Normal Modes of a Vibrating Membrane

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1 Abstract

We solved the wave equation in two dimensions for standing wave solutions which were then simulated using the matplotlib package of Python.These standing waves would emulate the vibrations of a membrane which is held fixed around a boundary. We took two such boundaries, a circular and a rectangular one. We also defined a function that would simulate the superposition of two given normal modes.

2 Circular Boundary

The wave equation is given by the following second order homogeneous partial differential equation in three variables.

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 \mathrm{u}}{\partial t^2}$$

Here v is the speed of propogation in the medium.

The vibrating membrane is held fixed around a circular region which is centered at the origin and has radius a, where a is a positive constant. The membrane lies in the area described by $r < a$ in polar coordinates.

2.1 Derivation

We now use separation of variables to separate u into its time dependent part (say $\mathrm{T}(t)$) and a spatial part (or the time independent part, say $\Omega(r, \theta)$).

Substituting $\mathrm{u}(r, \theta, t) = \Omega(r, \theta)\mathrm{T}(t)$ we get

$$\mathrm{T}\nabla^2\Omega = \frac{\Omega}{v^2}\frac{\partial^2 \mathrm{T}}{\partial t^2}$$

Rearranging, we get,

$$\frac{\nabla^2\Omega}{\Omega} = \frac{1}{v^2\mathrm{T}}\frac{\partial^2 \mathrm{T}}{\partial t^2}$$

Now Ω is a function of r and θ whereas T is a function of t. Since these are all independent variables, the expression above must equate to a constant. Say,

$$\frac{\nabla^2\Omega}{\Omega} = \frac{1}{v^2\mathrm{T}}\frac{\partial^2 \mathrm{T}}{\partial t^2} = C_1$$

Focusing on $\mathrm{T}(t)$,

$$\frac{\mathrm{d}^2\mathrm{T}}{\mathrm{d}t^2} = C_1 v^2 \mathrm{T}$$

Now if $C_1 > 0$, the solution would be $\mathrm{T}(t) = A_1\cosh(t) + B_1\sinh(t)$, (where A_1 and B_1 are constants) which would either shoot off to infiniy or decay to zero as time increases, neither of which is possible for our solution since there is no driving force or damping. Hence $C_1 < 0$. ($C_1 = 0$ would imply that T is a constant function, which absurd since T must oscillate with time.)

Substituting $C_1 = -K_1^2$, we now have to solve the following equations

$$\nabla^2\Omega = -K_1^2\Omega \tag{2.1}$$

$$\frac{\mathrm{d}^2\mathrm{T}}{\mathrm{d}t^2} = -K_1^2 v^2 \mathrm{T} \tag{2.2}$$

The solution for (2) directly as

$$\mathrm{T}(t) = A_2\sin(K_1 v t) \tag{2.3}$$

Where A_2 is a constant. We didn't add the phase shift because we want our normal mode to start from the xy plane.

In order to solve (1), we use separation of variables again, by writing $\Omega(r, \theta) = \Phi(\theta)\mathrm{R}(r)$ and substituting this in (1) and expanding the laplacian in polar coordinates we get,

$$\frac{\Phi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathrm{R}}{\partial r}\right)+\frac{\mathrm{R}}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}=-K_1^2\mathrm{R}\Phi$$

By separating the variables we get,

$$\frac{r}{\mathrm{R}}\frac{\partial}{\partial r}\left(r\frac{\partial \mathrm{R}}{\partial r}\right)+K_1^2r^2=-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\theta^2}$$

Now since the expression on the LHS contains terms only as a function of r and the expression on the RHS contains terms only in θ, they must both be equal to a constant. Similar to the previous argument, Φ tending to infinity or zero as θ increases is absurd (it must be a periodic function in order for the membrane to be continuous), said constant must be positve. So we now have to solve

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{dR}}{\mathrm{d}r}\right)+K_1^2r^2\mathrm{R}=K_2^2\mathrm{R} \tag{2.4}$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\theta^2}=-K_2^2\Phi \tag{2.5}$$

Recognizing (5) as our old friend, we can write the general solution directly as $\Phi(\theta)=A_3\cos(K_2\theta)$. We did not add the phase constant because that would just have the effect of rotating our base, which is not a matter of importance (the reason for using a cosine instead of a sine will become apparant soon).

