PHY208 Experiment 4

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Using a Spectrometer to find the Dispersive Power of a Prism

1 Aim

To use a prism spectrometer to find the dispersive power of a prism.

2 Apparatus

1. A prism spectrometer
2. A mercury lamp
3. Power source for the mercury lamp
4. Prism
5. Spirit level
6. Magnifying glass for taking the readings

3 Formulae

The dispersive power of a prism is given by:

$$\omega := \frac{n_1 - n_2}{\tilde{n} - 1} \quad \text{where} \quad \tilde{n} = \frac{n_1 + n_2}{2} \tag{1}$$

Here, n_1 is the refractive index with respect to air of light with the highest frequency, and

n_2 is the refractive index with respect to air of light with the lowest frequency,

Note that even though the dispersive power is defined using the refractive indices of the highest and the lowest frequency light, we can use any two pairs of refractive indecies to find ω for those two frequencies.

The refractive index with respect to air can be calculated by use of the following formula:

$$\mu = \frac{\sin \frac{A+\delta_m}{2}}{\sin \frac{A}{2}} \tag{2}$$

Where A is the angle of prism, and δ_m is the angle of minimum deviation for the frequency for which we are calculating the refractive index.

4 Theory

4.1 Measuring the Anlge of Prism

While measuring the angle of prism using a spectrometer, our measured angle is $2A$ instead of A. This is because we have the incident light hit the top of the prism (the point at which the two refracting surfaces meet), and measure the angle between the *reflected* rays.

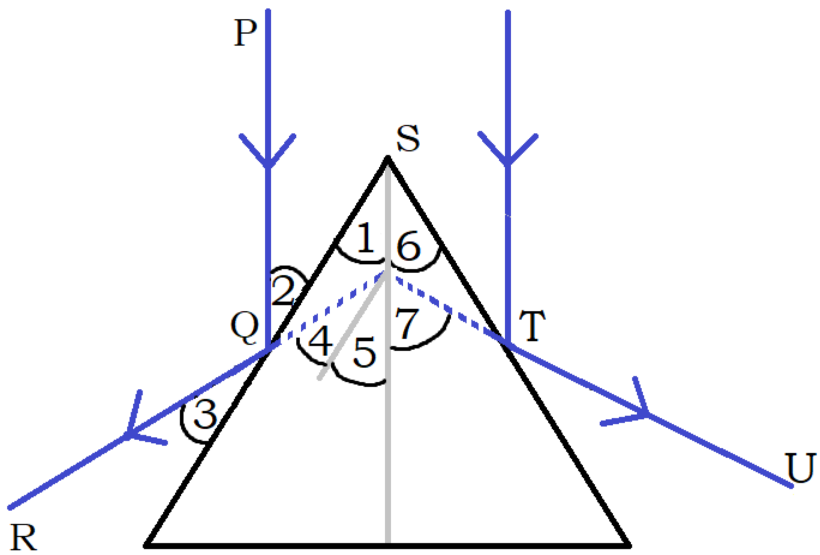

Figure 1 Measuring A

In Fig. 1, $\angle 1 = A/2$. Now $\angle 1 = \angle 2$ and $\angle 2 = \angle 3$ since the angle of incidence is equal to the angle of reflection. Now $\angle 3 = \angle 4$, and $\angle 1 = \angle 5$. Hence $\angle 5 = \angle 1 = \angle 4 = A/2$, or $\angle 4 + \angle 5 = A$. Similarly, $\angle 7 = A$. While doing the experiment, we measure the angle between the rays QR and TU. This angle would be $\angle 4 + \angle 5 + \angle 7 = 2A$.

4.2 Defining Some Angles

We define the angle of deviation and angle of prism by use of the figure below.

