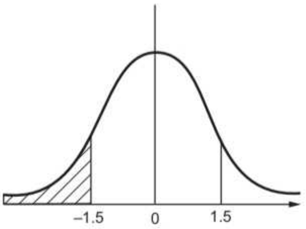

Fig. 6.26

$$= 0.5 - A\,(1.5)$$
$$= 0.5 - 0.4332 = 0.0668$$

Required number of bulbs

$$= 0.0668 \times 2000 = 1336 \text{ bulbs}$$

ii) For $z = 10\ \% = \dfrac{10}{100} = 0.1$ to find x

$$z = \frac{x - \mu}{\sigma}$$

$$\therefore \quad 0.1 = \frac{x - 1000}{200}$$

$$\therefore \quad x = 20 + 1000$$
$$x = 1020 \text{ hours}$$

Example 6.96 : *Fit a normal curve to the following data.*

Length of a line in cm	:	8.60	8.59	8.58	8.57	8.56	8.55	8.54	8.53	8.52
Frequency	:	2	3	4	9	10	8	4	1	1

Solution : Let A = 8.56
Frequency distribution

x_i	f_i	$d_i = x_i - A$	$f_i\,d_i$	$f_i\,d_i^2$
8.60	2	0.04	0.08	0.032
8.59	3	0.03	0.09	0.0027
8.58	4	0.02	0.08	0.0016
8.57	9	0.01	0.09	0.0009
8.56	10	0	0	0
8.55	8	– 0.01	– 0.08	0.0008
8.54	4	– 0.02	– 0.08	0.0016
8.53	1	– 0.03	– 0.03	0.0009
8.52	1	– 0.04	– 0.04	0.0016
	$\sum f_i = 42$		$\sum f_i\,d_i = 0.11$	$\sum f_i\,d_i^2 = 0.0133$

$\therefore$ Mean $\mu = a + \dfrac{\sum f_i d_i}{\sum f_i}$

$= 8.56 + \dfrac{0.11}{42} = 8.56262$

Standard deviation $\sigma = \sqrt{\dfrac{1}{N}\sum f_i d_i^2 - \left(\dfrac{\sum f_i d_i}{N}\right)^2}$

$= \sqrt{\dfrac{0.0133}{42} - \left(\dfrac{0.11}{42}\right)^2}$

$\sigma = 0.0176$

$\therefore$ Equation of normal curve

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty \le x \le \infty$$

where $\mu = 8.56262$, $\sigma = 0.0176$

Example 6.97 : *5000 candidates appeared in a certain paper carrying a maximum of 100 marks. It was found that marks are normally distributed with mean 39.5 and standard deviation 12.5. Determine approximately the number of candidates who secured a first class for which minimum of 60 marks is necessary.*

Solution : Given data $\mu = 39.5$, $\sigma = 12.5$

$z = \dfrac{x-\mu}{\sigma} = \dfrac{x-39.5}{12.5}$

when, $x = 60$, $z = \dfrac{60-39.5}{12.5} = 1.64$

$\therefore$ The required probability,

$= P(x \ge 60)$

$= P(z \ge 1.64)$

$= 0.5 -$ Area of ABCD

$= 0.5 - 0.4495$ (by Table) $= 0.0505$

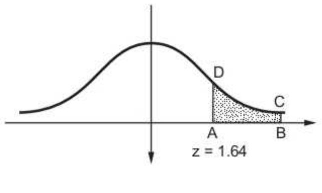

Fig. 6.27

$\therefore$ The number of students obtaining first class

$= 5000 \times 0.0505 \cong 253$ Approximately

Example 6.98 : *A fair coin is tossed 600 times. Using normal distribution. Find the probability of getting : i) Number of heads less than 270 ii) Number of heads between 280 to 360.*

Solution : Let x be the number of heads in 600 tosses,

$\therefore$ $P(H) = \dfrac{1}{2}$, $n = 600$

$\therefore$ $\mu =$ Mean $= np = 600 \times \dfrac{1}{2} = 300$

$\sigma = \sqrt{npq} = \sqrt{150}$

i) P (Number of heads less than 270) $= P(x < 270)$

$= P\left(z < \dfrac{270-300}{\sqrt{150}}\right) = P(z < -2.4495)$

$= P(z > 2.4495)$ ... (Due to symmetry)

$= 0.0071428$

ii) P (Number of heads between 280 and 350)

$= P(280 < x < 350)$

$= P(-1.633 < z < 4.0823)$

$= P(0 < z < 1.6) + P(0 < z < 4.0823)$

$= 1 - P(z > 1.633) - P(z > 4.0823)$

$= 1 - 0.51551 - 0.000022518 = \mathbf{0.4845}$

Example 6.99 : *For a normal distribution of sample size 1000, mean is 15 and S.D. is 2. Find the number of samples for which the normal variable X is such that*

i) $12 \le X \le 15$ ii) $X \ge 20$

Solution : Given $n = 1000$, $\mu = 15$, $\sigma = 2$

i) $P(12 \le X \le 15) = P\left(\dfrac{12-15}{2} \le z \le \dfrac{15-15}{2}\right)$

$= P(-1.5 < z < 0)$

$= P(0 \le z \le 1.5) = 0.4332$

$\therefore$ Required number of students $= 0.4332 \times 1000$

$\cong 433$

ii) $P(X \ge 20) = P\left(\dfrac{X-\mu}{\sigma} > \dfrac{20-15}{2}\right) = P(z \ge 2.5)$

$= 0.5 - P(0 \le z \le 2.5)$

$= 0.5 - 0.4938 = 0.0062$

$\therefore$ Required number of students $= 0.0062 \times 100 \cong 7$

Example 6.100 : *A random sample of 200 screws is drawn from a population which represents the size of screws. If a sample is distributed normally with a mean 3.15 cm of S.D. 0.025 cm. Find the expected number of screws whose size falls between 3.12 cm and 3.2 cm.*

Solution : Given that $\mu = 3.15$, $\sigma = 0.025$

$$X_1 = 3.12 \text{ and } X_2 = 3.2$$

$$z_1 = \frac{X_1 - \mu}{\sigma} = -1.2, \quad z_2 = \frac{X_2 - \mu}{\sigma} = 2$$

$$\therefore P(3.12 < X < 3.2) = P(-1.2 < z < 2)$$

$$= P(0 < z < 1.2) + P(0 < z < 2)$$

$$= 0.3849 + 0.4772 = 0.8621$$

$\therefore$ Expected number of screws $= 200 \times 0.8621 \cong$ **172**

Example 6.101 : *The average test marks in a particular class is 59 and S.D. is 9. If the marks are normally distributed, how many students in a class of 70 received marks below 50 ? More than 70 ?*

Solution : Given that $\mu = 59$, $\sigma = 9$

$$\text{i) } P(X < 50) = P\left(\frac{X-\mu}{\sigma} < \frac{50-59}{9}\right)$$

$$= P(z < 1) = P(0 < z < 1) = 0.3413$$

$\therefore$ The required number of students

$$= 0.3413 \times 70 \cong 24$$

$$\text{ii) } P(X > 70) = P\left(z > \frac{70-59}{9}\right)$$

$$= P\left(z > \frac{11}{9}\right) = P(z > 1.22)$$

$$= 0.5 - P(0 < z < 1.22)$$

$$= 0.5 - 0.3888 = 0.1112$$

$\therefore$ The required number of students $= 70 \text{ X} \cong$ **8**

Example 6.102 : *Suppose heights of students follows normal distribution with mean 190 cm and variance 80 cm^2. In a school of 1000 students, how many would you expect to be above 200 cm tall ? (Given that : $A_1(z > 1.1180) = 0.13136$).*

SPPU : Dec.-16, Marks 4

Solution : Given that Mean $= \mu = 190$ cm and Variance $= 80 \text{ cm}^2$ $\therefore$ S.D. $= \sqrt{80}$

Let x denote the height of student

and
$$Z = \frac{x-\mu}{\sigma} = \frac{x-190}{\sqrt{80}}$$

For $x = 200$, $z = \frac{200-190}{\sqrt{80}} = \frac{10}{\sqrt{80}} = 1.1180$

$$\therefore \quad p(x > 200) = p(z > 1.1180) = 0.13136$$

Thus $1000 \times 0.13136 = 131.6 \cong 132$ students

This is the required answer.

Example 6.103 : *The lifetime of a certain component has a normal distribution with mean of 400 hours and standard deviation of 50 hours. Assuming a normal sample of 1000 components whose lifetime lies between 340 to 465 hours. Given : Z = 1.2 Area = 0.3849, Z = 1.3 Area = 0.4032*

SPPU : May-17, Marks 4

Solution : Given that $\mu = 400$, $\sigma = 50$

$$n = 1000, \quad z = \frac{x-\mu}{\sigma} = \frac{x-400}{50}$$

When $x_1 = 340$, $z_1 = \frac{x_1 - 400}{50} = -\frac{60}{50} = -\frac{6}{5} = -1.2$

When $x_2 = 465$, $z_2 = \frac{x_2 - 400}{50} = \frac{65}{50} = 1.3$

$$\therefore \quad P(340 < x < 465) = P(-1.2 < z < 1.3)$$

$$= P(0 < z < 1.2) + P(0 < z < 1.3)$$

$$= 0.3849 + 0.4032$$

$$= 0.7881$$

$\therefore$ The number of components whose lifetime lie between 340 to 465 is $1000 \times 0.7881 = 788$.

Example 6.104 : *The average test marks in a particular class is 79 and standard deviation is 5. If the marks are normally distributed, how many students in a class of 200, did not receive marks between 75 and 82. Given z = 0.8, Area = 0.2881 and z = 0.6, Area = 0.2257.*

SPPU : Dec.-17, Marks 4

Solution : Given that $\mu = 79$ and $\sigma = 5$

$$P(75 < x < 82) = P\left(\frac{75-79}{5} < \frac{x-\mu}{\sigma} < \frac{82-79}{5}\right)$$

$$= P(-0.8 < z < 0.6)$$

$$= P(0 < z < 0.8) + P(0 < z < 0.6)$$

$$= 0.2881 + 0.2257 = 0.5138$$

$\therefore$ The number of students in a class receive marks between 75 and 82 is

$$= 200 \times 0.5138 = 102.76 \cong 103$$

Hence the number of students in a class not receiving marks between 75 and 82 is $200 - 103 = 97$ students.

Exercise 6.2

Problems on Binomial Distribution

1. *The probability that a man aged 60 years will live for 70 years is $\frac{1}{10}$. Find the probability that out of 5 men selected at random 2 will live for 70 years.*

   Hint : $p = \frac{1}{10}; q = \frac{9}{10}, n = 5, r = 3$, then use "$C_r (p)^r (q)^{n-r}$

   [Ans. : 0.054675]

2. *During war, one ship out of nine was sunk on an average in making a certain voyage. What was the probability that exactly '3' out of a convoy of 6 ships would arrive safely ?*

   [Ans. : $\frac{10240}{9^6}$]

3. *Eight dice are rolled. Calling a '5' or a '6' as success, find the probability of getting :*

   i) 3 successes **[Ans. : $\frac{8!}{5!3!}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^5$]**

   ii) At most 3 successes **[Ans. : $\frac{1}{3^3}\left[2^8 + 8(2)^7 + 28(2)^6 + 56(2)^5\right]$]**

4. *A bag contains 10 balls, each marked with one of the numbers 0 to 9. If four balls are drawn from the bag. Find the probability that none is marked '0'.* **[Ans. : $\left(\frac{9}{10}\right)^4$]**

5. *Following is the data for number of seeds germinating out of 10 on damp filter for 80 sets of seeds. Fit a binomial distribution to this data :*

x	0	1	2	3	4	5	6	7	8	9	10
f	6	20	28	12	8	6	0	0	0	0	0

   [Ans. : Here n = 10, N = 80 ∴ Mean nP = 2.175]

6. *The incidence of occupational in an industry is such that the workers have a 20 % chance of suffering from it. What is the probability that out of 6 workers chosen of random, four or more will suffer from the disease ?* **[Ans. : $\frac{53}{3125}$]**

7. *A manufacturer knows that the condensers he makes contain on an average 1 % of defective. He packs them in boxes of 100. What is the probability that a box picked at random will contain 4 or more faulty condensers ?* **[Ans. : 0.019]**

8. *If the probability that a man aged 70 will live to be 80 is 0.65. What is the probability that out of 10 men now 70, at least 7 will live to be 80 ?* **[Ans. : 0.514]**

9. *10 coins are thrown simultaneously. Find the probability that*

   i) Exactly 3 Heads will appear.

   ii) Three or less Heads will appear.

   [Ans. : i) 0.1172, ii) 0.1718]

10. *Fit a Binomial distribution to the following data.*

x	0	1	2	3	4	5
f	2	22	63	76	96	56

   [Ans. : $\left[315\,(0.34 + 0.66)^5\right]$]

11. *According to past record of one day internationals between India and Pakistan, India has won 15 matches and lost 10. If they decide to play a series of 6 matches now, what is the probability of India winning the series ? (Draw is rules out).* **[Ans. : 0.5443]**

12. *Two dices are thrown 100 times and the number of nines recorded. What is the probability that 'r' nines occur ? Find the probability that at least 3 nines occur.*

   [Ans. : 0.00045]

13. *Probability of man now aged 60 years will live upto 70 years of age is 0.65. Find the probability of out of 10 men sixty years old 6 or mote will live upto the age of 70 years.*

   [Ans. : 0.2377]

14. *In sampling the large numbers of parts manufactured by a machine, the mean number of defective in a sample of 20 is 2 out of 1000 such samples. How many would be expected to contain at least 3 defective parts ?* **[Ans. : 1]**

15. *A manufacturing company claims 90 % assurance that the capacitors manufactured by them will show a tolerance of better than 5 %. The capacitors are packed in lots of 10. Show that 26 % of the customers ought to complain that capacitors do not reach the specified standard.*

Problems on Poisson distribution

16. *Fit a Poisson distribution to the following :*

x	0	1	2	3	4
f	192	100	24	3	1

Hint : *Mean* $z = \frac{\sum fx}{\sum f} = \frac{|6|}{320} = 0.5$ *(Approximately)*

Then use $p(r) = \frac{e^{-z} \cdot z^r}{r!}$

[Ans. : $\frac{e^{-0.5}\,(0.5)^r}{r!}$**]**

17. *Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well shuffled cards at least once in 104 consecutive trials.*

Hint : $p = \frac{1}{52}$ *(Probability of ace of spades),* $n = 104$

$\therefore\ z = np = 2$, *use* $p(r) = \frac{e^{-z} \cdot x^r}{r!} = \frac{1}{e^2} \cdot \frac{(2)^r}{r!}$

P *(at least once)* $= P(1) + P(2) + \ldots + P(104)$

$= 1 - P(0) = 1 - \frac{1}{e^2} = 0.864$

18. *Fit a Poisson distribution to the following frequency distribution and compare the theoretical frequencies with observed frequencies.*

x	0	1	2	3	4	5
f	158	160	60	25	10	2

19. *A source of liquid is known to contain bacteria with the mean number of bacteria per cubic centimeter equal to 2. Five 1 c.c. test tubes are filled with the liquid, assuming that Poisson distribution is applicable, calculate the probability that all test tubes show growth.* **[Ans. : 0.036]**

20. *An insurance company found that only 0.01 % of the population is involved in a certain type of accident each year. If 1000 policy holders are randomly selected from the population. What is the probability that not more than two of its clients are involved in such an accident next year.*

[Ans. : 0.9998]

21. *A book of 600 pages contains 40 printing mistakes. Assuming that errors are randomly distributed throughout the book and x, the number of errors per page has a Poisson distribution what is the probability that 10 pages selected at random will be free of errors ?* **[Ans. : 0.51]**

22. *The probability that a man aged 35 years will die before reaching the age of 40 years may be taken as 0.018. Out of a group of 400 men now aged 35 years, what is the probability that 2 men will die within next 5 years ?* **[Ans. : 0.01936]**

23. *Fit a Poisson distribution to the following data :*

Number of deaths recorded in a day	0	1	2	3	4	5	6	7
Number of days	364	376	218	89	33	13	2	1

24. *If 3 % of the electric bulbs manufactured by a company are defective, find the probability that in a sample of 100 bulbs :*

i) 0 **[Ans. : 0.05]**

ii) 1 **[Ans. : 0.149]**

iii) 2 **[Ans. : 0.224]**

iv) 3 **[Ans. : 0.224]**

v) 4 and **[Ans. : 0.168]**

vi) 5 bulbs will be defective **[Ans. : 0.101]**

25. *In a Telephone exchange, the probability that any one call is wrongly connected is 0.02. What is the minimum number if calls required to ensure a probability 0.1 that at least one call is wrongly connected ?*
[Ans. : 6 calls approximately]

26. *A manufacturer of electronic goods has 4 % of his product defective. He sells the articles in packets of 300 and guarantees 90 % good quality. Determine the probability that a particular packet will violate the guarantee.*

[Ans. : $1 - \sum_{r=1}^{r} \frac{e^{-12}\,(12)^r}{r!}$**]**

27. *A manufacturer knows that the condensers he makes contain on the average 1 % of defective. He packs them in boxes of '100'. What is the probability that a box picked at random will contain 4 or more faulty condensers ?*

Hint : $P = 1\ \% = 0.01 = 100\ \therefore\ z = nP = 1$

$\therefore\ P(r) = \frac{e^{-z} \cdot z^r}{r!} = \frac{e^{-1}(1)^r}{r!} = \frac{e^{-1}}{r!}$

$\therefore$ P*(4 or more faulty condensers)* $= P(4) + P(5) + \ldots + P(100) = 1 - [P(0) + P(1) + P(2) + P(3)] = 0.019$

28. *A source of water is known to contain bacteria with mean number of bacteria per cc equal to 2. Five 1 cc test tubes were filled with the water. Assuming that Poisson distribution is applicable, calculate the probability that exactly 2 test tubes contain at least 1 bacterium each.*

[Ans. : 0.1083]

29. *If the probability that an individual suffers a bad reaction from a certain injection is 0.001 determine the probability that out of 2000 individuals : i) exactly 3 ii) more than 2 individuals will suffer a bad reaction.*

Hint : $P = 0.001,\ n = 2000\ \therefore\ z = np = 2$

$$P(r) = \frac{e^{-z} z^r}{r!} = \frac{1}{e^2}\frac{2^r}{r!}$$

$$\text{i) } P(3) = \frac{2^3}{e^2}\cdot\frac{1}{3!} = 0.18$$

$$\text{ii) } P(\text{more than } 2) = P(3) + P(4) + \ldots + P(2000)$$

$$= 1 - [P(0)\ P(1) + P(2)] = 0.32$$

30. *Suppose that book of 600 pages contains 40 printing mistakes, assume that these errors and the number of errors per page has a Poisson distribution. What is the probability that 10 pages selected at random will be free of errors ?*

Hint : $P = \frac{40}{600} = \frac{1}{15},\ z = np = 10 \times \frac{1}{15} = \frac{2}{3}$ *Find P(0)*

[Ans. : 0.51]

31. *Using Poisson's distribution, find the probability that the ace of spades will be drawn from a pack of well shaffled cards at least once in 104 consecutive trials.* **[Ans. : 0.864]**

32. *Out of 1000 balls, 50 are red and the rest white. If 60 balls are picked at random, what is the probability of picking up i) 3 red balls ii) Not more than 3 red balls in the sample.*

[Ans. : i) 0.2241 and ii) 0.6474]

33. *Between 2 p.m. and 3 p.m. the average number of phone calls per minute coming into the company are 2. Find the probability that during one particular minute, there will be*

i) No phone calls at all.

ii) 2 or less calls. **[Ans. : i) 0.1353; 6/65]**

34. *Prove that the following data represents Poisson distribution*

x	0	1	2	3	4
y	109	65	22	3	1

[Ans. : Mean = Variance = 0.61]

35. *An insurance company found that only 0.01 % of the population is involved in a certain type of accident each year. If its 100 policy holders randomly selected from the population, what is the probability that not more than two of its clients are involved in such an accidents next year ?*

Hint : $P = 0.01\ \% = \frac{1}{100} \times \frac{1}{100} = \frac{1}{10{,}000},\ n = 100$

$$\therefore\ z = nP = 01.$$

$$P(r) = \frac{e^{-z}\cdot z^r}{r!}$$

P (not more than 2) = P(0) + P(1) + P(2) = 0.9998

36. *The accidents per shift in a factory is given by the table :*

Accidents 'x' per shift	0	1	2	3	4	5
Frequency	142	158	67	27	5	1

Fit a Poisson distribution to the above and calculate theoretical frequencies.

Hint : $z = \frac{\sum f(x)}{\sum f} = 0.995$

$$P(r) = \frac{e^{-z}(z)^r}{r!}$$

Problems on Normal Distribution

37. *An aptitude test for selecting officers in a bank is conducted on 1000 candidates. The average score is 42 and the standard deviation of score is 24. Assuming normal distribution for the scores, find :*

i) The number of candidates whose scores exceed 60. **[Ans. : 252]**

ii) The number of candidates whose scores lie between 30 and 60. **[Ans. : 533]**

38. *The mean weight of 500 male students in a certain college is 151 lbs and the standard deviation is 15 lbs. Assuming the weights are normally distributed, in how many students weight between 120 and 155 lbs ?* **[Ans. : 300]**

39. *For a normal distribution, N = 300, μ = 75 and σ = 15. How many values lie between x = 60 and x = 70 ?*

The area under the normal curve for various values of 2 is given as,

z	Area
0.33	0.12930
0.34	0.13307
1.0	0.34134

[Ans. : 63 approximately]

40. *In a normal distribution N = 700, mean $\bar{x}$ = 95, σ = 15. How many values lie between 80 and 90 ?* **[Ans. : 149]**

41. *Assume that the weights of 1000 school children follow a normal law. The mean weight is 48 lbs, and the standard deviation is 6 lbs. Find the number of children having their weight :*

    i) Between 36 lbs and 42 lbs **[Ans. : 136]**

    ii) Between 42 lbs and 57 lbs **[Ans. : 624]**

    iii) Less than 30 lbs **[Ans. : 1]**

    iv) More than 60 lbs **[Ans. : 23]**

42. *A random sample 200 screws is drawn from a population which represents the size of screws. If a sample is distributed normally with a mean 3.15 cm and standard deviation 0.025 cm, find expected number of screws whose size falls between 3.12 cm and 3.2 cms.*

    Area corresponding to 1.2 $\to$ 0.3849

    Area corresponding to 2.0 $\to$ 0.4772
    [Ans. : 172 Approximately]

43. *In a certain examination, the percentage of passes and distinction were 48 and 10 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively.*
    [Ans. : 38.5772]

44. *5000 candidates appeared in a certain paper carrying a maximum of 100 marks. It was found that marks were normally distributed with mean 39.5 and standard deviation 12.5. Determine approximately the number of candidates who secured a first class for which a minimum of 60 marks is necessary.* **[Ans. : 253]**

45. *Obtain the equation of normal curve that may be fitted to the following distributed.*

x	50	60	70	80	90	100
f	5	20	120	250	240	5

Also obtain expected normal frequencies.

46. *X is normally distributed and the mean of X is 15 and standard deviation 3. Determine the probability of*

    i) $0 < X < 10$; ii) $X \geq 18$ **[Ans. : i) 0.10483, ii) 0.1587]**

47. *Assuming the resistance of the resistors, to be normal with mean 100 ohms and standard deviation 2 ohms, what percentage of resistors will have resistance between 987 ohms and 102 ohms if given that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1} e^{-z^2/2}\,dz = 0.8413$*

    [Ans. : P ($-1 < z < 1$) = 0.6826)]

48. *In a certain city, 2000 electric lamps are installed. If the lamps have average life of 1000 burning hours with standard deviation of 200 hours,*

    i) What number of lamps might be expected to fail in first 700 burning hours ?

    ii) After what period of burning hours, 10 % of lamps would still be burning ?

    Given that if $F(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}z^2}\,dz$

    then F (1.5) = 0.933

    and F (1.28) = 0.900 **[Ans. : i) 866, ii) 1256]**

6.23 Chi-square Distribution

Chi-square distribution is used mainly in 1) Testing hypothesis 2) Testing independence of attributes 3) Testing the goodness of fit of a model.

The Chi-square distribution variable or simply Chi-square variable is denoted by X_n^2, Here n is the parameter of the distribution also called as "n degrees of freedom".

6.24 Definition of χ_n^2

The variate χ_n^2 is defined as the sum of squares of n independent standard normal variables [N (0, 1)] i.e. If $X_1, X_2 \ldots X_n$ be n independent N (0, 1) variables then the Chi-square distribution is given by

$$\boxed{Y = \chi_n^2 = \sum_{i=1}^{n} (X_i)^2}$$

where n = Degrees of freedom.

6.25 Additive Property of Chi-square Distribution

If Y_1 and Y_2 are independent Chi-square variates with n_1 and n_2 degrees of freedom respectively, then $Y_1 + Y_2$ has chi-square distribution with $(n_1 + n_2)$ degrees of freedom.

6.26 Definition of Hypothesis

Hypothesis is the statement or assertion about the statistical distribution or unknown parameter of statistical distribution.

Hypothesis is a claim to be tested about the population parameters such as mean variance. For e.g. : 1) Proportion of fail students in two subjects in any college 2) Comparison of F.E. results in any two or more engineering colleges. These claims stated in terms of population parameters or distribution are called hypothesis.

6.27 Null Hypothesis

According to the famous statistician R.A. Fisher null hypothesis is denoted by H_0 and is defined as a hypothesis of "no difference".

For e.g. : 1) If there is no difference between the failure percentage of two subjects then $H_0 : \mu_1 = \mu_2$ i.e. there is no difference between two population means.

6.28 Alternative Hypothesis

Alternative hypothesis is denoted by H_1 and is defined as a complementary hypothesis to null hypothesis.

If null hypothesis is rejected then alternative hypothesis is accepted.

For e.g. if $H_0 : \mu_1 = \mu_2$ is the null hypothesis then the alternative hypothesis may be

$H_1 : \mu_1 \neq \mu_2$

or $H_1 : \mu_1 < \mu_2$

or $H_1 : \mu_1 > \mu_2$

6.29 One Sided or Two Sided Hypothesis OR (One Tailed or Two Tailed Hypothesis)

OR $H_1 : \mu_1 < \mu_2$

If $H_1 : \mu_1 > \mu_2$ or $H_1 : \mu_1 < \mu_2$ then there is only one possibility in which such a hypothesis is true. Such a hypothesis is called as one sided hypothesis.

If $H_1 : \mu_1 \neq \mu_2$ then there are two possibilities in which such a hypothesis is true i.e. $(\mu_1 < \mu_2$ OR $\mu_1 > \mu_2)$ such a hypothesis is called as two sided hypothesis.

6.30 Errors

There are two types of errors defined in testing of hypothesis

Type I error : When H_0 is true and it is rejected

Type II error : When H_0 is false and it is accepted.

For e.g. : Consider Null hypothesis regarding inspection of a Job.

H_0 : Job "A" is a good job.

In reality if job 'A' is a good job i.e. H_0 is true, but if the hypothesis is rejected i.e. if the job inspector says that it is a bad job then there is a mistake, such a mistake is called as Type - I error.

Again consider a null hypothesis

H_0 : Job "B" is a bad job.

And in reality if job 'B' is a good job then H_0 is false, but if the hypothesis is accepted i.e. if the job inspector says that it is a bad job then there is a mistake, such a mistake is called as Type II error.

The following table will clear the idea.

Actual situation	H_0 is Rejected	H_0 is Accepted
H_0 is true (H_1 is false)	Type I error	Correct decision
H_0 is false (H_1 is true)	Correct decision	Type II error

6.31 Some Important Definitions

a) Test of hypothesis :

It is rule which leads to the decision of acceptance or rejection of H_0 on the basis of random samples.

b) Test statistic :

A function of random sample observations which is used to test the null hypothesis H_0.

c) One sided or two sided tests :

The tests used for testing null hypothesis (H_0) are called one sided or two sided tests according as the corresponding alternative hypothesis (H_1) are one sided or two sided hypothesis.

d) Critical region :

The set of values of $x_1, x_2 \ldots x_n$ for which the null hypothesis H_0 is rejected is called Critical region. (or rejection region) critical region is denoted by W and the acceptance region is denoted by (W^c).

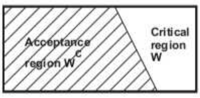

Fig. 6.28

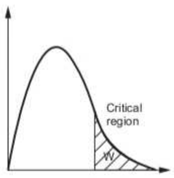

Fig. 6.29

e) Level of significance :

Probability of rejecting the null hypothesis H_0 when it is true is called as level of significance. It is denoted by α. Thus it is the probability of committing error of Type I.

If we try to minimize level of significance, the probability of error of Type II increases.

So level of significance cannot be made zero. However we can fix it in advance as 0.01 or 0.05 i.e. (1 % or 5 %). In most of the cases it is 5!.

Degrees of freedom	Distribution of χ^2	
	5 %	1 %
1	3.841	6.635
2	5.991	9.210
3	7.815	11.345
4	9.488	13.277
5	11.070	15.086
6	15.592	16.812
7	15.067	18.475
8	15.507	20.090
9	16.919	21.666
10	18.307	23.209
11	19.675	24.725
12	21.026	26.217
13	22.362	27.668
14	23.685	29.141
15	24.996	30.578
16	26.296	32.000
17	27.587	33.409
18	28.869	34.191
19	30.144	36.191
20	31.410	37.566
21	32.671	38.932
22	33.924	40.289
23	35.172	41.638
24	36.415	42.980
25	37.652	44.314
26	38.885	45.642
27	40.113	46.963
28	41.337	48.278
29	42.557	49.588
30	43.773	50.892
40	55.759	63.691
60	79.082	88.379
∞	–	–

Table 6.2

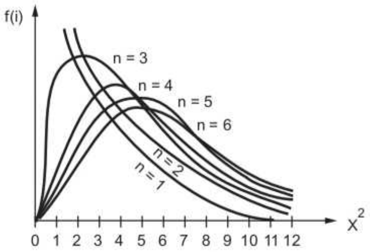

Fig. 6.30

f) Test for goodness of fit of χ^2 distribution :

Consider a frequency distribution, we try to fit some probability distribution.

Let H_0 : Fitting of the probability distribution to given data is proper.

As the test is based on χ^2 distribution. $\therefore$ known as χ^2 test of goodness of fit.

Suppose O_1 O_2 .... O_i .... O_k be the observed frequencies and e_1 e_2 ... e_i ... e_k be the expected frequencies or theoretical frequencies. There is no significant difference between observed and theoretical (expected) frequencies.

Let P = Number of parameters estimated for fitting the probability distribution.

$$N = \sum_{i=1}^{k} O_i = \sum_{i=k}^{k} e_i$$

If H_0 is true then the static

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$= \sum_{i=1}^{k} \frac{O_i^2 - 2 e_i O_i + e_i^2}{e_i}$$

$$= \sum_{i=1}^{k} \frac{O_i^2}{e_i} - 2 O_i + e_i$$

$$= \sum_{i=1}^{k} \frac{O_i^2}{e_i} - \sum_{i=1}^{k} 2 O_i + \sum_{i=1}^{k} e_i$$

$$= \sum_{i=1}^{k} \frac{O_i^2}{e_i} - 2N + N$$

$$\chi^2 = \sum_{i=1}^{k} \frac{O_i^2}{e_i} - N$$

has Chi-square distribution with (k–p–1) degrees of freedom. If

$$\underset{\text{(calculated)}}{\chi^2_{k-p-1}} \geq \underset{\text{(expected or table value)}}{\chi^2_{k-p-1,\ \alpha}}$$

Thus we reject H_0

Note :

1) If any parameters are not estimated while fitting a probability distribution or in finding theoretical frequencies (expected frequencies) then the value of p is zero.
2) We apply this test if the total cell frequencies are sufficiently large (greater than 50) and if expected frequencies are greater than or equal to 5 ($e_i \geq 5$).
3) When expected frequency of a class is less than 5 the class is merged into neighbouring class along with its observed and expected frequencies to get total of expected frequencies becomes ≥ 5. This procedure is called "pooling the class". In this case k is number of class frequencies after pooling.

6.32 Illustrations

Example 6.105 : *A die is tossed 60 times and frequency of each face is indicated below*

Face x_i	1	2	3	4	5	6
Frequency f_i	5	7	5	14	13	16

Assume that the die is fair and apply Chi-square test of goodness of fit at 0.05 level of significance.

Solution : Let H_0 : The die is fair.

If the die is fair then $P_i = \frac{1}{6}$

$$\therefore \quad E(x) = \text{Expected frequency}$$

$$= N \cdot P_i$$

$$= (60)\left(\frac{1}{6}\right) = 10$$

x_i	f_i (or O_i) or observed values	Expected frequency (e_i)	$O_i - e_i$
1	5	10	– 5
2	7	10	– 3
3	5	10	– 5
4	14	10	4
5	13	10	3
6	16	10	6

$$\chi^2_{k-p-1} = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$$

Here k = 6, p = 0 (no parameter)

$$\therefore \chi_5^2 = \sum_{i=1}^{6} \frac{(O_i - e_i)^2}{e_i}$$

$$= \frac{(-5)^2 + (-3)^2 + (-5)^2 + (4)^2 + (3)^2 + (6)^2}{10}$$

= 12 (Observed value)

$\chi^2_{5,\ 0.05} = 11.07$ (Table value / Expected value)

As $\chi_5^2 > \chi^2_{5,\ 0.05}$

i.e. (Observed value) > (Expected value/Table value)

$\therefore$ We reject this hypothesis.

$\therefore$ The die is not fair.

Example 6.106 : *A nationalized bank utilizes four teller windows. On a particular day 800 customers were observed inside the bank. They were given service at the different windows as following*

Window number	*1*	*2*	*3*	*4*
Customers	*150*	*250*	*170*	*230*

Test whether the customers are uniformly distributed over the windows.

Solution : Let H_0 : The distribution of customers over 4 windows is uniform.

As the distribution is even $\therefore\ p_i = \frac{1}{4}$

$\therefore$ E (x) = Expected frequency

$= N \cdot p_i$

$= 800 \times \frac{1}{4} = 200$

x_i	O_i	e_i	$O_i - e_i$	$(O_i - e_i)^2$
1	150	200	– 50	2500
2	250	200	50	2500
3	170	200	– 30	900
4	230	200	30	900
Total				6800

Here k = 4, p = 0

$$\therefore \quad \chi^2_{k-p-1} = \sum_{i=1}^{4} \frac{(O_i - e_i)^2}{e_i} = \frac{6800}{200}$$

$$\chi^2_3 = 34 \text{ (Calculated value)}$$

$$\chi^2_{3,\,0.05} = 7.815 \text{ (Table value)}$$

As $\chi^2_3 > \chi^2_{3,0.05}$

Calculated value > Table value

$\therefore$ We reject H_0

$\therefore$ The customers are not uniformly distributed over the windows.

Example 6.107 : *The table below gives the number of accidents that occurred in certain factory on various days of week.*

Days	*Sun*	*Mon*	*Tue*	*Wed*	*Thurs*	*Fri*	*Sat*
Accidents	*6*	*4*	*9*	*7*	*8*	*10*	*12*

Test at 5 % level of significance whether accidents are uniformly distributed over the days.

Solution : Let H_0 : The distribution of accidents over the days in uniform.

$$P_i = \frac{1}{7}$$

$$N = \sum O_i = 56$$

$\therefore$ E (x) = Expected accidents

$= N \cdot P_i = 56 \cdot \frac{1}{7}$

$= 8$

x_i	O_i	e_i	$O_i - e_i$	$(O_i - e_i)^2$
1	6	8	– 2	4
2	4	8	– 4	16
3	9	8	1	1
4	7	8	– 1	1
5	8	8	0	0
6	10	8	2	4
7	12	8	4	16
				42

Here p = 0, k = 7

$$\therefore \quad \chi^2_{k-p-1} = \sum_{i=0}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_6 = \sum_{i=0}^{7} \frac{(O_i - e_i)^2}{e_i} = \frac{42}{8}$$

$$= 5.25 \text{ (observed)}$$

Now $\chi^2_{6,\,0.05} = 15.592$ (Table value)

As $\chi^2_6 < \chi^2_{6,0.05}$

i.e. Observed value < Table value i.e. expected value

$\therefore$ We accept this hypothesis.

$\therefore$ The accidents are distributed uniformly over the days of week.

Example 6.108 : *The table below gives the number of books issued from a library on the various days of week.*

Days	*Mon*	*Tue*	*Wed*	*Thurs*	*Fri*	*Sat*
Books issued	*120*	*130*	*110*	*115*	*135*	*110*

Test 5 % level of significance whether the issuing is independent of day.

Solution : Let H_0 : The issuing is independent

$$N = \sum O_i = 720$$

$$P_i = \frac{1}{6}$$

$\therefore$ Expected frequency = E (x)

$= N \cdot P_i$

$= \frac{720}{6} = 120$

x_i	O_i	e_i	$O_i - e_i$	$(O_i - e_i)^2$
1	120	120	0	0
2	130	120	+ 10	100
3	110	120	– 10	100
4	115	120	– 5	25
5	135	120	15	225
6	110	120	– 10	100
				550

Here p = 0, k = 6

$$\therefore \quad \chi^2_{k-p-1} = \sum_{i=0}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_5 = \frac{550}{120} = 4.58 \text{ (observed)}$$

Now $\chi^2_{5,\,0.05} = 11.07$ (Table value)

$\therefore \quad \chi^2_5 < \chi^2_{5,\,0.05}$

observed value < expected value (Table value)

$\therefore$ We accept H_0

Example 6.109 : *A set of five similar coins is tossed 210 times and the result is*

No. of heads	0	1	2	3	4	5
Frequency	2	5	20	60	100	31

Test the hypothesis that the data follow a binomial distribution.

Solution : Let H_0 : The data follows binomial distribution

Here N = 210

$$p_i = \text{Probability of getting head} = \frac{1}{2}$$

$$q_i = \text{Probability of getting tail} = \frac{1}{2}$$

Now $p(r) = \beta(n, p, r) = {}^nC_r\, p^r\, q^{n-r}$

$$= {}^5C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r} = {}^5C_r \left(\frac{1}{2}\right)^5$$

$$\therefore \quad p(0) = {}^5C_0 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$$p(1) = {}^5C_1 \left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

$$p(2) = {}^5C_2 \left(\frac{1}{2}\right)^5 = \frac{10}{32}$$

$$p(3) = {}^5C_3 \left(\frac{1}{2}\right)^5 = \frac{10}{32}$$

$$p(4) = {}^5C_4 \left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

$$p(5) = {}^5C_5 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

As expected frequency $E(x_i) = N \cdot p_i$ thus we prepare the table.

x_i	O_i	P_i	$e_i = N \cdot P_i$	$O_i - e_i$
0	2	1/32	7	– 5
1	5	5/32	35	– 30
2	20	10/32	70	– 50
3	60	10/32	70	– 10
4	100	5/32	35	65
5	37	1/32	7	30
	224			

k = 6, p = 0

$$\therefore \quad \chi^2_{k-p-1} = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_5 = \frac{(-5)^2}{7} + \frac{(-30)^2}{35} + \frac{(-50)^2}{70} + \frac{(-10)^2}{70} + \frac{(+65)^2}{35} + \frac{(30)^2}{7}$$

$$= \frac{25}{7} + \frac{900}{35} + \frac{2500}{70} + \frac{100}{70} + \frac{4225}{35} + \frac{900}{7}$$

$$= 315.7142 \text{ (Calculated)}$$

$\chi^2_{5,\,0.05} = 11.07$ (Table value)

$\therefore \quad \chi^2_5 > \chi^2_{5,\,0.05}$

$\therefore$ H_0 is rejected.

