MATHEMATICS: QUESTION BANK
CHAPTER 7: INTEGRALS(INDEFINITE)

Standard forms

1mark questions:

Write an antiderivative for each of the following functions using differentiation

Question 1: i)sin $2x$

Soln: The anti derivative of sin $2x$ is a function of x whose derivative is sin $2x$.

It is known that,

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

$$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Therefore, the anti derivative of

$\sin 2x$ is $-\frac{1}{2}\cos 2x$

Question 2: Cos $3x$

The anti derivative of cos $3x$ is a function of x whose derivative is cos $3x$.

It is known that,

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x$$

$$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Therefore, the anti derivative of

$\cos 3x$ is $\frac{1}{3}\sin 3x$.

Question 3: e^{2x}

The anti derivative of e^{2x} is the function of x whose derivative is e^{2x}.

It is known that,

$$\frac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$$

$$\Rightarrow e^{2x} = \frac{1}{2}\frac{d}{dx}\left(e^{2x}\right)$$

$$\therefore e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Therefore, the anti derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

Evaluate the following integrals:

Question 4: $\int\left(4e^{3x}+1\right)dx$

$$\int\left(4e^{3x}+1\right)dx$$

$$= 4\int e^{3x}dx + \int 1dx$$

$$= 4\left(\frac{e^{3x}}{3}\right) + x + C$$

$$= \frac{4}{3}e^{3x} + x + C$$

Question 5: $\int x^2\left(1-\frac{1}{x^2}\right)dx$

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

$$= \int\left(x^2-1\right)dx$$

$$= \int x^2dx - \int 1dx$$

$$= \frac{x^3}{3} - x + C$$

Question 6: $\int\left(2x^2+e^x\right)dx$

$$\int\left(2x^2+e^x\right)dx$$

$$= 2\int x^2dx + \int e^xdx$$

$$= 2\left(\frac{x^3}{3}\right) + e^x + C$$

$$= \frac{2}{3}x^3 + e^x + C$$

Question 7: Find an anti-derivative of $\cot^2 x$ with respect to x.

Ans: $\cot^2 x = \operatorname{cosec}^2 x - 1$; antiderivative of $\cot^2 x$ is -cotx-x+c

Question 8: Find an anti-derivative of $\sqrt{1+\sin 2x}$ with respect to x.

$\sqrt{1+\sin 2x}$

$= \sqrt{\sin^2 x + \cos^2 x + 2sinx\,cosx}$

$= \sqrt{(sinx + cosx)^2}$= sinx+cosx

Antiderivative of sinx+cosx is cosx-sinx+c

Question 9: Evaluate $\int\left(\frac{d}{dx}e^{5x}\right)dx$.

Ans: e^{5x}+c

<u>TWO MARK QUESTIONS:</u> Evaluate the following integrals:

Write an antiderivative for each of the following functions using differentiation :

Question 1: $(ax+b)^2$

The anti derivative of $(ax+b)^2$ is the function of x whose derivative is $(ax+b)^2$.

It is known that,

$$\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2$$

$$\Rightarrow (ax+b)^2 = \frac{1}{3a}\frac{d}{dx}(ax+b)^3$$

$$\therefore (ax+b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Therefore, the anti derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$.

Question 2: $\sin 2x - 4e^{3x}$

The anti derivative of $(\sin 2x - 4e^{3x})$ is the function of x whose derivative is $(\sin 2x - 4e^{3x})$.

It is known that,

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$$

Therefore, the anti derivative of $(\sin 2x - 4e^{3x})$ is $\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right)$.

Evaluate the following integrals:

Question 3: $\int (ax^2+bx+c)\,dx$

$$\int (ax^2+bx+c)\,dx$$

$$= a\int x^2 dx + b\int x dx + c\int 1.dx$$

$$= a\left(\frac{x^3}{3}\right) + b\left(\frac{x^2}{2}\right) + cx + C$$

$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$$

Question 4: $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int \left(x + \frac{1}{x} - 2\right) dx$$

$$= \int x dx + \int \frac{1}{x} dx - 2\int 1.dx$$

$$= \frac{x^2}{2} + \log|x| - 2x + C$$

Question 5: $\int \frac{x^3+5x^2-4}{x^2} dx$

$$\int \frac{x^3+5x^2-4}{x^2} dx$$

$$= \int \left(x + 5 - 4x^{-2}\right) dx$$

$$= \int x dx + 5\int 1.dx - 4\int x^{-2} dx$$

$$= \frac{x^2}{2} + 5x - 4\left(\frac{x^{-1}}{-1}\right) + C$$

$$= \frac{x^2}{2} + 5x + \frac{4}{x} + C$$

Question 6: $\int \frac{x^3+3x+4}{\sqrt{x}} dx$

$$\int \frac{x^3+3x+4}{\sqrt{x}} dx$$

$$= \int \left(x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}\right) dx$$

$$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}} + \frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}} + C$$

$$= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C$$

$$= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C$$

Question 7: $\int \frac{x^3-x^2+x-1}{x-1} dx$

$$\int \frac{x^3-x^2+x-1}{x-1} dx$$

On dividing, we obtain

$$= \int (x^2+1) dx$$

$$= \int x^2 dx + \int 1 dx$$

$$= \frac{x^3}{3} + x + C$$

Question 8: $\int (1-x)\sqrt{x}dx$

$$\int (1-x)\sqrt{x}dx = \int\left(\sqrt{x}-x^{\frac{3}{2}}\right)dx$$
$$= \int x^{\frac{1}{2}}dx - \int x^{\frac{3}{2}}dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$
$$= \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C$$

Question 9: $\int \sqrt{x}\left(3x^2+2x+3\right)dx$

$$\int \sqrt{x}\left(3x^2+2x+3\right)dx$$
$$= \int\left(3x^{\frac{5}{2}}+2x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)dx$$
$$= 3\int x^{\frac{5}{2}}dx + 2\int x^{\frac{3}{2}}dx + 3\int x^{\frac{1}{2}}dx$$
$$= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right) + 2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) + 3\frac{\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}} + C$$
$$= \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$$

Question 10: $\int\left(2x-3\cos x+e^x\right)dx$

$$\int\left(2x-3\cos x+e^x\right)dx$$
$$= 2\int xdx - 3\int \cos xdx + \int e^x dx$$
$$= \frac{2x^2}{2} - 3(\sin x) + e^x + C$$
$$= x^2 - 3\sin x + e^x + C$$

Question 11: $\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$$
$$= 2\int x^2dx - 3\int \sin xdx + 5\int x^{\frac{1}{2}}dx$$
$$= \frac{2x^3}{3} - 3(-\cos x) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C$$

Question 12: $\int \sec x\left(\sec x+\tan x\right)dx$

$$\int \sec x\left(\sec x+\tan x\right)dx$$
$$= \int\left(\sec^2 x+\sec x\tan x\right)dx$$
$$= \int \sec^2 xdx + \int \sec x\tan xdx$$
$$= \tan x + \sec x + C$$

Question 13: $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$

$$\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}dx$$
$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}dx$$
$$= \int \frac{\sin^2 x}{\cos^2 x}dx$$
$$= \int \tan^2 xdx$$
$$= \int\left(\sec^2 x-1\right)dx$$
$$= \int \sec^2 xdx - \int 1dx$$
$$= \tan x - x + C$$

Question 14: $\int \frac{2-3\sin x}{\cos^2 x}dx$

$$\int \frac{2-3\sin x}{\cos^2 x}dx$$
$$= \int\left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right)dx$$
$$= \int 2\sec^2 xdx - 3\int \tan x\sec xdx$$
$$= 2\tan x - 3\sec x + C$$

Question 15:Find the anti derivative of $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$

Solution:

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$$
$$= \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

3 mark questions:

If $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$ such that $f(2) = 0$, find $f(x)$

Solution: It is given that $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$

∴Anti derivative of $4x^3-\frac{3}{x^4}=f(x)$

$$\therefore f(x)=\int 4x^3-\frac{3}{x^4}\,dx$$

$$f(x)=4\int x^3dx-3\int\left(x^{-4}\right)dx$$

$$f(x)=4\left(\frac{x^4}{4}\right)-3\left(\frac{x^{-3}}{-3}\right)+C$$

$$\therefore f(x)=x^4+\frac{1}{x^3}+C$$

Also,

$$f(2)=0$$

$$\therefore f(2)=(2)^4+\frac{1}{(2)^3}+C=0$$

$$\Rightarrow 16+\frac{1}{8}+C=0$$

$$\Rightarrow C=-\left(16+\frac{1}{8}\right)$$

$$\Rightarrow C=\frac{-129}{8}$$

$$\therefore f(x)=x^4+\frac{1}{x^3}-\frac{129}{8}$$

INTEGRATION BY SUBSTITUTION:

ONE MARK QUESTIONS:

1`. Evaluate $\int \tan^2(2x).dx$

$$\int \tan^2(2x).dx=\int(\sec^2 2x-1)dx$$

Solution: $=\frac{1}{2}\tan 2x-x+c$

2. Evaluate $\int \operatorname{cosec}^2\left(\frac{x}{2}\right)dx$.

$= -2\cot\frac{x}{2}+c$

TWO MARK QUESTIONS:

Integrate the following w.r.t x

1. $\frac{2x}{1+x^2}$

Hint: $1+x^2=t$ Ans: $\log(1+x^2)+c$

2. $\frac{(\log x)^2}{x}$

Hint: $\log|x|=t$ $\therefore \frac{1}{x}dx=dt$

$$\Rightarrow \int\frac{(\log|x|)^2}{x}dx=\int t^2dt$$

$$=\frac{t^3}{3}+C$$

$$=\frac{(\log|x|)^3}{3}+C$$

Question 3: $\frac{1}{x+x\log x}$

$$\frac{1}{x+x\log x}=\frac{1}{x(1+\log x)}$$

Let $1+\log x=t$

$$\therefore \frac{1}{x}dx=dt$$

$$\Rightarrow \int\frac{1}{x(1+\log x)}dx=\int\frac{1}{t}dt$$

$$=\log|t|+C$$

$$=\log|1+\log x|+C$$

Question 4: $\sin x\cdot\sin(\cos x)$

$\sin x\cdot\sin(\cos x)$

Let $\cos x=t$

$\therefore -\sin x\,dx=dt$

$$\Rightarrow \int\sin x\cdot\sin(\cos x)dx=-\int\sin t\,dt$$

$$=-[-\cos t]+C$$

$$=\cos t+C$$

$$=\cos(\cos x)+C$$

Question 5: $\sqrt{ax+b}$

Let $ax + b = t \Rightarrow adx = dt$

$$\therefore dx = \frac{1}{a}dt \Rightarrow \int(ax+b)^{\frac{1}{2}}dx = \frac{1}{a}\int t^{\frac{1}{2}}dt$$

$$= \frac{1}{a}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + C$$

Question 6: $x\sqrt{1+2x^2}$

Let $1 + 2x^2 = t \therefore 4xdx = dt$

$$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int\frac{\sqrt{t}\,dt}{4}$$

$$= \frac{1}{4}\int t^{\frac{1}{2}}dt$$

$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{6}(1+2x^2)^{\frac{3}{2}} + C$$

Question 7: $(4x+2)\sqrt{x^2+x+1}$

Let $x^2+x+1 = t \therefore (2x+1)dx = dt$

$$\int(4x+2)\sqrt{x^2+x+1}\;dx$$

$$= \int 2\sqrt{t}\,dt$$

$$= 2\int\sqrt{t}\,dt$$

$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{4}{3}(x^2+x+1)^{\frac{3}{2}} + C$$

Question 8: $\dfrac{1}{x-\sqrt{x}}$

$$\frac{1}{x-\sqrt{x}} = \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Let $\left(\sqrt{x}-1\right) = t$

$\therefore \dfrac{1}{2\sqrt{x}}dx = dt$

$$\Rightarrow \int\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx = \int\frac{2}{t}dt$$

$$= 2\log|t| + C$$

$$= 2\log\left|\sqrt{x}-1\right| + C$$

Question 9: e^{2x+3}

Let $2x+3 = t$

$\therefore 2dx = dt$

$$\Rightarrow \int e^{2x+3}dx = \frac{1}{2}\int e^t dt$$

$$= \frac{1}{2}\left(e^t\right) + C$$

$$= \frac{1}{2}e^{(2x+3)} + C$$

Question 10: $\dfrac{x^2}{\left(2+3x^3\right)^3}$

Let $2+3x^3 = t$

$\therefore 9x^2\,dx = dt$

$$\Rightarrow \int\frac{x^2}{\left(2+3x^3\right)^3}dx = \frac{1}{9}\int\frac{dt}{(t)^3}$$