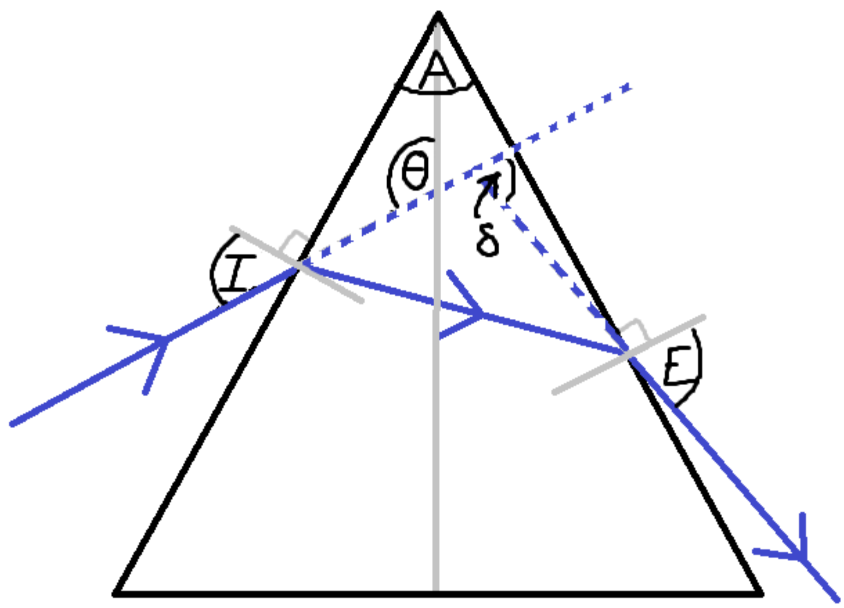

Figure 2 Path of Light in a Prism

In Fig. 2, we see that the angle of prism, A is defined as the angle between the two refracting surfaces of the prism. The angle of deviation, δ is defined as the angle by which the prism deviated the incident ray from the path it would have taken, had there been no prism. The angle of minimum deviation, δ_m is defined as the least possible value of δ. I and E are the angles of incidence and emergence, which are defined as the angle made by the incident ray and the emergent ray with respect to the perpedicular drawn at the point at which they enter and leave the prism respectively. Additionaly, I have defined an angle θ, which I am calling the tilting angle, and it is the angle between the 'intended' path of the incident ray and the angle bisector of A. Note that when the tilting angle is zero, the incident ray would hit the top end of the prism (the point where the two refracting surfaces meet), and increasing θ while not changing the direction of the incident ray would serve to further tilt the prism with respect to the incident ray.

4.3 Dependence of δ on Various Parameters

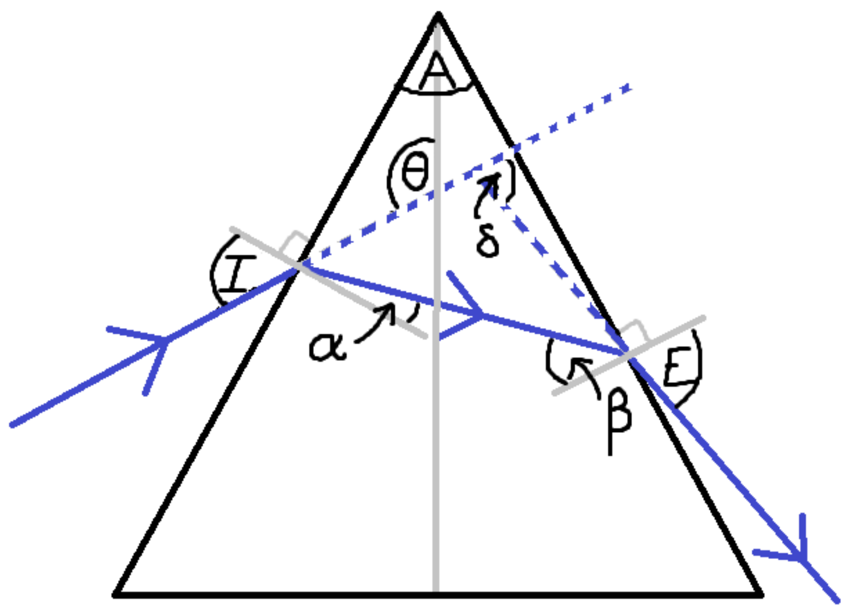

Figure 3 Path of Light in a Prism

In Fig. 3, by Snell's law, we have

$$\frac{\sin I}{\sin \alpha} = n$$

Where n represents the refractive index of the prism for that frequency of light with respect to air. Solving for α gives

$$\alpha = \arcsin\left(\frac{\sin I}{n}\right) \tag{3}$$

We don't have to worry about the different values of α which would also satisfy the equation, since I and α are always between 0 and $\pi/2$.

