TRIGONOMETRY

One mark questions

1. Define an angle.

Solution: An angle is an amount of rotation of a half-line in a plane about its end from its initial position to a terminal position.

2. What do you mean by measure of an angle?

Solution: The amount of rotation from initial position to terminal position is called measure of an angle.

3. Define one radian.

Solution: A Radian is the angle subtended at the centre of a circle by an arc, whose length is equal to its radius. 1 radian is denoted by 1^c. One radian is also called one circular measure.

4. Convert the following into radian measure:

a) $18°$ b) $7.5°$ c) $105°$ d) $\frac{1}{2}^{\circ}$ e) $36°32'34''$

Solution:

a) $1° = \left(\frac{\pi}{180}\right)^c \quad \therefore 18° = \left(\frac{\pi}{180} \times 18\right)^c = \left(\frac{\pi}{10}\right)^c$

b) $7.5° = \left(\frac{\pi}{180} \times 7.5\right)^c = \left(\frac{\pi}{180} \times \frac{15}{2}\right)^c = \left(\frac{\pi}{24}\right)^c$

c) $105° = \left(\frac{\pi}{180} \times 105\right)^c = \left(\frac{7\pi}{12}\right)^c$

d) $\frac{1}{2}^{\circ} = \left(\frac{\pi}{180} \times \frac{1}{2}\right)^c = \left(\frac{\pi}{360}\right)^c$

e) $36°32'34'' = \left(36 + \frac{32}{60} + \frac{34}{3600}\right)^{\circ} = (36.5427)^{\circ} = (36.5427) \times (0.0174)^c$

(since $1° = (0.0174)^c$) $\therefore\ 36°32'34'' \approx 0.6358$ radian.

5. Express the following in degree measure:

a) $\frac{\pi}{36}$ b) $\frac{7\pi}{6}$ c) $\frac{23\pi}{4}$ d) $\frac{7\pi}{24}$ e) $\frac{5\pi}{12}$

Solution:

a) $1^c = \left(\frac{180}{\pi}\right)^\circ \quad \therefore \frac{\pi}{36} = \left(\frac{180}{\pi} \times \frac{\pi}{36}\right)^\circ = 5°$

b) $\frac{7\pi}{6} = \left(\frac{180}{\pi} \times \frac{7\pi}{6}\right)^\circ = 210°$

c) $\frac{23\pi}{4} = \left(\frac{180}{\pi} \times \frac{23\pi}{4}\right)^\circ = 1035°$

d) $\frac{7\pi}{24} = \left(\frac{180}{\pi} \times \frac{7\pi}{24}\right)^\circ = \left(\frac{105}{2}\right)^\circ = (52.5)^\circ$

e) $\frac{5\pi}{12} = \left(\frac{180}{\pi} \times \frac{5\pi}{12}\right)^\circ = 75^\circ$.

6. Prove that $sin\,A \cdot cot\,A = cos\,A$

Solution: $LHS = sin\,A \cdot \frac{cos\ A}{sin\ A} = cos\,A = RHS$.

7. Prove that $cot\,A \cdot sec\,A \cdot \sin A = 1$

Solution: $LHS = \frac{cos\ A}{sin\ A} \cdot \frac{1}{\cos A} \cdot sin\,A = 1 = RHS$.

8. Prove that $cot^2\,\theta \cdot (1 - cos^2\,\theta) = cos^2\,\theta$

Solution: $LHS = \frac{cos^2\,\theta}{sin^2\,\theta} \cdot sin^2\,\theta = cos^2\,\theta$.

9. Prove that $cosec\,\alpha \cdot \sqrt{1 - sin^2\,\alpha} = \cot\alpha$

Solution: $LHS = \frac{1}{sin\ \alpha} \cdot cos\,\alpha = cot\,\alpha = RHS$.

10. Prove that $(1 + tan^2\,\theta).(1 - cos^2\,\theta) = tan^2\,\theta$

Solution: $LHS = sec^2\,\theta \cdot sin^2\,\theta = \frac{1}{cos^2\,\theta} \cdot sin^2\,\theta = tan^2\,\theta = RHS$.

11. Prove that $(1 + cot^2\,A) \cdot sin^2\,A = 1$

Solution: $LHS = cosec^2\,A \cdot sin^2\,A$

$= (cosec\,A \cdot sin\,A)^2 = 1^2 = 1 = RHS$.

12. Prove that $\cos\alpha \cdot cosec\,\alpha \cdot \sqrt{sec^2\alpha - 1} = 1.$

Solution: $LHS = \cos\alpha \cdot \frac{1}{\sin\alpha} \cdot \sqrt{\tan^2\alpha}$

$$= \cos\alpha \cdot \frac{1}{\sin\alpha} \cdot \tan\alpha = \frac{\cos\alpha}{\sin\alpha} \cdot \frac{\sin\alpha}{\cos\alpha} = 1 = RHS.$$

13. Prove that $\cos\theta \cdot \sqrt{\cot^2\theta - 1} = \sqrt{cosec^2\,\theta - 1}.$

Solution: $LHS = \cos\theta \cdot cosec\,\theta$

$$= \cos\theta \cdot \frac{1}{\sin\theta} = \cot\theta = \sqrt{cosec^2\,\theta - 1} = RHS.$$

14. Prove that $\sin^2 A \cdot \cot^2 A + \sin^2 A = 1.$

Solution: $LHS = \sin^2 A \cdot \frac{\cos^2 A}{\sin^2 A} + \sin^2 A$

$$= \cos^2 A + \sin^2 A = 1 = RHS.$$

15. Define coterminal angles. Give two examples.

Solution: If the initial and terminal sides of two angles are equal then the angles are called coterminal angles.

16. Prove that $cosec\,(180° - \theta) = cosec\,\theta.$

Solution: $cosec\,(180° - \theta) = \frac{1}{\sin(180° - \theta)} = \frac{1}{\sin\theta} = cosec\,\theta.$

17. Prove that $\sec(180° - \theta) = -\sec\theta.$

Solution: $\sec(180° - \theta) = \frac{1}{\cos(180° - \theta)} = \frac{1}{-\cos\theta} = -\sec\theta.$

18. Prove that $\cot(180° - \theta) = -\cot\theta.$

Solution: $\cot(180° - \theta) = \frac{1}{\tan(180° - \theta)} = \frac{1}{-\tan\theta} = -\cot\theta.$

19. Prove that $cosec\,(180° + \theta) = -\,cosec\,\theta.$

Solution: $cosec\,(180° + \theta) = \frac{1}{\sin(180° + \theta)} = \frac{1}{-\sin\theta} = -\,cosec\,\theta.$

20. Prove that $\sec(180° + \theta) = -\sec\theta.$

Solution: $\sec(180° + \theta) = \frac{1}{\cos(180° + \theta)} = \frac{1}{-\cos\theta} = -\sec\theta.$

21. Prove that $cot\,(180° + \theta) = -cot\,\theta$.

Solution: $cot\,(180° + \theta) = \frac{1}{tan\ (180° + \theta)} = \frac{1}{-tan\ \theta} = -cot\,\theta$.

22. Prove that $cosec\,(270° - \theta) = -sec\,\theta$.

Solution: $cosec\,(270° - \theta) = \frac{1}{sin\ (270° - \theta)} = \frac{1}{-cos\ \theta} = -sec\,\theta$.

23. Prove that $sec\,(270° - \theta) = -cosec\,\theta$.

Solution: $sec\,(270° - \theta) = \frac{1}{cos\ (270° - \theta)} = \frac{1}{-sin\ \theta} = -cosec\,\theta$.

24. Prove that $cot\,(270° - \theta) = tan\,\theta$.

Solution: $cot\,(270° - \theta) = cot\,(180° + 90° - \theta)$

$$= cot\,(90° - \theta) = tan\,\theta.$$

25. Prove that $cosec\,(270° + \theta) = -sec\,\theta$.

Solution: $cosec\,(270° + \theta) = \frac{1}{sin\ (270° + \theta)} = \frac{1}{-cos\ \theta} = -sec\,\theta$.

26. Prove that $sec\,(270° + \theta) = cosec\,\theta$.

Solution: $sec\,(270° + \theta) = \frac{1}{cos\ (270° + \theta)} = \frac{1}{sin\ \theta} = cosec\,\theta$.

27. Prove that $cot\,(270° + \theta) = -\ tan\,\theta$.

Solution: $cot\,(270° + \theta) = cot\,(180° + 90° + \theta)$

$$= cot\,(90° + \theta) = -\ tan\,\theta.$$

28. Define periodic function.

Solution: A function $f(x)$ is said to be periodic if

$$f(x + T) = f(x)$$

for some real number T.

If T is the smallest positive real number satisfying $f(x + T) = f(x)$ then T is called the period of $f(x)$.

29. Write the period of all the trigonometric functions (1 mark each).

Solution:
Period of $sin\, x$ is 2π ; Period of $cos\, x$ is 2π ;
Period of $tan\, x$ is π ; Period of $cot\, x$ is π ;
Period of $cosec\, x$ is π ; Period of $sec\, x$ is π.

30. Prove that $sin\,(45° + A) = \frac{1}{\sqrt{2}}(sin\, A + cos\, A)$

Solution: $sin(45° + A) = sin\, 45°. cos\, A + cos\, 45°. sin\, A$

$$= \frac{1}{\sqrt{2}} cos\, A + \frac{1}{\sqrt{2}} sin\, A) = \frac{1}{\sqrt{2}}(sin\, A + cos\, A).$$

31. Find the value of $sin\, 54°$

Solution: $sin\, 54° = sin\,(90° - 36°) = cos\, 36° = \frac{\sqrt{5}+1}{4}$.

32. Find the value of $cos\, 72°$

Solution: $cos\, 72° = cos\,(90° - 18°) = sin\, 18° = \frac{\sqrt{5}-1}{4}$.

33. Prove that $2\, sin\, 15°\; cos 15° = \frac{1}{2}$.

Solution: $2\, sin\, 15°\; cos 15° = sin\,(2 \cdot 15°) = sin 30° = \frac{1}{2}$

34. Prove that $sin\, \frac{\pi}{8} cos \frac{\pi}{8} = \frac{1}{2\sqrt{2}}$

Solution: $LHS = \frac{2}{2}. sin\, \frac{\pi}{8}. cos \frac{\pi}{8} = \frac{1}{2}\left(2 sin\, \frac{\pi}{8}. cos \frac{\pi}{8}\right)$

$$= \frac{1}{2} sin\, \left(2 \cdot \frac{\pi}{8}\right) = \frac{1}{2} sin\, \frac{\pi}{4} = \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

35. Prove that $\frac{sin\, x}{1 + cos\, x} = tan\, \frac{x}{2}$

Solution: $\frac{sin\, x}{1 + cos\, x} = \frac{2 sin\, \frac{x}{2}. cos\, \frac{x}{2}}{2 cos^2 \frac{x}{2}} = \frac{sin\, \frac{x}{2}}{cos\, \frac{x}{2}} = tan\, \frac{x}{2}$.

36. Prove that $\sqrt{\frac{1\ +\ cos\ \ 2x}{1\ -\ cos\ \ 2x}}\ = cot\, x$

Solution: $\sqrt{\frac{1\ +\ cos\ \ 2x}{1\ -\ cos\ \ 2x}}\ \ = \sqrt{\frac{2cos^2 x}{2sin^2 x}}\ \ = \frac{cos\ x}{sin\ x}\ \ = \ cot\ x.$

37. Express $2\,sin2A \cdot cosA$ as a sum.

Solution: $2\ sin\ 2A. cos\ A\ \ = sin\ (2A\ +A) + sin\ (2A\ -A)$

$= sin\ 3A\ + sin\ A.$

38. Express $cos\,7\theta \cdot cos\,5\theta$ as a sum.

Solution: $cos\,7\theta \cdot cos\,5\theta\ \ = \frac{1}{2}[cos\ (7\theta + 5\theta) + cos\ (7\theta - 5\theta)]$

$= \frac{1}{2}[cos\ 12\theta + cos\ 2\theta].$

39. Express $sin\ \ 4A\ +\ sin\ \ 2A$ as product.

Solution: $sin\,4A\ +\ sin\,2A\ = 2\,sin\left(\frac{4A\ +\ 2A}{2}\right).cos\left(\frac{4A -\ 2A}{2}\right)$

$= 2\,sin\,3A\,.\,cos\,A.$

40. Express $cos\,60° - cos\,20°$ as a product.

Solution: $cos\,60° - cos\,20°\ \ = -\ 2\,sin\left(\frac{60°\ +\ 20°}{2}\right).sin\left(\frac{60°\ -\ 20°}{2}\right)$

$= -\,2\ \ sin\,40° \cdot sin\,20°.$

41. Express $cos\,55° + sin\,55°$ as a product.

Solution: $cos\,55° + sin\,55° = cos\,55° + cos\ (90° - 55°)$

$= cos\,55° + cos\,35°\ \ \ = 2\,cos\left(\frac{55°\ -\ 35°}{2}\right).cos\left(\frac{55°\ -\ 35°}{2}\right)$

$= 2\,cos\,45°\,.\,cos\,10°\ \ \ = 2.\frac{1}{\sqrt{2}}.cos\,10°\ \ \ = \sqrt{2}.cos\,10°.$

42. Define a trigonometric equation.

Solution: An equation involving one or more trigonometric functions are called trigonometric equation.

43. What do you mean by general solution of a trigonometric equation.

Solution: The set of all values of the angle, θ (involved in a trigonometric equation) satisfying the trigonometric equation, $f(\theta) = 0$ is called general solution of the trigonometric equation.

44. Find the general solution of $sin\,\theta = 0$.

Solution: When $= 0\ ,\ \pm\pi\ ,\ \pm 2\pi\ ,\ \pm 3\pi, \ldots\ sin\,\theta = 0$.
$\therefore$ the general solution of $sin\,\theta = 0$ is $\theta = n\pi, n \in Z$.

45. Find the general solution of $cos\,\theta = 0$.

Solution: When $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\ cos\,\theta = 0$.
$\therefore$ the general solution of $cos\,\theta = 0$ is $\theta = (2n + 1)\frac{\pi}{2}\ , n \in Z$.

46. Find the general solution of $tan\,\theta = 0$.

Solution: When $\theta = 0\ ,\ \pm\pi\ ,\ \pm 2\pi\ ,\ \pm 3\pi, \ldots\ tan\,\theta = 0$.
$\therefore$ the general solution of $tan\,\theta = 0$ is $\theta = n\pi, n \in Z$.

47. Find the principal value of $cos\,x = \frac{\sqrt{3}}{2}$

Solution: $cos\,x = \frac{\sqrt{3}}{2} > 0\ \therefore x$ lies in the I or IV quadrant.
Principal value of $x \in [0,\ \pi]$. Since $cosx$ is positive, the principal value is in the I quadrant.
$cos\,x = \frac{\sqrt{3}}{2} = cos\frac{\pi}{6}$ and $\frac{\pi}{6} \in [0,\ \pi]$. $\therefore$ principal value of x is $\frac{\pi}{6}$.

48. Find the principal value of $sin\,x = \frac{1}{2}$.

Solution: $sin\,x = \frac{1}{2} > 0\ \therefore x$ lies in the I or II quadrant.
Principal value of $x \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$. Since $sinx$ is positive the principal value is in the I quadrant.

$sinx = \frac{1}{2} = sin\frac{\pi}{6}$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$. $\therefore$ principal value of x is $\frac{\pi}{6}$.

49. Find the principal value of $cosec\ \theta = -\frac{2}{\sqrt{3}}$.

Solution: $cosec\ \theta = -\frac{2}{\sqrt{3}} \quad \Rightarrow sin\ \theta = -\frac{\sqrt{3}}{2} < 0$

$\therefore x$ lies in the III or IV quadrant.

Principal value of $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Since $sin\theta$ is negative the principal value is in the IV quadrant.

$sin\ \theta = -\frac{\sqrt{3}}{2} = sin\left(-\frac{\pi}{3}\right)$ and $-\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore$ principal value of x is $-\frac{\pi}{3}$.

50. Find the principal value of $cos\ x = -\frac{\sqrt{3}}{2}$.

Solution: $cos\ x = -\frac{\sqrt{3}}{2} < 0 \ \therefore x$ lies in the II or III quadrant.

Principal value of $x \in [0, \pi]$. Since $cosx$ is negative the principal value is in the II quadrant.

$$cosx = -\frac{\sqrt{3}}{2} = -cos\frac{\pi}{6} = cos\ (180° - 30°)$$
$$= cos150° = cos\frac{5\pi}{6} \text{ and } \frac{5\pi}{6} \in [0, \pi].$$

$\therefore$ principal value of $x\ is\ \frac{5\pi}{6}$.

Two mark questions

1. Define sexagesimal system.

Solution: In this system, a right angle is divided into 90 equal parts. Each part is equal to one degree. One degree is divided into 60 equal parts. Each part is called one minute. One minute is divided into 60 equal parts. Each part is called one second.

One right angle $= 90$ degrees, written as $90°$;

one degree, $1° = 60$ minutes, written as $60'$;

one minute, $1' = 60$ seconds, written as $60''$;

2. Prove that one radian is a constant angle.

Solution: Since the angle subtended at the center of a circle of unit radius, by an arc of length one unit is one radian, the angle subtended at the center by the circumference of the circle is 2π radian. Thus $2\pi^c = 360°$.

$$\therefore \quad \pi^c = 180°.$$

$$\therefore \quad 1^c = \frac{180}{\pi}.$$

This gives the conclusion, 1^c is a constant angle.

3. With usual notations prove that $l = r\theta$.

Solution: We know that in a circle of radius r, an arc of length r, subtends an angle of one radian at the center of the circle. Since the angle subtended at the center by an arc of length r is one radian, the angle subtended by an arc of length l has the measure $= \frac{l}{r}$. Thus if θ is the angle subtended at the center by an arc of length l then $\frac{l}{r} = \theta \;\therefore\; l = r\theta$.

4. With usual notations prove that the area of a sector of a circle is given by $\frac{1}{2}r^2\theta$ or $\frac{1}{2}rl$

Solution: We know that in a circle of radius r, the angle 2π radian traces the area πr^2. Therefore the area of the sector tracing an angle θ radian at the center is $A = \frac{\pi r^2}{2\pi} \times \theta = \frac{1}{2}r^2\theta$.
Since $l = r\theta$ we get $A = \frac{1}{2}r^2\theta = \frac{1}{2}r.r\theta = \frac{1}{2}rl$.

5. An arc of a circle subtends $15°$ at the centre. If the radius is 4 cm, find the length of the arc and area of the sector formed.

