Multi Qubit Quantum Teleportation and Associated Phenomenon
Final Report for OUR 2023

Kaustubh Singh
(2010110343)
Project Advisor: Dr. Kenji Nishiwaki

August 10, 2023

Abstract
I will begin by briefly going over the rudiments of quantum computation, leading up to the protocol for single qubit quantum teleportation. I will then explain, in detail, a method for two qubit teleportation, and give a similar method for teleporting an arbitrary number of qubits, before moving on to the case when there is decoherence in the entangled state used for the teleportation of two qubits and see how that affects the quantum teleportation protocol.

1 Preliminaries

First, I will lay out the basics of the theory needed to understand the problem, and some of the techniques used to approach their solutions. I will be using the bra-ket notation, which is common to most physics textbooks throughout.

1.1 Axioms of quantum mechanics

The following axioms are as stated in Nielsen and Chuang's canonical book on Quantum Computation [1] .

1.1.1 State-vector formulation

This is the more well-known formulation, as this is the one which a physics student is usually first introduced to:

Axiom 1A: Associated with any isolated physical system is an inner product space[1] over the field of complex numbers, called the *state space* of the system. The system is completely described by its *state vector*, which is a unit vector in the system's state space.

Axiom 2A: The time evolution of a closed quantum system is described by a unitary transformation. That is, if the state of a system at time t_1 is $|\psi\rangle$, then at time t_2, the state of the system would be

$$|\psi'\rangle = U(t_2, t_1)\,|\psi\rangle\,, \tag{1.1}$$

where $U(t_2, t_1)$ is a unitary operator.

This unitary operator is given by[2]

$$U(t_2, t_1) = \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right]. \tag{1.2}$$

Here H is hermitian operator called the Hamiltonian, and $\hbar$ is the reduced Planck's constant.

Axiom 3A: Quantum measurements are described by a set

$$\left\{M_m : m \in A\right\}$$

of measurement operators. Here[3] $A \subseteq \mathbb{R}$. There is bijective relation between the elements of A and all the possible measurement outcomes from the experiment. These operators act on the state space of the system being measured.

[1]Here, this vector space will always be finite dimensional.

[2]This, is true only when the Hamiltonian is time independent. If the Hamiltonian is time dependent, then the unitary time evolution operator is somewhat more complicated [2].

[3]For my purposes here, the set A will always be a subset of $\mathbb{Z}$. This description of the measurement postulate differs from its presentation in most introductory quantum mechanics books since in quantum computation, a more precise description of quantum measurements is needed.

If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that the result m occurs is given by

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle . \tag{1.3}$$

.

The state of the system after the measurement has been made is

$$\frac{M_m |\psi\rangle}{\sqrt{\langle\psi| M_m^\dagger M_m |\psi\rangle}}. \tag{1.4}$$

Additionally, the measurement operators satisfy the completeness equation,

$$\sum_{m \in A} M_m^\dagger M_m = I, \tag{1.5}$$

where I is the identity operator for that vector space.

Axiom 4A: The state space of a composite physical system (a system having more than one particle) is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and the system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

1.1.2 Density operator formulation

There is an alternate formulation of quantum mechanics which is equivalent to the state vector formulation, called the density operator formulation. This might be tedious to work with occasionally, but offers an edge while analysing subsystems of a quantum system, or when the state of a quantum system is not known exactly by probabilistically.

The axioms of quantum mechanics written in terms of the density operator formulation are given below:

Axiom 1B: Associated with any isolated physical system is an inner product space[4] over the field of complex numbers, called the *state space* of the system. The system is described completely by its *density operator*, which is a positive operator with unit trace, acting on the state space of the system.

[4]Here, this vector space will always be finite dimensional.

If a quantum system is in the state ρ_i with probability p_i, then the density operator for the system is $\sum_i p_i \rho_i$.

Axiom 2B: The evolution of a closed quantum system is described by a unitary transformation. If the state is given by ρ at time t_1, and by ρ' at time t_2, and U is the unitary time evolution operator then

$$\rho' = U \rho U^\dagger. \tag{1.6}$$

Here the time evolutions operator U is the same one as in (1.2).

Axiom 3B: Quantum measurements are described by a set

$$\left\{ M_m : m \in A \right\} \tag{1.7}$$

of measurement operators where $A \subseteq \mathbb{R}$. There is bijective relation between the elements of A and all the possible measurement outcomes from the experiment.

If the state of the quantum system is ρ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = \mathrm{tr}(M_m^\dagger M_m \rho), \tag{1.8}$$

and the state of the system after the measurement is

$$\frac{M_m \rho M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)}. \tag{1.9}$$

The measurement operators satisfy the completeness equation,

$$\sum_{m \in A} M_m^\dagger M_m = I.$$

Note that these measurement operators are same as the ones in the state vector formulation.

Axiom 4B: The state space of a composite physical system is the tensor product of the state space of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state ρ_i, then the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$

The density operator formalism really shines when it is used to analyse subsystems of a quantum system. Say we have two physical systems A and B, whose combined states is described by the density operator ρ^{AB}. The reduced density operator for system A is defined by

$$\rho^A = \mathrm{tr}_B(\rho^{AB}), \tag{1.10}$$

where tr_B is a linear map of operators known as the *partial trace* over system B. The action of the partial trace on the state $|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|$ is given by

$$\begin{aligned}
\mathrm{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) &= |a_1\rangle\langle a_2| \, \mathrm{tr}(|b_1\rangle\langle b_2|) \\
&= |a_1\rangle\langle a_2| \, \mathrm{tr}\left(\sum_{n,m} |m\rangle\langle m|b_1\rangle\langle b_2|n\rangle\langle n| \right) \\
&= |a_1\rangle\langle a_2| \sum_n \langle n|b_1\rangle\langle b_2|n\rangle \\
&= |a_1\rangle\langle a_2| \langle b_2| \left(\sum_n |n\rangle\langle n| \right) |b_1\rangle \\
&= |a_1\rangle\langle a_2| \langle b_2|b_1\rangle .
\end{aligned} \tag{1.11}$$

Above, the states $\{|n\rangle\}_n$ form an orthonormal basis for the state space of system B.

1.1.3 Conversion between the two formulations

Say we are given a quantum system, but our knowledge of the state of the system is probabilistic; we know that it is in the state $|\psi_i\rangle$ with probability p_i where $i \in S$ with[5] $S \subseteq \mathbb{Z}$.

The set $\{(p_i, |\psi_i\rangle) : i \in S\}$ is called an *ensemble of pure states*, and the density operator corresponding to this ensemble of pure states is

$$\rho = \sum_{i \in S} p_i |\psi_i\rangle\langle\psi_i| . \tag{1.12}$$

If $j \in S$ and for every $i \in S$, $p_i = \delta_{ij}$ where δ_{ij} is the Kronecker delta, then the corresponding density operator is said to be a *pure state*, otherwise ρ is said to be a *mixed state*.

[5] For my purposes here, the set S will always be finite.

If we have two ensembles of pure states, $\{(p_i, |\psi_i\rangle) : i \in S_1\}$ and $\{(p'_i, |\phi_i\rangle) : i \in S_2\}$, then the density operator corresponding to both these ensembles is the same if and only if [3]

$$|\tilde{\psi}_i\rangle = \sum_{j \in S_2} u_{ij} |\tilde{\phi}_j\rangle ,$$

for every i in S_1. Here $|\tilde{\psi}_i\rangle := \sqrt{p_i}\,|\psi_i\rangle$, $|\tilde{\phi}_i\rangle := \sqrt{p'_i}\,|\phi_i\rangle$, and u_{ij} are elements of a unitary matrix[6] $S \subseteq \mathbb{Z}$, and we pad the ensemble corresponding to whichever set S_1 or S_2 has lesser elements with additional 0 vectors so that the two sets have the same number of elements.