Now in the triangle formed by the prism and the refracted ray, we have

$$A + \left(\frac{\pi}{2} - \alpha\right) + \left(\frac{\pi}{2} - \beta\right) = \pi$$

Solving for β gives

$$\beta = A - \alpha \tag{4}$$

Now applying Snell's law on the refracted and the emergent ray, we get

$$\frac{\sin E}{\sin \beta} = n$$

Writing E in terms of β, and substituting Eq. 4, yields

$$E = \arcsin\Big(n \sin(A - \alpha)\Big)$$

Now substituting Eq. 3, we get the angle of emergence in terms of the angle of incidence

$$E = \arcsin\left[n \sin\left(A - \arcsin\left(\frac{\sin I}{n}\right)\right)\right]$$

By simplifying, we obtain,

$$\boxed{E(I) = \arcsin\Big(\sin A\sqrt{n^2 - \sin^2 I} - \cos A \sin I\Big)} \tag{5}$$

Note that in the limiting case when $n = 1$ (when the prism is made of air), we get $E = A - I$, which can be checked by doing some simple geometry (in this case, the incident light would not be deviated at all and would continue on its original path). Also note that the expression inside the parenthesis looks like a tampered version of $\sin(A - I)$.

Now in the triangle formed by the extended incident ray, the refracted ray and the extended emergent ray, we have

$$(I - \alpha) + (E - \beta) + (\pi - \delta) = \pi$$

Substituting the result of Eq. 4, and solving for δ gives

$$\delta = I + E - A \tag{6}$$

Substituting Eq. 5, we get the angle of deviation as a funciton of the angle of incidence.

$$\boxed{\delta(I) = I - A + \arcsin\left(\sin A\sqrt{n^2 - \sin^2 I} - \cos A \sin I\right)} \tag{7}$$

Note that the angle of deviation varies with the refractive index (which is dependent on the material used and the frequency of incident light).

The variation of the angle of deviation with the refractive index is shown below.