Solution: Here $r = 4\,cm$ and $\theta = 15° = 15 \times \frac{\pi}{180} = \frac{\pi}{12}$

$\therefore l = r\theta \Longrightarrow l = 4 \times \frac{22}{7} \times \frac{1}{12} = \frac{22}{21}\ cm.$

Area of the sector $= \frac{1}{2}rl = \frac{1}{2} \times 4 \times \frac{22}{21} = \frac{44}{21}\ sq.cm.$

6. An arc of length $\frac{11}{3}$ cm subtends an angle of $30°$ at the centre of a circle. Find the area of the sector of the circle so formed.

Solution: Given $= \frac{11}{3}$, $\theta = 30° = \frac{\pi}{6}$.

We have $l = r\theta \Longrightarrow r = \frac{l}{\theta} = \frac{11}{3} \times \frac{6}{\pi} = \frac{11}{3} \times \frac{6}{22} \times 7 = 7cm$

$A = \frac{1}{2} r^2\theta = \frac{1}{2} (7)^2 \frac{\pi}{6} = \frac{1}{2} \times 49 \times \frac{22}{7} \times \frac{1}{6} = \frac{77}{6}\ sq.cm.$

7. If, in two circles arcs of the same length subtend angles of $60°\ and\ 75°$ at the centre, find the ratio of their radii.

Solution: Let r_1 and r_2 be the radii of the two circles.

Given $\theta_1 = 60° = \left(\frac{\pi}{180} \times 60\right) = \frac{\pi}{3}$ and $\theta_2 = 75° = \left(\frac{\pi}{180} \times 75\right) = \frac{5\pi}{12}$

Now the length of each of the arc, $l = r_1\theta_1 = r_2\theta_2$, which gives

$r_1 \times \frac{\pi}{3} = r_2 \times \frac{5\pi}{12} \Longrightarrow \frac{r_1}{r_2} = \frac{5}{4}$. Hence $r_1 : r_2 = 5:4$.

8. Prove the following:

a) $sin\,\theta . cosec\,\theta = 1$, $cosec\,\theta = \frac{1}{sin\ \theta}$;

b) $cos\,\theta . sec\,\theta = 1$, $sec\,\theta = \frac{1}{cos\ \theta}$;

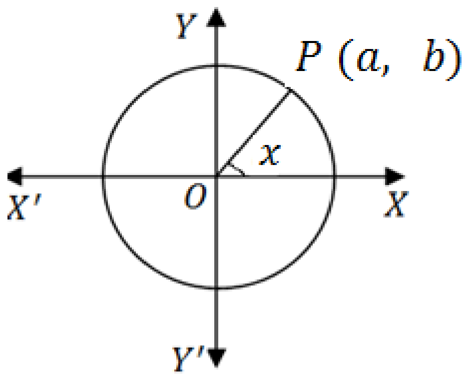
c) $tan\,\theta . cot\,\theta = 1$, $cot\,\theta = \frac{1}{tan\ \theta}$;

d) $\frac{sin\ \theta}{cos\ \theta} = tan\,\theta$; e) $\frac{cos\ \theta}{sin\ \theta} = cot\,\theta$.

f) $sin^2\theta + cos^2\theta = 1$;

g) $1 + tan^2\theta = sec^2\theta$; h) $1 + cot^2\theta = cosec^2\theta$.

Solution: Consider a circle of radius r centered at the origin of the Cartesian coordinate system. Let $P(x,\ y)$ be a point on the circle. Join OP. Define angle θ having OX has the initial direction and OP as the terminal direction. Consider a circle of radius r centered at the origin of



the Cartesian coordinate system. Let $P(x,\ y)$ be a point on the circle. Join OP. Define angle θ having OX has the initial direction and OP as the terminal direction. Then

$$sin\ \theta\ =\ \frac{y}{r}\,,os\ \theta\ =\ \frac{x}{r}\,,\ tan\ \theta\ =\ \frac{y}{x}\,,$$
$$cot\ \theta\ =\ \frac{x}{y},\ sec\ \theta\ =\ \frac{r}{x},\ cosec\ \theta\ =\ \frac{r}{y}.$$

a) $sin\ \theta.\,cosec\ \theta\ =\ \frac{y}{r}.\frac{r}{y}\ =\ 1.\ \therefore\ cosec\ \theta\ =\ \frac{1}{sin\ \theta}$;

b) $cos\ \theta.\,sec\ \theta = \frac{x}{r}.\frac{r}{x}\ =\ 1.\ \ \therefore\ sec\ \theta\ =\ \frac{1}{cos\ \theta}$;

c) $tan\ \theta.\,cot\ \theta = \frac{y}{x}.\frac{x}{y}\ =\ 1.\ \ \therefore\ \ cot\ \theta\ =\ \frac{1}{tan\ \theta}$;

d) $\frac{sin\ \theta}{cos\ \theta}\ =\ \frac{y/r}{x/r}\ =\ \frac{y}{x}\ = tan\ \theta$;

e) $\frac{cos\ \theta}{sin\ \theta}\ =\ \frac{x/r}{y/r}\ =\ \frac{x}{y}\ = cot\ \theta$;

f) $sin^2\ \theta\ +\ cos^2\ \theta\ = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{y^2}{r^2} + \frac{x^2}{r^2}\ =\ \frac{y^2+x^2}{r^2} = \frac{r^2}{r^2}\ = 1$;

g) $1\ +\ tan^2\ \theta\ =\ 1 + \left(\frac{y}{x}\right)^2 =\ 1\ + \frac{y^2}{x^2}\ =\ \frac{x^2+y^2}{x^2}\ =\ \frac{r^2}{x^2}\ =\ sec^2\ \theta$;

h) $1\ +\ cot\ ^2\theta\ =\ 1 + \left(\frac{x}{y}\right)^2\ =\ 1 + \frac{x^2}{y^2} = \frac{y^2+x^2}{y^2}\ =\ \frac{r^2}{y^2}\ =\ cosec^2\ \theta.$

9. Prove that $cosec^2\ A \cdot tan^2\ A - 1 = tan^2\ A$.

Solution: $LHS\ =\ (1 + cot^2 A)\cdot tan^2\ A - 1$
$= tan^2\ A + cot^2\ A \cdot tan^2\ A - 1$
$= tan^2\ A + (cot\ A \cdot tan\ A)^2 - 1$
$= tan^2\ A + 1 - 1 = tan^2\ A = RHS$.

10. Prove that $\frac{cos\ \alpha}{sec\ \alpha} + \frac{sin\ \alpha}{cosec\ \alpha} =\ 1.$

Solution: $LHS\ =\ \frac{cos\ \alpha}{\left(\frac{1}{cos\ \alpha}\right)}\ +\ \frac{sin\ \alpha}{\left(\frac{1}{sin\ \alpha}\right)}\ =\ cos^2\ \alpha\ +\ sin^2\ \alpha\ \ = 1 = RHS.$

11. Prove that $cos^4\ A - sin^4\ A = 2\ cos^2\ A -\ 1$.

Solution: $LHS\ =\ (cos^2\ A)^2 - (sin^2\ A)^2$
$=\ (cos^2\ A + sin^2\ A)(cos^2\ A - sin^2\ A)$
$= 1 \cdot (cos^2\ A - sin^2\ A)$

$$= cos^2 A - (1 - cos^2 A)$$
$$= cos^2 A - 1 + cos^2 A = 2cos^2 A - 1 = RHS .$$

12. Prove that $(sec\,\theta \cdot cot\,\theta)^2 - (cos\,\theta \cdot cosec\,\theta)^2 = 1.$

Solution: $LHS = \left(\frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta}\right)^2 - \left(\cos\theta \cdot \frac{1}{\sin\theta}\right)^2$
$$= \left(\frac{1}{\sin\theta}\right)^2 - \left(\frac{\cos\theta}{\sin\theta}\right)^2 = (\operatorname{cosec}\theta)^2 - (\cot\theta)^2 = 1 = RHS.$$

13. Prove that $sec\,\theta - tan\,\theta \cdot sin\,\theta = cos\,\theta.$

Solution: $LHS = \frac{1}{cos\,\theta} - \frac{sin\,\theta}{cos\,\theta} \cdot sin\,\theta$
$$= \frac{1}{cos\,\theta} - \frac{sin^2\,\theta}{cos\,\theta} = \frac{1 - sin^2\,\theta}{cos\,\theta} = \frac{(cos\,\theta)^2}{cos\,\theta} = cos\,\theta = RHS.$$

14. Prove that $tan\,\theta + cot\,\theta = sec\,\theta \cdot cosec\,\theta.$

Solution: $LHS = \frac{sin\,\theta}{cos\,\theta} + \frac{cos\,\theta}{sin\,\theta}$
$$= \frac{sin^2\,\theta + cos^2\,\theta}{cos\,\theta \cdot sin\,\theta}$$
$$= \left(\frac{1}{cos\,\theta}\right) \cdot \left(\frac{1}{sin\,\theta}\right) = sec\,\theta \cdot cosec\,\theta = RHS.$$

15. Prove that $(cos\,\theta + sin\,\theta)^2 + (cos\,\theta - sin\,\theta)^2 = 2.$

Solution:

$LHS = cos^2\,\theta + sin^2\,\theta + 2\,cos\,\theta\,.\,sin\,\theta$
$$+ cos^2\,\theta + sin^2\,\theta - 2\cos\theta\,.\sin\theta = 1 + 1 = 2 = RHS.$$

16. Prove that $tan^2\alpha + tan^4\alpha = sec^4\alpha - sec^2\alpha.$

Solution: $LHS = tan^2\alpha + tan^4\alpha = tan^2\alpha\,(1 + tan^2\alpha)$
$$= (sec^2\alpha - 1)sec^2\alpha = sec^4\alpha - sec^2\alpha = RHS.$$

17. Prove that $\frac{1}{1 - cos\,\alpha} + \frac{1}{1 + cos\,\alpha} = 2\,cosec^2\alpha$

Solution: $LHS = \frac{1 + cos\,\alpha + 1 - cos\,\alpha}{(1 - cos\,\alpha)(1 - cos\,\alpha)}$
$$= \frac{2}{1 - cos^2\,\alpha} = \frac{2}{sin^2\,\alpha} = 2\,cosec^2\alpha = RHS.$$

18. Prove that $\frac{1}{1 + cos^2\,\alpha} + \frac{1}{1 + sec^2\,\alpha} = 1.$

Solution: $LHS = \frac{1}{1 + cos^2 \alpha} + \frac{1}{1 + \frac{1}{cos^2 \alpha}}$

$= \frac{1}{1+ cos^2 \alpha} + \frac{cos^2 \alpha}{1+ cos^2 \alpha} = \frac{1 + cos^2 \alpha}{1 + cos^2 \alpha} = 1 = RHS.$

19. Prove that $sin^2 90° + cos^2 60° - 2\, sin^2 45° + \frac{3}{2} sin^2 30° = \frac{5}{8}$.

Solution: $LHS = 1^2 + \left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{3}{2}\left(\frac{1}{2}\right)^2$

$= 1 + \frac{1}{4} - 1 + \frac{3}{8} = \frac{1}{4} + \frac{3}{8} = \frac{5}{8} = RHS.$

20. Prove that $cot^2 \frac{\pi}{6} - 2cos^2 \frac{\pi}{3} - \frac{3}{4} sec^2 \frac{\pi}{4} - 4sin^2 \frac{\pi}{6} = 0.$

Solution: $LHS = \left(\sqrt{3}\right)^2 - 2\left(\frac{1}{2}\right)^2 - \frac{3}{4}\left(\sqrt{2}\right)^2 + 4\left(\frac{1}{2}\right)^2$

$= 3 - \frac{1}{2} - \frac{3}{2} - 1 = 0 = RHS.$

21. Prove that $cosec^2 45° \cdot cos 60° \cdot \sin 90° \cdot sec^2 30° = \frac{4}{3}$.

Solution: $LHS = \left(\sqrt{2}\right)^2 \cdot \frac{1}{2} \cdot 1 \cdot \left(\frac{2}{\sqrt{3}}\right)^2 = 2 \cdot \frac{1}{2} \cdot 1 \cdot \frac{4}{3} = \frac{4}{3} = RHS.$

22. Prove that $tan^2 60° + 2\, tan^2 45° = 5.$

Solution: $LHS = \left(\sqrt{3}\right)^2 + 2 = 3 + 2 = 5 = RHS.$

23. Prove that $tan^2 \frac{\pi}{6} + tan^2 \frac{\pi}{4} + tan^2 \frac{\pi}{3} = \frac{13}{3}$.

Solution: $LHS = \left(\frac{1}{\sqrt{3}}\right)^2 + (1)^2 + \left(\sqrt{3}\right)^2$

$= \frac{1}{3} + 1 + 3 = \frac{1 + 3 + 9}{3} = \frac{13}{3} = RHS.$

24. Prove that $\frac{1 - cos 30°}{1 + cos 30°} = \frac{2 - \sqrt{3}}{2 + \sqrt{3}}$

Solution: $LHS = \frac{1 - \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}} = \frac{\frac{2 - \sqrt{3}}{2}}{\frac{2 + \sqrt{3}}{2}} = \frac{2 - \sqrt{3}}{2 + \sqrt{3}} = RHS.$

25. Find x, $x \sin^2 \frac{\pi}{4} . \cos^2 \frac{\pi}{4} = \tan^2 \frac{\pi}{3}$

Solution: $x \left(\frac{1}{\sqrt{2}}\right)^2 \cdot \left(\frac{1}{\sqrt{2}}\right)^2 = \left(\sqrt{3}\right)^2 \implies \frac{x}{4} = 3 \quad \therefore x = 12.$

26. Prove that trigonometric ratios of $2\pi n + \theta$ are same as the trigonometric ratio of θ.

Solution: θ, $2\pi + \theta$, $4\pi + \theta$, ... are co-terminal angles. In general θ is co-terminal with the angle $2\pi n + \theta$, $\forall\, n \in Z$. Since the points corresponding to the coterminal angles are the same, the trigonometric ratios of coterminal angles are the same. Thus $\sin(2\pi n + \theta) = \sin\theta$, $\cos(2\pi n + \theta) = \cos\theta$ and $\tan(2\pi n + \theta) = \tan\theta$, $\forall\, n \in Z$.

27. Prove that trigonometric ratios of $2\pi n - \theta, \forall\, n \in Z$ are same as the trigonometric ratio of $-\theta$.

Solution: $-\theta$, $2\pi - \theta$, $4\pi - \theta$, ... are co-terminal angles. In general $-\theta$ is co-terminal with the angle $2\pi n - \theta$, $\forall\, n \in Z$. Since the points corresponding to the coterminal angles are the same, the trigonometric ratios of coterminal angles are the same. Thus $\sin(2\pi n - \theta) = \sin(-\theta)$, $\cos(2\pi n - \theta) = \cos(-\theta)$ and $\tan(2\pi n - \theta) = \tan(-\theta)$, $\forall\, n \in Z$.

28. Prove that $\sin(180° - \theta) = \sin\theta$.

Solution:
$$\begin{aligned} \sin(180° - \theta) &= \sin 180°.\cos\theta - \cos 180°.\sin\theta \\ &= 0.\cos\theta - (-1).\sin\theta = \sin\theta. \end{aligned}$$

29. Prove that $\cos(180° - \theta) = -\cos\theta$.

Solution:
$$\begin{aligned} \cos(180° - \theta) &= \cos 180°.\cos\theta + \sin 180°.\sin\theta \\ &= (-1).\cos\theta + 0.\sin\theta = -\cos\theta. \end{aligned}$$

30. Prove that $\tan(180° - \theta) = -\tan\theta$.

Solution:
$$\begin{aligned} \tan(180° - \theta) &= \frac{\tan 180° - \tan\theta}{1 + \tan 180°.\tan\theta} \\ &= \frac{0 - \tan\theta}{1 + 0.\tan\theta} = -\tan\theta. \end{aligned}$$

31. Prove that $sin\,(180° + \theta) = -\,sin\,\theta$.

Solution: $sin\,(180° + \theta) = sin\,180°.\,cos\,\theta + cos\,180°.\,sin\,\theta$
$= 0.\,cos\,\theta + (-1).\,sin\,\theta = -\,sin\,\theta.$

32. Prove that $cos\,(180° + \theta) = -\,cos\,\theta$.

Solution: $cos\,(180° + \theta) = cos\,180°.\,cos\,\theta - sin\,180°.\,sin\,\theta$
$= (-1).\,cos\,\theta - 0.\,sin\,\theta = -\,cos\,\theta.$

33. Prove that $tan\,(180° + \theta) = tan\,\theta$.

Solution: $tan\,(180° + \theta) = \frac{tan\,180° + tan\,\theta}{1 - tan\,180°.tan\,\theta}$
$= \frac{0 + tan\,\theta}{1 - 0.tan\,\theta} = tan\,\theta.$

34. Prove that $sin\,(270° - \theta) = -\,cos\,\theta.$

Solution: $sin\,(270° - \theta) = sin\,270°.\,cos\,\theta - cos\,270°.\,sin\,\theta$
$= -1.\,cos\,\theta - 0.\,sin\,\theta = -\,cos\,\theta.$

35. Prove that $cos\,(270° - \theta) = sin\,\theta.$

Solution: $cos\,(270° - \theta) = cos\,270°.\,cos\,\theta + sin\,270°.\,sin\,\theta$
$= 0.\,cos\,\theta + (-1).\,sin\,\theta = -\,sin\,\theta.$

36. Prove that $tan\,(270° - \theta) = cot\,\theta.$

Solution: $tan\,(270° - \theta) = tan\,(180° + 90° - \theta)$
$= tan\,(90° - \theta) = cot\,\theta.$

37. Prove that $sin\,(270° + \theta) = -\,cos\,\theta.$

Solution: $sin\,(270° + \theta) = sin\,270°.\,cos\,\theta + cos\,270°.\,sin\,\theta$
$= -1.\,cos\,\theta + 0.\,sin\,\theta = -\,cos\,\theta.$

38. Prove that $cos\,(270° + \theta) = sin\,\theta.$

Solution: $cos\,(270° + \theta) = cos\,270°.\,cos\,\theta - sin\,270°.\,sin\,\theta$
$= 0.\,cos\,\theta - (-1).\,sin\,\theta = sin\,\theta.$

39. Prove that $tan\,(270° + \theta) = -\,cot\,\theta.$

Solution: $tan\,(270° + \theta) = tan\,(180° + 90° + \theta)$
$= tan\,(90° + \theta) = -cot\,\theta.$

40. Prove that $tan\,(x + y) = \dfrac{tan\ x + tan\ y}{1 - tan\ x.tan\ y}$

Solution: $tan\,(x + y) = \dfrac{sin\ (x + y)}{cos\ (x + y)} = \dfrac{sin\ x.cos\ y + cos\ x.sin\ y}{cos\ x.cos\ y - sin\ x.sin\ y}.$

Dividing the numerator and the denominator by $cos\,x.cos\,y$ we get, $tan\,(x + y) = \dfrac{\frac{sin\ x.cos\ y}{cos\ x.cos\ y} + \frac{cos\ x.sin\ y}{cos\ x.cos\ y}}{\frac{cos\ x.cos\ y}{cos\ x.cos\ y} - \frac{sin\ x.sin\ y}{cos\ x.cos\ y}} = \dfrac{tan\ x + tan\ y}{1 - tan\ x.tan\ y}$.