$$= \frac{1}{9}\left[\frac{t^{-2}}{-2}\right] + C$$

$$= \frac{-1}{18}\left(\frac{1}{t^2}\right) + C$$

$$= \frac{-1}{18\left(2+3x^3\right)^2} + C$$

Question 11: $\dfrac{1}{x(\log x)^m}$, $x > 0$

Let $\log x = t$ $\therefore$ $\dfrac{1}{x}dx = dt$

$$\Rightarrow \int\frac{1}{x(\log x)^m}dx = \int\frac{dt}{(t)^m}$$

$$= \left(\frac{t^{-m+1}}{1-m}\right) + C$$

$$= \frac{(\log x)^{1-m}}{(1-m)} + C$$

Question 12: $\dfrac{x}{9-4x^2}$

Let $9-4x^2 = t$

$\therefore -8x\,dx = dt$

$$\Rightarrow \int\frac{x}{9-4x^2}dx = \frac{-1}{8}\int\frac{1}{t}dt$$

$$= \frac{-1}{8}\log|t| + C$$

$$= \frac{-1}{8}\log\left|9-4x^2\right| + C$$

Question 13: $\frac{x}{e^{x^2}}$

Let $x^2 = t$ $\therefore 2xdx = dt$

$\Rightarrow \int \frac{x}{e^{x^2}} dx = \frac{1}{2}\int \frac{1}{e^t} dt$

$= \frac{1}{2}\int e^{-t} dt$

$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C$

$= -\frac{1}{2}e^{-x^2} + C$

$= \frac{-1}{2e^{x^2}} + C$

Question 14: $\frac{x}{e^{x^2}}$

Let $x^2 = t$ $\therefore 2xdx = dt$

$\Rightarrow \int \frac{x}{e^{x^2}} dx = \frac{1}{2}\int \frac{1}{e^t} dt$

$= \frac{1}{2}\int e^{-t} dt$

$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C$

$= -\frac{1}{2}e^{-x^2} + C$

$= \frac{-1}{2e^{x^2}} + C$

Question 15: $\frac{e^{\tan^{-1} x}}{1+x^2}$

Let $\tan^{-1} x = t$ $\therefore \frac{1}{1+x^2} dx = dt$

$\Rightarrow \int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^t dt$

$= e^t + C$

$= e^{\tan^{-1} x} + C$

Question 16: $\frac{\cos\sqrt{x}}{\sqrt{x}}$

Let $\sqrt{x} = t$ $\therefore \frac{1}{2\sqrt{x}} dx = dt$

$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = 2\int \cos t\, dt$

$= 2\sin t + C$

$= 2\sin\sqrt{x} + C$

Question 17: $\sqrt{\sin 2x}\cos 2x$

Let sin $2x = t$ $\therefore$ $2\cos 2x\, dx = dt$

$\Rightarrow \int \sqrt{\sin 2x}\ \cos 2x\, dx = \frac{1}{2}\int \sqrt{t}\, dt$

$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$

$= \frac{1}{3}t^{\frac{3}{2}} + C$

$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$

Question 18: $\frac{\cos x}{\sqrt{1+\sin x}}$

Let $1+\sin x = t$

$\therefore \cos x\, dx = dt$

$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$

$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= 2\sqrt{t} + C$

$= 2\sqrt{1+\sin x} + C$

Question 19: cot x log sin x

Let log sin $x = t$

$\Rightarrow \frac{1}{\sin x}\cdot\cos x\ dx = dt$

$\therefore \cot x\ dx = dt$

$\Rightarrow \int \cot x\ \log\sin x\ dx = \int t\, dt$

$= \frac{t^2}{2} + C$

$= \frac{1}{2}(\log\sin x)^2 + C$

Question 20: $\frac{\sin x}{1+\cos x}$

Let $1 + \cos x = t$ $\therefore -\sin x\, dx = dt$

$\Rightarrow \int \frac{\sin x}{1+\cos x} dx = \int -\frac{dt}{t}$

$= -\log|t| + C$

$= -\log|1+\cos x| + C$

Question 21: $\frac{\sin x}{(1+\cos x)^2}$

Let 1 + cos x = t ∴ −sin x dx = dt

$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2}\,dx = \int -\frac{dt}{t^2}$

$= -\int t^{-2}dt$

$= \frac{1}{t} + C$

$= \frac{1}{1+\cos x} + C$

Question 22: $\frac{(1+\log x)^2}{x}$

Let 1 + log x = t ∴ $\frac{1}{x}dx = dt$

$\Rightarrow \int \frac{(1+\log x)^2}{x}\,dx = \int t^2 dt$

$= \frac{t^3}{3} + C$

$= \frac{(1+\log x)^3}{3} + C$

Question 23: $\int x^2 e^{x^3}\,dx$ equals

Let $I = \int x^2 e^{x^3}\,dx$ Also, let $x^3 = t$

$\Rightarrow 3x^2 dx = dt$

$\Rightarrow I = \frac{1}{3}\int e^t dt$

$= \frac{1}{3}(e^t) + C$

$= \frac{1}{3}e^{x^3} + C$

Question 24: Find $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\,dx$

Let $x^{10} + 10^x = t$ ∴ $(10x^9 + 10^x \log_e 10)dx = dt$

$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}\,dx = \int \frac{dt}{t}$

$= \log t + C$

$= \log(10^x + x^{10}) + C$

Question 25: $\int \frac{dx}{\sin^2 x \cos^2 x} =$

Let $I = \int \frac{dx}{\sin^2 x \cos^2 x}$

$= \int \frac{1}{\sin^2 x \cos^2 x}dx$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}dx$

$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}dx$

$= \int \sec^2 x dx + \int \text{cosec}^2 x dx$

$= \tan x - \cot x + C$

<u>THREE MARKS QUESTIONS</u>

Integrate the following :

Question 1: $\sin(ax+b)\cos(ax+b)$

$\sin(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2} = \frac{\sin 2(ax+b)}{2}$

Let $2(ax+b) = t$ ∴ $2a dx = dt$

$\Rightarrow \int \frac{\sin 2(ax+b)}{2}dx = \frac{1}{2}\int \frac{\sin t\ dt}{2a}$

$= \frac{1}{4a}[-\cos t] + C$

$= \frac{-1}{4a}\cos 2(ax+b) + C$

Question 2: $x\sqrt{x+2}$

Let $(x+2) = t$ ∴ $dx = dt$

$\Rightarrow \int x\sqrt{x+2}dx = \int (t-2)\sqrt{t}dt$

$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)dt$

$= \int t^{\frac{3}{2}}dt - 2\int t^{\frac{1}{2}}dt$

$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$

$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$

$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$

Question 3: $\dfrac{x}{\sqrt{x+4}}, x>0$

Let $x+4=t \quad \therefore dx = dt$

$$\int \frac{x}{\sqrt{x+4}}dx = \int \frac{(t-4)}{\sqrt{t}}dt$$

$$= \int\left(\sqrt{t}-\frac{4}{\sqrt{t}}\right)dt$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t \cdot t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t^{\frac{1}{2}}(t-12) + C$$

$$= \frac{2}{3}(x+4)^{\frac{1}{2}}(x+4-12) + C$$

$$= \frac{2}{3}\sqrt{x+4}(x-8) + C$$

Question 4: $(x^3-1)^{\frac{1}{3}}x^5$

Let $x^3-1=t \quad \therefore 3x^2dx = dt$

$$\Rightarrow \int (x^3-1)^{\frac{1}{3}}x^5dx = \int (x^3-1)^{\frac{1}{3}}x^3 \cdot x^2dx$$

$$= \int t^{\frac{1}{3}}(t+1)\frac{dt}{3}$$

$$= \frac{1}{3}\int\left(t^{\frac{4}{3}} + t^{\frac{1}{3}}\right)dt$$

$$= \frac{1}{3}\left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}} + \frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right] + C$$

$$= \frac{1}{3}\left[\frac{3}{7}t^{\frac{7}{3}} + \frac{3}{4}t^{\frac{4}{3}}\right] + C$$

$$= \frac{1}{7}(x^3-1)^{\frac{7}{3}} + \frac{1}{4}(x^3-1)^{\frac{4}{3}} + C$$

Question 5: $\dfrac{e^{2x}-1}{e^{2x}+1}$

$\dfrac{e^{2x}-1}{e^{2x}+1}$ Dividing numerator and denominator by e^x, we obtain

$$\frac{\frac{(e^{2x}-1)}{e^x}}{\frac{(e^{2x}+1)}{e^x}} = \frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Let $e^x+e^{-x}=t \quad \therefore (e^x-e^{-x})dx = dt$

$$\Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

$$= \int \frac{dt}{t}$$

$$= \log|t| + C$$

$$= \log\left|e^x+e^{-x}\right| + C$$

Question 6: $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

Let $e^{2x}+e^{-2x}=t \quad \therefore \left(2e^{2x}-2e^{-2x}\right)dx = dt$

$$\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx = dt$$

$$\Rightarrow \int\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)dx = \int\frac{dt}{2t}$$

$$= \frac{1}{2}\int\frac{1}{t}dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log\left|e^{2x}+e^{-2x}\right| + C$$

Question 7: $\tan^2(2x-3)$

$\tan^2(2x-3) = \sec^2(2x-3) - 1$

Let $2x-3=t \quad \therefore 2dx = dt$

$$\Rightarrow \int \tan^2(2x-3)dx = \int\left[\left(\sec^2(2x-3)\right)-1\right]dx$$

$$= \frac{1}{2}\int(\sec^2 t)dt - \int 1dx$$

$$= \frac{1}{2}\int \sec^2 t\,dt - \int 1dx$$

$$= \frac{1}{2}\tan t - x + C$$

$$= \frac{1}{2}\tan(2x-3) - x + C$$

Question 8: $\sec^2(7-4x)$

Let $7-4x=t \quad \therefore -4dx = dt$

$$\therefore \int \sec^2(7-4x)dx = \frac{-1}{4}\int \sec^2 t\,dt$$

$$= \frac{-1}{4}(\tan t) + C$$

$$= \frac{-1}{4}\tan(7-4x) + C$$

Question 9: $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

Let $\sin^{-1}x=t \quad \therefore \dfrac{1}{\sqrt{1-x^2}}dx = dt$

$$\Rightarrow \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx = \int t\,dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{(\sin^{-1}x)^2}{2} + C$$

Question 10: $\frac{2\cos x-3\sin x}{6\cos x+4\sin x}$

$$\frac{2\cos x-3\sin x}{6\cos x+4\sin x}=\frac{2\cos x-3\sin x}{2(3\cos x+2\sin x)}$$

Let $3\cos x+2\sin x=t$

$\therefore (-3\sin x+2\cos x)dx=dt$

$$\int\frac{2\cos x-3\sin x}{6\cos x+4\sin x}dx=\int\frac{dt}{2t}$$

$$=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

$$=\frac{1}{2}\log|2\sin x+3\cos x|+C$$

Question 11: $\frac{2\cos x-3\sin x}{6\cos x+4\sin x}$

$$\frac{2\cos x-3\sin x}{6\cos x+4\sin x}=\frac{2\cos x-3\sin x}{2(3\cos x+2\sin x)}$$

Let $3\cos x+2\sin x=t$ $\therefore$ $(-3\sin x+2\cos x)dx=dt$

$$\int\frac{2\cos x-3\sin x}{6\cos x+4\sin x}dx=\int\frac{dt}{2t}$$

$$=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

$$=\frac{1}{2}\log|2\sin x+3\cos x|+C$$

Question 12: $\frac{1}{\cos^2 x(1-\tan x)^2}$

$$\frac{1}{\cos^2 x(1-\tan x)^2}=\frac{\sec^2 x}{(1-\tan x)^2}$$

Let $(1-\tan x)=t$ $\therefore -\sec^2 x dx=dt$

$$\Rightarrow \int\frac{\sec^2 x}{(1-\tan x)^2}dx=\int\frac{-dt}{t^2}$$

$$=-\int t^{-2}dt$$

$$=+\frac{1}{t}+C$$

$$=\frac{1}{(1-\tan x)}+C$$

Question 13: $\frac{1}{1+\cot x}$

Let $I=\int\frac{1}{1+\cot x}dx$

$$=\int\frac{1}{1+\frac{\cos x}{\sin x}}dx$$

$$=\int\frac{\sin x}{\sin x+\cos x}dx$$

$$=\frac{1}{2}\int\frac{2\sin x}{\sin x+\cos x}dx$$

$$=\frac{1}{2}\int\frac{(\sin x+\cos x)+(\sin x-\cos x)}{(\sin x+\cos x)}dx$$

$$=\frac{1}{2}\int 1\,dx+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$$

$$=\frac{1}{2}(x)+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$$

Let $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\, dx = dt$

$$\therefore I=\frac{x}{2}+\frac{1}{2}\int\frac{-(dt)}{t}$$

$$=\frac{x}{2}-\frac{1}{2}\log|t|+C$$

$$=\frac{x}{2}-\frac{1}{2}\log|\sin x+\cos x|+C$$

Question 14: $\frac{1}{1-\tan x}$

Let $I=\int\frac{1}{1-\tan x}dx$

$$=\int\frac{1}{1-\frac{\sin x}{\cos x}}dx$$

$$=\int\frac{\cos x}{\cos x-\sin x}dx$$

$$=\frac{1}{2}\int\frac{2\cos x}{\cos x-\sin x}dx$$

$$=\frac{1}{2}\int\frac{(\cos x-\sin x)+(\cos x+\sin x)}{(\cos x-\sin x)}dx$$

$$=\frac{1}{2}\int 1\,dx+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x-\sin x}dx$$

$$=\frac{x}{2}+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x-\sin x}dx$$

Put $\cos x - \sin x = t \Rightarrow (-\sin x - \cos x)\, dx = dt$

$$\therefore I=\frac{x}{2}+\frac{1}{2}\int\frac{-(dt)}{t}$$

$$=\frac{x}{2}-\frac{1}{2}\log|t|+C$$

$$=\frac{x}{2}-\frac{1}{2}\log|\cos x-\sin x|+C$$

Question 15: $\frac{\sqrt{\tan x}}{\sin x \cos x}$

Let $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

$= \int \frac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx$

$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$

$= \int \frac{\sec^2 x \, dx}{\sqrt{\tan x}}$

Let $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$\therefore I = \int \frac{dt}{\sqrt{t}}$

$= 2\sqrt{t} + C$

$= 2\sqrt{\tan x} + C$

Question 16: $\frac{(x+1)(x+\log x)^2}{x}$

$\frac{(x+1)(x+\log x)^2}{x} = \left(\frac{x+1}{x}\right)(x+\log x)^2 = \left(1+\frac{1}{x}\right)(x+\log$

Let $(x+\log x) = t$

$\therefore \left(1+\frac{1}{x}\right) dx = dt$

$\Rightarrow \int \left(1+\frac{1}{x}\right)(x+\log x)^2 dx = \int t^2 dt$

$= \frac{t^3}{3} + C$

$= \frac{1}{3}(x+\log x)^3 + C$

Question 17: $\frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}$

Let $x^4 = t$

$\therefore 4x^3 \, dx = dt$

$\Rightarrow \int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8} dx = \frac{1}{4} \int \frac{\sin(\tan^{-1} t)}{1+t^2} dt$ ...(1)

Let $\tan^{-1} t = u$

$\therefore \frac{1}{1+t^2} dt = du$

From (1), we obtain

$\int \frac{x^3 \sin(\tan^{-1} x^4) dx}{1+x^8} = \frac{1}{4} \int \sin u \, du$

$= \frac{1}{4}(-\cos u) + C$

$= \frac{-1}{4} \cos(\tan^{-1} t) + C$

$= \frac{-1}{4} \cos(\tan^{-1} x^4) + C$

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INTEGRATION USING TRIGONOMETRIC IDENTITIES:

THREE MARKS QUESTIONS:

Integrate the following functions:

Question 1: $\sin^2(2x+5)$

$\sin^2(2x+5) = \frac{1-\cos 2(2x+5)}{2} = \frac{1-\cos(4x+10)}{2}$

$\Rightarrow \int \sin^2(2x+5) dx = \int \frac{1-\cos(4x+10)}{2} dx$

$= \frac{1}{2} \int 1 \, dx - \frac{1}{2} \int \cos(4x+10) \, dx$

$= \frac{1}{2} x - \frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right) + C$

$= \frac{1}{2} x - \frac{1}{8} \sin(4x+10) + C$

Question 2: $\sin 3x \cos 4x$

It is known that,

$\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

$\therefore \int \sin 3x \cos 4x \, dx = \frac{1}{2} \int \{\sin(3x+4x) + \sin(3x-4x)\} \, dx$

$= \frac{1}{2} \int \{\sin 7x + \sin(-x)\} \, dx$

$= \frac{1}{2} \int \{\sin 7x - \sin x\} \, dx$

$= \frac{1}{2} \int \sin 7x \, dx - \frac{1}{2} \int \sin x \, dx$

$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}(-\cos x) + C$

$= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$

Question 3: $\cos 2x \cos 4x \cos 6x$

It is known that,

$$\cos A\cos B=\frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$$

$$\therefore \int \cos 2x(\cos 4x\cos 6x)dx=\int \cos 2x\left[\frac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\}\right.$$

$$=\frac{1}{2}\int\{\cos 2x\cos 10x+\cos 2x\cos(-2x)\}\,dx$$

$$=\frac{1}{2}\int\{\cos 2x\cos 10x+\cos^2 2x\}\,dx$$

$$=\frac{1}{2}\int\left[\left\{\frac{1}{2}\cos(2x+10x)+\cos(2x-10x)\right\}+\left(\right.\right.$$

$$=\frac{1}{4}\int(\cos 12x+\cos 8x+1+\cos 4x)\,dx$$

$$=\frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+x+\frac{\sin 4x}{4}\right]+C$$

Question 4: $\sin^3(2x+1)$

Let $I=\int \sin^3(2x+1)$

$$\Rightarrow \int \sin^3(2x+1)dx=\int \sin^2(2x+1)\cdot\sin(2x+1)dx$$

$$=\int\left(1-\cos^2(2x+1)\right)\sin(2x+1)dx$$

Let $\cos(2x+1)=t$

$$\Rightarrow -2\sin(2x+1)dx=dt$$

$$\Rightarrow \sin(2x+1)dx=\frac{-dt}{2}$$

$$\Rightarrow I=\frac{-1}{2}\int(1-t^2)dt$$

$$=\frac{-1}{2}\left\{t-\frac{t^3}{3}\right\}$$

$$=\frac{-1}{2}\left\{\cos(2x+1)-\frac{\cos^3(2x+1)}{3}\right\}$$

$$=\frac{-\cos(2x+1)}{2}+\frac{\cos^3(2x+1)}{6}+C$$

Question 5: $\sin^3 x\cos^3 x$

$$\text{Let } I=\int \sin^3 x\cos^3 x\cdot dx$$

$$=\int \cos^3 x\cdot\sin^2 x\cdot\sin x\cdot dx$$

$$=\int \cos^3 x\left(1-\cos^2 x\right)\sin x\cdot dx$$

Let $\cos x=t$

$$\Rightarrow -\sin x\cdot dx=dt$$

$$\Rightarrow I=-\int t^3(1-t^2)dt$$

$$=-\int(t^3-t^5)dt$$

$$=-\left\{\frac{t^4}{4}-\frac{t^6}{6}\right\}+C$$

$$=-\left\{\frac{\cos^4 x}{4}-\frac{\cos^6 x}{6}\right\}+C$$

$$=\frac{\cos^6 x}{6}-\frac{\cos^4 x}{4}+C$$

Question 6: $\sin x\sin 2x\sin 3x$

It is known that $\sin A\sin B=\frac{1}{2}\{\cos(A-B)-\cos(A+B)\}$

$$\therefore \int \sin x\sin 2x\sin 3x\,dx=\int\left[\sin x\cdot\frac{1}{2}\{\cos(2x-3x)-\cos(2x+3x)\}\right]dx$$

$$=\frac{1}{2}\int(\sin x\cos(-x)-\sin x\cos 5x)\,dx$$

$$=\frac{1}{2}\int(\sin x\cos x-\sin x\cos 5x)\,dx$$

$$=\frac{1}{2}\int\frac{\sin 2x}{2}\,dx-\frac{1}{2}\int\sin x\cos 5x\,dx$$

$$=\frac{1}{4}\left[\frac{-\cos 2x}{2}\right]-\frac{1}{2}\int\left\{\frac{1}{2}\sin(x+5x)+\sin(x-5x)\right\}dx$$

$$=\frac{-\cos 2x}{8}-\frac{1}{4}\int(\sin 6x+\sin(-4x))\,dx$$

$$=\frac{-\cos 2x}{8}-\frac{1}{4}\left[\frac{-\cos 6x}{3}+\frac{\cos 4x}{4}\right]+C$$

$$=\frac{-\cos 2x}{8}-\frac{1}{8}\left[\frac{-\cos 6x}{3}+\frac{\cos 4x}{2}\right]+C$$

$$=\frac{1}{8}\left[\frac{\cos 6x}{3}-\frac{\cos 4x}{2}-\cos 2x\right]+C$$

Question 7: $\sin 4x\sin 8x$

It is known that $\sin A\sin B=\frac{1}{2}\cos(A-B)-\cos(A+B)$

$$\therefore \int\sin 4x\sin 8x\,dx=\int\left\{\frac{1}{2}\cos(4x-8x)-\cos(4x+8x)\right\}dx$$

$$=\frac{1}{2}\int(\cos(-4x)-\cos 12x)\,dx$$

$$=\frac{1}{2}\int(\cos 4x-\cos 12x)\,dx$$

$$=\frac{1}{2}\left[\frac{\sin 4x}{4}-\frac{\sin 12x}{12}\right]$$

Question 8: $\dfrac{1-\cos x}{1+\cos x}$

$$\frac{1-\cos x}{1+\cos x}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[2\sin^2\frac{x}{2}=1-\cos x \text{ and } 2\cos^2\frac{x}{2}=1+\cos x\right]$$

$$=\tan^2\frac{x}{2}$$

$$=\left(\sec^2\frac{x}{2}-1\right)$$

$$\therefore \int\frac{1-\cos x}{1+\cos x}dx=\int\left(\sec^2\frac{x}{2}-1\right)dx$$

$$=\left[\frac{\tan\frac{x}{2}}{\frac{1}{2}}-x\right]+C$$

$$=2\tan\frac{x}{2}-x+C$$

Question 9: $\frac{\cos x}{1+\cos x}$

$$\frac{\cos x}{1+\cos x}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2} \text{ and } \cos x = 2\cos\right.$$

$$=\frac{1}{2}\left[1-\tan^2\frac{x}{2}\right]$$

$$\therefore \int\frac{\cos x}{1+\cos x}dx=\frac{1}{2}\int\left(1-\tan^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\int\left(1-\sec^2\frac{x}{2}+1\right)dx$$

$$=\frac{1}{2}\int\left(2-\sec^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\left[2x-\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]+C$$

$$=x-\tan\frac{x}{2}+C$$

Question 10: $\sin^4 x$

$$\sin^4 x=\sin^2 x\sin^2 x$$

$$=\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$$

$$=\frac{1}{4}(1-\cos 2x)^2$$

$$=\frac{1}{4}\left[1+\cos^2 2x-2\cos 2x\right]$$

$$=\frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right)-2\cos 2x\right]$$

$$=\frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right]$$

$$=\frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right]$$

$$\therefore \int\sin^4 x\,dx=\frac{1}{4}\int\left[\frac{3}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right]dx$$

$$=\frac{1}{4}\left[\frac{3}{2}x+\frac{1}{2}\left(\frac{\sin 4x}{4}\right)-\frac{2\sin 2x}{2}\right]+C$$

$$=\frac{1}{8}\left[3x+\frac{\sin 4x}{4}-2\sin 2x\right]+C$$

$$=\frac{3x}{8}-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

Question 11: $\cos^4 2x$

$$\cos^4 2x=\left(\cos^2 2x\right)^2$$

$$=\left(\frac{1+\cos 4x}{2}\right)^2$$

$$=\frac{1}{4}\left[1+\cos^2 4x+2\cos 4x\right]$$

$$=\frac{1}{4}\left[1+\left(\frac{1+\cos 8x}{2}\right)+2\cos 4x\right]$$

$$=\frac{1}{4}\left[1+\frac{1}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]$$

$$=\frac{1}{4}\left[\frac{3}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]$$

$$\therefore \int\cos^4 2x\,dx=\int\left(\frac{3}{8}+\frac{\cos 8x}{8}+\frac{\cos 4x}{2}\right)dx$$

$$=\frac{3}{8}x+\frac{\sin 8x}{64}+\frac{\sin 4x}{8}+C$$

Question 12: $\frac{\sin^2 x}{1+\cos x}$

$$\frac{\sin^2 x}{1+\cos x}=\frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}} \qquad \left[\sin x=2\sin\frac{x}{2}\cos\frac{x}{2};\ \cos x=2\cos^2\frac{x}{2}-1\right]$$

$$=\frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$=2\sin^2\frac{x}{2}$$

$$=1-\cos x$$

$$\therefore \int\frac{\sin^2 x}{1+\cos x}dx=\int(1-\cos x)dx$$

$$=x-\sin x+C$$

Question 13: $\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}=\frac{-2\sin\frac{2x+2\alpha}{2}\sin\frac{2x-2\alpha}{2}}{-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}} \qquad \left[\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$=\frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=\frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$=2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right]$$

$$=2\left[\cos(x)+\cos\alpha\right]$$

$$=2\cos x+2\cos\alpha$$

$$\therefore \int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx=\int 2\cos x+2\cos\alpha$$

$$=2\left[\sin x+x\cos\alpha\right]+C$$

Question 14: $\frac{\cos x-\sin x}{1+\sin 2x}$

$$\frac{\cos x-\sin x}{1+\sin 2x}=\frac{\cos x-\sin x}{\left(\sin^2 x+\cos^2 x\right)+2\sin x\cos x}$$

$$\left[\sin^2 x+\cos^2 x=1;\ \sin 2x=2\sin x\cos x\right]$$

$$=\frac{\cos x-\sin x}{(\sin x+\cos x)^2}$$

Let $\sin x+\cos x=t$

$\therefore (\cos x-\sin x)\,dx=dt$

$$\Rightarrow \int\frac{\cos x-\sin x}{1+\sin 2x}dx=\int\frac{\cos x-\sin x}{(\sin x+\cos x)^2}dx$$

$$=\int\frac{dt}{t^2}$$

$$=\int t^{-2}dt$$

$$=-t^{-1}+C$$

$$=-\frac{1}{t}+C$$

$$=\frac{-1}{\sin x+\cos x}+C$$

Question 15: $\tan^3 2x \sec 2x$

$$\tan^3 2x\sec 2x = \tan^2 2x\tan 2x\sec 2x$$
$$= \left(\sec^2 2x - 1\right)\tan 2x\sec 2x$$
$$= \sec^2 2x\cdot\tan 2x\sec 2x - \tan 2x\sec 2x$$
$$\therefore \int\tan^3 2x\sec 2x\, dx = \int\sec^2 2x\tan 2x\sec 2x\, dx - \int\tan 2x\sec 2x\, dx$$
$$= \int\sec^2 2x\tan 2x\sec 2x\, dx - \frac{\sec 2x}{2} + C$$

Let $\sec 2x = t$

$\therefore 2\sec 2x\tan 2x\, dx = dt$

$$\therefore \int\tan^3 2x\sec 2x\, dx = \frac{1}{2}\int t^2 dt - \frac{\sec 2x}{2} + C$$
$$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$$
$$= \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C$$

Question 16: $\tan^4 x$

$\tan^4 x$

$= \tan^2 x\cdot\tan^2 x$

$= \left(\sec^2 x - 1\right)\tan^2 x$

$= \sec^2 x\tan^2 x - \tan^2 x$

$= \sec^2 x\tan^2 x - \left(\sec^2 x - 1\right)$

$= \sec^2 x\tan^2 x - \sec^2 x + 1$

$$\therefore \int\tan^4 x\, dx = \int\sec^2 x\tan^2 x\, dx - \int\sec^2 x\, dx + \int 1\cdot dx$$
$$= \int\sec^2 x\tan^2 x\, dx - \tan x + x + C \qquad ...(1)$$

Consider $\int\sec^2 x\tan^2 x\, dx$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \int\sec^2 x\tan^2 x dx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we obtain

$$\int\tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

Question 17: $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x\cos^2 x}$

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x\cos^2 x} = \frac{\sin^3 x}{\sin^2 x\cos^2 x} + \frac{\cos^3 x}{\sin^2 x\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$$
$$= \tan x\sec x + \cot x\,\mathrm{cosec}\, x$$
$$\therefore \int\frac{\sin^3 x + \cos^3 x}{\sin^2 x\cos^2 x}\, dx = \int\left(\tan x\sec x + \cot x\,\mathrm{cosec}\, x\right)dx$$
$$= \sec x - \mathrm{cosec}\, x + C$$

Question 18: $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$
$$= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \qquad \left[\cos 2x = 1 - 2\sin^2 x\right]$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x$$
$$\therefore \int\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\, dx = \int\sec^2 x\, dx = \tan x + C$$

Question 19: $\dfrac{1}{\sin x\cos^3 x}$

$$\frac{1}{\sin x\cos^3 x} = \frac{\sin^2 x + \cos^2 x}{\sin x\cos^3 x}$$
$$= \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x\cos x}$$
$$= \tan x\sec^2 x + \frac{\frac{1\cos^2 x}{\sin x\cos x}}{\cos^2 x}$$
$$= \tan x\sec^2 x + \frac{\sec^2 x}{\tan x}$$
$$\therefore \int\frac{1}{\sin x\cos^3 x}dx = \int\tan x\sec^2 x\, dx + \int\frac{\sec^2 x}{\tan x}\, dx$$

Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \int\frac{1}{\sin x\cos^3 x}dx = \int t dt + \int\frac{1}{t}dt$$
$$= \frac{t^2}{2} + \log|t| + C$$
$$= \frac{1}{2}\tan^2 x + \log|\tan x| + C$$