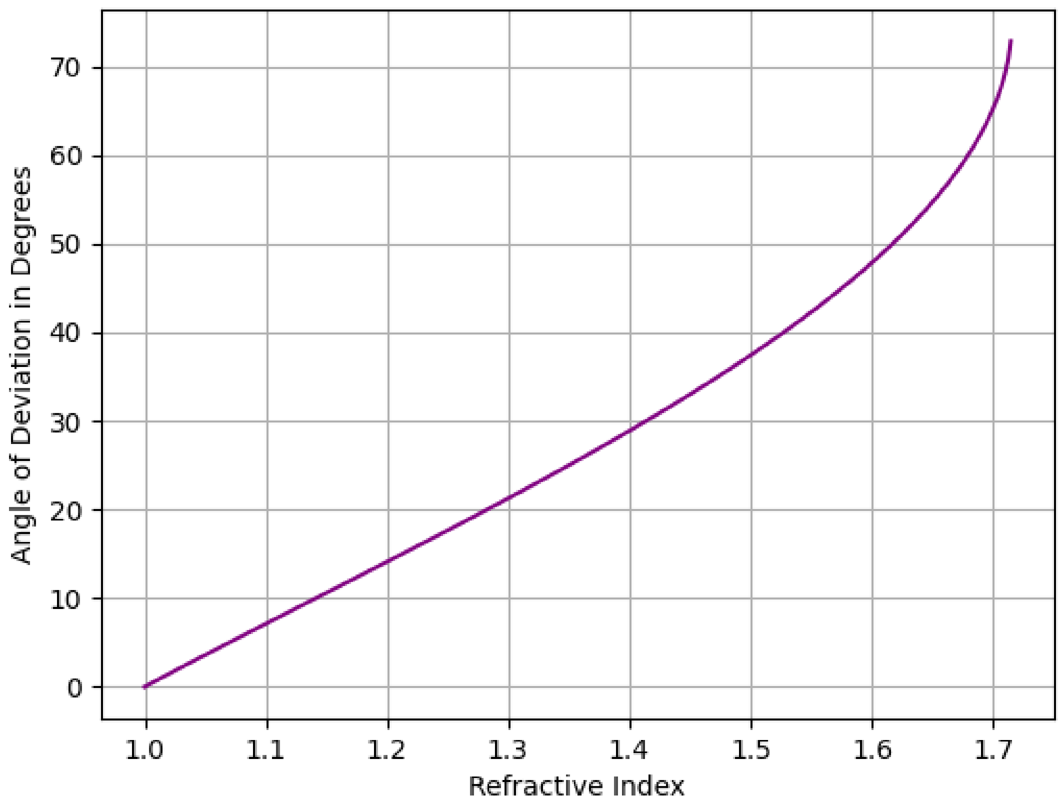

Figure 4 Variation of δ with n for a Fixed Angle of Incidence

Here the angle of prism and the angle of incidence were taken to be 60° and 45° respectively.

In the smallest top left triangle of Fig. 3, we have

$$\left(\frac{\pi}{2} - I\right) + \theta + \frac{A}{2} = \pi$$

Solving for I, we obtain

$$I = \frac{A}{2} + \theta - \frac{\pi}{2} \tag{8}$$

Substituting this into Eq. 7, we get

$$\boxed{\delta(\theta) = \theta - \frac{A}{2} - \frac{\pi}{2} + \arcsin\left[\sin A\sqrt{n^2 - \cos^2\left(\theta + \frac{A}{2}\right)} + \cos A \cos\left(\theta + \frac{A}{2}\right)\right]} \tag{9}$$

The variation of the angle of deviation with the tilting angle using Eq. 9 is shown below.

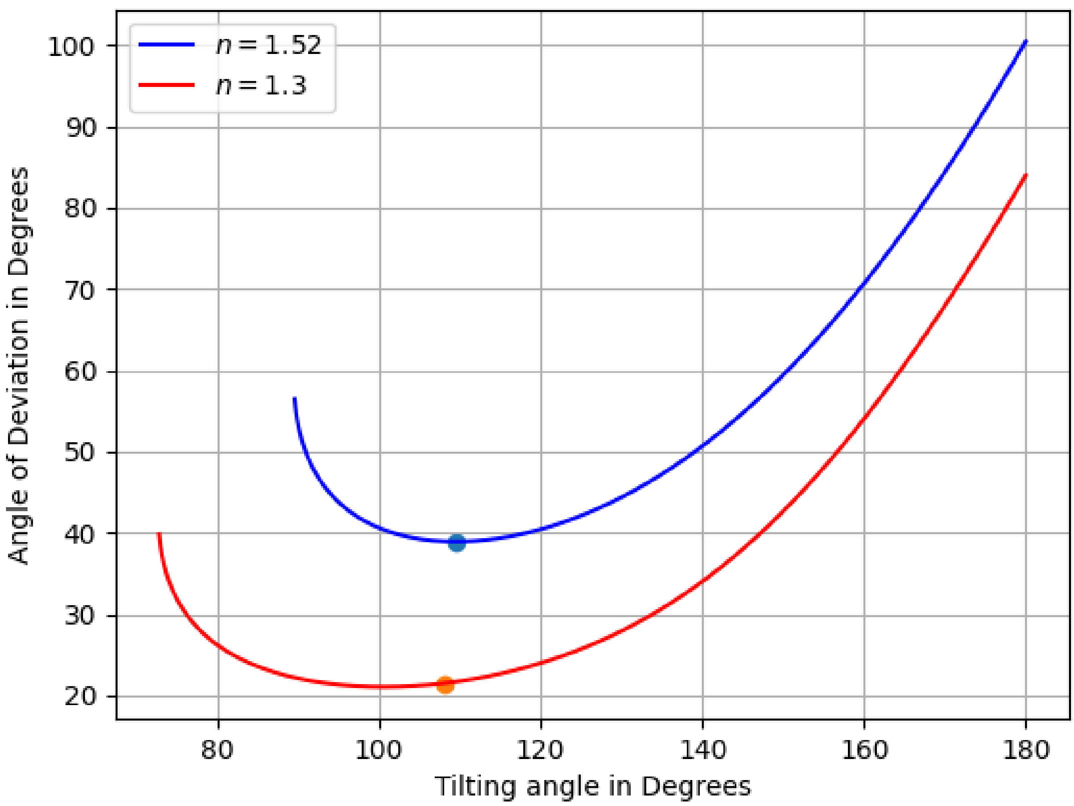

Figure 5 Variation of δ with θ for Two Refractive Indices

Here the A was taken to be $60°$. For $n = 1.52$, the angle of minimum deviation and the tilting angle at which it occurs is $38.93°$ and $109.55°$ respectively. For $n = 1.3$, the angle of minimum deviation and the tilting angle at which it occurs is $21.53°$ and $108.11°$ respectively. Note that the angle of minimum deviation and the corresponding tilting angle is different for different values of n.