On Φ we have the additional constraint that $\Phi(0)=\Phi(2\pi)$ so that the membrane is continuous, and there are no brakes on the surface. This implies that $\cos(2\pi K_2)=1$, or $K_2\in\mathbb{Z}$. . Had we taken a sine instead of a cosine, we would have had $K_2\in\mathbb{Z}/2$, which would give the same results, but would be less aesthetic.

Henceforth, we will replace K_2 by n, and it would be understood that n is an integer.

Hence the solution for (5) is

$$\Phi_n(\theta)=A_3\cos(n\theta) \tag{2.6}$$

Now we shall solve (4). We start by rewriting (4) as

$$r^2\frac{\mathrm{d}^2\mathrm{R}}{\mathrm{d}r^2}+r\frac{\mathrm{dR}}{\mathrm{d}r}+\left(K_1^2r^2-n^2\right)\mathrm{R}=0$$

Making a change of variable from r to $K_1 r$, (i.e. defining $s(r) = K_1 r$ and writing the differential operator $\frac{\mathrm{d}}{\mathrm{d}r}$ as $K_1 \frac{\mathrm{d}}{\mathrm{d}s}$), we get

$$s^2 \frac{\mathrm{d}^2 \mathrm{R}}{\mathrm{d}s^2} + s \frac{\mathrm{dR}}{\mathrm{d}s} + \left(s^2 - n^2\right) \mathrm{R} = 0$$

which is Bessel's differential equation of order n.

Hence $\mathrm{R}(s) = A_4 J_n(s) + B_4 Y_n(s)$ or $\mathrm{R}(s) = A_4 J_n(K_1 r) + B_4 Y_n(K_1 r)$where A_4 and B_4 are constants, $J_n(s)$ is the bessel function of the first kind of order n and $Y_n(s)$ is the bessel function of the second kind of order n. B_4 must be zero since $Y_n(K_1 r) = -\infty$ at $r = 0$, which happens to be at the center of our surface, which is not possible on physical grounds.

So we have $\mathrm{R}(r) = A_4 J_n(K_2 r)$. Now we require $\mathrm{R}(a) = 0$ because the membrane is held fixed at the circular rim which has radius a. Hence $J_n(K_2 a) = 0$, or $K_2 = k_{m,n}/a$ where $k_{m,n}$ represents the mth positive root of $J_n(x)$. Therefore the solution is

$$\mathrm{R}_{m,n}(r) = A_4 J_n \left(\frac{k_{m,n} r}{a} \right) \tag{2.7}$$

Combining (3),(6) and (7), we get the complete form of a normal mode as

$$\boxed{\mathrm{u}_{n,m}(r, \theta, t) = A_{n,m} J_n \left(\frac{k_{m,n} r}{a} \right) \cos(n\theta) \sin(k_{m,n} v t)} \tag{2.8}$$

where $A_{n,m}$ is a constant.

Here, replacing n by $-n$ would make no difference (other than changing the sign of A, which is irrelevant), since n is an integer, $J_{-n}(x) = (-1)^n J_n(x)$, and hence $k_{m,n} = k_{m,-n}$. So without any loss of generality, we can take n to be a non-negative integer. Note that every normal mode is specified by two numbers (in contrast to the one dimensional case where each normal mode was specified by one natural number.)

2.2 Visualizing the Solution

In the radial part of the solution, $k_{m,n}$ represents the mth positive root of $J_n(x)$, so there will be m radial nodes in $J_n \left(\frac{k_{m,n} r}{a} \right)$ (As we are including

