Q-1. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$ (Nov. 2018)

→ Let $\frac{d}{dx} = D$,

$$\therefore D^2y + 3Dy + 2y = e^{e^x}$$

$$(D^2 + 3D + 2)y = e^{e^x}$$

$\therefore \phi(D) = D^2 + 3D + 2$, $f(x) = e^{e^x}$

$= (D+2)(D+1)$

$$\therefore y_c = C_1 e^{-x} + C_2 e^{-2x}$$

$$y_p = \frac{1}{\phi(D)} \cdot f(x)$$

$$= \frac{1}{(D+2)(D+1)} \cdot e^{e^x} = \frac{1}{D+2}\left[\frac{1}{D+1} \cdot e^{e^x}\right]$$

$$= \frac{1}{D+2}\left[e^{-x}\int e^{x}.e^{e^{x}}.dx\right]$$

$$= \frac{1}{D+2}\left[e^{-x}.e^{e^{x}}\right]$$

$$= e^{-2x}\int e^{2x}.e^{-x}.e^{e^{x}}.dx$$

$$y_p = e^{-2x}.e^{e^{x}}$$

$$y = y_c + y_p$$

$$\boxed{y = C_1 e^{-x} + C_2 e^{-2x} + e^{-2x}.e^{e^{x}}}$$

Q-2. Solve $\frac{d^2y}{dx^2} + \frac{2dy}{dx} + y = xe^{-x}.\cos x$ (May 2017)

→ Let $\frac{d}{dx} = D$,

$$\therefore (D^2+2D+1)y = x.e^{-x}.\cos x$$

$$\phi(D) = D^2+2D+1$$
$$= (D+1)^2$$

$$\therefore y_c = (c_1 + xc_2)e^{-x} \quad \text{— } 1$$

$$f(x) = x.e^{-x}.\cos x = e^{-x}(x.\cos x).$$

$$\therefore y_p = \frac{1}{\phi(D)}.f(x) = \frac{1}{(D+1)^2}.e^{-x}.(x\cos x)$$

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Compare with $\frac{1}{\phi(D+a)} \cdot e^{+ax} \cdot V = \frac{1}{\phi(D)} \cdot e^{ax} \cdot V$

$$\therefore y_p = \frac{1}{(D+1)^2} \cdot e^{-x} \cdot (x \cos x)$$

$\because a = -1, \; V = x \cos x$

$$= \frac{1}{(D+1-1)^2} \cdot e^{-x} (x \cdot \cos x)$$

$$= e^{-x} \left[\frac{1}{D^2} \cdot (x \cdot \cos x) \right]$$

Here, $\phi(D) = D^2$, $V = \cos x$.

$$\therefore y_p = e^{-x} \left[x - \frac{\phi'(D)}{\phi(D)} \right] \frac{1}{\phi(D)} \cdot \cos x$$

$$= e^{-x} \left[x - \frac{2D}{D^2} \right] \left[\frac{1}{D^2} \cdot \cos x \right]$$

$\rightarrow a = 1, \; D^2 \rightarrow -1$

$$= e^{-x} \left[x - \frac{2}{D} \right] \left[\frac{\cos x}{-1} \right]$$

$$= e^{-x} \left[-x \cos x + 2 \int \cos x \right] \quad \because \left[1/D = \int \cos x \right]$$

$$y_p = e^{-x} \left[2 \sin x - x \cos x \right] \quad \text{—— } (2)$$

$$\therefore y = y_c + y_p$$

$$\boxed{y = e^{-x} \left[(c_1 + x c_2) + 2 \sin x - x \cos x \right]} \text{ — from (1) and (2)}$$

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Q-3. Solve $(D^2 - 2D + 2)y = e^x \cdot \tan x$ (Nov. 2017)

$\rightarrow$ Here, $\phi(D) = D^2 - 2D + 2$, $f(x) = e^x \cdot \tan x$.

$$\text{roots} = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= 1 \pm i$$

$$\therefore y_c = e^x [C_1 \cos x + C_2 \sin x] \quad \text{—} \quad (1)$$

In MVP, $y_c = C_1 y_1 + C_2 y_2$.

$$\therefore y_1 = e^x \cdot \cos x \quad , \quad y_2 = e^x \cdot \sin x.$$

$$\therefore y_1' = e^x \cos x - e^x \sin x \quad , \quad y_2' = e^x \sin x + e^x \cos x$$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^{2x} (\cancel{\sin x \cdot \cos x} + \cos^2 x - \cancel{\sin x \cdot \cos x} + \sin^2 x)$$

$$= e^{2x} (\cos^2 x + \sin^2 x)$$

$$= e^{2x} (1)$$

$$W = e^{2x} \quad \text{—} \quad (2)$$

$$u = \int \frac{-y_2}{W} \cdot f(x) \cdot dx = \int \frac{-\cancel{e^x} \cdot \sin x}{\cancel{e^{2x}}} \cdot \cancel{e^x} \cdot \tan x$$

$$u = -[\log(\sec x + \tan x) - \sin x] \quad \text{—} \quad (3)$$

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$$v = \int \frac{y_1}{W} \cdot f(x) \cdot dx = \int \frac{e^x \cdot \cos x}{e^{2x}} \cdot e^x \cdot \tan x \cdot dx$$
$$= \int \sin x \cdot dx$$
$$= -\cos x$$