4.4 Deriving Equation 2

The derivation of Eq. 2 can be found in most elementary physics textbooks where they begin by assuming that $I = E$ at the minimum deviation angle. Here I will present the mathematical proof of that fact, which is seldom found in literature.

At the angle of minimum deviation, we would have

$$\frac{d\delta}{dI} = 0$$

Substituting Eq. 7 and differentiating, we get

$$1 - \frac{\sin A \cos I \sin I (n^2 - \sin^2 I)^{-\frac{1}{2}} + \cos A \cos I}{\sqrt{1 - \left(\sin A \sqrt{n^2 - \sin^2 I} - \cos A \sin I\right)^2}} = 0$$

Rearranging and squaring gives

$$1 - \left(\sin A \sqrt{n^2 - \sin^2 I} - \cos A \sin I\right)^2$$

$$= \frac{\cos^2 I}{n^2 - \sin^2 I} \left(\sin^2 A \sin^2 I + \cos^2 A (n^2 - \sin^2 I) + 2 \sin A \cos A \sin I \sqrt{n^2 - \sin^2 I}\right)$$

$$= \frac{\cos^2 I}{n^2 - \sin^2 I} \Bigg(\sin^2 I - \cos^2 A \sin^2 I + (n^2 - \sin^2 I)$$

$$- \sin^2 A (n^2 - \sin^2 I) + 2 \sin A \cos A \sin I \sqrt{n^2 - \sin^2 I} \Bigg)$$

$$= \frac{\cos^2 I}{n^2 - \sin^2 I} \left(n^2 - \left(\sin^2 A (n^2 - \sin^2 I) + 2 \cos A \sin I \sin A \sqrt{n^2 - \sin^2 I} + \cos^2 A \sin^2 I \right) \right)$$

Now substituting the sine of Eq. 5 we get,

$$1 - \sin^2 E = \frac{\cos^2 I}{n^2 - \sin^2 I}\left(n^2 - \sin^2 E\right)$$

Cross multiplying gives

$$\sin^2 I \cos^2 E = \sin^2 E \cos^2 I$$

Taking the sqaure root and rearranging,

$$\sin(I - E) = 0$$

Which finally yields

$$I = E$$

We didn't take the negative square root because that would give $E = -I$, which is not possible since I and E are always between 0 and $\pi/2$.

