Graph Theory Answers

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Introduction

This document seeks to provide the author's original solutions to the exercises posed in the Springer publication "Graph Theory" attributed to J.A. Bondy and U.S.R. Murty. This list will be updated as and when the author progresses with the problems aforementioned. Corrections and amendments are welcome.

Throughout this document we define, as in the book, $n = |V(G)|$ and $m = |E(G)|$ for some graph G. We also assume, within the solution, the parameters already defined in the question.

Chapter 1

Graphs

1.1 Graphs and Their Representation

Answer 1.1.1. First, since G is simple, we have $E(G) \subseteq \binom{V(G)}{2}$. That is, the edge set of G is a subset of the set of unordered pairs of the elements of $V(G)$. We know that $\binom{|V(G)|}{2} = \binom{n}{2}$ counts the number of such unordered pairs, hence it follows that $m \leq \binom{n}{2}$. We must have equality when all unordered pairs of vertices are in the edge set. This occurs in a complete graph.

Answer 1.1.2.

(a) We will present a procedure for placing the edges between vertices to obtain the result. We begin with n vertices in a bipartition (X, Y) with $|X| = r$ and $|Y| = s$. Now for each vertex $v \in X$, we may place at most s edges between v and all vertices in Y since G is simple. So we may place at most $\underbrace{s + s + \ldots + s}_{r \text{ times}} = rs$ edges between the sets in the bipartition.

(b) We have $r + s = n$ since (X, Y) is a bipartition. If n is even, we obtain the maximum number of edges in G when $r = s = \frac{n}{2}$. So by part (a) we have $m \leq \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$. If n is odd, we obtain the maximum number of edges in G when, without loss of generality, we have $r = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ and $s = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. So by part (b) we have $m \leq \frac{n+1}{2} \cdot \frac{n-1}{2} = \frac{n^2-1}{4} \leq \frac{n^2}{4}$.

(c) Equality holds when we have $|X| = |Y| = \frac{n}{2}$ for even $n = |V|$.

Answer 1.1.3.

(a) Let G be a path with $n = |V(G)|$. Then will create a path isomorphic to G that is demonstrably bipartite. Consider the path $G' = (\{v_1, v_2, \ldots, v_n\}, \{v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n\})$. Then $(X, Y) = (\{v_i \in V(G') : i \text{ odd}\}, \{v_j \in V(G') : j \text{ even}\})$ is a bipartition of G' since we only have odd indexed vertices adjacent to even indexed vertices, hence G' is bipartite. Now we must present an isomorphism from G to G'. Consider a function $\iota : V(G) \to V(G')$ where first ι maps an arbitrary endpoint of G to an arbitrary endpoint of G'. We then construct ι by mapping a vertex $v \in V(G)$ to $v' \in V(G')$ if and only if v is adjacent to $u \in V(G)$ such that v' is adjacent to $\iota(u)$. By initialising this construction with only an endpoint, we have ensured that ι is a bijection under this rule and that ι preserves adjacency. Hence ι is an isomorphism. Thus we obtain that G is bipartite under the bipartition $(X', Y') = (\{v \in V(G) : \iota(v) \in X\}, \{u \in V(G) : \iota(u) \in Y\})$.

(b) Let G be a cycle. Assume G is bipartite under the bipartition (X, Y). Now suppose, towards a contradiction, that G has odd length. Then n is odd, since we may associate each edge in G with a single vertex, so either $|X|$ or $|Y|$ is odd. Assume, without loss of generality, that $|X|$ is odd and $|Y|$ is even. We have that $d(v) = 2$ for all $v \in V(G)$, so

counting all edges in G from X to Y, we have $2|X|$ edges in G. Clearly this is not odd, so we have a contradiction. Hence G must have even length. Now assume that G is a cycle with even length. Removing an arbitrary edge in G produces a path of odd length, which is bipartite from part (a). We obtain our bipartition (X, Y) using the method in (a) and then add in the previously removed edge. The endpoints of the path are in different parts of the partition, since there are an even number of vertices, so the extra edge is added between the parts, hence our bipartition is preserved. Thus, G is bipartite. So we have that a cycle is bipartite if and only if it has even length.

Answer 1.1.4. Let $G = (\{v_1, ..., v_n\}, E)$. For any vertex $u \in V(G)$, $\delta(G) \leq d_G(u)$ by definition, so we have $\delta(G) = \frac{n \cdot \delta(G)}{n} \leq \frac{d_G(v_1) + ... + d_G(v_n)}{n} = d(G)$. Similarly, for any vertex $u \in V(G)$, $d_G(u) \leq \Delta(G)$, so $d(G) = \frac{d(v_1) + ... + d(v_n)}{n} \leq \frac{n \cdot \Delta(G)}{n} = \Delta(G)$. Hence $\delta(G) \leq d(G) \leq \Delta(G)$.