Example 6.110 : *The following table of frequencies of seeds were observed in experiment on pea breeding*

Round and Green	*Wrinkled and Green*	*Round and Yellow*	*Wrinkled and Yellow*	*Total*
222	120	32	150	524

Theory predicts that the frequencies should be in proportion 8 : 2 : 2 : 1. Examine the correspondence between theory experiment.

Solution : Let H_0 : There is a good correspondence between theory and experiment.

Here N = 524

As the proportion is 8 : 2 : 2 : 1

$\therefore$ Their probabilities are

$\frac{8}{13}, \frac{2}{13}, \frac{2}{13}, \frac{1}{13}$

Thus

x_i	O_i	P_i	$e_i = N \times P_i$	$O_i - e_i$
1	222	8/13	323	– 101
2	120	2/13	81	39
3	32	2/13	81	– 49
4	150	1/13	40	110

Here k = 4, p = 0

$$\therefore \quad \chi^2_{k-p-1} = \sum \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_3 = \frac{(-101)^2}{323} + \frac{(39)^2}{81} + \frac{(-49)^2}{81} + \frac{(110)^2}{40}$$

$$= 382.5 \text{ (Calculated)}$$

Now $\chi^2_{3, 0.05} = 7.815$ (Table value)

$\therefore \quad \chi^2_3 > \chi^2_{3, 0.05}$

$\therefore$ We reject H_0

$\therefore$ There is a very low correspondence between theory and experiment.

Example 6.111 : *Among 64 offsprings of a certain cross between guinea pigs, the following observations were made*

red	*black*	*white*
34	*10*	*20*

According to genetic model, these numbers should be in the ratio 9 : 3 : 4. Are the data consistent with 5 % level ?

Solution : Let H_0 : The offsprings in colours red, black, white are in the ratio 9 : 3 : 4.

Here N = 64

As the ratio is 9 : 3 : 4

$\therefore$ their probabilities are $\frac{9}{16}, \frac{3}{16}, \frac{4}{16}$

O_i	P_i	$e_i = N \cdot P_i$	$O_i - e_i$
34	9/16	36	– 2
10	3/16	12	– 2
20	4/16	16	4

Here k = 3, p = 0

$$\chi^2_{k-p-1} = \sum_{i=0}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$= \frac{(-2)^2}{36} + \frac{(-2)^2}{12} + \frac{(4)^2}{16}$$

$$\chi^2_2 = 1.444 \text{ (Calculated value)}$$

Now $\chi^2_{2, 0.05} = 5.991$ (Table value)

as $\chi^2_2 < \chi^2_{2, 0.05}$ we accept H_0

Example 6.112 : *One hundred samples were drawn from a production process each after 5 hours. The number of defectives were noted from these samples. A Poisson distribution by estimating the parameter m was fitted to these data. The results obtained are as follows*

Number of defectives	*Number of samples observed*	*Number of samples expected*
0	*63*	*60.65*
1	*28*	*30.33*
2	*6*	*7.58*
3	*2*	*1.26*
4	*1*	*0.16*
5 and above	*0*	*0.02*

Test the goodness of fit of Poisson distribution (Use 5 % level of significance)

Solution : Let H_0 : Fitting of Poisson distribution is good.

Here as the expected frequencies are less than five

$\therefore$ We pool the data

Thus

O_i	e_i	$O_i - e_i$
63	60.65	2.35
28	30.33	– 2.33
9	9.02	– 0.02

Here k = 3, p = 1 = one parameter

$$\chi^2_{k-p-1} = \sum_{i=0}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_{3-1-1} = \frac{(2.35)^2}{60.65} + \frac{(-2.33)^2}{30.33} + \frac{(-0.02)^2}{9.02}$$

$$\chi^2_1 = 0.2701 \text{ (Calculated value)}$$

Now $\chi^2_{1,0.05} = 3.841$ (Table value)

$\therefore$ $\chi^2_1 < \chi^2_{1,0.05}$

$\therefore$ We accept H_0.

Example 6.113 : *The freshman math grades of 250 males and 210 females at a university were distributed as in the following table.*

Grades

	A	B	C	D	F	Totals
Male	35	42	85	48	40	250
Female	28	50	77	35	20	210
Totals	63	92	162	83	60	460

Use the Chi-square random variable test at 0.05 significance level, the hypothesis is

H_0 : The grade distribution of males and females are same.

Solution : Here $N_1 = 250$, $N_2 = 210$

The probabilities of each grade are

$$P_A = \frac{63}{460}, P_B = \frac{92}{460}, P_C = \frac{162}{460}, P_D = \frac{83}{460}, P_F = \frac{60}{460}$$

x_i	Grades	P_i	O_i (Males)	O_i (Females)	$N_1 \times P_i$ e_i (Males) $= 250 \cdot P_i$	$N_2 \cdot P_i$ e_i (Females) $= 210 \cdot P_i$
1	A	63/460	35	28	34.24	28.76
2	B	92/460	42	50	50	42
3	C	162/460	85	77	88.04	73.96
4	D	83/460	48	35	44.11	37.89
5	F	60/460	40	20	32.61	27.39

Here k = 5, p = 0

$$\chi^2_{k-p-1} = \sum_{i=0}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_4 = \left[\frac{(35-34.24)^2}{34.24} + \frac{(42-50)^2}{50} + \frac{(85-88.04)^2}{88.04} + \frac{(48-45.11)^2}{45.11} + \frac{(40-32.61)^2}{32.61}\right]$$

$$+ \left[\frac{(28-28.76)^2}{28.76} + \frac{(50-42)^2}{42} + \frac{(77-73.96)^2}{73.96} + \frac{(35-37.89)^2}{37.89} + \frac{(20-27.39)^2}{27.39}\right]$$

$$\chi^2_4 = 7.14 \text{ Calculated}$$

$$\chi^2_{4,0.05} = 9.488 \text{ (Table value)}$$

$\therefore$ $\chi^2_4 < \chi^2_{4,0.05}$

$\therefore$ We accept this hypothesis

i.e. the grade distribution for males is same as that of females.

Example 6.114 : *A random group of 40 people younger than 50 years was given a flu shot, and a second random group of 60 people 50 years or older was given the same flu shot. Each member of the groups was classified according to whether the member did not get the lfu (N), had a mild case of the flu (M), or had a severe case of the flu (S). The frequencies in each group are as indicated in the following table.*

Reaction

		N	M	S	Totals
Age	Under 50 years	30	6	4	40
	50 years of older	36	12	12	60
	Total	66	18	16	100

Use a Chi-square random variable to test, at the 0.05 significance level, the hypothesis that the reactions to the shot are the same in each group.

Solution : By pooling the subjects under 50 years and those 50 years and over in each reaction group, we get the following estimated probabilities.

$$P_N = \frac{66}{100}\ 0.66,\ P_M = \frac{18}{100} = 0.18,\ P_S = \frac{16}{100} = 0.16$$

The expected frequencies mP_j for the m = 40 subjects under 50 years are

$mP_N = 40 \times 0.66 = 26.4$, $mP_N = 40 \times 0.18 = 7.2$,

$mP_S = 40 \times 0.16 = 6.4$

and the expected frequencies nP_j for the n = 60 subjects 50 years and over are

$nP_N = 60 \times 0.66 = 39.6$, $nP_M = 60 \times 0.18 = 10.8$,

$nP_S = 60 \times 0.16 = 9.6$

The corresponding Chi-square test value is

$$\chi^2 = \frac{(30-26.4)^2}{26.4} + \frac{(6-7.2)^2}{7.2} + \frac{(4-6.4)^2}{6.4} + \frac{(36-39.6)^2}{39.6} + \frac{(12-10.8)^2}{10.8} + \frac{(12-9.6)^2}{9.6} = 2.65$$

Here k = 3, p = 0

$\therefore \quad \chi_2^2 = 2.65$ (Calculated)

$\chi_{2,0.05}^2 = 5.991$ (Table value)

As $\quad \chi_2^2 < \chi_{2,0.05}^2$

We accept the hypothesis that the reactions to the shot are same in each group.

Example 6.115 : *Salaries for 200 males and 300 females at a certain company are as indicated in the following frequency table, where the notation [a, b] means a salary greater than or equal to a but less than b.*

Salaries in thousands of dollars

		1 [20, 30)	2 [30, 40)	3 [40, 50)	4 [50, 60)	5 [60, –)	Totals
Gender	Male	20	34	46	60	40	200
	Female	45	78	90	62	25	300
	Totals	65	112	136	122	65	500

Use the Chi-square random variable to test, at the 0.05 significance level, the hypothesis the salary distributions are the same.

Solution : By pooling the male and female frequencies in each salary grade, we obtain the following estimated probabilities :

$P_1 = \frac{65}{500} = 0.13,\ P_2 = \frac{112}{500} = 0.224,\ P_3 = \frac{136}{500} = 0.272,$

$P_4 = \frac{122}{500} = 0.244,\ P_5 = \frac{65}{500} = 0.13$

The expected frequencies mP_j for the m = 200 males are :

$mP_1 = 200 \times 0.13 = 26,\ mP_2 = 200 \times 0.224 = 44.8,$

$mP_3 = 200 \times 0.272 = 54.4,$

$mP_4 = 200 \times 0.244 = 48.8,\ mP_5 = 200 \times 0.13 = 26$

and the expected frequencies nP_j for the n = 300 females are :

$nP_1 = 300 \times 0.13 = 39,\ nP_2 = 300 \times 0.224 = 67.2,$ $nP_3 = 300 \times 0.272 = 81.6,$

$nP_4 = 300 \times 0.244 = 73.2,\ nP_5 = 300 \times 0.13 = 39$

The corresponding Chi-square test value is

(Here $\quad$ k = 5, p = 0)

$$\chi_{k-p-1}^2 = \frac{(20-26)^2}{26} + \frac{(34-44.8)^2}{44.8} + \frac{(46-54.4)^2}{54.4} + \frac{(60-48.8)^2}{48.8} + \frac{(40-26)^2}{26} + \frac{(45-39)^2}{39} + \frac{(78-67.2)^2}{67.2} + \frac{(90-81.6)^2}{81.6} + \frac{(62-73.2)^2}{73.2} + \frac{(25-39)^2}{39}$$

$\chi_4^2 = 25.66$ (Calculated)

Now $\quad \chi_{4,0.05}^2 = 9.488$ (Table value)

$\therefore \quad \chi_4^2 > \chi_{4,0.05}^2$

$\therefore$ We reject the hypothesis that the salary distribution for males and females is same.

Example 6.116 : *The demand for a particular spare part in a factory was found to vary from day to day. In a sample study. The following information was obtained.*

Days	Mon	Tue	Wed	Thur	Fri	Sat
No. of parts demanded	1124	1125	1110	1120	1126	1115

Test the hypothesis that the number of parts demanded does not depend on the day of the week.

Solution : Let H_0 : The no. of parts demanded does not depend on the day of the week.

The total number of parts demanded during a week (six days) = 6720.

$\therefore$ Expected number of parts to be demanded per day is 1120.

By χ^2 test.

$$\chi_{k-p-1}^2 = \sum \frac{(O_i - e_i)^2}{e_i}$$

where $\quad O_i$ = Observed frequency

e_i = Expected frequency

Consider the following table.

Days	O_i	e_i	$(O_i - e_i)^2$	$(O_i - e_i)^2/e_i$
Mon	1124	1120	16	0.0142
Tue	1125	1120	25	0.0223
Wed	1110	1120	100	0.0892
Thur	1120	1120	0	0
Fri	1126	1120	36	0.0321
Sat	1115	1120	25	0.0223

$k = 6,\ p = 0$

$$\chi_5^2 = \sum_{i=1}^{6} \frac{(O_i - e_i)^2}{e_i}$$

$$= 0.0142 + 0.0223 + 0.0892 + 0 + 0.0321 + 0.0223$$

$\chi_5^2 = 0.1801$

But tabulated value of $\chi^2_{5,\,0.05}$ is 11.07

$\therefore \quad \chi_5^2 < \chi^2_{5,\,0.05}$

$\Rightarrow$ Accept H_0.

$\therefore$ The number of parts demanded does not depend on the day of the week.

Example 6.117 : *Fit a Poisson distribution to the following data and test the goodness of fit by applying the χ^2 test.*

x	*0*	*1*	*2*	*3*	*4*
f	*155*	*157*	*58*	*22*	*8*

Solution : Consider the following table.

x_i	f_i	$x_i f_i$	$P(r) = \frac{e^{-z} z^r}{r!}$	Expected frequency P(r) × 400
0	155	0	0.3955	158
1	157	157	0.3668	148
2	58	116	0.1701	68
3	22	66	0.0525	21
4	8	32	0.0121	5

Here z = mean = 0.9275

Now, we construct following table.

x_i	O_i	e_i	$(O_i - e_i)^2$	$(O_i - e_i)^2/e_i$
0	155	158	9	0.05696
1	157	148	81	0.5472
2	58	68	100	0.4705
3	22	21	1	0.0476
4	8	5	9	1.8
				3.9222

$k = 5,\ p = 0$

$$\therefore \quad \chi^2_{k-p-1} = \frac{\sum (O_i - e_i)^2}{e_i} = 3.9222$$

$$\chi_4^2 = 3.9222$$

But $\quad \chi^2_{4,\,0.05} = 9.488$ (by table)

Calculated value < Table value

$\therefore$ We accept this hypothesis.

$\therefore$ Given data is good for Poisson distribution.

Example 6.118 : *An urn contains 6 red balls and 3 white balls. Two balls are drawn at random from the urn their colours are noted and balls are replaced. This process is repeated 120 times and results obtained are shown in table.*

	0 Red 2 White	1 Red 1 White	2 Red 0 White	Total
No. of drawning	6	53	61	120

i) Determine the expected frequencies.

ii) Determine at 0.05 significance level, whether the results obtained consistent with these expected.

Solution : Let H_0 : Given data is consistent

N = 120, Expected frequency = $e_i = N P_i$.

Construct the following table.

O_i	P_i	$e_i = N P_i$	$(O_i - e_i)^2$	$(O_i - e_i)^2/e_i$
6	1/9	13	49	3.7692
53	4/9	53	0	0
61	4/9	53	64	1.2075

k = 3

$$\chi^2_{k-p-1} = \chi^2_2 = \sum \frac{(O_i - e_i)^2}{e_i} = 4.9767$$

But $\chi^2_{2,0.05} = 5.999$ (by table)

$$\chi^2_2 < \chi^2_{2,0.05}$$

we accept H_0.

Thus the results obtained are consistent with these expected frequencies.

Example 6.119 : *A die when tossed 300 times gave the following results.*

Score	1	2	3	4	5	6
Frequency	43	49	56	45	66	41

Are the data consistent at 5 % level of significance with the hypothesis that the die is unbiased ?

Solution : We construct the following table,

Score	1	2	3	4	5	6
Observed frequency	43	49	56	45	66	61
Expected frequency	50	50	50	50	50	50
$(O_i - e_i)^2$	49	1	36	25	256	81

H_0 : The die is unbiased.

k = 6

$$\chi^2_{k-p-1} = \chi^2_5 = \sum_i \frac{(O_i - e_i)^2}{e_i} = 8.96$$

By table $\chi^2_{5,0.05} = 11.07$

$$\chi^2_5 < \chi^2_{5,0.05}$$

$\therefore H_0$ is accepted.

6.33 Chi-square Test for Independent Attributes

Contingency table of probabilities :

Let X and Y be attributes associated with individuals in population. Suppose that X can be classified into mutually disjoint categories $A_1, A_2, ... A_m$ and Y can be classified into mutually disjoint categories $B_1, B_2 ... B_n$. The probability $P(A_i\ B_j)$ is denoted by P_{ij}. The following Table is an mxn contingency table of probabilities where P_{ij} is in the i^{th} and j^{th} column. Sum of all P_{ij}'s is 1.

	B_1	B_2	B_3	....	B_n	
A_1	P_{11}	P_{12}	P_{13}	....	P_{1n}	$P_{1.} = P(A_1)$
A_2	P_{21}	P_{22}	P_{23}	....	P_{2n}	$P_{2.} = P(A_2)$
A_3	P_{31}	P_{32}	P_{33}	....	P_{3n}	$P_{2.} = P(A_2)$
:	:	:	:		:	
:	:	:	:		:	
A_m	P_{m1}	P_{m2}	P_{m3}		P_{mn}	$P_{m.} = P(A_m)$
	$P_{.1}$	$P_{.2}$	$P_{.3}$		$P_{.n}$	
	$= P(B_1)$	$= P(B_2)$	$P(B_3)$		$P(B_n)$	

Here degrees of freedom = (m – 1) (n – 1).

Example 6.120 : *A random group of 300 males was cross-classified according to age and total cholesterol level, as indicated in the table below.*

		Total cholesterol			
		Under 200 Low	*200-239 Medium*	*240 or higher High*	*Totals* $(f_{i.})$
Age	*20-34*	*66*	*24*	*8*	*98*
	35-54	*54*	*48*	*22*	*124*
	55-75	*18*	*50*	*10*	*78*

Use the Chi-square random variable to test, at the 0.01 significance level, the hypothesis that the attributes of age and cholesterol level are independent.

Solution : The contingency table has m = 3 rows, where age bracket 20-34 corresponds to i = 1, 35-54 corresponds to i = 2, and 55-74 corresponds to i = 3. There are n = 5 columns, where j = 1 corresponds to low cholesterol level, j = 2 corresponds to medium, and j = 3 corresponds to high. The estimated row probabilities are :

$$P_{1.} = \frac{98}{300},\ P_{2.} = \frac{124}{300},\ P_{3.} = \frac{78}{300}$$

and the estimated column probabilities are :

$$P_{.1} = \frac{138}{300},\ P_{.2} = \frac{122}{300},\ P_{.3} = \frac{40}{300}$$

The expected cross-classification frequency estimates, where N = 300, are :

$N \times P_{1.} \times P_{.1} = 45.08$, $N \times P_{1.} \times P_{.2} = 39.853$,

$N \times P_{1.} \times P_{.3} = 13.067$,

$N \times P_{2.} \times P_{.1} = 57.04$, $N \times P_{2.} \times P_{.2} = 50.427$,

$N \times P_{2.} \times P_{.3} = 16.533$,

$N \times P_{3.} \times P_{.1} = 35.88$, $N \times P_{3.} \times P_{.2} = 31.72$,

$N \times P_{3.} \times P_{.3} = 10.4$

The test value of the Chi-square statistic is :

$$\chi^2 = \frac{(66 - 45.08)^2}{45.08} + \frac{(24 - 39.853)^2}{39.853} + \frac{(8 - 13.067)^2}{13.067} + \frac{(54 - 57.04)^2}{57.04} + \frac{(48 - 50.427)^2}{50.427} + \frac{(22 - 16.533)^2}{16.533} + \frac{(18 - 35.88)^2}{35.88} + \frac{(50 - 31.72)^2}{31.72} + \frac{(10 - 10.4)^2}{10.4}$$

$$= 39.53 \quad \text{(Calculated)}$$

There are $(m - 1)(n - 1) = 2 \times 2 = 4$ degrees of freedom.

Now $\chi^2_{4,\,0.05} = 9.488$

$\therefore \quad \chi^2_4 > \chi^2_{4,\,0.05}$

$\therefore$ We reject the hypothesis that the age and total cholesterol level are independent.

Example 6.121 : *A random group of 800 eligible voters was cross-classified according to annual income and party affiliation, as indicated in the following table. In the table, [20, 40] signifies income of at least \$20,000 but less than \$40,000; [40, 60] means at least \$40,000 but less than \$60,000 and [60, –] means \$60,000 over. Apply a Chi-square test for independent of annual income and party affiliation at the 0.05 significance level.*

Annual income

		[20, 40]	[40, 60]	[60, –]	Totals ($f_{i,}$)
Party	Democratic	125	225	70	420
	Republican	60	200	120	380
	Totals ($f_{,j}$)	185	425	190	800

Solution : The contingency table has m = 2, where Democratic affiliation corresponds to i = 1 and Republican affiliation corresponds to i = 2, there are n = 3 columns, where j = 1 corresponds to the salary range [20, 40], j = 2 corresponds to [40, 60], and j = 3 corresponds to [60, –]. The estimated row probabilities are :

$$P_{1.} = \frac{420}{800} = 0.525, \quad P_{2.} = \frac{380}{800} = 0.475$$

and the estimated column probabilities are :

$$P_{.1} = \frac{185}{800}, \quad P_{.2} = \frac{425}{800}, \quad P_{.3} = \frac{190}{800}$$

The expected frequency estimates are : N = 800

$$N \times P_{1.} \times P_{.1} = 800 \times 0.525 \times \frac{185}{800}, \quad N \times P_{1.} \times P_{.2} = 223.125,$$

$$N \times P_{1.} \times P_{.3} = 99.75$$

$$N \times P_{2.} \times P_{.1} = 87.875, \; N \times P_{2.} \times P_{.2} = 201.875,$$

$$N \times P_{2.} \times P_{.3} = 90.25$$

The test value of the Chi-square statistic is :

$$\chi^2 = \frac{(125 - 97.125)^2}{97.125} + \frac{(225 - 223.125)^2}{223.125} + \frac{(70 - 99.75)^2}{99.75} + \frac{(60 - 87.875)^2}{87.875} + \frac{(200 - 201.875)^2}{201.875} + \frac{(120 - 90.25)^2}{90.25}$$

$= 35.56$ Calculated

There are $(m - 1)(n - 1) = 1 \times 2 = 2$ degrees of freedom.

Here $\quad \chi^2_2 = 35.56$

Now $\quad \chi^2_{2,\,0.05} = 5.991$

$\therefore \quad \chi^2_2 > \chi^2_{2,\,0.05}$

$\therefore$ We reject the hypothesis that the annual income and party affiliation are independent.

6.34 The 't'-Distribution

Let $x_1, x_2, \ldots, x_n$ be the members of random sample drawn from a normal population with mean μ. If $\bar{x}$ be the mean of sample, then

$$t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)} \quad \text{where } s^2 = \frac{\sum (x - \bar{x})^2}{n - 1}$$

The student t-distribution is used to test the significance of (i) The mean of a small sample. (ii) The difference between the means of two small samples or to compare two small samples. (iii) The correlation coefficient.

Example 6.122 : *A machine which produces mica insulating washers for use in electric device to turn out washers having a thickness of 10 mm. A sample of 10 washers has an average thickness 9.52 mm with a standard deviation 0.6 mm. Find t.*

Solution : Given that $\bar{x} = 9.52$, $n = 10$, $\mu = 10$, $s = 0.6$

$$\therefore \quad t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)} = \frac{9.52 - 10}{\left(\frac{0.6}{\sqrt{10}}\right)} = \frac{-4}{5}\sqrt{10} = -2.528$$

6.35 Test of Hypothesis

a) Testing population mean (μ) equal to a particular value (μ_0)

Here the test statistic is

$$U = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

Under H_0, $U \to N(0, 1)$

Critical region : Value of $U > U_{\alpha/2} = 1.96$ at 5 % level of significance

Observation : If the calculated value of U is more than (1.96) or less than (– 1.96) then H_0 is rejected otherwise accepted.

Example 6.123 : *Score of Engineering C.E.T. in Maths average 60 with standard deviation 8.2 A class of 50 recently average 75 in this examination. Is this sufficient to prove that the good average is 60 ?*

Solution : To test $H_0 : \mu = 60$

Given $\mu_0 = 60$, $\sigma = 8.2$

Test statistic is

$$U = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{x} - 60}{\frac{8.2}{\sqrt{50}}} = \frac{75 - 60}{\frac{8.2}{\sqrt{50}}} = \frac{15 \times \sqrt{50}}{8.2} = 12.93$$

$\therefore$ $U > 1.96$

i.e. $U > U_{\alpha/2} = 1.96$

$\therefore$ We reject H_0

$\therefore$ The average marks are not 60.

Example 6.124 : *A random sample of 100 people in certain city gives average teenagers content 27 with standard deviation 8. Test whether the teenager content overall the city is 30 % with 5 % level of significance.*

Solution : To test $H_0 : \mu = \mu_0 = 30$

Given $\sigma = 8$

Test statistic is

$$U = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{27 - 30}{\frac{8}{\sqrt{100}}} = \frac{-3}{\frac{8}{10}} = \frac{-30}{8} = -3.75$$

$\therefore |U| > U_{\alpha/2} = 1.96$

$\therefore$ We reject this hypothesis.

i.e. the average teenager content of population in certain is not 30 %.

Exercise 6.3

1. *A pair of dice is tossed 360 times, and the frequency of each sum is indicated in the chart.*

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	8	24	35	37	44	65	51	42	26	14	14

Would you say that the dice are fair on the basis of the Chi-square test ?

Hint :

$P(2) = \frac{1}{36} = P(12),\quad P(3) = \frac{2}{36} = P(11),$

$P(4) = \frac{3}{36} = P(1)$

$P(5) = \frac{4}{36} = P(9),\quad P(6) = \frac{5}{36} = P(8),\quad P(7) = \frac{6}{36}$

where $n = 360$

Sum	2	3	4	5	6	7	8	9	10	11	12
Expected frequency	10	20	30	40	50	60	50	40	30	20	10

$$\chi^2 = \frac{(8-10)^2}{10} + \frac{(24-20)^2}{20} + \frac{(35-30)^2}{30} + \frac{(37-40)^2}{40} + \frac{(44-50)^2}{50} + \frac{(65-60)^2}{60} + \frac{(51-50)^2}{50} + \frac{(42-40)^2}{40} + \frac{(26-30)^2}{30} + \frac{(14-20)^2}{20} + \frac{(14-10)^2}{10} \approx 7.45 < \chi^2_{11, 0.05} \Rightarrow \textit{accepted}$$

2. *Over the years, the grades in a certain college professor's class are typically as follows : 10 percent As, 20 percent Bs, 50 percent Cs, 15 percent Ds, and 5 percent Fs. The grades for her current class of 100 are 16 As, 28 Bs, 46 Cs, 10 Ds, and 0 Fs. Test the hypothesis that the current class is typical by a Chi-square test at the 0.05 significance level.*

Hint :

Grade	A	B	C	D	F
H_0 : Probability =	0.1	0.2	0.5	0.15	0.05
Expected frequency	10	20	50	15	5
Actual frequency	16	28	46	10	0

The Chi-square test sum is

$$\chi_4^2 = \frac{(16-10)^2}{10} + \frac{(28-20)^2}{20} + \frac{(46-50)^2}{50} + \frac{(10-15)^2}{15}$$

$$+ \frac{(0-5)^2}{5} \approx 13.79 > \chi_{4,0.05}^2 = 9.48$$

with 5 – 1 = 4 degrees of freedom we reject the hypothesis that the class is typical.

3. *A bag is supposed to contain 20 percent red beans and 80 percent white beans. A random sample of 50 beans from the bag contains 16 red and 34 white. Apply the Chi-square test at the 0.05 significance level to either reject or not reject the hypothesis that the contents are as advertised.*

Hint :

If the contents are 20 percent red and 80 percent white, then P (red) = p_1 = 0.2, and P (white) = p_2 = 0.8; $np_1 = 50 \times 0.2 = 10$ and $np_2 = 50 \times 0.8 = 40$. The test Chi-square value is

$$\chi_1^2 = \frac{(16-10)^2}{10} + \frac{(34-40)^2}{40} = 4.5 > \chi_{1,0.05}^2 = 3.84$$

∴ *rejected.*

4. *A coin is tossed 100 times, resulting in 60 heads (H) and 40 tails (T). Apply the Chi-square test at the 0.05 significance level to either reject or not reject the hypothesis that the coin is fair.*

Hint : *P (H) = p_1 = 0.5, P (T) = p_2 = 0.5. We have $n = 100$, so $np_1 = 100 \times 0.5 = 50 = np_2$. The test Chi-square value is*

$$\chi_1^2 = \frac{(60-50)^2}{50} + \frac{(40-50)^2}{50} = 4 > \chi_{1,0.05}^2 = 3.84$$

∴ *rejected.*

5. *The table below gives the number of accidents that occurred in the certain factory on the various days of a particular week.*

Days of week	Sun	Mon	Tues	Wed	Thurs	Fri	Sat
No. of accidents	6	4	9	7	8	10	12

Test at 5 % level whether accidents are uniformly distributed over the different days.

[Ans. : $\chi_6^2 = 5.25$; χ_6^2; 0.05 = 15.592, accept H_0]

6. *The following is a 2×2 contingency table :*

Eye colour in father	Eye colour in son	
	Not light	Light
Not light	23	15
Light	15	47

Test whether the eye colour in sone is associated with the eye colour in father.

[Ans. : $\chi_1^2 = 13.2$, $\chi_{1,0.05}^2 = 3.841$, reject H_0]

7. *A die when tossed 300 times gave the following results :*

Score	1	2	3	4	5	6
Frequency	43	49	56	45	66	41

Are the data consistent at 5 % level of significance with the hypothesis that the die is true ?

[Ans. : $\chi_5^2 = 8.56$, $\chi_{5,0.05}^2 = 11.07$, accept H_0]

8. *In 150 tosses of a coin 90 heads and 60 tails were observed. Test the hypothesis that the coin is fair by Chi-square test at 0.05 significance level.* **[Ans. : 6.00 > 3.84, rejected]**

9. *The table below gives the number of books issued from a certain library on the various days of a week.*

Days	Mon	Tues	Wed	Thurs	Fri	Sat
No. of books issued	120	130	110	115	135	110

Test at 5 % l.o.s whether the issuing of books is independent of a day. **[Ans. : 4.58 < 11.07, accept]**

10. In a locality, 100 persons were randomly selected and asked for their educational achievements. The results are given as under.

Sex	Education		
	Primary school	High school	College
Male	10	15	25
Female	25	10	15

Test whether education depends on sex at 1 % level of significance. **[Ans. : 9.929 > 5.991, reject]**

11. In an experiment on pea breeding, a scientist obtained the following frequencies of seeds : 316 round and yellow, 102 wrinkled and yellow, 109 round and green and 33 wrinkled and green. Theory predicts that the frequencies of seeds should be in the proportion 9 : 3 : 3 : 1 respectively. Set a proper hypothesis and test it at 5 % l.o.s. **[Ans. : 0.355 < 7.815, accept]**

12. A newspaper publisher is interested in testing whether newspaper readership in the society is associated with reader's educational achievement. A related survey showed the followed results :

Type of readership	Level of Education			
	Post graduate	Graduate	Passed S.S.C	Not passed S.S.C
Never	9	12	30	60
Sometimes	25	20	15	20
Daily	68	48	40	10

Test whether type of newspaper readership depends on level of education. [Take $\alpha = 0.05$]

[Ans. : $\chi_6^2 = 97.65 > \chi_{6,\,0.05}^2 = 12.592$, rejected]

13. From the information given below, test whether the type of occupation and attitude towards the social laws are independent. [Use 1 % l.o.s.]

Occupation	Attitude towards Social Laws		
	Favourable	Neutral	Opposite
Blue-collar	29	26	37
White-collar	25	32	56
Professional	34	21	42

[Ans. : $\chi_4^2 = 5.415 < \chi_{4,0.01}^2 = 9.488$, accepted]

14. A random digit generator on a calculator gave the distribution of digits shown in the table. Test the hypothesis that the digits are random by a Chi-sqaure test at the 0.05 significance level.

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	11	11	9	8	8	11	9	11	13	9

[Ans. : $\chi_9^2 = 2.4 < \chi_{9,\,0.05}^2 = 16.9$, accepted]

Chi-square test for equal distributions

15. Independently obtained random samples of two independent multinomial random variables, X and Y, each with outcomes a_1, a_2, a_3, a_4 resulted in the following contingency table of frequencies. Apply a Chi-square test at the 0.05 significance level to the hypothesis that X and Y have the same probability distribution.

	a_1	a_2	a_3	a_4	Totals
X	25	45	15	15	100
Y	45	50	35	10	140
Totals	70	95	50	25	240

[Ans. : $\chi_3^2 = 8.55 > \chi_{3,\,0.05}^2$, rejected]

16. Each dice of pair of unbalanced dice, one red and one white, is tossed 200 times, resulting in the following frequency distribution for the faces of the dice, Apply a Chi-square test at the 0.01 significance level to the hypothesis that the dice have the same probability distribution.

Side landing face-up

	1	2	3	4	5	6	Totals
Red die	30	20	42	10	41	57	200
White die	44	30	24	20	50	32	200
Totals	74	50	66	30	91	89	400

[Ans. : $\chi_5^2 = 20.8 > \chi_{6,\,0.05}^2$, rejected]

6.36 University Questions

Dec. - 2016

Q.1 *Number of absent student in a class follow Poisson distribution with mean 5. Find the probability that in a certain month number of absent student in a class will be :*
i) More than 3 ii) Between 3 and 5. **[4]**

May - 2017

Q.2 *An unbiased coin is thrown 10 times. Find the probability of getting exactly 6 heads using bionomial distribution.* **[4]**

Dec. - 2017

Q.3 *In a certain examination test, 2000 students appeared in a subject of mathematics. Average marks obtained were 50 % with standard deviation 5 %. How many students do you expect to obtain more than 60 % of marks, supposing that marks are distributed normally. (Given A (z = 2) = 0.4772)* **[4]**

May - 2018

Q.4 *In a certain factory turning out razor blades, there is a small chance of* $\frac{1}{500}$ *for any blade to be defective. The blades are supplied in a packet of 10. Use Poisson distribution to calculate the approximate number of packets containing :*
i) No defective blades in a consignment
ii) Two defective blades in a cosignment of 10,000 packets. **[4]**

Dec. - 2018

Q.5 *200 students appeared in a certain examination obtained average marks 50 % with standard deviation 5 %. How many students do you expect to obtain more than 60 % of marks, supposing that the marks distributed normally.*
(Given : At z = 2, A = 0.4772) **[4]**

May - 2019

Q.6 *A dice is thrown five times. If getting an odd number is a success, then what is the probability of getting :*
i) Four successes
ii) At least four successes. **[4]**

Dec. - 2019

Q.7 *In a town, 10 accidents took place in a span of 50 days. Assuming that the number of accidents per day follows Poisson distribution, find the probability that there will be :*
i) At least 3 accidents in a day
ii) At least 2 accidents in a day. **[4]**

❑❑❑

Notes

UNIT - V

7 Numerical Methods

7.1 Introduction

In this chapter we shall deal with the methods for solving equations by using rough approximations and solution of system of linear equations by numerical approach.

Some rules to locate roots of $f(x) = 0$

Rule 1 : Let $f(x) = 0$ be any function such that f(a) and f(b) are of opposite signs then at least one or an odd number of real roots of the equation $f(x) = 0$ lie between a and b. If f(a) and f(b) are of the same signs then there is either no real root or an even number of roots lie between a and b.

Rule 2 : Every equation of an odd degree has at least one real root where as every equation of an even degree whose last term is negative has at least two real roots, one positive and the other negative.

If f(x) is purely a polynomial in x then $f(x) = 0$ is called as algebraic equation. If f(x) contains trignometric logarithmic exponential functions, etc. then $f(x) = 0$ is called a transcendental equation.

7.2 Bisection Method

1) If f(x) is continuous in the interval (a, b) and f(a) and f(b) have different signs then the equation $f(x) = 0$ has at least one root between $x = a$ to $x = b$

2 Bisection method : This method is based on the repeated application of intermediate value property.

Let f(x) be continuous in the interval [a, b] such that f(a) and f(b) have opposite signs. Without loss of generality, assume that $f(a) < 0$ and $f(b) > 0$,

then the first approximation to the root is

$$x_1 = \frac{1}{2}(a+b).$$

Now there are three possibilities :

i) If $f(x_1) = 0$, then x_1 is the root of $f(x) = 0$

ii) If $f(x_1) < 0$, then root lies between x_1 and b.

iii) If $f(x_1) > 0$, then root lies between a and x_1.

Suppose $f(x_1) > 0$ then root lies in the interval $[a, x_1]$

$\therefore$ The second approximation to the root is $x_2 = \frac{1}{2}(a + x_1)$.

Again there are three possibilities.

i) If $f(x_2) = 0$, then x_2 is the root of $f(x) = 0$

ii) If $f(x_2) < 0$, then root lies between x_2 and x_1.

iii) If $f(x_2) > 0$ then root lies between a and x_2.

Assume that $f(x_2) > 0$, then the root lies in the interval $[a, x_2]$

$\therefore$ The third approximation to the root is

$$x_3 = \frac{1}{2}(a + x_2) \text{ and so on.}$$

Repeat this procedure till we reached to required root.

Example 7.1 : *Find a real root of the equation $x^3 - x = 1$ between 1 and 2 by bisection method. Compute give iterations.*

Solution : Given : $x^3 - x - 1 = 0$

Let

$$f(x) = x^3 - x - 1 = 0$$

$$f(1) = 1^3 - 1 - 1 = -1 \ (-\text{ve})$$

$$f(2) = 2^3 - 2 - 1 = 5 \ (+\text{ve})$$

$\because$ $f(1) < 0$ and f (2) 0,

$\therefore$ By intermediate value theorem root lies between 1 and 2.

$$\therefore \quad x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$$f(1.5) = (1.5)^3 - 1.5 - 1$$

$$= 0.875 > 0 \quad (+\text{ve})$$

Hence, root lies between 1 and 1.5.

$$x_2 = \frac{1+1.5}{2} = 1.25$$

$$f(1.25) = -0.2968 < 0 \quad (-\text{ve})$$

$\therefore$ Root lies between 1.25 and 1.5

$$x_3 = \frac{1.25+1.5}{2} = 1.375$$

$$f(1.375) = 0.2246 > 0 \quad (+\text{ve})$$

$\therefore$ Root lies between 1.25 and 1.375

$$x_4 = \frac{1.25+1.375}{2} = 1.3125$$

$$\therefore \quad f(1.3125) = -0.0515 < 0 \quad (-\text{ve})$$

$\therefore$ Root lies between 1.3125 and 1.375.

$$x_5 = \frac{1.3125+1.375}{2} = 1.3437$$

After 5^{th} iteration the required root is 1.3437.

Example 7.2 : *Find the real root of equation $x \log_{10} x = 1.2$ by bisection method correct to four decimal places.*

Solution :

Let $\quad f(x) = x\log_{10}x - 1.2 = 0$

$$f(2) = -0.59794$$

$$f(3) = 0.23136$$

Hence a root lies between 2 and 3.

$\therefore$ First approximation to the root is

$$x_1 = \frac{1}{2}(2+3) = 2.5$$

$$\therefore \quad f(x_1) = f(2.5) = -0.205 \ (-)\text{ve}$$

Hence root lies between 2.5 and 3.

Second approximation to the root is

$$x_2 = \frac{1}{2}[2.5+3]$$

$$x_2 = 2.75$$

Now $\quad f(x_2) = 0.008 \ (+)\text{ve}$

Hence root lies between 2.5 and 2.75.

$\therefore$ Third approximation to the root is

$$x_3 = \frac{1}{2}(2.5+2.75)$$

$$x_3 = 2.625$$

Now $f(x_3) = -0.099$ (– ve).

Hence root lies between 2.625 and 2.75.

$\therefore$ Fourth approximation to the root is :

$$x_4 = \frac{1}{2}(2.625+2.75)$$

$$x_4 = 2.6875$$

Now $f(x_4) = f(2.6875) = -0.046$ (–)ve

Hence root lies between 2.6875 and 2.75.