41. Prove that $tan\,(x - y) = \dfrac{tan\ x - tan\ y}{1 + tan\ x.tan\ y}$

Solution: $tan\,(x - y) = \dfrac{sin\ (x - y)}{cos\ (x - y)} = \dfrac{sin\ x.cos\ y - cos\ x.sin\ y}{cos\ x.cos\ y + sin\ x.sin\ y}.$

Dividing the numerator and the denominator by $cos\,x.cos\,y$ we get, $tan\,(x - y) = \dfrac{\frac{sin\ x.cos\ y}{cos\ x.cos\ y} - \frac{cos\ x.sin\ y}{cos\ x.cos\ y}}{\frac{cos\ x.cos\ y}{cos\ x.cos\ y} + \frac{sin\ x.sin\ y}{cos\ x.cos\ y}} = \dfrac{tan\ x - tan\ y}{1 + tan\ x.tan\ y}$.

42. Prove that $cot\,(x + y) = \dfrac{cot\ x.cot\ y - 1}{cot\ x + cot\ y}$

Solution: We know that $tan\,(x + y) = \dfrac{tan\ x + tan\ y}{1 - tan\ x.tan\ y}$.

$\therefore\ cot\,(x + y) = \dfrac{1 - tan\ x.tan\ y}{tan\ x + tan\ y}$

$$= \frac{1 - \frac{1}{cot\ x.cot\ y}}{\frac{1}{cot\ x} + \frac{1}{cot\ y}} = \frac{\frac{cot\ x.cot\ y - 1}{cot\ x.cot\ y}}{\frac{cot\ y + cot\ x}{cot\ x.cot\ y}} = \frac{cot\ x.cot\ y - 1}{cot\ x + cot\ y}$$

43. Prove that $cot\,(x - y) = \dfrac{cot\ x.cot\ y + 1}{cot\ y - cot\ x}.$

Solution: We know that $tan\,(x - y) = \dfrac{tan\ x - tan\ y}{1 + tan\ x.tan\ y}$.

$\therefore\ cot\,(x - y) = \dfrac{1 + tan\ x.tan\ y}{tan\ x - tan\ y}$

$$= \frac{1 + \frac{1}{cot\ x.cot\ y}}{\frac{1}{cot\ x} - \frac{1}{cot\ y}} = \frac{\frac{cot\ x.cot\ y + 1}{cot\ x.cot\ y}}{\frac{cot\ y - cot\ x}{cot\ x.cot\ y}} = \frac{cot\ x.cot\ y + 1}{cot\ y - cot\ x}.$$

44. Prove that $sin\,(x + y).sin\,(x - y) = sin^2 x - sin^2 y$

Solution: $sin\,(x + y).sin\,(x - y)$

$= (sin\,x.cos\,y + cos\,x.sin\,y).(sin\,x.cos\,y - cos\,x.sin\,y)$

$= sin^2 x.cos^2 y - cos^2 x.sin^2 y$
$= sin^2 x(1 - sin^2 y) - (1 - sin^2 x)sin^2 y$
$= sin^2 x - sin^2 x.sin^2 y - sin^2 y + sin^2 x.sin^2 y$
$= sin^2 x - sin^2 y.$

45. Prove that $cos(x + y).cos(x - y) = cos^2 x - sin^2 y$

Solution: $cos(x + y).cos(x - y)$
$= (cos x.cos y - sin x.sin y).(cos x.cos y + sin x.sin y)$
$= cos^2 x.cos^2 y - sin^2 x.sin^2 y$
$= cos^2 x(1 - sin^2 y) - (1 - cos^2 x)sin^2 y$
$= cos^2 x - cos^2 x.sin^2 y - sin^2 y + cos^2 x.sin^2 y$
$= cos^2 x - sin^2 y.$

46. Find the values of the following
a) $sin\,15°$; b) $cos\,15°$; c) $tan\,15°$;
d) $sin\,75°$; e) $cos\,75°$; f) $tan\,75°$.

Solution: a) $sin\,15° = sin(45° - 30°)$
$= sin\,45°.cos\,30° - cos\,45°.sin\,30°$
$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$

b) $cos\,15° = cos(45° - 30°)$
$= cos\,45°.cos\,30° + sin\,45°.sin\,30°$
$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$

c) $tan\,15° = tan(45° - 30°)$
$= \frac{tan\,45° - tan\,30°}{1 + tan\,45°.tan\,30°} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1.\frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$
$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}.\frac{\sqrt{3} - 1}{\sqrt{3} - 1} = \frac{(\sqrt{3} - 1)^2}{\sqrt{3}^2 - 1} = \frac{3 + 1 - 2\sqrt{3}}{2}$
$= \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}.$

d) $sin\,75° = sin(45° + 30°)$
$= sin\,45°.cos\,30° + cos\,45°.sin\,30°$
$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$

e) $cos\,75° = cos(45° + 30°)$

$$= cos\, 45°. cos\, 30° - sin\, 45°. sin\, 30°$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

f) $tan\, 75° = tan\,(45° + 30°)$

$$= \frac{tan\ 45° + tan\ 30°}{1 - tan\ 45°.tan\ 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1.\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}.\frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2}{\sqrt{3}^2 - 1} = \frac{3 + 1 + 2\sqrt{3}}{2}$$

$$= \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}.$$

47. Find the values of the following

a) $cosec\, 15°$; b) $sec\, 15°$; c) $cot\, 15°$;

d) $cosec\, 75°$; e) $sec\, 75°$; f) $cot\, 75°$.

Solution: a) $cosec\, 15° = \frac{1}{sin\ 15°} = \frac{2\sqrt{2}}{\sqrt{3}-1} = \frac{2\sqrt{2}}{\sqrt{3}-1}.\frac{\sqrt{3}+1}{\sqrt{3}+1}$

$$= \frac{2\sqrt{2}(\sqrt{3}+1)}{3-1} = \sqrt{2}(\sqrt{3} + 1) = \sqrt{6} + \sqrt{2}.$$

b) $sec\, 15° = \frac{1}{cos\ 15°} = \frac{2\sqrt{2}}{\sqrt{3}+1} = \frac{2\sqrt{2}}{\sqrt{3}+1}.\frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{2\sqrt{2}(\sqrt{3}-1)}{3-1}$

$$= \sqrt{2}(\sqrt{3} - 1) = \sqrt{6} - \sqrt{2}.$$

c) $cot\, 15° = \frac{1}{tan\ 15°} = \frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}}.\frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3} = 2 + \sqrt{3}.$

d) $cosec\, 75° = \frac{1}{sin\ 75°} = \frac{2\sqrt{2}}{\sqrt{3}+1} = \frac{2\sqrt{2}}{\sqrt{3}+1}.\frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{2\sqrt{2}(\sqrt{3}-1)}{3-1}$

$$= \sqrt{2}(\sqrt{3} - 1) = \sqrt{6} - \sqrt{2}.$$

e) $sec\, 75° = \frac{1}{cos\ 75°} = \frac{2\sqrt{2}}{\sqrt{3}-1} = \frac{2\sqrt{2}}{\sqrt{3}-1}.\frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{2\sqrt{2}(\sqrt{3}+1)}{3-1}$

$$= \sqrt{2}(\sqrt{3} + 1) = \sqrt{6} + \sqrt{2}.$$

f) $cot\, 75° = \frac{1}{tan\ 75°} = \frac{1}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}}.\frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{4-3} = 2 + \sqrt{3}.$

48. Show that

$$cos\,(45° - \theta). cos\,(45° + \theta) - sin\,(45° - \theta). sin\,(45° + \theta) = 0$$

Solution:

$LHS = cos\,(45° - \theta). cos\,(45° + \theta) - sin\,(45° - \theta). sin\,(45° + \theta)$

$$= cos[(45° - \theta) + (45° + \theta)]$$

$$= cos[45° - \theta + 45° + \theta] = \cos 90° = 0.$$

49. Show that $tan\left(\frac{\pi}{4}+\theta\right)\cdot tan\left(\frac{\pi}{4}-\theta\right)=1$.

Solution: $LHS\ = tan\left(\frac{\pi}{4}\ +\theta\right).tan\left(\frac{\pi}{4}-\theta\right)$

$$=\left(\frac{tan\ \frac{\pi}{4}\ +\ tan\theta}{1-tan\ \frac{\pi}{4}.tan\theta}\right)\left(\frac{tan\ \frac{\pi}{4}\ -\ \ tan\theta}{1+\ tan\ \frac{\pi}{4}.tan\theta}\right)=\left(\frac{1\ +\ tan\theta}{1-tan\theta}\right)\left(\frac{1\ -\ tan\theta}{1+\ tan\theta}\right)\ =1=\text{RHS.}$$

50. Show that $\frac{sin\ (x+y)\ +sin\ \ (x+y)}{cos\ \ (x+y)+cos\ (x+y)}=tan\,x$.

Solution: LHS $=\frac{sin\ (x+y)+sin\ (x-y)}{cos\ \ (x+y)+cos\ \ (x-y)}$

$$=\frac{sinx\ cosy\ \ +\ cosx\ siny\ \ +\ sinx\ cosy\ \ -\ cosx\ siny}{cosx\ cosy\ \ -\ sinx\ siny\ +\ cosx\ cosy\ \ +\ sinx\ siny}\ =\ \frac{2sinx\ cosy}{2cosx\ cosy}=\tan x.$$

51. If $tanA\ =\frac{1}{2}$, $tanB=\frac{1}{3}$ show that $A\ +\ B\ =\frac{\pi}{4}$

Solution: $tan\ (A+B)=\frac{tan\ A\ +\ tan\ B}{1-tan\ A\ tan\ B}\ =\ \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}$

$$=\ \frac{5/6}{1-1/6}\ =\ \frac{5/6}{5/6}\ =1=\tan\frac{\pi}{4}\ \ \therefore\ A\ +B\ =\ \frac{\pi}{4}$$

52. Prove that $tan\,85°-tan\,25°=\sqrt{3}(1\ +\ tan\,85°\cdot tan\,25°$

Solution:

$tan\,60°\ =\ \sqrt{3}\quad \Longrightarrow tan\,(85°-25°)=\sqrt{3}$

$$\Longrightarrow\frac{tan\ 85°-tan\ 25°}{1+tan\ 85°\cdot tan\ 25°}\ =\sqrt{3}$$

$$\Longrightarrow tan\,85°-tan\,25°=\sqrt{3}(1+tan\,85°\cdot tan\,25°)$$

53. Prove that a) $\frac{tan\ 69°+tan\ 66°}{1-tan\ 69°\cdot tan\ 66°}\ =\ -1$; b) $\frac{cos\ 17°+sin\,17°}{cos\ 17°-sin\,17°}=tan62°$

Solution:

a) $LHS\ =\frac{tan\ 69°+tan\ 66°}{1-tan\ 69°\cdot tan\ 66°}\quad =tan\,(69°+66°)\quad =tan\,135°$

$$=tan\,(90°+45°)\ \ =-\,cot\,45°\ \ \ =\ -1\ \ =RHS\ .$$

b) Dividing the numerator and denominator by $cos\,17°$, we get,

$LHS\ =\frac{cos\ 17°\ +\ sin\ 17°}{cos\ 17°-sin\ 17°}\ \ =\ \frac{1\ +\ ta\,n17°}{1-\ tan\ \ 17°}$

$$=\ \frac{tan\ 45°\ +\ tan\ 17°}{1-tan\ 45°\cdot tan\ 17°}\ \ =\ tan\,(45°+17°)\ =\ tan\,62°\ =\ RHS.$$

54. Prove that $tan\,3A\ -\ tan\,2A\ -\ tan\,A=tan\,A\cdot tan\,2A\cdot tan\,3A$

Solution: $tan\,3A = tan\,(A + 2A) = \frac{tan\,A + tan\,2A}{1 - tan\,A\,tan\,2A}$

$\Longrightarrow tan\,3A(1 - tan\,A\,tan\,2A) = tan\,A + tan\,2A$

$\Longrightarrow tan\,3A - tan\,A.tan\,2A.tan\,3A = tan\,A + tan\,2A$

$\Longrightarrow tan\,3A - tan\,2A - tan\,A = tan\,A \cdot tan\,2A \cdot tan\,3A.$

55. Prove that $sin\,2A = 2sin\,A.cos\,A.$

Solution: We know that $sin\,(x + y) = sin\,x.cos\,y + cos\,x.sin\,y.$
By taking $x = y = A$, we get,

$$sin\,(A + A) = sin\,A.cos\,A + cos\,A.sin\,A$$

$$\Rightarrow sin\,2A = 2sin\,A.cos\,A.$$

56. Prove that $cos\,2A = cos^2A - sin^2\,A.$

Solution: We know that $cos\,(x + y) = cos\,x.cos\,y - sin\,x.sin\,y.$
By taking $x = y = A$, we get,

$$cos\,(A + A) = cos\,A.cos\,A - sin\,A.sin\,A$$

$$\Rightarrow cos\,2A = cos^2A - sin^2A.$$

57. Prove that $cos\,2A = 2.cos^2A - 1$

Solution: We know that $cos\,(x + y) = cos\,x.cos\,y - sin\,x.sin\,y.$
By taking $x = y = A$, we get,

$$cos\,(A + A) = cos\,A.cos\,A - sin\,A.sin\,A$$

$$\Rightarrow cos\,2A = cos^2A - sin^2A$$

$$= cos^2A - (1 - cos^2A)$$

$$= cos^2A - 1 + cos^2A = cos^2A - 1.$$

58. Prove that $cos\,2A = 1 - 2.sin^2A$

Solution: We know that $cos\,(x + y) = cos\,x.cos\,y - sin\,x.sin\,y.$
By taking $x = y = A$, we get,

$$cos\,(A + A) = cos\,A.cos\,A - sin\,A.sin\,A$$

$$\Rightarrow cos\,2A = cos^2A - sin^2A$$

$$= (1 - sin^2A) - sin^2A = 1 - 2sin^2A.$$

59. Prove that $tan\, 2A = \frac{2tan\, A}{1 - tan^2 A}$.

Solution: We know that $tan\,(x + y) = \frac{tan\, x + tan\, y}{1 - tan\, x.tan\, y}$.

By taking $x = y = A$, we get, $tan\,(A + A) = \frac{tan\, A + tan\, A}{1 - tan\, A.tan\, A}$

$\Rightarrow \; tan\, 2A = \frac{2tan\, A}{1 - tan^2 A}$.

60. Prove that $sin\, 2A = \frac{2tan\, A}{1 + tan^2 A}$.

Solution: $\frac{2.tan\, A}{1 + tan^2 A} = \frac{2.tan\, A}{sec^2 A} = 2.tan\, A.cos^2 A = 2\frac{sin\, A}{cos\, A}cos^2 A$
$= 2.sin\, A.cos\, A = sin\, 2A$.

61. Prove that $cos\, 2A = \frac{1 - tan^2 A}{1 + tan^2 A}$.

Solution:

$\frac{1 - tan^2 A}{1 + tan^2 A} = \frac{1 - tan^2 A}{sec^2 A} = \frac{1}{sec^2 A} - \frac{tan^2 A}{sec^2 A} = cos^2 A - tan^2 A.cos^2 A$
$= cos^2 A - \frac{sin^2 A}{cos^2 A}cos^2 A = cos^2 A - sin^2 A = cos\, 2A$.

62. Prove that $tan^2 A = \frac{1 - cos\, 2A}{1 + cos\, 2A}$.

Solution:

We know that $cos\, 2A = 1 - 2.sin^2 A$. $\therefore\; 2.sin^2 A = 1 - cos\, 2A$;

Also $cos\, 2A = 2.cos^2 A - 1 \;\therefore\; 2.cos^2 A = 1 + cos\, 2A$.

$\therefore \; \frac{1 - cos\, 2A}{1 + cos\, 2A} = \frac{2.sin^2 A}{2.cos^2 A} = tan^2 A$.

63. Find the values of a. $sin\, 22\frac{1}{2}°$; b. $cos\, 22\frac{1}{2}°$; c. $tan\, 22\frac{1}{2}°$.

Solution:

a. We know that $2.sin^2\left(\frac{\theta}{2}\right) = 1 - cos\,\theta$.

By taking $\theta = 45°$ we get

$2.sin^2\left(22\frac{1}{2}°\right) = 1 - cos\, 45° = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2} - 1}{\sqrt{2}}$.

$\therefore sin^2\left(22\frac{1}{2}°\right) = \frac{\sqrt{2} - 1}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{2} - 1)}{2\sqrt{2}.\sqrt{2}} = \frac{2 - \sqrt{2}}{4}$

$\therefore sin\left(22\frac{1}{2}°\right) = \sqrt{\frac{2 - \sqrt{2}}{4}} = \frac{\sqrt{2 - \sqrt{2}}}{2}$

b. We know that $2.cos^2\left(\frac{\theta}{2}\right) = 1 + cos\,\theta$.

By taking $\theta = 45°$ we get

$$2.cos^2\left(22\frac{1}{2}°\right) = 1 + cos\,45° = 1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2} + 1}{\sqrt{2}}.$$

$$\therefore cos^2\left(22\frac{1}{2}°\right) = \frac{\sqrt{2} + 1}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{2} + 1)}{2\sqrt{2}.\sqrt{2}} = \frac{2 + \sqrt{2}}{4}.$$

$$\therefore cos\left(22\frac{1}{2}°\right) = \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{2 + \sqrt{2}}}{2}.$$

c. We know that, $tan^2\left(\frac{\theta}{2}\right) = \frac{1 - cos\ \theta}{1 + cos\ \theta}$.

By taking $\theta = 45°$ we get,

$$tan^2\left(22\frac{1}{2}°\right) = \frac{1 - cos\ 45°}{1 + cos\ 45°} = \frac{1 - 1/\sqrt{2}}{1 + 1/\sqrt{2}} = \frac{\sqrt{2} - 1}{\sqrt{2} + 1}$$

$$= \frac{\sqrt{2} - 1}{\sqrt{2} + 1}.\frac{\sqrt{2} - 1}{\sqrt{2} - 1} = \frac{\left(\sqrt{2} - 1\right)^2}{2 - 1} = \left(\sqrt{2} - 1\right)^2.$$

$$\therefore tan\left(22\frac{1}{2}°\right) = \sqrt{2} - 1.$$

64. If $sin\,A = \frac{3}{8}$ and A is acute angle, find $sin\,2A$.

Solution: Given $sin\,A = \frac{3}{8}$.

