4. PRINCIPLE OF MATHEMATICAL INDUCTION

Ex. Prove the following by principle of mathematical induction.

1. $1 + 2 + 3 + \ldots\ldots\ldots\ldots + n = \frac{n(n+1)}{2}$.
2. $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
3. $1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$.
4. $(1) + (1 + 3) + (1 + 3 + 5) + \ldots\ldots\ldots$ up to n terms $= \frac{n(n+1)(2n+1)}{6}$.
5. Prove by the principle of mathematical induction that the sum of the first n odd natural numbers is n^2.
6. $1.2 + 3.4 + 5.6 + \ldots\ldots\ldots$ up n terms $= \frac{n(n+1)(4n-1)}{3}$.
7. $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots\ldots\ldots\ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$.
8. $1.2.3 + 2.3.4 + 3.4.5 + \ldots\ldots\ldots + n.(n + 1).(n + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$.
9. $1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \ldots\ldots\ldots\ldots + \frac{1}{(1+2+3+\cdots\ldots+n)} = \frac{2n}{(n+1)}$.
10. $1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots + (2n - 1)^2 = \frac{n(4n^2-1)}{3}$.
11. $a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$.
12. $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots\ldots\ldots\ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$.
13. $1.2 + 2.2^2 + 3.2^3 + \ldots\ldots\ldots + n.2^n = (n - 1)2^{n+1} + 2$.
14. $(1 + \frac{3}{1})(1 + \frac{5}{4})(1 + \frac{7}{9})\ldots\ldots\ldots\ldots(1 + \frac{2n+1}{n^2}) = (n + 1)^2$.
15. $1 + 3 + 3^2 + \ldots\ldots\ldots\ldots + 3^{n-1} = \frac{3^n-1}{2}$.
16. $\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$.
17. $1 + 2 + 3 + \ldots\ldots\ldots\ldots + n < \frac{(2n+1)^2}{8}$.
18. $10^{2n-1} + 1$ is divisible by 11.

19. x^{2n} - y^{2n} is divisible by (x + y).

20. $3^{2n+2} - 8n - 9$ is divisible by 8 .

21. $(2n + 7) < (n + 3)^2$.

SOLUTIONS TO THE EXAMPLES

1. Let P(n) : $1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$

For n=1, P(1) : $1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$, which is true.

Assuming that P(k) is true for some +ve integer k, we have ,

$1 + 2 + 3 + \ldots\ldots + k = \frac{k(k+1)}{2}$ ------------------- (1)

We shall now prove that P(k+1) is also true , now we have

$1 + 2 + 3 + \ldots\ldots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1)$ (using (1))

$$= (k + 1)\left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+1+1)}{2} .$$

Thus P(k+1) is true whenever P(k) is true.

Hence from Principal of Mathematical Inductiion (PMI) the statement P(n) is true for all natural numbers n.

2. Let P(n) : $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

For n = 1 , P(1) : $1^2 = \frac{1(1+1)(2.1+1)}{6} = \frac{1(2)(3)}{6} = 1 \Rightarrow 1 = 1$, which is true.

Assume that P(k) is true for some positive integer k , we have ,

$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$ .... (1)

Now we shall prove that P(k+1) is also true , now we have ,

$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + k^2 + (k + 1)^2 = \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$ **(using (1))**

$$= (k + 1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$= (k + 1)\left[\frac{2k^2+7k+6}{6}\right]$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}.$$

Thus P(k+1) is true and the inductive proof is completed.

Hence P(n) is true for all positive integers of n.