Using MVP, $y_P = u y_1 + v y_2$

$$= [\sin x - \log(\sec x + \tan x)] e^x \cdot \cos x + [-\cos x] e^x \cdot \sin x$$

$$y_P = -e^x \cdot \cos x \cdot \log(\sec x + \tan x) \quad \text{---} \quad 4$$

$$y = y_c + y_P$$

$$\therefore y = e^x [c_1 \cos x + c_2 \sin x - \cos x \cdot \log(\sec x + \tan x)]$$

Q-4. $x^2 \cdot \frac{d^2 y}{dx^2} + (-x) \cdot \frac{dy}{dx} + 4y = \cos(\log x) + x \sin(\log x)$ [Nov. 2018]

$\longrightarrow$ Let $D, \frac{d}{dz}$, $x = e^z$, $z = \log x$.

$$x \cdot \frac{dy}{dx} = Dy \quad , \quad x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

Given equation becomes,

$$D(D-1)y - Dy + 4y = \cos(z) + e^z \sin(z)$$

$$(D^2 - 2D + 4)y = \cos(z) + e^z \cdot \sin(z) \quad , \; y \text{ depends on } z.$$

$\phi(D) = D^2 - 2D + 4$, $f(z) = \cos z + e^z . \sin z$

$\rightarrow 1 \pm \sqrt{3}\, i$

$\therefore y_c = e^z (C_1 \cos\sqrt{3}z + C_2 \sin\sqrt{3}z)$

$y_p = \frac{1}{\phi(D)} . f(z) = \frac{1}{(D^2-2D+4)} . [\cos z + e^z \sin z]$

$= \frac{1}{(D^2-2D+4)} . \cos z + \frac{1}{(D^2-2D+4)} . e^z \sin z$

$a = 1,\ D^2 = -1$ $\qquad$ $a=1,\ V = \sin z,\ D \to D+1$

$= -\frac{1}{2D-3} . \cos z + e^z \left[\frac{1}{D^2+3} . \sin z \right]$

$= -\frac{(2D+3)}{4D^2-9} . \cos z + e^3 \left[\frac{1}{D^2+3} . \sin z \right]$

$a = 1,\ D^2 \to -1$ $\qquad$ $a = 1,\ D^2 \to -1$

$= \frac{(2D+3) \cos z}{13} + \frac{e^3 \sin z}{2}$

$\therefore y_p = \frac{3\cos(\log x) - 2\sin(\log x)}{13} + \frac{x \sin(\log x)}{2}$ $[\because z = \log x]$

$\therefore y = y_c + y_p$

$$\boxed{y = x[C_1 \cos(\sqrt{3}\log x) + C_2 \sin(\sqrt{3}\log x)] + \frac{x \sin \log x}{2} + \frac{3\cos(\log x)}{13} - \frac{2\sin}{13}}$$

Q-5. Solve the following simultaneously $\frac{dy}{dt} + x = \sin t$, $\frac{dx}{dt} + y = \cos t$

→ Let $\frac{d}{dt} = D$ $\therefore$ $Dy + x = \sin t$ — (1)

$Dx + y = \cos t$ — (2)

Differentiate eq. 2, we get

$$D^2 x + Dy = D \cos t \quad — (3)$$

(3) - (1), we get

$$D^2 x - x = -2 \sin t$$
$$(D^2 - 1) x = -2 \sin t$$

$\phi(D) = D^2 - 1 = (D+1)(D-1)$, $f(t) = -2 \sin t$.

$$\therefore x_c = C_1 e^{-t} + C_2 e^{t} \quad — (4)$$

$$x_p = \frac{1}{\phi(D)} \cdot f(t)$$
$$= \frac{1}{D^2 - 1} \cdot (-2 \sin t)$$

$a = 1, D^2 \to -1$

$$x_p = \sin t \quad — (5)$$

$$\boxed{x = x_c + x_p = C_1 e^{-t} + C_2 e^{t} + \sin t} \quad — (6)$$

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Subtitute ⑥ in ②, we get

$$y = \cancel{\cos t} + C_1 e^{-t} - C_2 e^{t} - \cancel{\cos t}$$

$$\boxed{y = C_1 e^{-t} - C_2 e^{t}}$$

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