$\therefore$ Fifth approximation :

$$x_5 = \frac{1}{2}(2.6875+2.75) = 2.71875$$

$$f(x_5) = f(2.71875) = -0.019 \ (-\text{ve})$$

Hence root lies between 2.71875 and 2.75

$\therefore$ Sixth approximation :

$$x_6 = \frac{1}{2}(2.71875+2.75)$$

$$x_6 = 2.734375$$

$$f(x_6) = f(2.734375) = -0.00546 \ (-)\text{ve}$$

Hence roots lies between 2.73475 and 2.75

$\therefore$ Seventh approximation :

$$x_7 = \frac{1}{2}(2.73475+2.75)$$

$$x_7 = 2.74218175$$

$$f(x_7) = f(2.74218175) = 0.0013 \ (+)\text{ve}$$

Hence root lies between 2.734375 and 2.7421875

$\therefore$ Eighth approximation :

$$x_8 = \frac{1}{2}(2.734375+2.7421875)$$

$$x_8 = 2.73828125$$

$$f(x_8) = f(2.73828125) = -0.002 \ (-\text{ve})$$

Hence root lies between 2.73828125 and 2.7421875.

$\therefore$ Ninth approximation :

$$x_9 = \frac{1}{2}(2.73828125+2.7421875)$$

$$x_9 = 2.740234$$

$$f(x_9) = f(2.740234) = -0.00035 \ (-)\text{ve}$$

Hence root lies between 2.7402 and 2.7421

$\therefore$ Tenth approximation :

$$x_{10} = \frac{1}{2}(2.7402+2.7421)$$

$$x_{10} = 2.74115$$

$$f(x_{10}) = f(2.74115) = -0.00043 \text{ (+)ve}$$

Hence the root lies between 2.7402 and 2.74115.

∴ Eleventh approximation :

$$x_{11} = \frac{1}{2}(2.7402 + 2.74115)$$

$$x_{11} = 2.740675$$

$$f(x_{11}) = f(2.740675) = 0.000025 \text{ (+)ve}$$

Hence root lies between 2.7402 and 2.740675.

∴ Twelfth approximation :

$$x_{12} = \frac{1}{2}(2.7402 + 2.740675)$$

$$x_{12} = 2.7404$$

$$f(x_{12}) = f(2.7404) = -0.00021 \text{ (–)ve}$$

Hence root lies between 2.7404 and 2.70675.

∴ Thirteen approximation :

$$x_{13} = \frac{1}{2}(2.7404 + 2.70675)$$

$$x_{13} = 2.7405$$

$$f(x_{13}) = f(2.7405) = -0.00012 \text{ (–)ve}$$

Hence root lies between 2.7405 and 2.740675.

∴ Fourteenth approximation :

$$x_{14} = \frac{1}{2}(2.7405 + 2.740675)$$

$$x_{14} = 2.74058$$

$$f(x_{14}) = f(2.74058) = -0.00005 \text{ (–)ve}$$

Hence root lies between 2.74058 and 2.740675

∴ Fifteenth approximation :

$$x_{15} = \frac{1}{2}(2.74058 + 2.740675)$$

$$x_{15} = 2.7406$$

Since x_{14} and x_{15} are same upto four decimal places. Hence the approximate real root is 2.7406.

Example 7.3 : *Find root of equation $x^3 - 2x - 5 = 0$ which lies between 2 and 3 using bisection method is five step.*

Solution :

Here, $$f(x) = x^3 - 2x - 5 = 0$$

$$f(2) = -1$$

$$f(3) = 16$$

Root lies between 2 and 3. So, here a = 2 and b = 3.

1^{st} approximation :

$$x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$$

$$f(2.5) = 5.625 \text{ (+ve)}$$

Root lies between 2 and 2.5.

2^{nd} approximation :

$$x_2 = \frac{2+2.5}{2} = 2.25$$

$$f(2.25) = 1.890625 \text{ (+ve)}$$

Root lies between 2 and 2.25.

3^{rd} approximation :

$$x_3 = \frac{2+2.25}{2} = 2.125$$

$$f(2.125) = 0.345 \text{ (+ve)}$$

Root lies between 2 and 2.125.

4^{th} approximation :

$$x_4 = \frac{2+2.125}{2} = 2.0625$$

$$f(2.0625) = -0.351 \text{ (–ve)}$$

Root lies between 2.125 and 2.0625.

5^{th} approximation :

$$x_5 = \frac{2.125+2.0625}{2} = 2.09375$$

Example 7.4 : *Find root of equation $x^3 - 4x - 9 = 0$ using bisection method correct to 3 decimal places.*

Solution :

$$f(x) = x^3 - 4x - 9 = 0$$

$$f(0) = -9$$

$$f(1) = -12$$

$$f(2) = -9 \text{ (–ve)}$$

$$f(3) = 6 \text{ (+ve)}$$

Root lies between 2 and 3.

So, $a = 2, b = 3$

1^{st} approximation :

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(2.5) = -3.375 \text{ (–ve)}$$

Root lies between 2.5 and 3.

2^{nd} approximation :

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$$f(2.75) = 0.79 \text{ (+ve)}$$

Root lies between 2.5 and 2.75.

3^{rd} approximation :

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

$$f(2.625) = -1.412 \text{ (–ve)}$$

Root lies between 2.625 and 2.75.

4^{th} approximation :

$$x_4 = \frac{2.75+2.625}{2} = 2.6875$$

$$f(2.6875) = -0.339 \text{ (–ve)}$$

Root lies between 2.75 and 2.6875.

5^{th} approximation :

$$x_5 = \frac{2.75+2.6875}{2} = 2.7185$$

$$f(2.7185) = 0.22 \text{ (+ve)}$$

Root lies between 2.6875 and 2.7185.

6^{th} approximation :

$$x_6 = \frac{2.6875+2.7185}{2} = 2.703$$

$$f(2.703) = -0.063$$

Root lies between 2.703 and 2.7185.

7^{th} approximation :

$$x_7 = \frac{2.703+2.7185}{2} = 2.7094$$

$$f(2.7094) = 0.051 \text{ (+ve)}$$

Root lies between 2.703 and 2.7094.

8^{st} approximation :

$$x_8 = \frac{2.703+2.7094}{2} = 2.7062$$

$$f(2.7062) = -5.89\times10^{-3} \text{ (–ve)}$$

Root lies between 2.7062 and 2.7094.

9^{th} approximation :

$$x_9 = \frac{2.7062+2.7094}{2} = 2.7078$$

$$f(2.7078) = 0.022 \text{ (+ve)}$$

Root lies between 2.7062 and 2.7078.

10^{th} approximation :

$$x_{10} = \frac{2.7062+2.7078}{2} = 2.707$$

$$f(2.707) = 4.48\times10^{-3} \text{ (+ve)}$$

Root lies between 2.7062 and 2.707.

11^{th} approximation :

$$x_{11} = \frac{2.7062+2.707}{2} = 2.7066$$

7.3 Regula-Flasi Method

The bisection method guarantees that the iterative process will converge. It is however slow. Thus attempt have been made to speed up bisection method retaining its guaranteed convergence. A method of doing this is called the method of false position.

It is sometimes known as method of linear interpolation.

This is oldest method for finding the real root of numerical equation and closely resembles the bisection method. In this method.

Let $$f(x) = 0 \qquad \text{...(7.1)}$$

Let y = f(x) be represented by the curve AB cuts the X - axis at P.

The real root of (1) is OP.

The false position of curve AB is taken as the chord AB.

The chord AB cuts the X-axis at Q. The approximates root of f(x) = 0 is OQ. By this method, we find OQ.

Let A[a, f(a)], B[b, f(b)] be the extremities of the chord AB. The equation of the chord AB is

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

To find OQ, put y = 0

$$0 - f(a) = \frac{f(b)+f(a)}{b-a}(x-a)$$

$$(x - a) = -\frac{(b-a)f(a)}{f(b)-f(a)}$$

$$x = a+\frac{(a-b)f(a)}{f(b)-f(a)}$$

$$\boxed{x = \frac{a\,f(b)-b\,f(a)}{f(b)-f(a)}}$$

We continue the literation till the root is found to the desired accuracy. This method is also known as method of linear interpolation.

Example 7.5 : *Find the real root of the equation $x \log_{10} x - 1.2 = 0$ by the method of false position method correct to four decimal places.*

Solution : Here $f(x) = x\log_{10}x-1.2 = 0$

$$f(2) = 2\log_{10}2 - 1.2 = -0.59794 < 0$$
$$f(3) = 3\log_{10}3 - 1.2 = 0.23136 > 0$$

one root lies between 2 and 3.

taking a = 2, b = 3

$$f(2) = -0.59794, \quad f(3) = 0.23136$$

By method of false position; we have

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$
$$= \frac{2\,f(3) - 3\,f(2)}{f(3) - f(2)}$$
$$= \frac{2\,(0.23136) - 3(-0.59794)}{0.23136 - (-0.59794)}$$
$$x_1 = 2.72102$$
$$f(2.72102) = 2.72102\log_{10}2.72102 - 1.2$$
$$= -0.01709 < 0$$

Since f(2.72102) and f(3) are of opposite sign. So the root lies between 2.72102 and 3.

$$x_2 = \frac{2.72102\,f(3) - 3\,f(2.72102)}{f(3) - f(2.72102)}$$
$$= \frac{2.72102\,(0.23136) - 3(-0.01709)}{0.23136 - (-0.01709)}$$
$$x_2 = 2.74021$$

Now f(2.74021) = – 0.00038

Since f(2.74021) and f(3) are of opposite sign. So the root lies between 2.74021 and 3.

$$x_3 = \frac{2.74021\,f(3) - 3\,f(2.74021)}{f(3) - f(2.74021)}$$
$$x_3 = \frac{2.74021(0.23136) - 3(-0.00038)}{0.23136 - (-0.00038)}$$
$$x_3 = 2.74064$$

Again $\quad f(x_3) = 2.74064\log_{10}2.74064 - 1.2$

$$f(3) = -0.00001$$

root lies between 2.74064 and 3.

$$x_4 = \frac{2.74064\,f(3) - 3\,f(2.74064)}{f(3) - f(2.74064)}$$
$$= \frac{2.74064\,(0.23136) - 3(-0.00001)}{0.23136 - (-0.00001)}$$
$$x_4 = 2.74065$$

Here the root correct to four decimal place is 2.7406.

Example 7.6 : *Find the root of the equation $x^3 - 5x - 7 = 0$ which lies between 2 and 3 by method of false position.*

Solution : Here we have

$$f(x) = x^3 - 5x - 7 = 0$$
$$f(2) = 8 - 10 - 7 = -9$$
$$f(3) = 27 - 15 - 7 = +5$$

As f(2) and f(3) are of opposite sign so the root lies between 2 and 3.

First iteration :

$$x_1 = \frac{af(b) - bf(a)}{f(b) - af(a)}$$

Here $\quad a = 2, b = 3$

$$f(2) = -9, \ f(3) = 5$$
$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$
$$= \frac{2(5) - 3(-9)}{5 - (-9)} = \frac{37}{14} = 2.6429$$
$$x_1 = 2.6429$$

Now $\quad f(2.6429) = (2.6429)^3 - 5(2.6429) - 7$

$$= -1.7541$$

Again f(2.6429) and f(3) are of opposite sign. Root lies between 2.6429 and 3.

$$a = 2.6429, \ b = 3$$

Second iteration :

$$x_2 = \frac{2.6429f(3) - 3f(2.6429)}{f(3) - f(2)}$$
$$= \frac{2.6429(5) - 3(-1.7541)}{5 - (-1.7541)}$$
$$= \frac{18.4768}{6.7541} = 2.7356$$

Now $\quad f(2.7356) = (2.7356)^3 - 5(2.7356) - 7$

$$= -0.2061$$

Root lies between 2.7356 and 3

Third iteration :

$$a = 2.7356, \ b = 3$$
$$x_3 = \frac{2.7356(f)3 - 3f(2.7356)}{f(3) - f(2.7356)}$$
$$= \frac{2.7356(5) - 3(-0.2061)}{5 - (-0.2061)}$$

$$= \frac{14.2963}{5.2061} = 2.7461$$

Now $f(2.7461) = (2.7461)^3 - 5(2.7461) - 7$

$$= -0.02198$$

roots lies between 2.7461 and 3

Four iteration :

$$x_4 = \frac{2.7461 f(3) - 3f(2.7461)}{f(3) - f(2.7461)}$$

$$= \frac{2.7461(5) - 3(-0.02198)}{5 - (-0.02198)}$$

$$= \frac{13.79644}{5.02198} = 2.7472$$

Hence root of given equation is 2.7472

Example 7.7 : *Find a real root of equation $x^3 - 2x - 5 = 0$ by method of Regula-Falsi method correct to three decimal place.*

Solution :

Let $f(x) = x^3 - 2x - 5 = 0$

$$f(1) = 1 - 2 - 5 = -6$$

$$f(2) = 8 - 4 - 5 = -1 \ (-\text{ ve})$$

$$f(3) = 27 - 6 - 5 = 16 \ (+\text{ ve})$$

The root lies between 2 and 3.

Let $a = 2, b = 3$

$$f(2) = -1, f(3) = 16$$

First iteration :

$$x_1 = \frac{af(b) - bf(a)}{f(b) - af(a)}$$

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$

$$= \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{35}{17} = 2.0588$$

$$x_1 = 2.0588$$

Now $f(2.0588) = (2.0588)^3 - 5(2.0588) - 5$

$$= -0.3911 \ (-) \text{ ve}$$

Root lies between 2.0588 and 3

Second iteration :

$$x_2 = \frac{2.0588 f(3) - 3f(2.0588)}{f(3) - f(2.0588)}$$

$$x_2 = \frac{2.0588(16) - 3(-0.3911)}{16 - (-0.3911)}$$

$$= \frac{34.1141}{16.3911} = 2.0813$$

Now $f(2.0813) = (2.0813)^3 - 2(2.0813) - 5$

$$= -0.1468 \ (-\text{ ve})$$

Root lies between 2.0813 and 3

Third iteration :

$$x_3 = \frac{2.0813 f(3) - 3f(2.0813)}{f(3) - f(2.0813)}$$

$$= \frac{2.0813(16) - 3(-0.1468)}{16 - (-0.1468)}$$

$$= \frac{33.7412}{16.1468} = 2.0897$$

Now $f(2.0897) = (2.0897)^3 - 2(2.0813) - 5$

$$= -0.0540 \ (-\text{ ve})$$

Roots lies between 2.0897 and 3

Fourth iteration :

$$x_4 = \frac{2.0897 f(3) - 3f(2.089)}{f(3) - f(2.0897)}$$

$$= \frac{2.0897(16) - 3(-0.0540)}{16 - (-0.0540)}$$

$$x_4 = \frac{33.5972}{16.0540} = 2.0928$$

Now $f(2.0928) = (2.0928)^3 - 2(2.0928) - 5$

$$= -0.0195$$

Root lies between 2.0928 and 3

Fifth iteration :

$$x_5 = \frac{2.0928 f(3) - 3f(2.0928)}{f(3) - f(2.0928)}$$

$$x_5 = \frac{2.0928(16) - 3(-0.0195)}{16 - (-0.0195)}$$

$$= \frac{33.5433}{16.0195} = 2.0939$$

Now $f(2.0939) = (2.0939)^3 - 2(2.0939) - 5$

$$= -0.0073$$

Root of lies between 2.0939 and 3

Sixth iteration :

$$x_6 = \frac{2.0939 f(3) - 3f(2.0939)}{f(3) - f(2.0939)}$$

$$= \frac{2.0939(16) - 3(-0.0073)}{16 - (-0.0073)}$$

$$= \frac{33.5243}{16.0073} = 2.0943$$

Now $f(2.0943) = (2.0943)^3 - 2(2.0943) - 5$

$= -0.0028$

Seventh iteration :

$$x_7 = \frac{2.0943f(3) - 3f(2.0943)}{f(3) - f(2.0943)}$$

$$x_7 = \frac{2.0943(16) - 3(-0.0028)}{16 - (-0.0028)}$$

$$= \frac{33.5172}{16.0028} = 2.0945$$

Hence $x_6 = x_7$ are correct to three decimal places. So required root of the given equation is 2.094

7.4 Secant Method

The secant method requires two initial approximations x_0 and x_1, preferably both reasonably close to the solution x^*. From x_0 and x_1, we can determine that the points $(x_0, y_0) = f(x_0)$ and $(x_1, y_1 = f(x_1))$ both lie on the graph of f. Connecting these points given the (secant) line.

$$y - y_1 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_1)$$

Since, we want f(x) = 0, we set y = 0, solve for x and use that as our next approximation. Repeating this process gives us the iteration.

$$x_{i+1} = x_i - \frac{x_i - x_{i-1}}{y_i - y_{i-1}} y_i \qquad \ldots(7.2)$$

with $y_i = f(x_i)$. Below Fig. 7.1 is for an illustration.

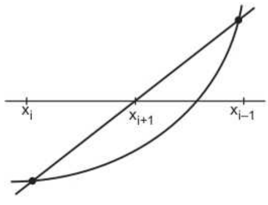

Fig. 7.1 The secant method in the case when root is bracketed

- For example suppose $f(x) = x^4 - 5$, which has a solution $x^* = \sqrt[4]{5} \approx 1.5$. Choose $x_0 = 1$ and $x_1 = 2$ as initial approximation, we have, $y_0 = f(1) = -4$ and $y_1 = f(2) = 11$. We may then calculate x_2 from the formulae.

$$x_2 = 2 - \frac{2-1}{11-(-4)}11 = \frac{19}{15} \approx 1.2666$$

Plugging $x_2 = \frac{19}{15}$ into f(x) we obtain $y_2 = f\left(\frac{19}{15}\right)$ ≈ -2.425.

In the next step we would use $x_1 = 2$ and $x_2 = \frac{19}{15}$ in the formula (7.2) find x_3 and so on.

7.5 Newton's Raphson's Method

- The solution to an equation f(x) = 0 may often be found by a simple procedure called Newton-Raphson method.
- Let $x = x_0$ be the initial or starting value of the root. This method is generally used to improve the result obtained by the previous method.
- This method consists of refacing the part of the curve between points $[x_0, f(x_0)]$ and the x-axis by means of the tangent to the curve at the point and is described graphically in Fig. 7.2.

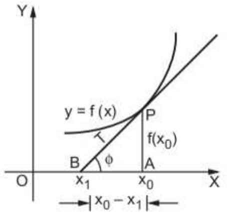

Fig. 7.2

- The interrupt OT on the x-axis of the tangent to the curve at the point P is taken as the first approximation. From the figure,

$$\tan\phi = \frac{AD}{BA}$$

$$\tan\phi = \frac{f(x_0)}{x_0 - x_1}$$

But $\tan\phi = \frac{dy}{dx} = f'(x_0) \therefore y = f(x)$

this gives $f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$

$$(x_0 - x_1)f'(x_0) = f(x_0)$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Repeating the process replacing x_0 by x_1 we get second iteration as.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \text{ and so on}$$

In general, after (n + 1) iterations, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $n = 0, 1, 2, 3 \ldots$

Example 7.8 : *Find a real root of equation $x^4 - x - 10 = 0$ correct to three decimal place by using Newton Raphson method.*

Solution : $f(x) = x^4 - x - 10 = 0 \quad f(0) = -10$

$$\left.\begin{aligned} f(1) &= -10(-ve) \\ f(2) &= 4(+ve) \end{aligned}\right\} \text{Root lies between 1 and 2}$$

$$x_0 = \frac{1+2}{2} = 1.5$$

1^{st} approximation,

(n = 0)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1.5 - \left[\frac{(1.5)^4 - 1.5 - 10}{4(1.5)^3 - 1}\right] = 2.015$$

Note : Continue finding root until value upto 3 decimal will be equal.

2^{nd} approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.015 - \left[\frac{(12.015)^4 - 2.015 - 10}{4(2.015)^3 - 1}\right]$$

$$x_2 = 1.87409$$

3^{rd} approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 1.87409 - \left[\frac{(1.87409)^4 - 1.87409 - 10}{4(1.87409)^3 - 1}\right]$$

$$x_3 = 1.85586$$

4^{th} approximation

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 1.85586 - \left[\frac{(1.85586)^4 - 1.85586 - 10}{4(1.85586)^3 - 1}\right]$$

$$x_4 = 1.85586$$

3^{rd} and 4^{th} approximation are same.

Hence, one root is 1.8558.

Example 7.9 : *Find a root by N-R method of equation $x^3 - 3x + 1 = 0$ correct to 3 decimal place.*

Solution : $f(x) = x^3 - 3x + 1$

$$x(0) = 0,\ f(0) = 1$$

$$x = 1,\ f(1) = -1,\ x_0 = \frac{0+1}{2} = 0.5$$

1^{st} approximation, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$x_1 = 0.5 - \frac{(0.5)^3 + 3 \times 0.5 - 1}{3(0.5)^2 - 3}$$

$$x_1 = 0.3333$$

2^{nd} approximation,

$$x_2 = 0.3333 - \frac{(0.3333)^3 + 3 \times 0.3333 - 1}{3(0.3333)^2 - 3}$$

$$x_2 = 0.34721$$

3^{rd} approximation,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.34721 - \frac{(0.34721)^3 + 3 \times 0.34721 - 1}{3(0.34721)^2 - 3}$$

$$x_3 = 0.34729$$

2^{nd} and 3^{rd} approximation is name.

Hence are root is 0.3472

Example 7.10 : *Apply Newton Raphson method to solve $3x - \cos x - 1 = 0$ correct to three decimal place.*

Solution : $f(x) = 3x - \cos x - 1 = 0$

$$f(0) = -2\ (-\text{ve})$$

$$f(1) = 1.459\ (+\text{ve}) \Rightarrow x_0 = 0.5$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \frac{3(0.5) - \cos(0.5) - 1}{3 + \sin(0.5)}$$

$$x_1 = 0.6085$$

$$x_2 = 0.6085 - \frac{3(0.6085) - \cos(0.6085) - 1}{3 + \sin(0.6085)}$$

$$x_2 = 0.6071$$

$$x_3 = 0.6071 - \frac{3(0.6071) - \cos(0.6071) - 1}{3 + \sin(0.6071)}$$

$$x_3 = 0.6071$$

2^{nd} and 3^{rd} approximation is same.

Hence, one root is 0.6071.

Example 7.11 : *Using Newton Raphson method find real root of $x \log_{10} x = 1.2 = 0$ correct to five decimal places.*

Solution :

$$f(x) = x \log_{10} x - 1.2 = 0$$
$$= x(0.4343)\log_e x - 1.2$$
$$f(2) = -0.597940 \; (-\text{ ve})$$
$$f(3) = 0.2313 \; (+\text{ ve})$$

Root lies between 2 and 3.

$$f(x) = x \log_{10} x - 1.2 = 0$$
$$= x\left(\frac{\log_e 10}{\log e}\right) - 1.2$$
$$f(x) = x[0.4343 \log_e x] - 1.2$$
$$f'(x) = 0.4343\left[x \cdot \frac{1}{x} + \log_e x\right] - 0$$
$$f'(x) = 0.4343 + 0.4343 \log_e x$$
$$= 0.4343(1 + \log_e x)$$
$$x_0 = \frac{2+3}{2} = 2.5$$

1^{st} approximation,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{[(2.5)\log_{10}(2.5) - 1.2]}{[0.4343(1 + \log_e 25)]}$$

$$x_1 = 2.74650$$

2^{nd} approximation,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.74650 - \left(\frac{(274650)\log_{10} 2.74650 - 1.2}{0.4343(1 + \log_e 2.74650)}\right)$$

$$x_2 = 2.740649$$

3^{rd} approximation,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 2.740649 - \left(\frac{(2.740649)\log_{10} 2.740649 - 1.2}{0.4343(1 + \log_e 2.740649)}\right)$$

$$x_3 = 2.74064$$

Root locus 2.74064.

Example 7.12 : *Find a root of equation $x\,e^x - \cos x$ by N - R method upto three decimal place.*

Solution :

$$f(x) = x \cdot e^x - \cos x$$
$$f(0) = 0.1 = -1$$
$$f(1) = 1.77$$
$$x_0 = \frac{0+1}{2} = 0.5$$
$$\delta'(x) = x \cdot e^x + e^x \cdot 1 - \sin x$$
$$= (x+1)e^x + \sin x$$
$$x_1 = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$n = 0$$
$$x_1 = 0.5 - \frac{(0.5)e^{0.5} - \cos 0.5}{(0.5+1)e^{0.5} + \sin 0.5}$$
$$= 0.57075537$$

Example 7.13 : *Evaluate $\sqrt{12}$ to four decimal place by Newton - Raphson method.*

Ans. : Let

$$x = \sqrt{12}$$
$$x^2 = 12$$
$$f(x) = x^2 - 12 = 0$$
$$f(3) = -3$$
$$f(4) = 4$$

Root lies between + 3 and 4

$$x_0 = \frac{3+4}{2} = 3.5$$

1^{st} approximation,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.5 - \left(\frac{(3.5)^2 - 12}{2(3.5)}\right)$$

$$x_1 = 3.46429$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 3.46429 - \left(\frac{(3.464290)^2 - 12}{2(3.46429)}\right)$$
$$= 3.46410$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 3.46410 - \left(\frac{(3.46410)^2 - 12}{2(3.46410)}\right)$$
$$= 3.4640$$

Example 7.14 : *Find the negative root of equation $x^3 - 21x + 3500 = 0$ correct to three decimal place by Newton - Raphson method.*

Solution :

$$f(x) = x^3 - 21x + 3500$$

$$x' = 3x^2 - 21$$

$$f(-15) = 440$$

$$f(-10) = -260$$

Root lies between -15 and -16.

$$x_0 = \frac{-15-16}{2} = -15.5$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= -15.5 - \left[\frac{(-15.5)^3 - 21(-15.5) + 3500}{3(-15.5)^2 - 21}\right]$$

$$x_1 = -15.6452$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -15.6452 - \left(\frac{(-15.6452)^3 - 21(-15.6452) + 3500}{3(-15.6452)^2 - 21}\right)$$

$$= -15.6438$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -15.6438 - \left(\frac{(-15.6438)^3 - 21(-15.6430) + 3500}{3(-15.6438)^2 - 21}\right)$$

$$= -15.6438$$

7.6 Numerical Solutions of System of Linear Equations

7.6.1 Gauss Elimination Method

This is the elementary elimination method and it reduces the given system of linear equations to an equivalent upper triangular system of linear equations which is then solved by backward substitution. To describe the method, we consider the system of three equations in three unknowns for the sake of clarity and simplicity.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \qquad \ldots(7.3)$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The **Gauss elimination** method consists of the following steps.

Step 1 : Elimination of x_1 : Assuming $a_{11} \neq 0$ we divide coefficients of the first equation of system (1) by a_{11}, we get

$$x_1 + a'_{12}x_2 + a'_{13}x_3 = b'_1 \qquad \ldots(7.4)$$

Where $\quad a'_{12} = \frac{a_{12}}{a_{11}}, \; a'_{13} = \frac{a_{13}}{a_{11}}$ and $b'_1 = \frac{b_1}{a_{11}}$

Using equation (7.4), we now eliminate x_1 from the remaining equations [i.e. From second and third equations of (7.3)] by subtracting a_{21} times equation (7.4) from second equation of (7.3), and a_{31} times equation (7.4) from third equation of (7.3). We thus get system consisting of two equations in two unknowns, namely x_2 and x_3 as,

$$a'_{22}x_2 + a'_{23}x_3 = b'_2$$
$$a'_{32}x_2 + a'_{33}x_3 = b'_3 \qquad \ldots(7.5)$$

where $\quad a'_{22} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}, \; a'_{23} = a_{23} - \frac{a_{21}}{a_{11}}a_{13},$

$$b'_2 = b_2 - \frac{a_{21}}{a_{11}}b_1, \; a'_{32} = a_{32} - \frac{a_{31}}{a_{11}}a_{12},$$

$$a'_{33} = a_{33} - \frac{a_{31}}{a_{11}}a_{13},$$

$$b'_3 = b_3 - \frac{a_{31}}{a_{11}}b_1.$$

Step 2 : Elimination of x_2 : Assuming $a'_{22} \neq 0$, we divide the coefficients of the first equation of system (7.5), we get

$$x_2 + a''_{23}x_3 = b''_2 \qquad \ldots(7.6)$$

Where $\quad a''_{23} = \frac{a'_{23}}{a'_{22}}, \; b''_2 = \frac{b'_2}{a'_{22}}$

Using equation (7.6), we next eliminate x_2 from second equation of (7.5) by subtracting a'_{32} times equation (7.6) from second equation of (7.5), we thus get

$$a''_{33}x_3 = b''_3 \qquad \ldots(7.7)$$

Where $\quad a''_{33} = a'_{33} - \frac{a'_{32}}{a'_{22}}a'_{23}, \; b''_3 = b'_3 - \frac{a'_{32}}{a'_{22}}b'_2$

Step 3 : Determination of unknowns x_3, x_2, x_1 : Collecting the first equation from each stage i.e. from equations (7.4), (7.6) and (7.7), we obtain

$$x_1 + a'_{12}x_2 + a'_{13}x_3 = b_1$$
$$x_2 + a''_{23}x_3 = b''_3 \qquad \ldots(7.8)$$

$$x_3 = \frac{b_3''}{a_{33}''}$$

The system (6) is an upper triangular system and can be solved by a process called back substitution. The last equation determine x_3, which is then substituted in the next last equation to determine x_2 and finally on substituting x_3 and x_2 in the first equation, we obtain x_1.

Remark 1 : The elements a_{11}, a'_{22} and a''_{33} which have been assumed to be non-zero are called **pivot elements** and the equations containing these pivot elements are called **pivotal equations.**

Remark 2 : The process of dividing by a_{11} to first equation of system (1) for making coefficient of x_1 unity is called **normalization.**

Remark 3 : A necessary and sufficient condition for using Gauss elimination method is that all leading elements (pivot elements) be non-zero.

In the elimination process, if any one of the pivot elements a_{11}, a'_{22} ... vanishes or becomes very small compared to other elements in that equation then we attempt to rearrange the remaining equation so as to obtain a non-vanishing pivot or to avoid the division by small pivot element (equivalently multiplication by large number) to every coefficient in the pivotal equation. This procedure is a called **partial pivoting.** If the matrix A is diagonally dominant or real, symmetric and positive definite then no pivoting is necessary.

We note that, if the numerators of any fractions contain rounding errors, the effects of such errors are diminished when the denominator is large pivot.

I) Partial Pivoting : In the first step of elimination, the numerically largest coefficient of x_1 is chosen from all the equations and brought as first pivot by interchanging the first equation with equation having the largest coefficient of x_1. In the second step of elimination the largest coefficient of x_2 in magnitude is chosen from the remaining equations (leaving the first equation) and brought as second pivot by interchanging the second equation with the equation having the largest coefficient of x_2. This process is continued till we arrive at the equation with the single variable. In other words, partial pivoting involves searching for largest coefficient of an unknown quantity amongst a systems of equations at each step of the elimination.

Complete Pivoting : A slightly better result may be obtain by disregarding the order of elimination of x_1, x_2, x_3 and solving at each step of elimination process, the equation in which largest coefficient in the entire set (or in the entire matrix of coefficient) occurs. This requires not only an interchange of equations but also interchange of the position of the variables. The equation is then solved for that unknown to which the largest coefficient is attached. This process is called as **complete pivoting** and is rarely employed as it changes the order of unknowns and consequently adds complexity.

Remark 4 : The Gauss elimination method can be interpreted in matrix form as reducing the augmented matrix to upper triangular from whose leading diagonal elements are unity by using row operations only. Thus.

$$[A \mid B] \xrightarrow[\text{Elimination}]{\text{Gauss}} [U \mid B']$$

Where $[A \mid B]$ is the augmented matrix.

II) III-conditioned Linear Systems : In practical applications, one encounters systems in which small change in the coefficients of the system or right-hand side terms result in very large changes in the solution. Such systems are said to be **III-conditioned.** If the corresponding changes in the solution are also small, then the system is **well-conditioned.**

Example 7.15 : *Use Gauss elimination method to solve the following system of equations.*

$$x_1 + 4x_2 - x_3 = -5$$
$$x_1 + x_2 - 6x_3 = -12$$
$$3x_1 - x_2 - x_3 = 4$$

SPPU : Dec.-16

Solution : Given system is

$$\left.\begin{aligned} x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 6x_3 &= -12 \\ 3x_1 - x_2 - x_3 &= 4 \end{aligned}\right\} \quad \text{... (1)}$$

Step 1 : Since coefficient of x_1 is unity in the first equation, we eliminate x_1 from second and third equations of (1) by subtracting first equation from second equation and 3 times first equation from third equation of (1), we obtain

$$-3x_2 - 5x_3 = -7 \quad \text{... (2)}$$
$$-13x_2 + 2x_3 = 19$$

Step 2 : Dividing the first equation (2) by -3, we get

$$x_2 + \frac{5}{3}x_3 = \frac{7}{3} \quad \text{... (3)}$$

Using equation (3), we next eliminate x_2 from second equation of (2) by adding 13 times equation (3) in second equation of (2), we get

$$\frac{71}{3}x_3 = \frac{148}{3}$$

Or
$$x_3 = \frac{148}{71} \quad \dots (4)$$

Step 3 : Collecting first equation of (1) and equations (3) and (4), we have

$$\begin{aligned} x_1 + 4x_2 - x_3 &= -5 \\ x_2 + \frac{5}{3}x_3 &= \frac{7}{3} \\ x_3 &= \frac{148}{71} \end{aligned} \quad \dots (5)$$

The system (5) is equivalent upper triangular system. Using backward substitution, the solution is

$$x_1 = \frac{117}{71},\ x_2 = \frac{-81}{71} \text{ And } x_3 = \frac{148}{71}$$

Example 7.16 : *Solve the following system of equations by Gauss elimination method :*

$10x + 2y + z = 9$

$2x + 20y - 22 = -44$

$-2x + 3y + 10z = 22$ **SPPU : May-16**

Solution : Step 1 : Given that

$$10x + 2y + z = 9 \quad \dots (1)$$

$$2x + 20y - 22 = -44 \quad \dots (2)$$

$$-2x + 3y + 10z = 22 \quad \dots (3)$$

Dividing equation (1) by 10, we get

$$x + \frac{2}{10}y + \frac{1}{10}z = \frac{9}{10} \quad \dots (4)$$

Eequation (2) – Equation (4) × 2

$$\Rightarrow \quad \frac{196}{10}y - \frac{22}{10}z = \frac{-458}{10} \quad \dots (5)$$

Eequation (3) + Equation (4) × 2

$$\Rightarrow \quad \frac{32}{10}y + \frac{102}{10}z = \frac{238}{10} \quad \dots (6)$$

Dividing equation (5) by $\frac{196}{10}$, we get

$$y - \frac{22}{196}z = -\frac{458}{196} \quad \dots (7)$$

$\therefore$ Equation (6) $- \frac{32}{10}$ Equation (7)

$$\Rightarrow \quad \left(\frac{102}{10} + \frac{22 \times 32}{1960}\right)z = \frac{238}{10} + \frac{458 \times 32}{1960}$$

$$\frac{20696}{1960}z = \frac{61304}{1960}$$

$$\therefore \quad z = \frac{61304}{20696} = 2.9621$$

$$\text{Equation (6)} \Rightarrow y = \frac{238}{10} - \frac{102}{10}z = -6.41342$$

$$\text{equation (4)} \Rightarrow x = -\frac{2}{10}y - \frac{1}{10}z + \frac{9}{10}$$

$$= 1.886474$$

Thus the solution is $x = 1.8864$, $y = -6.41342$,

$$z = 2.9621$$

Example 7.17 : *Solve the system*

$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1$$
$$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = 0$$
$$\frac{1}{3}x_1 + \frac{1}{4}x_2 + \frac{1}{5}x_3 = 0$$

Solution : Given system is

$$\left.\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 &= 1 \\ \frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 &= 0 \\ \frac{1}{3}x_1 + \frac{1}{4}x_2 + \frac{1}{5}x_3 &= 0 \end{aligned}\right\} \quad \dots (1)$$

Step 1 : Since coefficient of x_1 in the first equation of the given system is unity, we eliminate x_1 from second and third equations by multiplying first equation of (1) by $\frac{1}{2}$ and $\frac{1}{3}$ and then subtracting from second and third equations respectively, we get

$$\begin{aligned} \frac{1}{12}x_2 + \frac{1}{12}x_3 &= -\frac{1}{2} \\ \frac{1}{12}x_2 + \frac{4}{45}x_3 &= -\frac{1}{3} \end{aligned} \quad \dots (2)$$

Step 2 : Divide the first equation by $\frac{1}{12}$ i.e. Multiply by 12, we get

$$x_2 + x_3 = -6 \quad \dots (3)$$

Using equation (3), we next eliminate x_2 from second equation of (2), we get

$$\frac{1}{180}x_3 = \frac{1}{6}$$

Or
$$x_3 = 30 \quad \dots (4)$$

Now collecting first equation of system (1), equation (3) and equation (4), we obtain

$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1$$

$$x_2 + x_3 = -6 \quad \dots (5)$$

$x_3 = 30$

Which is equivalent upper triangular system. Using backward substitution, the solution is

$x_1 = 9$, $x_2 = -36$, $x_3 = 30$

7.6.2 The Gauss Seidel Iteration Method

The Gauss-Seidel method is a modification of the Jacobi's iteration method. As in Jacobi's iteration method, consider a system of equations in which each equation is first solved for unknown having large coefficient, thereby expressing it explicitly in terms of other unknowns as

$$\left.\begin{aligned}
x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots\ldots - a_{1n}x_n)\\
x_2 &= \frac{1}{a_{22}}(b_2 - a_{11}x_1 - a_{13}x_3 \ldots\ldots\ldots - a_{2n}x_n)\\
x_3 &= \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}x_2 - a_{34}x_4 \ldots\ldots - a_{3n}x_n)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
x_4 &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 \ldots\ldots - a_{nn-1}x_{n-1})
\end{aligned}\right\} \quad \ldots (7.9)$$

We again start with initial approximations ($x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots x_n^{(0)} = 0$). However, this time we substitute these only in the first equation on the right hand side of system (7.9), we substitute $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots, x_n^{(0)})$ and denote the improved results $x_2^{(1)}$. In the third equation of (7.9), we substitute $(x_1^{(1)}, x_2^{(1)}, x_3^{(0)}, \ldots, x_n^{(0)})$ and call the result $x_3^{(1)}$. Proceeding like this we find first iteration values as $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)}$. This completes the first stage of iteration and the entire process is repeated till the values of $x_1, x_2, \ldots x_n$ are obtained to desired accuracy.

Thus, if the values of the variables in k^{th} iteration are $x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}$ then the values in the next $(k+1)^{th}$ iteration are given by

$$\left.\begin{aligned}
x_1^{(k+1)} &= \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} \ldots\ldots\ldots - a_{1n}x_n^{(k)})\\
x_2^{(k+1)} &= \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} \ldots\ldots\ldots - a_{2n}x_n^{(k)})\\
x_3^{(k+1)} &= \frac{1}{a_{33}}(b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - a_{34}x_4^{(k)} \ldots\ldots - a_{3n}x_n^{(k)})\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
x_n^{(k+1)} &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} \ldots\ldots\ldots - a_{nn-1}x_{n-1}^{(k+1)})
\end{aligned}\right\} \quad \ldots (7.10)$$

Above system of approximation (7.10) can be considered as a general formula for Gauss-Seidel iterative method.