Answer 1.1.5. Let G be a k-regular graph. If $k = 0$, then we have no edges in G since the degree of each vertex is 0. If $k = 1$, then G consists of any number of components of 2 vertices joined by an edge. If $k = 2$, then G consists of any number of components that are cycles.

Answer 1.1.6.

(a) We represent this problem with a simple graph $G = (V, E)$ where the vertices represent people in the group and two vertices are adjacent if the represented people are friends. Hence we seek to prove that $d(u) = d(v)$ for distinct $u, v \in V$. Given a vertex $v \in V$, we must have $d(v) \in \{0, 1, ..., n - 1\}$ since G is simple. However, if there exists a vertex of degree 0 in G then we cannot have a vertex of degree $n - 1$ in G, and vice versa. Hence we have n vertices in G with only $n - 1$ options for the degree of each vertex. Thus, by the pigeonhole principle there exist at least two vertices of G with equal degree.

(b) We present a graphic representation of the group of friends below.

Answer 1.1.7.

(a)

Q_1

Q_2

Q_3

Q_4
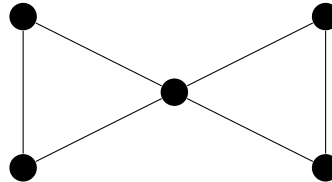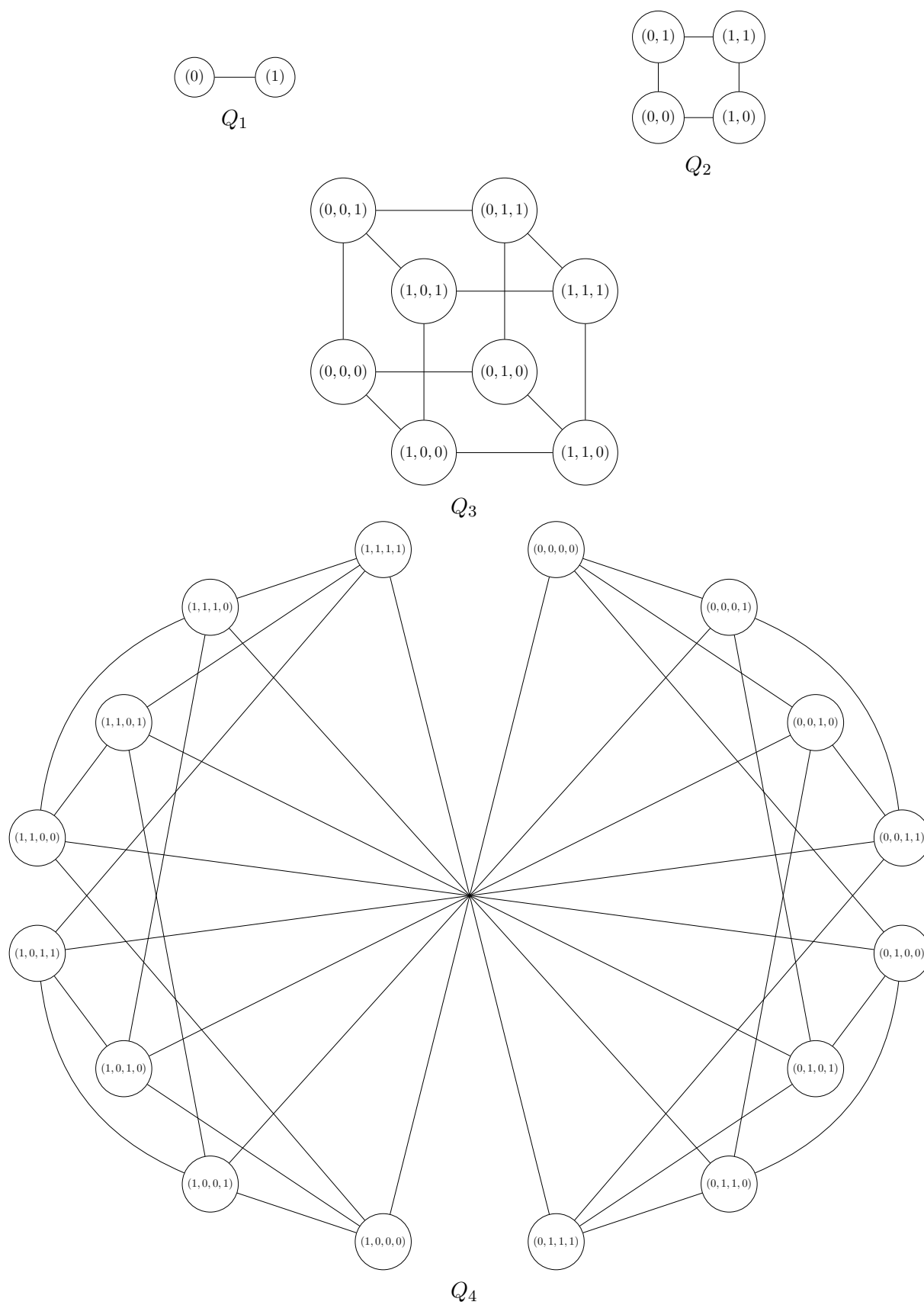
(b) Considering Q_n, each vertex is an n-tuple with 2 options for each entry. Hence $v(Q_n) = 2^n$. For each vertex of Q_n, there are n vertices that have an n-tuple that differ by a single entry, so $d(v) = n$ for $v \in V(Q_n)$. So by the degree-sum formula,