Since A is acute,

$$cos\,A = \sqrt{1 - sin^2 A} = \sqrt{1 - \frac{9}{64}} = \sqrt{\frac{64 - 9}{64}} = \frac{\sqrt{55}}{8}.$$

$$\therefore\ sin\ 2A = 2\,sin\,A.cos\,A = 2 \cdot \frac{3}{8} \cdot \frac{\sqrt{55}}{8} = \frac{3\sqrt{55}}{32}.$$

65. Prove that $cos^4 A - sin^4 A = \cos 2A$.

Solution: $LHS = (cos^2 A)^2 - (sin^2 A)^2$

$$= (cos^2 A + sin^2 A)(cos^2 A - sin^2 A)$$

$$= 1 \cdot \cos 2A = \cos 2A = RHS.$$

66. Prove that $\sqrt{2 + \sqrt{2 + 2\,cos\,4\theta}} = 2.cos\,\theta$

Solution: $LHS = \sqrt{2 + \sqrt{2(1 + cos\,4\theta)}} = \sqrt{2 + \sqrt{2.2cos^2 2\theta}}$

$$= \sqrt{2 + 2\,cos\,2\theta} = \sqrt{2(1 + cos\,2\theta)}$$

$$= \sqrt{2 \cdot 2cos^2 2\theta} = 2cos\,\theta = RHS.$$

67. Prove that $\frac{1 + tan^2(45° - \theta)}{1 - tan^2(45° - \theta)} = cosec\,2\theta$.

Solution: $\frac{1 + tan^2(45° - \theta)}{1 - tan^2(45° - \theta)} = \frac{1 + tan^2 t}{1 - tan^2 t} = \frac{1}{cos\,2t}$ where $t = 45° - \theta$.

$= sec\,2t \;=\; sec\,2(45° - \theta) \;=\; sec\,(90° - 2\theta) \;=\; cosec\,2\theta$.

68. Prove that $8\,cos^3\frac{\pi}{9} - 6\,cos\frac{\pi}{9} = 1$.

Solution: $8\,cos^3\frac{\pi}{9} - 6\,cos\frac{\pi}{9} = 2\left(4\,cos^3\frac{\pi}{9} - 3\,cos\frac{\pi}{9}\right)$

$= 2\,cos\left(3\cdot\frac{\pi}{9}\right) = 2\,cos\left(\frac{\pi}{3}\right) = 2\cdot\frac{1}{2} = 1$.

69. Prove that $\frac{1 + sin\,\theta - cos\,\theta}{1 + sin\,\theta + cos\,\theta} = tan\,\frac{\theta}{2}$

Solution: $\frac{1 + sin\,\theta - cos\,\theta}{1 + sin\,\theta + cos\,\theta} = \frac{1 - cos\,\theta + sin\,\theta}{1 + cos\,\theta + sin\,\theta}$

$= \frac{2sin^2\left(\frac{\theta}{2}\right) + 2\,sin\left(\frac{\theta}{2}\right)\cdot cos\left(\frac{\theta}{2}\right)}{2cos^2\left(\frac{\theta}{2}\right) + 2\,sin\left(\frac{\theta}{2}\right)\cdot cos\left(\frac{\theta}{2}\right)} = \frac{2\,sin\left(\frac{\theta}{2}\right)\left\{cos\left(\frac{\theta}{2}\right) + sin\left(\frac{\theta}{2}\right)\right\}}{2\,cos\left(\frac{\theta}{2}\right)\left\{cos\left(\frac{\theta}{2}\right) + sin\left(\frac{\theta}{2}\right)\right\}} = tan\,\frac{\theta}{2}$.

70. Prove that $cos\,10° - cos\,50° = sin\,20°$

Solution: $LHS = -2\,sin\left(\frac{10° + 50°}{2}\right) sin\left(\frac{10° - 50°}{2}\right)$

$= -2\,sin\,30°\,sin\,(-20°) = -(-\,2\cdot\frac{1}{2}sin\,20°) = sin20°$.

71. Find the general solution of $sin\,\theta = \frac{1}{2}$

Solution: $sin\,\theta = \frac{1}{2} = sin\,\frac{\pi}{6}$. Principal angle $\alpha = \frac{\pi}{6}$.

∴ the general solution is $x = n\pi + (-1)^n\cdot\frac{\pi}{6}$ $, n \in Z$.

72. Find the general solution of $sec\,\theta = -\sqrt{2}$.

Solution: $sec\,\theta = -\sqrt{2} \Longrightarrow \cos\theta = -\frac{1}{\sqrt{2}}$.

Principal angle $\alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

∴ general solution is $\theta = 2n\pi \pm \frac{3\pi}{4}, n \in Z$.

73. Find the general solution of $sin\,2x = 4\,cos\,x$.

Solution: $sin\ 2x = 4\ cos\ x \implies 2sin\ x\ .cos\ x = 4\ cos\ x$

$\implies 2sin\ x.cos\ x - 4cos\ x = 0 \implies 2cos\ x(sin\ x - 2) = 0$

$\implies cos\ x = 0\ \ or\ \ sin\ x = 2.$

Since $sin\ x = 2 > 1$, the equation $sin\ x = 2$ has no solution.

$cos\ x = 0 \implies x = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}.$

This is the general solution of the given equation.

Three mark questions

1. The angles of a triangle are in the ratio $4:5:6$. Find them in radian and degree.

Solution: Let A, B and C be the angles of the triangle.

Now $A:B:C = 4:5:6 \implies A = 4\theta,\ B = 5\theta, C = 6\theta$

$A + B + C = \pi \implies 4\theta + 5\theta + 6\theta = \pi \implies 15\theta = \pi \quad \therefore \theta = \frac{\pi}{15}$

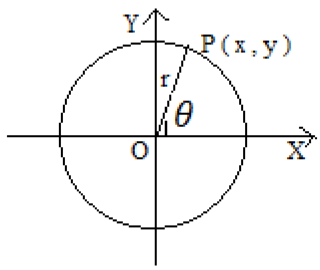
$\therefore\ A = 4\theta = 4 \times \frac{\pi}{15} = \frac{4\pi}{15},\quad B = 5\theta = 5 \times \frac{\pi}{15} = \frac{\pi}{3},\quad C = 6\theta = 6 \times \frac{\pi}{15} = \frac{2\pi}{3}.$

$A + B + C = 180° \implies 15\theta = 180° \quad \therefore \theta = \frac{180°}{15} = 12°$

$\therefore A = 4\theta = 4 \times 12° = 48°,\ B = 5 \times 12° = 60°,\ C = 6\theta = 6 \times 12° = 72°$

2. Define the six trigonometric functions.

Solution: Consider a circle of radius r centered at the origin of the Cartesian coordinate system. Let $P(x,\ y)$ be a point on the circle. Join OP. Define angle θ having OX has the initial direction and OP as the terminal direction.



Then for any position of the point $P(x,\ y)$ on the circle,

i. sine of the angle θ is denoted by $sin\ \theta$ and is defined by $sin\ \theta = \frac{y}{r}$.

ii. cosine of the angle θ is denoted by $cos\ \theta$ and is defined by $cos\ \theta = \frac{x}{r}$.

iii. tangent of the angle θ is denoted by $tan\ \theta$ and is defined by

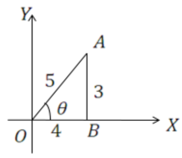
$tan\,\theta \;=\; \frac{y}{x}$.

iv. cotangent of the angle θ is denoted by $cot\,\theta$ and is defined by $cot\,\theta \;=\; \frac{x}{y}$.

v. secant of the angle θ is denoted by $sec\,\theta$ and is defined by $sec\,\theta \;=\; \frac{r}{x}$.

vi. cosecant of the angle θ is denoted by $cosec\,\theta$ or $csc\,\theta$ and is defined by $cosec\,\theta \;=\; \frac{r}{y}$.

3. If $\sin\theta = \frac{3}{5}$ and θ is acute, find all the other trigonometric ratios.



Solution: Given $\sin\theta = \frac{3}{5} = \frac{AB}{OA}$.

Therefore $AB = 3$ and $OA = 5$

Now $OB^2 = OA^2 - AB^2 = 25 - 9 = 16$.

Therefore $OB = 4$.

We have, $\cos\theta = \sqrt{1 - sin^2\theta} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$;

$\tan\theta = \frac{\sin\,\theta}{\cos\,\theta} = \frac{3/5}{4/5} = \frac{3}{4}$;

$\cot\theta = \frac{\cos\,\theta}{\sin\,\theta} = \frac{4/5}{3/5} = \frac{4}{3}$;

$sec\theta = \frac{1}{\cos\,\theta} = \frac{1}{4/5} = \frac{5}{4}$;

$cosec\theta = \frac{1}{\sin\,\theta} = \frac{1}{3/5} = \frac{5}{3}$.

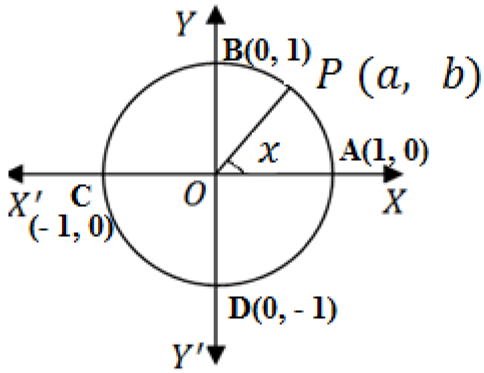
4. Prove that $(1 - sin\,A + cosA)^2 = 2(1 - sin\,A)(1 + cos\,A)$

Solution:

$$
\begin{aligned}
LHS \;&=\; [(1 - sin\,A) + cosA]^2 \\
&= (1 - sin\,A)^2 + \; cos^2\,A + 2(1 - sin\,A)\cdot cosA \\
&= (1 - sin\,A)^2 + \; (1 - sin^2\,A) + 2(1 - sin\,A)\cdot cosA \\
&= (1 - sin\,A)^2 + \; (1 - sin\,A)\cdot(1 + sin\,A) + 2(1 - sin\,A)\cdot cosA \\
&= (1 - sin\,A)[1 - sin\,A + 1 + sin\,A + 2\,cos\,A] \\
&= (1 - sin\,A)[2 + 2\,cos\,A] = 2(1 - sin\,A)(1 + cos\,A) = RHS.
\end{aligned}
$$

5. Find the trigonometric ratios of the quadrantal angles
a) $0°$; b) $90° \left(\frac{\pi}{2}\right)$; c) $180° \,(\pi)$; d) $270° \left(\frac{3\pi}{2}\right)$ and e) $360° \,(2\pi)$.

Solution: Consider the points, $A(1,0)$, $B(0,1)$, $C(-1,0)$ and $D(0,\ -1)$ corresponding to the quadrantal angles $\frac{\pi}{2}$ $(90°)$, π $(180°)$, $\frac{3\pi}{2}$ $(270°)$ and 2π $(360°)$, lying on a circle of unit radius centered at the origin of a Cartesian co-ordinate system. Let $P(a,\ b)$ be the point on the circle such that angle $POX = x$.



Then $\sin x = b$ and $\cos x = a$.

a)When $P(a,\ b)$ coincides with the point $A(1,0)$, the angle x becomes $0°$.

$\therefore\ \cos 0 = 1$ and $\sin 0 = 0$.

$\therefore\ \tan 0 = \frac{0}{1} = 0,\ \cot 0 = \frac{1}{0} = \infty,\ \operatorname{cosec} 0 = \frac{1}{0} = \infty,\ \sec 0 = \frac{1}{1} = 1.$

b) When $P(a,\ b)$ coincides with the point $B(0,1)$, the angle x becomes $90°$.

$\therefore\ \cos 90° = 0$ and $\sin 90° = 1$.

$\therefore\ \tan 90° = \frac{1}{0} = \infty,\ \cot 90° = \frac{0}{1} = 0,$

$\operatorname{cosec} 90° = \frac{1}{1} = 1,\ \sec 90° = \frac{1}{0} = \infty.$

c) When $P(a,\ b)$ coincides with the point $C(-1,0)$, the angle x becomes $180°$.

$\therefore\ \cos 180° = -1$ and $\sin 180° = 0$.

$\therefore\ \tan 180° = \frac{0}{-1} = 0,\ \cot 180° = \frac{-1}{0} = \infty,$

$\operatorname{cosec} 180° = \frac{1}{0} = \infty,\ \sec 180° = \frac{1}{-1} = -1.$

d) When $P(a,\ b)$ coincides with the point $D(0,\ -1)$, the angle x becomes $270°$.

$\therefore\ \cos 270° = 0$ and $\sin 270° = -1$.

$\therefore\ tan\,270° = \frac{-1}{0} = \infty,\ \ cot\,270° = \frac{0}{-1} = 0,$

$cosec\,270° = \frac{1}{-1} = -1,\ \ sec\,270° = \frac{1}{0} = \infty.$

e) When $P(a,\ b)$ coincides with the point $A(1,0)$, after one complete rotation, the angle x becomes $360°$.

$\therefore\ cos\,360° = 1$ and $sin\,360° = 0.$

$\therefore\ tan\,360° = \frac{0}{1} = 0,\ \ cot\,360° = \frac{1}{0} = \infty,$

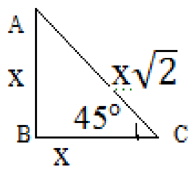
$cosec\,360° = \frac{1}{0} = \infty,\ \ sec\,360° = \frac{1}{1} = 1.$

6. Find the trigonometric ratios of a) $45°$; b) $30°$ and $60°$.

Solution:

a) The trigonometric ratios of $45°$.

Consider a right angled triangle ABC, right angled at B. Now angle $ACB = 45°$.



Let $AB = x$. Then $BC = x$.

Now $AC^2 = AB^2 + BC^2 = x^2 + x^2 = 2x^2$.

$\therefore\ AC = x\sqrt{2}$. Now,

$$sin\,45° = \frac{AB}{AC} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}\ ;$$

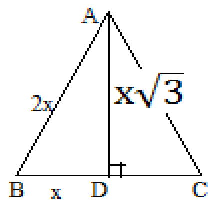
$$cos\,45° = \frac{BC}{AC} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}\ ;$$

$$tan\,45° = \frac{AB}{BC} = \frac{x}{x} = 1.$$

Similarly, $cot\,45° = 1,\ \ cosec\,45° = \sqrt{2},\ \ sec\,45° = \sqrt{2}.$

b) The trigonometric ratios of $30°$ and $60°$.

Consider an equilateral triangle ABC. Draw AD perpendicular to BC. Then D is the midpoint of BC. Let $BD = x$. Then $AB = 2x$.



Now $AD^2 = AB^2 - BD^2 = (2x)^2 - x^2 = 4x^2 - x^2 = 3x^2$. $\therefore\ AD = x\sqrt{3}$.

Also, $\angle ABC = 60°$ and $\angle BAD = 30°$.

$$sin\,60° = \frac{AD}{AB} = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}\ ;$$

$$cos\,60° = \frac{BD}{AB} = \frac{x}{2x} = \frac{1}{2}\ ;$$

$tan\,60° \;=\; \frac{AD}{BD} = \frac{x\sqrt{3}}{x} = \sqrt{3}.$

Similarly, $cot\,60° \;=\; \frac{1}{\sqrt{3}},\;\; cosec\,60° \;=\; \frac{2}{\sqrt{3}},\;\; sec\,60° \;=\; 2.$

$sin\,30° \;=\; \frac{BD}{AB} \;=\; \frac{x}{2x} \;=\; \frac{1}{2}\;;$

$cos\,30° \;=\; \frac{AD}{AB} \;=\; \frac{x\sqrt{3}}{2x} \;=\; \frac{\sqrt{3}}{2}\;;$

$tan\,30° \;=\; \frac{BD}{AD} \;=\; \frac{x}{x\sqrt{3}} \;=\; \frac{1}{\sqrt{3}}.$

Similarly, $cot\,30° \;=\; \sqrt{3},\;\; cosec\,30° \;=\; 2,\;\; sec\,30° \;=\; \frac{2}{\sqrt{3}}.$

7. Prove that $\frac{tan^2 60° \;-\; 2\,tan^2 45°}{3\,sin^2 45° \cdot sin\,90° \;+\; cos^2 60° \cdot cos^2\, 0°} \;=\; \frac{4}{7}$

Solution: $LHS \;=\; \frac{\left(\sqrt{3}\right)^2 - \;2\,(1)^2}{3\left(\frac{1}{\sqrt{2}}\right)^2 \cdot 1 \;+\; \left(\frac{1}{2}\right)^2 \cdot (1)^2} = \frac{3\;-\;2}{\frac{3}{2}\;+\;\frac{1}{4}} \;=\; \frac{1}{\frac{7}{4}} \;=\; \frac{4}{7} = RHS.$

8. Find x, $\;x\sin 30° \cdot cos^2 45° = \frac{cot^2 30° \cdot sec\,60° \cdot tan\,45°}{cosec^2 45° \cdot cosec\,30°}$

Solution: $x \cdot \frac{1}{2} \cdot \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\left(\sqrt{3}\right)^2 \cdot 2 \cdot 1}{\left(\sqrt{2}\right)^2 \cdot 2} \;\Longrightarrow\; x \cdot \frac{1}{4} \;=\; \frac{6}{4}.\;\; \therefore x = 6.$

9. Find x, $\;x\,[sin^2 30° \;+\; sin^2 45° \;+\; sin^2 60°] - \;cos^2 30° = 0$

Solution: $x\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right] - \left(\frac{\sqrt{3}}{2}\right)^2 = 0$

$\Longrightarrow x\left[\frac{1}{4} + \frac{1}{2} + \frac{3}{4}\right] - \frac{3}{4} = 0 \quad \Longrightarrow x\left[\frac{1+\;2\;+3}{4}\right] = \frac{3}{4} \quad \therefore x \;=\; \frac{3}{6} \;=\; \frac{1}{2}.$

10. Find the domain of $f(x) \;=\; sin\,x$ and $f(x) \;=\; cosec\,x$.

Solution: Recall the definition, $sin\,\theta \;=\; \frac{y}{r}$.

This is defined for any $y \in R$.

$\therefore$ domain of the function $y \;=\; sin\,x$ is the set of all reals, R.

We know that $sin\,0 \;=\; 0,\; sin\,\pi = \;0$ and $sin\,2\pi \;=\; 0.$

By using $sin\,(2\pi n \;+\; \theta) \;=\; sin\,\theta$ we get

$sin\,(2\pi n \;+\pi) \;=\; 0,\; \forall\, n \;\in\; Z.$

This gives $sin\,n\pi \;=\; 0, \forall\, n \;\in\; Z.$

Thus $sin\,x \;=\; 0$ when $x \;=\; n\pi,\; n \in Z.$

Since $cosec\,x \;=\; \frac{1}{sin\;x}$, the function $y \;=\; cosec\,x$ is defined only

when $sin\, x \neq 0$. Thus the domain of the function,
$y = cosec\, x$, is $\{x : x \neq n\pi,\ n \in Z\}$.