Remark (1) : For any choice of the first (initial) approximation, Gauss–Seidel iterative method converges. Condition for convergence of iteration process is same as discussed in Jacobi's iteration method. That is "if in each equation of the system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients in that equation." In this method of iteration, the result of any stage within a step is used in succeeding stages of the same step, the method is also called method of **successive correction.**

Illustrations on Gauss–Seidel Method

Example 7.18 : *Solve the following system of equations by Gauss-Seidel method :*

$$13x_1 + 5x_2 - 3x_3 + x_4 = 18$$
$$2x_1 + 12x_2 + x_3 - 4x_4 = 13$$
$$3x_1 - 4x_2 + 10x_3 + x_4 = 29$$
$$2x_1 + x_2 - 3x_3 + 9x_4 = 31$$

Solution : Given system is

$$\begin{aligned} 13x_1 + 5x_2 - 3x_3 + x_4 &= 18 \\ 2x_1 + 12x_2 + x_3 - 4x_4 &= 13 \\ 3x_1 - 4x_2 + 10x_3 + x_4 &= 29 \\ 2x_1 + x_2 - 3x_3 + 9x_4 &= 31 \qquad \ldots (1) \end{aligned}$$

Since these equations meet the requirement for iteration (i.e. given system of equations is diagonally dominant), hence it can be solved by Gauss-Seidel iteration method. We now use successive equations to express for each unknown in terms of the others. Thus we obtain,

$$\begin{aligned} x_1 &= \frac{1}{13}[18 - 5x_2 + 3x_3 - x_4] \\ x_2 &= \frac{1}{12}[13 - 2x_1 - x_3 + 4x_4] \\ x_3 &= \frac{1}{10}[29 - 3x_1 + 4x_2 - x_4] \qquad \ldots (2) \\ x_4 &= \frac{1}{9}[31 - 2x_1 - x_2 + 3x_3] \end{aligned}$$

First iteration : Let us start iteration with trial (initial) approximation to the solution $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$.

In the Gauss–Seidel method, we substitute these only in the first equation of (2) (working to three decimals), which gives

$$x_1^{(1)} = \frac{1}{13}[18 - 5(0) + 3(0) - (0)] = 1.385$$

We then substitute $x_1^{(1)} = 1.385$, and $x_3^{(0)} = 0$, $x_4^{(0)} = 0$ in the second equation of (2), we get

$$x_2^{(1)} = \frac{1}{12}[13 - 2(1.385) - (0) + 4(0)] = 0.853$$

Putting $x_1^{(1)} = 1.385$,

$x_2^{(1)} = 0.853$ and $x_4^{(0)} = 0$ in the equation,

we get

$$x_3^{(1)} = \frac{1}{10}[29 - 3(1.385) + 4(0.853) - (0)] = 2.826$$

Finally, putting $x_1^{(1)} = 1.385$, $x_2^{(1)} = 0.853$

and $x_3^{(1)} = 2.826$ in the fourth equation, we get

$$\begin{aligned} x_4^{(1)} &= \frac{1}{9}[31 - 2(1.385) - (0.853) + 3(2.826)] \\ &= 3.984 \end{aligned}$$

Second iteration : The set of values $x_1^{(1)} = 1.385$, $x_2^{(1)} = 0.853$, $x_3^{(1)} = 2.826$ and $x_4^{(1)} = 3.984$ now becomes our second of trial solution.

Using these values, we have

$$\begin{aligned} x_1^{(2)} &= \frac{1}{13}[18 - 5(0.853) + 3(2.826) - (3.984)] = 1.402 \\ x_2^{(2)} &= \frac{1}{12}[13 - 2(1.402) - (2.826) + 4(3.984)] = 1.942 \\ x_3^{(2)} &= \frac{1}{10}[29 - 3(1.402) + 4(1.942) - (3.984)] = 2.858 \\ x_4^{(2)} &= \frac{1}{9}[31 - 2(1.402) - (1.942) + 3(2.858)] = 3.870 \end{aligned}$$

By continuing in this manner, successive iterations can be tabulated as

n	x_1	x_2	x_3	x_4
1	1.385	0.853	2.826	3.984
2	1.402	1.942	2.858	3.870
3	1.000	1.969	3.001	4.004
4	1.012	1.999	2.996	3.996
5	1.000	1.999	3.000	4.000

It can be easily checked that the correct answer is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$

Example 7.19 : *Solve the following system of equations by the Gauss–Seidel method :*

$$10x_1 + x_2 + x_3 = 12$$
$$2x_1 + 10x_2 + x_3 = 13$$
$$2x_1 + 2x_2 + 10x_3 = 14$$

Solution : Since these equations meet the requirement for iteration, we now use successive equations to express for each unknown in terms of others and thus get the system

$$\left.\begin{aligned} x_1 &= \frac{1}{10}[12 - x_2 - x_3] \\ x_2 &= \frac{1}{10}[13 - 2x_1 - x_3] \\ x_3 &= \frac{1}{10}[14 - 2x_1 - 2x_2] \end{aligned}\right\} \quad \dots (1)$$

First iteration : Let us start iteration with initial (zeroth) approximation to the solution as

$$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$$

We now substitute these only in the first equation of (1), we have

$$x_1^{(1)} = \frac{1}{10}[12 - (0) - (0)] = 1.2$$

We then substitute $x_1^{(1)} = 1.2$, $x_3^{(0)} = 0$ in the second equation of (1), we get

$$x_2^{(1)} = \frac{1}{10}[13 - 2(1.2) - (0)] = 1.06$$

We next put $x_1^{(1)} = 1.2$, $x_2^{(1)} = 1.06$ in third equation of (1), we get

$$x_3^{(1)} = \frac{1}{10}[14 - 2(1.2) - 2(1.06)] = 0.948$$

Second iteration : Using $x_1^{(1)} = 1.2$, $x_2^{(1)} = 1.06$ and $x_3^{(1)} = 0.948$ as second set of trial solution, we obtain

$$x_1^{(2)} = \frac{1}{10}[12 - 1.06 - 0.948] = 0.9992$$

$$x_2^{(2)} = \frac{1}{10}[13 - 2(0.9992) - (0.948)] = 1.00536$$

$$x_3^{(2)} = \frac{1}{10}[14 - 2(0.9992) - 2(1.00536)] = 0.999098$$

By continuing in this manner, the successive iterations compound correct to four decimal places are tabulated as

n	x_1	x_2	x_3
1	1.2	1.06	0.948
2	0.9992	1.0054	0.9991
3	0.9996	1.0001	1.0001
4	1.000	1.000	1.000

The solution of the given system is therefore

$x_1 = 1$, $x_2 = 1$ and $x_3 = 1$.

Example 7.20 : *Solve the following system of equations by Gauss Seidel method* $20x + 4y - z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24$

SPPU : Dec.-14, 15

Solution : The given linear system is diagonally dominant

$$\therefore \quad x = \frac{1}{20}(32 - 4y + z) \quad \dots (1)$$

$$y = \frac{1}{17}(35 - 2x - 4z) \quad \dots (2)$$

$$z = \frac{1}{10}(24 - x - 3y) \quad \dots (3)$$

First Iteration : Let us start iteration with initial approximation, $x^{(0)} = y^{(0)} = z^{(0)} = 0$

$$\therefore \quad x^{(1)} = \frac{1}{20}(32) = 1.6$$

Substitute $x^{(1)}$, $z^{(0)}$ in equation (2), we get

$$y^{(1)} = \frac{1}{17}(35 - 3.2) = 1.8705$$

Substitute $x^{(1)}$, $y^{(1)}$, in equation (3)

$$\therefore \quad z^{(1)} = \frac{1}{10}(24 - 1.6 - 5.6117) = 1.6788$$

Second Iteration : Substitute $y^{(1)}$, $z^{(1)}$ in equation (1)

$$\therefore \quad x^{(2)} = \frac{1}{20}(32 - 4y^{(1)} + z^{(1)}) = 1.3098$$

$$y^{(2)} = \frac{1}{17}(35 - 2x^{(2)} - 4z^{(1)}) = 1.5097$$

$$z^{(2)} = \frac{1}{10}(24 - x^{(2)} - 3y^{(2)}) = 1.8161$$

By continuing in this way, the successive iterations can be tabulated as

n	x	y	z
1	1.6	1.8705	1.6788
2	1.3098	1.5097	1.8161
3	1.3888	1.4681	1.8206
4	1.3974	1.4660	1.8204
5	1.3978	1.4660	1.8204

Thus, the solution is $x = 1.3978$, $y = 1.4660$, $z = 1.8204$.

7.6.3 Cholesky Method

Let AX = B be a Linear System.

If the coefficient matrix A is symmetric and positive definite then matrix A can be expressed as

$A = LL^t$ where L is the lower triangular matrix

or $A = UU^t$ where U is the upper triangular matrix

Putting $A = LL^t$ in AX = B, we get ... (7.11)

$$LL^tX = B$$

Let $$L^tX = Z \quad \dots (7.12)$$

$$\therefore \quad L(L^tX) = LZ = B \quad \dots (7.13)$$

Solving equation (7.13), we get Z and using it to solve equation (7.12), we get X.

Example 7.21 : *Solve the following system of equations by Cholesky's method* $2x_1 - x_2 = 1$

$-x_1 + 3x_2 + x_3 = 0$

$x_2 + 2x_3 = 0$ **SPPU : Dec.-14**

Solution : Given linear system can be written in matrix form as

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

i.e. $$AX = B \quad \dots (1)$$

We can express matrix A as $A = LL^t$

$$\therefore \quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Multiplying the right hand side matrices and equating corresponding elements of equation

we get
$$l_{11}^2 = 2 \Rightarrow l_{11} = \sqrt{2}$$
$$l_{11}\,l_{21} = -1 \Rightarrow l_{21} = -\frac{1}{\sqrt{2}}$$
$$l_{11}\,l_{31} = 0 \Rightarrow l_{31} = 0$$
$$l_{21}^2 + l_{22}^2 = 3 \Rightarrow l_{22} = \sqrt{\frac{5}{2}}$$
$$l_{21}\,l_{31} + l_{22}\,l_{32} = 1 \Rightarrow l_{32} = \sqrt{\frac{2}{5}}$$
$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 2 \Rightarrow l_{33} = \sqrt{\frac{8}{5}}$$

Thus, $$L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{5}{2}} & 0 \\ 0 & \sqrt{\frac{2}{5}} & \sqrt{\frac{8}{5}} \end{bmatrix}$$

Now, we have $AX = B \Rightarrow L(L^tX) = B$

Let $$L^tX = Z \quad \dots (2)$$

$$\therefore \quad LZ = B \quad \dots (3)$$

Now solve equation (3)

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{5}{2}} & 0 \\ 0 & \sqrt{\frac{2}{5}} & \sqrt{\frac{8}{5}} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \Rightarrow \sqrt{2}\,z_1 = 1 \Rightarrow z_1 = \frac{1}{\sqrt{2}}$$

$$R_2 \Rightarrow -\frac{1}{\sqrt{2}}z_1 + \sqrt{\frac{5}{2}}\,z_2 = 0 \Rightarrow z_2 = \frac{1}{\sqrt{10}}$$

$$R_3 \Rightarrow \sqrt{\frac{2}{5}}\,z_2 + \sqrt{\frac{8}{5}}\,z_3 = 0 \Rightarrow z_3 = -\frac{1}{\sqrt{40}}$$

Now, the solution of $L^tX = z$ is

$$\begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{\frac{5}{2}} & \sqrt{\frac{2}{5}} \\ 0 & 0 & \sqrt{\frac{8}{5}} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{40}} \end{bmatrix}$$

$$\therefore \quad R_3 \Rightarrow \sqrt{\frac{8}{5}}\,x_3 = -\frac{1}{\sqrt{40}} \Rightarrow x_3 = -\frac{1}{8}$$

$$R_2 \Rightarrow \sqrt{\frac{5}{2}}\,x_2 + \sqrt{\frac{2}{5}}\,x_3 = \frac{1}{\sqrt{10}} \Rightarrow x_2 = \frac{1}{4}$$

$$R_1 \Rightarrow \sqrt{2}\,x_1 - \frac{1}{\sqrt{2}}\,x_2 = \frac{1}{\sqrt{2}} \Rightarrow x_1 = \frac{5}{8}$$

Thus, the required solution is

$$x_1 = \frac{5}{8},\ x_2 = \frac{1}{4},\ x_3 = -\frac{1}{8}$$

Example 7.22 : *Solve the following system of equations by Cholesky method* $2x + 3y + z = 0$

$x + 2y - z = -2$

$-x + y + 2z = 0$ **SPPU : May-15**

Solution : Given linear system can be written in matrix form as

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \quad \dots (1)$$

i.e. $AX = B$

We can express matrix A as $A = LL^t$

$$\therefore \quad \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Multiplying the right hand side matrices and equating corresponding elements of equation, we get

$$l_{11}^2 = 2 \Rightarrow l_{11} = \sqrt{2}$$

$$l_{11}\, l_{21} = 3 \Rightarrow l_{21} = \frac{3}{\sqrt{2}}$$

$$l_{11}\, l_{31} = 1 \Rightarrow l_{31} = \frac{\sqrt{2}}{3}$$

$$l_{21}^2 + l_{22}^2 = 2 \Rightarrow l_{22} = \frac{1}{\sqrt{2}}$$

$$l_{21}\, l_{31} + l_{22}\, l_{32} = -1 \Rightarrow l_{32} = (-1-1)\sqrt{2} = -2\sqrt{2}$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 2 \Rightarrow l_{33}^2 = 2 - \frac{2}{9} - 8 = \frac{-56}{9}$$

$$l_{33} = \sqrt{\frac{-56}{9}}$$

Thus
$$L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{2}}{3} & -2\sqrt{2} & \frac{\sqrt{-56}}{3} \end{bmatrix}$$

Now, we have $AX = B \Rightarrow (LL^t)X = B$

Let $L^tX = Z \quad \dots (2)$

$LZ = B \quad \dots (3)$

Now solve equation (3)

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{2}}{3} & -2\sqrt{2} & \frac{\sqrt{-56}}{3} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$R_1 \Rightarrow \sqrt{2}\, z_1 = 0 \Rightarrow z_1 = 0$$

$$R_2 \Rightarrow \frac{3}{\sqrt{2}} z_1 + \frac{1}{\sqrt{2}} z_2 = -2 \Rightarrow z_2$$

$$= -2\sqrt{2}$$

$$R_3 \Rightarrow \frac{\sqrt{2}}{3} z_1 - 2\sqrt{2}\, z_2 + \frac{\sqrt{-56}}{3} z_3 = 0$$

$$z_3 = 8 \times \frac{3}{\sqrt{-56}} = \frac{24}{\sqrt{-56}}$$

Now the solution of $L^tX = Z$

$$\begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \frac{\sqrt{2}}{3} \\ 0 & \frac{1}{\sqrt{2}} & -2\sqrt{2} \\ 0 & 0 & \frac{\sqrt{-56}}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2\sqrt{2} \\ \frac{24}{\sqrt{-56}} \end{bmatrix}$$

$$\therefore \quad R_3 \Rightarrow \frac{\sqrt{-56}}{3} z = \frac{24}{\sqrt{-56}} \Rightarrow z = \frac{72}{-56} = -\frac{9}{7}$$

$$R_2 \Rightarrow \frac{1}{\sqrt{2}} y - 2\sqrt{2}\, z = -2\sqrt{2}$$

$$\Rightarrow y = \sqrt{2}\left[-2\sqrt{2} - \frac{18\sqrt{2}}{7}\right] = -\frac{64}{7}$$

$$R_1 \Rightarrow \sqrt{2}\, x + \frac{3}{\sqrt{2}} y + \frac{\sqrt{2}}{3} z = 0$$

$$x = \frac{1}{\sqrt{2}}\left[\frac{192}{7\sqrt{2}} + \frac{9\sqrt{2}}{21}\right] = \frac{1}{7\sqrt{2}}\left[\frac{192}{\sqrt{2}} + \frac{9\sqrt{2}}{3}\right]$$

$$= \frac{1}{7\sqrt{2}}\left[\frac{576+18}{3\sqrt{2}}\right] = \frac{594}{42} = \frac{297}{21} = \frac{99}{7}$$

Thus, the solution is $x = \frac{99}{7}$, $y = -\frac{64}{7}$, $z = \frac{-9}{7}$

7.6.4 Gauss Jacobi's Methods

Consider the equations,

$$\left.\begin{aligned} a_{11}x + a_{12}y - a_{13}z = d_1 \\ a_{21}x + a_{22}y - a_{23}z = d_2 \\ a_{31}x + a_{32}y - a_{33}z = d_3 \end{aligned}\right\} \quad \dots(7.14)$$

If a_1, b_2, c_3 are large as compared to other coefficients, then solving these for x, y, z respectively, the system can be written in the form

$$\left.\begin{aligned} x = k_1 - l_1 y - m_1 z \\ y = k_2 - l_2 x - m_2 z \\ z = k_3 - l_3 x - m_3 y \end{aligned}\right\} \quad \dots(7.15)$$

Let us start with the initial approximations $x_0 = 0$, $y_0 = 0$. $z_0 = 0$, Substituting these values on the right sides of (2) we get the first approximations $x^{(1)} = k_1$, $y^{(1)} = k_2$, $z^{(1)} = k_3$.

Now substituting values of $x^{(1)}, y^{(1)}, z^{(1)}$ in equation (7.15) we will get the second approximation say $x^{(2)}$, $y^{(2)}$ and $z^{(2)}$,

Again, substituting $x^{(2)}$, $y^{(2)}$ and $z^{(2)}$ in equation (7.15), we will get the third approximations,

$\therefore$ say $x^{(3)}$, $y^{(3)}$ and $z^{(3)}$.

This process is repeated till the difference between two consecutive approximations is negligible.

Example 7.23 : *Solve Gauss Jacobi Method*

$$27x + 6y - z = 85$$
$$6x + 15y - 2z = 72$$
$$x + y + 54z = 110$$

Solution : We have

$$\left.\begin{array}{l} |a_{11}| > |a_{12}| + |a_{13}| \\ |a_{22}| > |a_{21}| + |a_{23}| \\ |a_{23}| > |a_{31}| + |a_{32}| \end{array}\right\} \quad \ldots(1)$$

$$\left.\begin{array}{l} x = \frac{1}{27}[85 - 6y + z] \\ y = \frac{1}{15}[72 - 6x - 3z] \\ z = \frac{1}{54}[110 - x - y] \end{array}\right\} \quad \ldots(2)$$

Starting with initial approximation

$x = y = z = 0$

i) Substituting $x = y = 0$ in equation (2), we get the first approximations

$$x^{(1)} = \frac{85}{27} = 3.148$$

$$y^{(1)} = \frac{72}{15} = 4.8$$

$$z^{(1)} = \frac{110}{54} = 2.037$$

ii) Substituting $x^{(1)}$, $y^{(1)}$ and $z^{(1)}$ in equation (2), We will get the second approximations

$$x^{(2)} = \frac{1}{27}[85 - 6(4.8) + 2.037]$$
$$= 2.1569$$

$$y^{(2)} = \frac{1}{15}[72 - 6\times 3.148 - 2\times 2.037]$$
$$= 3.2692$$

$$z^{(2)} = \frac{1}{54}[110 - 3.148 - 4.8]$$
$$= 1.889$$

iii) Substituting $x^{(2)}$, $y^{(2)}$ & $z^{(2)}$ in equation (2), We will get the third approximations,

$$x^{(3)} = \frac{1}{27}[85 - 6\times 3.2622 + 1.889] = 2.4915$$

$$y^{(3)} = \frac{1}{15}[72 - 6\times 2.1569 - 2\times 1.839]$$
$$= 3.68537$$

$$z^{(3)} = \frac{1}{54}[110 - 2.1565 - 3.2692]$$
$$= 1.93655$$

iv) $$x^{(4)} = \frac{1}{27}[85 - 6\times 3.68537 + 1.93655]$$
$$= 2.4009$$

$$y^{(4)} = \frac{1}{15}[72 - 6\times 2.4915 - 2\times 1.93655]$$
$$= 3.54519.$$

$$z^{(4)} = \frac{1}{54}[110 - 2.4915 - 3.68534]$$
$$= 1.92265$$

v) $$x^{(5)} = \frac{1}{27}[85 - 6\times 3.54519 + 1.92265]$$
$$= 2.4315$$

$$y^{(5)} = \frac{1}{15}[72 - 6\times 2.4009 - 2\times 1.92265]$$
$$= 3.58328.$$

$$z^{(5)} = \frac{1}{54}[110 - 2.4009 - 3.54519]$$
$$= 1.92692$$

vi) $$x^{(6)} = \frac{1}{27}[85 - 6\times 3.5325 + 1.92692]$$
$$= 2.42323$$

$$y^{(6)} = \frac{1}{15}[72 - 6\times 2.4315 - 2\times 1.92692]$$
$$= 3.5704$$

$$z^{(6)} = \frac{1}{54}[110 - 2.431 - 3.58328]$$
$$= 1.92763$$

vii) $$x^{(7)} = \frac{1}{27}[85 - 6\times 3.5704 + 1.9258]$$
$$= 2.426$$

$$y^{(7)} = \frac{1}{15}[72 - 6\times 2.42323 - 2\times 1.9258]$$
$$= 3.573$$

$$z^{(7)} = \frac{1}{54}[110 - 2.42323 - 3.5704] = 1.926$$

Here, 6^{th} and 7^{th} approximation are same

so, $x = 2.42$

$y = 3.57$

$z = 1.926$

Example 7.24 : *Solve Gauss Jacobi method*

$10x - 5y - 2z = 3$

$4x - 10y + 3z = -3$

$x + 6y + 10z = -3$

Solution : We have

$$\left.\begin{aligned} x &= \frac{1}{10}[3+5y+2z] \\ y &= \frac{1}{10}[3+4x+3z] \\ z &= \frac{1}{10}[-3-x-6y] \end{aligned}\right\} \qquad \ldots(1)$$

Starting with initial approximation $x = y = z = 0$

i) Substituting $x_0 = y_0 = z_0 = 0$ in equation (1) we will get the first approximation,

$$x^{(1)} = \frac{1}{10}[3+5(0)+2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3+4(0)+3(0)] = 0.3$$

$$z^{(1)} = \frac{1}{10}[-3-0-6(0)] = -0.3$$

ii) $$x^{(2)} = \frac{1}{10}[3+5(0.3)+2(-0.3)] = 0.39$$

$$y^{(2)} = \frac{1}{10}[-3+4(0.3)+3(-0.3)]$$

$$= 0.33$$

$$z^{(2)} = \frac{1}{10}[-3-0.3-6(0.3)] = -0.51$$

iii) $$x^{(3)} = \frac{1}{10}[3+5(0.73)+2(-0.51)] = 0.363$$

$$y^{(3)} = \frac{1}{10}[3+4(0.39)+3(-0.51)] = 0.303$$

$$z^{(3)} = \frac{1}{10}[-3-0.39-6(0.33)] = -0.537$$

iv) $$x^{(4)} = \frac{1}{10}[3+5(0.303)+2(-0.537)]$$

$$= 0.3441$$

$$y^{(4)} = \frac{1}{10}[3+4(0.361)+3(-0.537)]$$

$$= 0.2841$$

$$z^{(4)} = \frac{1}{10}[-3-0.763-6(0.303)]$$

$$= -0.5181$$

$x^{(5)} = 0.34$

$y^{(5)} = 0.28$

$z^{(5)} = -0.50$

vi) $x^{(6)} = 0.34$

$y^{(6)} = 0.28$

$z^{(6)} = -0.50$

5^{th} & 6^{th} approximation are, same,

$x = 0.34$

$y = 0.28$

$z = -0.50$

Notes

UNIT - VI

8 Interpolation, Numerical Differentiation and Integration

8.1 Introduction

The most of the experimental or observed data is in the form of ordered pairs (x_0, y_0), (x_1, y_1), (x_2, y_2) ... (x_n, y_n) which is the tabular form of an unknown function $y = f(x)$. The process of determining the value of y for any $x \in [x_0, x_n]$ is known as interpolation. The points $x_0, x_1, x_2, \ldots x_n$ are known as interpolation or mesh points. The process of computing the value of y at any outside point of a given range is called as extrapolation.

The theme of interpolation is to construct a new function $\phi(x)$ which coincide with the unknown function f(x) at the set of tabulated points. The process of finding $\phi(x)$ is the interpolation. If the $\phi(x)$ is a polynomial then process is called the polynomial interpolation. The process of finding $\phi(x)$ is done with the help of calculus of finite differences which deals with the changes in the dependent variable due to changes in the independent variable. In this chapter we will study the Newton-Gregory forward, backward and Lagranges interpolation formulae.

8.2 Finite Differences

Consider the function $y = f(x)$. The values of y corresponding to different values of x are obtained by substituting the various values of x. Let the consecutive values of x differing by h be $x_0, x_0 + h = x_1, \quad x_0 + 2h = x_2, \ldots x_0 + nh = x_n$ and corresponding values of $y = f(x)$ be $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2) \ldots y_n = f(x_n)$. The values of the independent variable x are called arguments and corresponding values of dependent variable y are called entries.

8.2.1 Forward Differences

The first forward difference is defined by $\Delta f(x) = f(x+h) - f(x)$ where Δ is called the forward or descending difference operator. (Δ = Delta)

In particular $\Delta f(x_0) = f(x_0 + h) - f(x_0)$

$$\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ \Delta f(x_0 + h) &= f(x_0 + 2h) - f(x_0 + h) \\ \Delta y_1 &= y_2 - y_1 \\ &\ldots\ldots \\ \Delta y_{n-1} &= y_n - y_{n-1} \end{aligned}$$

The differences of first forward differences are called the second forward differences.

Thus

$$\begin{aligned} \Delta^2 f(x) &= \Delta(\Delta f(x)) = \Delta\, (f(x+h) - f(x)) \\ &= \Delta f(x+h) - \Delta f(x) \\ &= f(x+2h) - f(x+h) - f(x+h) + f(x) \\ \Delta^2 f(x) &= f(x+2h) - 2f(x+h) + f(x) \end{aligned}$$

In particular

$$\Delta^2 f(x_1) = f(x_1 + 2h) - 2f(x_1 + h) + f(x_1)$$

i.e. $\quad \Delta^2 y_1 = y_3 - 2y_2 + y_1$

and $\quad \Delta^2 y_2 = y_4 - 2y_3 + y_2$

The third forward difference is given by

$$\begin{aligned} \Delta^3 f(x) &= \Delta(\Delta^2 f(x)) = \Delta[f(x+2h) - 2f(x+h) + f(x)] \\ &= f(x+3h) - f(x+2h) - 2[f(x+2h) - f(x+h)] \\ &\quad + f(x+h) - f(x) \\ &= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x) \end{aligned}$$

In particular

$$\Delta^3 f(x_1) = f(x_1+3h)-3f(x_1+2h)+3f(x_1+h)-f(x_1)$$

$$\Delta^3 y_1 = y_4-3y_3+3y_2-y_1$$

In general, n^{th} forward difference is defined by

$$\Delta^n f(x) = \Delta^{n-1} f(x+h)-\Delta^{n-1} f(x)$$

These forward differences are shown in the following table.

Forward difference table

Argument x	Entry f(x)	First forward differences Δ f(x)	2^{nd} forward differences Δ^2 f(x)	3^{rd} forward differences Δ^3 f(x)
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
x_0+h	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_0+2h	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$
x_0+3h	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$	
		$\Delta y_3 = y_4 - y_3$		
x_0+4h	y_4			

8.2.2 Backward Differences

The first backward difference is defined by

$$\nabla f(x) = f(x)-f(x-h)$$

where ∇ is called the backward or ascending difference operator and h is interval difference.

(∇ = Del)

In particular

$$\nabla y_1 = y_1-y_0$$

$$\nabla y_2 = y_2-y_1$$

$$\vdots$$

$$\nabla y_n = y_n-y_{n-1}$$

The second backward difference is defined by

$$\nabla^2 f(x) = \nabla(\nabla f(x)) = \nabla(f(x)-f(x-h))$$

$$= f(x)-2f(x-h)+f(x-2h)$$

In particular

$$\nabla^2 y_2 = y_2-2y_1+y_0$$

$$\nabla^2 y_3 = y_3-2y_2+y_1$$

$$\vdots$$

$$\nabla^2 y_n = y_n-2y_{n-1}+y_{n-2}$$

These backward differences are shown in the following table.

Argument x	Entry f(x)	1^{st} backward differences ∇ f(x)	2^{nd} backward differences ∇^2 f(x)	3^{rd} backward differences ∇^3 f(x)
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
x_0+h	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_0+2h	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$
x_0+3h	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$	
		$\nabla y_4 = y_4 - y_3$		
x_0+4h	y_4			

Properties of differences :

1) If C is a constant then Δ C = 0.

2) $\Delta\{f(x)\cdot g(x)\} = f(x+h)\Delta g(x)+g(x)\Delta f(x)$

3) $\Delta\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x+h)\, g(x)}$

4) If $\Delta f(x)=0$ then it does not mean that either $\Delta=0$ or f(x) = 0.

8.2.3 The Central Difference

The first order central difference is defined by

$$\delta\, f(x) = f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)$$

where δ is the central difference operator.

In particular

$$\delta\, y_{1/2} = y_1 - y_0$$
$$\delta\, y_{3/2} = y_2 - y_1$$

The second order central difference is given by

$$\delta^2 f(x) = \delta\left(f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)\right)$$
$$= f\left(x+\frac{h}{2}+\frac{h}{2}\right)-f\left(x+\frac{h}{2}-\frac{h}{2}\right)$$
$$-f\left(x-\frac{h}{2}+\frac{h}{2}\right)+f\left(x-\frac{h}{2}-\frac{h}{2}\right)$$
$$= f(x+h)-2f(x)+f(x-h)$$

In particular

$$\delta^2 y_1 = y_2 - 2y_1 + y_0 = \delta y_{3/2} - \delta y_{1/2}$$
$$\delta^2 y_2 = y_3 - 2y_2 + y_1$$

These central differences are shown in the following table.

Argument x	Entry y = f(x)	1st central differences δy	2nd central differences $\delta^2 y$	3rd central differences $\delta^3 y$
x_0	y_0			
		$\delta y_{1/2} = y_1 - y_0$		
$x_0 + h$	y_1		$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$	
		$\delta y_{3/2} = y_2 - y_1$		$\delta^3 y_{3/2} = \delta^2 y_2 - \delta^2 y_1$
$x_0 + 2h$	y_2		$\delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$	
		$\delta y_{5/2} = y_3 - y_2$		$\delta^3 y_{5/2} = \delta^2 y_3 - \delta^2 y_2$
$x_0 + 3h$	y_3		$\delta^2 y_3 = \delta y_{7/2} - \delta y_{5/2}$	
		$\delta y_{7/2} = y_4 - y_3$		
$x_0 + 4h$	y_4			

8.2.4 The Shift Operator E

The shift operator E is defined by

$$E\, f(x) = f(x+h)$$

$$\therefore \quad E^2 f(x) = E(E\, f(x)) = E\,(f\,(x+h)) = f(x+2h)$$
$$E^3 f(x) = f(x+3h)$$
$$\vdots$$
$$E^n f(x) = f(x+n\,h)$$

The inverse shift operator E^{-1} is defined as

$$E^{-1} f(x) = f(x-h)$$
$$E^{-2} f(x) = f(x-2h)$$
$$\vdots$$
$$E^{-n} f(x) = f(x-nh)$$

8.2.5 The Average Operator μ

The average operator μ is defined as

$$\mu\, f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]$$

and $$\mu^2 f(x) = \mu\,[\mu\, f(x)] = \mu\left[\frac{f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)}{2}\right]$$

$$= \frac{1}{4}\left\{f\left(x+\frac{h}{2}+\frac{h}{2}\right)-f\left(x+\frac{h}{2}-\frac{h}{2}\right)\right.$$
$$\left.+f\left(x-\frac{h}{2}+\frac{h}{2}\right)-f\left(x-\frac{h}{2}-\frac{h}{2}\right)\right\}$$
$$= \frac{1}{4}\,[f(x+h)-f(x-h)]$$

8.2.6 Symbolic Relations

1) Show that $E[f(x)] = (1+\Delta)\, f(x)$

We have $E[f(x)] = f\,(x+h)$

and $(1+\Delta)\, f(x) = f(x)+f\,(x+h)-f(x)$

$$= f\,(x+h) = E\,[f(x)]$$

Thus $\boxed{E \equiv 1+\Delta}$ or $\Delta \equiv E-1$

$\therefore$ $E^2 \equiv (1+\Delta)^2$ and $\boxed{E^n \equiv (1+\Delta)^n}$

2) $E\nabla\,[f(x)] = \nabla\, E\,[f(x)] = \Delta\, f(x)$

We have $E\,\{\nabla\,[f(x)]\} = E\,\{f(x)-f(x-h)\}$

$$= f(x+h)-f(x) = \Delta\, f(x)$$

and $\nabla\,\{E\, f(x)\} = \nabla\, f(x+h)$

$$= f(x+h)-f(x) = \Delta\, f(x)$$

Thus $E\nabla\,[f(x)] = \nabla\, E\,[f(x)] = \Delta\, f(x)$

$\therefore$ $\boxed{E\,\nabla \equiv \nabla\, E \equiv \Delta}$

3) Relation between Δ and ∇

$\{(1+\Delta)\,(1-\nabla)\}\,[f(x)] = f(x)$

$\Rightarrow$ We have $\{(1+\Delta)\,(1-\nabla)\}\,f(x) = (1+\Delta)\,\{(1-\nabla)f(x)\}$

$= (1+\Delta)\,\{f(x) - f(x) + f(x-h)\}$

$= (1+\Delta)\,f(x-h)$

$= E\,f\,(x-h) = f(x) = 1 \cdot f(x)$

$\therefore$ $\boxed{(1+\Delta)(1-\nabla) \equiv 1}$

4) Relation between the operator E and D

From differential calculus, we have

$$\frac{d}{dx} f(x) = D\,f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \dots (1)$$

Now by Taylor's series, we have

$$f\,(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2\,!} f''(x) + \dots$$

$$E\,f(x) = f(x) + h\,D\,f(x) + \frac{h^2}{2\,!} D^2 f(x) + \dots$$

$$= \left\{1 + h\,D + \frac{h^2}{2\,!} D^2 + \dots\right\} f(x)$$

$$E\,f(x) = e^{hD}\,f(x)$$

$$E \equiv e^{hD}$$

$$\therefore \quad h\,D \equiv \log E \equiv \log\,(1+\Delta)$$

$$D \equiv \frac{1}{h}\left\{\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots\right\}$$

5) Relation between δ and E and μ and E

$\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$ and $\mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$

$\Rightarrow$ We have

$$\delta\,f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) = E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$$

$$\delta\,f(x) = \left[E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right] f(x)$$

$$\therefore \quad \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

Now $\mu\,f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$

$$= \frac{1}{2}\left[E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)\right]$$

$$= \left(\frac{E^{1/2} + E^{-1/2}}{2}\right) f(x)$$

$\therefore$ $\boxed{\mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)}$

6) Relation between μ and δ : $\mu = \left[1 + \frac{\delta^2}{4}\right]^{1/2}$

We have $\delta^2 = \left[E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right]^2 = E + E^{-1} - 2$

$$\therefore \quad \left[1 + \frac{\delta^2}{4}\right]^{\frac{1}{2}} = \left[1 + \frac{1}{4}\left(E + E^{-1} - 2\right)\right]^{\frac{1}{2}}$$

$$= \frac{1}{2}\,[E + E^{-1} + 2]^{\frac{1}{2}} = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right] = \mu$$

7) Prove that $\Delta\nabla \equiv \nabla\Delta = \delta^2$

Consider $\Delta\nabla\,f(x) = \Delta[\nabla f(x)]$

$$= \Delta[f(x) - f(x-h)]$$

$$= \Delta f(x) - \Delta f(x-h)$$

$$= f(x+h) - f(x) - f(x) + f(x-h)$$

$$\Delta\nabla f(x) = f(x+h) - 2f(x) + f(x-h) \qquad \dots (1)$$

$$\nabla\Delta f(x) = \nabla[\Delta f(x)] = \nabla[f(x+h) - f(x)]$$

$$= \nabla f(x+h) - \nabla f(x)$$

$$= f(x+h) - f(x) - f(x) + f(x-h)$$

$$\nabla\Delta f(x) = f(x+h) - 2f(x) + f(x-h) \qquad \dots (2)$$

Now $\delta^2 f(x) = \delta[\delta\,f(x)] = \delta\left[f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right]$

$$= \delta f\left(x+\frac{h}{2}\right) - \delta f\left(x-\frac{h}{2}\right)$$

$$= f(x+h) - f(x) - f(x) + f(x-h)$$

$$= f(x+h) - 2f(x) + f(x-h) \qquad \dots (3)$$

From (1), (2), (3), we get

$$\Delta\nabla f(x) = \nabla\Delta f(x) = \delta^2 f(x)$$

$\boxed{\Delta\nabla \equiv \nabla\Delta \equiv \delta^2}$

Relationship between the operators

	Δ	∇	E
Δ	Δ	$(1-\nabla)^{-1}-1$	$E-1$
∇	$1-(1+\Delta)^{-1}$	∇	$1-E^{-1}$
E	$\Delta+1$	$(1-\nabla)^{-1}$	E
δ	$\Delta(1+\Delta)^{-\frac{1}{2}}$	$\nabla(1-\nabla)^{-\frac{1}{2}}$	$E^{\frac{1}{2}}-E^{-\frac{1}{2}}$
μ	$\left(1+\frac{1}{2}\Delta\right)(1+\Delta)^{\frac{1}{2}}$	$\left(1-\frac{1}{2}\nabla\right)(1-\nabla)^{-\frac{1}{2}}$	$\frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)$

8.2.7 Generalized Power or Factorial Function

The first factor is x and the successive factors decrease by a constant difference h, such a product of n consecutive factors is known as a factorial function and it is denoted by $x^{(n)}$ where n is a positive integer.

$$\therefore \qquad x^{(n)} = x(x-h)(x-2h)\ldots(x-(n-1)h)$$

For h = 1, $\quad x^{(n)} = x(x-1)(x-2)\ldots(x-n+1)$

and for h = 0, $\quad x^{(n)} = x(x-0)(x-0)\ldots(x-0) = x^n$

Fundamental theorem : The n^{th} difference of a polynomial function f(x) of n^{th} degree is constant i.e. $\Delta^n f(x) = \text{constant}$ and $\Delta^{n+1} f(x) = 0$.

Illustrative Examples

Example 8.1 : *Prove that*

$$\Delta \log f(x) = \log\left\{1+\frac{\Delta f(x)}{f(x)}\right\}$$

Solution : We have

$$\text{L.H.S.} = \Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{E\, f(x)}{f(x)}\right]$$

$$= \log\left[\frac{(1+\Delta) f(x)}{f(x)}\right] = \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$$

$$= \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

$$= \text{R.H.S.}$$

Example 8.2 : *Evaluate i)* $\Delta^n (e^{3x+4})$

ii) $\left(\frac{\Delta^2}{E}\right) x^4$

iii) $\Delta^{14}[(1-ax^5)(1-bx^4)(1-cx^3)(1-dx^2)]$

iv) $\Delta[e^{ax}\sin bx]$ *(Take h = 1)*

Solution : i) We have

$$\Delta(e^{3x+4}) = e^{3(x+1)+4} - e^{3x+4} = e^{3x+4}(e^3-1)$$

$$\Delta^2(e^{3x+4}) = (e^3-1)\Delta e^{3x+4} = (e^3-1)^2 e^{3x+4}$$

By induction

$$\Delta^n(e^{3x+4}) = (e^3-1)^n e^{3x+4}$$

ii) $$\left(\frac{\Delta^2}{E}\right)x^4 = \left[\frac{(E-1)^2}{E}\right]x^4 = \left[\frac{E^2-2E+1}{E}\right]x^4$$

$$= [E-2+E^{-1}]x^4 = E[x^4] - 2x^4 + E^{-1}[x^4]$$

$$= (x+1)^4 - 2x^4 + (x-1)^4 = 12x^2$$

iii) Here $f(x) = (1-ax^5)(1-bx^4)(1-cx^3)(1-dx^2)$ is a polynomial of degree 14 and coefficient of x^{14} is abcd.