Question 20: $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$

$$\frac{\cos 2x}{(\cos x + \sin x)^2} = \frac{\cos 2x}{\cos^2 x + \sin^2 x + 2\sin x\cos x} = \frac{\cos 2x}{1 + \sin 2x}$$
$$\therefore \int\frac{\cos 2x}{(\cos x + \sin x)^2}\, dx = \int\frac{\cos 2x}{(1 + \sin 2x)}\, dx$$

Let $1 + \sin 2x = t$

$\Rightarrow 2\cos 2x\, dx = dt$

$$\therefore \int\frac{\cos 2x}{(\cos x + \sin x)^2}dx = \frac{1}{2}\int\frac{1}{t}dt$$
$$= \frac{1}{2}\log|t| + C$$
$$= \frac{1}{2}\log|1 + \sin 2x| + C$$
$$= \frac{1}{2}\log\left|(\sin x + \cos x)^2\right| + C$$
$$= \log|\sin x + \cos x| + C$$

Question 21: $\sin^{-1}(\cos x)$

$\sin^{-1}(\cos x)$

Let $\cos x = t$

Then, $\sin x = \sqrt{1-t^2}$

$\Rightarrow (-\sin x)dx = dt$

$dx = \frac{-dt}{\sin x}$

$dx = \frac{-dt}{\sqrt{1-t^2}}$

$$\therefore \int \sin^{-1}(\cos x)dx = \int \sin^{-1} t \left(\frac{-dt}{\sqrt{1-t^2}}\right)$$
$$= -\int \frac{\sin^{-1} t}{\sqrt{1-t^2}} dt$$

Let $\sin^{-1} t = u$

$\Rightarrow \frac{1}{\sqrt{1-t^2}} dt = du$

$$\therefore \int \sin^{-1}(\cos x)dx = \int 4du$$
$$= -\frac{u^2}{2} + C$$
$$= \frac{-(\sin^1 t)^2}{2} + C$$
$$= \frac{-[\sin^{-1}(\cos x)]^2}{2} + C \qquad \dots(1)$$

It is known that,

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)$

Substituting in equation (1), we obtain

$$\int \sin^{-1}(\cos x)\, dx = \frac{-\left[\frac{\pi}{2} - x\right]^2}{2} + C$$
$$= -\frac{1}{2}\left(\frac{\pi^2}{2} + x^2 - \pi x\right) + C$$
$$= -\frac{\pi^2}{8} - \frac{x^2}{2} + \frac{1}{2}\pi x + C$$
$$= \frac{\pi x}{2} - \frac{x^2}{2} + \left(C - \frac{\pi^2}{8}\right)$$
$$= \frac{\pi x}{2} - \frac{x^2}{2} + C_1$$

Question 22: $\frac{1}{\cos(x-a)\cos(x-b)}$

$$\frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$
$$= \frac{1}{\sin(a-b)}\left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\right]$$
$$= \frac{1}{\sin(a-b)} \frac{[\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)]}{\cos(x-a)\cos(x-b)}$$
$$= \frac{1}{\sin(a-b)}[\tan(x-b) - \tan(x-a)]$$

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}dx = \frac{1}{\sin(a-b)}\int[\tan(x-b) - \tan(x-a)]dx$$
$$= \frac{1}{\sin(a-b)}\left[-\log|\cos(x-b)| + \log|\cos(x-a)|\right]$$
$$= \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right] + C$$

Question 23: $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx$

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int\left(\frac{\sin^2 x}{\sin^2 x\cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x}\right) dx$$
$$= \int(\sec^2 x - \operatorname{cosec}^2 x)\, dx$$
$$= \tan x + \cot x + C$$

Question 24: $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$

$\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$ Let $e^x x = t$

$\Rightarrow (e^x \cdot x + e^x \cdot 1)dx = dt$

$e^x(x+1)dx = dt$

$$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)}\, dx = \int \frac{dt}{\cos^2 t}$$
$$= \int \sec^2 t\ dt$$
$$= \tan t + C$$
$$= \tan(e^x \cdot x) + C$$

INTEGRALS OF SOME PARTICULAR FUNCTIONS

TWO MARK QUESTIONS:

Integrate the following w.r.t x

Question 1: $\frac{3x^2}{x^6+1}$

Let $x^3 = t \quad \therefore 3x^2\,dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$
$$= \tan^{1} t + C$$
$$= \tan^{-1}\left(x^3\right) + C$$

Question 2: $\frac{1}{\sqrt{1+4x^2}}$

Let $2x = t \quad \therefore 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$
$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right] + C$$
$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + C$$

Question 3: $\frac{1}{\sqrt{(2-x)^2+1}}$

Let $2 - x = t \Rightarrow -dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$
$$= -\log\left|t+\sqrt{t^2+1}\right| + C \qquad \left[\int \frac{}{\sqrt{x}}\right.$$
$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right| + C$$
$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right| + C$$

Question 4: $\frac{1}{\sqrt{9-25x^2}}$

Let $5x = t \quad \therefore 5dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}\,dx = \frac{1}{5}\int \frac{1}{9-t^2}\,dt$$
$$= \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}\,dt$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$
$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

Question 5: $\frac{3x}{1+2x^4}$

Let $\sqrt{2}x^2 = t$

$\therefore\ 2\sqrt{2}x\ dx = dt$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$
$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$$
$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

Question 6: $\frac{x^2}{1-x^6}$

Let $x^3 = t \quad \therefore 3x^2\,dx = dt$

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

Question 7: $\frac{x-1}{\sqrt{x^2-1}}$

$$\int \frac{x-1}{\sqrt{x^2-1}}\,dx = \int \frac{x}{\sqrt{x^2-1}}\,dx - \int \frac{1}{\sqrt{x^2-1}}\,dx \qquad \ldots(1)$$

For $\int \frac{x}{\sqrt{x^2-1}}dx$, let $x^2-1=t \Rightarrow 2x\ dx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$
$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$
$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$
$$= \sqrt{t}$$
$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad \left[\int \frac{1}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$
$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

Question 8: $\frac{x^2}{\sqrt{x^6+a^6}}$

Let $x^3 = t \quad \Rightarrow 3x^2\,dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$
$$= \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$
$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

Question 9: $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

Let $\tan x = t$ $\therefore \sec^2 x\, dx = dt$

$$\Rightarrow \int\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int\frac{dt}{\sqrt{t^2+2^2}}$$
$$= \log\left|t+\sqrt{t^2+4}\right|+C$$
$$= \log\left|\tan x+\sqrt{\tan^2 x+4}\right|+C$$

Question 10: $\frac{1}{\sqrt{x^2+2x+2}}$

$$\int\frac{1}{\sqrt{x^2+2x+2}}dx = \int\frac{1}{\sqrt{(x+1)^2+(1)^2}}dx$$

Let $x+1=t$

$\therefore dx = dt$

$$\Rightarrow \int\frac{1}{\sqrt{x^2+2x+2}}dx = \int\frac{1}{\sqrt{t^2+1}}dt$$
$$= \log\left|t+\sqrt{t^2+1}\right|+C$$
$$= \log\left|(x+1)+\sqrt{(x+1)^2+1}\right|+C$$
$$= \log\left|(x+1)+\sqrt{x^2+2x+2}\right|+C$$

Question 11: $\frac{1}{9x^2+6x+5}$

$$\int\frac{1}{9x^2+6x+5}dx = \int\frac{1}{(3x+1)^2+(2)^2}dx$$

Let $(3x+1)=t$

$\therefore 3dx = dt$

$$\Rightarrow \int\frac{1}{(3x+1)^2+(2)^2}dx = \frac{1}{3}\int\frac{1}{t^2+2^2}dt$$
$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$
$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

Question 12: $\frac{1}{\sqrt{7-6x-x^2}}$

$7-6x-x^2$ can be written as $7-\left(x^2+6x+9-9\right)$.

Therefore,

$$7-\left(x^2+6x+9-9\right)$$
$$=16-\left(x^2+6x+9\right)$$
$$=16-(x+3)^2$$
$$=(4)^2-(x+3)^2$$
$$\therefore \int\frac{1}{\sqrt{7-6x-x^2}}dx = \int\frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

Let $x+3=t$

$\Rightarrow dx = dt$

$$\Rightarrow \int\frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int\frac{1}{\sqrt{(4)^2-(t)^2}}dt$$
$$= \sin^{-1}\left(\frac{t}{4}\right)+C$$
$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

Question 13: $\int\frac{dx}{x^2+2x+2}$

$$\int\frac{dx}{x^2+2x+2} = \int\frac{dx}{\left(x^2+2x+1\right)+1}$$
$$= \int\frac{1}{(x+1)^2+(1)^2}dx$$
$$= \left[\tan^{-1}(x+1)\right]+C$$

Question 14: $\int\frac{dx}{\sqrt{9x-4x^2}}$

$$\int\frac{dx}{\sqrt{9x-4x^2}}$$
$$= \int\frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$
$$= \int\frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}dx$$
$$= \int\frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$
$$= \frac{1}{2}\int\frac{1}{\sqrt{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}}dx$$
$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \qquad \left(\int\frac{dy}{\sqrt{a^2-y^2}}=\sin^{-1}\frac{y}{a}+C\right)$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

THREE MAKRS QUESTIONS:

Question 1: $\frac{1}{\sqrt{(x-1)(x-2)}}$

$(x-1)(x-2)$ can be written as x^2-3x+2.

Therefore,

x^2-3x+2

$=x^2-3x+\frac{9}{4}-\frac{9}{4}+2$

$=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$

$=\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$

$\therefore \int\frac{1}{\sqrt{(x-1)(x-2)}}dx=\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$\Rightarrow\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$

$=\log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+C$

$=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$

Question 2: $\frac{1}{\sqrt{8+3x-x^2}}$

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$.

Therefore,

$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$

$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$

$\Rightarrow\int\frac{1}{\sqrt{8+3x-x^2}}dx=\int\frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$

Let $x-\frac{3}{2}=t$

$\therefore\ dx=dt$

$\Rightarrow\int\frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int\frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}}dt$

$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$

$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$

$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$

Question 3: $\frac{1}{\sqrt{(x-a)(x-b)}}$

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$.

Therefore,

$x^2-(a+b)x+ab$

$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$

$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$

$\Rightarrow\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx$

Let $x-\left(\frac{a+b}{2}\right)=t$

$\therefore\ dx=dt$

$\Rightarrow\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a-b}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$

$=\log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$

$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$

Question 4: $\frac{4x+1}{\sqrt{2x^2+x-3}}$

Let $4x+1=A\frac{d}{dx}(2x^2+x-3)+B$

$\Rightarrow 4x+1=A(4x+1)+B$

$\Rightarrow 4x+1=4Ax+A+B$

Equating the coefficients of x and constant term on both sides, we obtain

$4A=4\Rightarrow A=1$

$A+B=1\Rightarrow B=0$

Let $2x^2+x-3=t$

$\therefore (4x+1)\,dx=dt$

$\Rightarrow\ \int\frac{4x+1}{\sqrt{2x^2+x-3}}dx=\int\frac{1}{\sqrt{t}}dt$

$=2\sqrt{t}+C$

$=2\sqrt{2x^2+x-3}+C$

Question 5: $\frac{x+2}{\sqrt{x^2-1}}$

Let $x+2=A\frac{d}{dx}(x^2-1)+B \qquad ...(1)$

$\Rightarrow x+2=A(2x)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$2A=1\Rightarrow A=\frac{1}{2}$

$B=2$

From (1), we obtain

$$(x+2)=\frac{1}{2}(2x)+2$$

$$\text{Then, }\int\frac{x+2}{\sqrt{x^2-1}}dx=\int\frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$$

$$=\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx+\int\frac{2}{\sqrt{x^2-1}}dx \qquad ...(2)$$

In $\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx$, let $x^2-1=t \Rightarrow 2xdx=dt$

$$\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx=\frac{1}{2}\int\frac{dt}{\sqrt{t}}$$

$$=\frac{1}{2}\left[2\sqrt{t}\right]$$

$$=\sqrt{t}$$

$$=\sqrt{x^2-1}$$

$$\text{Then, }\int\frac{2}{\sqrt{x^2-1}}dx=2\int\frac{1}{\sqrt{x^2-1}}dx=2\log\left|x+\sqrt{x^2-1}\right|$$

From equation (2), we obtain

$$\int\frac{x+2}{\sqrt{x^2-1}}dx=\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+C$$

Question 6: Integrate $\frac{x+2}{\sqrt{x^2+2x+3}}$ with respect to x.

$$I=\int\frac{x+2}{\sqrt{x^2+2x+3}}dx=\int\frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx$$

$$=\frac{1}{2}\int\frac{2x+2}{\sqrt{x^2+2x+3}}dx+\int\frac{1}{\sqrt{x^2+2x+3}}dx$$

$$=\frac{1}{2}\cdot 2\sqrt{x^2+2x+3}+\int\frac{1}{\sqrt{(x+1)^2+2}}dx$$

$$=\sqrt{x^2+2x+3}+\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+c$$

ADDITIONAL 4 TO 5 MARK QUESTIONS: Integrate the following:

Question 1: $\frac{5x-2}{1+2x+3x^2}$

Let $5x-2=A\frac{d}{dx}\left(1+2x+3x^2\right)+B$

$\Rightarrow 5x-2=A(2+6x)+B$

Equating the coefficient of *x* and constant term on both sides, we obtain

$5=6A\Rightarrow A=\frac{5}{6}$

$2A+B=-2\Rightarrow B=-\frac{11}{3}$

$\therefore 5x-2=\frac{5}{6}(2+6x)+\left(-\frac{11}{3}\right)$

$$\Rightarrow \int\frac{5x-2}{1+2x+3x^2}dx=\int\frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int\frac{2+6x}{1+2x+3x^2}dx-\frac{11}{3}\int\frac{1}{1+2x+3x^2}dx$$

Let $I_1=\int\frac{2+6x}{1+2x+3x^2}dx$ and $I_2=\int\frac{1}{1+2x+3x^2}dx$

$$\therefore \int\frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}I_1-\frac{11}{3}I_2 \qquad ...(1)$$

$$I_1=\int\frac{2+6x}{1+2x+3x^2}dx$$

Let $1+2x+3x^2=t$

$\Rightarrow (2+6x)dx=dt$

$\therefore I_1=\int\frac{dt}{t}$

$I_1=\log|t|$

$$I_1=\log\left|1+2x+3x^2\right| \qquad ...(2)$$

$$I_2=\int\frac{1}{1+2x+3x^2}dx$$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$.