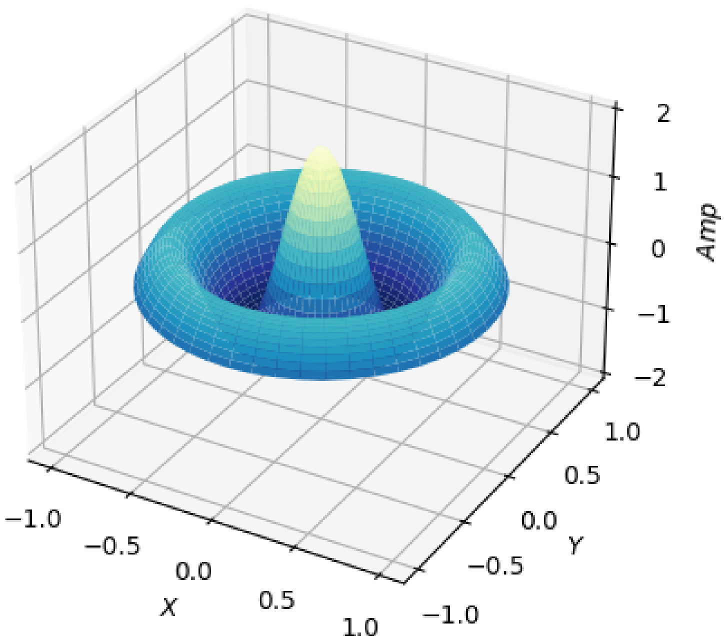


Figure 1: $u_{0,3}$.

the mth root, which lies at the boundary). On a radial node, the value of the normal mode would be zero around a circle. Similarly, by looking at the angular part, there will be n angular nodes. On an angular node, the value of the normal mode would be zero on a straight line. So by we can interpret the indices of each normal mode as the number of angular and radial nodes present in it. That is, $u_{n,m}$ would have n angular nodes and m radial nodes (including the boundary). Let us look at some examples.

Above, we see $u_{0,3}$, and as can be observed, there are no angular nodes and there are three radial nodes.

Below we have the plot for $u_{1,3}$, and we see three radial nodes and one angluar nodes.

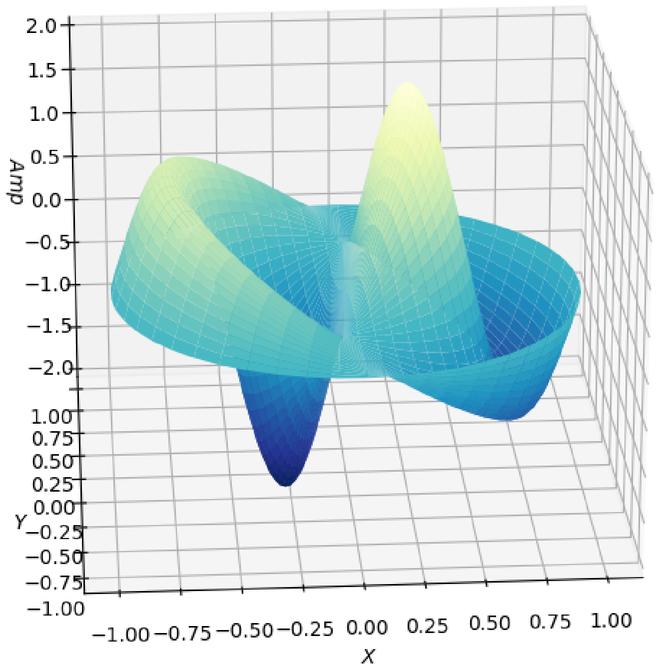


Figure 2: $u_{1,3}$.

3 Rectangular Boundary

Just as for the circular boundary, we solve the wave equation by using separation of variables in cartesian coordinates. Here the membrane is held fixed around a rectangular boundary and lies in the area described by $x < a$, $x > 0$ and $y > 0$, $y < b$, where a and b are positive constants

3.1 Derivation

Let $\mathrm{u}(x, y, t)$ be the functional form of a normal mode. Then writing $\mathrm{u}(x, y, t) = \Omega(x, y)\mathrm{T}(t)$, and substituing in the wave equation (just like we did in the circular boundary case), we obtain

$$\frac{\nabla^2\Omega}{\Omega} = \frac{1}{v^2\mathrm{T}}\frac{\partial^2\mathrm{T}}{\partial\mathrm{T}^2} = -K_1^2$$

Solving for T we get

$$\mathrm{T}(t) = A_5 \sin(K_3 v t) \tag{3.1}$$

Where A_5 is a constant.

Now writing $\Omega = \mathrm{X}(x)\mathrm{Y}(y)$, and substituting we get,

$$\nabla^2 \Omega = -K_3^2 \Omega$$

or

$$\mathrm{Y}\frac{\partial^2 \mathrm{X}}{\partial x^2} + \mathrm{X}\frac{\partial^2 \mathrm{Y}}{\partial y^2} = -K_3^2 \mathrm{XY}$$

or

$$\frac{1}{\mathrm{X}}\frac{\partial^2 \mathrm{X}}{\partial x^2} + K_3^2 = -\frac{1}{\mathrm{Y}}\frac{\partial^2 \mathrm{Y}}{\partial y^2}$$

Since the expression on the LHS is exclusively a function of x and the function on the RHS is exclusively a function of y, and x and y are independent variables, both of them can't vary with x or y, and hence must equate to a constant.