$\therefore$ By fundamental theorem, we have

$$\Delta^{14} f(x) = abcd\,(14\,!)$$

iv) Let $\quad f(x) = e^{ax}$ and $g(x) = \sin bx$

$$\Delta f(x) = e^{ax+a} - e^{ax} = e^{ax}(e^a-1)$$

$$\Delta g(x) = \sin(bx+b) - \sin bx$$

$$= 2\cos\left(bx+\frac{b}{2}\right)\sin\frac{b}{2}$$

$$\therefore \qquad \Delta[f(x)\cdot g(x)] = f(x+1)\Delta g(x) + g(x)\Delta f(x)$$

$$= e^{ax+a}\, 2\cos\left(bx+\frac{b}{2}\right)\sin\frac{b}{2} + \sin bx\, e^{ax}(e^a-1)$$

$$= e^{ax}\left[2\cos\left(bx+\frac{b}{2}\right)\sin\frac{b}{2} + (\sin bx)(e^a-1)\right]$$

Example 8.3 : *Prove that*

a) $\Delta^n x^{(n)} = n!\,h^n$ *and* $\Delta^{n+1} x^{(n)} = 0$

b) $x^{(-n)} = \dfrac{1}{(x+n)^{(n)}}$ *where h = 1*

Solution : a) We have

$$\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$$

$$= [(x+h)(x+h-h)(x+h-2h)\ldots(x+h-(n-1)h)] - [x(x-h)(x-2h)\ldots(x-(n-1)h)]$$

$\therefore\ \Delta x^{(n)} = [(x+h)\, x\, (x-h)\, (x-2h) \ldots (x-(n-2)\, h)]$

$\quad - [x\, (x-h)\, (x-2h) \ldots (x-(n-2)\, h)\, (x-(n-1)\, h)]$

$= x\, (x-h) \ldots (x-(n-2)\, h)\, [(x+h)-(x+h-n\, h)]$

$= x\, (x-h)\, (x-2h) \ldots (x-(n-2)\, h)\, (nh)$

$$\Delta x^{(n)} = nh\, x^{(n-1)} \quad \ldots (1)$$

Now $\Delta^2 x^{(n)} = \Delta\, (\Delta x^{(n)}) = \Delta\, (nh\, x^{(n-1)})$ ... by (1)

$= nh\, \Delta x^{(n-1)} = nh\, [(x+h)^{(n-1)} - x^{(n-1)}]$

$\Delta^2 x^{(n)} = nh\, \{(x+h)\, (x+h-h)\, (x+h-2h) \ldots (x+h-(n-2)\, h)]$

$\quad - [x\, (x-h)\, (x-2h) \ldots (x-(n-2)\, h)]\}$

$= nh\, \{[(x+h)\, x\, (x-h) \ldots (x-(n-3)\, h)]$

$\quad - [x\, (x-h)\, (x-2h) \ldots (x-(n-2)\, h)]\}$

$= nh\, x\, (x-h)\, (x-2h) \ldots (x-(n-3)\, h)\, [x+h-x+nh-2h]$

$= nh\, x^{(n-2)}\, (n-1)\, h$

$$\Delta^2 x^{(n)} = n\, (n-1)\, h^2 x^{(n-2)}$$

Similarly, we get

$$\Delta^{n-1} x^{(n)} = n\, (n-1)\, (n-2) \ldots 2\, h^{n-1} x$$

$$\Delta^n x^{(n)} = n\, (n-1)\, (n-2) \ldots 2\, h^{n-1}\, \Delta x = n\,!\, (x+h-x)\, h^{n-1}$$

$$\Delta^n x^{(n)} = n\,!\, h^{n-1} = \text{constant}$$

hence $\Delta^{n+1} x^{(n)} = 0$

b) We have

$$x^{(n)} = x\, (x-h)\, (x-2h) \ldots (x-(n-1)\, h)$$

$$x^{(n)} = [x-(n-1)\, h]\, x^{(n-1)} \quad \ldots (1)$$

Substituting n = 0, we get

$$x^{(0)} = [x-(0-1)h]\, x^{(-1)}$$

$$1 = (x+h)\, x^{(-1)}$$

$$\therefore\quad x^{(-1)} = \frac{1}{x+h}$$

Substituting n = – 1, we get

$$x^{(-1)} = (x+2h)\, x^{(-2)}$$

$$\therefore\quad x^{(-2)} = \frac{1}{(x+h)\, (x+2h)}$$

In general $x^{(-n)} = \dfrac{1}{(x+h)\, (x+2h) \ldots (x+nh)} = \dfrac{1}{(x+nh)^{(n)}}$

But h = 1

$$\therefore\quad x^{(-n)} = \frac{1}{(x+n)^{(n)}}$$

Example 8.4 : *Prove that*

i) $u_x - \Delta^n u_{x-n} = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \ldots + \Delta^{n-1} u_{x-n}$

ii) $u_1 + u_2 + \ldots + u_n = n_{C_1} u_1 + n_{C_2} \Delta u_1 + n_{C_3} \Delta^2 u_1 + \ldots + \Delta^{n-1} u_1$

Solution : i) Consider

L.H.S. $= u_x - \Delta^n u_{x-n} = u_x - \Delta^n\, \{E^{-n} u_x\}$

$= u_x - \Delta^n E^{-n} u_x = (1-\Delta^n E^{-n})\, u_x = \left(1-\dfrac{\Delta^n}{E^n}\right) u_x$

$= \dfrac{1}{E^n}\left[\dfrac{E^n-\Delta^n}{1}\right] u_x = E^{-n}\, [E^n - \Delta^n]\, u_x$

$= E^{-n} (E-\Delta)\, [E^{n-1} + \Delta\, E^{n-2} + \Delta^2 E^{n-3} + \ldots + \Delta^{n-1}]\, u_x$

$(\because\ 1 \equiv E-\Delta)$

$= E^{-n}\, [E^{n-1} + \Delta\, E^{n-2} + \Delta^2\, E^{n-3} + \ldots + \Delta^{n-1}]\, u_x$

$= [E^{-1} + \Delta\, E^{-2} + \Delta^2 E^{-3} + \ldots + \Delta^{n-1} E^{-n}]\, u_x$

$= u_{x-1} + \Delta\, u_{x-2} + \Delta^2 u_{x-3} + \ldots + \Delta^{n-1} u_{x-n}$

= R.H.S.

ii) We have consider L.H.S. $= u_1 + u_2 + u_3 + \ldots + u_n$

$= u_1 + E\, u_1 + E^2 u_1 + \ldots + E^n u_1$

$= (1+E+E^2+\ldots+E^n)\, u_1$

$= \left\{\dfrac{E^n-1}{E-1}\right\} u_1 = \left\{\dfrac{(1+\Delta)^n-1}{\Delta}\right\} u_1$

$= \left\{\dfrac{({}^nC_0 + {}^nC_1\, \Delta + {}^nC_2 \Delta^2 + \ldots + {}^nC_n \Delta^n)-1}{\Delta}\right\} u_1$

$= \left\{\dfrac{{}^nC_1 \Delta + {}^nC_2 \Delta^2 + \ldots + {}^nC_n \Delta^n}{\Delta}\right\} u_1$

$= \{{}^nC_1 + {}^nC_2 \Delta + {}^nC_3 \Delta^2 + \ldots + {}^nC_n\, \Delta^{n-1}\}\, u_1$

$= {}^nC_1 u_1 + {}^nC_2\, \Delta\, u_1 + {}^nC_3\, \Delta^2 u_1 + \ldots + \Delta^{n-1} u_1$

= R.H.S.

Example 8.5 : *Find $f(6)$ and $f(7)$ if $f(x) = x^3 + 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$.*

Solution : We construct the following table.

x	f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)	Δ^4 f(x)
– 1	– 13				
		6			
0	– 7		0		
		6		6	
1	– 1		6		0
		12		6	
2	11		12		0
		24		6	
3	35		18		0
		42		6	
4	77		24		
		66			
5	143				

Now

$$f(6) = E^7 f(-1) = (1+\Delta)^7 f(-1)$$
$$= (1 + {}^7C_1\Delta + {}^7C_2\Delta^2 + {}^7C_3\Delta^3$$
$$+ {}^7C_4\Delta^4 + {}^7C_5\Delta^5 + {}^7C_6\Delta^6 + {}^7C_7\Delta^7) f(-1)$$
$$= f(-1) + 7\Delta f(-1) + 21\Delta^2 f(-1) + 35\Delta^3 f(-1)$$
$$= -13 + 7\times 6 + 21\times 0 + 35\times 6$$
$$f(6) = 239$$

Now $f(7) = f(8-1) = E^8 f(-1) = (1+\Delta)^8 f(-1)$

$$= (1 + {}^8C_1\Delta + {}^8C_2\Delta^2 + {}^8C_3\Delta^3) f(-1)$$
$$= f(-1) + 8\Delta f(-1) + 28\Delta^2 f(-1) + 56\Delta^3 f(-1)$$
$$= -13 + 8(6) + 28(0) + 56(6)$$
$$f(7) = 371$$

Example 8.6 : *Find the function whose first difference is $6x^2 + 3x + 9$.*

Solution : Let f(x) be the required function of x

$\therefore \Delta f(x) = 6x^2 + 3x + 9 \equiv 6x(x-1) + bx + c$ ... (1)

Substituting x = 0 in equation (1) we get c = 9

And substituting x = 1 and c = 9 in equation (1), we get

$$b + c = 18 \Rightarrow b = 18 - 9 = 9$$

$$\therefore \quad \Delta f(x) = 6x^{(2)} + 9x^{(1)} + 9$$

Hence

$$f(x) = \frac{6x^{(3)}}{3} + \frac{9x^{(2)}}{2} + 9x^{(1)} + k$$
$$= 2x^{(3)} + \frac{9}{2}x^{(2)} + 9x^{(1)} + k$$
$$= 2x(x-1)(x-2) + \frac{9}{2}x(x-1) + 9x + k$$
$$= 2x^3 - 6x^2 + 4x + \frac{9}{2}x^2 - \frac{9}{2}x + 9x + k$$
$$f(x) = 2x^3 - \frac{3}{2}x^2 + \frac{17}{2}x + k$$

Example 8.7 : *Represent the function $f(x) = x^4 - 12x^3 + 24x^2 - 28x + 8$ into factorial notations and then into Δ.*

Solution : The factorial representation of f(x) is

$$f(x) = x^4 - 12x^3 + 24x^2 - 28x + 8$$
$$= A\,x(x-1)(x-2)(x-3) + B\,x(x-1)(x-2) + C\,x(x-1) + D\,x + E \quad \text{... (1)}$$

where A, B, C, D, E are constants to be determined.

Substituting x = 0 in equation (1), we get E = 8.

Substituting x = 1, we get $-7 = D + E \Rightarrow D = -15$

Substituting x = 2, we get $-32 = 3C + 2D + E \Rightarrow C = -5$

Substituting x = 3, we get $-103 = 6B + 6C + 3D + E \Rightarrow B = -6$

And substituting x = 4, we get

$$-232 = 24A + 24B + 12C + 4D + E$$
$$\Rightarrow \quad A = 1$$
$$\therefore \quad f(x) = x^{(4)} - 6x^{(3)} - 5x^{(2)} - 15x^{(1)} + 8$$
$$\therefore \quad \Delta f(x) = 4x^{(3)} - 18x^{(2)} - 10x^{(1)} - 15$$
$$\Delta^2 f(x) = 12x^{(2)} - 36x^{(1)} - 10$$
$$\Delta^3 f(x) = 24x^{(1)} - 36$$
$$\Delta^4 f(x) = 24 \text{ and } \Delta^5 f(x) = 0$$

Example 8.8 : *Give an estimate of the population in 1971 from the following table*

Year	1941	1951	1961	1971	1981	1991
Population in Lakhs	363	391	421	–	467	501

Solution :

Let x be the population in 1971 and let

$u_0 = 363$, $u_1 = 391$, $u_2 = 421$, $u_3 = x$, $u_4 = 467$, $u_5 = 501$.

As five entries are given, $\Delta^5 u_0 = 0$

$\therefore (E-1)^5 u_0 = 0 \Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)u_0 = 0$

$$\therefore \quad u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0 = 0$$

$$501 - 5(467) + 10(x) - 10(421) + 5(391) - 363 = 0$$

$\Rightarrow$ x = 445.2 lakhs

Hence the estimated population is 1971 is 445.2 Lakhs.

Example 8.9 : *Given that*

$u_0 + u_8 = 80,\ u_1 + u_7 = 10,\ u_2 + u_6 = 5,\ u_3 + u_5 = 10$ find u_4.

Solution : As 8 values of u are given $\therefore \Delta^8 u_0 = 0$

$$\therefore \quad (E-1)^8 u_0 = 0$$

$$E^8 u_0 - {}^8C_1 E^7 u_0 + {}^8C_2 E^6 u_0 - {}^8C_3 E^5 u_0 + {}^8C_4 E^4 u_0 - {}^8C_5 E^3 u_0 + {}^8C_6 E^2 u_0 - {}^8C_7 E\, u_0 + {}^8C_8 u_0 = 0$$

$$u_8 - 8 u_7 + 28 u_6 - 56 u_5 + 70 u_4 - 56 u_3 + 28 u_2 - 8 u_1 + u_0 = 0$$

$$\therefore (u_8 + u_0) - 8(u_1 + u_7) + 28(u_6 + u_2) - 56(u_3 + u_5) + 70 u_4 = 0$$

$$\Rightarrow \quad 70 u_4 = 420$$

$$\Rightarrow \quad u_4 = 6$$

Example 8.10 : *Find the a and b in the following table*

x	1	2	3	4	5	6	7	8
f(x)	1	8	a	64	b	216	343	512

Solution : As six values of f(x) are given

$$\therefore \quad \Delta^6 f(x) = 0 \quad \forall x$$

$$\therefore \quad (E-1)^6 f(x) = 0$$

$$\therefore [E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1] f(x) = 0$$

$$\therefore f(x+6) - 6f(x+5) + 15f(x+4) - 20f(x+3) + 15f(x+2) - 6f(x+1) + f(x) = 0 \quad \text{... (1)}$$

Substituting x = 1 in (1) we get

$$f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) - 6f(2) + f(1) = 0$$

$$\Rightarrow \quad 15f(5) + 15f(3) = 2280$$

$$\therefore \quad a + b = 152 \quad \text{... (2)}$$

Substituting x = 2 in equation (1) we get

$$f(8) + 6f(7) + 15f(6) - 20f(5) + 15f(4) - 6f(3) + f(2) = 0$$

$$\Rightarrow \quad -20f(5) - 6f(3) = -2662$$

$$3a + 10b = 1331 \quad \text{... (3)}$$

Solving equation (3) and equation (2) we get

a = f(3) = 27 and b = f(5) = 125

Exercise 8.1

1. *Prove that* $\Delta\left(\dfrac{1}{f(x)}\right) = \dfrac{\Delta f(x)}{f(x) \cdot f(x+1)}$
2. *Evaluate* $\Delta^{11}(1-ax)(1-bx^3)(1-x^5)(1+cx^7)$
3. *P.T.* $\Delta \tan^{-1}\left(\dfrac{x-1}{x}\right) = \tan^{-1}\left(\dfrac{1}{2x^2}\right)$
4. *Show that* $\dfrac{\Delta^2 x^2}{E(x+\log x)} = \dfrac{2}{x+1+\log(x+1)}$
5. *Find* $\Delta^{-1}[x(x+1)(x+2)]$
6. *Find and correct a misprint in the sequence* $y(x) = 1, 3, 11, 31, 69, 113, 223, 351, 521, 739$
7. *Prove that*
   i) $\Delta^3 y_2 = \nabla^3 y_3$ ii) $\Delta = \mu\delta + \dfrac{1}{2}\delta^2$
   iii) $\Delta + \nabla = \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$ iv) $\nabla = 1 - e^{-hD}$
8. *Obtain the function whose first difference is*
   i) $x^3 + 3x^2 + 5x + 12$ ii) $4x^{(3)} - 18x^{(2)} - 10x^{(1)} - 17$
9. *Express in terms of factorial notations and hence find their differences*
   i) $2x^3 - 3x^2 + 5x + 2$ ii) $x^3 - 8x + 5$
   iii) $x^4 - 8x^2 + 5x - 10$ iv) $x^3 - 3x^2 + 3x - 1$

8.3 Newton's Formulae for Interpolation

8.3.1 Newton-Gregory Formula for Forward Interpolation

Consider the set of (n+1) equidistant values of the function y = f(x) viz [a, f(a)], [a+h, f(a+h)], [a+2h, f(a+2h)], ... [a+nh, f(a+nh)].

Let $P_n(x)$ be a polynomial in x of degree n.

We have

$$P_n(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + A_3(x-a)(x-a-h)(x-a-2h) + \ldots + A_n(x-a)(x-a-h)\ldots(x-a-(n-1)h) \quad \text{... (1)}$$

where $A_0, A_1, A_2, \ldots A_n$ are constants and are chosen such that $P_n(a) = f(a)$, $P_n(a+h) = f(a+h)$, $P_n(a+2h) = f(a+2h) \ldots P_n(a+nh) = f(a+nh)$

Substituting x = a in equation (1) we get

$$P_n(a) = A_0 \Rightarrow A_0 = f(a)$$

Substituting $x = a+h$ in equation (1) we get

$$P_n(a+h) = A_0 + h A_1$$

$$\therefore \quad f(a+h) = f(a) + h A_1$$

$$\therefore \quad A_1 = \frac{f(a+h)-f(a)}{h} = \frac{\Delta f(a)}{h}$$

Again substituting $x = a+2h$ in equation (1),

we get

$$P_n(a+2h) = A_0 + 2h A_1 + 2h^2 A_2$$

$$\therefore \quad f(a+2h) = f(a) + 2(f(a+h)-f(a)) + 2h^2 A_2$$

$$\therefore \quad 2h^2 A_2 = f(a+2h) - 2f(a+h) + f(a) = \Delta^2 f(a)$$

$$\therefore \quad A_2 = \frac{\Delta^2 f(a)}{2!\,h^2}$$

Similarly, we get $A_3 = \dfrac{\Delta^3 f(a)}{3!\,h^3} \ldots A_n = \dfrac{\Delta^n f(a)}{n!\,h^n}$

Now, substituting values of $A_0, A_1, A_2, \ldots A_n$ in equation (1) we get

$$P_n(x) = f(a) + \frac{\Delta f(a)}{h}(x-a) + \frac{\Delta^2 f(a)}{2!\,h^2}(x-a)(x-a-h)$$

$$+ \frac{\Delta^3 f(a)}{3!\,h^3}(x-a)(x-a-h)(x-a-2h) + \ldots$$

$$\ldots + \frac{\Delta^n f(a)}{n!\,h^n}(x-a)(x-a-h)\ldots(x-a-(n-1)h) \qquad \ldots (2)$$

which is Newton-Gregory formula for forward interpolation

Substituting $\dfrac{x-a}{h} = u$ or $x = a+hu$ in equation (2)

We get,

$$f(a+uh) = P_n(x) \equiv f(x)$$

$$= f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a)$$

$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots$$

$$\ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n f(a)$$

In factorial notation, we get,

$$P_n(x) = f(a) + u^{(1)}\Delta f(a) + u^{(2)}\frac{\Delta^2 f(a)}{2!} + \ldots + u^{(n)}\frac{\Delta^n f(a)}{n!}$$

where $u^{(n)} = u(u-1)(u-2)\ldots(u-(n-1))$

Note : This formula is useful for interpolation near the beginning of a set of tabular values.

8.3.2 Newton-Gregory Formula for Backward Interpolation

Consider the set of $n+1$ equidistant values of the function $y = f(x)$ viz $[a, f(a)], [a+h, f(a+h)], [a+2h, f(a+2h)], \ldots [a+nh, f(a+nh)]$. Let $P_n(x)$ be a polynomial in x of degree n.

We have

$$P_n(x) = A_0 + A_1(x-a-nh) + A_2(x-a-nh)(x-a-nh+h) + A_3(x-a-nh)(x-a-nh+h)(x-a-nh+2h) + \ldots + A_n(x-a-nh)(x-a-nh+h)\ldots(x-a-h) \qquad \ldots (1)$$

where $A_0, A_1, A_2, \ldots A_n$ are constants and are chosen such that $P_n(a+nh) = f(a+nh) \ldots$

$$\ldots\ldots P_n(a) = f(a)$$

Substituting $x = a+nh$ in equation (1) we get

$$P_n(a+nh) = A_0 \quad \therefore A_0 = f(a+nh)$$

Substituting $x = a+nh-h$ in equation (1)

we get

$$P_n(a+nh-h) = A_0 - h A_1$$

$$\therefore \quad f(a+nh-h) = f(a+nh) - h A_1$$

$$\therefore \quad A_1 = \frac{f(a+nh)-f(a+nh-h)}{h} = \frac{\Delta f(a+nh)}{h}$$

Substituting $x = a+nh-2h$ in equation (1)

we get

$$P_n(a+nh-2h) = A_0 + A_1(-2h) + A_2(-2h)(-h)$$

$$\therefore \quad 2h^2 A_2 = -f(a+nh) + 2[f(a+nh) - f(a+nh-h)] + f(a+nh-2h)$$

$$\therefore \quad 2h^2 A_2 = f(a+nh) - 2f(a+nh-h) + f(a+nh-2h)$$

$$A_2 = \frac{\nabla^2 f(a+nh)}{2!\,h^2}$$

Similarly we get

$$A_3 = \frac{1}{3!\,h^3}\nabla^3 f(a+nh)\ldots$$

$$\ldots\ldots A_n = \frac{1}{n!\,h^n}\nabla^n f(a+nh)$$

Substituting values of $A_0, A_1, A_2, \ldots A_n$ in equation (1) we get

$$P_n(x) = f(a+nh)+\frac{\nabla f(a+nh)}{h}(x-a-nh)$$
$$+\frac{\nabla^2 f(a+nh)}{2!\,h^2}(x-a-nh)(x-a-nh+h)$$
$$+\ldots+\frac{\nabla^n f(a+nh)}{n!\,h^n}(x-a-nh)\ldots$$
$$(x-a-h) \qquad \ldots (2)$$

which is the Newton-Gregory formula for backward interpolation

Substituting $u = \frac{x-(a+nh)}{h}$ i.e. $x = a+nh+uh$ in equation (2) we get,

$$P_n(x) = P_n(a+nh+uh) = f(a+nh)+u\nabla f(a+nh)$$
$$+\frac{u(u+1)}{2!}\nabla^2 f(a+nh)+\ldots$$
$$\ldots+\frac{u(u+1)(u+2)\ldots(u+n-1)}{n!}\nabla^n f(a+nh)$$

Note : This formula is useful for interpolation near the end of a set of tabular values.

Example 8.11 : *From the following table find*

i) y when x = 7 ii) y when x = 17

iii) y when x = 19

x :	8	10	12	14	16	18
f(x) = y :	10	19	32.5	54	89.5	15.4

Solution : Consider the following forward difference table.

x	y = f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
8	10					
		9				
10	19		4.5			
		13.5		3.5		
12	32.5		8		2.5	
		21.5		6		6.5
14	54		14		9	
		35.5		15		
16	89.5		29			
		64.5				
18	15.4					

i) To find y at x = 7 :

As x = 7 is at the beginning of the table but x = 7 is the outside point of the given range of x.

$\therefore$ Use Newton-Gregory forward interpolation formula for extrapolation at x = 7.

We have

$$f(x) = f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)$$
$$+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$
$$+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$
$$+\frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 f(a)+\ldots \qquad \ldots (1)$$

Here a = 8 and $u = \frac{x-a}{h} = \frac{x-8}{2} = \frac{7-8}{2} = -\frac{1}{2}$

$\therefore$ Equation (1) becomes

$$f(7) = 10-\frac{9}{2}+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{2!}\right)(4.5)$$
$$+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{3.5}{6}$$
$$+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\frac{2.5}{24}$$
$$+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)\frac{6.5}{120}$$
$$f(7) = 5.1777$$

ii) To find y at x = 17

As x = 17 is at the end of the given table, we use N.G. Backward interpolation formula at x = 17,

We have

$$f(x) = f(a+nh)+u\nabla f(a+nh)$$
$$+\frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$
$$+\frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh)$$
$$+\frac{u(u+1)(u+2)(u+3)}{4!}\Delta^4 f(a+nh)$$
$$+\frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^4 f(a+nh)$$
$$+0 \qquad \ldots (2)$$

Here x = 17, $u = \frac{x-(a+nh)}{h} = \frac{17-18}{2} = -\frac{1}{2}$

Equation (2) becomes

$$\therefore f(17) = 154+\left(-\frac{1}{2}\right)(64.5)+\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\frac{29}{2}$$

$$+\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)\frac{15}{6}$$

$$+\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)\left(-\frac{1}{2}+3\right)\frac{9}{24}$$

$$+\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)\left(-\frac{1}{2}+3\right)\left(-\frac{1}{2}+4\right)\frac{6.5}{120}$$

$$\therefore f(17) = 126.841$$

iii) To find y at x = 19

As x = 19 is at the end of the table but x = 19 is the outside point of the given range of x.

∴ We use NGBIF for extrapolation at x = 19

$$\therefore \quad u = \frac{19-18}{2} = \frac{1}{2}$$

∴ Equation (2) becomes

$$f(19) = 154+\left(\frac{1}{2}\right)(64.5)+\frac{1}{2}\left(\frac{3}{2}\right)\frac{29}{2}$$

$$= +\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{15}{6}\right)+\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\frac{9}{24}$$

$$= +\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\left(\frac{9}{2}\right)\frac{6.5}{120}$$

$$\therefore \quad f(19) = 219.208$$

Example 8.12 : *From the following data, find the number of students who obtained less than 45 marks.*

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

Solution : We construct the following table.

Marks x	No. of students f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)	Δ^4 f(x)
below 40	31				
		42			
below 50	73		9		
		51		– 25	
below 60	124		– 16		37
		35		12	
below 70	159		– 4		
		31			
below 80	190				

As 45 lies at the beginning of the given table.

∴ We use N.G.F. for forward interpolation.

We have

$$f(x) = f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)$$

$$+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)+0 \quad ..(1)$$

Here $u = \frac{x-a}{h} = \frac{45-40}{10} = \frac{1}{2}$

∴ Equation (1) becomes

$$f(45) = f(40)+\frac{1}{2}\Delta f(40)+\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{\Delta^2 f(40)}{2}$$

$$+\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{\Delta^3 f(40)}{6}$$

$$+\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\frac{\Delta^4 f(40)}{120}$$

$$f(45) = 47.868 \simeq 48$$

Thus approximately 48 students obtained marks less than 45.

Example 8.13 : *Find value of y for x = 0.5 for the following set of values of x and y using Newton's forward difference formula.*

x	0	1	2	3	4
y	1	5	25	100	250

Solution : Consider following forward difference table.

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
0	1				
		4			
1	5		16		
		20		39	
2	25		55		–19
		75		20	
3	100		75		
		150			
4	250				

We have h = 1, x = 0.5, x_0 = 0, y_0 = 1

$$u = \frac{x-x_0}{h} = \frac{0.5-0}{1} = 0.5$$

By Newton's forward interpolation formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots$$

$$= 1+(0.5)4+(0.5)(0.5-1)\frac{16}{2} + \frac{(0.5)(0.5-1)(0.5-2)}{6}39 + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}\times(-19)$$

$$\boxed{y = 4.1796}$$

Example 8.14 : *From the following data find the cubic polynomial by using Newton's Gregory forward interpolation formula. Use it to find f(4).*

x	*0*	*1*	*2*	*3*
f(x)	*1*	*2*	*1*	*10*

Solution : Consider the following difference table.

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)
0	1			
		1		
1	2		–2	
		–1		12
2	1		10	
		9		
3	10			

Take a = 0, h = 1, $u = \frac{x-a}{h} = \frac{x}{1} = x$

By Newton's forward interpolation formula, we get

$$f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{6}\Delta^3 f(0) + \ldots.$$

$$= 1+x+(x^2-x)(-1)+x(x^2-3x+2)(2)$$

$$= 2x^3 - 7x^2 + 6x + 1$$

which is the required polynomial.

Now $f(4) = 2(4)^3 - 7(4)^2 + 6(4) + 1$

$$\boxed{f(4) = 37}$$

Example 8.15 : *From the tabulated values of x and y given below, prepare forward difference table. Find polynomial passing through the points and find it's slope at x = 1.5*

x	*0*	*2*	*4*	*6*	*8*
y	*5*	*29*	*125*	*341*	*725*

Solution : Consider the following forward difference table.

x	y	Δ y	Δ^2 y	Δ^3 y	Δ^4 y
0	5				
		24			
2	29		72		
		96		48	
4	125		120		0
		216		48	
6	341		168		
		384			
8	725				

We have $u = \frac{x-x_0}{h} = \frac{x-0}{2} = \frac{x}{2}$

By Newton's forward difference formula we have

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$y = f(x) = 5 + \frac{x}{2}(24) + \frac{x}{2}\left(\frac{x}{2}-1\right)\frac{72}{2} + \frac{x}{2}\left(\frac{x}{2}-1\right)\left(\frac{x_2}{2}-2\right)\frac{68}{6}$$

$$y = x^3 + 3x^2 + 2x + 5$$

∴ At x = 1.5, y = 18.125

And $\frac{dy}{dx} = 3x^2 + 6x + 2$

∴ $\left(\frac{dy}{dx}\right)_{x=1.5}$ = Slope at (1.5) = **17.75**

Example 8.16 : *From the following data find the cubic polynomial and hence find f(4).*

x	0	1	2	3
f(x)	1	2	1	10

Solution : We construct the following difference table.

x	f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)
0	1			
		1		
1	2		– 2	
		– 1		12
2	1		10	
		+ 9		
3	10			

Take a = 0, $\therefore$ $u = \frac{x-a}{h} = \frac{x-0}{1} = x$

$\therefore$ Using NGFIF we get

$$f(x) = f(0)+x\Delta f(0)+\frac{x(x-1)}{2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{6}\Delta^3 f(0)+0$$

$$= 1+x+(x^2-x)(-1)+(x^3-3x^2+2x)(2)$$

$$f(x) = 2x^3-7x^2+6x+1$$

which is the required polynomial.

We have, if a tabulated function is a polynomial then interpolation and extrapolation are obtained by using polynomial.

$$\therefore \quad f(4) = 2(4)^3-7(4)^2+6(4)+1$$

$$= 128-112+25 = 16 + 25 = 37$$

Note : We get same value of f(4) by using Newton-Gregory backward interpolation formula.

8.4 (A) Lagrange's Interpolation Formula

The Newton-Gregory backward or forward interpolation formulae are applicable only when argument x is equally spaced. But the Lagranges interpolation formula is more general and can be applied for unequally spaced argument.

Let y = f(x) be continuous and differentiable (n+1) times in the interval (a, b). Given n+1 values $[x_0, f(x_0)], [x_1, f(x_1)], [x_2, f(x_2)] \ldots [x_n, f(x_n)]$ where arguments $x_0, x_1, x_2, \ldots x_n$ are not necessarily equally spaced. As (n+1) values of f(x) are considered so $(n+1)^{th}$ differences are zero. Let $P_n(x)$ be a polynomial in x of degree n. We write $P_n(x)$ in the following form.

$$P_n(x) = A_0(x-x_1)(x-x_2)\ldots(x-x_n) + A_1(x-x_0)(x-x_2)\ldots(x-x_n)+\ldots + A_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \quad \ldots (1)$$

where $A_0, A_1, A_2, \ldots A_n$ are constants and are chosen such that $P_n(x_0) = f(x_0), P_n(x_1) = f(x_1) \ldots P_n(x_n) = f(x_n)$

Substituting $x = x_0$ in equation (1), we get

$$P_n(x_0) = f(x_0) = A_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)$$

$$\therefore \quad A_0 = \frac{1}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$

Substituting $x = x_1$ in equation (1), we get

$$f(x_1) = A_1(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)$$

$$\therefore \quad A_1 = \frac{1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1)$$

Similarly, we get

$$A_2 = \frac{1}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} f(x_2)$$

$$\vdots$$

$$A_n = \frac{1}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n)$$

Substituting values of $A_0, A_1, A_2, \ldots A_n$ in equation (1), we get

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1) + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n)$$

Which is the Lagranges interpolation formula. We may write above formula as

$$P_n(x) = f(x) = L_0 f(x_0)+L_1 f(x_1)+\ldots+L_i f(x_i) + \ldots + L_n f(x_n)$$

where $L_i = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$

For $0 \le i \le n$

(B) Inverse Lagranges interpolation formula

In this formula we have to find x for given value of y by interchanging roles of x and y in the Lagranges interpolation formula.

Thus

$$x = \frac{(y-y_1)(y-y_2)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_n)} x_0 + \dots$$

$$\dots + \frac{(y-y_0)(y-y_1)\dots(y-y_{n-1})}{(y_n-y_0)(y_0-y_1)\dots(y_n-y_{n-1})} x_n$$

i.e. to find x at $y = y^*$, we substitute $y = y^*$ in above equation and then simplify.

Example 8.17 : *Use Lagranges interpolation formula to express the function* $\frac{3x^2+x+2}{(x-1)(x-2)(x-3)}$ *as sums of partial fractions.*

Solution : In this type of examples we take numerator as

$$f(x) = 3x^2+x+2$$

We find values of f(x) for x = 1, 2, 3

x	1	2	3
f(x)	6	16	32

Here $x_0 = 1$, $x_1 = 2$, $x_2 = 3$

The Lagranges interpolation formula is

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)} f(1) + \frac{(x-1)(x-3)}{(2-1)(2-3)} f(2) + \frac{(x-1)(x-2)}{(3-1)(3-2)} f(3)$$

$$= 3(x-2)(x-3) - 16(x-1)(x-3) + 16(x-1)(x-2)$$

$$\therefore \frac{f(x)}{(x-1)(x-2)(x-3)} = \frac{3}{x-1} - \frac{16}{x-2} + \frac{16}{x-3}$$

Example 8.18 : *Find f(5) by using Lagranges interpolation formula, given that f(1) = 2, f(2) = 4, f(3) = 8, f(4) = 16, f(7) = 128.*

Solution : The given data will be tabulated as follows

x	1	2	3	4	7
f(x)	2	4	8	16	128

Here $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 7$ and $x = 5$

By Lagranges formula we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

$$f(5) = \frac{(-12)}{36}(2) + \frac{(-16)}{(-10)}(4) + \frac{(-24)}{8}(8) + \frac{(-48)}{-18}(16) + \frac{24}{360}(128)$$

$$f(5) = 33.13$$

Example 8.19 : *Find Lagranges polynomial passing through set of points (0, 2), (2, –2), (3, –1) Use it to find the* $\frac{dy}{dx}$ *at x = 2.*

Solution : We have

$$x_0 = 0, \quad x_1 = 2, \quad x_2 = 3,$$
$$y_0 = 2, \quad y_1 = -2, \quad y_2 = -1,$$

By Lagranges formula,

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$y = \frac{(x-2)(x-3)}{(0-2)(0-3)}(2) + \frac{(x-0)(x-3)}{(2-0)(2-3)}(-2) + \frac{(x-0)(x-2)}{(3-0)(3-2)}(-1)$$

$y = x^2 - 4x + 2$ which is the required polynomial

Now

$$\frac{dy}{dx} = 2x - 4$$

$$\left[\frac{dy}{dx}\right]_{x=2} = 2 \times 2 - 4 = 0$$

Example 8.20 : *Using Lagranges interpolation formula to evaluate y for x = 1.07 for the following set of values.*

x	1.0	1.2	1.3	1.5
y	1.0	1.728	2.197	3.375

Solution : We have

$x_0 = 1, \quad x_1 = 1.2, \quad x_2 = 1.3, \quad x_3 = 1.5$

$y_0 = 1, \quad y_1 = 1.728, \quad y_2 = 2.197, \quad y_3 = 3.375$

By Lagranges interpolation formula

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Putting values of x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3 and x = 1.07, we get,

$$\boxed{y = 0.35816}$$

Example 8.21 : *Find Lagrange's interpolating polynomial passing through set of points. Use it to find y at x = 1.5*

x	0	1	2
y	4	3	6

Solution : Given that

$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2$

$y_1 = 4, \quad y_2 = 3, \quad y_4 = 6,$

By Lagranges interpolation formula we get,

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_2 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_3$$

$$y = f(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)}(4) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(3) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(6)$$

$$y = f(x) = 2(x^2-3x+2) - 3(x^2-2x) + 3(x^2-x)$$

$$y = f(x) = 2x^2 - 3x + 4$$

$\therefore \quad y(1.5) = f(1.5) = 1.75$

Example 8.22 : *The velocity distribution of fluid near a flat surface is given below.*

x	0.1	0.3	0.6	0.8
v	0.72	1.81	2.73	3.47

where x is the distance from the surface (mm) and v is the velocity (mm/sec). Use Lagrangues interpolation formula to obtain velocity at x = 0.4

Example 8.23 : *The velocity distribution of fluid near a flat surface is given below.*

x	0.1	0.3	0.6	0.8
v	0.72	1.81	2.73	3.47

where x is the distance from the surface (mm) and v is the velocity (mm/sec). Use Lagrangues interpolation formula to obtain velocity at x = 0.4

Solution : We have

$x_0 = 0.1, \quad x_1 = 0.3, \quad x_2 = 0.6, \quad x_3 = 0.8$

$v_0 = 0.72, \quad v_1 = 1.81, \quad v_2 = 2.73, \quad v_3 = 3.47$

By Lagranges interpolation formula, we have

$$v = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}v_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}v_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}v_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}v_3$$

Substituting values of $x_0, x_1, x_2, x_3, v_0, v_1, v_2, v_3$ and x = 0.4, we get,

$$\boxed{v = 2.16028}$$

Example 8.24 : *Compute the value of x when y = 8 by inverse Lagranges interpolation formula.*

x	– 2	– 1	1	2
y	– 7	2	0	11

Solution : Here $x_0 = -2$, $x_1 = -1$, $x_2 = 1$, $x_3 = 2$ and $y_0 = -7$, $y_1 = 2$, $y_2 = 0$, $y_3 = 11$ and $y = 8$.

By inverse Lagranges interpolation formula, we have

$$x(y) = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} x_1 + \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3$$

$$\therefore \quad x(8) = \frac{6(8)(-3)}{(-9)(-7)(-18)}(-2) + \frac{15(8)(-3)}{9(2)(-9)}(-1) + \frac{(15)(6)(-3)}{7(-2)(-11)}(1) + \frac{(15)(6)(8)}{18(9)(11)}(2)$$

$$x(8) = -3.5483$$

Thus $x = -3.5483$ at $y = 8$

Example 8.25 : *Find the form of the function y for the following data. Hence find y at x = 4.*

x	0	1	2	5
y	2	3	12	147

Solution : Applying Lagranges formula, we get

$$y = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$y = \frac{x^3-8x^2+17x-10}{-5} + 3\frac{(x^3-7x^2+10x)}{4} - 2(x^3-6x^2+5x) + \frac{x^3-3x^2+2x}{60} \times 147$$

$$y = x^3+x^2-x+2$$

$\therefore$ At $x = 4$, $y(4) = 4^3+4^2-4+2 = 78$

Exercise 8.2

1. *Find the number of students from the following who scored marks not more than 45.*

Marks range	30-40	40-50	50-60	60-70	70-80
No. of students	35	48	70	40	22

2. *Find f(0.2) if $f(0) = 176$, $f(1) = 185$, $f(2) = 194$, $f(3) = 203$, $f(4) = 212$, $f(5) = 220$, $f(6) = 229$.*

3. *The following temperature readings were taken on a day*

Time	2 a.m.	6 a.m.	10 a.m.	2 p.m.
Temp.	40.2°	42.4°	51.0°	72.4°

Find the temperature at 3 a.m. and 4 a.m.