1.2 Entanglement

A *qubit* is a quantum system, whose state space can be spanned by two orthonormal vectors, usually denoted by $|0\rangle$ and $|1\rangle$. Equivalently, we can define a qubit as a quantum system, for which there exists a set of measurement operators (1.7), with only two elements[7]. These measurement operators would be $M_0 := |0\rangle \langle 0|$ and $M_1 := |1\rangle \langle 1|$. Qubits are used to express quantum information.

Now if we have a system composed of two qubits, then the state space of the composite system will be a tensor product of the individual state spaces. That is, it will be spanned by the set $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Here, $|01\rangle := |0\rangle \otimes |1\rangle$, denotes the tensor product of $|0\rangle$ and $|1\rangle$. I will be omitting the $\otimes$ while writing the tensor product of states for the sake of brevity.

So in general, the state of a quantum system could be expressed in the form

$$|\psi\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle ,$$

with $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$.

Sometimes, it is possible to write $|\psi\rangle$ as

$$|\psi\rangle = (a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle).$$

[6]Here, it should be understood that both the sets S_1 and S_2 have their least element as 1, and there should only be one $i \in S_1$ (or S_2) such that $i + 1 \notin S_1$ (or S_2).

[7]An example of an of a quantum system which has an experimentally observable property with only two possible outcomes is the electron, the property being the spin angular momentum.

Which can be interpreted as having the first qubit in the state $a|0\rangle + b|1\rangle$ and the second qubit in the state $c|0\rangle + d|1\rangle$.

However, this is not always possible. For instance, consider the states

$$|\Omega_\pm\rangle = \frac{1}{\sqrt{2}}\big(\,|00\rangle \pm |11\rangle\,\big), \tag{1.13}$$

$$|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}\big(\,|01\rangle \pm |10\rangle\,\big). \tag{1.14}$$

These states, called the *Bell states*[8] or EPR states, cannot be separated as the tensor product of two single particle states (qubits) like how we did above.

States such as these, which cannot be expressed as a tensor product of two or more states are called entangled states.

To give a better description of entangled quantum states, I will first define a *maximally mixed state*. A state ρ, acting on a d dimensional state space is said to be maximally mixed, if and only if $\rho = I/d$. Crudely, this means there is an equal probability for it to be in any one of the basis states, so this is the most 'mixed' a state can be in the sense of (1.12).

A bipartite state, ρ^{AB} (a state which can be looked at as being composed of two subsystems) is said to be *maximally entangled* if the reduced density operator ρ^A and ρ^B are maximally mixed. Roughly, if a quantum system is in an entangled state, it is not possible to know the state of its subsystems independently of each other; the state of the subsystems is known only probabilistically until maybe a measurement is made on one of the subsystems. Even then, sometimes the post measurement state can be entangled[9]. Examples of maximally mixed states are the Bell states in (1.13) and (1.14).

2 Quantum teleportation for one qubit

Here I will explain the procedure for teleporting one qubit, which will serve as a starting off point for us to delve into multi qubit quantum teleportation.

The problem of single qubit quantum teleportation was first published in 1993 [5]. The problem of single qubit quantum teleportation will be explained using two fictional characters Alice and Bob. Say Alice and Bob start out with each having a qubit from the Bell state $|\Omega_+\rangle$ (defined in (1.13)). Now

[8]Bell states are crucial in single qubit quantum teleportation and give an idea for how multi qubit quantum teleportation can be carried out. They can also be shown to form a basis for the state space they reside in.

[9]For instance, the W state [4].

Alice and Bob are far apart, and can only communicate through classical communication channels, that is they can only send classical information, bits to each other. If Alice has a qubit $|\psi\rangle$, is it possible for her to send this exact qubit to Bob? Note that unlike classical information, it is not always possible to copy quantum information [6,7], so she won't be able to keep her qubit intact; her qubit would be destroyed, while Bob would get an exact copy of it.

The solution to the problem hinges on a measurement in the Bell basis. Say

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle ,$$

with $|\alpha|^2 + |\beta|^2 = 1$.

Then the state of the entire system would be given by

$$|\Psi\rangle = |\psi\rangle |\Omega_+\rangle .$$

Expanding this out, we get

$$\begin{aligned}
|\Psi\rangle &= |\psi\rangle |\Omega_+\rangle \\
&= \frac{1}{\sqrt{2}}(\alpha |0\rangle_A + \beta |1\rangle_A)(|00\rangle_{AB} + |11\rangle_{AB}) \\
&= \frac{\alpha}{\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\beta}{\sqrt{2}}(|100\rangle + |111\rangle) \\
&= \frac{\alpha}{2}\Big[\Big(|\Omega_+\rangle + |\Omega_-\rangle \Big) |0\rangle + \Big(|\Phi_+\rangle + |\Phi_-\rangle \Big) |1\rangle \Big] \\
&\quad + \frac{\beta}{2}\Big[\Big(|\Phi_+\rangle - |\Phi_-\rangle \Big) |0\rangle + \Big(|\Omega_+\rangle - |\Omega_-\rangle \Big) |1\rangle \Big] \\
&= \frac{1}{2}\Big[|\Omega_+\rangle_{AA} \Big(\alpha |0\rangle_B + \beta |1\rangle_B \Big) + |\Omega_-\rangle_{AA} \Big(\alpha |0\rangle_B - \beta |1\rangle_B \Big) \\
&\quad + |\Phi_+\rangle_{AA} \Big(\beta |0\rangle_B + \alpha |1\rangle_B \Big) + |\Phi_-\rangle_{AA} \Big(- \beta |0\rangle_B + \alpha |1\rangle_B \Big)\Big]
\end{aligned} \tag{2.1}$$

The subscripts clarify which qubit belongs to whom. Now when Alice makes a measurement in the Bell basis (the set of measurement operators being $\{|\Omega_+\rangle \langle\Omega_+| , |\Omega_-\rangle \langle\Omega_-| , |\Phi_+\rangle \langle\Phi_+| , |\Phi_-\rangle \langle\Phi_-|\}$), she will get one of the four Bell states with equal probability, and Bob's qubit will collapse into whichever state is tensored with the Alice's post measurement Bell state in $|\Psi\rangle$ above. Depending on which Bell state Alice gets upon making a measurement, she can send Bob two numbers, which will tell him which unitary operations to apply to 'fix up' his state to get $|\psi\rangle$. The unitary operators which Bob will use can be the Pauli spin operators, σ_3 and σ_1[10]. A table specifying which operators Bob should use depending on Alice's outcome is given below

Measurement Outcome	Operator
Ω_+	I
Ω_-	σ_3
Φ_+	σ_1
Φ_-	$\sigma_3\sigma_1$

Table 1:
The unitary operators Bob would have to apply to his qubit to get to become $|\psi\rangle$ depending on Alice's measurement outcome.

3 Quantum teleportation for multiple qubits

There seem to be several ways one could teleport multiple qubits, each varying in either fidelity, the entangled states used, the kind of states which can be teleported or reliability [8–13].

Here, I will present a method for teleporting two qubits in an arbitrary state with unit fidelity and probability.

