Graph Theory Answers

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Introduction

This document seeks to provide the author's original solutions to the exercises posed in the Springer publication "Graph Theory" attributed to J.A. Bondy and U.S.R Murty. This list will be updated as and when the author progresses with the problems aforementioned. Corrections and amendments are welcome.

Throughout this document we define, as in the book, $n = |V(G)|$ and $m = |E(G)|$ for some graph G. We also assume, within the solution, the parameters already defined in the question.

Chapter 1

Graphs

1.1 Graphs and Their Representation

Answer 1.1.1. First, since G is simple, we have $E(G) \subseteq \binom{V(G)}{2}$. That is, the edge set of G is a subset of the set of unordered pairs of the elements of $V(G)$. We know that $\binom{|V(G)|}{2} = \binom{n}{2}$ counts the number of such unordered pairs, hence it follows that $m \leq \binom{n}{2}$. We must have equality when all unordered pairs of vertices are in the edge set. This occurs in a complete graph.

Answer 1.1.2.

(a) We will present a procedure for placing the edges between vertices to obtain the result. We begin with n vertices in a bipartition (X, Y) with $|X| = r$ and $|Y| = s$. Now for each vertex $v \in X$, we may place at most s edges between v and all vertices in Y since G is simple. So we may place at most $\underbrace{s + s + ... + s}_{r \text{ times}} = rs$ edges between the sets in the bipartition.

(b) We have $r + s = n$ since (X, Y) is a bipartition. If n is even, we obtain the maximum number of edges in G when $r = s = \frac{n}{2}$. So by part (a) we have $m \leq \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$. If n is odd, we obtain the maximum number of edges in G when, without loss of generality, we have $r = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ and $s = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. So by part (b) we have $m \leq \frac{n+1}{2} \cdot \frac{n-1}{2} = \frac{n^2-1}{4} \leq \frac{n^2}{4}$.

(c) Equality holds when we have $|X| = |Y| = \frac{n}{2}$ for even $n = |V|$.