$$\sum_{v \in V(Q_n)} d(v) = n \cdot 2^n = 2 \cdot e(Q_n) \implies e(Q_n) = \frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$$

(c) We partition $V(Q_n)$ into (X, Y) where X is the set of vertices containing an even number of 1's and Y is the set of vertices containing an odd number of 1's. Clearly each vertex of Q_n is in either X or Y. Now take an arbitrary vertex $v \in X$. v has an even number of 1's, so for any other vertex $u \in X$, we cannot have $uv \in E(Q_n)$ since we are required to change at least 2 numbers in the n-tuple of v to attain the n-tuple of u, as u also has an even number of 1's. This logic also holds for Y. Hence (X, Y) is indeed a bipartition of $V(Q_n)$ and so Q_n is bipartite.

Answer 1.1.8.

(a)

BL_1

BL_2

BL_3

BL_4
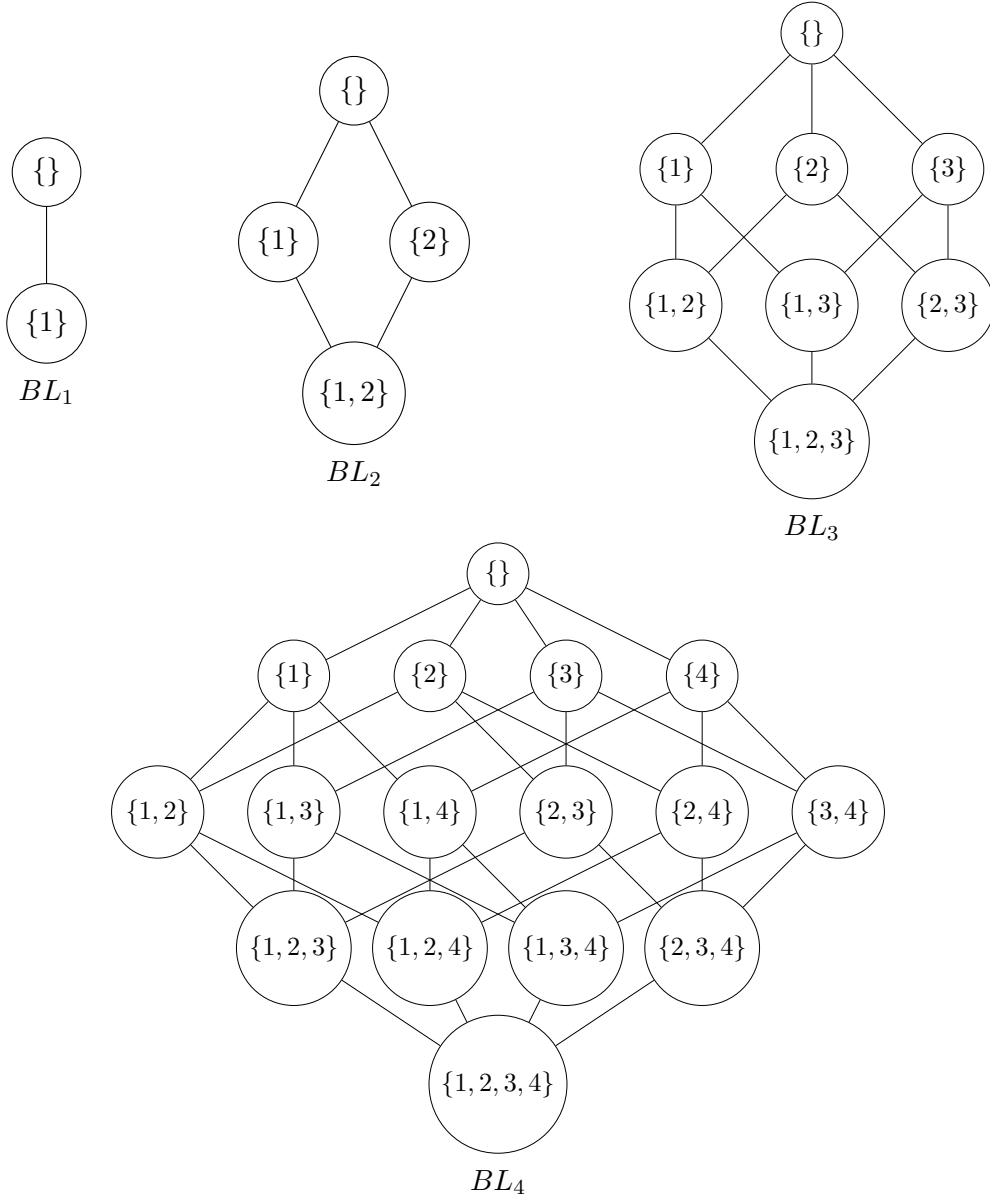
(b) $v(BL_n)$ is given by the number of subsets of $\{1, ..., n\}$, so $v(BL_n) = 2^n$. To determine $e(BL_n)$, we consider each "level" of vertices in BL_n, that is the set of vertices representing sets of size m where $1 \leq m \leq n$. First note that for any "level" m in BL_n, there are $C(n, m)$ vertices in that level by definition of BL_n. Now, for each vertex v in "level" m, consider its symmetric difference with the vertices in the previous "level" $m - 1$. There exist m vertices in "level" $m - 1$ whose symmetric difference with v is of size 1, since we may isolate each element in v by selecting the correct set (of size $m - 1$) in the previous "level." Hence, summing over each "level" in BL_n, we determine