11. Find the domain of $f(x) = cos\, x$ and $f(x) = sec\, x$.

Solution: Recall the definition, $cos\, \theta = \frac{x}{r}$.
This is defined for any $x \in R$.
∴ domain of the function $y = cos\, x$ is the set of all reals, R.
We know that $cos\, \frac{\pi}{2} = 0$, $cos\, \frac{3\pi}{2} = 0$.
By using $cos\,(2\pi n + \theta) = cos\, \theta$ we get
$$cos\,(2\pi n + \frac{\pi}{2}) = 0,\ \forall\, n \in Z.$$
This gives $cos\,(2n + 1)\frac{\pi}{2} = 0,\ \forall\, n \in Z$.
Thus $cos\, x = 0$ when $x = (2n + 1)\frac{\pi}{2},\ n \in Z$.
Since $sec\, x = \frac{1}{cos\ x}$, the function $y = sec\, x$ is defined only when $cos\, x \neq 0$. Thus the domain of the function,
$y = sec\, x$, is $\{x : x \in R,\ x \neq (2n + 1)\frac{\pi}{2},\ n \in Z\}$.

12. Find the domain of $f(x) = tan\, x$.

Solution: Recall the definition, $tan\, \theta = \frac{y}{x}$. This is defined for any $x,\ y \in R,\ x \neq 0$. We know that, $tan\, \frac{\pi}{2} = \infty$, $tan\, \frac{3\pi}{2} = \infty$.
By using $tan\,(2\pi n + \theta) = tan\, \theta$ we get $tan\,(2\pi n + \frac{\pi}{2}) = \infty$, $\forall\, n \in Z$. This gives $tan\,(2n + 1)\frac{\pi}{2} = \infty,\ \forall\, n \in Z$.
Thus $tan\, x = \infty$ when $x = (2n + 1)\frac{\pi}{2},\ n \in Z$. Thus the domain of the function, $y = tan\, x$, is $\{x : x \in R,\ x \neq (2n + 1)\frac{\pi}{2},\ n \in Z\}$.

13. Find the domain of $f(x) = cot\, x$.

Solution: Recall the definition, $cot\, \theta = \frac{x}{y}$. This is defined for any $x, y \in R,\ y \neq 0$. We know that $cot\, \pi = \infty$, $cot\, 3\pi = \infty$.
By using $cot\,(2\pi n + \theta) = cot\, \theta$ we get $cot\,(2\pi n + \pi) = \infty,\ \forall\, n \in Z$. This gives $cot\, n\pi = \infty,\ \forall\, n \in Z$. Thus $cot\, x = \infty$ when

$x = n\pi,\ n \in Z$. Thus the domain of the function, $y = cot\, x$, is $\{x : x \in R,\ x \neq n\pi,\ n \in Z\}$.

14. Prove that $sin\,(x + y) = sin\, x. cos\, y + cos\, x. sin\, y$. Hence prove that $sin\,(x - y) = sin\, x. cos\, y - cos\, x. sin\, y$.

Solution: $sin\,(x + y) = cos\left(\frac{\pi}{2} - (x + y)\right)$

$$= cos\left(\left(\frac{\pi}{2} - x\right) - y\right)$$

$$= cos\left(\frac{\pi}{2} - x\right). cos\, y + sin\left(\frac{\pi}{2} - x\right). sin\, y$$

$$= sin\, x. cos\, y + cos\, x. sin\, y.$$

By replacing y by $-y$ we get,

$$sin\,(x - y) = sin\, x. cos\,(-y) + cos\, x. sin\,(-y)$$

$$= sin\, x. cos\, y + cos\, x. (-sin\, y)$$

$$= sin\, x. cos\, y - cos\, x. sin\, y.$$

15. Draw the graph of $y = sin\, x$.

Solution: $sin\, 0 = 0$ and $sin\,\pi/2 = 1$.

$\therefore$ $sin\, x$ increases from 0 to 1 as x increases from 0 to $\pi/2$.

$sin\,\pi/2 = 1$ and $sin\,\pi = 0$.

$\therefore$ $sin\, x$ decreases from 1 to 0 as x increases from $\pi/2$ to π.

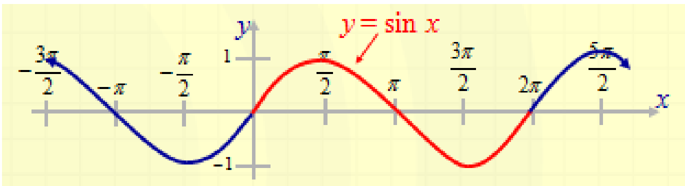
$sin\,\pi = 0$ and $sin\, 3\pi/2 = -1$.

$\therefore$ $sin\, x$ decreases from 0 to -1 as x increases from π to $3\pi/2$.

$sin\, 3\pi/2 = -1$ and $sin\, 2\pi = 0$.

$\therefore$ $sin\, x$ increases from -1 to 0 as x increases from $3\pi/2$ to 2π.

Since $sin\, x$ is periodic with the period 2π we can make the same observation in the intervals



$\ldots, [-4\pi, -2\pi], [-2\pi, 0], [0, 2\pi], [2\pi, 4\pi], [4\pi, 6\pi],$ ...The graph is shown in the adjacent figure.

16. Draw the graph of $y = \cos x$.

Solution: $\cos 0 = 1$ and $\cos \pi/2 = 0$.

$\therefore$ $\cos x$ decreases from 1 to 0 as x increases from 0 to $\pi/2$.

$\cos \pi/2 = 0$ and $\cos \pi = -1$.

$\therefore$ $\cos x$ decreases from 0 to -1 as x increases from $\pi/2$ to π.

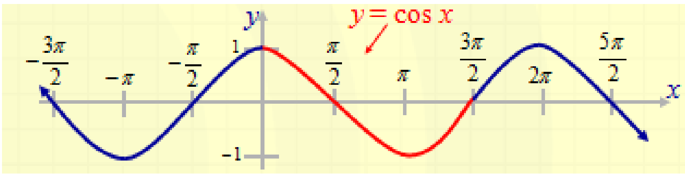
$\cos \pi = -1$ and $\cos 3\pi/2 = 0$.

$\therefore$ $\cos x$ increases from -1 to 0 as x increases from π to $3\pi/2$.

$\cos 3\pi/2 = 0$ and $\cos 2\pi = 1$.

$\therefore$ $\cos x$ increases from 0 to 1 as x increases from $3\pi/2$ to 2π.

Since $\cos x$ is periodic with the period 2π, we can make the same observation in the intervals



$\ldots, [-4\pi, -2\pi], [-2\pi, 0], [0, 2\pi], [2\pi, 4\pi], [4\pi, 6\pi], \ldots$ The graph is shown in the adjacent figure.

17. Draw the graph of $y = \tan x$.

Solution: $\tan 0 = 0$ and $\tan \pi/2 = \infty$.

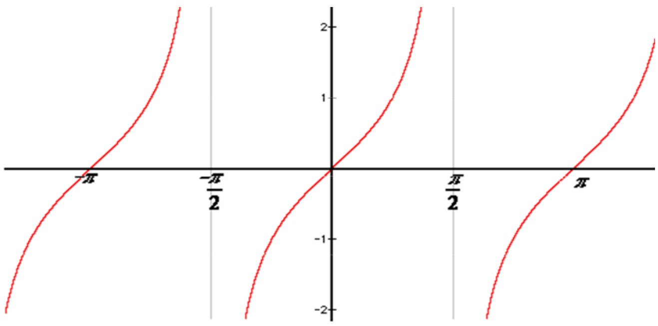
$\therefore$ $\tan x$ increases from 0 to ∞ as x increases from 0 to $\pi/2$.

$\tan \pi/2 = \infty$ and $\tan \pi = 0$.

Between $x = \pi/2$ and $x = \pi$, $\tan x$ is negative.

$\therefore$ $\tan x$ increases from $-\infty$ to 0 as x increases from $\pi/2$ to π.

Since $\tan x$ is periodic with the period π, we can make the same observations in the intervals, $\ldots [-2\pi, -\pi], [-\pi, 0], [0, \pi], [\pi, 2\pi], \ldots$ The graph is shown in the adjacent figure.



18. Draw the graph of $y = cot\, x$.

Solution: $cot\, 0 = \infty$ and $cot\, \pi/2 = 0$.

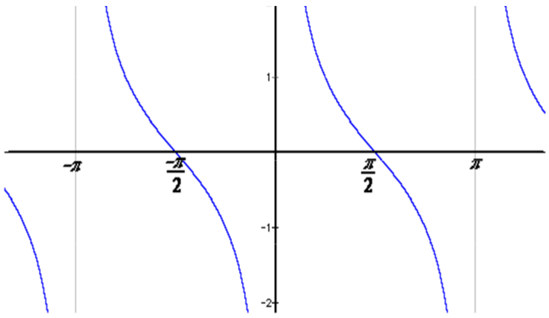
$\therefore$ $cot\, x$ decreases from ∞ to 0 as x increases from 0 to $\pi/2$.

$cot\, \pi/2 = 0$ and $cot\, \pi = \infty$.

Between $x = \pi/2$ and $x = \pi$ $cot\, x$ is negative.

$\therefore$ $cot\, x$ decreases from 0 to $-\infty$ as x increases from $\pi/2$ to π.

Since $cot\, x$ is periodic with the period π, we can make the same observations in the intervals, ... $[-2\pi, -\pi]$, $[-\pi, 0]$, $[0, \pi]$, $[\pi, 2\pi]$, ... The graph is shown in the adjacent figure.



19. Draw the graph of $y = cosec\, x$.

Solution: $cosec\, 0 = \infty$ and $cosec\, \pi/2 = 1$.

$\therefore$ $cosec\, x$ decreases from ∞ to 0 as x increases from 0 to $\pi/2$.

$cosec\, \pi/2 = 1$ and $cosec\, \pi = \infty$.

$\therefore$ $cosec\, x$ increases from 1 to ∞ as x increases from $\pi/2$ to π.

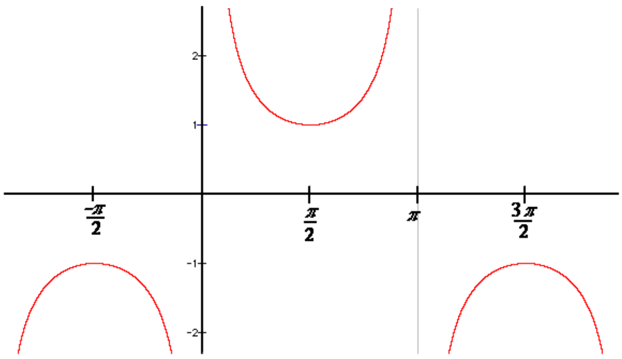
$cosec\, \pi = \infty$ and $cosec\, 3\pi/2 = -1$.

Between $x = \pi$ and $x = 3\pi/2$, $cosec\, x$ is negative.

$\therefore$ $cosec\, x$ increases from $-\infty$ to -1 as x increases from π to $3\pi/2$.

$cosec\, 3\pi/2 = -1$ and $cosec\, 2\pi = \infty$.

Between $x = 3\pi/2$ and $x = 2\pi$, $cosec\, x$ is negative.



$\therefore$ $cosec\, x$ decreases from -1 to $-\infty$ as x increases from $3\pi/2$ to 2π.

Since $cosec\, x$ is periodic with the period 2π, we can make the same observations in the intervals, ..., $[-4\pi, -2\pi]$, $[-2\pi, 0]$, $[0, 2\pi]$, $[2\pi, 4\pi]$, ... The graph is shown in the adjacent figure.

20. Draw the graph of $y = sec\,x$.

Solution: $sec\,0 = 1$ and $sec\,\pi/2 = \infty$.

$\therefore$ $sec\,x$ increases from 1 to ∞ as x increases from 0 to $\pi/2$.

$sec\,\pi/2 = \infty$ and $sec\,\pi = -1$.

Between $x = \pi/2$ and $x = \pi$, $sec\,x$ is negative.

$\therefore$ $sec\,x$ increases from $-\infty$ to -1 as x increases from $\pi/2$ to π.

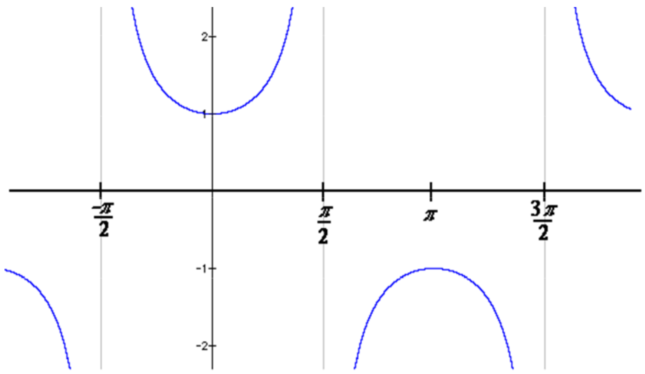
$sec\,\pi = -1$ and $sec\,3\pi/2 = \infty$.

Between $x = \pi$ and $x = 3\pi/2$, $sec\,x$ is negative. $\therefore$ $sec\,x$ decreases from -1 to $-\infty$. as x increases from π to $3\pi/2$.

$sec\,3\pi/2 = \infty$ and $sec\,2\pi = 1$.

Between $x = 3\pi/2$ and $x = 2\pi$, $sec\,x$ is positive.

$\therefore$ $sec\,x$ decreases from ∞ to 1 as x increases from $3\pi/2$ to 2π.



Since $sec\,x$ is periodic with the period 2π, we can make the same observations in the intervals, ..., $[-4\pi, -2\pi]$, $[-2\pi, 0]$, $[0, 2\pi]$, $[2\pi, 4\pi]$, ... The graph is shown in the adjacent figure.

21. If $cos\,\theta = \frac{4}{5}$ and θ is acute, find the value of $\frac{3cos\,\theta + 2cosec\,\theta}{4\,sin\,\theta - cot\,\theta}$.

Solution: Given $\cos\theta = \frac{4}{5}$. Let $adj = 4$ and $hyp = 5$. Then $opp = 3$. Then $cosec\,\theta = \frac{hyp}{opp} = \frac{5}{3}$ and $cot\,\theta = \frac{adj}{opp} = \frac{4}{3}$.

Therefore $\frac{3cos\,\theta + 2cosec\,\theta}{4\,sin\,\theta - cot\,\theta} = \frac{3\left(\frac{4}{5}\right) + 2\left(\frac{5}{3}\right)}{4\left(\frac{3}{5}\right) - \left(\frac{4}{3}\right)} = \frac{\frac{12}{5} + \frac{10}{3}}{\frac{12}{5} - \frac{4}{3}} = \frac{\frac{86}{15}}{\frac{16}{15}} = \frac{86}{16} = \frac{43}{8}$.

22. If $sec\,\theta = \frac{13}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find the value of $\frac{2sin\ \theta - 3cos\ \theta}{4\,sin\ \theta\ +9\ cos\ \theta}$.

Solution: Given $sec\,\theta = \frac{13}{5}$. Let $hyp = 13$ and $adj = 5$.

Then $opp = 12$. In the fourth quadrant $sin\,\theta = -\frac{opp}{hyp} = -\frac{12}{13}$;

Thus $\frac{2sin\ \theta - 3cos\ \theta}{4\,sin\ \theta\ +9\ cos\ \theta} = \frac{2\sin\,\theta - 3\cos\,\theta}{4\sin\,\theta\ 9\ \cos\,\theta} = \frac{2\left(\frac{-12}{13}\right) - 3\left(\frac{5}{13}\right)}{4\left(\frac{-12}{13}\right) + 9\left(\frac{5}{13}\right)} = \frac{\frac{-24}{13} - \frac{15}{13}}{\frac{-48}{13} + \frac{45}{13}} = \frac{-39}{-3} = 13.$

23. If A, B are acute angles, $sin\,A = \frac{3}{5}$; $cos\,B = \frac{12}{13}$, find $cos\ (A+B)$

Solution: We have $cos(A+B) = cosA\,cosB - sinA\,sinB$.

Given $sin\,A = \frac{3}{5}$; $cos\,B = \frac{12}{13}$.

$\therefore cos\,A = \sqrt{1 - sin^2 A} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25 - 9}{25}} = \frac{4}{5}$ and

$sinB = \sqrt{1 - cos^2 B} = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}$.

$\therefore cos\,(A+B) = \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13} = \frac{48 - 15}{65} = \frac{33}{65}$.

24. If $A + B = 45°$ then prove that $(1 + tan\,A)(1 + tan\,B) = 2$ and hence deduce that $tan\left(22\frac{1}{2}\right)^{\circ} = \sqrt{2} - 1$.

Solution: Given $A + B = 45° \Rightarrow tan\,(A + B) = tan\,45°$

$\Rightarrow \frac{tan\ A\ +\ tan\ B}{1 - tan\ A\ tan\ B} = 1 \qquad \Rightarrow tan\,A + tan\,B = 1 - tan\,A.tan\,B$

$\Rightarrow tan\,A + tan\,B + tan\,A.tan\,B = 1$ (by adding 1 to both sides)

$\Rightarrow 1 + tan\,A + tan\,B + tan\,A.tan\,B = 2$

$\Rightarrow (1 + tan\,A) + tan\,B.(1 + tan\,A) = 2$

$\Rightarrow (1 + tan\,A).(1 + tan\,B) = 2$.

By taking $A = B = \left(22\frac{1}{2}\right)^{\circ}$ we get,

$$\left[1 + tan\left(22\tfrac{1}{2}\right)^{\circ}\right]^2 = 2 \quad \Rightarrow 1 + tan\left(22\tfrac{1}{2}\right)^{\circ} = \pm\sqrt{2}$$

$$\Rightarrow tan\left(22\tfrac{1}{2}\right)^{\circ} = \pm\sqrt{2} - 1.$$

Since $\left(22\frac{1}{2}\right)^{\circ}$ is acute, $tan\left(22\frac{1}{2}\right)^{\circ}$ is positive. Therefore

$tan\left(22\frac{1}{2}\right)^{\circ} = \sqrt{2} - 1$.

25. Prove that $\sin 3x = 3\sin x - 4\sin^3 x$

Solution: $\sin 3x = \sin(2x + x) = \sin 2x.\cos x + \cos 2x.\sin x$

$= (2\sin x.\cos x)\cos x + (1 - 2\sin^2 x)\sin x$

$= 2.\sin x.\cos^2 x + \sin x - 2\sin^3 x$

$= 2.\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x$.

26. Prove that $\cos 3x = 4\cos^3 x - 3\cos x$

Solution: $\cos 3x = \cos(2x + x) = \cos 2x.\cos x - \sin 2x.\sin x$

$= (2\cos^2 x - 1)\cos x - (2\sin x.\cos x)\sin x$

$= 2\cos^3 x - \cos x - 2\sin^2 x.\cos x$

$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x = 4\cos^3 x - 3\cos x$.