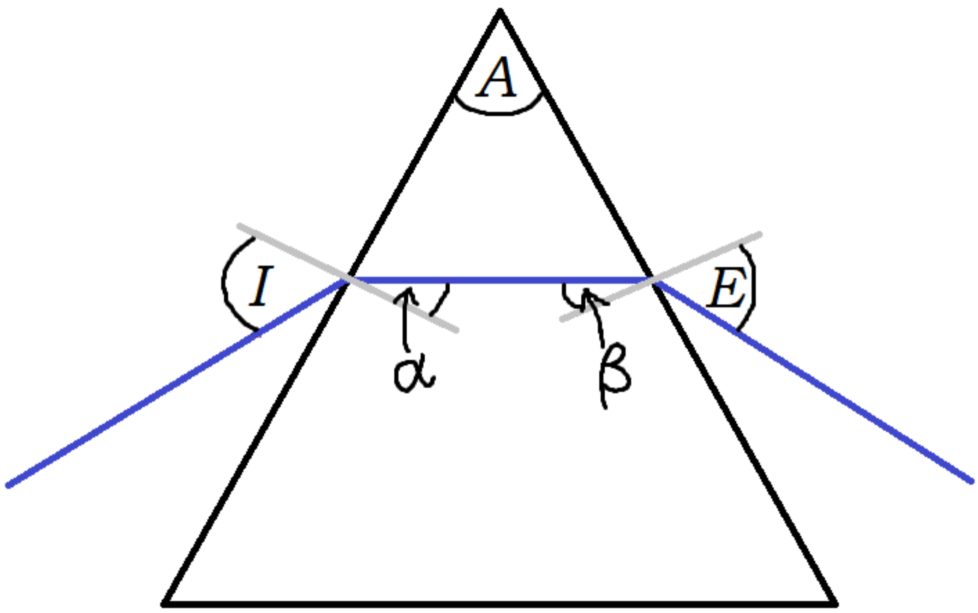

Figure 6 Minimum Deviation in a Prism

By Snell's law, we have

$$n = \frac{\sin I}{\sin \alpha}$$

By Eq. 6,

$$\delta_m = 2I - A$$

where δ_m represents the angle of minimum deviation. Also, Eq. 4 gives

$$\alpha = A - \alpha$$

since $\beta = \alpha$ (this can be proved by using Snell's law on the incident, refracted and refracted, emergent rays in conjunction with the fact that $I = E$ in this case). Now we have

$$I = \frac{\delta_m + A}{2} \quad \text{and} \quad \alpha = \frac{A}{2}$$

Substituting this into the equation obtained via Snell's law gives the Eq. 2

$$n = \frac{\sin\left(\frac{\delta_m + A}{2}\right)}{\sin\frac{A}{2}}$$

5 Observation Tables

14.5°on the main scale corresponded with 30 vernier scale divisions. Hence the least count is 1'.

5.1 Angle of Prism

S.no.	Angle 1			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	139	30	319	30
2	139	17	319	19
3	112	15	292	16

S.no.	Angle 2			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	19.5	16	199.5	17
2	19.5	16	199.5	16
3	352	10	172	8

5.2 Angle of Minimum Deviation

I took two sets of measurments for the anlge of minimum deviation, on different days. Since these sets had different angles of zero deviation, they are presented separately.

5.2.1 First Set of Measurments

Angle of Zero Deviation			
Vernier 1		Vernier 2	
MSR	VSR	MSR	VSR
113.5	30	294	5

S.no.	Angle of Minimum Deviation for Red			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	164.5	15	344.5	16
2	164.5	14	344.5	16

S.no.	Angle of Minimum Deviation for Yellow			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	165	1	345	15
2	165	2	345	14

S.no.	Angle of Minimum Deviation for Green			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	165	25	345	14
2	165.5	1	345.5	2

S.no.	Angle of Minimum Deviation for Blue			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	167.5	0	347.5	0
2	167.5	3	347.5	2

S.no.	Angle of Minimum Deviation for Violet			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	168.5	12	348.5	15
2	168.5	1	348.5	4

5.2.2 Second Set of Measurments

Angle of Zero Deviation			
Vernier 1		Vernier 2	
MSR	VSR	MSR	VSR
29	15	209	16

S.no.	Angle of Minimum Deviation for Red			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	79.5	30	260	1
2	79.5	25	260	0

S.no.	Angle of Minimum Deviation for Yellow			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	80	28	260	30
2	80	30	260	20

S.no.	Angle of Minimum Deviation for Green			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	80.5	15	260.5	17
2	80.5	16	260.5	18

S.no.	Angle of Minimum Deviation for Blue			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	82.5	20	262.5	26
2	82.5	21	262.5	25

S.no.	Angle of Minimum Deviation for Violet			
	Vernier 1		Vernier 2	
	MSR	VSR	MSR	VSR
1	83.5	10	263.5	9
2	83.5	20	264	1

6 Calculations

6.1 Angle of Prism

Here we have taken the difference of the corresponding measurments in the tables Angle 1 and Angle 2.

$2A$	
Vernier 1	Vernier 2
119°44'	119°43'
119°31'	119°33'
120°35'	120°8'

The half of the average of all these measurments comes out to be 59°56.2'
Hence $A = 59°56.2'$

6.2 Angle of Minimum Deviation

The angle of minimum deviation for a set of a particular colour was calculated by taking the average of all the measurments for that colour and substracting the angle of zero deviation from this average.