Say the state to be teleported is

$$|\psi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle\,,$$

with $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. I will use two Bell states, specifically two $|\Omega_+\rangle$ states. Alice and Bob will each have one qubit from both of these Bell states. The complete state of all the qubits would be

$$|\Psi\rangle = |\psi\rangle_{A_1A_2}\,|\Omega_+\rangle_{A_3B_1}\,|\Omega_+\rangle_{A_4B_2}\,.$$

I have labelled the qubits to specify which qubits belong to whom. Qubits with A as their subscripts belong to Alice and with B belong to Bob. Expanding this out,

$$|\Psi\rangle = \frac{1}{2}(a\,|00\rangle_{A_1A_2}+b\,|01\rangle_{A_1A_2}+c\,|10\rangle_{A_1A_2}+d\,|11\rangle_{A_1A_2})(|00\rangle_{A_3B_1}+|11\rangle_{A_3B_1})(|00\rangle_{A_4B_2}+|11\rangle_{A_4B_2}) \tag{3.1}$$

[10] $\sigma_1\,|0\rangle = |1\rangle$, $\sigma_1\,|1\rangle = |0\rangle$ and $\sigma_3\,|0\rangle = |0\rangle$, $\sigma_3\,|1\rangle = -\,|1\rangle$.

Now, as we did for single qubit teleportation, we want to group Alice's qubits and Bob's qubits together. So rearranging a bit, $|\Psi\rangle$ is equal to

$$\frac{1}{2}\Bigg[a\Bigg(|0000\rangle|00\rangle+|0001\rangle|01\rangle+|0010\rangle|10\rangle+|0011\rangle|11\rangle\Bigg)$$
$$+b\Bigg(|0100\rangle|00\rangle+|0101\rangle|01\rangle+|0110\rangle|10\rangle+|0111\rangle|11\rangle\Bigg)$$
$$+c\Bigg(|1000\rangle|00\rangle+|1001\rangle|01\rangle+|1010\rangle|10\rangle+|1011\rangle|11\rangle\Bigg)$$
$$+d\Bigg(|1100\rangle|00\rangle+|1101\rangle|01\rangle+|1110\rangle|10\rangle+|1111\rangle|11\rangle\Bigg)\Bigg].$$

Above, the qubits are rearranged in the form $|A_1A_2A_3A_4\rangle|B_1B_2\rangle$. It is of importance to note which qubits are entangled. Looking at (3.1) we can tell that qubits A_3 and B_1 are entangled and qubits A_4 and B_2 are entangled. We will be making two measurements here in the Bell basis, and we need to ensure that in each of these measurement, we measure one of the qubits of $|\psi\rangle$ (which would be in the A_1 and A_2 slots) and one of the entangled qubits (the ones in the A_3 and A_4 slots). Hence it would be beneficial for us to write the qubits in the A_1 and A_3 slots together and the qubits in the A_2 and A_4 slots together. Making this change, $|\Psi\rangle$ is equal to

$$\begin{aligned}\frac{1}{2}\Bigg[&a\Bigg(|00\rangle|00\rangle|00\rangle+|00\rangle|01\rangle|01\rangle+|01\rangle|00\rangle|10\rangle+|01\rangle|01\rangle|11\rangle\Bigg)\\&+b\Bigg(|00\rangle|10\rangle|00\rangle+|00\rangle|11\rangle|01\rangle+|01\rangle|10\rangle|10\rangle+|01\rangle|11\rangle|11\rangle\Bigg)\\&+c\Bigg(|10\rangle|00\rangle|00\rangle+|10\rangle|01\rangle|01\rangle+|11\rangle|00\rangle|10\rangle+|11\rangle|01\rangle|11\rangle\Bigg)\\&+d\Bigg(|10\rangle|10\rangle|00\rangle+|10\rangle|11\rangle|01\rangle+|11\rangle|10\rangle|10\rangle+|11\rangle|11\rangle|11\rangle\Bigg)\Bigg].\end{aligned} \tag{3.2}$$

Above, the qubits are arranged in the form $|A_1A_3\rangle|A_2A_4\rangle|B_1B_2\rangle$. Now we will rewrite the the $|A_1A_3\rangle$ qubits and the $|A_2A_4\rangle$ qubits in terms of the Bell states. Then $|\Psi\rangle$ is equal to

$$\frac{1}{4}\Bigg[a\Bigg\{\Bigg(|\Omega_+\rangle+|\Omega_-\rangle\Bigg)\Bigg(|\Omega_+\rangle+|\Omega_-\rangle\Bigg)|00\rangle+\Bigg(|\Omega_+\rangle+|\Omega_-\rangle\Bigg)\Bigg(|\Phi_+\rangle+|\Phi_-\rangle\Bigg)|01\rangle$$
$$+\Bigg(|\Phi_+\rangle+|\Phi_-\rangle\Bigg)\Bigg(|\Omega_+\rangle+|\Omega_-\rangle\Bigg)|10\rangle+\Bigg(|\Phi_+\rangle+|\Phi_-\rangle\Bigg)\Bigg(|\Phi_+\rangle+|\Phi_-\rangle\Bigg)|11\rangle\Bigg\}$$

$$b\Bigg\{\Bigg(|\Omega_+\rangle+|\Omega_-\rangle\Bigg)\Bigg(|\Phi_+\rangle-|\Phi_-\rangle\Bigg)|00\rangle+\Bigg(|\Omega_+\rangle+|\Omega_-\rangle\Bigg)\Bigg(|\Omega_+\rangle-|\Omega_-\rangle\Bigg)|01\rangle$$
$$+\Bigg(|\Phi_+\rangle+|\Phi_-\rangle\Bigg)\Bigg(|\Phi_+\rangle-|\Phi_-\rangle\Bigg)|10\rangle+\Bigg(|\Phi_+\rangle+|\Phi_-\rangle\Bigg)\Bigg(|\Omega_+\rangle-|\Omega_-\rangle\Bigg)|11\rangle\Bigg\}$$

$$c\Bigg\{\Bigg(|\Phi_+\rangle-|\Phi_-\rangle\Bigg)\Bigg(|\Omega_+\rangle+|\Omega_-\rangle\Bigg)|00\rangle+\Bigg(|\Phi_+\rangle-|\Phi_-\rangle\Bigg)\Bigg(|\Phi_+\rangle+|\Phi_-\rangle\Bigg)|01\rangle$$
$$+\Bigg(|\Omega_+\rangle-|\Omega_-\rangle\Bigg)\Bigg(|\Omega_+\rangle+|\Omega_-\rangle\Bigg)|10\rangle+\Bigg(|\Omega_+\rangle-|\Omega_-\rangle\Bigg)\Bigg(|\Phi_+\rangle+|\Phi_-\rangle\Bigg)|11\rangle\Bigg\}$$

$$d\Bigg\{\Bigg(|\Phi_+\rangle-|\Phi_-\rangle\Bigg)\Bigg(|\Phi_+\rangle-|\Phi_-\rangle\Bigg)|00\rangle+\Bigg(|\Phi_+\rangle-|\Phi_-\rangle\Bigg)\Bigg(|\Omega_+\rangle-|\Omega_-\rangle\Bigg)|01\rangle$$
$$+\Bigg(|\Omega_+\rangle-|\Omega_-\rangle\Bigg)\Bigg(|\Phi_+\rangle-|\Phi_-\rangle\Bigg)|10\rangle+\Bigg(|\Omega_+\rangle-|\Omega_-\rangle\Bigg)\Bigg(|\Omega_+\rangle-|\Omega_-\rangle\Bigg)|11\rangle\Bigg\}\Bigg].$$

Now I will open all the parentheses and group all terms with the same Bell state products together. Above, qubits are arranged in the same order as in

(3.2).