4. *From the following table find tan 17 °*

θ°	0	4	8	12	16	20	24
tan θ°	0	0.0699	0.1405	0.2126	0.2167	0.3640	0.4402

5. *The population of a town in the census is given below. Estimate the population in the year 1995 and 1925.*

Year x	1891	1901	1911	1921	1931
Population (in 1000's)	46	66	81	93	101

6. *From the following table find tan 0.12 and tan 0.26.*

x	0.10	0.15	0.20	0.25	0.30
tan x	0.1003	0.1511	0.2027	0.2553	0.3093

7. *From the following table estimate sin 28° 24'. Also find the error.*

θ	15°	26°	27°	28°	29°	30°
sin θ	0.42262	0.43837	0.45399	0.46947	0.48481	0.50000

8.5 Numerical Differentiation

Numerical differentiation is the process of calculating the derivative of the function for some particular value of independent variable without knowning the actual function.

8.5.1 Derivatives using Newton-Gregory Forward Interpolation Formula

Consider the Newton forward interpolation formula

$$y = f(x) = f(a) + u\,\Delta\, f(a) + \frac{u\,(u-1)}{2!}\Delta^2 f(a) + \frac{u\,(u-1)\,(u-2)}{3!}\Delta^3 f(a) + \frac{u\,(u-1)\,(u-2)\,(u-3)}{4!}\Delta^4 f(a) + \ldots \quad \ldots (1)$$

where $u = \frac{x-a}{h}$ where $a = x_0$ and h is interval difference

$$\therefore \quad y = y_0 + u\,\Delta\, y_0 + \frac{u\,(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \frac{u(u-1)\,(u-2)\,(u-3)}{24}\Delta^4 y_0 + \ldots\ldots \quad (2)$$

where $u = \frac{x-x_0}{h}$

Here $y \to u \to x$

$\therefore$ Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\frac{dy}{du}$$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left\{\Delta\, y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18\,u^2+22\,u-6}{24}\Delta^4 y_0 + \ldots\right\} \quad \ldots (3)$$

At $x = x_0$, $u = \frac{x-x_0}{h} = 0$

$$\therefore \left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta\, y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

Differentiating equation (3) with respect to x, we get

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left[\frac{dy}{dx}\right]\cdot\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left[\frac{dy}{dx}\right]$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6u-6}{6}\Delta^3 y_0 + \frac{12\,u^2-36\,u+22}{24}\Delta^4 y_0 + \ldots\right] \quad \ldots (4)$$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right]$$

Now differentiating equation (2) with respect to x we get

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d}{du}\left[\frac{d^2y}{dx^2}\right]\cdot\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left[\frac{d^2y}{dx^2}\right]$$

$$\therefore \quad \frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3\, y_0 + \frac{24\,u-36}{24}\Delta^4\, y_0 + \ldots\right]$$

$$\therefore \left[\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4\, y_0 + \ldots\right]$$

8.5.2 Derivatives using Newton-Gregory Backward Interpolation Formula

Consider the Newton's backward interpolation formula

$$y = y_n + u\,\nabla\, y_n + \frac{u\,(u+1)}{2!}\nabla^2 y_n + \frac{u\,(u+1)\,(u+2)}{3!}\nabla^3 y_n + \frac{u\,(u+1)\,(u+2)\,(u+3)}{4!}\Delta^4 y_n + \ldots$$

$$\therefore \quad y = y_n + u\,\nabla\, y_n + \frac{u^2+u}{2}\nabla^2 y_n + \frac{u^3+3u^2+2u}{6}\nabla^3 y_n + \frac{u^4+6u^3+11u^2+6u}{24}\nabla^4 y_n + \ldots \quad \ldots (1)$$

where $u = \frac{x-x_n}{h}$ Here $y \to u \to x$

Differentiating equation (1) with respect to x, we get

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\frac{dy}{du}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla\, y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{24}\nabla^4 y_n + \ldots\right] \quad \ldots (2)$$

At $x = x_n$, $u = 0$

$$\therefore \left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right]$$

Differentiating equation (2) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{du}\left[\frac{dy}{dx}\right]\cdot\frac{du}{dx} = \frac{1}{h}\frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6u+6}{6}\Delta^3 y_n + \frac{12u^2+36u+22}{24}\nabla^4 y_n + ...\right] \quad ... (3)$$

At $x = x_n$, $u = 0$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...\right]$$

Differentiating equation (3) with respect to x, we get

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{du}\left(\frac{d^2y}{dx^2}\right)\cdot\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left(\frac{d^2y}{dx^2}\right)$$

$$\therefore \frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{24u+36}{24}\nabla^4 y_n + ...\right]$$

At $x = x_n$, $u = 0$

$$\therefore \left[\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + ...\right]$$

8.5.3 Applications of Derivatives to Find Maxima and Minima of a Tabulated Function

Consider Newton's forward interpolation formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ...$$

Differentiating with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + ...$$

Consider $\frac{dy}{du} = 0$

$$\therefore \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 = 0$$

Solving above equation for u and then find x by using $x = x_0 + uh$ at which y is maximum or minimum.

Example 8.26 : *Given that*

x	1	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at i)* $x = 1.1$ *ii)* $x = 1$ *iii)* $x = 1.6$

Solution : Consider the difference table

x	y	Δy	Δ^2 y	Δ^3 y	Δ^4 y	Δ^5 y	Δ^6 y
1	7.989						
		0.414					
1.1	8.403		– 0.036				
		0.378		0.006			
1.2	8.781		– 0.030		– 0.002		
		0.348		0.004		0.002	
1.3	9.129		– 0.026		0.000		– 0.003
		0.322		0.004		– 0.001	
1.4	9.451		– 0.023		– 0.001		
		0.299		0.005			
1.5	9.750		– 0.018				
		0.281					
1.6	10.031						

i) We have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...\right]$$

Here h = 0.1 and $x_1 = 1.1$

$$\therefore \left(\frac{dy}{dx}\right)_{x_1=1.1} = \frac{1}{h}\left[\Delta y_1 - \frac{1}{2}\Delta^2 y_1 + \frac{1}{3}\Delta^3 y_1 - \frac{1}{4}\Delta^4 y_1 + \frac{1}{5}\Delta^5 y_1\right]$$

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(0) + \frac{1}{5}(-0.001)\right] = 3.941$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_1} = \frac{1}{h^2}\left[\Delta^2 y_1 - \Delta^3 y_1 + \frac{11}{12}\Delta^4 y_1 - \frac{5}{6}\Delta^5 y_1 + ...\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(0) - \frac{5}{6}(-0.001)\right]$$

$$= -3.3167$$

ii) $$\left(\frac{dy}{dx}\right)_{x_0=1} = \frac{1}{0.1}\left[0.414 - \frac{1}{2}(-0.036) + \frac{1}{3}(0.006) - \frac{1}{4}(-0.002) + \frac{1}{5}(0.002) + \frac{1}{6}(-0.003)\right]$$

$$= +2.9$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0=1} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + ...\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.036 - 0.006 + \frac{11}{12}(-0.002) - \frac{5}{6}(0.002) + ...\right] = -2.9827$$

iii) By using backward difference table, we have

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^5 y_n + \frac{1}{5}\nabla^6 y_n + ...\right]$$

Here $h = 0.1$, $x_n = 1.6$, $\nabla y_6 = 0.281$, $\nabla^2 y_6 = -0.018$ etc.

$$\therefore \left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(-0.001) + \frac{1}{5}(-0.001)\right] = 2.732$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(-0.001) + \frac{5}{6}(-0.001)\right] = -1.475$$

Example 8.27 : *Find first two derivatives of f(x) at x = 1.1 from the following table.*

x	1	1.2	1.4	1.6	1.8	2.0
y	0	0.1280	0.5440	1.2960	2.4320	4

Solution : Here x = 1.1 lies at the begining of the argument so we use Newton's forward interpolation difference formula.

Consider the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0				
		0.1280			
1.2	0.1280		0.2880		
		0.4160		0.0480	
1.4	0.544		0.3360		0
		0.7520		0.0480	
1.6	1.2960		0.3840		0
		1.1360		0.0480	
1.8	2.4320		0.4320		
		1.5680			
2	4				

Here $h = 0.2$, $x_0 = 1$, $x = 1.1$

$$\therefore \quad u = \frac{x - x_0}{h} = \frac{1.1-1}{0.2} = \frac{0.1}{0.2} = 0.5$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^2 y_0 + 0\right]$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1.1} = \frac{1}{0.2}\left[0.1280 + \frac{(2(0.5)-1)}{2}(0.2880) + \frac{3(0.5)^2 - 6(0.5)+2}{6}(0.048) + 0\right]$$

$$= 0.630$$

And the formula for second order is

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6u-6}{6}\Delta^3 y_0 + ...\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.1} = \frac{1}{(0.2)^2}\left[0.2880 + \frac{6(0.5)-6}{6}(0.0480) + 0\right]$$

$$= 6.6$$

Example 8.28 : *The population of a certain city is shown in the following table*

Year (x)	1931	1941	1951	1961	1971
Population (y) (in Lakhs)	46.62	60.80	79.95	103.56	132.65

Find the rate of growth of the population in 1961.

Solution : Here h = 10, the rate of growth of population is dy/dx.

Here find $\frac{dy}{dx}$ at x = 1961 which lies near to the end of tabular values so we use Newton's backward interpolation formula for derivatives.

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2 + 6u + 2}{6}\nabla^3 y_n + \frac{4u^3 + 18u^2 + 22u + 6}{24}\nabla^4 y_n + ...\right]$$

$$\text{Here } u = \frac{x - x_n}{h} = \frac{1961 - 1971}{10} = -1$$

Consider the backward difference table

x	y	Δ y	Δ^2 y	Δ^3 y	Δ^4 y
1931	40.62				
		20.18			
1941	60.80		– 1.03		
		19.15		5.49	
1951	79.95		4.46		– 4.47
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

Thus

$$\left(\frac{dy}{dx}\right)_{x=1961} = \frac{1}{10}\left[29.09 - \frac{1}{2}(5.48) + \frac{3(-1)^2+6(-1)+2}{6}\right.$$
$$\left.(1.02) + \frac{2(-1)^3+9(-1)^2+11(-1)+3}{12}(-4.47)\right]$$
$$= 2.6553$$

$\therefore$ The rate of growth of population in the year 1961 is 2.6553 Lakhs.

Example 8.29 : *Find the maximum and minimum value of y from the following table.*

x	0	1	2	3	4	5
y	0	0.25	0	2.25	16	56.25

Solution : Consider the forward difference table

x	y	Δy	Δ^2y	Δ^3y	Δ^4y	Δ^5y
0	0					
		0.25				
1	0.25		– 0.5			
		– 0.25		3		
2	0		2.5		6	
		2.25		9		0
3	2.25		11.5		6	
		13.75		15		
4	16		26.5			
		40.25				
5	56.25					

Here $x_0 = 0$ and $h = 1$, $\therefore$ $u = \frac{x - x_0}{h} = x$

We have

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0\right.$$
$$\left. + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \ldots\right]$$
$$= 0.25 + \frac{2x-1}{2}(-0.50) + \frac{3x^2-6x+2}{6}(3)$$
$$+ \frac{4x^3-18x^2+22x-6}{24}(6) + 0$$
$$= 4x^3 - 12x^2 + 8x$$

Now $\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 12x^2 + 8x = 0$

$\Rightarrow 4x(x-2)(x-1) = 0$

$\Rightarrow x = 0, 1, 2$

Also $\frac{d^2y}{dx^2} = 12x^2 - 24x + 8$

At $x = 0$, $\frac{d^2y}{dx^2} = 8 > 0$

At $x = 1$, $\frac{d^2y}{dx^2} = -4 < 0$

At $x = 2$, $\frac{d^2y}{dx^2} = 8 > 0$

$\therefore$ y is maximum at x = 1 and minimum at x = 0 and x = 2.

$\therefore$ The maximum value of y at u = 1 (x = 1) is

$$y(1) = y_0 + \Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$
$$y(1) = 0.25$$

And minimum value of y at x = 0 and x = 2 is

$y(0) = y(2) = 0$

Example 8.30 : *A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has rotated for various values of time t (seconds).*

t	0	0.2	0.4	0.6	0.8	1.0
θ	0	0.12	0.49	1.12	2.02	3.20

Find angular velocity and angular acceleration of the rod when t = 0.2 seconds.

Solution : Here h = 0.2

The angular velocity = $\left(\frac{d\theta}{dt}\right)$ and angular acceleration $= \frac{d^2\theta}{dt^2}$

As t = 0.2 is near to the begining value of t so we use Newton's forward formula for derivative

$$\frac{d\theta}{dt} = \frac{1}{h}\left[\Delta\theta_0 + \frac{(2u-1)}{2}\Delta^2\theta_0 + \frac{3u^2-6u+2}{6}\Delta^3\theta_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4\theta_0 + \ldots\right] \quad \ldots (1)$$

and

$$\frac{d^2\theta}{dt^2} = \frac{1}{h^2}\left[\Delta^2\theta_0 + \frac{6u-6}{6}\Delta^3\theta_0 + \frac{12u^2-36u+22}{24}\Delta^4\theta_0 + \ldots\right] \quad \ldots (2)$$

Consider the difference table

t	θ	Δθ	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	0.49		0.26		0
		0.63		0.01	
0.6	1.12		0.27		0
		0.90		0.01	
0.8	2.02		0.28		
		1.18			
1.0	3.20				

Here $u = \frac{t-t_0}{h} = \frac{0.2-0}{0.2} = \frac{0.2}{0.2} = 1$

∴ Equation (1) becomes

$$\left[\frac{d\theta}{dt}\right]_{t=0.2} = \frac{1}{0.2}\left[0.12 + \frac{1}{2}(0.25) - \frac{1}{6}(0.01) + 0\right] = 0.056$$

$$\left[\frac{d^2\theta}{dt^2}\right]_{t=0.2} = \frac{1}{0.2}[0.25+0+0] = 0.125$$

Exercise 8.3

1. *Obtain the first and second order derivatives at $x = 1.5, 1.7, 3.7, 4$.*

x	1.5	2	2.5	3	3.5	4
y	2	3.5	4.5	6	8.5	10

2. *Find $f'(1)$ for $f(x) = \frac{1}{1+x^2}$*

x	1	1.1	1.2	1.3	1.4
f(x)	0.25	0.2268	0.2066	0.1890	1.1736

3. *Obtain y' and y'' at $x = 5$, $x = 6.5$, $x = 12$ and 12.5. if*

x	6	7	9	12
y	1.556	1.690	1.908	2.158

8.6 Numerical Integration

The computation of the definite integral $\int_{x_0}^{x_n} f(x)\,dx$ becomes difficult or practically impossible if i) any formula of integral calculus is not applicable. ii) when integrand f(x) is specified in tabular form. To overcome this we find first approximating function f(x) by interpolation and then integrating this approximation between the desired limits. The process of evaluating a definite integral from a set of tabulated values of the integrand f(x) is called Numerical integration.

If this process is applied to the integration of a function of a single variable, it is known as quadrature.

8.6.1 A General Quadrature Formula

Let $I = \int_a^b y\,dx$ where $y = f(x)$ be given for certain equidistant values of arguments say $x_0, x_0+h, \ldots$ Let the range b-a be divided into n equal parts each of width h.

$$\therefore \quad h = \frac{b-a}{n} \quad \therefore \; b-a = nh$$

Let $x_0 = a,\; x_1 = a+h,\; x_2 = a+2h, \ldots x_n = a+nh = b$

Put $u = \frac{x-a}{h} \quad \therefore \quad dx = h\,du$

and $x = a+uh$

At $x = a$, $u = 0$ and at $x = a+nh$ $u = n$

$$\therefore \quad I = \int_a^b y\,dx = \int_0^n y(u)\,h\,du$$

By using Newton's forward difference formula in the integrand, we get

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u^2-u}{2}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{6}\Delta^3 y_0 + \ldots\right] du$$

$$= h\left[u\,y_0 + \frac{u^2}{2}\Delta y_0 + \left(\frac{u^3}{3}-\frac{u^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{u^4}{4}-\frac{3u^3}{3}+\frac{2u^2}{2}\right)\frac{\Delta^3 y_0}{6}+\ldots\right]_0^n$$

$$= h\left[n\,y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3}-\frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4}-n^3+n^2\right)\frac{\Delta^3 y_0}{6}+ \ldots \text{upto } (n+1) \text{ terms}\right] \quad \ldots (1)$$

which is known as a General quadrature formula.

We can obtain number of quadrature formulae from (1) by substituting n = 1, 2, 3, ...

8.6.2 Trapezoidal Rule

Substitute n = 1 in General quadrature formula.

$\therefore$ The interval of integration will be from x_0 to x_0+h. As there are only two functional values y_0 and y_1. So there are no differences of orders two and higher than two. $\therefore$ we get

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{y_1-y_0}{2}\right] = h\left[\frac{y_0+y_1}{2}\right]$$

Similarly $\displaystyle\int_{x_0+h}^{x_0+2h} y\,dx = h\left[\frac{y_1+y_2}{2}\right]$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = h\left[\frac{y_{n-1}+y_n}{2}\right]$$

Adding these n integrals, we get

$$I = \int_{x_0}^{x_0+nh} y\,dx$$

$$= h\left[\frac{1}{2}(y_0+y_n) + (y_1+y_2+y_3+\ldots+y_{n-1})\right]$$

= Distance between 2 consecutive ordinates

[Mean of first and last ordinate + Sum of all the intermediate ordinates]

This is known as Trapezoidal rule.

8.6.3 Simpson's $\left(\frac{1}{3}\right)^{rd}$ Rule

Substitute n = 2 in general quadrature formula $\therefore$ the interval of integration will be from x_0 to x_0+2h. As there are only three functional values y_0, y_1, y_2. So there are no differences of orders 3 and higher than 3.

$\therefore$ We get

$$\int_{x_0}^{x_0+2h} y\,dx = h\left[2y_0 + 2\Delta y_0 + \left(\frac{8}{3}-2\right)\frac{\Delta^2 y_0}{2}+0\right]$$

$$= h\left[2y_0+2y_1-2y_0+\frac{1}{3}(y_2-2y_1+y_0)\right]$$

$$= \frac{h}{3}[y_0+4y_1+y_2]$$

Similarly

$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}[y_2+4y_3+y_4]$$

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}[y_{n-2}+4y_{n-1}+y_n]$$

Adding all these n integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}[(y_0+y_n)+4(y_1+y_3+\ldots+y_{n-1}) + 2(y_2+y_4+\ldots+y_{n-2})]$$

$= \left(\frac{1}{3}\right)^{rd}$ of h [(Sum of extreme ordinates)

+ 4 (Sum of odd ordinates)

+ 2 (Sum of even ordinates)]

This is known as Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule.

8.6.4 Simpson's $\left(\frac{3}{8}\right)^{rd}$ Rule

Substitute n = 3 in general quadrature formula ∴ The interval of integration will be from x_0 to x_0+3h. As there are only 4 functional values y_0, y_1, y_2, y_3. So there are no differences of orders 4 and higher than 4.

∴ We get

$$\int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0+\frac{9}{2}\Delta y_0+\left(\frac{27}{3}-\frac{9}{2}\right)\frac{\Delta^2 y_0}{2}\right.$$

$$\left.+\left(\frac{81}{4}-27+9\right)\frac{\Delta^3 y_0}{6}\right]$$

$$= h\left[3y_0+\frac{9}{2}(y_1-y_0)\right.$$

$$+\frac{9}{4}(y_2-2y_1+y_0)$$

$$\left.+\frac{3}{8}(y_3-3y_2+3y_1-y_0)\right]$$

$$= \frac{3h}{8}[y_0+3y_1+3y_2+y_3]$$

Similarly

$$\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3h}{8}[y_3+3y_4+3y_5+y_6]$$

$$\vdots$$

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3h}{8}[y_{n-3}+3y_{n-2}+3y_{n-1}+y_n]$$

Adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4$$

$$+y_5+...+y_{n-1})$$

$$+2(y_3+y_6+y_9+...+y_{n-3})]$$

$$= \left(\frac{3}{8}\right)^{th} \text{ of h [(Sum of extreme ordinates)}$$

+ 2 (Sum of multiple of 3 ordinates)

+ 3 (Sum of all remaining ordinates)]

This is known as Simpson's $\left(\frac{3}{8}\right)^{th}$ rule.

Note :

1) There is no restriction for the number of intervals in Trapezoidal rule.
2) In Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule, the number of subintervals must be even.
3) In Simpson's $\left(\frac{3}{8}\right)^{th}$ rule, the number of subintervals must be multiple of 3.
4) To get more accuracy, divide the given interval into maximum number of subintervals.

Example 8.31 : *Use trapezoidal rule to evaluate* $I = \int_0^1 xe^{x^2}dx$ *by taking* $h = 0.1$.

Solution : Let $y = xe^{x^2}$, h = 0.1

Consider the following table

x	0	0.1	0.2	0.3	0.4	0.5
y	0	0.101005	0.2081621	0.3282552	0.4694043	0.6420127
	y_0	y_1	y_2	y_3	y_4	y_5

x	0.6	0.7	0.8	0.9	1.0
y	0.8599976	1.142614	1.1571847	2.0231172	2.7182818
	y_6	y_7	y_8	y_9	y_{10}

We have

$$I = \frac{h}{2}[(y_0+y_{10})+(y_1+y_2+.....+y_9)]$$

$$= \frac{0.1}{2}[(0 + 2.7182818) + (0.101005 + 0.2081621$$

$$+ 0.3282552 + 0.4694043 + 0.6420127$$

$$+ 0.8599976 + 1.142614 + 1.1571847$$

$$+ 2.0231172)]$$

$$I = 0.8650898$$

Actual value :

$$I = \int_0^1 xe^{x^2}dx = \int_0^1 (2x)e^{x^2}dx$$

$$= \frac{1}{2}[e^{x^2}]_0^1 = \frac{1}{2}[e-1]$$

$$= \mathbf{0.8591409}$$

Example 8.32 : *Evaluate* $\int_0^1 \frac{1}{1+x} dx$ *by Simpson's* $\left(\frac{1}{3}\right)^{rd}$ *and* $\left(\frac{3}{8}\right)^{th}$ *rule.*

Compare the result with actual value.

Solution : To apply Simpson's $\left(\frac{1}{3}\right)^{rd}$ and $\left(\frac{3}{8}\right)^{th}$ rules, number of subintervals must even and multiple of 3 so we take 6 subintervals

$\therefore \quad h = \frac{1-0}{6} = \frac{1}{6}$

The values of y corresponding to each value of x are given as

x	0	1/6	2/6	3/6	4/6	5/6	1
y	1	6/7	6/8	6/9	6/10	6/11	6/12

i) By Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule

$$I = \int_0^1 \frac{1}{1+x} dx = \frac{h}{3}[(y_0+y_6)+4(y_2+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{18}\left[\left(1+\frac{6}{12}\right)+4\left(\frac{6}{7}+\frac{6}{9}+\frac{6}{11}\right)+2\left(\frac{6}{8}+\frac{6}{10}\right)\right]$$

$= 0.6931$

ii) By Simpson's $\left(\frac{3}{8}\right)^{th}$ rule

$$\int_0^1 \frac{dx}{1+x} = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$

$$= \frac{3}{8}\left(\frac{1}{6}\right)\left[\left(1+\frac{6}{12}\right)+3\left(\frac{6}{7}+\frac{6}{8}+\frac{6}{10}+\frac{6}{11}\right)+2\left(\frac{6}{9}\right)\right]$$

$= 0.6931$

iii) $I = \int_0^1 \frac{1}{1+x} dx = [\log(1+x)]_0^1 = \log 2 - \log 1$

$= 0.6930$

The actual value is very near to the values obtained by Simpson's rules.

Example 8.33 : *Find the area bounded by the curve f(x) and the x-axis and x = 7.47 to x = 7.52 from following data.*

x	7.47	7.48	7.49	7.50	7.51	7.52
f(x)	1.92	1.95	1.98	2.01	2.03	2.06

Solution : By Trapezoidal rule, we have

$$\text{Required area} = \int_{x_0}^{x_0+nh} f(x)\, dx$$

$$= h\left[\frac{1}{2}(y_0+y_5)+(y_1+y_2+y_3+y_4)\right] = 0.0996$$

Example 8.34 : *A curve passes through the given points. Find*

i) Area bounded by the curve, x axis, x = 1 and x = 9.

ii) The volume of the solid generated by revolving this area about the x-axis.

x	1	2	3	4	5	6	7	8	9
y	0.2	0.7	1	1.3	1.5	1.7	1.9	2.1	2.3

Solution : i) Here h = 1, By Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule

$$\text{Area} = \int_1^9 y\, dx$$

$$= \frac{h}{3}[(y_0+y_8)+2(y_2+y_4+y_6)+4(y_1+y_3+y_5+y_7)]$$

$$= \frac{1}{3}[(0.2+2.3)+2(1+1.5+1.9)+4(0.7+1.3+1.7+2.1)]$$

$= 11.5$ sq. units

ii) The required volume $= V = \pi\int_1^9 y^2\, dx$

$$= \frac{\pi}{3}[(0.2^2+2.3^2)+2(1^2+1.5^2+1.9^2)$$

$$+4(0.7^2+1.3^2+1.7^2+2.1^2)]$$

$$= \frac{\pi}{3}(5697) = 59.6588 \text{ cubic units}$$

Example 8.35 : *Following table gives speeds of an electric train at various times after leaving one station until it stops at the next station are as*

Speed in mph	0	13	33	79/2	40	40	36	15	0
Time in minutes	0	1/2	1	3/2	2	5/2	3	13/4	7/2

Find the distance between two stations.

Solution : Let $v = \frac{ds}{dt}$ be the velocity of train at any time t. Then s the distance between the two stations (I^{st} and last stations) is

$$s = \int_{t=0}^{t=7/2} \frac{ds}{dt}\, dt = \int_0^{7/2} v\, dt = \int_0^3 v\, dt + \int_3^{7/2} v\, dt$$

$$= s_1 + s_2 \qquad \dots (1)$$

($\because$ width of time sub intervals is not same).

Consider the following table.

Speed in mph	0	13	33	79/2	40	40	36	15	0
Time in hours	0	$\frac{1}{120}$	$\frac{1}{60}$	$\frac{1}{40}$	$\frac{1}{30}$	$\frac{1}{24}$	$\frac{1}{20}$	$\frac{13}{240}$	$\frac{7}{480}$

$\therefore$ Equation (1) becomes (by Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule).

If $\quad s = s_1 + s_2$

$$= \frac{1}{3}\frac{1}{120}\left[(0+36)+4\left(13+\frac{79}{2}+40\right)\right.$$

$$\left. +2(33+40)+2(33+40)\right]$$

$$+\frac{1}{3}\frac{1}{240}[36+0+4(15)]$$

$$s = \frac{23}{15}+\frac{2}{15} = 1.666 \text{ miles.}$$

Thus the distance between I[st] and last station is 1.666 miles.

Example 8.36 : *Use Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule to obtain $\int_0^{\pi/2}\left(\frac{\sin x}{x}\right)dx$ by dividing the interval into 4 parts.*

Solution : Take $h = \frac{\pi}{8}$, $y = \frac{\sin x}{x}$

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	0	0.974495	0.9003163	0.784233	0.636619

$\therefore$ By Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule

$$\int_0^{\pi/2}\left(\frac{\sin x}{x}\right)dx = \frac{h}{3}[(y_0+y_4)+4(y_1+y_3)+2y_2]$$

$$= \frac{\pi}{24}[(1 + 0.636619) + 4(0.974495 + 0.789233)$$

$$+ 2(0.9003163)]$$

$$I = 1.3707929$$

Example 8.37 : *Evaluate $\int_4^{5.2} \log_e x\, dx$ take $h = 0.2$ use Simpson's $\frac{1}{3}$ rule. Compare with exact value.*

Solution : Let $y = \log x_e$, $h = 0.2$

x	4	4.2	4.4	4.6	4.8	5	5.2
y	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

By Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule

$$I = \int_4^{5.2} \log x_e dx$$

$$= \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{0.2}{3}[(1.3863 + 1.6487) + 4(1.4351 + 1.5261$$

$$+ 1.6094) + 2(1.4816 + 1.568)]$$

$$I = 1.827853$$

Exact value :

$$I = \int_4^{5.2} \log x_e\, dx = [x \log x_e - x]_4^{5.2}$$

$$= [(5.2) \log 5.2 - 5.2] - [4 \log 4 - 4]$$

$$= 1.827847$$

Result is correct upto four decimal places.

Example 8.38 : *A solid of revolution is formed by rotating about x - axis. The area between x-axis, lies x = 0 and x = 1 and curve through the axis is given as*

x	*0*	*0.25*	*0.5*	*0.75*	*1*
y	*1*	*0.9886*	*0.9589*	*0.8489*	*0.9415*

Estimate the volume of solid formed.

Solution : The volume of the solid =

$$V = \int_0^{2\pi} A\, d\theta \text{ where}$$

$$A = \text{Area} = \frac{h}{3}[(y_0+y_4)+2(y_2)+4(y_1+y_3)]$$

$$= \frac{h}{3}[(1+0.9415)+2(0.9589)+4(0.9886)+0.8489)]$$

$$= 0.93410$$

$$\therefore V = \int_0^{2\pi} A\, d\theta = 0.9341(2\pi) = \mathbf{5.87\ m^3}$$

Example 8.39 : *Use Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule with 10 intervals to evaluate $\int_1^2 \frac{1}{x} dx$.*

Solution : Let $I = \int_1^2 \frac{1}{x} dx,$

$$y = \frac{1}{x} \text{ and } h = \frac{2-1}{10} = \frac{1}{10} = 0.1$$

Consider

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
y	1	0.909	0.833	0.769	0.714	0.667	0.625	0.588	0.556	0.526	0.5

By Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule we have

$$I = \frac{h}{3}[(y_0+y_{10})+2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)]$$

$$= \frac{0.1}{3}[(1+0.5)+2(0.833+0.714+0.625+0.556)+4(0.909+0.769+0.667+0.588+0.526)$$

$$= \frac{0.1}{3}[1.5+5.456+13.836]$$

$$I = 0.693067$$

Example 8.40 : *Evaluate :*

$\int_0^{0.8} [\log_e(x+1)+\sin(2x)]\, dx$ where x is in radians, by using Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule, divide interval into 8 strips.

Solution : Let $y = \log_e(x+1)+\sin 2x$

$$h = \frac{x_n - x_0}{n} = \frac{0.8-0}{8} = 0.1 = \text{Strip width}$$

Consider following table

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
y	0	0.2939	0.5717	0.827	0.0538	1.2469	1.402	1.516	1.5873
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule

$$I = \int_0^{0.8} y\, dx = \frac{h}{3}[(y_0+y_8)+2(y_2+y_4+y_6) + 4(y_1+y_3+y_5+y_7)]$$

$$= \frac{0.1}{3}[(0+1.5873)+2(0.5717+0.0538+1.402) + 4(0.2939+0.827+1.2469+1.516)]$$

$$\boxed{I = 0.7725823}$$

Example 8.41 : *Evaluate : $\int_0^1 \frac{1}{1+x^2} dx$ taking $h = \frac{1}{6}$ by using Simpson's $\left(\frac{3}{8}\right)^{th}$ rule. Compare result with actual value.*

Solution : Let $y = \frac{1}{1+x^2}$, $h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}=1$
y	1	0.97297	0.9	0.8	0.69231	0.59016	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\left(\frac{3}{8}\right)^{th}$ rule

$$I = \int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5+2y_3]$$

$$= \frac{1}{16}[(1+0.5)+3(0.97297+0.9+0.69231+0.59016+2(0.8)]$$

$$I = 0.785395$$

Actual value :

$$I = \int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1 = \frac{\pi}{4} = 0.7853981$$

Result is correct upto five decimal places.

Example 8.42 : *Evaluate : $\int_0^3 \frac{dx}{1+x}$ with 7 ordinates by using Simpson's $\left(\frac{3}{8}\right)^{th}$ rule and hence calculate log 2.*

Solution : Here $h = \frac{3-0}{6} = \frac{1}{2} = 0.5, \quad y = \frac{1}{1+x}$

Considering the following table

x	0	0.5	1	1.5	2	2.5	3
y	1	0.6667	0.5	0.4	0.333	0.2857	0.25
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\left(\frac{3}{8}\right)^{th}$ rule, we have

$$I = \frac{3h}{8}[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)]$$

$$= \frac{3(0.5)}{8}[(1+0.5)+2(0.4)+3(0.6667+0.5+0.333+0.2857)]$$

$$\boxed{I = 1.3888}$$

i.e. $$\int_0^3 \frac{dx}{1+x} = 1.3888$$

By integrating

$$[\log(1+x)]_0^3 = 1.3888$$

$$\log 4 - \log 1 = 1.3888$$

$$\log 2^2 = 1.3888$$

$$2 \log 2 = 1.3888$$

$$\boxed{\log 2 = 0.6944}$$

Example 8.43 : *Evaluate :* $\int_0^{\pi} \frac{\sin^2\theta}{5+4\cos\theta} d\theta$ *by Simpson's* $\left(\frac{3}{8}\right)^{th}$ *rule, taking* $h = \frac{\pi}{6}$.

Solution : Let $y = \frac{\sin^2\theta}{5+4\cos\theta} d\theta, \; h = \frac{\pi}{6}$

θ	0	π/6	π/3	π/2	2π/3	5π/6	π
y	0	0.0295	0.1071	0.2	0.25	0.1627	0

By Simpson's $\left(\frac{3}{8}\right)^{th}$ rule, we have

$$I = \int_0^{\pi} y\, d\theta \quad \frac{3h}{8}[(y_0+y_6)+2y_3 + 3(y_1+y_2+y_4+y_5)]$$

$$= \frac{3}{8}\frac{\pi}{6}[0+2(0.2)+3(0.0295+0.1071+0.25+0.1627)]$$

$$= \frac{3}{8}\frac{\pi}{6}[2.04835] = 0.40219$$

$$\therefore \quad I = 0.40219$$

Exercise 8.4

1. *Evaluate using Trapezoidal rule* $\int_{-2}^{2} \frac{x}{5+x^2} dx$.

2. *Find approximate value of* $\int_0^{\pi/2} \sqrt{\cos x}\, dx$ *by Trapezoidal rule Simpson's* $\frac{1}{3}$ *and* $\left(\frac{3}{8}\right)^{th}$ *rules.*

3. *Compute by Simpson's rule, the value of integral* $\int_{200}^{1000} \frac{dx}{\log_{10} x}$ *taking eight subintervals.*

4. *By Simpson's rule find the area of the cross section of a river 80 metres wide, the depth y (in meters) at a distance x from one bank is given by*

x	0	10	20	30	40	50	60	70	80
y	1	4	7	9	12	15	14	8	3

5. *Find value of log 2 from the formula* $\log_2 = \int_1^2 \frac{1}{x} dx$ *by Simpson's* $\left(\frac{1}{3}\right)^{rd}$ *rule.*

6. *The velocities of vehicle at intervals of 2 minutes are given as*

Time (minutes)	0	2	4	6	8	10	12
Velocity (km/hr)	0	22	30	27	18	7	0

Apply Simpson's rule to obtain the distance covered by vehicle.

7. *A solid of revolution is formed by rotating about the x-axis the area between the x-axis the lines x = 0 and x = 1 and the curve through the points with the following co-ordinates.*

x	0	0.25	0.5	0.75	1
y	1	0.9896	0.5589	0.9089	0.8415

Compute the volume of solid formed giving the answer to three decimal places.

8. *Evaluate* $\int_0^{\pi/4} \tan x\, dx$ *dy Trapezoidal rule and Simpson's rules.*

Notes

UNIT - VI

9 Numerical Solution of Ordinary Differential Equations

9.1 Introduction

The differential equation is the most important mathematical model for physical phenomena such as motion of object, bending and cracking of materials, vibrations, fluids, heat flow chemical reactions. Many of these models can be reduced to the problem of solving differential equation with certain given conditions. By applying analytical methods we can solve several standard types of differential equations. But the D.E. appearing in some physical problems are quite complex and are not solved by any analytic method. In such cases we apply special Numerical methods.

A number of numerical methods are available for the solution of first ordered differential equation of the form

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0$$

This problem is known as initial value problem.

9.2 Taylor's Series Method

Consider the first order differential equation

$$y' = \frac{dy}{dx} = f(x,y) \qquad \text{... (9.1)}$$

with $\quad y(x_0) = y_0$

Differentiating equation (9.1) w.r.t. x, we get

$$y'' = \frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\therefore \quad y'' = f_x + f_y\, y' \qquad \text{... (9.2)}$$

Differentiating equation (9.2) successively, we get $y''', y^{iv}, \ldots$

Substituting $x = x_0$ and $y = y_0$, we get $y'_0, y''_0, y'''_0 \ldots$

Let y(x) be the solution of equation (9.1)

$\therefore$ Taylor's series expansion of y(x) about $x = x_0$ is

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \ldots$$

$$\therefore \quad y(x) = y_0 + (x - x_0)\, y'_0 + \frac{(x-x_0)^2}{2!} y''_0 + \frac{(x-x_0)^3}{3!} y'''_0 + \ldots \qquad \text{... (9.3)}$$

Substituting the values of $y_0, y'_0, y''_0, \ldots$ in equation (9.3), we get required solution y(x)

Let $\quad x_1 = x_0 + h$

$$\therefore \quad y_1 = y(x_1) = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \ldots$$

After getting y_1 we can find y'_1, y''_1, y'''_1. Then y can be expanded in taylor's series about $x = x_1$ as

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y'_1 + \frac{h^2}{2!} y''_1 + \frac{h^3}{3!} y'''_1 + \ldots$$

Continuing in this way we can find the solution y(x).

Example 9.1 : *Solve the equation $\frac{dy}{dx} = xy + 1$ with $y(0) = 2$. Find i) $y(0.1)$ ii) $y(0.2)$ iii) $y(0.3)$ by Taylor's series method.*

Solution : i) By Taylor's series, we have

$$y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \ldots \qquad \text{... (1)}$$

with $x_0 = 0, y_0 = 2$, take $h = x - x_0$ and $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$.