Therefore,

$$1+3\left(x^2+\frac{2}{3}x\right)$$

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2 = \frac{1}{3}\int \frac{1}{\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$= \frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$= \frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C$$

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

Question 2: $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

Let $6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right)+B$

$\Rightarrow 6x+7 = A(2x-9)+B$

Equating the coefficients of x and constant term, we obtain $2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3(2x - 9) + 34$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$

Let $I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$\therefore \int \frac{6x+7}{\sqrt{x^2-9x+20}} = 3I_1 + 34I_2$

Then,

$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$

Let $x^2-9x+20 = t$

$\Rightarrow (2x-9)dx = dt$

$\Rightarrow I_1 = \frac{dt}{\sqrt{t}}$

$I_1 = 2\sqrt{t}$

$I_1 = 2\sqrt{x^2-9x+20}$

and $I_2 = \int \frac{1}{\sqrt{x^2-9x+20}}dx$

$x^2-9x+20$ can be written as $x^2-9x+20+\frac{81}{4}-\frac{81}{4}$.

Therefore,

$x^2-9x+20+\frac{81}{4}-\frac{81}{4}$

$= \left(x-\frac{9}{2}\right)^2 - \frac{1}{4}$

$= \left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}dx$$

$$I_2 = \log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| \qquad ...(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}}dx = 3\left[2\sqrt{x^2-9x+20}\right] + 34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right] + C$$

$$= 6\sqrt{x^2-9x+20} + 34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right] + C$$

Question 3: $\frac{x+2}{\sqrt{4x-x^2}}$

Let $x+2 = A\frac{d}{dx}\left(4x-x^2\right)+B$

$\Rightarrow x+2 = A(4-2x)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$-2A = 1 \Rightarrow A = -\frac{1}{2}$

$4A + B = 2 \Rightarrow B = 4$

$\Rightarrow (x+2) = -\frac{1}{2}(4-2x)+4$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

Let $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$ and $I_2 \int \frac{1}{\sqrt{4x-x^2}}dx$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$

Let $4x-x^2 = t$

$\Rightarrow (4-2x)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \quad ...(2)$$

$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$

$\Rightarrow 4x-x^2 = -(-4x+x^2)$

$= (-4x+x^2+4-4)$

$= 4-(x-2)^2$

$= (2)^2-(x-2)^2$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2-(x-2)^2}}dx = \sin^{-1}\left(\frac{x-2}{2}\right) \quad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right)+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

$$= -\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

Question 4: $\frac{x+2}{\sqrt{x^2+2x+3}}$

$$\int \frac{(x+2)}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \quad ...(1$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \quad ...(2)$$

$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$

$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2+\left(\sqrt{2}\right)^2$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right|$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right]+\log\left|(x+1)+\sqrt{x^2+2x+\right.$$

$$= \sqrt{x^2+2x+3}+\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$$

Question 5: $\frac{x+3}{x^2-2x-5}$

Let $(x+3) = A\frac{d}{dx}\left(x^2-2x-5\right)+B$

$(x+3) = A(2x-2)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$2A = 1 \Rightarrow A = \frac{1}{2}$

$-2A+B = 3 \Rightarrow B = 4$

$\therefore (x+3) = \frac{1}{2}(2x-2)+4$

$$\Rightarrow \int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

Let $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$ and $I_2 = \int \frac{1}{x^2-2x-5}dx$

$$\therefore \int \frac{x+3}{(x^2-2x-5)}dx = \frac{1}{2}I_1 + 4I_2 \quad ...(1)$$

Then, $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$

Let $x^2-2x-5 = t$

$\Rightarrow (2x-2)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log\left|x^2-2x-5\right| \quad ...(2)$$

$I_2 = \int \frac{1}{x^2-2x-5}dx$

$= \int \frac{1}{(x^2-2x+1)-6}dx$

$= \int \frac{1}{(x-1)^2+\left(\sqrt{6}\right)^2}dx$

$$= \frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \quad ...(3)$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right|+\frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|+C$$

$$= \frac{1}{2}\log\left|x^2-2x-5\right|+\frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|+C$$

Question 6: $\frac{5x+3}{\sqrt{x^2+4x+10}}$

Let $5x+3 = A\frac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow 5x+3 = A(2x+4)+B$

Equating the coefficients of x and constant term, we obtain

$2A = 5 \Rightarrow A = \frac{5}{2}$

$4A + B = 3 \Rightarrow B = -7$

$\therefore 5x + 3 = \frac{5}{2}(2x+4) - 7$

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$

$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2$

Then, $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2 + 4x + 10 = t$

$\therefore (2x+4)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \quad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+6}}dx$$

$$= \int \frac{1}{(x+2)^2 + (\sqrt{6})^2}dx$$

$$= \log\left|(x+2)\sqrt{x^2+4x+10}\right| \quad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\left[2\sqrt{x^2+4x+10}\right] - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

$$= 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

INTEGRATION BY PARTIAL FRACTIONS

TWO MARK QUESTIONS:

Question 1: $\frac{x}{(x+1)(x+2)}$

Let $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$\Rightarrow x = A(x+2) + B(x+1)$

Equating the coefficients of *x* and constant term, we obtain

$A + B = 1$; $2A + B = 0$

On solving, we obtain $A = -1$ and $B = 2$

$\therefore \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$

$$\Rightarrow \int \frac{x}{(x+1)(x+2)}dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)}dx$$

$$= -\log|x+1| + 2\log|x+2| + C$$

$$= \log(x+2)^2 - \log|x+1| + C$$

$$= \log\frac{(x+2)^2}{(x+1)} + C$$

Question 2: $\frac{1}{x^2-9}$

Let $\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$

$1 = A(x-3) + B(x+3)$

Equating the coefficients of *x* and constant term, we obtain

$A + B = 0$; $-3A + 3B = 1$

On solving, we obtain

$A = -\frac{1}{6}$ and $B = \frac{1}{6}$

$\therefore \frac{1}{(x+3)(x-3)} = \frac{-1}{6(x+3)} + \frac{1}{6(x-3)}$

$$\Rightarrow \int \frac{1}{(x^2-9)}dx = \int \left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)}\right)dx$$

$$= -\frac{1}{6}\log|x+3| + \frac{1}{6}\log|x-3| + C$$

$$= \frac{1}{6}\log\left|\frac{(x-3)}{(x+3)}\right| + C$$

<u>THREE MARK QUESTIONS</u>:

Question 1: $\frac{3x-1}{(x-1)(x-2)(x-3)}$

Let

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$3x - 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad ...(1)$$

Substituting *x* = 1, 2, and 3 respectively in equation (1), we obtain

$A = 1$, $B = -5$, and $C = 4$

$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}$

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)}dx = \int \left\{\frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)}\right\}dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

Question 2: $\frac{x}{(x-1)(x-2)(x-3)}$

Let

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad ...(1)$$

Substituting *x* = 1, 2, and 3 respectively in equation (1), we obtain

$A = \frac{1}{2}$, $B = -2$, and $C = \frac{3}{2}$

$\therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$

$\Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)} dx = \int \left\{ \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)} \right\} dx$

$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-$

Question 3: $\frac{2x}{x^2+3x+2}$

Let $\frac{2x}{x^2+3x+2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$2x = A(x+2) + B(x+1) \quad ...(1)$

Substituting $x = -1$ and -2 in equation (1), we obtain $A = -2$ and $B = 4$

$\therefore \frac{2x}{(x+1)(x+2)} = \frac{-2}{(x+1)} + \frac{4}{(x+2)}$

$\Rightarrow \int \frac{2x}{(x+1)(x+2)} dx = \int \left\{ \frac{4}{(x+2)} - \frac{2}{(x+1)} \right\} dx$

$= 4\log|x+2| - 2\log|x+1| + C$

Question 4: $\frac{1-x^2}{x(1-2x)}$

It can be seen that the given integrand is not a proper fraction. Therefore, on dividing $(1 - x^2)$ by $x(1 - 2x)$, we obtain

$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)$

Let $\frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{(1-2x)}$

$\Rightarrow (2-x) = A(1-2x) + Bx \quad ...(1)$

Substituting $x = 0$ and $\frac{1}{2}$ in equation (1), we obtain $A = 2$ and $B = 3$

$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{1-2x}$

Substituting in equation (1), we obtain

$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2}\left\{\frac{2}{x} + \frac{3}{(1-2x)}\right\}$

$\Rightarrow \int \frac{1-x^2}{x(1-2x)} dx = \int \left\{ \frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{1-2x}\right) \right\} dx$

$= \frac{x}{2} + \log|x| + \frac{3}{2(-2)}\log|1-2$

$= \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C$

Question 4: $\frac{x}{(x^2+1)(x-1)}$

Let $\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)}$

$x = (Ax+B)(x-1) + C(x^2+1)$

$x = Ax^2 - Ax + Bx - B + Cx^2 + C$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + C = 0 \quad ; -A + B = 1 \quad ; \ -B + C = 0$

On solving these equations, we obtain

$A = -\frac{1}{2}, B = \frac{1}{2}$, and $C = \frac{1}{2}$

From equation (1), we obtain

$\therefore \frac{x}{(x^2+1)(x-1)} = \frac{\left(-\frac{1}{2}x + \frac{1}{2}\right)}{x^2+1} + \frac{\frac{1}{2}}{(x-1)}$

$\Rightarrow \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$

$= -\frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$

Consider $\int \frac{2x}{x^2+1}dx$, let $(x^2+1) = t \Rightarrow 2x\,dx = dt$

$\Rightarrow \int \frac{2x}{x^2+1}dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1|$

$\therefore \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$

$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$

Question 5: $\frac{x}{(x-1)^2(x+2)}$

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

Substituting $x = 1$, we obtain $B = \frac{1}{3}$

Equating the coefficients of x^2 and constant term, we obtain

$A + C = 0 \quad ; -2A + 2B + C = 0$

On solving, we obtain $A = \frac{2}{9}$ and $C = \frac{-2}{9}$

$\therefore \frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$

$\Rightarrow \int \frac{x}{(x-1)^2(x+2)} dx = \frac{2}{9}\int \frac{1}{(x-1)}dx + \frac{1}{3}\int \frac{1}{(x-1)^2}dx - \frac{2}{9}\int \frac{1}{(x+2)}dx$

$= \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{9}\log|x+2| + C$

$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$

Question 6: $\frac{3x+5}{x^3-x^2-x+1}$

$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x-1)^2(x+1)}$

Let $\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$

$3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

$3x+5 = A(x^2-1) + B(x+1) + C(x^2+1-2x) \qquad ...(1)$

Substituting $x = 1$ in equation (1), we obtain
$B = 4$

Equating the coefficients of x^2 and x, we obtain $\quad A + C = 0; \quad B - 2C = 3$

On solving, we obtain $A = -\frac{1}{2}$ and $C = \frac{1}{2}$

$\therefore \frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$

$$\Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)}dx = -\frac{1}{2}\int\frac{1}{x-1}dx + 4\int\frac{1}{(x-1)^2}dx + \frac{1}{2}\int\frac{1}{(x+1)}dx$$

$$= -\frac{1}{2}\log|x-1| + 4\left(\frac{-1}{x-1}\right) + \frac{1}{2}\log|x+1| + C$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{(x-1)} + C$$

Question 7: $\frac{2x-3}{(x^2-1)(2x+3)}$

$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x+1)(x-1)(2x+3)}$

Let

$\frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}$

$\Rightarrow (2x-3) = A(x-1)(2x+3) + B(x+1)(2x+3) + C(x+1)(x-$

$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$

$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$

Equating the coefficients of x^2 and x, we obtain

$B = -\frac{1}{10}$, $A = \frac{5}{2}$, and $C = -\frac{24}{5}$

$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$

$$\Rightarrow \int\frac{2x-3}{(x^2-1)(2x+3)}dx = \frac{5}{2}\int\frac{1}{(x+1)}dx - \frac{1}{10}\int\frac{1}{x-1}dx - \frac{24}{5}\int\frac{}{(2x}$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2$$

$$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+$$

Question 8: $\frac{5x}{(x+1)(x^2-4)}$

$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$

Let

$\frac{5x}{(x+1)(x+2)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$

$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2)$

Substituting $x = -1, -2$, and 2 respectively in equation (1), we obtain

$A = \frac{5}{3}$, $B = -\frac{5}{2}$, and $C = \frac{5}{6}$

$\therefore \frac{5x}{(x+1)(x+2)(x-2)} = \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)}$

$$\Rightarrow \int\frac{5x}{(x+1)(x^2-4)}dx = \frac{5}{3}\int\frac{1}{(x+1)}dx - \frac{5}{2}\int\frac{1}{(x+2)}dx + \frac{5}{6}\int\frac{1}{(x-2)}dx$$

$$= \frac{5}{3}\log|x+1| - \frac{5}{2}\log|x+2| + \frac{5}{6}\log|x-2| + C$$

Question 9: $\frac{x^3+x+1}{x^2-1}$

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(x^3 + x + 1)$ by $x^2 - 1$, we obtain $\frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1}$

Let $\frac{2x+1}{x^2-1} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$

$2x+1 = A(x-1) + B(x+1) \qquad ...(1)$

Substituting $x = 1$ and -1 in equation (1), we obtain $A = \frac{1}{2}$ and $B = \frac{3}{2}$

$\therefore \frac{x^3+x+1}{x^2-1} = x + \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$

$$\Rightarrow \int\frac{x^3+x+1}{x^2-1}dx = \int x\,dx + \frac{1}{2}\int\frac{1}{(x+1)}dx + \frac{3}{2}\int\frac{1}{(x-1)}dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$$