3. **Let P(n) :** $1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$

For n=1, P(1) : $1^3 = \frac{1^2(1+1)^2}{4} = \frac{1(2)^2}{4} = \frac{4}{4} = 1 \Rightarrow 1 = 1$ **, is true.**

Assume that P(k) is true for some positive integer k , we have ,

$1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots + k^3 = \frac{k^2(k+1)^2}{4}$ ---------------------(1)

Now we shall prove that P(k+1) is also true , now we have ,

$1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots + k^3 + (k + 1)^3 = \frac{k^2(k+1)^2}{4} + (k + 1)^3$ **(using (1))**

$$= (k + 1)^2\left[\frac{k^2}{4} + (k+1)\right]$$

$$= (k + 1)^2\left[\frac{k^2+4k+4}{4}\right]$$

$$= \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2(k+1+1)^2}{4}.$$

Thus P(k+1) is also true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

4. Let P(n) : (1) + (1+3) + (1+3+5) +…………up to n terms = $\frac{n(n+1)(2n+1)}{6}$

Here nth term is 1 + 3 + 5 +…………….+ (2n-1) = n^2**, the equation becomes,**

P(n) : (1) + (1 + 3) + (1 + 3 + 5) +…………+ n^2 **=** $\frac{n(n+1)(2n+1)}{6}$

For n=1, P(1) : 1= $\frac{1(1+1)(2.1+1)}{6}$ **=** $\frac{1(2)(3)}{6}$ **= 1 , which is true.**

Assume that P(k) is true for some positive integer k , we have ,

(1) + (1 + 3) + (1+ 3 + 5) +…………+ k^2 **=** $\frac{k(k+1)(2k+1)}{6}$ **---------------(1)**

Now we shall prove that P(k+1) is also true , now we have ,

(1) + (1 + 3) + (1+ 3 + 5)+…………+ $k^2 + (k + 1)^2 = \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$ **(using(1))**

$$= (k + 1)\left[\frac{k(2k+1)}{6} + (k + 1)\right]$$

$$= (k + 1)\,\frac{(2k^2+k+6k+6)}{6}$$

$$= (k + 1)\,\frac{(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}.$$

Thus P(k+1) is also true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

5. Let P(n) : 1+ 2 + 3 + ……………..+ (2n – 1) = n^2

For n = 1 , P(n) : 1 = 1^2 **= 1 , which is true.**

Assume that P(k) is true for some positive integer k , we have ,

1+ 2 + 3 + ……………..+ (2k – 1) = k^2 …………………….(1)

We shall now prove that P(k+1) is true, now we have,

1+ 2 + 3 + ……………..+ (2k – 1) + (2k + 1) = k^2 + (2k + 1) (using (1))

= $k^2 + 2k + 1 = (k + 1)^2$.

Thus P(k+1) is also true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

6. Let P(n) : 1.2 + 3.4 + 5.6+…………up n terms = $\frac{n(n+1)(4n-1)}{3}$

Here the nth term is = (2n-1)(2n) , so the equation becomes ,

P(n) : 1.2 + 3.4 + 5.6 +……………+ (2n – 1)(2n) = $\frac{n(n+1)(4n-1)}{3}$

For n = 1, P(1) : 1.2 = $\frac{1(1+1)(4.1-1)}{3} = \frac{1(2)(3)}{3} = 2 \Rightarrow 2 = 2$, which is true.

Assume that P(k) is true for some positive integer k , we have ,

1.2 + 3.4 + 5.6+……………+ (2k – 1)(2k) = $\frac{k(k+1)(4k-1)}{3}$ ……………..(1)

We shall now prove that P(k+1) is true, now we have,

1.2 + 3.4 + 5.6+……………+ (2k – 1)(2k) + (2k + 1)2(k + 1)

$= \frac{k(k+1)(4k-1)}{3} + (2k + 1)2(k + 1)$ (using (1))

$= (k + 1)\left[\frac{k(4k-1)}{3} + 2(2k+1)\right]$

$= (k + 1)\left[\frac{(4k^2-k+12k+6)}{3}\right] = (k + 1)\left[\frac{(4k^2+11k+6)}{3}\right]$

$= \frac{(k+1)(k+2)(4k+3))}{3} = \frac{(k+1)(k+1+1)(4k-1)}{3}$.