$$
\begin{aligned}
|\Psi\rangle = \frac{1}{4}\Bigg[& |\Omega_+\rangle\,|\Omega_+\rangle \Big(a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle \Big) \\
& + |\Omega_+\rangle\,|\Omega_-\rangle \Big(a\,|00\rangle - b\,|01\rangle + c\,|10\rangle - d\,|11\rangle \Big) \\
& + |\Omega_+\rangle\,|\Phi_+\rangle \Big(a\,|01\rangle + b\,|00\rangle + c\,|11\rangle + d\,|10\rangle \Big) \\
& + |\Omega_+\rangle\,|\Phi_-\rangle \Big(a\,|01\rangle - b\,|00\rangle + c\,|11\rangle - d\,|10\rangle \Big) \\
\\
& + |\Omega_-\rangle\,|\Omega_+\rangle \Big(a\,|00\rangle + b\,|01\rangle - c\,|10\rangle - d\,|11\rangle \Big) \\
& + |\Omega_-\rangle\,|\Omega_-\rangle \Big(a\,|00\rangle - b\,|01\rangle - c\,|10\rangle + d\,|11\rangle \Big) \\
& + |\Omega_-\rangle\,|\Phi_+\rangle \Big(a\,|01\rangle + b\,|00\rangle - c\,|11\rangle - d\,|10\rangle \Big) \\
& + |\Omega_-\rangle\,|\Phi_-\rangle \Big(a\,|01\rangle - b\,|00\rangle - c\,|11\rangle + d\,|10\rangle \Big) \\
\\
& + |\Phi_+\rangle\,|\Omega_+\rangle \Big(a\,|10\rangle + b\,|11\rangle + c\,|00\rangle + d\,|01\rangle \Big) \\
& + |\Phi_+\rangle\,|\Omega_-\rangle \Big(a\,|10\rangle - b\,|11\rangle + c\,|00\rangle - d\,|01\rangle \Big) \\
& + |\Phi_+\rangle\,|\Phi_+\rangle \Big(a\,|11\rangle + b\,|10\rangle + c\,|01\rangle + d\,|00\rangle \Big) \\
& + |\Phi_+\rangle\,|\Phi_-\rangle \Big(a\,|11\rangle - b\,|10\rangle + c\,|01\rangle - d\,|00\rangle \Big) \\
\\
& + |\Phi_-\rangle\,|\Omega_+\rangle \Big(a\,|10\rangle + b\,|11\rangle - c\,|00\rangle - d\,|01\rangle \Big) \\
& + |\Phi_-\rangle\,|\Omega_-\rangle \Big(a\,|10\rangle - b\,|11\rangle - c\,|00\rangle + d\,|01\rangle \Big) \\
& + |\Phi_-\rangle\,|\Phi_+\rangle \Big(a\,|11\rangle + b\,|10\rangle - c\,|01\rangle - d\,|00\rangle \Big) \\
& + |\Phi_-\rangle\,|\Phi_-\rangle \Big(a\,|11\rangle - b\,|10\rangle - c\,|01\rangle + d\,|00\rangle \Big)\Bigg].
\end{aligned}
\tag{3.3}
$$

Now Alice would make two Bell measurements on her four qubits. The act of measurement would cause Bob's qubits on the left of (3.3) to collapse into the corresponding state, which could be fixed up depending upon Alice's measurement result, just like in the case for single qubit teleportation, using Pauli spin operators.

Alice's Measurement Result	Bob's operation
$\|\Omega_+\rangle \|\Omega_+\rangle$	I
$\|\Omega_+\rangle \|\Omega_-\rangle$	σ_2^z
$\|\Omega_+\rangle \|\Phi_+\rangle$	σ_2^x
$\|\Omega_+\rangle \|\Phi_-\rangle$	$\sigma_2^z\sigma_2^x$
$\|\Omega_-\rangle \|\Omega_+\rangle$	σ_1^z
$\|\Omega_-\rangle \|\Omega_-\rangle$	$\sigma_1^z\sigma_2^z$
$\|\Omega_-\rangle \|\Phi_+\rangle$	$\sigma_1^z\sigma_2^x$
$\|\Omega_-\rangle \|\Phi_-\rangle$	$\sigma_1^z\sigma_2^z\sigma_2^x$
$\|\Phi_+\rangle \|\Omega_+\rangle$	σ_1^x
$\|\Phi_+\rangle \|\Omega_-\rangle$	$\sigma_1^x\sigma_2^z$
$\|\Phi_+\rangle \|\Phi_+\rangle$	$\sigma_1^x\sigma_2^x$
$\|\Phi_+\rangle \|\Phi_-\rangle$	$\sigma_1^x\sigma_2^z\sigma_2^x$
$\|\Phi_-\rangle \|\Omega_+\rangle$	$\sigma_1^z\sigma_1^x$
$\|\Phi_-\rangle \|\Omega_-\rangle$	$\sigma_1^z\sigma_1^x\sigma_2^z$
$\|\Phi_-\rangle \|\Phi_+\rangle$	$\sigma_1^z\sigma_1^x\sigma_2^x$
$\|\Phi_-\rangle \|\Phi_-\rangle$	$\sigma_1^z\sigma_1^x\sigma_2^z\sigma_2^x$

Table 2:
The unitary operators Bob would have to apply to his qubit to get to become $|\psi\rangle$ depending on Alice's measurement outcome. The subscript specifies the qubit on which the operator will act on (from the left). The superscript tells which Pauli operator to apply. There is a simply way to implement this. $|\Omega_+\rangle$ corresponds to I, $|\Omega_-\rangle$ corresponds to σ^z, $|\Phi_+\rangle$ corresponds to σ^x, and $|\Phi_-\rangle$ corresponds to $\sigma^z\sigma^x$.

This same procedure could be applied to teleport N number of qubits in an arbitrary state. We would need N pairs of qubits in the one of the Bell states, with N of those qubits[11] being with Alice (in addition to the N qubits she wants to send to Bob, so a total of $2N$ qubits) and the rest of the entangled N qubits would be with Bob. Then Alice would make pairs out of the qubits which are part of a Bell state and the qubits she want to teleport before performing a Bell measurement on each of those pairs. These

[11] These qubits would not be entangled with each other. For every Bell state, Alice and Bob will each share a qubit.

measurements would destroy the entanglement between Alice's and Bob's qubits, and would simultaneously force Bob's qubits into an N qubits state which could be converted into the state which they want to teleport by the application of Pauli's spin operators. The operators to be applied, would, as before be dependent on what Alice's measurement outcomes are.

4 Quantum Teleportation for two qubits with an environmental interaction

Here I will explore the case when there is decoherence in the Bell states used for bidirectional two qubit teleportation. The decoherence will be introduced in the form of an interaction with the environment with a pure dephasing Hamiltonian [14]. In my analysis, I will look at the state of the qubits as well as the environment to keep track of the transfer of entanglement and information, as done in [15] for a single qubit. A significant difference in my approach would be to use the state-vector formulation instead of density operators since it is more convenient when dealing with a greater number of qubits.

4.1 Teleportation using a decohered state

I will assume that the pure-dephasing Hamiltonian acts only on one Bell state pair, for the sake of simplicity. The interacting Hamiltonian is taken to be

$$H = \sum_{i,j=0,1} |ij\rangle_{A_3B_1} \langle ij|_{A_3B_1} \otimes V_{ij}, \tag{4.1}$$

where V_{ij} is a Hermitian operator acting on the state space of the environment. I am using the same notation for labelling the qubits as in (3.1). Now I will calculate the unitary time evolution operator.