Therefore we have

$$\frac{1}{\mathrm{X}}\frac{\mathrm{d}^2 \mathrm{X}}{\mathrm{d}x^2} + K_3^2 = C_2 \tag{3.2}$$

and

$$\frac{1}{\mathrm{Y}}\frac{\mathrm{d}^2 \mathrm{Y}}{\mathrm{d}y^2} = -C_2 \tag{3.3}$$

Now C_2 must be a positive constant, since we can't expect Y to vary exponentially (which would happen when $C_2 < 0$) or linearly (when $C_2 = 0$) with space, since $\mathrm{Y}(y)$ must have at least two zeros, (at $y = \pm b$) we can rule out $C_2 < 0$ and $C_2 = 0$ as both of them would give at most one zero for Y. Hence $C_2 > 0$, so we can write $C_2 = K_4^2$ for some $K_4 \in \mathbb{R}$.

So solving for Y, we get $\mathrm{Y}(y) = A_6 \sin(K_4 y + \delta)$. Now we want $\mathrm{Y}(b) = Y(0) = 0$, So by substituting the values we get,

$$\sin(K_4 b + \delta) = 0 \text{ and } \sin(\delta) = 0$$
$$\Leftrightarrow \; K_4 = \frac{n_1 \pi - \delta}{b} \text{ and } \delta = n_2 \pi \text{ where } n_1, n_2 \in \mathbb{Z}$$

Now taking $0 \leq \delta < 2\pi$, (there would be no loss in generality in our solution since the $\sin(2\pi + \epsilon) = \sin(\epsilon)$ for every $\epsilon \in \mathbb{R}$.)
Hence the possible values of δ are 0 and π. If $\delta = \pi$, $\sin(K_4 y + \delta) = \sin(K_4 y + \pi) = -\sin(K_4 y)$. Absorbing the $-$ sign in the constant would give the same solution as what we would get by taking $\delta = 0$. So we can take $\delta = 0$ without any loss in generality in our solution upto a constant factor outside the sine term.Substituing $\delta = 0$ to get the value of K_4, we get $K_4 = \frac{n_1 \pi}{b}$ where $n_1 \in \mathbb{Z}$.

Therefore

$$\mathrm{Y}_n(y) = A_n \sin\left(\frac{n\pi y}{b}\right) \tag{3.4}$$

where A_n is any non-zero real number and n is an integer.

Now we shall solve (3.2). Substituting $C_2 = K_4^2$, we get

$$\frac{\mathrm{d}^2\mathrm{X}}{\mathrm{d}x^2} = -(K_3^2 - K_4^2)\mathrm{X}$$

Here K_3 must be greater than K_4 because X cannot exhibit exponential or linear behavior (the arguemnt is just like what it was for Y), and the rest of the argument is similar to what was done in the previous paragraph. So we would have $\sqrt{K_3^2 - K_4^2} = \frac{m\pi}{a}$ and

$$\mathrm{X}_m(y) = A_m \sin\left(\frac{m\pi x}{a}\right) \tag{3.5}$$

where A_m is any non-zero real number and m is an integer. Now we solve for K_3. Since $\sqrt{K_3^2 - K_4^2} = \frac{m\pi}{a}$ and $K_4 = \frac{n\pi}{b}$, we have $K_3 = \pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$
By (3.1), (3.4), and (3.5) we finally obtain the general form of a normal mode as

$$\boxed{\mathrm{u}_{n,m}(x, y, t) = A_{n,m} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\pi v t\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\right)} \tag{3.6}$$

3.2 Visualizatin of the Solution

Just like in the circular boundary case, the indices of a given normal mode tell us something about the number of nodal lines. By looking at (3.6), we