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

7. Let P(n) : $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10}$ **+……………+** $\frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$

For n = 1 , P(1) : $\frac{1}{1.4} = \frac{1}{(3.1+1)} = \frac{1}{(4)} \Rightarrow \frac{1}{4} = \frac{1}{4}$ **, is true.**

Assume that P(k) is true for some positive integer k , we have ,

$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10}$ **+……………+** $\frac{1}{(3k-2)(3k+1)} = \frac{k}{(3k+1)}$

We shall now prove that P(k+1) is true, now we have,

$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10}$ **+……………+** $\frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)} = \frac{k}{(3k+1)} + \frac{1}{(3k+1)(3k+4)}$

$$= \frac{1}{(3k+1)}\left[k + \frac{1}{3k+4}\right]$$

$$= \frac{1}{(3k+1)}\left(\frac{3k^2+4k+1}{3k+4}\right)$$

$$= \frac{1}{(3k+1)} \frac{(k+1)(3k+1)}{3k+4} = \frac{(k+1)}{(3(k+1)+1)}.$$

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

8. Let P(n) : 1.2.3 + 2.3.4 + 3.4.5 +……….+ n(n+1)(n+2) = $\frac{n(n+1)(n+2)(n+3)}{4}$

For n = 1 , P(1) : 1.2.3 = $\frac{1(1+1)(1+2)(1+3)}{4} = \frac{1(2)(3)(4)}{4} \Rightarrow$ **6 = 6, is true.**

Assume that P(k) is true for some positive integer k , we have ,

1.2.3 + 2.3.4 + 3.4.5 +……….+ k(k + 1)(k + 2) = $\frac{k(k+1)(k+2)(k+3)}{4}$ **…………(1)**

We shall now prove that P(k+1) is true, now we have,

1.2.3 + 2.3.4 + 3.4.5 +……….+ k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3)

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k + 1)(k + 2)(k + 3) \quad \text{(using (1))}$$

$$= (k + 1)(k + 2)(k + 3)\left[\frac{k}{4} + 1\right]$$

$$= (k + 1)(k + 2)(k + 3)\left[\frac{k+4}{4}\right]$$

$$= \frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}.$$

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

9. Let $P(n) : 1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \ldots\ldots\ldots\ldots\ldots + \frac{1}{(1+2+3+\cdots\ldots+n)} = \frac{2n}{(n+1)}$

Here the nth term is = $\frac{2}{n(n+1)}$ **so the equation becomes**

$$1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \ldots\ldots\ldots\ldots\ldots + \frac{2}{n(n+1)} = \frac{2n}{(n+1)}$$

For n = 1 , $P(1) : 1 = \frac{2.1}{(1+1)} = \frac{2}{(2)} = 1$, **is true .**

Assume that P(k) is true for some positive integer k , we have ,

$$1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \ldots\ldots\ldots\ldots\ldots + \frac{2}{k(k+1)} = \frac{2k}{(k+1)} \quad \ldots\ldots\ldots\ldots(1)$$

We shall now prove that P(k+1) is true, for we have,

$$1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \ldots\ldots\ldots\ldots\ldots + \frac{2}{k(k+1)} + \frac{2}{(k+1)(k+2)}$$

$$= \frac{2k}{(k+1)} + \frac{2}{(k+1)(k+2)} \quad \text{(using (1)),}$$

$$= \frac{2}{(k+1)}\left[k + \frac{1}{k+2}\right]$$

$$= \frac{2}{(k+1)}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$= \frac{2(k+1)^2}{(k+1)(k+2)} = \frac{2(k+1)}{(k+2)} = \frac{2(k+1)}{(k+1+1)}.$$

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

10. **Let P(n) :** $1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$

For n=1 , P(1) : $1^2 = \frac{1.(4.1^2-1)}{3} = \frac{1.(3)}{3} = 1 \Rightarrow 1 = 1$, **is true.**

Assume that P(k) is true for some positive integer k , we have ,

$$1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots + (2k-1)^2 = \frac{k(4k^2-1)}{3} \ldots\ldots\ldots\ldots(1)$$

We shall now prove that P(k+1) is true, for we have,

$$1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots + (2k-1)^2 + (2k+1)^2 = \frac{k(4k^2-1)}{3} + (2k+1)^2 \quad \text{(using (1))}$$