Colour	First Set	Second Set
Red	50°42.8'	50°43.5'
Yellow	51°5.5'	51°11.5'
Green	51°23'	51°31'
Blue	53°28.74'	53°37.5'
Violet	54°35.5'	54°32'

Taking the average of the measurments in the two columns gives the angles of minimum deviation for our measurments

Colour	δ_m
Red	50°43.1'
Yellow	51°8.5'
Green	51°27'
Blue	53°33.1'
Violet	54°33.8'

6.3 Refractive Index

The refractive indices for each colour calculated using Eq. 2, on the measured values of A and δ_m are shown below.

Colour	Refractive Index
Red	1.6464
Yellow	1.6506
Green	1.6537
Blue	1.6741
Violet	1.6837

6.4 Dispersive Power

The dispersive power for various colours is calculated below using Eq. 1. The first letter of each colour is used to denote the pair of colours for which the dispersive power is calculated. For instance, ω_{vr} would be the dispersive power for the prism when calculated for the violet and the red colours.

Colour Pair	Dispersive Power
ω_{vr}	0.0561
ω_{vg}	0.0449
ω_{by}	0.0354
ω_{gr}	0.0111

7 Error Analysis

In the formulas that follow, γ represents the least count of the spectrometer, which is 1'.

The error in a function $f(x_1, x_2, \ldots, x_k)$ where $x_1, x_2, \ldots, x_k$ are independent variables with errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_k$ respectively has been estimated as

$$\Delta f = \sqrt{\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \Delta x_i \right)^2} \tag{10}$$

Here I am assuming that all errors are independent and random.

7.1 Angle of Prism

Now

$$A = \frac{1}{2N}\left(\sum_{i=1}^{N}(u_{1i} - u_{2i}) + \sum_{i=1}^{N}(v_{1i} - v_{2i})\right) \tag{11}$$

where u_{1r} and u_{2r} represent the r^{th} readings of vernier 1, when the prism is in position 1 and 2 respectively, and v_{1r} and v_{2r} represent the r^{th} readings of vernier 2, when the prism is in position 1 and 2 respectively. N is the number of measurments taken for each vernier scale.

Using Eq. 10 on Eq. 11 gives the error in A as

$$\Delta A = \frac{\gamma}{2\sqrt{N}} \tag{12}$$

Taking $N = 3$ gives $\Delta A \approx 0.28'$. Hence we have,

$$A = 59°56.2' \pm 0.3'$$

The standard deviation in the value of A from the average of A is 11.3'.

The error percentage of A is 0.008%. The error percentage was calculated as $|\Delta x| \times 100/|x|$ where x is a measured variable.

7.2 Angle of Minimum Deviation

Setting up an equation for δ_m in terms of the measurements along similar lines as that in section 7.1, and calculating the error in δ_m gives

$$\Delta\delta_m = \frac{\gamma}{2N}\sqrt{N+1} \tag{13}$$

Taking $N = 3$ gives $\Delta\delta_m \approx 0.43'$. Hence we have

Colour	δ_m	Error Percentage
Red	50°43.1'±0.4′	0.0131%
Yellow	51°8.5'±0.4′	0.0130%
Green	51°27'±0.4′	0.0126%
Blue	53°33.1'±0.4′	0.0124%
Violet	54°33.8'±0.4′	0.0122%

The standard deviation of the angle of minimum deviation for each colour is shown below.

Colour	Standard Deviation of δ_m
Red	2.0'
Yellow	5.2'
Green	7.1'
Blue	5.1'
Violet	7.7'

7.3 Refractive Index

The following formula was derived using Eq. 10 to calculate the error in the refractive index.

$$\Delta n = \frac{1}{2}\sqrt{\csc^4\left(\frac{A}{2}\right)\sin^2\left(\frac{\delta_m}{2}\right)\Delta A^2 + \left[\cos\left(\frac{\delta_m}{2}\right)\cot\left(\frac{A}{2}\right) - \sin\left(\frac{\delta_m}{2}\right)\right]^2 \Delta\delta_m^2} \tag{14}$$

Using the measured values of A and δ_m, the error in the refractive indices is shown below.