Question 10: $\frac{3x-1}{(x+2)^2}$

Let $\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$

$\Rightarrow 3x-1 = A(x+2) + B$

Equating the coefficient of x and constant term, we obtain $\quad A = 3$

$2A + B = -1 \Rightarrow B = -7$

$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$

$$\Rightarrow \int\frac{3x-1}{(x+2)^2}dx = 3\int\frac{1}{(x+2)}dx - 7\int\frac{x}{(x+2)^2}dx$$

$$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$$

$$= 3\log|x+2| + \frac{7}{(x+2)} + C$$

Question 11: $\frac{1}{x\left(x^n+1\right)}$

[Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

$\frac{1}{x\left(x^n+1\right)}$ Multiplying numerator and denominator by x^{n-1}, we obtain

$$\frac{1}{x\left(x^n+1\right)}=\frac{x^{n-1}}{x^{n-1}x\left(x^n+1\right)}=\frac{x^{n-1}}{x^n\left(x^n+1\right)}$$

Let $x^n = t \Rightarrow x^{n-1}dx = dt$

$$\therefore \int\frac{1}{x\left(x^n+1\right)}dx=\int\frac{x^{n-1}}{x^n\left(x^n+1\right)}dx=\frac{1}{n}\int\frac{1}{t\left(t+1\right)}dt$$

Let $\frac{1}{t\left(t+1\right)}=\frac{A}{t}+\frac{B}{\left(t+1\right)}$

$$1=A\left(1+t\right)+Bt \qquad ...(1)$$

Substituting $t = 0, -1$ in equation (1), we obtain

$A = 1$ and $B = -1$

$$\therefore \frac{1}{t\left(t+1\right)}=\frac{1}{t}-\frac{1}{\left(1+t\right)}$$

$$\Rightarrow \int\frac{1}{x\left(x^n+1\right)}dx=\frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{\left(t+1\right)}\right\}dx$$

$$=\frac{1}{n}\left[\log\left|t\right|-\log\left|t+1\right|\right]+C$$

$$=-\frac{1}{n}\left[\log\left|x^n\right|-\log\left|x^n+1\right|\right]+C$$

$$=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$$

Question 12: $\frac{\cos x}{\left(1-\sin x\right)\left(2-\sin x\right)}$ [Hint: Put $\sin x = t$]

$\frac{\cos x}{\left(1-\sin x\right)\left(2-\sin x\right)}$

Let $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\therefore \int\frac{\cos x}{\left(1-\sin x\right)\left(2-\sin x\right)}dx=\int\frac{dt}{\left(1-t\right)\left(2-t\right)}$$

Let $\frac{1}{\left(1-t\right)\left(2-t\right)}=\frac{A}{\left(1-t\right)}+\frac{B}{\left(2-t\right)}$

$$1=A\left(2-t\right)+B\left(1-t\right) \qquad ...(1)$$

Substituting $t = 2$ and then $t = 1$ in equation (1), we obtain $A = 1$ and $B = -1$

$$\therefore \frac{1}{\left(1-t\right)\left(2-t\right)}=\frac{1}{\left(1-t\right)}-\frac{1}{\left(2-t\right)}$$

$$\Rightarrow \int\frac{\cos x}{\left(1-\sin x\right)\left(2-\sin x\right)}dx=\int\left\{\frac{1}{1-t}-\frac{1}{\left(2-t\right)}\right\}dt$$

$$=-\log\left|1-t\right|+\log\left|2-t\right|+C$$

$$=\log\left|\frac{2-t}{1-t}\right|+C$$

$$=\log\left|\frac{2-\sin x}{1-\sin x}\right|+C$$

Question 13: $\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}$

$\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}$ Let $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \int\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx=\int\frac{dt}{\left(t+1\right)\left(t+3\right)} \qquad ...(1)$$

Let $\frac{1}{\left(t+1\right)\left(t+3\right)}=\frac{A}{\left(t+1\right)}+\frac{B}{\left(t+3\right)}$

$$1=A\left(t+3\right)+B\left(t+1\right) \qquad ...(1)$$

Substituting $t = -3$ and $t = -1$ in equation (1), we obtain $A=\frac{1}{2}$ and $B=-\frac{1}{2}$

$$\therefore \frac{1}{\left(t+1\right)\left(t+3\right)}=\frac{1}{2\left(t+1\right)}-\frac{1}{2\left(t+3\right)}$$

$$\Rightarrow \int\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx=\int\left\{\frac{1}{2\left(t+1\right)}-\frac{1}{2\left(t+3\right)}\right\}dt$$

$$=\frac{1}{2}\log\left|\left(t+1\right)\right|-\frac{1}{2}\log\left|t+3\right|+C$$

$$=\frac{1}{2}\log\left|\frac{t+1}{t+3}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$$

Question 14: $\frac{1}{x\left(x^4-1\right)}$

$\frac{1}{x\left(x^4-1\right)}$

Multiplying numerator and denominator by x^3, we obtain

$$\frac{1}{x\left(x^4-1\right)}=\frac{x^3}{x^4\left(x^4-1\right)}$$

$$\therefore \int\frac{1}{x\left(x^4-1\right)}dx=\int\frac{x^3}{x^4\left(x^4-1\right)}dx$$

Let $x^4 = t \Rightarrow 4x^3dx = dt$

$$\therefore \int\frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int\frac{dt}{t\left(t-1\right)}$$

Let $\frac{1}{t\left(t-1\right)}=\frac{A}{t}+\frac{B}{\left(t-1\right)}$

$$1=A\left(t-1\right)+Bt \qquad ...(1)$$

Substituting $t = 0$ and 1 in (1), we obtain $A = -1$ and $B = 1$

$$\Rightarrow \frac{1}{t\left(t+1\right)}=\frac{-1}{t}+\frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{x(x^4-1)}dx = \frac{1}{4}\int\left\{\frac{-1}{t}+\frac{1}{t-1}\right\}dt$$
$$= \frac{1}{4}\left[-\log|t| + \log|t-1|\right] + C$$
$$= \frac{1}{4}\log\left|\frac{t-1}{t}\right| + C$$
$$= \frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + C$$

Question 15: $\frac{1}{(e^x-1)}$

[Hint: Put $e^x = t$]

$\frac{1}{(e^x-1)}$ Let $e^x = t \Rightarrow e^x\,dx = dt$

$$\Rightarrow \int \frac{1}{e^x-1}dx = \int \frac{1}{t-1}\times\frac{dt}{t} = \int \frac{1}{t(t-1)}dt$$

Let $\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$

$1 = A(t-1) + Bt$ ...(1)

Substituting $t = 1$ and $t = 0$ in equation (1), we obtain $A = -1$ and $B = 1$

$$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$
$$\Rightarrow \int \frac{1}{t(t-1)}dt = \log\left|\frac{t-1}{t}\right| + C$$
$$= \log\left|\frac{e^x-1}{e^x}\right| + C$$

Question 16:

$\int \frac{xdx}{(x-1)(x-2)}$ equals

Let $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$x = A(x-2) + B(x-1)$ ...(1)

Substituting $x = 1$ and 2 in (1), we obtain

$A = -1$ and $B = 2$

$$\therefore \frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$$
$$\Rightarrow \int \frac{x}{(x-1)(x-2)}dx = \int\left\{\frac{-1}{(x-1)} + \frac{2}{(x-2)}\right\}dx$$
$$= -\log|x-1| + 2\log|x-2| + C$$
$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Question 17: $\int \frac{dx}{x(x^2+1)}$ equals

Let $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$1 = A(x^2+1) + (Bx+C)x$

Equating the coefficients of x^2, x, and constant term, we obtain

$A + B = 0$; $C = 0$; $A = 1$

On solving these equations, we obtain

$A = 1$, $B = -1$, and $C = 0$

$$\therefore \frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$$
$$\Rightarrow \int \frac{1}{x(x^2+1)}dx = \int\left\{\frac{1}{x} - \frac{x}{x^2+1}\right\}dx$$
$$= \log|x| - \frac{1}{2}\log\left|x^2+1\right| + C$$

ADDITIONAL QUESTIONS: 4 TO 5 MARKS:

Question 1: $\frac{2}{(1-x)(1+x^2)}$

Let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$

$2 = A(1+x^2) + (Bx+C)(1-x)$

$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$

Equating the coefficient of x^2, x, and constant term, we obtain

$A - B = 0$; $B - C = 0$; $A + C = 2$

On solving these equations, we obtain

$A = 1$, $B = 1$, and $C = 1$

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$
$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)}dx = \int \frac{1}{1-x}dx + \int \frac{x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$
$$= -\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$
$$= -\log|x-1| + \frac{1}{2}\log\left|1+x^2\right| + \tan^{-1}x + C$$

Question 2: $\frac{1}{x^4-1}$

$$\frac{1}{(x^4-1)}=\frac{1}{(x^2-1)(x^2+1)}=\frac{1}{(x+1)(x-1)(1+x^2)}$$

Let $\frac{1}{(x+1)(x-1)(1+x^2)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{Cx+D}{(x^2+1)}$

$$1=A(x-1)(x^2+1)+B(x+1)(x^2+1)+(Cx+D)(x^2-1)$$

$$1=A(x^3+x-x^2-1)+B(x^3+x+x^2+1)+Cx^3+Dx^2-Cx-D$$

$$1=(A+B+C)x^3+(-A+B+D)x^2+(A+B-C)x+(-A+B-$$

Equating the coefficient of x^3, x^2, x, and constant term, we obtain

$A+B+C=0$

$-A+B+D=0$

$A+B-C=0$

$-A+B-D=1$

On solving these equations, we obtain

$A=-\frac{1}{4},B=\frac{1}{4},C=0,$ and $D=-\frac{1}{2}$

$$\therefore \frac{1}{x^4-1}=\frac{-1}{4(x+1)}+\frac{1}{4(x-1)}-\frac{1}{2(x^2+1)}$$

$$\Rightarrow \int\frac{1}{x^4-1}dx=-\frac{1}{4}\log|x-1|+\frac{1}{4}\log|x-1|-\frac{1}{2}\tan^{-1}x+C$$

$$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$$

Question 18: $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=1-\frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

Let $\frac{4x^2+10}{(x^2+3)(x^2+4)}=\frac{Ax+B}{(x^2+3)}+\frac{Cx+D}{(x^2+4)}$

$$4x^2+10=(Ax+B)(x^2+4)+(Cx+D)(x^2+3)$$

$$4x^2+10=Ax^3+4Ax+Bx^2+4B+Cx^3+3Cx+Dx^2+3D$$

$$4x^2+10=(A+C)x^3+(B+D)x^2+(4A+3C)x+(4B+3D)$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

$A+C=0$; $B+D=4$;$4A+3C=0$

$4B+3D=10$

On solving these equations, we obtain

$A=0$, $B=-2$, $C=0$, and $D=6$

$$\therefore \frac{4x^2+10}{(x^2+3)(x^2+4)}=\frac{-2}{(x^2+3)}+\frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=1-\left(\frac{-2}{(x^2+3)}+\frac{6}{(x^2+4)}\right)$$

$$\Rightarrow \int\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx=\int\left\{1+\frac{2}{(x^2+3)}-\frac{6}{(x^2+4)}\right\}dx$$

$$=\int\left\{1+\frac{2}{x^2+(\sqrt{3})^2}-\frac{6}{x^2+2^2}\right\}$$

$$=x+2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right)-6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)+C$$

$$=x+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}+C$$

INTEGRATION BY PARTS

TWO MARKS QUESTIONS:

Question 1: $x\sin x$

Let $I=\int x\sin x\,dx$

Taking x as first function and $\sin x$ as second function and integrating by parts, we obtain

$$I=x\int\sin x\,dx-\int\left\{\left(\frac{d}{dx}x\right)\int\sin x\,dx\right\}dx$$

$$=x(-\cos x)-\int 1\cdot(-\cos x)dx$$

$$=-x\cos x+\sin x+C$$

Question 2: $x\sin 3x$

Let $I=\int x\sin 3x\,dx$

Taking x as first function and $\sin 3x$ as second function and integrating by parts, we obtain

$$I=x\int\sin 3x\,dx-\int\left\{\left(\frac{d}{dx}x\right)\int\sin 3x\,dx\right\}$$

$$=x\left(\frac{-\cos 3x}{3}\right)-\int 1\cdot\left(\frac{-\cos 3x}{3}\right)dx$$

$$=\frac{-x\cos 3x}{3}+\frac{1}{3}\int\cos 3x\,dx$$

$$=\frac{-x\cos 3x}{3}+\frac{1}{9}\sin 3x+C$$

Question 3: $\int logx\ dx$.

Given Integral

$=\log x\cdot\int 1\,dx-\int 1.\frac{d}{dx}\log x\,dx$.