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= (2k+1)\left[\frac{k(2k-1)}{3} + (2k+1)\right]$$

$$= (2k+1)\left[\frac{2k^2-k+6k+3}{3}\right]$$

$$= (2k+1)\left[\frac{2k^2+5k+3}{3}\right] = \frac{(2k+1)(k+1)(2k+3)}{3}$$

$$= \frac{(k+1)(4k^2+8k+3)}{3} = \frac{(k+1)(4(k^2+2k+1)-1)}{3}$$

$$= \frac{(k+1)(4(k+1)^2-1)}{3}.$$

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

11. Let P(n) : $a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-1} = \frac{a(r^n-1)}{r-1}$

For n = 1 , P(1) : $a = \frac{a(r^1-1)}{r-1} = a$, which is true .

Assume that P(k) is true for some positive integer k , we have ,

$a + ar + ar^2 + \ldots\ldots\ldots + ar^{k-1} = \frac{a(r^k-1)}{r-1}$

We shall now prove that P(k+1) is true, for we have,

$$a + ar + ar^2 + \ldots\ldots\ldots + ar^{k-1} + ar^k = \frac{a(r^k-1)}{r-1} + ar^k$$

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r-1}$$

$$= \frac{a(r^{k+1}-1)}{r-1}, \quad (r \neq 1).$$

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

12. Let P(n) : $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots\ldots\ldots\ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

For n = 1, P(1) : $\frac{1}{1.2.3} = \frac{1(1+3)}{4(1+1)(1+2)} = \frac{1(4)}{4(2)(3)} = \frac{1}{6} \implies \frac{1}{6} = \frac{1}{6}$, which is true.

Assume that P(k) is true for some positive integer k , we have ,

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots\ldots\ldots\ldots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We shall now prove that P(k+1) is true, for we have,

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots\ldots\ldots\ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \quad \text{(using (1))}$$

$$= \frac{1}{(k+1)(k+2)}\left[\frac{k(k+3)}{4} + \frac{1}{k+3}\right]$$

$$= \frac{1}{(k+1)(k+2)} = \left[\frac{k(k^2+6k+9)+4}{4(k+1)(k+2)(k+3)}\right]$$

$$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)} = \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)} = \frac{(k+1)(k+1+3)}{4(k+1+1)(k+1+2)} .$$

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

13. **Let P(n) : $1.2 + 2.2^2 + 3.2^3 + \ldots\ldots\ldots\ldots + n.2^n = (n-1)2^{n+1} + 2$**

For n = 1 , P(1) : $1.2 = (1-1)2^{1+1} + 2 = 0 + 2 = 2 \Rightarrow 2 = 2$, which is true.

Assume that P(k) is true for some positive integer k , we have ,

$1.2 + 2.2^2 + 3.2^3 + \ldots\ldots\ldots\ldots + k.2^k = (k-1)2^{k+1} + 2$ ……………..(1)

We shall now prove that P(k+1) is true, for we have,

$1.2 + 2.2^2 + 3.2^3 + \ldots\ldots\ldots\ldots + k.2^k + (k+1).2^{k+1}$

$$= (k-1)2^{k+1} + 2 + (k+1).2^{k+1} \quad \text{(using (1))}$$

$$= (k-1 + k+1)2^{k+1} + 2$$

$$= 2k.2^{k+1} = k.2^{k+2} + 2$$

$$= (k+1-1)2^{k+1+1} + 2 .$$

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

14. **Let P(n) : $(1 + \frac{3}{1})(1 + \frac{5}{4})(1 + \frac{7}{9})$.............$(1 + \frac{2n+1}{n^2}) = (n + 1)^2$**

For n = 1 , P(1) : $(1 + \frac{3}{1}) = (1+1)^2 = 4 \Rightarrow 4 = 4$, which is true .