27. Prove that $\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

Solution: $\tan 3x = \tan(2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x.\tan x}$

$$= \frac{\frac{2\tan x}{1 - \tan^2 x} + \tan x}{1 - \left(\frac{2\tan x}{1 - \tan^2 x}\right).\tan x} = \frac{2\tan x + \tan x(1 - \tan^2 x)}{1 - \tan^2 x - 2\tan^2 x}$$

$$= \frac{2\tan x + \tan x - \tan^3 x)}{1 - \tan^2 x - 2\tan^2 x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$$

28. If $\tan A = \frac{1 - \cos B}{\sin B}$, prove that $\tan 2A = \tan B$, where A and B are acute angles.

Solution: $RHS = \frac{1 - \cos B}{\sin B} = \frac{2\sin^2\left(\frac{B}{2}\right)}{2\sin\left(\frac{B}{2}\right)\cdot\cos\left(\frac{B}{2}\right)} = \frac{\sin\left(\frac{B}{2}\right)}{\cos\left(\frac{B}{2}\right)} = \tan\left(\frac{B}{2}\right)$.

$\therefore \tan A = \tan\left(\frac{B}{2}\right)$.

Since A and B are acute angles we get, $A = \frac{B}{2}$. $\therefore 2A = B$

$\therefore \tan 2A = \tan B$.

29. Show that $4\,sin\,A.sin\,(60° + A).sin\,(60° - A) = sin\,3A$.

Solution: $LHS = 4\,sin\,A.\{sin\,(60° + A).sin\,(60° - A)\}$

$= 4\,sin\,A\,\{sin^2\,60° - sin^2\,A\} = 4\,sin\,A\left\{\left(\frac{\sqrt{3}}{2}\right)^2 - sin^2\,A\right\}$

$= 4\,sin\,A\,\left\{\frac{3}{4} - sin^2 A\right\} = 3sin\,A - 4sin^3 A = sin\,3A = RHS.$

30. Show that $4\,cos\,A.cos\,(60° + A).cos\,(60° - A) = cos\,3A$.

Solution: $LHS = 4\,cos\,A.cos\,(60° + A).cos\,(60° - A)$

$= 4\,cos\,A\,\{cos^2\,60° - sin^2\,A\}$

$= 4\,cos\,A\,\left\{\frac{1}{4} - (1 - cos^2 A)\right\} = 4\,cos\,A\,\left\{-\frac{3}{4} + cos^2 A\right\}$

$= -3cos\,A + 4cos^3 A = 4cos^3 A - 3cos\,A = cos\,3A = RHS.$

31. Prove that $\frac{2\,sin\,A - sin\,2A}{2\,sin\,A + sin\,2A} = tan^2\,\frac{A}{2}$.

Solution: $\frac{2\,sin\,A - sin\,2A}{2\,sin\,A + sin\,2A} = \frac{2\,sin\,A - 2sin\,A.cos\,A}{2\,sin\,A + 2sin\,A.cos\,A}$

$= \frac{2\,sin\,A(1 - cos\,A)}{2\,sin\,A(1 + cos\,A)} = \frac{2sin^2\frac{A}{2}}{2cos^2\frac{A}{2}} = tan^2\,\frac{A}{2}.$

32. Prove that $\frac{sin\,4x\,cos\,2x}{(1 + cos\,4x)(1 + cos\,2x)} = tan\,x$

Solution: $\frac{sin\,4x\,cos\,2x}{(1 + cos\,4x)(1 + cos\,2x)} = \frac{2sin\,2x.cos\,2x.cos\,2x}{2cos^2\,2x.2cos^2\,x} = \frac{sin\,2x}{2cos^2\,x}$

$= \frac{2sin\,x.cos\,x}{2cos\,x.cos\,x} = tan\,x.$

33. If $sin\,A = \frac{3}{5}$ and A is acute angle, find the values of $sin\,2A$ and $cos\,2A$.

Solution: $sin\,A = \frac{3}{5} \Rightarrow opp = 3$ and $hyp = 5$. $\therefore adj = 4$.

$\therefore cos\,A = \frac{adj}{hyp} = \frac{4}{5}.$

$\therefore sin\,2A = 2sin\,A.cos\,A = 2\frac{3}{5}.\frac{4}{5} = \frac{24}{25}$;

$cos\,2A = 1 - 2sin^2 A = 1 - 2\left(\frac{3}{5}\right)^2 = 1 - 2\left(\frac{9}{25}\right) = 1 - \frac{18}{25} = \frac{7}{25}.$

34. If $\tan A = \frac{1}{3}$, $\tan B = \frac{1}{7}$ prove that $2A + B = \frac{\pi}{4}$.

Solution: $\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{2(1/3)}{1 - (1/3)^2} = \frac{2/3}{1 - 1/9} = \frac{2/3}{8/9} = \frac{3}{4}$.

$\therefore \tan(2A + B) = \frac{\tan 2A + \tan B}{1 - \tan 2A.\tan B} = \frac{\frac{3}{4} + \frac{1}{7}}{1 - \left(\frac{3}{4}\right)\left(\frac{1}{7}\right)} = \frac{\frac{21 + 4}{28}}{\frac{28 - 3}{28}} = \frac{25}{25} = 1.$

Since $\tan A$ and $\tan B$ are positive and since $\tan(2A + B)$ is positive, we get, $2A + B = \frac{\pi}{4}$.

35. Show that $\tan A = \frac{1 - \cos 2A}{\sin 2A}$ and hence deduce the value of $\tan 15°$.

Solution: $\frac{1 - \cos 2A}{\sin 2A} = \frac{2\sin^2 A}{2\sin A.\cos A} = \tan A$. $\therefore \tan A = \frac{1 - \cos 2A}{\sin 2A}$.

By taking $A = 15°$ we get,

$$\tan 15° = \frac{1 - \cos 2(15°)}{\sin 2(15°)} = \frac{1 - \cos 30°}{\sin 30°} = \frac{1 - \sqrt{3}/2}{1/2} = \frac{\frac{2 - \sqrt{3}}{2}}{1/2} = 2 - \sqrt{3}.$$

36. Prove that $\cos 20° + \cos 100° + \cos 140° = 0$.

Solution: $LHS = \cos 20° + (\cos 100° + \cos 140°)$

$$= \cos 20° + 2\cos\left(\frac{100° + 140°}{2}\right).\cos\left(\frac{100° - 140°}{2}\right)$$

$$= \cos 20° + 2\cos 120°.\cos(-20°)$$

$$= \cos 20° + 2\left(-\frac{1}{2}\right).\cos 20° = \cos 20° - \cos 20° = 0.$$

37. Prove that $\sin 50° - \sin 70° + \sin 10° = 0$

Solution: $LHS = \sin 50° - \sin 70° + \sin 10°$

$$= 2\cos\left(\frac{50° + 70°}{2}\right).\sin\left(\frac{50° - 70°}{2}\right) + \sin 10°$$

$$= 2\cos 60° \cdot \sin(-10°) + \sin 10°$$

$$= -2 \times \frac{1}{2}\sin 10° + \sin 10° = -\sin 10° + \sin 10° = 0.$$

38. Prove that $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

Solution: $LHS = \sin 60°.\sin 20° \sin(60° - 20°)\sin(60° + 20°)$

$$= \frac{\sqrt{3}}{2}\sin 20° \sin(60° - 20°)\sin(60° + 20°)$$

$$= \frac{\sqrt{3}}{2}.\frac{1}{4}.\sin(3\times\ 20°) \quad = \frac{\sqrt{3}}{2}.\frac{1}{4}.\sin 60°$$

$$= \frac{\sqrt{3}}{8}[sin\ 60°] = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2} = \frac{3}{16}.$$

39. Show that $\frac{sin\ 5A\ +\ sin\ 2A\ -\ sin\ A}{cos\ 5A\ +\ cos\ 2A\ +\ cos\ A} = tan\ 2A$

Solution: $\frac{sin\ 5A\ +\ sin\ 2A\ -\ sin\ A}{cos\ 5A\ +\ cos\ 2A\ +\ cos\ A} = \frac{sin\ 2A\ +\ sin\ 5A\ -\ sin\ A}{cos\ 2A\ +\ cos\ 5A\ +\ cos\ A}$

$$= \frac{sin\ 2A\ +\ 2\,cos\ \left(\frac{5A\ +\ A}{2}\right).sin\ \left(\frac{5A\ -\ A}{2}\right)}{cos\ 2A\ +\ 2\,cos\ \left(\frac{5A\ +A}{2}\right).cos\ \left(\frac{5A\ -\ A}{2}\right)}$$

$$= \frac{sin\ 2A\ +\ 2\,cos\ 3A\ .\,sin\ 2A}{cos\ 2A\ +2\,cos\ 3A\ .\,cos\ 2A}$$

$$= \frac{sin\ 2A[1+2\,cos\ 3A]}{cos\ 2A[1+2\,cos\ 3A]} = \frac{sin\ 2A}{cos\ 2A} = tan\ 2A.$$

40. Find the general solution of $sin\,\theta\ =\ k,\ -1\ \le\ k\ \le\ 1.$

Solution: Let α be an angle such that

$$sin\,\alpha\ =\ k,\ \ -\frac{\pi}{2}\ \le\ \alpha\ \le\ \frac{\pi}{2}.$$

Then $sin\,\theta\ =\ sin\,\alpha.$

Now $sin\,\theta\ =\ sin\,\alpha\ \Rightarrow\ sin\,\theta\ -\ sin\,\alpha\ =\ 0$

$$\Rightarrow\ 2cos\ \frac{\theta\ +\ \alpha}{2}.sin\ \frac{\theta\ -\ \alpha}{2}\ =\ 0$$

$$\Rightarrow\ cos\ \frac{\theta + \alpha}{2}\ =\ 0\ \text{ or }\ sin\ \frac{\theta\ -\ \alpha}{2}\ =\ 0.$$

Now $cos\ \frac{\theta\ +\ \alpha}{2}\ =\ 0\ \Rightarrow\ \frac{\theta + \alpha}{2}\ =\ (2n\ +\ 1)\frac{\pi}{2},\ n \in\ Z.$

$$\Rightarrow \theta\ +\ \alpha =\ (2n\ +\ 1)\pi$$

$$\Rightarrow\ \ \theta = (2n\ +\ 1)\pi\ -\ \alpha \qquad ...(1)$$

$sin\ \frac{\theta\ -\ \alpha}{2}\ =\ 0\ \Rightarrow\ \frac{\theta\ -\ \alpha}{2}\ =\ n\pi,\ \ n\ \in\ Z.$

$$\Rightarrow\ \theta - \alpha\ =\ 2n\pi\quad \Rightarrow\ \theta\ =\ 2n\pi\ +\ \alpha \qquad ...(2).$$

∴ combining (1) and (2),we get, $\theta = n\pi\ + (-1)^n\alpha,\ \ n \in Z.$

This is the general solution of $sin\,\theta\ =\ k,\ \ -1\ \le\ k\ \le\ \ 1.$

41. Find the general solution of $cos\,\theta\ =\ k,\ \ -1\ \le\ k\ \le\ \ 1.$

Solution: Let α be an angle such that $cos\,\alpha\ =\ k,\ 0 \le\ \alpha\ \le\ \pi$

Then $cos\,\theta\ =\ cos\,\alpha.$

Now $cos\,\theta\ =\ cos\,\alpha\ \ \Rightarrow\ cos\,\theta - cos\,\alpha\ =\ 0$

$$\Rightarrow -2sin\,\frac{\theta+\alpha}{2}.sin\,\frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow sin\frac{\theta+\alpha}{2} = 0 \text{ or } sin\,\frac{\theta-\alpha}{2} = 0.$$

Now $sin\,\frac{\theta+\alpha}{2} = 0 \Rightarrow \frac{\theta+\alpha}{2} = n\pi,\ n \in Z$

$$\Rightarrow \theta+\alpha = 2n\pi \quad \Rightarrow \theta = 2n\pi - \alpha \quad ...(1)$$

$sin\,\frac{\theta-\alpha}{2} = 0 \Rightarrow \frac{\theta-\alpha}{2} = n\pi,\ n \in Z$

$$\Rightarrow \theta - \alpha = 2n\pi \quad \Rightarrow \theta = 2n\pi + \alpha \quad ...(2).$$

∴ combining (1) and (2), we get, $\theta = 2n\pi \pm \alpha,\ n \in Z.$

This is the general solution of $cos\,\theta = k,\ -1 \le k \le 1.$

42. Find the general solution of $tan\,\theta = k, -\infty < k < \infty.$

Solution: Let α be an angle such that

$$tan\,\alpha = k,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}.$$

Then $tan\,\theta = tan\,\alpha.$

Now $tan\,\theta = tan\,\alpha \Rightarrow \frac{sin\theta}{cos\theta} = \frac{sin\alpha}{cos\alpha}$

$$\Rightarrow sin\,\theta.cos\,\alpha = cos\,\theta.sin\,\alpha$$

$$\Rightarrow sin\,\theta.cos\,\alpha - cos\,\theta.sin\,\alpha = 0$$

$$\Rightarrow sin(\theta - \alpha) = 0 \quad \Rightarrow \theta - \alpha = n\pi,\ n \in Z.$$

$$\Rightarrow \theta = n\pi + \alpha,\ n \in Z.$$

This is the general solution of $tan\,\theta = k, -\infty < k < \infty.$

43. Find the general solution of $tan\,2\theta.tan\,3\theta = 1.$

Solution: $tan\,2\theta.tan\,3\theta = 1 \implies \frac{sin\,2\,\theta}{cos\,2\,\theta}.\frac{sin\,3\,\theta}{cos\,3\,\theta} = 1$

$$\implies sin\,2\theta.sin\,3\theta = cos\,2\theta.cos\,3\theta$$

$$\implies cos\,2\theta.cos\,3\theta - sin\,2\theta.sin\,3\theta = 0$$

$$\implies cos\,(2\theta + 3\theta) = 0 \quad \implies cos\,5\theta = 0.$$

$\implies 5\,\theta = (2n+1)\frac{\pi}{2},\ n \in Z \quad \implies \theta = (2n+1)\frac{\pi}{10},\ n \in Z.$

This is the general solution of the given equation.

44. Find the general solution of

$$\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta.\tan 2\theta = \sqrt{3}.$$

Solution: $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta.\tan 2\theta = \sqrt{3}$

$\Rightarrow \tan\theta + \tan 2\theta = \sqrt{3}(1 - \tan\theta.\tan 2\theta)$

$\Rightarrow \frac{\tan\theta + \tan 2\theta}{(1 - \tan\theta \cdot \tan 2\theta)} = \sqrt{3} \quad \Rightarrow \tan 3\theta = \sqrt{3} = \tan\frac{\pi}{3}$

$\Rightarrow 3\theta = n\pi + \frac{\pi}{3}, n \in Z \quad \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{9}, n \in Z.$

This is the general solution of the given equation.

45. Find the general solution of $\sin 2x + \sin 4x + \sin 6x = 0$.

Solution: $\sin 2x + \sin 4x + \sin 6x = 0$

$\Rightarrow \sin 4x + \sin 6x + \sin 2x = 0$

$\Rightarrow \sin 4x + 2\sin\left(\frac{6x + 2x}{2}\right) \cdot \cos\left(\frac{6x - 2x}{2}\right) = 0$

$\Rightarrow \sin 4x + 2\sin 4x.\cos 2x = 0 \quad \Rightarrow \sin 4x(1 + 2\cos 2x) = 0$

$\Rightarrow \sin 4x = 0$ or $1 + 2\cos 2x = 0$.

$\sin 4x = 0 \quad \Rightarrow 4x = n\pi \quad \therefore x = \frac{n\pi}{4}, n \in Z.$

$1 + 2\cos 2x = 0 \quad \Rightarrow \cos 2x = -\frac{1}{2} = \cos\frac{2\pi}{3}.$

$\therefore 2x = 2n\pi \pm \frac{2\pi}{3}, n \in Z. \quad \therefore x = n\pi \pm \frac{\pi}{3}, n \in Z.$

This is the general solution of the given equation.

46. Find the general solution of $4\sin^2 x - 8\cos x + 1 = 0$.

Solution: $4\sin^2 x - 8\cos x + 1 = 0$

$\Rightarrow 4(1 - \cos^2 x) - 8\cos x + 1 = 0$

$\Rightarrow -4\cos^2 x - 8\cos x + 5 = 0$

$\Rightarrow 4\cos^2 x + 8\cos x - 5 = 0$

$\Rightarrow 4\cos^2 x + 10\cos x - 2\cos x - 5 = 0$

$\Rightarrow 2\cos x(2\cos x + 5) - 1(2\cos x + 5) = 0$

$\Rightarrow (2\cos x - 1)(2\cos x + 5) = 0$

$\Rightarrow 2\cos x - 1 = 0$ or $2\cos x + 5 = 0$

$\Rightarrow \cos x = \frac{1}{2}$ or $\cos x = -\frac{5}{2}$

Now $\cos x = -\frac{5}{2} < -1$. This has no solution.

$cos\, x = \frac{1}{2} \implies x = 2n\pi \pm \frac{\pi}{3}, n \in Z.$

This is the general solution of the given equation.

47. Find the general solution of $2sin^2 x + \sqrt{3}cos\, x + 1 = 0$.

Solution: $2sin^2 x + \sqrt{3}cos\, x + 1 = 0$

$$\implies 2(1 - cos^2 x) + \sqrt{3}cos\, x + 1 = 0$$

$$\implies 2cos^2 x - 2\sqrt{3}cos\, x + \sqrt{3}cos\, x - 3 = 0$$

$$\implies 2cos\, x\left(cos\, x - \sqrt{3}\right) + \sqrt{3}(cos\, x - \sqrt{3}) = 0$$

$$\implies (2cos\, x + \sqrt{3})\left(cos\, x - \sqrt{3}\right) = 0$$

$\implies cos\, x = -\frac{\sqrt{3}}{2}$ or $cos\, x = \sqrt{3}$.

$cosx = \sqrt{3} > 1$. This has no solution.

Now $cosx = -\frac{\sqrt{3}}{2} = \cos\left(\pi - \frac{\pi}{6}\right) = cos\, \frac{5\pi}{6}$. $\therefore x = 2n\pi \pm \frac{5\pi}{6}, n \in Z.$

This is the general solution of the given equation.

48. Find the general solution of $2cos^2 x + 3sin\, x = 0$.

Solution: $2cos^2 x + 3sin\, x = 0$

$$\implies 2(1 - sin^2 x) + 3sin\, x = 0 \implies 2 - 2sin^2 x + 3sin\, x = 0$$

$$\implies 2sin^2 x - 3\, sin\, x - 2 = 0 \implies (2sin\, x + 1)(sin\, x - 2) = 0$$

$\implies 2sin\, x + 1 = 0$ or $sin\, x - 2 = 0$

$\implies sin\, x = -\frac{1}{2}$ or $sin\, x = 2$

$sinx = 2 > 1$ has no solution.