$$e(BL_n) = \sum_{m=1}^{n} m \cdot C(n, m) = 2^{n-1} \cdot n$$

(c) Given $n \in \mathbb{N}$, we formulate the pair (X, Y) where $X = \{v \in V(BL_n) : |v| \text{ is even}\}$ and $Y = \{v \in V(BL_n) : |v| \text{ is odd}\}$ and wish to prove that (X, Y) forms a bipartition of BL_n. Take arbitrary vertices $u, v \in X$, so $|u|$ and $|v|$ are even. We first note that

$$u \Delta v = (u \setminus v) \cup (v \setminus u)$$
$$\implies |u \Delta v| = |(u \setminus v) \cup (v \setminus u)|$$
$$= |u \setminus v| + |v \setminus u| - |(u \setminus v) \cap (v \setminus u)|$$
$$= |u \setminus v| + |v \setminus u|$$

Assume, towards a contradiction, that there exists an edge between u and v in $BL_n[X, Y]$, so we must have $|u \Delta v| = 1$, so either $(|u \setminus v|, |v \setminus u|) = (1, 0)$ or $(|u \setminus v|, |v \setminus u|) = (0, 1)$. Without loss of generality, assume that $|u \setminus v| = 1$ and $|v \setminus u| = 0$. Then there exists precisely one element of u not in v and there exist no elements of v not in u. Hence we must have $u \supset v$ and $|u| = |v| + 1$, a contradiction since both $|u|$ and $|v|$ are even. This argument clearly holds when u and v are exchanged. Hence we may not have an edge between any two vertices in X. Repeating this logic for arbitrary vertices in Y, we determine that (X, Y) indeed gives a bipartition of BL_n.

Answer 1.1.9.

(a) By the degree-sum formula, we have $\sum_{v \in V(G)} d(v) = 2m$. So, since each edge has one endpoint in X and another in Y, we immediately see that $\sum_{v \in X} d(v) = \sum_{v \in Y} d(v) = m$.

(b) By our previous deduction, if G is k-regular with $k > 0$, we have

$$k \cdot |X| = \sum_{v \in X} d(v) = \sum_{v \in Y} d(v) = k \cdot |Y|$$

$$\implies |X| = |Y|$$

Answer 1.1.10. First consider an arbitrary part of G, say X, with size a_j. Since G is k-partite, we immediately see that, for any vertex in X, the upper limit of edges in G with that vertex as

an endpoint is given by $a_1 + ... + a_k - a_j = n - a_j$. Hence we have, for some vertex $v \in X$,

$$d(v) \leq n - a_j \implies \sum_{u \in X} d(u) \leq |X| \cdot (n - a_j) = a_j(n - a_j)$$

$$\implies \sum_{w \in V(G)} d(w) \leq \sum_{i=1}^{k} a_i(n - a_i)$$

$$\implies 2m \leq \sum_{i=1}^{k} a_i(n - a_i) \text{ by the degree-sum formula}$$

$$\implies m \leq \frac{1}{2} \sum_{i=1}^{k} a_i(n - a_i)$$