Assume that P(k) is true for some positive integer k , we have ,

$(1 + \frac{3}{1})(1 + \frac{5}{4})(1 + \frac{7}{9})$.............$(1 + \frac{2k+1}{k^2}) = (k + 1)^2$ ....(1)

Now we shall prove that P(k+1) is also true , now we have ,

$(1 + \frac{3}{1})(1 + \frac{5}{4})(1 + \frac{7}{9})$.............$(1 + \frac{2k+1}{k^2})\,(1 + \frac{2k+3}{(k+1)^2})$

$= (k + 1)^2 (1 + \frac{2k+3}{(k+1)^2})$ (using (1))

$= (k + 1)^2 [\frac{(k+1)^2+2k+3}{(k+1)^2}]$

$= k^2 + 2k + 1 + 2k + 3$

$= k^2 + 4k + 4$

$= (k + 2)^2 = (k + 1 + 1)^2$.

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

15. **Let P(n) : $1 + 3 + 3^2 +$ ....$+ 3^{n-1} = \frac{3^n-1}{2}$**

For n = 1 , P(1) : $1 = \frac{3^1-1}{2} = \frac{2}{2} = 1$, is true .

Assume that P(k) is true for some positive integer k , we have ,

$1 + 3 + 3^2 +$ ....$+ 3^{k-1} = \frac{3^k-1}{2}$ ....(1)

Now we shall prove that P(k+1) is also true , now we have ,

$1 + 3 + 3^2 +$ ....$+ 3^{k-1} + 3^k = \frac{3^k-1}{2} + 3^k$ (using (1))

$$= \frac{3^k - 1 + 2.3^k}{2}$$

$$= \frac{3.3^k - 1}{2} = \frac{3^{k+1} - 1}{2}$$

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

16. Let P(n) : $\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots\dots\dots\dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$

For n = 1 , P(1) : $\frac{1}{2.5} = \frac{1}{6.1+4} = \frac{1}{10} \Longrightarrow \frac{1}{10} = \frac{1}{10}$ **, is true .**

Assume that P(k) is true for some positive integer k , we have ,

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots\dots\dots\dots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4} \dots\dots\dots\dots(1)$$

Now we shall prove that P(k+1) is also true , now we have ,

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots\dots\dots\dots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)} \qquad \text{(using (1))}$$

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{1}{3k+2}\left[\frac{k}{2} + \frac{1}{3k+5}\right]$$

$$= \frac{1}{3k+2}\left[\frac{3k^2+5k+2}{2(3k+5)}\right] = \frac{(k+1)(3k+2)}{2(3k+2)(3k+5)}$$

$$= \frac{k+1}{2(3k+5)} = \frac{k+1}{6k+10} = \frac{k+1}{6(k+1)+4} .$$

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

17. **Let P(n) : 1 + 2 + 3 + …………..+ n < $\frac{(2n+1)^2}{8}$**

For n = 1 , P(1) : 1 < $\frac{(2.1+1)^2}{8}$ = $\frac{9}{8}$ $\Rightarrow$ 1 < $\frac{9}{8}$, which is true .

Assume that P(k) is true for some positive integer k , we have ,

1 + 2 + 3 + …………..+ k < $\frac{(2k+1)^2}{8}$ …………………(1).

Now we shall prove that P(k+1) is also true , now we have ,

1 + 2 + 3 + …………..+ k + (k+1) < $\frac{(2k+1)^2}{8}$ + (k+ 1) (using (1)),

$$< \frac{1}{8}[4k^2 + 4k + 1 + 8k + 8]$$

$$< \frac{1}{8}[4k^2 + 12k + 9]$$

$$< \frac{1}{8}[(2k + 3)^2]$$

$$< \frac{1}{8}[(2(k + 1) + 1)^2] .$$

Thus P(k+1) is true whenever P(k) is true.

Hence by the principle of mathematical induction(PMI) the statement P(n) is true for all positive integers of n.

18. **Let P(n) : 10^{2n-1} + 1 is divisible by 11.**

For n = 1, P(1) : $(10)^{2.1-1}$ +1 = 10 + 1 = 11, is divisible by 11 .

∴ P(1) is true.

Assume that P(k) is true for some positive integer k , we have ,

P(k) : 10^{2k-1} + 1 is divisible by 11.

$\Rightarrow$ 10^{2k-1} + 1 = 11d…………….(1), for some d$\in$ N.