First, we define the denote the eigenvectors and eigenvalues of V_{ij} by $|n_{ij}\rangle$ and E_{ij} respectively,

$$V_{ij} |n_{ij}\rangle = E_{ij} |n_{ij}\rangle . \tag{4.2}$$

By (1.2), and the definition of functions of operators as given in Nielsen and

Chuang's book, we get the time evolution operator to be

$$\begin{aligned}U(t) &= \sum_{i,j}\sum_{n_{ij}} e^{-iE_{ij}t/\hbar} \left|ij\right\rangle_{A_3B_1} \left\langle ij\right| \otimes \left|n_{ij}\right\rangle \left\langle n_{ij}\right| \\ &= \sum_{i,j} \left|ij\right\rangle \left\langle ij\right| \otimes \sum_{n_{ij}} e^{-iE_{ij}t/\hbar} \left|n_{ij}\right\rangle \left\langle n_{ij}\right| \\ &= \sum_{i,j} \left|ij\right\rangle \left\langle ij\right| \otimes \sum_{n_{ij}} e^{-iV_{ij}t/\hbar} \left|n_{ij}\right\rangle \left\langle n_{ij}\right| \\ &= \sum_{i,j} \left|ij\right\rangle \left\langle ij\right| \otimes e^{-iV_{ij}t/\hbar} \sum_{n_{ij}} \left|n_{ij}\right\rangle \left\langle n_{ij}\right| \\ &= \sum_{i,j} \left|ij\right\rangle \left\langle ij\right| \otimes e^{-iV_{ij}t/\hbar}.\end{aligned} \tag{4.3}$$

Above, we have used the fact that the eigenvectors of any observable span the entire state space. Also it is understood that the outer product is composed of the A_3B_1 qubits, as shown explicitly in the first line.

The initial state of the system and the environment is given by

$$\left|\Psi\right\rangle = \left|\psi\right\rangle_{A_1A_2} \left|\Omega_+\right\rangle_{A_3B_1} \left|\Omega_+\right\rangle_{A_4B_2} \left|\chi\right\rangle_E . \tag{4.4}$$

The subscript E denotes the state of the environment. We have assumed that the environment is in a product state with the qubits. That is, initially the correlations between the system of qubits and the environment are entirely classical. Now we let the A_3B_1 qubits interact with the environment via the pure-dephasing Hamiltonian for time τ. Then,

$$\begin{aligned}\left|\Psi(\tau)\right\rangle &= \left|\psi\right\rangle_{A_1A_2} \otimes \left(\sum_{i,j} \left|ij\right\rangle_{A_3B_1} \left\langle ij|\Omega_+\right\rangle_{A_3B_1} e^{-iV_{ij}\tau/\hbar} \left|\chi\right\rangle_E \right) \otimes \left|\Omega_+\right\rangle_{A_4B_2} \\ &= \left|\psi\right\rangle_{A_1A_2} (\left|00\right\rangle \left|\chi_0\right\rangle + \left|11\right\rangle \left|\chi_1\right\rangle)_{A_3B_1E} \left|\Omega_+\right\rangle_{A_4B_2} .\end{aligned} \tag{4.5}$$

Above, I have defined

$$\left|\chi_i\right\rangle := e^{-iV_{ii}\tau/\hbar} \left|\chi\right\rangle . \tag{4.6}$$

Now I will reorder the qubits so that they are written in the following order:

$|A_1A_3\rangle\,|A_2A_4\rangle\,|B_1B_2\rangle\,|E\rangle$.

$$|\Psi(\tau)\rangle = \frac{1}{2}\Big[a(|00\rangle\,|00\rangle\,|00\rangle\,|\chi_0\rangle + |00\rangle\,|01\rangle\,|01\rangle\,|\chi_0\rangle + |01\rangle\,|00\rangle\,|10\rangle\,|\chi_1\rangle + |01\rangle\,|01\rangle\,|11\rangle\,|\chi_1\rangle)$$
$$+b(|00\rangle\,|10\rangle\,|00\rangle\,|\chi_0\rangle + |00\rangle\,|11\rangle\,|01\rangle\,|\chi_0\rangle + |01\rangle\,|10\rangle\,|10\rangle\,|\chi_1\rangle + |01\rangle\,|11\rangle\,|11\rangle\,|\chi_1\rangle)$$
$$+c(|10\rangle\,|00\rangle\,|00\rangle\,|\chi_0\rangle + |10\rangle\,|01\rangle\,|01\rangle\,|\chi_0\rangle + |11\rangle\,|00\rangle\,|10\rangle\,|\chi_1\rangle + |11\rangle\,|01\rangle\,|11\rangle\,|\chi_1\rangle)$$
$$+d(|10\rangle\,|10\rangle\,|00\rangle\,|\chi_0\rangle + |10\rangle\,|11\rangle\,|01\rangle\,|\chi_0\rangle + |11\rangle\,|10\rangle\,|10\rangle\,|\chi_1\rangle + |11\rangle\,|11\rangle\,|11\rangle\,|\chi_1\rangle)\Big]. \tag{4.7}$$

Now rewriting the $|A_1A_3\rangle$ qubits and the $|A_2A_4\rangle$ qubits in terms of the Bell states, and rearranging we get

$$
\begin{aligned}
|\Psi(\tau)\rangle = \frac{1}{4}\Bigg[\, & |\Omega_+\rangle\,|\Omega_+\rangle \Big(a\,|00\rangle\,|\chi_0\rangle + b\,|01\rangle\,|\chi_0\rangle + c\,|10\rangle\,|\chi_1\rangle + d\,|11\rangle\,|\chi_1\rangle \Big) \\
& + |\Omega_+\rangle\,|\Omega_-\rangle \Big(a\,|00\rangle\,|\chi_0\rangle - b\,|01\rangle\,|\chi_0\rangle + c\,|10\rangle\,|\chi_1\rangle - d\,|11\rangle\,|\chi_1\rangle \Big) \\
& + |\Omega_+\rangle\,|\Phi_+\rangle \Big(a\,|01\rangle\,|\chi_0\rangle + b\,|00\rangle\,|\chi_0\rangle + c\,|11\rangle\,|\chi_1\rangle + d\,|10\rangle\,|\chi_1\rangle \Big) \\
& + |\Omega_+\rangle\,|\Phi_-\rangle \Big(a\,|01\rangle\,|\chi_0\rangle - b\,|00\rangle\,|\chi_0\rangle + c\,|11\rangle\,|\chi_1\rangle - d\,|10\rangle\,|\chi_1\rangle \Big) \\
\\
& + |\Omega_-\rangle\,|\Omega_+\rangle \Big(a\,|00\rangle\,|\chi_0\rangle + b\,|01\rangle\,|\chi_0\rangle - c\,|10\rangle\,|\chi_1\rangle - d\,|11\rangle\,|\chi_1\rangle \Big) \\
& + |\Omega_-\rangle\,|\Omega_-\rangle \Big(a\,|00\rangle\,|\chi_0\rangle - b\,|01\rangle\,|\chi_0\rangle - c\,|10\rangle\,|\chi_1\rangle + d\,|11\rangle\,|\chi_1\rangle \Big) \\
& + |\Omega_-\rangle\,|\Phi_+\rangle \Big(a\,|01\rangle\,|\chi_0\rangle + b\,|00\rangle\,|\chi_0\rangle - c\,|11\rangle\,|\chi_1\rangle - d\,|10\rangle\,|\chi_1\rangle \Big) \\
& + |\Omega_-\rangle\,|\Phi_-\rangle \Big(a\,|01\rangle\,|\chi_0\rangle - b\,|00\rangle\,|\chi_0\rangle - c\,|11\rangle\,|\chi_1\rangle + d\,|10\rangle\,|\chi_1\rangle \Big) \\
\\
& + |\Phi_+\rangle\,|\Omega_+\rangle \Big(a\,|10\rangle\,|\chi_1\rangle + b\,|11\rangle\,|\chi_1\rangle + c\,|00\rangle\,|\chi_0\rangle + d\,|01\rangle\,|\chi_0\rangle \Big) \\
& + |\Phi_+\rangle\,|\Omega_-\rangle \Big(a\,|10\rangle\,|\chi_1\rangle - b\,|11\rangle\,|\chi_1\rangle + c\,|00\rangle\,|\chi_0\rangle - d\,|01\rangle\,|\chi_0\rangle \Big) \\
& + |\Phi_+\rangle\,|\Phi_+\rangle \Big(a\,|11\rangle\,|\chi_1\rangle + b\,|10\rangle\,|\chi_1\rangle + c\,|01\rangle\,|\chi_0\rangle + d\,|00\rangle\,|\chi_0\rangle \Big) \\
& + |\Phi_+\rangle\,|\Phi_-\rangle \Big(a\,|11\rangle\,|\chi_1\rangle - b\,|10\rangle\,|\chi_1\rangle + c\,|01\rangle\,|\chi_0\rangle - d\,|00\rangle\,|\chi_0\rangle \Big) \\
\\
& + |\Phi_-\rangle\,|\Omega_+\rangle \Big(a\,|10\rangle\,|\chi_1\rangle + b\,|11\rangle\,|\chi_1\rangle - c\,|00\rangle\,|\chi_0\rangle - d\,|01\rangle\,|\chi_0\rangle \Big) \\
& + |\Phi_-\rangle\,|\Omega_-\rangle \Big(a\,|10\rangle\,|\chi_1\rangle - b\,|11\rangle\,|\chi_1\rangle - c\,|00\rangle\,|\chi_0\rangle + d\,|01\rangle\,|\chi_0\rangle \Big) \\
& + |\Phi_-\rangle\,|\Phi_+\rangle \Big(a\,|11\rangle\,|\chi_1\rangle + b\,|10\rangle\,|\chi_1\rangle - c\,|01\rangle\,|\chi_0\rangle - d\,|00\rangle\,|\chi_0\rangle \Big) \\
& + |\Phi_-\rangle\,|\Phi_-\rangle \Big(a\,|11\rangle\,|\chi_1\rangle - b\,|10\rangle\,|\chi_1\rangle - c\,|01\rangle\,|\chi_0\rangle + d\,|00\rangle\,|\chi_0\rangle \Big)\Bigg].
\end{aligned}
\tag{4.8}
$$