$sin\, x = -\frac{1}{2} = -sin\frac{\pi}{6} = sin\left(-\frac{\pi}{6}\right)$.

$\therefore x = n\pi + (-1)^n \left(-\frac{\pi}{6}\right), \quad n \in Z.$

This is the general solution of the given equation.

49. Find the general solution of $2cos^2 \theta + sin\, \theta = sin\, 2\theta + cos\, \theta$.

Solution: $2cos^2 \theta + sin\, \theta = sin\, 2\theta + cos\, \theta$

$$\implies 2cos^2 \theta - cos\, \theta = 2sin\, \theta cos\, \theta - sin\, \theta$$

$$\implies cos\, \theta(2cos\, \theta - 1) - sin\, \theta(2cos\, \theta - 1) = 0$$

$$\implies (2cos\, \theta - 1)(cos\, \theta - sin\, \theta) = 0$$

$\implies 2cos\, \theta - 1 = 0$ or $cos\, \theta - sin\, \theta = 0$.

Now $2cos\,\theta - 1 = 0 \implies cos\,\theta = \frac{1}{2} = cos\frac{\pi}{3}\ \therefore \theta = 2n\pi \pm \frac{\pi}{3}\ \ ,n \in Z.$

$cos\,\theta - sin\,\theta = 0 \implies sin\,\theta = cos\,\theta.$

Dividing by $cos\,\theta$ we get, $tan\ \theta = 1 = tan\frac{\pi}{4}.\ \therefore \theta = n\pi + \frac{\pi}{4},$

$n \in Z.$ $\therefore$ the general solution is $\theta = 2n\pi \pm \frac{\pi}{3}, \theta = n\pi + \frac{\pi}{4},\ n \in Z.$

50. Find the general solution of $cos\,x + cos\,2x + cos\,3x = 0.$

Solution: $cos\,x + cos\,2x + cos\,3x = 0$

$\implies 2\,cos\left(\frac{x+3x}{2}\right).cos\left(\frac{x-3x}{2}\right) + cos\,2x = 0 \implies cos\,2x(2cos\,x + 1) = 0$

$\implies cos\,2x = 0$ or $(2cos\,x + 1) = 0.$

Now $cos\,2x = 0 \implies 2x = (2n+1)\frac{\pi}{2}\ \ \therefore\ x = (2n+1)\frac{\pi}{4}\ \ ,n \in Z.$

$2cos\,x + 1 = 0 \implies cos\,x = -\frac{1}{2} = cos\left(\pi - \frac{\pi}{3}\right) = cos\,\frac{2\pi}{3}$

$\therefore x = 2n\pi \pm \frac{2\pi}{3}\ \ ,n \in Z.$

$\therefore$ the general solution is $x = (2n+1)\frac{\pi}{4},\ \ x = 2n\pi \pm \frac{2\pi}{3}\ \ n \in Z.$

51. Prove that $2cos\,\frac{\pi}{13}.cos\,\frac{9\pi}{13} + cos\,\frac{3\pi}{13} + cos\,\frac{5\pi}{13} = 0.$

Solution: $2cos\,\frac{\pi}{13}.cos\,\frac{9\pi}{13} + cos\,\frac{3\pi}{13} + cos\,\frac{5\pi}{13}$

$= 2cos\,\frac{\pi}{13}.cos\,\frac{9\pi}{13} + 2cos\,\frac{\frac{3\pi}{13}+\frac{5\pi}{13}}{2}.cos\,\frac{\frac{5\pi}{13}-\frac{3\pi}{13}}{13}$

$= 2cos\,\frac{\pi}{13}.cos\,\frac{9\pi}{13} + 2cos\,\frac{\frac{8\pi}{13}}{2}.cos\,\frac{\frac{2\pi}{13}}{2}$

$= 2cos\,\frac{\pi}{13}.cos\,\frac{9\pi}{13} + 2cos\,\frac{4\pi}{13}.cos\,\frac{\pi}{13}$

$= 2cos\,\frac{\pi}{13}.cos\left(\frac{13\pi - 4\pi}{13}\right) + 2cos\,\frac{4\pi}{13}.cos\,\frac{\pi}{13}$

$= 2cos\,\frac{\pi}{13}.cos\left(\pi - \frac{4\pi}{13}\right) + 2cos\,\frac{4\pi}{13}.cos\,\frac{\pi}{13}$

$= -2cos\,\frac{\pi}{13}.cos\,\frac{4\pi}{13} + 2cos\,\frac{4\pi}{13}.cos\,\frac{\pi}{13} = 0.$

52. Prove that $cos\,6x = 32cos^6 x - 48cos^4 x + 18cos^2 x - 1$

Solution: $cos\,6x = 2cos^2 3x - 1$

$= 2(4cos^3 x - 3cos\,x)^2 - 1$

$= 2\{(4cos^3 x)^2 + (3cos\,x)^2 - 24cos^3 x.cos\,x\} - 1$

$= 2\{16cos^6 x + 9cos^2 x - 24cos^4 x\} - 1$
$= 32cos^6 x - 48cos^4 x + 18cos^2 x - 1$

53. Prove that $cot\, x.cot\, 2x - cot\, 2x.cot\, 3x - cot\, 3x.cot\, x = 1.$

Solution: We have $x + 2x = 3x.$

$\therefore\ cot\, 3x = cot\,(x + 2x) = \frac{cot\, x.cot\, 2x - 1}{cot\, 2x + cot\, x}.$

$\therefore\ cot\, 3x(cot\, 2x - cot\, x) = cot\, x.cot\, 2x + 1$

$\therefore\ cot\, 3x.cot\, 2x + cot\, 3x.cot\, x = cot\, x.cot\, 2x - 1$

Thus $cot\, 3x.cot\, 2x + cot\, 3x.cot\, x - cot\, x.cot\, 2x = -1.$

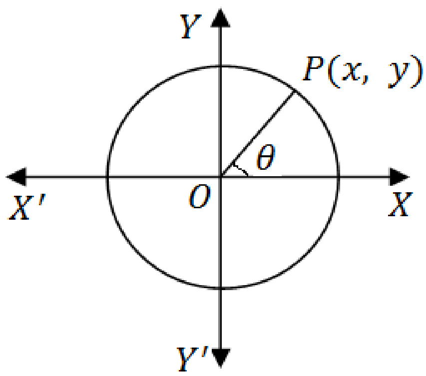
By changing the sign we get

$cot\, x.cot\, 2x - cot\, 2x.cot\, 3x - cot\, 3x.cot\, x = 1.$

Five mark questions

1. Discuss the signs of trigonometric functions in the four quadrants.

Solution: Consider a circle of radius r, centered at the origin of a Cartesian coordinate system. Let $P(x, y)$ be a point on the circle. Suppose that the angle made by the radius OP with the positive direction of the x-axis is θ. Then



$sin\, \theta = \frac{y}{r}, \quad cos\, \theta = \frac{x}{r}, \quad tan\, \theta = \frac{y}{x}.$

a) If $0 < \theta < \frac{\pi}{2}$ then $P(x, y)$ is a point in the first quadrant.

Here $x > 0$ and $y > 0$. $\therefore\ sin\, \theta > 0,\ cos\, \theta > 0,\ tan\, \theta > 0$

$\therefore\ cosec\, \theta > 0,\ sec\, \theta > 0,\ cot\, \theta > 0.$

In the first quadrant ***all*** the trigonometric functions are positive.

b) If $\frac{\pi}{2} < \theta < \pi$ then $P(x, y)$ is a point in the second quadrant.

Here $x < 0$ and $y > 0$. $\therefore\ sin\, \theta > 0,\ cos\, \theta < 0,\ tan\, \theta < 0$

$\therefore\ cosec\,\theta\ > 0,\ sec\,\theta\ < 0,\ \ cot\,\theta\ < 0.$

In the second quadrant $\boldsymbol{sin\,\theta}$ (and also $\boldsymbol{osec\,\theta}$) is positive.

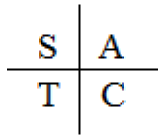
c) If $\pi < \theta < \frac{3\pi}{2}$ then $P(x,y)$ is a point in the third quadrant.

Here $x\ < 0$ and $y\ < 0.\quad \therefore\ sin\,\theta\ < 0,\ \ cos\,\theta\ < 0,\ \ tan\,\theta\ > 0$

$\therefore\ cosec\,\theta\ < 0,\quad sec\,\theta\ < 0,\quad cot\,\theta\ > 0.$

In the third quadrant $\boldsymbol{tan\,\theta}$ **(** and also $\boldsymbol{cot\,\theta}$ **)** is positive.

d) If $\frac{3\pi}{2} < \theta < 2\pi$ then $P(x,y)$ is a point in the fourth quadrant. Here $x\ > 0$ and $y\ < 0.$



$\therefore\ \ sin\,\theta\ < \ 0,\ \ cos\,\theta\ > 0$ and $\ tan\,\theta\ < \ 0.$

$\therefore\ \ cosec\,\theta\ < 0,\quad sec\,\theta\ > 0\ ,\ cot\,\theta\ \ < \ 0.$

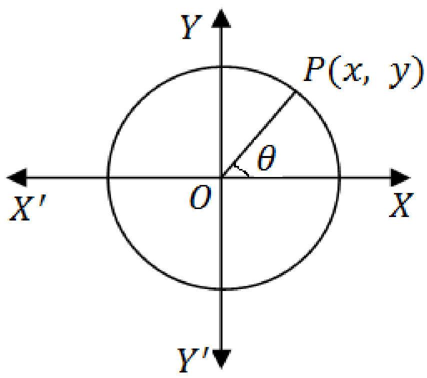
In the fourth quadrant $\boldsymbol{cos\,\theta}$ **(**and also $\boldsymbol{ec\,\theta}$ **)** is positive.

2. Find the range of values of the trigonometric functions

Solution: Consider a circle of radius r, centered at the origin of a Cartesian coordinate system.

Let $P(x,\ y)$ be a point on the circle.

Suppose that the angle made by the radius OP with the positive direction of the x-axis is θ.



Then $\quad sin\,\theta\ = \frac{y}{r}\ ,\ \ cos\,\theta\ = \frac{x}{r}\ ,$

$tan\,\theta\ = \frac{y}{x}\ ,$

$$cosec\,\theta\ = \frac{r}{y}\ ,\ \ sec\,\theta\ = \frac{r}{x}\ ,\quad cot\,\theta\ = \frac{x}{y}.$$

For any position of the point $P(x,\ y)$ on the circle,

$$-r\ \le\ y\ \le\ r\ \Rightarrow -1\ \le\ y/r\ \le\ 1\ \Rightarrow -1\ \le\ \sin\theta\ \le\ 1.$$

Similarly

$$-r\ \le\ x\ \le\ r\ \Rightarrow\ -1\ \le\ x/r\ \le\ 1 \Rightarrow\ -1\ \le\ \cos\theta\ \le\ 1.$$

Since $\ -\infty < y/x < \infty$ and $\ -\infty\ <\ x/y\ <\ \infty,$ we get,

$-\infty\ < \tan\theta < \infty$ and $\ -\infty < \cot\theta < \infty.$

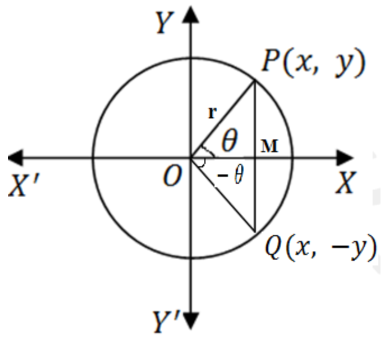
Also, $\ -r\ \le\ y\ \le\ r\ \ \Rightarrow\ r/y\ \ge\ 1$ or $\ r/y\ \le\ -1$

$\therefore\ |\,r/y\ |\ \ \ge\ 1.\quad \therefore\ |\sec\theta\,|\ \ \ge\ 1.$

Similarly $-r \leq x \leq r \Rightarrow r/x \geq 1$ or $r/x \leq -1$
$\therefore |r/x| \geq 1$. $\therefore |cosec\,\theta| \geq 1$.

3. Express the trigonometric ratios of $-\theta$ with those of θ.

Solution: Consider a circle of radius r centered at the origin of the Cartesian coordinate system. Let $P(x, y)$ be a point on the circle. Suppose that the angle made by the radius OP with the positive direction of the x-axis is θ. Then



$sin\,\theta = \frac{y}{r}$, $cos\,\theta = \frac{x}{r}$, $tan\,\theta = \frac{y}{x}$.

Draw PM perpendicular to the x-axis and produce to meet the circle at Q.

Here Q is the image of P. $\therefore Q \equiv (x, -y)$. Here angle QOX is $-\theta$.

By definition,

$sin\,(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -sin\,\theta$;

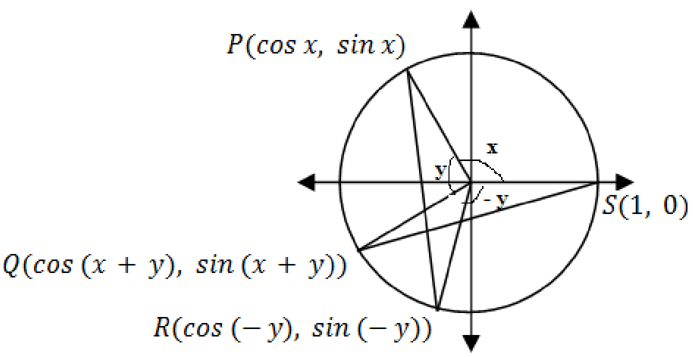
$cos\,(-\theta) = \frac{x}{r} = cos\,\theta$;

$tan\,(-\theta) = \frac{-y}{x} = -\frac{y}{x} = -tan\,\theta$.

By writing the reciprocals we get $cosec\,(-\theta) = -cosec\,\theta$; $sec\,(-\theta) = sec\,\theta$; $cot\,(-\theta) = -cot\,\theta$.

4. Prove geometrically $cos\,(x + y) = cos\,x.cos\,y - sin\,x.sin\,y$. Hence prove that $cos\,(x - y) = cos\,x.cos\,y + sin\,x.sin\,y$

Solution: Consider the unit circle centered at the origin of the Cartesian coordinate system.



Let P and Q be points on the unit circle such that angle $X = x$, angle $QOP = y$ and angle $QOX = x + y$.
Then $P \equiv (\cos x, \sin x)$ and $Q \equiv (\cos(x + y), \sin(x + y))$.
Let R be a point on the unit circle such that angle $QOX = -y$.
Then $R \equiv (\cos(-y), \sin(-y))$. Let $S \equiv (1, 0)$.
In ΔPOR and ΔQOS, $OP = OQ = OR = OS$.
Also angle $POR = QOS$.
$\therefore$ triangles are congruent $\Rightarrow PR = QS \Rightarrow PR^2 = QS^2$.

$$\Rightarrow [\cos x - \cos(-y)]^2 + [\sin x - \sin(-y)]^2$$
$$= [\cos(x + y) - 1]^2 + [\sin(x + y) - 0]^2$$
$$\Rightarrow [\cos x - \cos y]^2 + [\sin x + \sin y]^2$$
$$= [\cos(x + y) - 1]^2 + [\sin(x + y)]^2$$
$$\Rightarrow \cos^2 x + \cos^2 y - 2.\cos x.\cos y + \sin^2 x + \sin^2 y + 2.\sin x.\sin y$$
$$= \cos^2(x + y) + 1 - 2.\cos(x + y) + \sin^2(x + y)$$
$$\Rightarrow 2 - 2.\cos x.\cos y + 2.\sin x.\sin y = 2 - 2.\cos(x + y)$$
$$\Rightarrow -\cos x.\cos y + \sin x.\sin y = -\cos(x + y)$$
$$\Rightarrow \cos(x + y) = \cos x.\cos y - \sin x.\sin y.$$

By replacing y by $-y$ we get

$$\cos(x + (-y)) = \cos x.\cos(-y) - \sin x.\sin(-y)$$
$$\Rightarrow \cos(x - y) = \cos x.\cos y - \sin x.(-\sin y)$$
$$= \cos x.\cos y + \sin x.\sin y.$$

5. Express the trigonometric ratios of $90° - \theta$ with those of θ

Solution: We know that $\cos(x + y) = \cos x.\cos y - \sin x.\sin y$.

a) By taking $x = 90°$ and $y = -\theta$ we get,

$$\cos(90° - \theta) = \cos 90°.\cos(-\theta) - \sin 90°.\sin(-\theta)$$
$$= (0).\cos\theta - (1)(-\sin\theta) = \sin\theta.$$
$$\therefore \cos(90° - \theta) = \sin\theta.$$

b) Replacing θ by $90° - \theta$ in $\sin\theta = \cos(90° - \theta)$ we get,

$$\sin(90° - \theta) = \cos(90° - (90° - \theta)) = \cos\theta.$$

c) $\tan(90° - \theta) = \frac{\sin(90° - \theta)}{\cos(90° - \theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$

d) $\cot(90° - \theta) = \frac{\cos(90° - \theta)}{\sin(90° - \theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

e) $cosec\,(90° - \theta) = \frac{1}{sin\,(90° - \theta)} = \frac{1}{cos\,\theta} = sec\,\theta$.

f) $sec\,(90° - \theta) = \frac{1}{cos\,(90° - \theta)} = \frac{1}{sin\,\theta} = cosec\,\theta$.

6. Express the trigonometric ratios of $90° + \theta$ with those of θ.

Solution: We know that $cos\,(x + y) = cos\,x.cos\,y - sin\,x.sin\,y$.
By taking $x = 90°$ and $y = \theta$ we get,
a) $cos\,(90° + \theta) = cos\,90°.cos\,\theta - sin\,90°.sin\,\theta$
$= (0).cos\,\theta - (1).sin\,\theta \quad = -\,sin\,\theta$.
$\therefore\ cos\,(90° + \theta) = -\,sin\,\theta$.
b) $sin\,(90° + \theta) = cos\,(90° - (90° + \theta)) = cos\,(-\,\theta) = cos\,\theta$.
c) $tan\,(90° + \theta) = \frac{sin\,(90° + \theta)}{cos\,(90° + \theta)} = \frac{cos\,\theta}{-\,sin\,\theta} = -\,cot\,\theta$.
d) $cot\,(90° + \theta) = \frac{cos\,(90° + \theta)}{sin\,(90° + \theta)} = \frac{-\,sin\,\theta}{cos\,\theta} = -\,tan\,\theta$
e) $cosec\,(90° + \theta) = \frac{1}{sin\,(90° + \theta)} = \frac{1}{cos\,\theta} = sec\,\theta$.
f) $sec\,(90° + \theta) = \frac{1}{cos\,(90° + \theta)} = \frac{1}{-\,sin\,\theta} = -\,cosec\,\theta$.