Now we shall prove that P(k+1) is divisible by 11, we have,

$10^{2(k+1)-1} + 1 = 10^{2k-1+2} + 1 = 10^{2k-1}. 10^2 + 1 = 10^{2k-1}.100 + 1$

$= (11d - 1)100 + 1$ ($\because$ from (1))

$= 1100d - 99 = 11(100d - 9) = 11m$, where $m = 100d - 9 \in N$.

Thus P(k+1) is also true.

Hence by PMI , P(n) is divisible by 11 for all n$\in$ N.

19. Let P(n) : x^{2n} - y^{2n} is divisible by (x + y)

For n = 1 , P(1) : $x^{2.1} - y^{2.1} = x^2 - y^2 = (x + y)(x - y)$ is divisible by (x + y).

$\therefore$ P(1) is true.

Assume that P(k) is true for some positive integer k , we have ,

P(k) : $x^{2k} - y^{2k}$, is divisible by (x + y) .

P(k) : $x^{2k} - y^{2k} = (x + y)d$ ....(1), for some $d \in N$.

Now we shall prove that P(k+1) is divisible by (x + y) , we have ,

$x^{2(k+1)} - y^{2(k+1)} = x^{2k+2} - y^{2k+2} = x^{2k}.x^2 - y^{2k}.y^2$

$= x^{2k}.x^2 - x^2.y^{2k} + x^2.y^{2k} - y^{2k}.y^2$ (add & sub. $x^2.y^{2k}$)

$= x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2)$

$= x^2. (x + y)d + y^{2k} (x - y)(x + y)$ ($\because$ from (1)),

$= (x + y) [x^2d + y^{2k}(x - y)]$

$= (x + y) m$, where $m = x^2d + y^{2k}(x - y) \in N$.

Thus P(k+1) is also true .

Hence by PMI , P(n) is divisible by all n $\in$ N.

20. Let P(n) : $3^{2n+2} - 8n - 9$ is divisible by 8 .

For n = 1, P(1) : $3^{2.1+2} - 8.1 - 9 = 81 - 8 - 9 = 64$, is divisible by 8.

$\therefore$ P(1) is true.

Assume that P(k) is true for some positive integer k , we have ,

P(k) : $3^{2k+2} - 8k - 9$, is divisible by 8.

i.e. P(k) : $3^{2k+2} - 8k - 9 = 8d$ …………….(1), for some $d \in N$.

Now we shall prove that P(k+1) is divisible by 8 , we have ,

$$3^{2(k+1)+2} - 8(k+1) - 9 = 3^{2k+2+2} - 8k - 8 - 9 = 3^2.3^{2k+2} - 8k - 8 - 9$$

$$= 9.3^{2k+2} - 8k - 17 = 9(8d + 8k + 9) - 8k - 17 \quad (\because \text{using } (1))$$

$$= 72d + 72k + 81 - 8k - 17$$

$$= 72d + 64k - 64 = 8 (9d + 8k + 8)$$

$$= 8m \text{ , where } m = 9d + 8k + 8 \in N.$$

Thus P(k+1) is also divisible by 8 .

Hence by PMI , P(n) is divisible by all positive integers of n.

21. Let P(n) : $(2n + 7) < (n+3)^2$.

For n = 1 , P(1) : $(2.1 + 7) < (1+3)^2 \Rightarrow 9 < 16$, is true .

Assume that P(k) is true for some positive integer k , we have ,

Assume that P(k) is true for some positive integer k , we have ,

P(k) : $(2k + 7) < (k + 3)^2$ ………………(1).

Now we shall prove that P(k+1) is also true , for we have ,

We have to show that $2(k + 1) < [(k + 1) + 3]^2$.

Consider , $2(k + 1) + 7 = 2k + 2 + 7 = 2k + 7 + 2 < (k + 3)^2 + 2$, ($\because$ from (1))

$$< k^2 + 6k + 9 + 2 < k^2 + 8k + 16 = (k + 4)^2$$

$\therefore$ $2(k + 1) + 7 < [(k + 1) + 3]^2$.

Thus P(k+1) is also true whenever P(k) is true.

Hence by PMI the statement P(n) is true for all $n \in N$.