We have $\frac{dy}{dx} = y' = xy + 1$

$$\therefore \quad y'' = \frac{d^2y}{dx^2} = xy' + y$$

$$y''' = xy'' + 2y'$$

$$\therefore \quad y_0' = y'_{(x_0, y_0)} = x_0 y_0 + 1 = 1$$

$$y_0'' = y''_{(x_0, y_0)} = x_0 y_0' + y_0 = 2$$

$$y_0''' = y'''_{(x_0, y_0)} = x_0 y_0'' + 2y_0' = 2$$

Substituting $y_0', y_0'', y_0''' ....$ in equation (1) we get

$$y_1 = 2 + 0.1\,(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \ldots$$

$$y_1 = 2.1103$$

ii) We have $x_1 = 0.1$ and $y_1 = 2.1103$, $h = 0.1$

By Taylor's series,

$$y(0.2) = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots \quad \ldots (2)$$

$$y_1' = y'_{(x_1, y_1)} = 1 + x_1 y_1 = 1 + (0.1)\,(2.1103)$$

$$= 1.21103$$

$$y_1'' = y''_{(x_1, y_1)} = y_1 + x_1 y_1' = 2.2314$$

$$y_1''' = 2y_1' + x_1\, y_1'' = 2.6452$$

$\therefore$ Equation (2) becomes,

$$y_2 = y(0.2)$$

$$= 2.1103 + (0.1)\,(1.21103) + \frac{(0.1)^2}{2}(2.2314)$$

$$+ \frac{(0.1)^3}{6}(2.6452) = 2.2430$$

$$y(0.2) = 2.2430$$

iii) We have $x_2 = 0.2$, $y_2 = 2.2430$ and $h = 0.1$

$\therefore$ By Taylor's series,

$$y_3 = y(0.3)$$

$$= y_2 + h\,y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \ldots \quad \ldots (3)$$

$$y_2' = y'_{(x_2, y_2)}$$

$$= 1 + x_2 y_2 = 1.4486$$

$$y_2'' = y_2 + x_2 y_2' = 2.53272$$

$$y_2''' = 2y_2' + x_2 y_2'' = 3.4037$$

$\therefore$ Equation (3) becomes,

$$y_3 = 2.243 + (0.1)\,(1.4486) + \frac{(0.1)^2}{2!}(2.53272)$$

$$+ \frac{(0.1)^3}{3!}(3.4037) = 2.4011$$

$$y(0.3) = 2.4011$$

Example 9.2 : *Solve $\frac{dy}{dx} = x - y^2$ by Taylor's series method to calculate y at x = 0.4 with y(0) = 1 take h = 0.2.*

Solution : i) By Taylor's series, we have

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0'''$$

$$+ \frac{h^4}{4!} y_0^{iv} + \ldots \quad \ldots (1)$$

Here $x_0 = 0$, $h = 0.2$ and $y_0 = 1$

given that

$$y' = \frac{dy}{dx} = x - y^2$$

$$\therefore \quad y'' = 1 - 2yy', \quad y''' = -2yy'' - 2y'^2$$

$$y^{iv} = -2yy''' - 6y'y''$$

$$\therefore \quad y_0' = x_0 - y_0^2 = -1, \quad y_0'' = 1 - 2y_0 y_0' = 3$$

$$y_0''' = -8, \quad y_0^{iv} = 34$$

Using all these values in equation (1), we get

$$y_1 = y\,(0.2) = 1 + (0.2)\,(-1) + \frac{(0.2)^2}{2}(3)$$

$$+ \frac{(0.2)^3}{6}(-8) + \frac{(0.2)^4}{24}(34) + \ldots$$

$$y(0.2) = 0.8516$$

ii) To find y at x = 0.4.

we have $x_1 = 0.2$, $y_1 = 0.8516$, $h = 0.2$

$$\therefore \quad x_2 = 0.4$$

By Taylor's series

$$y_2 = y(0.4) = y_1 + hy_1' + \frac{h^2}{2!} y_1''$$

$$+ \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \ldots \quad \ldots (2)$$

Here $y_1 = 0.8516$

$$y_1' = x_1 - y_1^2 = -0.525$$

$$y_1'' = 1 - 2y_1 y_1' = 1.8942$$

$$y_1''' = -2y_1 y_1'' - 2(y_1')^2 = -3.78$$

$$y_1^{iv} = -2y_1 y_1''' - 6y_1' y_1'' = 12.405$$

∴ Equation (2) becomes,

$$y_2 = y(0.4) = 0.8516 + (0.2)(-0.525) + \frac{(0.2)^2}{2}(1.8942) + \frac{(0.2)^3}{6}(-3.78) + \frac{(0.2)^4}{24}(12.405) + \ldots$$

$$\mathbf{y(0.4) = 0.7803}$$

Example 9.3 : *Find value of y at $x = 0.3$ by Taylor's series method if $\frac{dy}{dx} = 2y + 3e^x$ with $y(0) = 1$. Take $h = 0.1$ and compare the result with the analytical solution.*

Solution : i) By Taylor's series, we have

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \ldots \quad \ldots (1)$$

Here $x_0 = 0,\ y_0 = 1,\ h = 0.1,\ x_1 = 0.1$

We have

$$y' = 2y + 3e^x \quad \therefore y_0' = 2y_0 + 3e^{x_0} = 5$$

$$y'' = 2y' + 3e^x, \quad y_0'' = 2y_0' + 3e^{x_0} = 13$$

$$y''' = 2y'' + 3e^x, \quad y_0''' = 2y_0'' + 3e^{x_0} = 29$$

$$y^{iv} = 2y''' + 3e^x, \quad y_0^{iv} = 2y_0''' + 3e^{x_0} = 61$$

Thus equation (1) becomes,

$$y_1 = y(0.1) = 1 + (0.1)\,5 + \frac{(0.1)^2}{2}(13) + \frac{(0.1)^3}{6}(29) + \frac{(0.1)^4}{24}(61) + \ldots$$

$$y(0.1) = 1.5701$$

ii) Here $x_1 = 0.1,\ y_1 = 1.5701,\ h = 0.2,\ \therefore x_2 = 0.2$

$$y_2 = y(0.2) = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \ldots$$

We have

$$y_1' = 2(1.5701) + 3.3155 = 6.4557 \quad \ldots (2)$$

$$y_1'' = 2(6.4557) + 3.3155 = 16.2270$$

$$y_1''' = 2(16.2270) + 3.3155 = 35.7695$$

$$y_1^{iv} = 2(35.7695) + 3.3155 = 74.8545$$

∴ Equation (2) becomes,

$$y_2 = y(0.2) = 1.5701 + (0.1)(6.4557) + \frac{(0.1)^2}{2!}(16.2270) + \frac{(0.1)^3}{6}(35.7695) + \frac{(0.1)^4}{24}(74.8545) + \ldots$$

$$\therefore y(0.2) = 2.3025$$

iii) Here $x_2 = 0.2,\ y_2 = 2.3025,\ h = 0.1\ \therefore x_3 = 0.3$

By Taylor's series, we have

$$y_3 = y(0.3) = y_2 + h\,y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \frac{h^4}{4!} y_2^{iv} + \ldots \quad \ldots (3)$$

Here $y_2 = 2.3025$

$$y_2' = 2(2.3025) + 3.6642 = 8.2692$$

$$y_2'' = 20.2026$$

$$y_2''' = 44.0694$$

$$y_2^{iv} = 91.8030$$

Using all these in equation (3), we get

$$y_3 = y(0.3) = 2.3025 + (0.1)(8.2692) + \frac{(0.1)^2}{2}(20.2026) + \frac{(0.1)^3}{6}(44.0694) + \frac{(0.1)^4}{24}\,91.8030$$

$$\therefore y(0.3) = 3.2382$$

iv) Analytical solution : we have

$$\frac{dy}{dx} - 2y = 3e^x \text{ which is linear D.E. in y.}$$

$$\therefore \quad \text{It's I.F.} = e^{\int -2dx} = e^{-2x}$$

and it's G.S. is

$$y \cdot e^{-2x} = 3\int e^x e^{-2x} dx + C = -3e^{-x} + C$$

$$y = -3e^x + Ce^{2x} \quad \ldots (4)$$

given that $y(0) = 1$

$\therefore \quad y(0) = -3 + C \Rightarrow C = 4.$

$\therefore \quad y(x) = 4e^{2x} - 3e^{x}$

$\therefore \quad$ At $x = 0.3$

$$y(0.3) = 4e^{2(0.3)} - 3e^{0.3} = 3.2389$$

Comparing these two solutions we conclude that numerical solution is corrected upto 3 decimal places.

Exercise 9.1

Solve by Taylor's series method.

1) *Find* $y(0.3),\ y' = 1 + xy,\ y(0) = 2,\ h = 0.1$

2) *Find* $y(1.2),\ y' = 1 - 2xy,\ y(0) = 0,\ h = 0.4$

3) *Find* $y(0.5),\ y' = x^2 + y,\ y(0) = 1,\ h = 0.1$

4) *Find* $y(0.06),\ y' = x^2 + y,\ y(0) = 1,\ h = 0.02$

5) *Find* $y(6),\ y' = \dfrac{x^2+1}{y^2+1},\ y(0) = 2,\ h = 2.$

9.3 Euler's Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \dots (9.4)$$

with $\quad y(x_0) = y_0$

Suppose we want to solve equation (9.4) at $x = x_1 = x_0 + h,\ x = x_2 = x_0 + 2h,\ \dots\ x = x_0 + nh.$

Integrating equation (9.4) within $[x_0, x_1]$, we get

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$\therefore \quad y_1 - y_0 = \int_{x_0}^{x_1} f(x, y) dx$$

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\, dx$$

Assume that $f(x,y) \simeq f(x_0, y_0)\ \forall\ x \in [x_0, x_1]$

$\therefore$ We have $\quad y_1 \simeq y_0 + h\, f(x_0, y_0) \qquad \dots (9.5)$

Thus value y at $x = x_1$ is calculated by using equation (9.5).

Similarly for the interval $[x_1, x_2]$

We have $\quad y_2 \simeq y_1 + \int_{x_1}^{x_2} f(x, y)\, dx$

$$y_2 \simeq y_1 + h\, f(x_1, y_1)$$

Proceeding in this way, we obtain general formula as

$$y_{n+1} \simeq y_n + h\, f(x_n, y_n) \quad \because n = 0, 1, 2, \dots$$

To obtain the solution with desired accuracy, we have to take a smaller value of h. Hence the solution is obtained very slowly. Due to this Euler's method is rarely used. The more accurate results will be obtained by the modified method.

9.4 Modified Euler's Method

To start we use Euler's formula to find value of y at $x = x_1$

$$\therefore \quad y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

To find more correct approximation at $x = x_1$, we use Euler's Modified iteration formula as follows.

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right] \qquad \dots (9.6)$$

$$\because n = 0, 1, 2$$

In particular $\quad y_1^{(1)} = y_0 + \dfrac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

By using formula (1), we can obtain approximations $y_1^{(1)}, y_1^{(2)}, \dots, y_1^{(n)}$. This process is repeated till no significant change occurs. Suppose $y_1^{(n-1)} \simeq y_1^{(n)}$ we call this as y_1. To find the value of y at $x = x_2$, the above procedure is repeated in the interval $[x_1, x_2]$.

$$\therefore \quad y_2^{(n+1)} = y_n + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(n)})\right]$$

Example 9.4 : *Using Euler's method solve* $\dfrac{dy}{dx} = 1 + y^2$ *given* $y(0) = 0$. *Take* $h = 0.05$ *and obtain* $y(0.05),\ y(0.1),\ y(0.15)$.

Solution : We have

$$\frac{dy}{dx} = 1 + y^2 = f(x, y),$$

$$h = 0.05,\ x_0 = y_0 = 0$$

Iteration (1) Euler's formula for y at $x = 0.05$

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 0 + (0.05)(1 + 0^2) = 0.05$$

Iteration (2) Approximate value of y at $x_1 = 0.1$ is given by

$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$= 0.05 + 0.05\,(1.0025) = 0.1001$$

Iteration (3) Approximate value of y at $x_2 = 0.15$ is given by Euler's formula as,

$$y_3 = y_2 + h\, f(x_2, y_2) = 0.1506$$

Thus $y(0.05) = 0.05,\ y(0.1) = 0.1001$ and $y(0.15) = 0.1506$.

Example 9.5 : *Using Euler's method solve* $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$ *with* $y(0) = 1$. *Find* $y(4)$ *take* $h = 0.5$.

Solution : We have

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5 = f(x, y)$$

$h = 0.5$ and $x_0 = 0,\ y_0 = 1$

Euler's formula is

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

Iteration (1) Approximate value of y at x = 0.5, is given by Euler's formula as

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 1 + (0.5)\, f(0, 1) = 5.25.$$

Iteration (2) Approximate value of y at x = 1 is given by Euler's formula as $[x_1 = 0.5,\ y_1 = 5.25]$

$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$= 5.25 + (0.5)\, f(0.5, 5.25) = 5.875$$

Iteration (3) Approximate value of y at x = 1.5 is given by Euler's theorem as $[x_2 = 1,\ y_2 = 5.875]$.

$$y_3 = y_2 + h\, f(x_2, y_2)$$
$$= 5.875 + (0.5)\, f(1.5, 5.875)$$
$$= 5.125$$

Iteration (4) Approximate value of y at x = 2 is given by Euler's theorem as $[x_3 = 1.5,\ y_3 = 5.125]$

$$y_4 = y_3 + h\, f(x_3, y_3)$$
$$= 5.125 + (0.5)\, f(1.5, 5.125) = 4.5$$

Iteration (5) Approximate value of y at x = 2.5 is given by Euler's theorem as $[x_4 = 2,\ y_4 = 4.5]$

$$y_5 = y_4 + h\, f(x_4, y_4) = 4.75$$

Iteration (6) Approximate value of y at x = 3 is given by Euler's formula as $[x_5 = 2.5,\ y_5 = 4.75]$

$$y_6 = y_5 + h\, f(x_5, y_5) = 5.875$$

Iteration (7) Approximate value of y at x = 3.5 is given by Euler's formula as $[x_6 = 3,\ y_6 = 5.875]$

$$y_7 = y_6 + h\, f(x_6, y_6) = 7.125$$

Iteration (8) Approximate value of y at x = 4 is given by Euler's formula as $[x_7 = 3.5,\ y_7 = 7.125]$

$$y_8 = y_7 + h\, f(x_7, y_7) = 7$$

$$\therefore \quad \boxed{y(0.4) = 7}$$

Example 9.6 : *Given that* $\frac{dy}{dx} = \frac{y-x}{y+x}$ *with initial condition* $y(0) = 1$. *Find y for* $x = 0.1$. *Also find the better approximation at* $x = 0.1$ *by dividing the interval* $[0, 0.1]$ *into five steps.*

Solution : We have

$$\frac{dy}{dx} = \frac{y-x}{y+x} \text{ with } x_0 = 0,\ y_0 = 1$$
$$= f(x, y)$$

i) If we take h = 0.1 then the approximate value of y at x = 0.1 is given by Euler's theorem as

$$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0.1\, f(0, 1)$$
$$= 1 + 0.1\left(\frac{1-0}{1+0}\right) = 1.1$$

ii) For better approximation we divide the interval into five steps.

$$\therefore \quad h = \frac{0.1-0}{5} = 0.02$$

Iteration (1) The approximate value of y at x = 0.02 is given by Euler's formula as

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 1 + (0.02)\left(\frac{1-0}{1+0}\right) = 1.02$$

Iteration (2) The approximate value of y at x = 0.04 is given by Euler's formula as $(x_1 = 0.02,\ y_1 = 1.02)$

$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$= 1.02 + (0.02)\left\{\frac{1.02 - 0.02}{1.02 + 0.02}\right\} = 1.0392$$

Iteration (3) The approximate value of y at x = 0.06 is given by Euler's formula as

$y_3 = y_2 + h\, f(x_2, y_2)$. Here $x_2 = 0.04,\ y_2 = 1.0392$

$$= 1.0392 + (0.02)\, f(0.04, 1.0392) = 1.0577$$

Iteration (4) The approximate value of y at x = 0.08 is given by Euler's formula as

$y_4 = y_3 + h\,f(x_3, y_3)$ (Here $x_3 = 0.06$, $y_3 = 1.0577$)

$= 1.0577 + (0.02)\ f(0.06,\ 1.0577) = 1.0755$

Iteration (5) The approximate value of y at x = 1 is given by Euler's formula as

$$y_5 = y_4 + h\,f(x_2, y_4)$$

(Here $x_4 = 0.08$, $y_4 = 1.0755$)

$$y_5 = 1.0755 + h\ f(0.08,\ 1.0755) = 1.0927$$

Thus the value of y at x = 0.1 is **1.0927**.

Example 9.7 : *Using Euler's Method find y at x = 0.1 and 0.4 if* $\frac{d^2y}{dx^2} = \frac{3}{2}x\frac{dy}{dx} - \frac{9}{2}y + \frac{9}{2}$ *with* $y(0) = 1$ *and* $y'(0) = -2$.

Solution : We have

$$\frac{d^2y}{dx^2} = \frac{3}{2}x\frac{dy}{dx} - \frac{9}{2}y + \frac{9}{2} \qquad \text{... (1)}$$

Let $\quad z = \frac{dy}{dx} \quad \therefore \frac{dz}{dx} = \frac{d^2y}{dx^2}$

$\therefore$ Equation (1) becomes,

$$\frac{dz}{dx} = \frac{3}{2}xz - \frac{9}{2}y + \frac{9}{2} \qquad \text{... (2)}$$

$$= f(x, y, z)$$

and $\quad \frac{dy}{dx} = z = g(z)$

with $\quad x_0 = 0,\ y_0 = 1,\ z_0 = y'(0) - 2$

Here we take h = 0.1.

Iteration (1) The approximate values of y and z at x = 0.1 are given by Euler's theorem are as

$$y_1 = y_0 + h\,z_0 = 1 + (0.1)(-2) = 0.8$$

and $\quad z_1 = z_0 + h\,f(x_0, y_0, z_0)$

$$= -2 + h\,f(0, 1, -2) = -2$$

Iteration (2) The approximate value of y and z at x = 0.2 are given by Euler's formula as

$$y_2 = y_1 + h\,z_1 = 0.8 + (0.1)\ (-2) = 0.6$$

$$z_2 = z_1 + h\,f(x_1, y_1, z_1) = -2 + (0.1)\ [1.5\ (0.1)\ (-2) - 4.5\ (0.8) + 4.5] = -1.94$$

Iteration (3) The approximate values of y and z at x = 0.3 given by Euler's formula as

$$y_3 = y_2 + h\,z_2 = 0.6 + (0.1)\ (-1.94) = 0.406.$$

$$z_3 = z_2 + h\,f(x_2, y_2, z_2) = (-1.94) + (0.1)\ f\ (0.2,\ 0.6,\ -1.94) = -1.8182$$

Iteration (4) The approximate values of y and z at x = 0.4 are given by Euler's formula as

$$y_4 = y_3 + h\,z_3 = 0.406 + (0.1)\ (-1.8182) = 0.23636$$

$$z_4 = z_3 + h\,f(x_3, y_3, z_3) = -1.6327$$

Thus $\ y(0.4) = 0.023636$

Example 9.8 : *Using Euler's modified method, find value of y when x = 0.1 given that* $\frac{dy}{dx} = x^2 + y,\ y(0) = 1.$

Solution : We have

$$\frac{dy}{dx} = x^2 + y = f(x, y) \quad \therefore x_0 = 0,\ y_0 = 1$$

Take h = 0.05, $f(x_0, y_0) = 1$

By Euler's formula $y_1 = y_0 + h\,f(x_0, y_0) = 1.05$

$$f(x_1, y_1) = x_1^2 + y_1 = (0.05)^2 + 1.05 = 1.0525$$

Step a) Using Euler's modified method, the improved values of y are given by,

Iteration (1)

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(1)} = 1 + \frac{0.05}{2}\,[x_0^2 + y_0 + x_1^2 + y_1]$$

$$= 1 + \frac{0.05}{2}[1 + 1.0525] = 1.0513$$

Iteration (2)

Now $\quad y_1^{(2)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$$y_1^{(2)} = 1 + \frac{0.05}{2}\,[1 + 1.0538] = 1.0513$$

Hence we take $y_1 = 1.0513$ which is correct upto 4 decimal places.

Step b) To obtain y_2 i.e. y at x = 0.1, first we use Euler's formula, here $x_1 = 0.05$. $y_1 = 1.0513$

$$y_2 = y_1 + h\,f(x_1, y_1) = y_1 + h\,(x_1^2 + y_1)$$

$$y_2 = 1.0513 + (0.05)\ ((0.05)^2 + 1.0513)$$

$$= 1.1040$$

$$f(x_2, y_2) = x_2^2 + y_2 = 1.114$$

Iteration (1) By modified Euler's formula, we have

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

$$= 1.1055$$

Iteration (2)

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1.1055$$

$\therefore$ The value of $y_2 = 1.1055$ correct upto 4 decimal.

Thus the value of y at x = 0.1 is 1.1055.

Example 9.9 : *Using modified Euler's method find $y(1.3)$ if $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ which $y(1) = 1$.*

Solution : We have

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x} = \frac{1-xy}{x^2}$$

$$\therefore \quad f(x, y) = \frac{1-xy}{x^2} \text{ and } x_0 = 1, y_0 = 1$$

We take $h = 0.1$, $f(x_0, y_0) = \frac{1-1}{1} = 0$

a) To find y at x = 1.1, we first Euler's formula.

$$y_1 = y_0 + h f(x_0, y_0) = 1 + (0.1)\, 0 = 1$$

Iteration (1) By using modified Euler's formula, improved values of y_1 are given by,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.1}{2}[0 + f(1.1, 1)] = 0.9959$$

Iteration (2)

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.1}{2}[0 + f(1.1, 0.9959)] = 0.9961$$

Iteration (3)

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.1}{2}[0 + f(1.1, 0.9961)] = 0.9960$$

Iteration (4) $$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= 1 + \frac{0.1}{2}[0 + f(1.1, 0.9960)]$$

$$= 0.9960$$

Thus the value of y_1 is 0.9960 correct upto 4 decimal places.

b)We have,

$$x_1 = 1.1, \; y_1 = 0.9960$$

$$f(x_1, y_1) = \frac{1-(1.1)(0.9960)}{(1.1)^2} = -0.0790$$

To find y_2, we first use Euler's formula

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 0.9960 + (0.1)\, f(1.1, 0.9960)$$

$$= 0.9881$$

Iteration (1) By using modified Euler's formula, improved values of y_2 are given by,

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 0.9960 + \frac{0.1}{2}[-0.0790 - 0.1290]$$

$$= 0.9856$$

Iteration (2) $$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 0.9960 + \frac{0.1}{2}[-0.0790 - 0.1269]$$

$$= 0.9857$$

Iteration (3) $$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 0.9960 + \frac{0.1}{2}[-0.0790 - 0.1020]$$

$$= 0.9857$$

Thus the value of y_2 is 0.9857 correct upto 4 decimal places.

c) We have $x_2 = 1.2$, $y_2 = 0.9857$

$$\therefore \quad f(x_2, y_2) = \frac{1-(1.2)(0.9857)}{(1.2)^2} = 0.1269$$

To find y_3, we first use Euler's formula

$$y_3 = y_2 + h f(x_2, y_2)$$

$$= 0.9857 + (0.1)(-0.1269)$$

$$= 0.9730$$

Iteration (1) By using modified Euler's formula improved values of y_3 are given by,

$$y_3^{(1)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3)]$$
$$= 0.9857 + \frac{0.1}{2}[-0.1269 - 0.1567]$$
$$= 0.9715$$

Iteration (2) $\quad y_3^{(2)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(1)})]$

$$y_3^{(2)} = 0.9857 + \frac{0.1}{2}[-0.1269 - 0.1556]$$
$$= 0.9716$$

Iteration (3) $\quad y_3^{(3)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(2)})]$

$$= 0.9716$$

Thus the value of y_3 is 0.9716 correct upto 4 decimal places.

Example 9.10 : *Using Euler's modified method find the value of y if* $\frac{dy}{dx} = \log(x+y)$ *with* $y(1) = 2$ *for* $x = 1.2$ *and* $x = 1.4$ *correct to three decimal places. Take* $h = 0.2$.

Solution : We have

$$\frac{dy}{dx} = \log(x+y)$$

$\therefore \quad f(x, y) = \log(x+y)$, $x_0 = 1$, $y_0 = 2$, $h = 0.2$

$$f(x_0, y_0) = 1.0986$$

a) To find y at $x_1 = 1.2$, we first use Euler's formula

$$y_1 = y_0 + h\, f(x_0, y_0) = 2.2197$$

Iteration (1) Applying Euler's modified formula, we get,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$
$$= 2 + \frac{0.2}{2}[1.0986 + 1.2296]$$
$$= 2.2328$$

Iteration (2) $\quad y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$$= 2 + \frac{0.2}{2}[1.0986 + 1.2334]$$
$$= 2.2332$$

Iteration (3) $\quad y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$$= 2 + \frac{0.2}{2}[1.0986 + 1.2335]$$
$$= 2.2332$$

Thus the value of y at $x_1 = 1.2$ is 2.2332 correct upto 3 decimal places.

b) To find y at $x_2 = 1.4$, we have

$$x_1 = 1.2, \quad y_1 = 2.2332$$
$$f(x_1, y_1) = 1.2335$$

We first use Euler's formula,

$$\therefore \quad y_2 = y_1 + h\, f(x_1, y_1)$$
$$= (2.2332) + (0.2)(1.2335)$$
$$y_2 = 2.4799$$

Iteration (1) By using Euler's modified formula, we get

$$\therefore \quad y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$
$$= 2.2332 + \frac{0.2}{2}[1.2335 + 1.3558]$$
$$= 2.4921$$

Iteration (2) $\quad y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$$y_2^{(2)} = 2.492$$

Thus the value of y_2 (i.e. y at $x = x_2$) is 2.492 correct upto 3 decimal places.

Example 9.11 : *Using modified Euler's method solve equation* $\frac{dy}{dx} = x - y^2$; $y(0) = 1$ *to calculate y at* $x = 0.2$ *take* $h = 0.2$.

Solution : We have,

$$\frac{dy}{dx} = x - y^2 = f(x, y), \; x_0 = 0 \text{ and } y_0 = 1$$

$h = 0.2$, $x_1 = 0.2$, $f(x_0, y_0) = 0 - 1 = -1$

Euler's formula is

$$y_1 = y_0 + hf(x_0, y_0)$$
$$y_1 = 1 + 0.2(-1) = 0.8$$

Iteration (1)

$$y_1 = 1 + 0.2(-1) = 0.8$$

Euler's modified formula is

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1+\frac{0.2}{2}[-1+x_1-y_1^2]$$

$$y_1^{(1)} = 1+\frac{0.2}{2}[-1+0.2-(0.8)^2]$$

$$= 0.8560$$

Iteration (2)

$$y_1^{(2)} = y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(1)})]$$

$$= 1+\frac{0.2}{2}[-1-0.5327] = 0.8467$$

Iteration (3)

$$y_1^{(3)} = y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(2)})]$$

$$= 1+\frac{0.2}{2}[-1-0.5169] = 0.8483$$

Iteration (4)

$$y_1^{(4)} = y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(3)})]$$

$$= 1+\frac{0.2}{2}[-1-0.5196] = 0.8480$$

Iteration (5)

$$y_1^{(5)} = y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(4)})]$$

$$= 1+\frac{0.2}{2}[-1-0.5191] = 0.8481$$

Iteration (6)

$$y_1^{(6)} = y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(5)})]$$

$$= 1+\frac{0.2}{2}[-1-0.5193] = 0.8481$$

Here, $y_1^{(5)} = y_1^{(6)}$

$\therefore$ $[y]_{x=0.2} = y(0.2) = \mathbf{0.8481}$

Exercise 9.2

1) *Solve by Euler's method*

a) Find y(2), $y' = 2+\sqrt{x}\,y$, $y(1) = 1$, $h = 0.2$.

b) Find y(0.2), $y' = 1-y$, $y(0) = 0$, $h = 0.1$.

c) Find y(0.06), $y' = x^2+y$, $y(0) = 1$, $h = 0.02$.

d) Find y(0.3), $y' = x^2+y^2$, $y(0) = 1$, $h = 0.05$.

2) *Solve by Euler's modified method*

a) Find y(0.3), $y' = x+y$, $y(0) = 1$, $h = 0.1$.

b) Find y(1.6), $y' = \log(x+y)$, $y(0) = 2$, $h = 0.2$.

c) Find y(0.1), $y' = x+y+xy$, $y(0) = 1$, $h = 0.025$.

d) Find y(1.5), $y' = y^2-\frac{y}{x}$; $y(1) = 1$, $h = 0.1$.

9.5 Runge Kutta Methods

The Runge Kutta Methods do not required the calculations of higher order derivatives like Taylor's series and it is designed to give greater accuracy with the advantage of requiring only the function values at some selected points on the subinterval. Thus this method is widely used as compare with Taylor's and Euler's methods.

A) Consider the Euler's modified formula in the following form.

$$y_1 = y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1)]$$

We put $x_1 = x_0+h$, $y_1 = y_0+h\,f(x_0,y_0)$ in the right side of above equation.

$$\therefore \quad y_1 = y_0+\frac{h}{2}[f(x_0,y_0)+f(x_0+h,\, y_0+h\,f(x_0,y_0))]$$

$$y_1 = y_0+\frac{h}{2}[f_0+f(x_0+h,\, y_0+hf_0)] \quad \text{... (9.7)}$$

where $f_0 = f(x_0,y_0)$

Now, put $k_1 = h\,f_0$ and $k_2 = h\,f(x_0+h,\, y_0+k_1)$

$\therefore$ The equation (9.7) becomes,

$$y_1 = y_0 = \frac{1}{2}(k_1+k_2) = y_0+k$$

where $k = \frac{1}{2}(k_1+k_2)$ and $k_1 = hf_0$

$$k_2 = h\,f(x_0+h,\, y_0+k_1)$$

This is known as the second order Runge-Kutta formula.

B) The most commonly used fourth order Runge Kutta formula is given as,

$$y_1 = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= y_0+k$$

where $k = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$$k_1 = h\,f(x_0,y_0)$$

$$k_2 = h\,f\left(x_0+\frac{1}{2}h,\, y_0+\frac{1}{2}k_1\right)$$

$$k_3 = h f\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_2\right)$$

$$k_4 = h f(x_0+h, y_0+k_3)$$

C) Runge-Kutta method for solution of simultaneous first order differential equations :

Consider the simultaneous first order D.E. as

$$\frac{dy}{dx} = f(x, y, z) \text{ and } \frac{dz}{dx} = g(x, y, z)$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. Here x is independent variable and y, z are dependent variables.

The starting point is (x_0, y_0, z_0) and increments (step sizes) for x, y, z be h, k, l respectively.

i) Runge Kutta formula of second order :

$$k_1 = h f(x_0, y_0, z_0) \quad l_1 = h g(x_0, y_0, z_0)$$

$$k_2 = h f\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_1, z_0+\frac{1}{2}l_1\right)$$

$$l_2 = h g\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_1, z_0+\frac{1}{2}l_1\right)$$

Then $y_1 = y_0+\frac{1}{2}(k_1+k_2)$

and $z_1 = z_0+\frac{1}{2}(l_1+l_2), \; x_1 = x_0+h$

ii) Runge Kutta formula of fourth order

$$k_1 = h f(x_0, y_0, z_0), \; l_1 = hg(x_0, y_0, z_0)$$

$$k_2 = h f\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_1, z_0+\frac{1}{2}l_1\right)$$

$$l_2 = h g\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_1, z_0+\frac{1}{2}l_1\right)$$

$$k_3 = h f\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_2, z_0+\frac{1}{2}l_2\right)$$

$$l_3 = h g\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_2, z_0+\frac{1}{2}l_2\right)$$

$$k_4 = h f(x_0+h, y_0+k_3, z_0+l_3)$$

$$l_4 = h g(x_0+h, y_0+k_3, z_0+l_3)$$

then $y_1 = y_0+\frac{h}{6}[k_1+2k_2+2k_3+k_4]$

and $z_1 = z_0+\frac{h}{6}[l_1+2l_2+2l_3+l_4]$

Example 9.12 : *Use Runge-Kutta formulae of 2^{nd} and 4^{th} orders to find y when x = 0.1 and x = 0.2 given that $\frac{dy}{dx} = x+y$ with y(0) = 1.*

Solution : i) By Range Kutta formula of 2^{nd} order : Runge Kutta 2^{nd} order formula is

$$y_1 = y_0+\frac{1}{2}(k_1+k_2)$$

where $k_1 = h f(x_0, y_0)$

and $k_2 = h f(x_0+h, y_0+k_1)$

To determine y(0.1) we take h = 0.1

Here $x_0 = 0$, $y_0 = 1$

$\therefore$ $k_1 = h f_0 = h f(x_0, y_0)$

$= 0.1\ (0+1) = 0.1$

$k_2 = 0.1\ [f(0 + 0.1, 1 + 0.1)]$

$= 0.1\ (0.1 + 1.1) = 0.12$

$\therefore$ $y_1 = y_0+\frac{1}{2}(k_1+k_2)$

$= 1+\frac{1}{2}(0.1+0.12) = 1.11$

To determine $y_2 = y\ (0.2)$, we take

$x_0 = 0.1$, $y_0 = 1.11$ and $h = 0.1$

$\therefore$ $k_1 = (0.1)\ f(0.1, 1.11)$

$= 0.1\ (0.1 + 1.11) = 0.121$

$k_2 = (0.1)\ f(0.1 + 0.1, 1.11 + 0.121)$

$= 0.1\ (0.2 + 1.231) = 0.1431$

$\therefore$ $y_2 = y(0.2) = 1.11+\frac{1}{2}(0.121+0.1431)$

$\therefore$ $y_2 = \mathbf{1.24205}$

ii) By Runge Kutta formula of 4^{th} order : This formula is

$$y_1 = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

where $k_1 = h f(x_0, y_0)$

$$k_2 = h f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$k_4 = h f(x_0+h, y_0+k_3)$$

To find $y(0.1) = y_1$ we take h = 0.1

$\therefore$ $k_1 = h f(x_0, y_0) = (0.1)\ f(0, 1) = 0.1$

$$k_2 = (0.1)\ f(0.05,\ 1.05) = 0.11$$

$$k_3 = (0.1)\ f(0.05,\ 1.055) = 0.1105$$

$$k_4 = (0.1)\ f(0.1,\ 1.1105) = 0.12105$$

$$\therefore \quad y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}[0.1 + 0.22 + 0.2210 + 0.12105]$$

$$= 1.11034$$

To find $y(0.2) = y_2$ we take

$x_0 = 0.1$, $y_0 = 1.11034$, and $h = 0.1$

$$\therefore \quad k_1 = (0.1)\ f(0.1,\ 1.11034) = 0.12103$$

$$k_2 = (0.1)\ (0.15 + 1.17085) = 0.13208$$

$$k_3 = (0.1)\ (0.15 + 1.17638) = 0.13263$$

$$k_4 = (0.1)\ (0.2 + 1.24297) = 0.14429$$

$$\therefore \quad y_2 = y(0.2) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.11034 + \frac{1}{6}[0.12103 + 2\ (0.13208) + 2\ (0.13263) + 0.14429]$$

$$y_2 = 1.24279$$

Thus $\quad y(0.2) = 1.24279$

Example 9.13 : *Using fourth order Runge Kutta method solve the equation $\frac{dy}{dx} = \sqrt{x+y}$ with $y(0) = 1$ and find $y\ (0.2)$ taking $h = 0.2$.*

Solution : The 4^{th} order Runge Kutta formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = h\ f(x_0, y_0)$

$$k_2 = h\ f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\ f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h\ f(x_0 + h, y_0 + k_3)$$

Here

$$f(x,\ y) = \sqrt{x+y},\ x_0 = 0,\ y_0 = 1,\ h = 0.2$$

$$\therefore \quad k_1 = 0.2(\sqrt{0+1}) = 0.2$$

$$k_2 = (0.2)\sqrt{0.1 + 1.1} = 0.2191$$

$$k_3 = (0.2)\sqrt{0.1 + 1.10955} = 0.2120$$

$$k_4 = (0.2)\sqrt{0.2 + 1.2120} = 0.2377$$

$$\therefore \quad y_1 = 1 + \frac{1}{6}[0.2 + 2\ (0.2191) + 2\ (0.2120) + 0.2377]$$

$$y(0.2) = \mathbf{1.2167}$$

Example 9.14 : *Use Runge Kutta method of 4^{th} order to solve $\frac{dy}{dx} = \frac{1}{x+y}$, $x_0 = 0$, $y_0 = 1$ to find y at $x = 0.4$ taking $h = 0.2$.*

Solution : Runge Kutta formula of 4^{th} order is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf\ (x_0 + h,\ y_0 + k_3)$$

To find $y(0.2) = y_1$ we take $h = 0.2$,

$$f(x,\ y) = \frac{1}{x+y}$$

$$\therefore\ k_1 = hf(x_0,\ y_0) = (0.2)\frac{1}{0+1} = 0.2$$

$$k_2 = (0.2)\left(\frac{1}{0.1+1.1}\right) = 0.167$$

$$k_3 = 0.2f(0.1,\ 1.0835) = 0.169$$

$$k_4 = 0.2f(0.2,\ 1.169) = 0.1461$$

$$\therefore\ y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.2 + 2 \times 0.167 + 2 \times 0.169 + 0.1461)$$

$$\boxed{y_1 = 1.1697}$$

For calculating y at $x = 0.4$ take $x_0 = 0.2$,

$$y_0 = 1.1697$$

$$h = 0.2,\ k_1 = 0.2f(0.2,\ 1.1697) = 0.146$$

$$k_2 = 0.2\ f(0.3,\ 1.2497) = 0.1296$$

$$k_3 = 0.2\ f(0.3,\ 1.2345) = 0.1303$$

$$k_4 = 0.2\ f(0.4,\ 1.3) = 0.1176$$

$$\therefore \quad y_2 = y(0.4) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.1697+\frac{1}{6}[0.146+2\times 0.1296+2\times 0.1303$$

$$+ 0.1176]$$

$$\boxed{y(0.4) = 1.30027}$$

Example 9.15 : *Use the 4^{th} order Runge Kutta formula to find the value of y when x = 1. Given that $\frac{dy}{dx}=\frac{y-x}{y+x}$ and y(0) = 1, take h = 1.*

Solution : We have

$$f(x, y) = \frac{y-x}{y+x},\ x_0 = 0,\ y_0 = 1,\ h = 1$$

$$k_1 = h\, f(x_0, y_0) = 1\, f(0, 1) = 1$$

$$k_2 = h\, f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$= f(0.5, 1.5) = 0.5$$

$$k_3 = h\, f\left(k_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$= f(0.5, 1.25) = 0.42857$$

$$k_4 = h\, f(x_0+h, y_0+k_3)$$

$$= f(1, 1.42857) = 0.17647$$

$$\therefore \quad y_1 = y(1) = y_0+\frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$= 1+\frac{1}{6}[1+2\,(0.5)+2(0.42857)+0.17647]$$

$$= 1.5056$$

Thus $y(1) =$ **1.5056**

Example 9.16 : *Use R.K. method of 4^{th} order to obtain the numerical solution of $y' = x^2+y^2$ with y(1) = 1.5 in the interval (1, 1.2) with h = 0.1.*

Solution : We have

$$y' = x^2+y^2 \quad \therefore f(x, y) = x^2+y^2$$

and $\quad x_0 = 1,\ y_0 = 1.5 \quad$ and $\quad h = 0.1$

Iteration (1) To find y(1.1)

$$f(x_0, y_0) = f(1, 1.5) = 1 + 2.25 = 3.25$$

$$k_1 = h\, f(x_0, y_0) = (0.1)\,(3.25) = 0.325$$

$$k_2 = h\, f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$= (0.1)\,[(1.05)^2+(1.6625)^2]$$

$$k_2 = 0.38664$$

$$k_3 = h\, f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$= (0.1)\,[(1.05)^2+(1.6933)^2]$$

$$k_3 = 0.39698$$

$$k_4 = h\, f\,(x_0+h, y_0+k_3)$$

$$= (0.1)\,[(1.1)^2+(1.89698)^2]$$

$$= 0.48086$$

$$\therefore \quad y_1 = y(1.1) = y_0+\frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$= 1.5+\frac{1}{6}[0.325+2\,(0.38664)$$

$$+ 2\,(0.39698)+0.48086]$$

$$y_1 = 1.8955$$

Iteration (2) To find $y_1 = y(1.2)$ we take

$$x_0 = 1.1,\ y_0 = 1.8955,\ h = 0.1$$

$$\therefore \quad k_1 = h\, f(x_0, y_0)$$

$$= (0.1)\,[(1.1)^2+(1.8955)^2]$$

$$= 0.48029$$

$$k_2 = h\, f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$= (0.1)\,[(1.15)^2+(2.13565)^2]$$

$$k_2 = 0.58835$$

$$k_3 = h\, f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$= (0.1)\,[(1.15)^2+(2.18968)^2]$$

$$k_3 = 0.61172$$

$$k_4 = h\, f(x_0+h, y_0+k_3)$$

$$= (0.1)\,[(1.2)^2+(2.51722)^2]$$

$$k_4 = 0.77262$$

$$\therefore \quad y_1 = y(1.2) = y_0+\frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$y_1 = 1.8955+\frac{1}{6}[0.48029+2\,(0.58835)$$

$$+ 2(0.61172)+0.77262]$$

$$y_1 = 2.50432$$

Thus $y(1.2) =$ **2.50432**

Example 9.17 : *Find y(0.2) by R.K. 4^{th} order formula for $y' = xy+y^2$ with y(0.1) = 1.1169.*

Solution : To find y(0.2) :

We have $f(x, y) = xy + y^2$

$x_0 = 0.1,\ y_0 = 1.1169$ take $h = 0.1$.

$$\therefore \quad k_1 = h\, f(x_0, y_0)$$

$$= (0.1)\,[(0.1)\,(1.1169) + (1.1169)^2]$$

$$k_1 = 0.1359$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1582$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1610$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = 0.1889$$

$$\therefore \quad y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.1169 + \frac{1}{6}[0.1359 + 2\,(0.1582) + 2\,(0.1610) + 0.1889]$$

$$y_1 = 1.2774$$

Thus $\quad y(0.2) = \mathbf{1.2774}$

Example 9.18 : *Using fourth order R.K. method find y when x = 1.1 given that* $\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$ *with* $y(1) = 1$ *take* $h = 0.1$.