Now we can see clearly that once Alice makes her Bell measurements, two things will occur simultaneously. The qubit she wanted to teleport would be teleported to Bob's qubits and the environment and Bob's qubits and the environment state would no longer be separable.

4.2 Teleportation of decohered state

Assuming, for the sake of simplicity, that the post measurement state of Alice's qubits is $|\Omega_+\rangle|\Omega_+\rangle$, the post measurement state of $|\Psi(\tau)\rangle$ would be

$$|\Psi_{PM}\rangle = |\Omega_+\rangle_{A_1A_3}|\Omega_+\rangle_{A_2A_4}\Big(a|00\rangle|\chi_0\rangle+b|01\rangle|\chi_0\rangle+c|10\rangle|\chi_1\rangle+d|11\rangle|\chi_1\rangle\Big)_{B_1B_2E}. \tag{4.9}$$

Now we will perform teleportation in the reverse direction, sending this decohered state back to Alice, without having the dephasing Hamiltonian act on our Bell states. Reordering the qubits of $|\Psi_{PM}(\tau)\rangle$ yet again so that the order of the qubits is $|A_1A_2\rangle|A_3B_1\rangle|A_4B_2\rangle|E\rangle$, we get

$$\begin{aligned}|\Psi_{PM}\rangle = \frac{1}{2}\Big[&|00\rangle\Big(a|00\rangle|00\rangle|\chi_0\rangle+b|00\rangle|01\rangle|\chi_0\rangle+c|01\rangle|00\rangle|\chi_1\rangle+d|01\rangle|01\rangle|\chi_1\rangle\Big)\\ &+|01\rangle\Big(a|00\rangle|10\rangle|\chi_0\rangle+b|00\rangle|11\rangle|\chi_0\rangle+c|01\rangle|10\rangle|\chi_1\rangle+d|01\rangle|11\rangle|\chi_1\rangle\Big)\\ &+|10\rangle\Big(a|10\rangle|00\rangle|\chi_0\rangle+b|10\rangle|01\rangle|\chi_0\rangle+c|11\rangle|00\rangle|\chi_1\rangle+d|11\rangle|01\rangle|\chi_1\rangle\Big)\\ &+|11\rangle\Big(a|10\rangle|10\rangle|\chi_0\rangle+b|10\rangle|11\rangle|\chi_0\rangle+c|11\rangle|10\rangle|\chi_1\rangle+d|11\rangle|11\rangle|\chi_1\rangle\Big)\Big].\end{aligned} \tag{4.10}$$

Expressing the A_3B_1 qubits and the A_4B_2 qubits in terms of the Bell states,

$$
\begin{aligned}
|\Psi_{PM}\rangle = \frac{1}{4}\Bigg[& \bigg(a\,|00\rangle\,|\chi_0\rangle + b\,|01\rangle\,|\chi_0\rangle + c\,|10\rangle\,|\chi_1\rangle + d\,|11\rangle\,|\chi_1\rangle \bigg)\,|\Omega_+\rangle\,|\Omega_+\rangle \\
& + \bigg(a\,|00\rangle\,|\chi_0\rangle - b\,|01\rangle\,|\chi_0\rangle + c\,|10\rangle\,|\chi_1\rangle - d\,|11\rangle\,|\chi_1\rangle \bigg)\,|\Omega_+\rangle\,|\Omega_-\rangle \\
& + \bigg(b\,|00\rangle\,|\chi_0\rangle + a\,|01\rangle\,|\chi_0\rangle + d\,|10\rangle\,|\chi_1\rangle + c\,|11\rangle\,|\chi_1\rangle \bigg)\,|\Omega_+\rangle\,|\Phi_+\rangle \\
& + \bigg(b\,|00\rangle\,|\chi_0\rangle - a\,|01\rangle\,|\chi_0\rangle + d\,|10\rangle\,|\chi_1\rangle - c\,|11\rangle\,|\chi_1\rangle \bigg)\,|\Omega_+\rangle\,|\Phi_-\rangle \\[1em]
& + \bigg(a\,|00\rangle\,|\chi_0\rangle + b\,|01\rangle\,|\chi_0\rangle - c\,|10\rangle\,|\chi_1\rangle - d\,|11\rangle\,|\chi_1\rangle \bigg)\,|\Omega_-\rangle\,|\Omega_+\rangle \\
& + \bigg(a\,|00\rangle\,|\chi_0\rangle - b\,|01\rangle\,|\chi_0\rangle - c\,|10\rangle\,|\chi_1\rangle + d\,|11\rangle\,|\chi_1\rangle \bigg)\,|\Omega_-\rangle\,|\Omega_-\rangle \\
& + \bigg(b\,|00\rangle\,|\chi_0\rangle + a\,|01\rangle\,|\chi_0\rangle - d\,|10\rangle\,|\chi_1\rangle - c\,|11\rangle\,|\chi_1\rangle \bigg)\,|\Omega_-\rangle\,|\Phi_+\rangle \\
& + \bigg(b\,|00\rangle\,|\chi_0\rangle - a\,|01\rangle\,|\chi_0\rangle - d\,|10\rangle\,|\chi_1\rangle + c\,|11\rangle\,|\chi_1\rangle \bigg)\,|\Omega_-\rangle\,|\Phi_-\rangle \\[1em]
& + \bigg(c\,|00\rangle\,|\chi_1\rangle + d\,|01\rangle\,|\chi_1\rangle + a\,|10\rangle\,|\chi_0\rangle + b\,|11\rangle\,|\chi_0\rangle \bigg)\,|\Phi_+\rangle\,|\Omega_+\rangle \\
& + \bigg(c\,|00\rangle\,|\chi_1\rangle - d\,|01\rangle\,|\chi_1\rangle + a\,|10\rangle\,|\chi_0\rangle - b\,|11\rangle\,|\chi_0\rangle \bigg)\,|\Phi_+\rangle\,|\Omega_-\rangle \\
& + \bigg(d\,|00\rangle\,|\chi_1\rangle + c\,|01\rangle\,|\chi_1\rangle + b\,|10\rangle\,|\chi_0\rangle + a\,|11\rangle\,|\chi_0\rangle \bigg)\,|\Phi_+\rangle\,|\Phi_+\rangle \\
& + \bigg(d\,|00\rangle\,|\chi_1\rangle - c\,|01\rangle\,|\chi_1\rangle + b\,|10\rangle\,|\chi_0\rangle - a\,|11\rangle\,|\chi_0\rangle \bigg)\,|\Phi_+\rangle\,|\Phi_-\rangle \\[1em]
& + \bigg(c\,|00\rangle\,|\chi_1\rangle + d\,|01\rangle\,|\chi_1\rangle - a\,|10\rangle\,|\chi_0\rangle - b\,|11\rangle\,|\chi_0\rangle \bigg)\,|\Phi_-\rangle\,|\Omega_+\rangle \\
& + \bigg(c\,|00\rangle\,|\chi_1\rangle - d\,|01\rangle\,|\chi_1\rangle - a\,|10\rangle\,|\chi_0\rangle + b\,|11\rangle\,|\chi_0\rangle \bigg)\,|\Phi_-\rangle\,|\Omega_-\rangle \\
& + \bigg(d\,|00\rangle\,|\chi_1\rangle + c\,|01\rangle\,|\chi_1\rangle - b\,|10\rangle\,|\chi_0\rangle - a\,|11\rangle\,|\chi_0\rangle \bigg)\,|\Phi_-\rangle\,|\Phi_+\rangle \\
& + \bigg(d\,|00\rangle\,|\chi_1\rangle - c\,|01\rangle\,|\chi_1\rangle - b\,|10\rangle\,|\chi_0\rangle + a\,|11\rangle\,|\chi_0\rangle \bigg)\,|\Phi_-\rangle\,|\Phi_-\rangle \Bigg].
\end{aligned}
\tag{4.11}
$$