Colour	Refractive Index	Error Percentage
Red	1.6464±0.0057	0.347%
Yellow	1.6506±0.0058	0.352%
Green	1.6537±0.0062	0.378%
Blue	1.6741±0.0104	0.623%
Violet	1.6837±0.0103	0.614%

7.4 Dispersive Power

The following formula was used to calculate the error in the dispersive power.

$$\Delta\omega = \frac{1}{(\tilde{n}-1)^2}\sqrt{(n_2-1)^2\Delta n_1^2 + (n_1-1)^2\Delta n_2^2} \quad \text{where} \quad \tilde{n} = \frac{n_1+n_2}{2} \tag{15}$$

Colour Pair	Dispersive Power	Error Percentage
ω_{vr}	0.056±0.018	32.15%
ω_{vg}	0.045±0.018	40.62%
ω_{by}	0.035±0.018	51.40%
ω_{gr}	0.011±0.013	117.60%

8 Precautions and Sources of Errors

1. Do not move the circular table on which the prism is placed. The circular vernier scale which is attached to the circular table should be moved if need be.

2. The measurments are extremely prone to parallax error, so try to take the measurments from right at the top of the vernier scales.

3. Make sure that while taking your readings, the MSRs are not too close to $0°$or $360°$, as in that case there is a possibility of the vernier scale crossing the 0 point and getting into the $360°$or $0°$range respectively, which would require you to add or substract $360°$from your readings.

4. Make sure that the collimator and the telescope are focused properly.

5. Do not try and clean the telescope eye piece from the inside as that is where the cross wires are.

6. Ensure that you are looking at the spectrum of mercury lamp and not looking at dispersed light coming from tubelights or windows. This can be checked by blocking the incident ray and seeing if the spectrum disappears. The spectrum of the mercury lamp is also if far more intense that these other sources of light. 7. Make sure that the prism table is of appropriate hight.

8. Once the angle of minimum deviation for a particular colour has been achieved, lock the prism table and measure the angles of minimum deviation for every other colour, without moving the prism table (or the prism).

9. Make sure that your hair don't touch the prism when you bend to read the vernier scale, as that may move the prism slightly.

9 Conclusion

We have measured the angle of prism and the anlges of minimum deviation for the colours red, yellow, green, blue and violet. The values of refractive indices are not self contradictory as they increase with the increase in frequency of light. The errors in the measured values of A and δ_m are quiet low.

We see a genral rising trend in the standard deviation of δ_m with the frequency of light. This might be because while taking my measurments, I started with red and ended at violet, dur to which the prism could have been moved slightly in the meantime (δ_m is extremely sensitive to any disturbances), resulting in higher variation.

We see that there is extremely high error in the dispersive powers of the higher frequency colours. These errors arise because the dispersive power formula requires us to take the difference of two quantities (the refractive indices) that are not too significantly different. Taking the difference of two comparable measurments increases the percentage error significantly, which is what we have seen here. This can not be avoided by taking more measurements with the same spectrometer. It can only be reduced by using a spectrometer with a lower least count.

10 Remarks

While measuring the angle of minimum deviation, we are asked to find move the prism table till one of the colours obtains its angle of minimum deviation, and measure the angles of all other colours subsequently *without* moving the prism table. The angle of zero deviation is measured once all the angles of minimum deviations have been measured this way.

This procedure is based on the incorrect assumption that all colours attain minimum deviation simultaneously (at the same angle of incidence). To see that different colours (with different refractive indices) attain minimum deviation at different we need only to look at Eq. 7 or Eq. 9. The angle of minimum deviation can be derived theoretically (as a funciton of the tilting angle or the incidence angle, depending upon which equation you choose) by taking the first derivative of δ with respect to I or θ (again, depends upon which equation you choose), setting it to zero, and solving the resultant equation for I or θ. We observe that there will always be a n in both the equations, meaning that the angle of incidence (or tilting angle) at which the minimum deviation occurs would depend on n, the refractive index, which depends upon the colour.

This can be seen more directily in a plot like the one in Fig. 5, where I have shown the angle of minimum deviation as a function of the tilting angle. The angle of minimum deviation (the minimum of these functions) would vary with n, as has been pointed out in section 4.3, right below Fig. 5.

To perform this experiment correctly, one would have to pick a colour, adjust the prism table till the angle of minimum deviation is attained, and then remove the prism and measure the angle of zero deviation and repeat the whole process again for a different colour. One would also have to take several measurments for a particular colour to reduce the random error as much as possible. This method (the correct method), however, is too time consuming to be performed within the time constraints imposed upon a student in a course lab.