Solution : We have

$$f(x, y) = \frac{1}{x^2} - \frac{y}{x} = \frac{1 - xy}{x^2}$$

and $\quad x_0 = 1,\ y_0 = 1,\ h = 0.1$

$$\therefore \quad f(x_0, y_0) = \frac{1-1}{1} = 0$$

$$k_1 = h\, f(x_0, y_0) = 0$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.1)\, f(1.05, 1) = -\,0.00454$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.1)\, f(1.05, 0.9977)$$

$$k_3 = -\,0.00432$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3)$$

$$= (0.1)\, f(1.1, 0.9956)$$

$$k_4 = -\,0.00788$$

$$\therefore \quad y_1 = y(1.1)$$

$$= y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$y_1 = 1 + \frac{1}{6}[0 - (0.00454)\,2 + 2\,(-\,0.00432) - 0.00788]$$

$$\therefore \quad y_1 = 0.9957$$

$$\therefore \quad y(1.1) = \mathbf{0.9957}$$

Example 9.19 : *Given* $y' = \sin x + \cos y$ *with* $y(0) = 2.5$. *Find y(0.5) by* 4^{th} *order R.K. method* $(h = 0.5)$.

Solution : We have $f(x, y) = \sin x + \cos y$ and $x_0 = 0,\ y_0 = 2.5,\ h = 0.5$.

$$f(x_0, y_0) = \sin 0 + \cos 2.5 = -\,0.80114$$

$$k_1 = h\, f(x_0, y_0)$$

$$= (0.5)\,(-\,0.80114) = -\,0.40057$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.5)\, f(0.25, 2.29972)$$

$$= -\,0.2093$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.5)\, f(0.25, 2.3954)$$

$$= -\,0.24343$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3)$$

$$= (0.5)\, f(0.5, 2.25657)$$

$$= -\,0.07692$$

$$\therefore \quad y_1 = y(0.5)$$

$$= y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$y_1 = 2.5 + \frac{1}{6}[-\,0.40057 + 2\,(-\,0.2093) + 2\,(-\,0.24343) + (-\,0.07692)]$$

$$\therefore \quad y_1 = 2.2695$$

Thus $\quad y(0.5) = \mathbf{2.2695}$

Example 9.20 : *Find the y(0.1), z(0.1) from the following D.E.* $\frac{dy}{dx} = x + z$ *and* $\frac{dz}{dx} = x - y^2$ *with* $y(0) = 2,\ z(0) = 1$. *Use R.K. method of* 4^{th} *order. Take* $h = 0.1$.

Solution : We have

$$\frac{dy}{dx} = x + z \quad \text{and} \quad \frac{dz}{dx} = x - y^2$$

$\therefore$ $f(x, y, z) = x + z$ and $g(x, y, z) = x - y^2$

$x_0 = 0,\ y_0 = 2,\ z_0 = 1$ and $h = 0.1$

Now
$$k_1 = h\,f(x_0, y_0, z_0) = (0.1)\,f(0, 2, 1) = 0.1$$
$$l_1 = h\,g(x_0, y_0, z_0) = (0.1)\,g(0, 2, 1) = -0.4$$
$$k_2 = h\,f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right)$$
$$k_2 = (0.1)\,f(0.05, 2.05, 0.8) = 0.085$$
$$l_2 = h\,g\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right)$$
$$= (0.1)\,g(0.05, 2.05, 0.8)$$
$$l_2 = -0.41525$$
$$k_3 = h\,f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$$
$$= (0.1)\,f(0.05, 2.0425, 0.7923)$$
$$k_3 = +0.084238$$
$$l_3 = h\,g\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$$
$$= (0.1)\,g(0.05, 2.0425, 0.7923)$$
$$= -0.4122$$
$$k_4 = h\,f(x_0+h, y_0+k_3, z_0+l_3)$$
$$= (0.1)\,f(0.1, 2.0842, 0.5878)$$
$$k_4 = 0.06878$$
$$l_4 = h\,g(x_0+h, y_0+k_3, z_0+l_3)$$
$$= (0.1)\,g(0.1, 2.0842, 0.5878)$$
$$l_4 = -0.4244.$$

$\therefore$
$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$y_1 = 2 + \frac{1}{6}[0.1 + 2\,(0.085) + 2(0.08423) + 0.06878]$$
$$y_1 = 2.0845$$

And
$$z_1 = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$
$$= 1 + \frac{1}{6}[-0.4 + 2\,(-0.41525) + 2\,(-0.4122) - 0.4244] = 0.5868$$

$\therefore$ $y(0.1) = 2.0845$ and $z(0.1) = 0.5868$

Example 9.21 : *Using R.K. method of 4^{th} order find $x(0.1)$ and $y(0.1)$ for the following system $\frac{dx}{dt} = 2x + y,\ \frac{dy}{dt} = x - 3y$ with $x(0) = 1,\ y(0) = 0.5$.*

Solution : We have
$$\frac{dx}{dt} = 2x + y = f(t, x, y)$$
$$\frac{dy}{dt} = x - 3y = g(t, x, y)$$

with $t_0 = 0,\ x_0 = 1,\ y_0 = 0.5$, Take $h = 0.1$

$\therefore$
$$k_1 = h\,f(t_0, x_0, y_0) = 0.1\,(2x_0 + y_0) = 0.25$$
$$l_1 = h\,g(t_0, x_0, y_0) = 0.1\,(x_0 - 3y_0) = -0.05$$
$$k_2 = h\,f\left(t_0+\frac{h}{2}, x_0+\frac{k_1}{2}, y_0+\frac{l_1}{2}\right)$$
$$= (0.1)\left[z\left(x_0+\frac{k_1}{2}\right)+\left(y_0+\frac{l_1}{2}\right)\right] = 0.2725$$
$$l_2 = hg\left[t_0 + h/2, x_0+\frac{k_1}{2}, y_0+\frac{l_1}{2}\right] = -0.03$$
$$k_3 = h\,f\left[t_0+\frac{h}{2}, x_0+\frac{k_2}{2}, y_0+\frac{l_2}{2}\right] = 0.2758$$
$$l_3 = hg\left[t_0+\frac{h}{2}, x_0+\frac{k_2}{2}, y_0+\frac{l_2}{2}\right] = -0.0319$$
$$k_4 = h\,f[t_0+h, x_0+k_3, y_0+l_3] = 0.302$$
$$l_4 = h\,g[t_0+h, x_0+k_3, y_0+l_3] = -0.0129$$

$\therefore$
$$x_1 = x_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= 1 + \frac{1}{6}[0.25 + 2\,(0.2725) + 2(0.2758) + 0.302]$$
$$= 0.2748 + 1 = 1.2748$$
$$y_1 = y_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$
$$= 0.5 + \frac{1}{6}[-0.05 + 2(-0.03) + 2\,(-0.0319) - 0.0129]$$
$$= -0.0311 + 0.5 = 0.4689$$

Thus $x(0.1) = 1.2748,\ y(0.1) = 0.4689$

Example 9.22 : *Given $y'' + xy' + y = 0,\ y(0) = 1,\ y'(0) = 0$. Find the value of $y(0.1)$ by using R.K. method of 4^{th} order.*

Solution : We have

$$\frac{d^2y}{dx^2} = -x\frac{dy}{dx} - y \quad \text{let } \frac{dy}{dx} = z$$

$$\therefore \quad \frac{dz}{dx} = -xz - y = f(x, y, z)$$

and $$\frac{dy}{dx} = z = g(x, y, z)$$

with $y = 1, z = 0$ at $x = 0$ take $h = 0.1$.

$$k_1 = h\,g(x_0, y_0, z_0)$$
$$= (0.1)\ g\ (0, 1, 0) = 0$$
$$l_1 = h\,f(x_0, y_0, z_0)$$
$$= (0.1)\ f(0, 1, 0) = -0.1$$
$$k_2 = h\,g\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$$
$$= -0.005$$
$$l_2 = h\,f\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l}{2}\right]$$
$$= -0.09975$$
$$k_3 = h\,g\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$$
$$= -0.00499$$
$$l_3 = h\,f\left[x_0 + h/2, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$$
$$= -0.09950$$
$$k_4 = h\,g[x_0 + h, y_0 + k_3, z_0 + l_3]$$
$$= -0.00995$$
$$l_4 = h\,f[x_0 + h, y_0 + k_3, z_0 + l_3]$$
$$= -0.0985$$

$$\therefore \quad y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= 0.9950$$

and $$z_1 = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$
$$= 0.9875$$

Thus $y(0.1) = 0.9950$ and $z(0.1) = 0.9875$

Exercise 9.3

1) *Given* $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$ *find* $y(0.2)$ *by using Runge Kutta method of* 4^{th} *order take* $h = 0.1$.

2) *Find* $y(0.2)$ *with* $h = 0.1$, $y' = y - \frac{2x}{y}$ *by R.K. method of* 2^{nd} *and* 4^{th} *order* $y(0) = 1$.

3) *Use the R.K. method to solve*

$10y' = x^2 + y^2$, $y(0) = 1$ *for the interval* $0 < x \leq 0.4$, $h = 0.1$.

4) *Using R.K. method of* 4^{th} *order find* $y(0.2)$ *and* $y(0.4)$ *if* $y' = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$.

❑❑❑

Notes

SOLVED MODEL QUESTION PAPER - 1 [In Sem]

Engineering Mathematics - III

S.E. (Comp / IT) Sem - IV (As Per 2019 Pattern)

Time : 1 Hour] **[Maximum Marks : 30**

N. B. :

i) Attempt Q.1 or Q.2, Q.3 or Q.4.

ii) Neat diagrams must be drawn wherever necessary.

iii) Figures to the right side indicate full marks.

iv) Assume suitable data, if necessary.

Q.1 a) *Solve any two* **[8]**

i) $\operatorname{cosec} x \dfrac{d^4y}{dx^4} + (\operatorname{cosec} x)y = \sin 2x$. **(Refer example 1.17)**

ii) $(D^2 + a^2)y = \sec ax$ *(by variation of parameters).* **(Refer example 1.111)**

iii) $x^3 \dfrac{d^2y}{dx^2} + 3x^2 \dfrac{dy}{dx} + xy = \sin(\log x)$. **(Refer example 1.61)**

b) $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + 3x = e^{-t}, \dfrac{d^2y}{dt^2} - 4\dfrac{dx}{dt} + 3y = \sin 2t$ **(Refer example 2.3)** **[7]**

OR

Q.2 a) *Solve any two :* **[8]**

i) $(D^4+1)y = e^x \cos 2x$ **(Refer example 1.36)**

ii) $(D^2+3D+2)y = e^{e^x}$ *[by variation of parameters]* **(Refer example 1.114)**

iii) $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$ **(Refer example 1.53)**

b) $\dfrac{dx}{1} = \dfrac{dy}{3} = \dfrac{dz}{5z + \tan(y-3x)}$ **(Refer example 2.9)** **[7]**

Q.3 a) *Find Fourier sine and cosine transform of* $f(x) = e^{-x} + e^{-2x}$ **(Refer example 3.3)** **[5]**

b) *Solve the integral equation* $\int_0^\infty f(x) \sin \lambda x \, dx = e^{-\lambda}, \lambda > 0$ **(Refer example 3.27)** **[3]**

c) *Solve any one* **[7]**

i) *Find the Z-transform of the following by showing the region of convergence*

$$f(k) = \begin{cases} -\left(-\dfrac{1}{3}\right)^k, & k < 0 \\ \left(-\dfrac{1}{4}\right)^k, & k \geq 0 \end{cases}$$ **(Refer example 4.5)**

ii) Find $Z^{-1}\left[\dfrac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{5}\right)}\right]$ *if*

i) $|z| > \dfrac{1}{4}$ ii) $\dfrac{1}{5} < |z| < \dfrac{1}{4}$ iii) $|z| < \dfrac{1}{5}$ **(Refer example 4.53)**

OR

Q.4 a) *Using Fourier integral representation show that*

$$\int_0^{\infty} \frac{\lambda^3 \sin \lambda x}{\lambda^4 + 4}\, d\lambda = \frac{\pi}{2} e^{-x} \cos x$$ **(Refer example 3.22)** **[5]**

b) *Solve the integral equation* $\int_0^{\infty} f(x) \cos \lambda x \, dx = e^{-\lambda},\ \lambda > 0$ **(Refer example 3.26)** **[3]**

c) *Solve any one* **[7]**

i) Find $Z\{f(k)\}$ *if* $f(k) = k e^{-2k} \cos 3k,\ k \geq 0$. **(Refer example 4.30)**

ii) Solve : Find : $Z^{-1}\left\{\frac{1}{(z-4)(z-5)}\right\}$ *by inversion integral method.* **(Refer example 4.67)**

SOLVED MODEL QUESTION PAPER - 2 [In Sem]

Engineering Mathematics - III

S.E. (Comp / IT) Sem - IV (As Per 2019 Pattern)

Time : 1 Hour] **[Maximum Marks : 30**

Q.1 a) *Solve any two* **[8]**

i) $(D^2 + D)\, y = \frac{1}{1+e^x}$. **(Refer example 1.46)**

ii) $(D^2 - 1)y = \frac{1}{(1+e^{-x})^2}$ *(by variation of parameters).* **(Refer example 1.113)**

iii) $\left(\frac{d^2}{dx^2} - \frac{2}{x^2}\right)^2 y = x^2$. **(Refer example 1.57)**

b) *Solve* $\frac{dx}{dt} - \omega y = a \cos pt,\ \frac{dy}{dt} + \omega x = a \sin pt\ (\omega \neq p)$ **(Refer example 2.4)** **[7]**

OR

Q.2 a) *Solve any two :* **[8]**

i) $(D^4 + 10D^2 + 9)y = 96 \sin 2x \cos x$. **(Refer example 1.20)**

ii) $(D^2 + 4)y = \tan 2x$ *[by variation of parameters].* **(Refer example 1.110)**

iii) $(x^2 D^2 + 3xD + 1)\, y = (1-x)^{-2}$. **(Refer example 1.60)**

b) $\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - zx^4)} = \frac{dz}{z(x^4 - y^4)}$. **(Refer example 2.15)** **[7]**

Q.3 a) *Using Fourier integral representation show that*

$$\int_0^{\infty} \frac{1 - \cos \lambda\pi}{\lambda} \sin \lambda x \, d\lambda = \begin{cases} \pi/2 & 0 < x < \pi \\ 0 & x > \pi \end{cases}$$ **(Refer example 3.17)** **[5]**

b) *Solve the integral equation.*

$$\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda & 0 \le \lambda \le 1 \\ 0 & \lambda > 1 \end{cases}$$ *Hence show that* $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$. **(Refer example 3.24)** **[3]**

c) *Solve any one*

i) Find the Z-transform of the following. Specify the ROC.

$f(k) = e^{-3k} \sin 4k,\ k \ge 0$. **(Refer example 4.20)**

ii) Find the Inverse Z-transform of $\frac{3z^2+2z}{z^2+3z+2}$, $1 < |z| < 2$. **(Refer example 4.56)** **[7]**

OR

Q.4 a) *i) Find Fourier transform of* e^{-x^2}

ii) Show that the Fourier transform of e^{-x^2} *is* $\frac{1}{\sqrt{2}} e^{-\lambda^2/4}$. **(Refer example 3.33)** **[5]**

b) *Using Fourier integral representation show that*

$$\int_0^\infty \frac{\lambda\cos\lambda\pi/2}{1-\lambda^2}\sin\lambda x\,d\lambda = \begin{cases} \frac{\pi}{2}\sin x & 0 \le x \le \frac{\pi}{2} \\ 0 & x > \frac{\pi}{2} \end{cases}$$ **(Refer example 3.19)** **[3]**

c) *Solve any one* **[7]**

i) Verify Convolution Theorem for $f_1(k) = k$ *and* $f_2(k) = k$. **(Refer example 4.42)**

ii) Use Inversion Integral Method to find $Z^{-1}\left[\frac{10z}{(z-1)(z-2)}\right]$. **(Refer example 4.73)**

SOLVED MODEL QUESTION PAPER - 3 [In Sem]

Engineering Mathematics - III

S.E. (Comp / IT) Sem - IV (As Per 2019 Pattern)

Time : 1 Hour] [Maximum Marks : 30

Q.1 a) *Solve any two* **[8]**

i) $(D^2 - 2D + 4)^2 = xe^x \cos(\sqrt{3}x + \alpha)$ **(Refer example 1.40)**

ii) $(D^3 + D)y = \operatorname{cosec} x$ **(Refer example 1.62)**

iii) $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$. **(Refer example 1.52)**

b) *The currents x and y in coupled circuits are given by*

$$L\frac{dx}{dt} + Rx + R(x-y) = E$$

$$L\frac{dy}{dt} + Ry - R(x-y) = 0$$

where L, R, E are constants. Find x and y in terms of t given x = 0, y = 0, when t = 0.

(Refer example 2.1) **[7]**

OR

Q.2 a) *Solve any two :* **[8]**

i) $(D^2-1)y = e^{-x}\sin e^{-x} + \cos e^{-x}$ **(Refer example 1.43)**

ii) $\left(D^2-4D+4\right)y = e^{2x}\sec^2 x$ **(Refer example 1.66)**

iii) $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=\dfrac{x^3}{1+x^2}$ **(Refer example 1.59)**

b) $\dfrac{dx}{x^2-yz}=\dfrac{dy}{y^2-zx}=\dfrac{dz}{z^2-xy}$ **(Refer example 2.20)** **[7]**

Q.3 a) *Find Fourier transform of* $e^{-|x|}$ *hence show that* $\int_{-\infty}^{\infty}\dfrac{e^{i\lambda x}}{1+\lambda^2}d\lambda = \pi e^{-|x|}$. **(Refer example 3.40)** **[5]**

b) *Using inverse sine transform, find f(x) if* $F_s(\lambda)=\dfrac{1}{\lambda}e^{-a\lambda}$. **(Refer example 3.38)** **[3]**

c) *Solve any one* **[7]**

i) *Find the Z-transform of the following by showing the region of convergence.*

$\left(-\dfrac{1}{2}\right)^{k+1}+3\left(\dfrac{1}{2}\right)^{k+1}$, $k \geq 0$. **(Refer example 4.11)**

ii) *Find inverse Z-transform of the following* $\dfrac{z^2}{z^2+a^2}$, $|z|>|a|$. **(Refer example 4.61)**

OR

Q.4 a) *Solve the integral equation* $\int_0^{\infty} f(x)\sin\lambda x\,dx = e^{-\lambda}$, $\lambda > 0$ **(Refer example 3.27)** **[5]**

b) *Solve the integral equation* $\int_0^{\infty} f(x)\sin\lambda x\,dx = \begin{cases} 1 & 0<\lambda<1 \\ 2 & 1<\lambda<2 \\ 0 & \lambda>2 \end{cases}$ **(Refer example 3.28)** **[3]**

c) *Solve any one* **[7]**

i) *Find the Z-transform of the following*

$f(k)=\dfrac{\sin\alpha k}{k}$, $k>0$ **(Refer example 4.33 (i))**

ii) *Obtain* $f(k)$ *given that* $12f(k+2)-7f(k+1)+f(k)=0$, $k\geq 0$, $f(0)=0$, $f(1)=3$ **(Refer example 4.80)**

SOLVED MODEL QUESTION PAPER - 1 [End Sem]

Engineering Mathematics - III

S.E. (Comp / IT) Sem - IV (As Per 2019 Pattern)

Time : $2\frac{1}{2}$ Hours] [Maximum Marks : 70

N. B. :

i) Attempt Q.1 or Q.2, Q.3 or Q.4, Q.5 or Q.6, Q.7 or Q.8.

ii) Neat diagrams must be drawn wherever necessary.

iii) Figures to the right side indicate full marks.

iv) Assume suitable data, if necessary.

Q.1 a) *Goals scored by two teams A and B in a football season were as follows. Determine which team is more consistent.* **(Refer example 5.4)** **[5]**

Number of goals scored	Number of matches	
	Team A	Team B
0	27	17
1	9	9
2	8	6
3	5	5
4	4	3

b) *Using method of least squares, derive the normal equations to fit the curve $y = ax^2 + bx$. Hence fit this curve to the following data :* **(Refer example 5.22)** **[6]**

x	1	2	3	4	5	6	7	8
y	1	1.2	1.8	2.5	3.6	4.7	6.6	9.1

c) *Compute the coefficient of correlation for the following data.* **(Refer example 5.27)** **[6]**

x	10	14	18	22	26	30
y	18	12	24	6	30	36

OR

Q.2 a) *Calculate the first four moments of the following distribution about the mean and find skewness and kurtosis.* **(Refer example 5.9)** **[5]**

x	0	1	2	3	4	5	6	7	8
f	1	8	28	56	70	56	28	8	1

b) *Use the method of least squares to fit the curve* $y = \frac{C_0}{x} + C_1\sqrt{x}$ *to the following table.*

(Refer example 5.25) **[6]**

x	0.1	0.2	0.4	0.5	1	2
y	21	11	7	6	5	6

c) *Calculate the coefficient of correlation and obtain the regression line of y on x for the following data :*

x :	1	2	3	4	5	6	7	8	9
y :	9	8	10	12	11	13	14	16	15

Also find an estimate of y which correspond to x = 6.2. **(Refer example 5.38)** **[6]**

Q.3 a) *A can hit the target 1 out of 4 times*
B can hit the target 2 out of 3 times
C can hit the target 3 out of 4 times
Find the probability that at least two hit the target. **(Refer example 6.3)** **[5]**

b) *An unbiased coin is thrown 10 times. Find the probability of getting exactly 6 heads, at least 6 heads.* **(Refer example 6.39)** **[6]**

c) *One percent of articles from a certain machine are defective. What is the probability of i) No defective ii) One defective iii) Two defectives iv) Two or more defective in a sample of 100 ?* **(Refer example 6.69)** **[6]**

OR

Q.4 a) *A die is tossed thrice. A success is getting 1 or 6' on a toss. Find the mean and the variance of the number of successes.* **(Refer example 6.35)** **[5]**

b) *In a sample of 1000 cases, the mean of a certain test is '14' and standard deviation is 2.5. Assuming the distribution to be normal find ;*
i) How many students score between 12 and 15 ?
ii) How many score above 18 ?
iii) How many score below 8 ?
iv) How may score 16 ? **(Refer example 6.88)** **[6]**

c) *A die is tossed 60 times and frequency of each face is indicated below*

Face x_i	1	2	3	4	5	6
Frequency f_i	5	7	5	14	13	16

Assume that the die is fair and apply Chi-square test of goodness of fit at 0.05 level of significance. **(Refer example 6.105)** **[6]**

Q.5 a) *Find a real root of equation* $x^4 - x - 10 = 0$ *correct to three decimal place by using Newton Raphson method.* **(Refer example 7.8)** **[6]**

b) *Find the real root of the equation* $x \log_{10} x - 1.2 = 0$ *by the method of false position method correct to four decimal places.* **(Refer example 7.5)** **[6]**

c) *Solve the following system of equations by Gauss elimination method :*
$10x + 2y + z = 9$
$2x + 20y - 22 = -44$
$-2x + 3y + 10z = 22$ **(Refer example 7.16)** **[6]**

OR

Q.6 a) *Solve the following system of equations by Cholesky method* $2x + 3y + z = 0$
$x + 2y - z = -2$
$-x + y + 2z = 0$ **(Refer example 7.22)** **[6]**

b) *Solve the following system of equations by Gauss-Seidel method :*
$13x_1 + 5x_2 - 3x_3 + x_4 = 18$
$2x_1 + 12x_2 + x_3 - 4x_4 = 13$
$3x_1 - 4x_2 + 10x_3 + x_4 = 29$
$2x_1 + x_2 - 3x_3 + 9x_4 = 31$ **(Refer example 7.18)** **[6]**

c) *Find the real root of equation* $x \log_{10} x = 1.2$ *by bisection method correct to four decimal places.* **(Refer example 7.2)** **[6]**

Q.7 a) *Prove that*
$\Delta \log f(x) = \log\left\{1 + \frac{\Delta f(x)}{f(x)}\right\}$ **(Refer example 8.1)** **[6]**

b) *Evaluate*
$\int_0^1 \frac{1}{1+x} dx$ *by Simpson's* $\left(\frac{1}{3}\right)^{rd}$ *and* $\left(\frac{3}{8}\right)^{th}$ *rule. Compare the result with actual value.*
(Refer example 8.32) **[6]**

c) *Using Euler's modified method, find value of y when* $x = 0.1$ *given that* $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$.
(Refer example 9.8) **[6]**

OR

Q.8 a) *Use Runge-Kutta formulae of* 2^{nd} *and* 4^{th} *orders to find y when* $x = 0.1$ *and* $x = 0.2$ *given that* $\frac{dy}{dx} = x + y$ *with* $y(0) = 1$. **(Refer example 9.12)** **[6]**

b) *Using Euler's method solve* $\frac{dy}{dx} = 1 + y^2$ *given* $y(0) = 0$. *Take* $h = 0.05$ *and obtain* $y(0.05)$, $y(0.1)$, $y(0.15)$.
(Refer example 9.4) **[6]**

c) *From the tabulated values of x and y given below, prepare forward difference table. Find polynomial passing through the points and find it's slope at* $x = 1.5$ **(Refer example 8.15)** **[6]**

x	0	2	4	6	8
y	5	29	125	341	725

SOLVED MODEL QUESTION PAPER - 2 [End Sem]

Engineering Mathematics - III

S.E. (Comp / IT) Sem - IV (As Per 2019 Pattern)

Time : $2\frac{1}{2}$ Hours] **[Maximum Marks : 70**

Q.1 a) *The first four moments of a distribution about the value of 4 of the variable are* -1.5, 17, -30 *and* 108. *Find the moments about mean and* β_1 *and* β_2. **(Refer example 5.6)** **[5]**

b) *Obtain a relation of the form $y = ab^x$ for the following data by the method of least squares :* **(Refer example 5.23)** **[6]**

x	2	3	4	5	6
y	8.3	15.4	33.1	65.2	127.4

c) *Two examiners A and B award marks to seven students.* **(Refer example 5.33)** **[6]**

Roll No.	1	2	3	4	5	6	7
Marks by A	40	44	28	30	44	38	31
Marks by B	32	39	26	30	38	34	28

OR

Q.2 a) *The following data gives according to age the frequency of marks obtained by 100 students in an intelligence test.* **(Refer example 5.35)** **[5]**

Marks \ Age in yrs	18	19	20	21	Total
10 - 20	4	2	2	-	8
20 - 30	5	4	6	4	19
30 - 40	6	8	10	11	35
40 - 50	4	4	6	8	22
50 - 60	-	2	4	4	10
60 - 70	-	2	3	1	6
Total	19	22	31	28	100

Calculate the coefficient of correlation between age and intelligence.

b) *Fit a second degree parabola to the following data.* **(Refer example 5.26)** **[6]**

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	8	13	5

c) *Given* $n = 6$, $\sum (x-18.5) = -3$, $\sum (y-50) = 0$, $\sum (x-18.5)^2 = 19$, $\sum (y-50)^2 = 850$, $\sum (x-18.5)(y-50) = -120$. *Calculate coefficient of correlation.*
(Refer example 5.42) **[6]**

Q.3 a) *A random variable X has the following probability function :*

Values of x, x : 0 1 2 3 4 5 6 7

$P(x)$: 0 K 2K 2K 3K K^2 $2K^2$ $7K^2 + K$

i) Find K, ii) Evaluate $P(X < 6)$, $P(X \geq 6)$, $P(3 < x \leq 6)$, *iii) Find the minimum value of x so that* $P(X \leq x) > \frac{1}{2}$. **(Refer example 6.34)** **[5]**

b) *In a certain factory turning out razor blades there is a small chance of 1/500 for any blade to be defective. The blades are supplied in a packet of 10. Calculate approximate number of packets containing no defective and 2 defective blades in a consignment of 10,000 packets.* **(Refer example 6.62)** **[6]**

c) *Assuming that the diagram of 1000 brass plugs taken consecutively from machine from a normal distribution with mean 0.7515 cm and standard deviation 0.0020 cm how many of the plugs are likely to be approved if the acceptable diagram is 0.752 ± 0.004 cm.* **(Refer example 6.83)** **[6]**

OR

Q.4 a) *A, B play a game of alternate tossing a coin one who gets head first wins the game. Find the probability that B wins the game if A has a start.* **(Refer example 6.21)** **[5]**

b) *A normal distribution has a mean 15.73 and a standard deviation of 2.08. Find the percentage of cases that fall between 17.81 and 13.65. Find also the % of cases lying above 18.85.* **(Refer example 6.90)** **[6]**

c) *The table below gives the number of accidents that occurred in certain factory on various days of week.*

Days	Sun	Mon	Tue	Wed	Thurs	Fri	Sat
Accidents	6	4	9	7	8	10	12

Test at 5 % level of significance whether accidents are uniformly distributed over the days.
(Refer example 6.107) **[6]**

Q.5 a) *Apply Newton Raphson method to solve* $3x - \cos x - 1 = 0$ *correct to three decimal place.*
(Refer example 7.10) **[6]**

b) *Find root of equation* $x^3 - 2x - 5 = 0$ *which lies between 2 and 3 using bisection method is five step.*
(Refer example 7.3) **[6]**

c) *Solve Gauss Jacobi method*

$10x - 5y - 2z = 3$

$4x - 10y + 3z = -3$

$x + 6y + 10z = -3$ **(Refer example 7.24)** **[6]**

OR

Q.6 a) *Solve the following system of equations by Cholesky's method* $2x_1 - x_2 = 1$

$-x_1 + 3x_2 + x_3 = 0$

$x_2 + 2x_3 = 0$ **(Refer example 7.21)** **[6]**

b) *Solve the following system of equations by Gauss Seidel method* $20x + 4y - z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24$ **(Refer example 7.20)** **[6]**

c) *Find a real root of equation* $x^3 - 2x - 5 = 0$ *by method of Regula-Falsi method correct to three decimal place.*
(Refer example 7.7) **[6]**

Q.7 a) *From the following table find :*
i) y *when* $x = 7$ *ii)* y *when* $x = 17$ *iii)* y *when* $x = 19$ **(Refer example 8.11)** **[6]**

x :	8	10	12	14	16	18
f(x) = y :	10	19	32.5	54	89.5	15.4

b) *A curve passes through the given points. Find*
i) Area bounded by the curve, x axis, $x = 1$ and $x = 9$.
ii) The volume of the solid generated by revolving this area about the x-axis. **(Refer example 8.34)** **[6]**

x	1	2	3	4	5	6	7	8	9
y	0.2	0.7	1	1.3	1.5	1.7	1.9	2.1	2.3

c) *Using Euler's modified method find the value of y if* $\frac{dy}{dx} = \log(x+y)$ *with* $y(1) = 2$ *for* $x = 1.2$ *and* $x = 1.4$ *correct to three decimal places. Take* $h = 0.2$. **(Refer example 9.10)** **[6]**

OR

Q.8 a) *Using fourth order Runge Kutta method solve the equation* $\frac{dy}{dx} = \sqrt{x+y}$ *with* $y(0) = 1$ *and find* $y(0.2)$ *taking* $h = 0.2$. **(Refer example 9.13)** **[6]**

b) *Given that* $\frac{dy}{dx} = \frac{y-x}{y+x}$ *with initial condition* $y(0) = 1$. *Find y for* $x = 0.1$. *Also find the better approximation at* $x = 0.1$ *by dividing the interval [0, 0.1] into five steps.* **(Refer example 9.6)** **[6]**

c) *Given that :* $u_0 + u_8 = 80$, $u_1 + u_7 = 10$, $u_2 + u_6 = 5$, $u_3 + u_5 = 10$ *find* u_4. **(Refer example 8.9)** **[6]**

SOLVED MODEL QUESTION PAPER - 3 [End Sem]

Engineering Mathematics - III

S.E. (Comp / IT) Sem - IV (As Per 2019 Pattern)

Time : $2\frac{1}{2}$ Hours] **[Maximum Marks : 70**

Q.1 a) *Calculate the first four moments about the mean of the given distribution. Also find* β_1 *and* β_2.
(Refer example 5.8) **[5]**

x_i	2.0	2.5	3.0	3.5	4.0	4.5	5.0
f_i	4	36	60	90	70	40	10

b) *Obtain the least squares fit of the form.* $f(t) = a\,e^{-3t} + b\,e^{-2t}$ *for the data :* **(Refer example 5.24)** **[6]**

t :	0.1	0.2	0.3	0.4
f(t) :	0.76	0.58	0.44	0.35

c) *The following table gives, according to age the frequency of marks obtained by 200 students in a test to determine talent in mathematics :* **(Refer example 5.36)** **[6]**

Age in yrs / Marks	20	21	22	23	24	Total
0 - 10	10	8	6	10	4	38
10 - 20	8	10	8	-	11	37
20 - 30	-	11	7	8	5	31

30 - 40	20	-	10	12	10	52
40 - 50	2	6	7	15	12	42
Total	40	35	38	45	42	200

OR

Q.2 a) *The first three moments about the value 2 of a distribution are 1, 16 and – 40. Find the mean, standard deviation and β_1 of the distribution.* **(Refer example 5.14)** **[5]**

b) *By the method of least squares, find the straight line that best fits the following data :* **(Refer example 5.21)** **[6]**

x :	1	2	3	4	5
y :	14	27	40	55	68

c) *From record of analysis of correlation data the following results are available variance of x = 9 and lines of regression are given by*

$8x - 10y + 66 = 0$

$40\,x - 18y = 214$

Find out a) Mean values for x and y services.

b) Standard deviation of y services.

c) Coefficient of correlation between x and y services. **(Refer example 5.45)** **[6]**

Q.3 a) *An aircraft gun can take a minimum of four shots at an enemy plane moving away from it. The probability of hitting plane at 1^{st}, 2^{nd}, 3^{rd}, 4^{th} shot are 0.4, 0.3, 0.2, and 0.1 respectively. What is the probability that gun hits the plane ?* **(Refer example 6.8)** **[5]**

b) *A dice is thrown 5 times. If getting an odd number is a success, what is the probability of (i) 4 successes (ii) At least 4 success.* **(Refer example 6.46)** **[6]**

c) *In a certain examination test 200 students appeared in subject of statistics. Average marks obtained were 50 % with standard deviation 5 %. How many students do you expect to obtain more than 60 % of marks, supposing that marks are distributed normally.* **(Refer example 6.85)** **[6]**

OR

Q.4 a) *A can hit the target 1 out of 4 times B can hit the target 2 out of 3 times, C can hit the target 3 out of 4 times. Find the probability of at least two hit the target.* **(Refer example 6.28)** **[5]**

b) *The average test marks in a particular class is 79 and standard deviation is 5. If the marks are normally distributed, how many students in a class of 200, did not receive marks between 75 and 82. Given z = 0.8, Area = 0.2881 and z = 0.6, Area = 0.2257.* **(Refer example 6.104)** **[6]**

c) *The following table of frequencies of seeds were observed in experiment on pea breeding*

Round and Green	Wrinkled and Green	Round and Yellow	Wrinkled and Yellow	Total
222	120	32	150	524

Theory predicts that the frequencies should be in proportion 8 : 2 : 2 : 1. Examine the correspondence between theory experiment. **(Refer example 6.110)** **[6]**

Q.5 a) *Evaluate $\sqrt{12}$ to four decimal place by Newton - Raphson method.* **(Refer example 7.13)** **[6]**

b) *Find root of equation $x^3 - 4x - 9 = 0$ using bisection method correct to 3 decimal places.* **(Refer example 7.4)** **[6]**

c) *Solve the following system of equations by the Gauss–Seidel method :*

$10x_1 + x_2 + x_3 = 12$

$2x_1 + 10x_2 + x_3 = 13$

$2x_1 + 2x_2 + 10x_3 = 14$ **(Refer example 7.19)** **[6]**

OR

Q.6 a) *Use Gauss elimination method to solve the following system of equations.*

$x_1 + 4x_2 - x_3 = -5$

$x_1 + x_2 - 6x_3 = -12$

$3x_1 - x_2 - x_3 = 4$ **(Refer example 7.15)** **[6]**

b) *Solve Gauss Jacobi Method*

$27x + 6y - z = 85$

$6x + 15y - 2z = 72$

$x + y + 54z = 110$ **(Refer example 7.23)** **[6]**

c) *Find the negative root of equation $x^3 - 21x + 3500 = 0$ correct to three decimal place by Newton - Raphson method.* **(Refer example 7.14)** **[6]**

Q.7 a) *Give an estimate of the population in 1971 from the following table* **(Refer example 8.8)** **[6]**

Year	1941	1951	1961	1971	1981	1991
Population in Lakhs	363	391	421	–	467	501

b) *A solid of revolution is formed by rotating about x - axis. The area between x-axis, lies x = 0 and x = 1 and curve through the axis is given as*

x	0	0.25	0.5	0.75	1
y	1	0.9886	0.9589	0.8489	0.9415

Estimate the volume of solid formed. **(Refer example 8.38)** **[6]**

c) *Using modified Euler's method find y(1.3) if $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ which y(1) = 1.* **(Refer example 9.9)** **[6]**

OR

Q.8 a) *Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$. Find the value of y(0.1) by using R.K. method of 4^{th} order.* **(Refer example 9.22)** **[6]**

b) *Using Euler's Method find y at x = 0.1 and 0.4 if $\frac{d^2y}{dx^2} = \frac{3}{2}x\frac{dy}{dx} - \frac{9}{2}y + \frac{9}{2}$ with $y(0) = 1$ and $y'(0) = -2$.*

(Refer example 9.7) **[6]**

c) *The velocity distribution of fluid near a flat surface is given below.*

x	0.1	0.3	0.6	0.8
v	0.72	1.81	2.73	3.47

where x is the distance from the surface (mm) and v is the velocity (mm/sec). Use Lagrangues interpolation formula to obtain velocity at x = 0.4. **(Refer example 8.23)** **[6]**

❑❑❑