Above, the qubits are ordered as $|A_1A_2\rangle\,|E\rangle\,|A_3B_1\rangle\,|A_4B_2\rangle$. Looking at the $|\Omega_+\rangle\,|\Omega_+\rangle$ term, we see that the qubit-environment entanglement is preserved from the pre teleportation state and has been successfully transferred to Alice's qubits (earlier, in (4.9) Bob's qubits and the environment were in the exact same entangled state).

4.3 Simultaneous transfer and teleportation of correlations

Now we are starting from the post measurement state (4.9), and allowing the qubits A_1A_3 to undergo pure dephasing under the Hamiltonian given below for time t before teleporting Bob's qubits back to Alice.

$$\tilde{H} = \sum_{i,j=0,1} |ij\rangle_{A_1A_3} \langle ij|_{A_1A_3} \otimes \tilde{V}_{ij}. \tag{4.12}$$

The corresponding unitary time evolution operator would be

$$\tilde{U}(t) = \sum_{i,j} |ij\rangle_{A_1A_3} \langle ij|_{A_1A_3} \otimes e^{-i\tilde{V}_{ij}t/\hbar}. \tag{4.13}$$

First we define

$$|\chi_i^j\rangle := e^{-i\tilde{V}_{jj}t/\hbar}\,|\chi_i\rangle\,, \tag{4.14}$$

and we have defined $|\chi_i\rangle$ in (4.6). Now

$$|\Psi_{PM}(t)\rangle = \tilde{U}(t)\,|\Psi_{PM}\rangle\,, \tag{4.15}$$

where $|\Psi_{PM}\rangle$ is taken from (4.9). Hence

$$\begin{aligned} |\Psi_{PM}(t)\rangle = \frac{1}{2}(|00\rangle + |11\rangle)_{A_2A_4}\Big[& a\,|00\rangle_{B_1B_2}\Big(|00\rangle\,|\chi_0^0\rangle + |11\rangle\,|\chi_0^1\rangle\Big)_{A_1A_3E} \\ & +b\,|01\rangle_{B_1B_2}\Big(|00\rangle\,|\chi_0^0\rangle + |11\rangle\,|\chi_0^1\rangle\Big)_{A_1A_3E} \\ & +c\,|10\rangle_{B_1B_2}\Big(|00\rangle\,|\chi_1^0\rangle + |11\rangle\,|\chi_1^1\rangle\Big)_{A_1A_3E} \\ & +d\,|11\rangle_{B_1B_2}\Big(|00\rangle\,|\chi_1^0\rangle + |11\rangle\,|\chi_1^1\rangle\Big)_{A_1A_3E}\Big]. \end{aligned} \tag{4.16}$$

Reordering the above equation so that the qubits are ordered like $|A_1A_2\rangle\,|A_3B_1\rangle\,|A_4B_2\rangle\,|e\rangle$, just as in (4.10), we get

$$\begin{aligned}|\Psi_{PM}(t)\rangle = \frac{1}{2}\Bigg[\, & |00\rangle\left(a\,|00\rangle\,|00\rangle\,|\chi_0^0\rangle + b\,|00\rangle\,|01\rangle\,|\chi_0^0\rangle + c\,|01\rangle\,|00\rangle\,|\chi_1^0\rangle + d\,|01\rangle\,|01\rangle\,|\chi_1^0\rangle\right)\\ & + |01\rangle\left(a\,|00\rangle\,|10\rangle\,|\chi_0^0\rangle + b\,|00\rangle\,|11\rangle\,|\chi_0^0\rangle + c\,|01\rangle\,|10\rangle\,|\chi_1^0\rangle + d\,|01\rangle\,|11\rangle\,|\chi_1^0\rangle\right)\\ & + |10\rangle\left(a\,|10\rangle\,|00\rangle\,|\chi_0^1\rangle + b\,|10\rangle\,|01\rangle\,|\chi_0^1\rangle + c\,|11\rangle\,|00\rangle\,|\chi_1^1\rangle + d\,|11\rangle\,|01\rangle\,|\chi_1^1\rangle\right)\\ & + |11\rangle\left(a\,|10\rangle\,|10\rangle\,|\chi_0^1\rangle + b\,|10\rangle\,|11\rangle\,|\chi_0^1\rangle + c\,|11\rangle\,|10\rangle\,|\chi_1^1\rangle + d\,|11\rangle\,|11\rangle\,|\chi_1^1\rangle\right)\Bigg].\end{aligned} \tag{4.17}$$