7. Find the value of $sin\,18°$.
Hence find $cos\,36°$.

Solution: Let $\theta = 18°$. Then $5\theta = 90°$.
$\therefore\ 3\theta + 2\theta = 90°$ or $2\theta = 90° - 3\theta$.
$\therefore\ sin\,2\theta = sin\,(90° - 3\theta) = cos\,3\theta$.
$\therefore\ 2sin\,\theta.cos\,\theta = 4cos^3\,\theta - 3cos\,\theta$.
Dividing by $cos\,\theta$, we get, $2sin\,\theta = 4cos^2\,\theta - 3$

$\Rightarrow 2sin\,\theta = 4(1 - sin^2\,\theta) - 3$

$\Rightarrow 2sin\,\theta = 4 - 4sin^2\,\theta - 3$

$\Rightarrow 2sin\,\theta = 1 - 4sin^2\,\theta$

$\Rightarrow 4sin^2\,\theta + 2sin\,\theta - 1 = 0$

$\Rightarrow sin\,\theta = \frac{-2 \pm \sqrt{4 - 4(4)(-1)}}{2 \times 4} = \frac{-2 \pm \sqrt{4 + 16}}{8} = \frac{-2 \pm \sqrt{20}}{8}$
$= \frac{-2 \pm 2\sqrt{5}}{8} = \frac{2(-1 \pm \sqrt{5})}{8} = \frac{-1 \pm \sqrt{5}}{4}$.

Since $\theta = 18°$, $sin\,\theta = sin\,18°$ is positive. $\therefore\ sin\,18° = \frac{\sqrt{5} - 1}{4}$.
By using $cos\,2A = 1 - 2sin^2 A$, we get,

$$cos\, 36° = 1 - 2sin^2\, 18° \quad = 1 - 2\left(\frac{\sqrt{5} - 1}{4}\right)^2$$

$$= 1 - 2\left(\frac{5 + 1 - 2\sqrt{5}}{16}\right) = 1 - \left(\frac{6 - 2\sqrt{5}}{8}\right) = \frac{8 - (6 - 2\sqrt{5})}{8}$$

$$= \frac{2 + 2\sqrt{5}}{8} = \frac{1 + \sqrt{5}}{4}.$$

$\therefore\ cos\, 36° = \frac{\sqrt{5} + 1}{4}.$

8. Find the value of $sin\, 18°$.
Hence find $cos\, 18°$

Solution: Let $\theta = 18°$. Then $5\theta = 90°$.

$\therefore\ 3\theta + 2\theta = 90°$ or $2\theta = 90° - 3\theta$.

$\therefore\ sin\, 2\theta = sin\,(90° - 3\theta) = cos\, 3\theta$.

$\therefore\ 2sin\,\theta.cos\,\theta = 4cos^3\,\theta - 3cos\,\theta$.

Dividing by $cos\,\theta$, we get, $2sin\,\theta = 4cos^2\,\theta - 3$

$$\Rightarrow 2sin\,\theta = 4(1 - sin^2\,\theta) - 3$$

$$\Rightarrow 2sin\,\theta = 4 - 4sin^2\,\theta - 3$$

$$\Rightarrow 2sin\,\theta = 1 - 4sin^2\,\theta$$

$$\Rightarrow 4sin^2\,\theta + 2sin\,\theta - 1 = 0$$

$$\Rightarrow sin\,\theta = \frac{-2 \pm \sqrt{4 - 4(4)(-1)}}{2 \times 4} = \frac{-2 \pm \sqrt{4 + 16}}{8} = \frac{-2 \pm \sqrt{20}}{8}$$

$$= \frac{-2 \pm 2\sqrt{5}}{8} = \frac{2(-1 \pm \sqrt{5})}{8} = \frac{-1 \pm \sqrt{5}}{4}.$$

Since $\theta = 18°$, $sin\,\theta = sin\, 18°$ is positive. $\therefore\ sin\, 18° = \frac{\sqrt{5} - 1}{4}$.

By using $cos\, 2A = 1 - 2sin^2 A$, we get,

$$cos\, 36° = 1 - 2sin^2\, 18° \quad = 1 - 2\left(\frac{\sqrt{5} - 1}{4}\right)^2$$

$$= 1 - 2\left(\frac{5 + 1 - 2\sqrt{5}}{16}\right) = 1 - \left(\frac{6 - 2\sqrt{5}}{8}\right) = \frac{8 - (6 - 2\sqrt{5})}{8}$$

$$= \frac{2 + 2\sqrt{5}}{8} = \frac{1 + \sqrt{5}}{4}.$$

$\therefore\ cos\, 36° = \frac{\sqrt{5} + 1}{4}.$

By using $2cos^2 A = 1 + cos\, 2A$, we get,

$$2cos^2 18° = 1 + cos\, 36° \quad = 1 + \frac{\sqrt{5} + 1}{4} = \frac{4 + \sqrt{5} + 1}{4} = \frac{5 + \sqrt{5}}{4}$$

$\therefore\ cos^2 18° = \frac{5 + \sqrt{5}}{8}$

$\therefore\ cos\ 18° = \sqrt{\frac{5 + \sqrt{5}}{8}} = \sqrt{\frac{2(5 + \sqrt{5})}{16}} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$.

9. Find the value of $sin\ 18°$.
Hence find $sin\ 36°$.

Solution: Let $\theta = 18°$. Then $5\theta = 90°$.

$\therefore\ 3\theta + 2\theta = 90°$ or $2\theta = 90° - 3\theta$.

$\therefore\ sin\ 2\theta = sin\ (90° - 3\theta) = cos\ 3\theta$.

$\therefore\ 2sin\ \theta . cos\ \theta = 4cos^3\ \theta - 3cos\ \theta$.

Dividing by $cos\ \theta$, we get, $2sin\ \theta = 4cos^2\ \theta - 3$

$\Rightarrow 2sin\ \theta = 4(1 - sin^2\ \theta) - 3$

$\Rightarrow 2sin\ \theta = 4 - 4sin^2\ \theta - 3$

$\Rightarrow 2sin\ \theta = 1 - 4sin^2\ \theta$

$\Rightarrow 4sin^2\ \theta + 2sin\ \theta - 1 = 0$

$\Rightarrow sin\ \theta = \frac{-2 \pm \sqrt{4 - 4(4)(-1)}}{2 \times 4} = \frac{-2 \pm \sqrt{4 + 16}}{8} = \frac{-2 \pm \sqrt{20}}{8}$

$= \frac{-2 \pm 2\sqrt{5}}{8} = \frac{2(-1 \pm \sqrt{5})}{8} = \frac{-1 \pm \sqrt{5}}{4}$.

Since $\theta = 18°$, $sin\ \theta = sin\ 18°$ is positive. $\therefore\ sin\ 18° = \frac{\sqrt{5} - 1}{4}$.

By using $cos\ 2A = 1 - 2sin^2 A$, we get,

$cos\ 36° = 1 - 2sin^2\ 18° = 1 - 2\left(\frac{\sqrt{5} - 1}{4}\right)^2$

$= 1 - 2\left(\frac{5 + 1 - 2\sqrt{5}}{16}\right) = 1 - \left(\frac{6 - 2\sqrt{5}}{8}\right) = \frac{8 - (6 - 2\sqrt{5})}{8}$

$= \frac{2 + 2\sqrt{5}}{8} = \frac{1 + \sqrt{5}}{4}$.

$\therefore\ cos\ 36° = \frac{\sqrt{5} + 1}{4}$.

$\therefore\ sin^2\ 36° = 1 - cos^2\ 36° = 1 - \left(\frac{\sqrt{5} + 1}{4}\right)^2 = 1 - \frac{5 + 1 + 2\sqrt{5}}{16}$

$= 1 - \frac{6 + 2\sqrt{5}}{16} = \frac{16 - 6 - 2\sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16}$

$\therefore\ sin\ 36° = \frac{\sqrt{10 - 2\sqrt{5}}}{4}$.

10. Show that $cos\ A\ +\ cos\ (120° + A)\ +\ cos\ (120° - A)\ = 0$.

Solution: We have

$cos\ (120° + A) = cos\ 120°. cos\ A\ - sin\ 120°. sin\ A$ and

$cos(120° - A) = cos\ 120°. cos\ A\ +\ sin\ 120°. sin\ A$

$\therefore\ cos\ (120° + A) + cos\ (120° - A)\ =\ 2\ cos\ 120°\ cos\ A$

$=\ 2\ cos\ (180° - 60°)\ cos\ A$

$=\ -2cos\ (60°). cos\ A\ =\ -2\left(\frac{1}{2}\right) cos\ A\ =\ -cos\ A.$

$\therefore\ cos\ A\ +\ cos\ (120°\ +\ A)\ +\ cos\ (120°\ -\ A)$

$=\ cos\ A\ -\ cos\ A\ =\ 0\ =\ RHS.$

11. Prove that $cos\ 20°\ cos\ 40°\ cos\ 60°\ cos\ 80° = \frac{1}{16}$.

Solution: $LHS = \frac{1}{2}[cos\ (20° + 40°) + cos\ (20° - 40°)]\ \frac{1}{2}. cos\ 80°.$

$= \frac{1}{4}[cos\ 60°\ + cos\ (-20°)]. cos\ 80°$

$= \frac{1}{4}\left[\frac{1}{2}\ + cos\ 20°\right]. cos\ 80°\ \ = \frac{1}{4}\left[\frac{1 + 2\ cos\ \ 20°}{2}\right]. cos\ 80°$

$= \frac{1}{8}[cos\ 80° + 2\ cos\ 80°\ cos\ 20°]$

$= \frac{1}{8}[cos\ 80°\ + cos\ (80° + 20°)\ +\ cos\ (80° - 20°)]$

$= \frac{1}{8}[cos\ 80° + cos\ 100° + cos\ 60°]$

$= \frac{1}{8}\left[cos\ 80° + cos\ (180° - 80°) + \frac{1}{2}\right]$

$= \frac{1}{8}\left[cos\ 80° - cos\ 80° + \frac{1}{2}\right]\ = \frac{1}{16}$

II method:

$cos\ 20°\ cos\ 40°\ cos\ 60°\ cos\ 80°$

$= cos\ 60°. cos\ 20°\ cos(60°\ -\ 20°)\ cos(60°\ +\ 20°)$

$= cos\ 60°. cos\ 20°\ \{cos^2\ 60°\ -\ sin^2\ 20°\}$

$= \frac{1}{2}. cos\ 20°\left\{\left(\frac{1}{2}\right)^2\ -\ (1\ -\ cos^2\ 20°)\right\}$

$= \frac{1}{2}. cos\ 20°\left\{\frac{1}{4} -\ 1\ +\ cos^2\ 20°\right\}$

$= \frac{1}{2}. cos\ 20°\left\{cos^2\ 20°\ -\ \frac{3}{4}\right\}$

$= \frac{1}{2}. cos\ 20°\left\{\frac{4cos^2\ 20°\ -\ 3}{4}\right\}$

$$= \frac{1}{2}.\left\{\frac{4cos^3 20° - 3 cos 20°}{4}\right\}$$

$$= \frac{1}{2}.\frac{1}{4}.cos\,(3 \times 20°) = \frac{1}{2}.\frac{1}{4}.cos\,60° = \frac{1}{8}.\frac{1}{2} = \frac{1}{16}.$$

12. Show that $sin\,\theta = \frac{sin\,3\theta}{1 + 2cos\,2\theta}$ and hence prove that $sin15° = \frac{\sqrt{3} - 1}{2\sqrt{2}}$.

Solution: $\frac{sin\,3\theta}{1 + 2cos\,2\theta} = \frac{3sin\,\theta - 4sin^3\,\theta}{1 + 2(1 - 2sin^2\,\theta)} = \frac{sin\,\theta(3 - 4sin^2\,\theta)}{3 - 4sin^2\,\theta} = sin\,\theta.$

$\therefore\ sin\,\theta = \frac{sin\,3\theta}{1 + 2cos\,2\theta}$.

By taking $\theta = 15°$ we get,

$$sin\,15° = \frac{sin\,3(15°)}{1 + 2cos\,2(15°)} = \frac{sin\,45°}{1 + 2cos\,30°} = \frac{1/\sqrt{2}}{1 + 2(\sqrt{3}/2)} = \frac{1}{\sqrt{2}(1 + \sqrt{3})}$$

$$= \frac{1}{\sqrt{2}(\sqrt{3} + 1)} = \frac{1}{\sqrt{2}(\sqrt{3} + 1)}.\frac{\sqrt{3} - 1}{\sqrt{3} - 1} = \frac{\sqrt{3} - 1}{\sqrt{2}(3 - 1)} = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

13. Show that $\frac{cos\,7x - cos\,5x + cos\,3x - cos\,x}{sin\,7x - sin\,5x - sin\,3x + sin\,x} = cot\,2x.$

Solution:

$$\frac{cos\,7x - cos\,5x + cos\,3x - cos\,x}{sin\,7x - sin\,5x - sin\,3x + sin\,x} = \frac{cos\,7x + cos\,3x - cos\,5x - cos\,x}{sin\,7x - sin\,3x - sin\,5x + sin\,x}$$

$$= \frac{cos\,7x + cos\,3x - (cos\,5x + cos\,x)}{sin\,7x - sin\,3x - (sin\,5x - sin\,x)}$$

$$= \frac{2\,cos\left(\frac{7x+3x}{2}\right).cos\left(\frac{7x-3x}{2}\right) - 2\,cos\left(\frac{5x+x}{2}\right).cos\left(\frac{5x-x}{2}\right)}{2\,cos\left(\frac{7x+3x}{2}\right).sin\left(\frac{7x-3x}{2}\right) - 2\,cos\left(\frac{5x+x}{2}\right).sin\left(\frac{5x-x}{2}\right)}$$

$$= \frac{2cos\,5x\cdot cos\,2x - 2cos\,3x\cdot cos\,2x}{2cos\,5x\cdot sin\,2x - 2cos\,3x\cdot sin\,2x}$$

$$= \frac{2cos\,2x(cos\,5x - cos\,3x)}{2sin\,2x(cos\,5x - cos\,3x)} = \frac{cos\,2x}{sin\,2x} = cot\,2x.$$

14. Prove that $(cos\,x + cos\,y)^2 + (sin\,x - sin\,y)^2 = 4cos^2\left(\frac{x + y}{2}\right)$.

Solution: $(cos\,x + cos\,y)^2 + (sin\,x - sin\,y)^2$

$$= cos^2x + cos^2\,y + 2cos\,x.cos\,y + sin^2\,x + sin^2\,y - 2sin\,x.sin\,y$$

$$= cos^2x + sin^2\,x + cos^2\,y + sin^2\,y + 2(cos\,x.cos\,y - sin\,x.sin\,y)$$

$$= 1 + 1 + 2cos\,(x + y)$$

$$= 2 + 2cos\,(x + y) = 2\{1 + cos\,(x + y)\} = 2.2cos^2\left(\frac{x + y}{2}\right)$$

$$= 4cos^2\left(\frac{x + y}{2}\right).$$

15. If $tan\,x = -\frac{4}{3}$, $\frac{\pi}{2} < x < \pi$ find $sin\,\frac{x}{2}$, $cos\,\frac{x}{2}$ and $tan\,\frac{x}{2}$.

Solution: Let $opp = 4$ and $adj = 3$. Then $hyp = 5$.
Since x is an angle lying in the II quadrant, $cos\,x$ is negative.
$\therefore\ cos\,x = -\frac{adj}{hyp} = -\frac{4}{5}$.
Now $2sin^2\,\frac{x}{2} = 1 - cos\,x = 1 - \left(-\frac{4}{5}\right) = \frac{5+4}{5} = \frac{9}{5}$.
$\therefore\ sin^2\,\frac{x}{2} = \frac{9}{10}$. $\therefore\ sin\,\frac{x}{2} = \pm\frac{3}{\sqrt{10}}$.
Since $\frac{\pi}{2} < x < \pi$ we get, $\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$.
$\therefore\ sin\,\frac{x}{2} > 0$. $\therefore\ sin\,\frac{x}{2} = \frac{3}{\sqrt{10}}$.
Now $2cos^2\,\frac{x}{2} = 1 + cos\,x = 1 + \left(-\frac{4}{5}\right) = \frac{5-4}{5} = \frac{1}{5}$.
$\therefore\ cos^2\,\frac{x}{2} = \frac{1}{10}$. $\therefore\ cos\,\frac{x}{2} = \pm\frac{1}{\sqrt{10}}$.
Since $\frac{\pi}{2} < x < \pi$ we get, $\frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$.
$\therefore\ cos\,\frac{x}{2} > 0$. $\therefore\ cos\,\frac{x}{2} = \frac{1}{\sqrt{10}}$.
Also $tan\,\frac{x}{2} = \frac{sin\,\frac{x}{2}}{cos\,\frac{x}{2}} = \frac{3/\sqrt{10}}{1/\sqrt{10}} = 3$.

16. Prove that $sin^2\,x + sin^2\left(x + \frac{2\pi}{3}\right) + sin^2\left(x - \frac{2\pi}{3}\right) = \frac{3}{2}$.

Solution: $sin^2\,x + sin^2\left(x + \frac{2\pi}{3}\right) + sin^2\left(x - \frac{2\pi}{3}\right)$

$$= \frac{1 - cos\,2x}{2} + \frac{1 - cos\,2\left(x + \frac{2\pi}{3}\right)}{2} + \frac{1 - cos\,2\left(x - \frac{2\pi}{3}\right)}{2}$$

$$= \frac{1 - cos\,2x}{2} + \frac{1 - cos\left(2x + \frac{4\pi}{3}\right)}{2} + \frac{1 - cos\left(2x - \frac{4\pi}{3}\right)}{2}$$

$$= \frac{1 - cos\,2x + 1 - cos\left(2x + \frac{4\pi}{3}\right) + 1 - cos\left(2x - \frac{4\pi}{3}\right)}{2}$$

$$= \frac{3 - \left\{cos\,2x + cos\left(2x + \frac{4\pi}{3}\right) + cos\left(2x - \frac{4\pi}{3}\right)\right\}}{2}$$

$$= \frac{3 - \left\{cos\,2x + 2\,cos\,2x.cos\,\frac{4\pi}{3}\right\}}{2}$$

$$\left\{cos\,\frac{4\pi}{3} = cos\,\frac{3\pi + \pi}{3} = cos\left(\pi + \frac{\pi}{3}\right) = -cos\,\frac{\pi}{3} = -\frac{1}{2}\right\}$$

$$= \frac{3 - \left\{cos\,2x + 2\,cos\,2x.\left(-\frac{1}{2}\right)\right\}}{2} = \frac{3 - \{cos\,2x - cos\,2x\}}{2} = \frac{3}{2}.$$