$==x\log x-x+c$

Question 4: $x \log x$

Let $I = \int x \log x dx$

Taking $\log x$ as first function and x as second function and integrating by parts, we obtain

$$I = \log x \int x \; dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x \; dx \right\} dx$$

$$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$$

$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + \text{C}$$

Question 5: $x \log 2x$

Let $I = \int x \log 2x dx$

Taking $\log 2x$ as first function and x as second function and integrating by parts, we obtain

$$I = \log 2x \int x \; dx - \int \left\{ \left(\frac{d}{dx} 2 \log x \right) \int x \; dx \right\} dx$$

$$= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$$

$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + \text{C}$$

Question 6: $x^2 \log x$

Let $I = \int x^2 \log x \; dx$

Taking $\log x$ as first function and x^2 as second function and integrating by parts, we obtain

$$I = \log x \int x^2 \; dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 dx \right\} dx$$

$$= \log x \left(\frac{x^3}{3} \right) - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$$

$$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$$

$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + \text{C}$$

Question 7: $x \sec^2 x$

Let $I = \int x \sec^2 x dx$

Taking x as first function and $\sec^2 x$ as second function and integrating by parts, we obtain

$$I = x \int \sec^2 x \, dx - \int \left\{ \left\{ \frac{d}{dx} x \right\} \int \sec^2 x \, dx \right\} dx$$

$$= x \tan x - \int 1 \cdot \tan x dx$$

$$= x \tan x + \log |\cos x| + \text{C}$$

<u>THREE MARKS QUESTIONS:</u>

Integrate the following w.r.t. x

Question 1: $x^2 e^x$

Let $I = \int x^2 e^x dx$

Taking x^2 as first function and e^x as second function and integrating by parts, we obtain

$$I = x^2 \int e^x \; dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx$$

$$= x^2 e^x - \int 2x \cdot e^x dx$$

$$= x^2 e^x - 2 \int x \cdot e^x dx$$

Again integrating by parts, we obtain

$$= x^2 e^x - 2 \left[x \cdot \int e^x dx - \int \left\{ \left(\frac{d}{dx} x \right) \cdot \int e^x dx \right\} dx \right]$$

$$= x^2 e^x - 2 \left[x e^x - \int e^x dx \right]$$

$$= x^2 e^x - 2 \left[x e^x - e^x \right]$$

$$= x^2 e^x - 2x e^x + 2e^x + \text{C}$$

$$= e^x \left(x^2 - 2x + 2 \right) + \text{C}$$

Question 2: $x \sin^{-1} x$

Let $I = \int x \sin^{-1} x \; dx$

Taking $\sin^{-1} x$ as first function and x as second function and integrating by parts, we obtain

$$I = \sin^{-1} x \int x \; dx - \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int x \; dx \right\} dx$$

$$= \sin^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \int \sqrt{1-x^2} dx - \int \frac{1}{\sqrt{1-x^2}} dx \right\}$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right\} + \text{C}$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \text{C}$$

$$= \frac{1}{4} \left(2x^2 - 1 \right) \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + \text{C}$$

Question 3: $x\tan^{-1}x$

Let $I = \int x\tan^{-1}x\ dx$

Taking $\tan^{-1}x$ as first function and x as second function and integrating by parts, we obtain

$$I = \tan^{-1}x\int x\,dx - \int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int x\,dx\right\}dx$$

$$= \tan^{-1}x\left(\frac{x^2}{2}\right) - \int\frac{1}{1+x^2}\cdot\frac{x^2}{2}dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\left(x - \tan^{-1}x\right) + C$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + C$$

Question 4: Evaluate: $\int \tan^{-1}x\,dx.$

$$\int \tan^{-1}x\,dx = tan^{-1}x\,.\int 1\,dx - \int\frac{d}{dx}tan^{-1}x.\int 1\,dx\,.dx.$$

$$= x\tan^{-1}x - \int\frac{x}{1+x^2}dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}$$

$$= x\tan^{-1}x - \frac{1}{2}\log|1+x^2| + c$$

Question 5: $x\cos^{-1}x$

Let $I = \int x\cos^{-1}x\,dx$

Taking $\cos^{-1}x$ as first function and x as second function and integrating by parts, we obtain

$$I = \cos^{-1}x\int x\,dx - \int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int x\,dx\right\}dx$$

$$= \cos^{-1}x\frac{x^2}{2} - \int\frac{-1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}dx$$

$$= \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}\int\left\{\sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}dx$$

$$= \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}\int\sqrt{1-x^2}\,dx - \frac{1}{2}\int\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}I_1 - \frac{1}{2}\cos^{-1}x \qquad \ldots(1)$$

where, $I_1 = \int\sqrt{1-x^2}\,dx$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int\frac{d}{dx}\sqrt{1-x^2}\int x\,dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int\frac{-2x}{2\sqrt{1-x^2}}.x\,dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{\int\sqrt{1-x^2}\,dx + \int\frac{-dx}{\sqrt{1-x^2}}\right\}$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{I_1 + \cos^{-1}x\right\}$$

$$\Rightarrow 2I_1 = x\sqrt{1-x^2} - \cos^{-1}x$$

$$\therefore I_1 = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1}x$$

Substituting in (1), we obtain

$$I = \frac{x\cos^{-1}x}{2} - \frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1}x\right) - \frac{1}{2}\cos^{-1}x$$

$$= \frac{\left(2x^2-1\right)}{4}\cos^{-1}x - \frac{x}{4}\sqrt{1-x^2} + C$$

Question 6: $\left(\sin^{-1}x\right)^2$

Let $I = \int\left(\sin^{-1}x\right)^2\cdot 1\,dx$

Taking $\left(\sin^{-1}x\right)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I=\left(\sin^{-1}x\right)\int 1dx-\int\left\{\frac{d}{dx}\left(\sin^{-1}x\right)^2\cdot\int 1\cdot dx\right\}dx$$
$$=\left(\sin^{-1}x\right)^2\cdot x-\int\frac{2\sin^{-1}x}{\sqrt{1-x^2}}\cdot x\,dx$$
$$=x\left(\sin^{-1}x\right)^2+\int\sin^{-1}x\cdot\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx$$
$$=x\left(\sin^{-1}x\right)^2+\left[\sin^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$
$$=x\left(\sin^{-1}x\right)^2+\left[\sin^{-1}x\cdot 2\sqrt{1-x^2}-\int\frac{1}{\sqrt{1-x^2}}\cdot 2\sqrt{1-x^2}\,dx\right]$$
$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-\int 2dx$$
$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$$

Question 7: $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Let $I=\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

$$I=\frac{-1}{2}\int\frac{-2x}{\sqrt{1-x^2}}\cdot\cos^{-1}x\,dx$$

Taking $\cos^{-1}x$ as first function and $\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ as second function and integrating by parts, we obtain

$$I=\frac{-1}{2}\left[\cos^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$
$$=\frac{-1}{2}\left[\cos^{-1}x\cdot 2\sqrt{1-x^2}-\int\frac{-1}{\sqrt{1-x^2}}\cdot 2\sqrt{1-x^2}\,dx\right]$$
$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+\int 2\,dx\right]$$
$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+2x\right]+C$$
$$=-\left[\sqrt{1-x^2}\cos^{-1}x+x\right]+C$$

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INTEGRAL OF THE FORM $\int e^x[f(x)+f'(x)]dx$

TWO MARK QUESTIONS

Question 1: $e^x(\sin x+\cos x)$

Let $I=\int e^x(\sin x+\cos x)dx$

Let $f(x)=\sin x$

$\Rightarrow f'(x)=\cos x$

$\therefore I=\int e^x\{f(x)+f'(x)\}dx$

It is known that,

$$\int e^x\{f(x)+f'(x)\}dx=e^xf(x)+C$$
$$\therefore I=e^x\sin x+C$$

Question 2: $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

Let $I=\int e^x\left[\frac{1}{x}-\frac{1}{x^2}\right]dx$

Also, let $\frac{1}{x}=f(x)$ $\Rightarrow$ $f'(x)=\frac{-1}{x^2}$

It is known that,

$$\int e^x\{f(x)+f'(x)\}dx=e^xf(x)+C$$
$$\therefore I=\frac{e^x}{x}+C$$

2. Evaluate: $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$.

$$I=\int e^x\left(\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx$$
$$=\int e^x\left(\frac{1}{2\cos^2\frac{x}{2}}+\tan\frac{x}{2}\right)dx$$
$$=\int e^x\left(\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)dx=\int e^x\left(f'(x)+f(x)\right)dx$$
$$=e^x\tan\frac{x}{2}$$

THREE MARK QUESTIONS

Integrate the following w.r.t x

Question 1: $\dfrac{xe^x}{(1+x)^2}$

Let $I=\int\frac{xe^x}{(1+x)^2}dx=\int e^x\left\{\frac{x}{(1+x)^2}\right\}dx$

$= \int e^x \left\{ \frac{1+x-1}{(1+x)^2} \right\} dx$

$= \int e^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx$

Let $f(x) = \frac{1}{1+x}$ $\Rightarrow$ $f'(x) = \frac{-1}{(1+x)^2}$

$\Rightarrow \int \frac{xe^x}{(1+x)^2} dx = \int e^x \{ f(x) + f'(x) \} dx$

It is known that,

$\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$

$\therefore \int \frac{xe^x}{(1+x)^2} dx = \frac{e^x}{1+x} + C$

Question 2: $e^x \left(\frac{1+\sin x}{1+\cos x} \right)$

$e^x \left(\frac{1+\sin x}{1+\cos x} \right)$

$= e^x \left(\frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}} \right)$

$= \frac{e^x \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2}{2\cos^2 \frac{x}{2}}$

$= \frac{1}{2} e^x \cdot \left(\frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2}} \right)^2$

$= \frac{1}{2} e^x \left[\tan \frac{x}{2} + 1 \right]^2$

$= \frac{1}{2} e^2 \left(1 + \tan \frac{x}{2} \right)^2$

$= \frac{1}{2} e^x \left[1 + \tan^2 \frac{x}{2} + 2\tan \frac{x}{2} \right]$

$= \frac{1}{2} e^x \left[\sec^2 \frac{x}{2} + 2\tan \frac{x}{2} \right]$

$\frac{e^x (1+\sin x) dx}{(1+\cos x)} = e^x \left[\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right]$

Let $\tan \frac{x}{2} = f(x)$ $\Rightarrow$ $f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$

It is known that,

$\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$

From equation (1), we obtain

$\int \frac{e^x (1+\sin x)}{(1+\cos x)} dx = e^x \tan \frac{x}{2} + C$

Question 3: $\frac{(x-3)e^x}{(x-1)^3}$

$\int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx = \int e^x \left\{ \frac{x-1-2}{(x-1)^3} \right\} dx$

$= \int e^x \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} dx$

Let $f(x) = \frac{1}{(x-1)^2}$ $\Rightarrow$ $f'(x) = \frac{-2}{(x-1)^3}$

It is known that,

$\int e^x \{ f(x) + f'(x) \} dx = e^x f(x) + C$

$\therefore \int e^x \left\{ \frac{(x-3)}{(x-1)^2} \right\} dx = \frac{e^x}{(x-1)^2} + C$

Question 3: $e^{2x} \sin x$

Let $I = \int e^{2x} \sin x \, dx \qquad \ldots(1)$

Integrating by parts, we obtain

$I = \sin x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \sin x \right) \int e^{2x} dx \right\} dx$

$\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} dx$

$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$

Again integrating by parts, we obtain

$I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\cos x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \cos x \right) \int e^{2x} dx \right\} dx \right]$

$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} dx \right]$

$\Rightarrow I = \frac{e^{2x} \cdot \sin x}{2} - \frac{1}{2} \left[\frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x dx \right]$

$\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I$ [From (1)]

$\Rightarrow I + \frac{1}{4} I = \frac{e^{2x} \cdot \sin x}{2} - \frac{e^{2x} \cos x}{4}$

$\Rightarrow \frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$

$\Rightarrow I = \frac{4}{5} \left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + C$

$\Rightarrow I = \frac{e^{2x}}{5} [2\sin x - \cos x] + C$

Question 4: $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$

Let $x = \tan\theta$ $\Rightarrow$ $dx = \sec^2 \theta \, d\theta$

$\therefore \sin^{-1} \left(\frac{2x}{1+x^2} \right) = \sin^{-1} \left(\frac{2\tan\theta}{1+\tan^2\theta} \right) = \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$\Rightarrow \int \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx = \int 2\theta \cdot \sec^2 \theta \, d\theta$

$= 2 \int \theta \cdot \sec^2\theta \, d\theta$

Integrating by parts, we obtain

$$2\left[\theta\cdot\int\sec^2\theta d\theta-\int\left\{\left(\frac{d}{d\theta}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]$$

$$=2\left[\theta\cdot\tan\theta-\int\tan\theta d\theta\right]$$

$$=2\left[\theta\tan\theta+\log\left|\cos\theta\right|\right]+C$$

$$=2\left[x\tan^{-1}x+\log\left|\frac{1}{\sqrt{1+x^2}}\right|\right]+C$$

$$=2x\tan^{-1}x+2\log\left(1+x^2\right)^{-\frac{1}{2}}+C$$

$$=2x\tan^{-1}x+2\left[-\frac{1}{2}\log\left(1+x^2\right)\right]+C$$

$$=2x\tan^{-1}x-\log\left(1+x^2\right)+C$$

Question 5:

$\int e^x\sec x\left(1+\tan x\right)dx$ equals

$$\int e^x\sec x\left(1+\tan x\right)dx$$

Let

$I=\int e^x\sec x\left(1+\tan x\right)dx=\int e^x\left(\sec x+\sec x\tan x\right)d$

Also, let $\sec x=f(x)\Rightarrow\sec x\tan x=f'(x)$

It is known that,

$$\int e^x\left\{f(x)+f'(x)\right\}dx=e^xf(x)+C$$

$$\therefore I=e^x\sec x+C$$

INDEFINITE INTEGRALS (FIVE MARK QUESTIONS)

1. Find the integral of $\frac{1}{\sqrt{a^2-x^2}}$ with respect to x and hence evaluate $\int\frac{1}{\sqrt{7-6x-x^2}}\,dx$.

Solution: Let $x=a\sin\theta$ then

$dx=a\cos\theta\ d\theta$

$\therefore\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{a\cos\theta\ d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$

$\int 1\ d\theta=\theta+c=\sin^{-1}\left(\frac{x}{a}\right)+c$

Consider $I=\int\frac{1}{\sqrt{7-6x-x^2}}\ dx$

$$=\int\frac{1}{\sqrt{16-(x+3)^2}}\ dx$$

$$=\sin^{-1}\left(\frac{x+3}{4}\right)+c$$

2. Find the integral of $\frac{1}{\sqrt{x^2+a^2}}$ with respect to x and hence evaluate $\int\frac{1}{\sqrt{x^2+2x+2}}\,dx$.

Solution: Let $I=\frac{1}{\sqrt{x^2+a^2}}\ dx$

Let $x=a\tan\theta$ then

$dx=a\sec^2\theta\ d\theta$

$\therefore I=\int\frac{a\sec^2\theta\ d\theta}{\sqrt{a^2\tan^2\theta+a^2}}=\int\sec\theta\,d\theta$

$=\log|\sec\theta+\tan\theta|+c_1$

$$=\log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right|+c_1$$

$=\log\left|x+\sqrt{x^2+\ a^2}\right|-\log|a|+c_1$

$=\log\left|x+\sqrt{x^2+\ a^2}\right|+c$,

$c=c_1-\log|a|$

Consider $\int\frac{1}{\sqrt{x^2+2x+2}}\ dx$

$$=\int\frac{1}{\sqrt{(x+1)^2+1}}\ dx$$

$=\log\left|(x+1)+\sqrt{(x+1)^2+1}\right|+c.$

3. Find the integral of $\frac{1}{x^2-a^2}$ with respect to x and evaluate $\int\frac{x}{x^4-16}\,dx$.

Solution: Let $I=\int\frac{1}{x^2-a^2}\ dx$

Consider $\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}$

$$=\frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right]$$

$$=\frac{1}{2a}\left[\frac{1}{(x-a)}\right]-\frac{1}{2a}\left[\frac{1}{(x+a)}\right]$$

$\therefore I=\int\frac{1}{2a}\left[\frac{1}{(x-a)}\right]dx-\int\frac{1}{2a}\left[\frac{1}{(x-a)}\right]\ dx$

$$=\frac{1}{2a}[\log|x-a|-\log|x+a|]+c$$

$$=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$$

Consider $\int\frac{x}{x^4-16}\ dx=\int\frac{x}{(x^2)^2-16}\ dx$ Let $x^2=t$ then $x\,dx\ =\frac{dt}{2}$

$\therefore I=\frac{1}{2}\int\frac{dt}{(t)^2-4^2}=\frac{1}{2}\times\frac{1}{2\times4}\log\left|\frac{t-4}{t+4}\right|+c$.