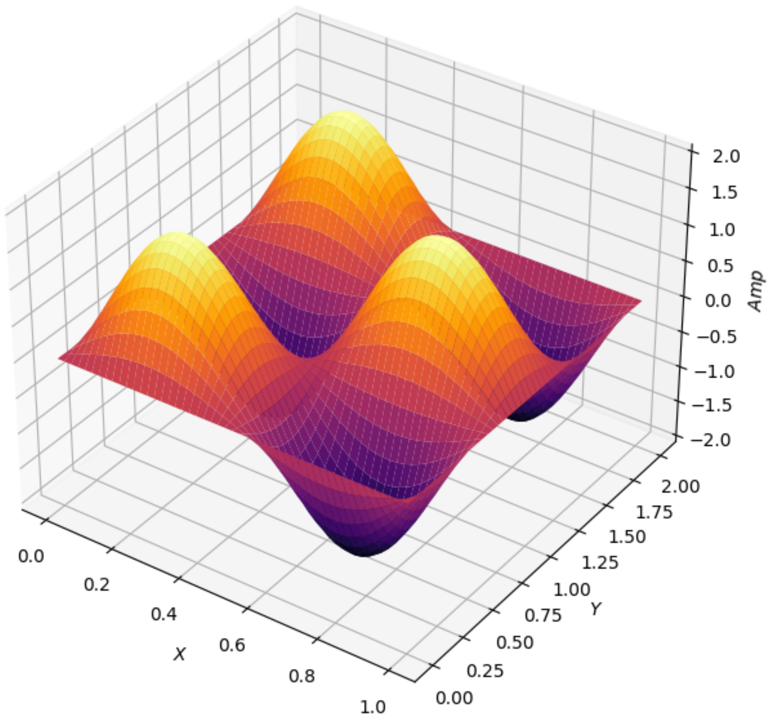
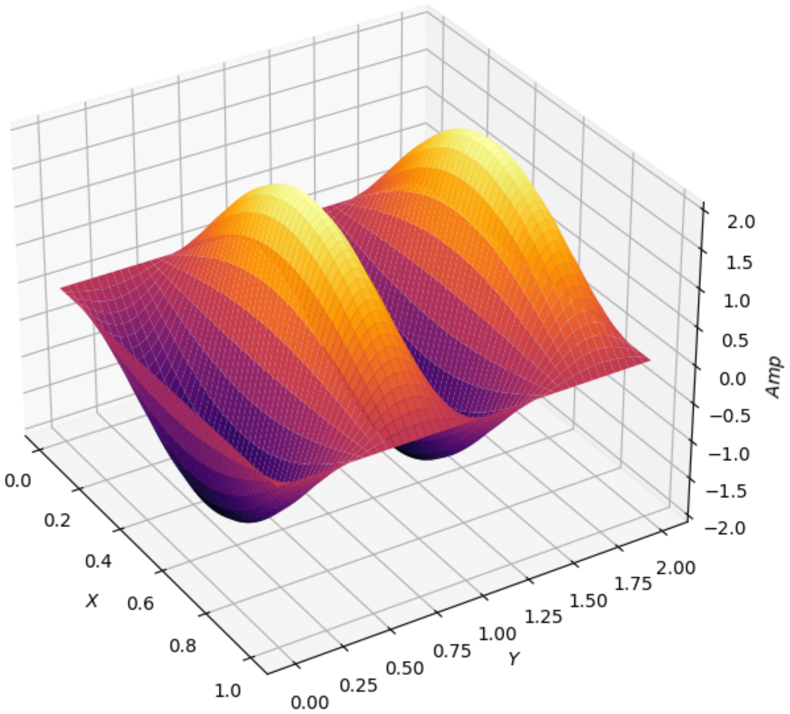


Figure 3: $u_{2,3}$.

can say that the normal mode described by the indices n and m would have $n+1$ nodal lines along the x axis and $m+1$ nodal lines along the y axis (when we include the nodes at the boundaries) by 'along x axis' I mean it lies on a line which can be obtained by fixing a particular value of x and varying y. Let us look at some examples.

In Figure 3, we see the normal mode $u_{2,3}$. Note that there are 3 nodal lines (2 on the boundaries and 1 wedged between the 'hills') along the x axis, and 4 nodal lines along the y axis (2 on the boundary and 2 wedged between the 'hills').

Below, in Figure 4, we see the normal mode $u_{1,4}$. Note that there are 2 nodal lines along the x axis and 5 nodal lines along the y axis.



Amp
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
X
0.0
0.2
0.4
0.6
0.8
1.0
Y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

Figure 4: $u_{1,4}$.

4 References

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