Now rewriting the A_3B_1 qubits and the A_4B_2 qubits in terms of the Bell states, we get

$$
\begin{aligned}
|\Psi_{PM}\rangle = \frac{1}{4}\Bigg[&\Big(a|00\rangle|\chi_0^0\rangle + b|01\rangle|\chi_0^0\rangle + c|10\rangle|\chi_1^1\rangle + d|11\rangle|\chi_1^1\rangle \Big)|\Omega_+\rangle|\Omega_+\rangle \\
&+\Big(a|00\rangle|\chi_0^0\rangle - b|01\rangle|\chi_0^0\rangle + c|10\rangle|\chi_1^1\rangle - d|11\rangle|\chi_1^1\rangle \Big)|\Omega_+\rangle|\Omega_-\rangle \\
&+\Big(b|00\rangle|\chi_0^0\rangle + a|01\rangle|\chi_0^0\rangle + d|10\rangle|\chi_1^1\rangle + c|11\rangle|\chi_1^1\rangle \Big)|\Omega_+\rangle|\Phi_+\rangle \\
&+\Big(b|00\rangle|\chi_0^0\rangle - a|01\rangle|\chi_0^0\rangle + d|10\rangle|\chi_1^1\rangle - c|11\rangle|\chi_1^1\rangle \Big)|\Omega_+\rangle|\Phi_-\rangle \\
&\\
&+\Big(a|00\rangle|\chi_0^0\rangle + b|01\rangle|\chi_0^0\rangle - c|10\rangle|\chi_1^1\rangle - d|11\rangle|\chi_1^1\rangle \Big)|\Omega_-\rangle|\Omega_+\rangle \\
&+\Big(a|00\rangle|\chi_0^0\rangle - b|01\rangle|\chi_0^0\rangle - c|10\rangle|\chi_1^1\rangle + d|11\rangle|\chi_1^1\rangle \Big)|\Omega_-\rangle|\Omega_-\rangle \\
&+\Big(b|00\rangle|\chi_0^0\rangle + a|01\rangle|\chi_0^0\rangle - d|10\rangle|\chi_1^1\rangle - c|11\rangle|\chi_1^1\rangle \Big)|\Omega_-\rangle|\Phi_+\rangle \\
&+\Big(b|00\rangle|\chi_0^0\rangle - a|01\rangle|\chi_0^0\rangle - d|10\rangle|\chi_1^1\rangle + c|11\rangle|\chi_1^1\rangle \Big)|\Omega_-\rangle|\Phi_-\rangle \\
&\\
&+\Big(c|00\rangle|\chi_1^0\rangle + d|01\rangle|\chi_1^0\rangle + a|10\rangle|\chi_0^1\rangle + b|11\rangle|\chi_0^1\rangle \Big)|\Phi_+\rangle|\Omega_+\rangle \\
&+\Big(c|00\rangle|\chi_1^0\rangle - d|01\rangle|\chi_1^0\rangle + a|10\rangle|\chi_0^1\rangle - b|11\rangle|\chi_0^1\rangle \Big)|\Phi_+\rangle|\Omega_-\rangle \\
&+\Big(d|00\rangle|\chi_1^0\rangle + c|01\rangle|\chi_1^0\rangle + b|10\rangle|\chi_0^1\rangle + a|11\rangle|\chi_0^1\rangle \Big)|\Phi_+\rangle|\Phi_+\rangle \\
&+\Big(d|00\rangle|\chi_1^0\rangle - c|01\rangle|\chi_1^0\rangle + b|10\rangle|\chi_0^1\rangle - a|11\rangle|\chi_0^1\rangle \Big)|\Phi_+\rangle|\Phi_-\rangle \\
&\\
&+\Big(c|00\rangle|\chi_1^0\rangle + d|01\rangle|\chi_1^0\rangle - a|10\rangle|\chi_0^1\rangle - b|11\rangle|\chi_0^1\rangle \Big)|\Phi_-\rangle|\Omega_+\rangle \\
&+\Big(c|00\rangle|\chi_1^0\rangle - d|01\rangle|\chi_1^0\rangle - a|10\rangle|\chi_0^1\rangle + b|11\rangle|\chi_0^1\rangle \Big)|\Phi_-\rangle|\Omega_-\rangle \\
&+\Big(d|00\rangle|\chi_1^0\rangle + c|01\rangle|\chi_1^0\rangle - b|10\rangle|\chi_0^1\rangle - a|11\rangle|\chi_0^1\rangle \Big)|\Phi_-\rangle|\Phi_+\rangle \\
&+\Big(d|00\rangle|\chi_1^0\rangle - c|01\rangle|\chi_1^0\rangle - b|10\rangle|\chi_0^1\rangle + a|11\rangle|\chi_0^1\rangle \Big)|\Phi_-\rangle|\Phi_-\rangle \Bigg].
\end{aligned}
\tag{4.18}
$$

Above, the qubits are ordered as $|A_1A_2\rangle\,|E\rangle\,|A_3B_1\rangle\,|A_4B_2\rangle$. Note that the kind of entanglement of Alice's qubits with the environment depends upon Bob's measurement results.

5 Conclusion

We started by giving a brief introduction to the tools used to analyse quantum systems. We then moved on to apply those tools to understand single qubit quantum teleportation without resorting to the pictorial description of quantum circuits, since a more analytical approach arguably gives a better understanding of how maximally entangled states are special, and how fundamental a role entanglement plays in quantum teleportation. I also studied a few different ways of teleporting multiple qubits, before presenting one here that seemed to me the simplest and the easiest to extend to a greater number of qubits. I read a paper [15] on bidirectional single qubit teleportation with an interaction with the environment. I used the same techniques to analyse two qubit quantum teleportation where there is an interaction with the environment[12].

There is still much that can be done in this area. The close relation between entanglement and teleportation could be exploited to come up with different ways of quantifying the entanglement of a quantum state. I wasn't able to give a concrete example of an environment which is interacting with our system as the authours in [15] did. We could look at more physical realisations of qubits and try to identify the causes of decoherence and focus on those kind of Hamiltonians. We could compare different methods of multi qubit teleportation based on how well they are able to preserve the fidelity while the entangled qubits are undergoing decoherence. This could be used to classify different multi qubit teleportation algorithms.

References

[1] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*, page 80. Cambridge University Press, 2010.

[12] The authors in that paper use the density operator formulation, however that was too cumbersome to work with for a greater number of qubits so I resorted to the state-vector formulation

[2] J. J. Sakurai and Jim Napolitano. *Modern Quantum Mechanics*. Cambridge University Press, 2 edition, 2017.

[3] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*, page 103. Cambridge University Press, 2010.

[4] W. Dür, G. Vidal, and J. I. Cirac. Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A*, 62:062314, Nov 2000.

[5] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Phys. Rev. Lett.*, 70:1895–1899, Mar 1993.

[6] W. K. Wootters and W. H. Zurek. A single quantum cannot be cloned. *Nature*, 299:802–803, Oct 1982.

[7] Juan Ortigoso. Twelve years before the quantum no-cloning theorem. *American Journal of Physics*, 86(3):201–205, mar 2018.

[8] Gustavo Rigolin. Quantum teleportation of an arbitrary two-qubit state and its relation to multipartite entanglement. *Phys. Rev. A*, 71:032303, Mar 2005.

[9] Ming-Jing Zhao, Zong-Guo Li, Xianqing Li-Jost, and Shao-Ming Fei. Multiqubit quantum teleportation. *Journal of Physics A: Mathematical and Theoretical*, 45(40):405303, sep 2012.

[10] L. Henderson, L. Hardy, and V. Vedral. Two-state teleportation. *Phys. Rev. A*, 61:062306, May 2000.

[11] Dan Song, Chen He, Zhengwen Cao, and Geng Chai. Quantum Teleportation of Multiple Qubits Based on Quantum Fourier Transform. *IEEE Communications Letters*, 22(12):2427–2430, 2018.

[12] Gao Ting, Wang Zhi-Xi, and Yan Feng-Li. Quantum Logic Network for Probabilistic Teleportation of Two-Particle State in a General Form. *Chinese Physics Letters*, 20(12):2094–2097, dec 2003.

[13] Jinhyoung Lee, Hyegeun Min, and Sung Dahm Oh. Multipartite entanglement for entanglement teleportation. *Phys. Rev. A*, 66:052318, Nov 2002.

[14] Katarzyna Roszak. Criteria for system-environment entanglement generation for systems of any size in pure-dephasing evolutions. *Phys. Rev. A*, 98:052344, Nov 2018.

[15] Tytus Harlender and Katarzyna Roszak. Transfer and teleportation of system-environment entanglement. *Phys. Rev. A*, 105:012407, Jan 2022.