4. Find the integral of $\frac{1}{a^2-x^2}$ with respect to x and hence evaluate $\int \frac{1}{16-(2x+3)^2}\,dx$.

Solution: Let $I = \int \frac{1}{a^2-x^2}\,dx$

Consider $\frac{1}{a^2-x} = \frac{1}{(a-x)(a+x)}$

$$= \frac{1}{2a}\left[\frac{(a+x)+(a-x)}{(a-x)(a+x)}\right]$$

$$= \frac{1}{2a}\left[\frac{1}{(a-x)}\right] + \frac{1}{2a}\left[\frac{1}{(a+x)}\right]$$

$\therefore I = \int \frac{1}{2a}\left[\frac{1}{(a-x)}\right]dx + \int \frac{1}{2a}\left[\frac{1}{(a+x)}\right]dx$

$$= \frac{1}{2a}\left[-\log|a-x| + \log|a+x|\right] + c$$

$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

Consider $I = \int \frac{1}{16-(2x+3)^2}\,dx$

Let $(2x+3) = t$ then $dx = \frac{dt}{2}$

$\therefore I = \frac{1}{2}\int \frac{dt}{4^2-t^2} = \frac{1}{2}\times\frac{1}{2\times4}\log\left|\frac{4+t}{4-t}\right| + c$.

$= \frac{1}{16}\log\left|\frac{4+(2x+3)}{4-(2x+3)}\right| + c$

5.Find the integral of $\frac{1}{\sqrt{x^2-a^2}}$ with respect to x and hence evaluate $\int \frac{1}{\sqrt{5x^2-2x}}dx$

Substituting $x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$\therefore \int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{a\sec\theta.\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}d\theta$

$= \int \frac{a\sec\theta.\tan\theta}{\sqrt{a^2(\sec^2\theta-1)}}d\theta = \int \frac{a\sec\theta.\tan\theta}{\sqrt{a^2\tan^2\theta}}d\theta$

$= \int \sec\theta\, d\theta$

$=\log|\sec\theta+\tan\theta| + C_1$

$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2}-1}\right| + C_1$

$= \log\left|x+\sqrt{x^2-a^2}\right| - \log|a| + C_1$

$= \log\left|x+\sqrt{x^2-a^2}\right| + C$

where $C = C_1 - \log|a|$

Now $\int \frac{1}{\sqrt{x^2+2x+2}}dx$

$= x^2+2x+1 = (x^2+2x+1)+1$

$= (x+1)^2+(1)^2$

$\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{(x+1)^2+1}}dx$

$= \log\left|x+1+\sqrt{x^2+2x+2}\right| + c$

6.Find the integral of $\frac{1}{x^2+a^2}$ with respect to x and hence evaluate $\int \frac{1}{x^2+2x+2}dx$

Let $I = \int \frac{1}{x^2+a^2}dx$; Put $x = a\tan\theta$,

then $dx = a\sec^2\theta$;

$x^2+a^2 = a^2\tan^2\theta + a^2$

$= a^2(\tan^2\theta+1) = a^2\sec^2\theta$

$$I = \int \frac{1}{x^2+a^2}dx = \int \frac{a\sec^2\theta d\theta}{a^2\sec^2\theta} = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

Consider $\int \frac{1}{x^2+2x+2}dx$

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{1}{(x+1)^2+(1)^2}dx$$

$$= \left[\tan^{-1}(x+1)\right] + C$$

7.**Find the integral of $\sqrt{x^2+a^2}$ with respect to x and evaluate $\int\sqrt{x^2+9}\,dx$.**

Solution: Let $I = \int \sqrt{x^2+a^2}\,dx$ applying integration by parts

We get $I = x\sqrt{x^2+a^2} - \frac{1}{2}\int \frac{2x^2}{\sqrt{x^2+a^2}}\,dx$

$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\,dx$

$= x\sqrt{x^2+a^2} - I + \int \frac{a^2}{\sqrt{x^2+a^2}}\,dx$

$2I = x\sqrt{x^2+a^2} + a^2 \log\left|x+\sqrt{x^2+a^2}\right| + c$

$I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

Consider $\int \sqrt{x^2+9}\, dx = \frac{x}{2}\sqrt{x^2+9}$

$$+\frac{9}{2}\log\left|x+\sqrt{x^2+9}\right| + c$$

8. Find the integral of $\sqrt{a^2-x^2}$ with respect to x and evaluate $\int \sqrt{1+4x-x^2}\, dx$.

Solution: Let $I = \int \sqrt{a^2-x^2}\, dx$ applying integration by parts

We get $I = x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\, dx$

$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\, dx$

$I = x\sqrt{a^2-x^2} - \int \frac{a^2}{\sqrt{a^2-x^2}}\, dx - I$

$2I = x\sqrt{a^2-x^2} - a^2\sin^{-1}\left(\frac{x}{a}\right) + c$

$I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

Consider $\int \sqrt{1+4x-x^2}\, dx$

$= \int \sqrt{5-(x-2)^2}\, dx$

$= \frac{(x-2)}{2}\sqrt{5-(x-2)^2} - \frac{5}{2}\sin^{-1}\left(\frac{x-2}{\sqrt{5}}\right) + c$

Find the integral of $\sqrt{x^2-a^2}$ with respect to x and hence evaluate $\int x^2+2x+5\, dx$

Solution: Let $I = \int \sqrt{x^2-a^2}\, dx$ applying integration by parts

We get $I = x\sqrt{x^2-a^2} - \frac{1}{2}\int \frac{2x^2}{\sqrt{x^2-a^2}}\, dx$

$= x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}\, dx$

$= x\sqrt{x^2-a^2} - \int \left[\sqrt{x^2-a^2} + \frac{a^2}{\sqrt{x^2-a^2}}\right] dx$

$= x\sqrt{x^2-a^2} - I - \int \frac{a^2}{\sqrt{x^2-a^2}}\, dx$

$2I = x\sqrt{x^2-a^2} - a^2\log\left|x+\sqrt{x^2-a^2}\right| + c$

$I = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

Now consider $I = \int \boldsymbol{x^2+2x+5\, dx}$

$x^2+2x+5=(x^2+2x+1)+4=(x+1)^2+4$

$\therefore I=\int[(x+1)^2+2^2]\ dx$

Put $x+1=t \Rightarrow dx=dt$

$I = \sqrt{t^2+2^2}dt$

$= \frac{t}{2}\sqrt{t^2+2^2} + \frac{4}{2}\log\left|t+\sqrt{t^2+2^2}\right|$

$= \frac{x+1}{2}\sqrt{x^2+2x+5}$

$+ 2\log\left|x+1+\sqrt{x^2+2x+1}\right|$

Note: The above questions is for 5 mark questions in part D of the question paper for second PUC.

Note: In this chapter "Indefinite Integrals" Some of the solved examples given in the text book are not included in the question bank. The students are advised to go through the those questions also.

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Assignments

(i) Integration by substitution

LEVEL I

1. $\int \frac{\sec^2(\log x)}{x}dx$ 2. $\int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$ 3. $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx$

LEVEL II

1. $\int \frac{1}{\sqrt{x}+x}dx$ 2. $\int \frac{1}{x\sqrt{x^6-1}}dx$ 3. $\int \frac{1}{e^x-1}dx$

LEVEL III

1. $\int \frac{\sqrt{\tan x}}{\sin x.\cos x}dx$ 2. $\int \frac{\tan x}{\sec x+\cos x}dx$ 3. $\int \frac{1}{\sin x.\cos^3 x}dx$

(ii) Application of trigonometric function in integrals

LEVEL I

1. $\int \sin^3 x.dx$ 2. $\int \cos^2 3x.dx$ 3. $\int \cos x.\cos 2x.\cos 3x.dx$

LEVEL II

1. $\int \sec^4 x.\tan x.dx$ 2. $\int \frac{\sin 4x}{\sin x}dx$

LEVEL III

1. $\int \cos^5 x.dx$ 2. $\int \sin^2 x.\cos^3 x.dx$

(iii) Integration using standard results

LEVEL I

1. $\int \frac{dx}{\sqrt{4x^2-9}}$ 2. $\int \frac{1}{x^2+2x+10}dx$ 3. $\int \frac{dx}{9x^2+12x+13}$

LEVEL II

1. $\int \frac{x}{x^4+x^2+1}dx$ 2. $\int \frac{\cos x}{\sin^2 x+4\sin x+5}dx$ 3. $\int \frac{dx}{\sqrt{7-6x-x^2}}$

LEVEL III

1. $\int \frac{2x}{\sqrt{1-x^2-x^4}}dx$ 2. $\int \frac{x^2+x+1}{x^2-x+1}dx$ 3. $\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$

4. $\int \sqrt{\frac{1-x}{1+x}}\,dx$ 5. $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}$ **[CBSE 2011]**

(iv) Integration using Partial Fraction

LEVEL I

1. $\int \frac{2x+1}{(x+1)(x-1)}dx$ 2. $\int \frac{x^2}{(x-1)(x-2)(x-3)}dx$ 3. $\int \frac{3x-2}{(x+1)^2(x+3)}dx$

LEVEL II

1. $\int \frac{x^2+2x+8}{(x-1)(x-2)}dx$ 2. $\int \frac{x^2+x+1}{x^2(x+2)}dx$ 3. $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$

LEVEL III

1. $\int \frac{8}{(x+2)(x^2+4)}dx$ 2. $\int \frac{dx}{\sin x+\sin 2x}$ 3. $\int \frac{1}{1+x^3}dx$

(v) Integration by Parts

LEVEL I

1. $\int x.\sec^2 x.dx$ 2. $\int \log x.dx$ 3.

$\int e^x(\tan x+\log\sec x)dx$

LEVEL II

1. $\int \sin^{-1}x.dx$ 2. $\int x^2.\sin^{-1}x.dx$ 3. $\int \frac{x.\sin^{-1}x}{\sqrt{1-x^2}}dx$

4. $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right).dx$ 5. $\int \sec^3 x.dx$

LEVEL III

1. $\int \cos(\log x)dx$ 2. $\int \frac{e^x(1+x)}{(2+x)^2 dx}$ 3. $\int \frac{\log x}{(1+\log x)^2}dx$

4. $\int \frac{2+\sin x}{1+\cos 2x}e^x.dx$ 5. $\int e^{2x}.\cos 3x.dx$

(vi) Some Special Integrals

LEVEL I

1. $\int \sqrt{4+x^2}.dx$ 2. $\int \sqrt{1-4x^2}.dx$

LEVEL II

1. $\int \sqrt{x^2+4x+6}.dx$ 2. $\int \sqrt{1-4x-x^2}.dx$

LEVEL III

1. $\int (x+1)\sqrt{1-x-x^2}.dx$ 2. $\int (x-5)\sqrt{x^2+x}\,dx$

(vii) Miscellaneous Questions

LEVEL II

1. $\int \frac{dx}{4\sin^2 x+5\cos^2 x}$ (Hint: Divide the Numerator and Denominator by $\cos^2x$ and use the relation $\sec^2x=1+\tan^2x$; and put tanx=t

LEVEL III

1. $\int \frac{1}{1+tanx}dx$

SOLUTIONS: ASSIGNMENTS: INDEFINITE INTEGRALS

(i) Integration by substitution

LEVEL I 1. $\tan(\log_e x) + C$ 2. $\frac{1}{m}e^{m\tan^{-1}x} + C$ 3. $e^{\sin^{-1}x} + C$

LEVEL II 1. $2\log_e\left|1+\sqrt{x}\right| + C$ 2. $\frac{1}{3}\sec^{-1}x^3 + C$ 3. $\log_e\left|1-e^x\right| + C$

LEVEL III 1. $2\sqrt{\tan x} + C$ 2. $-\tan^{-1}(\cos x) + C$ 3. $\frac{\tan^2 x}{2} + \log_e\left|\tan x\right| + C$

(ii)) Application of trigonometric function in integrals

LEVEL I 1. $-\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$ 2. $\frac{1}{2}\left[x + \frac{\sin 6x}{6}\right] + C$

3. $\frac{x}{4} + \frac{1}{4}\sin 6x + \frac{1}{16}\sin 4x + \frac{1}{8}\sin 2x + C$

LEVEL II 1. $\frac{1}{4}\sec^4 x + C$ OR $\frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C$ 2. $\frac{2}{3}\sin 3x + 2\sin x + C$

LEVEL III 1. $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$ 2. $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

(iii) Integration using Standard results

LEVEL I 1. $\frac{1}{2}\log_e\left|x + \frac{1}{2}\sqrt{4x^2-9}\right| + C$ 2. $\frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + C$ 3. $\frac{1}{9}\tan^{-1}\left(\frac{3x+2}{3}\right) + C$

LEVEL II 1. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C$ 2. $\tan^{-1}(\sin x + 2) + C$ 3. $\sin^{-1}\left(\frac{2x-1}{5}\right) + C$

LEVEL III 1. $\sin^{-1}\left(\frac{2x^2-1}{5}\right) + C$ 2. $x + \log\left|x^2 - x + 1\right| + \frac{2}{\sqrt{3}}\log\left|\frac{2x-1}{\sqrt{3}}\right| + C$

3. $\sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + C$

4. $\sin^{-1}x + \sqrt{1-x^2} + C$ [Hint: Put x=cos2θ]

5. $6\sqrt{x^2-9x+20} + 34\log\left|\left(\frac{2x-9}{2}\right) + \sqrt{x^2-9x+20}\right| + C$

(iv) Integration using Partial Fraction

LEVEL I 1. $\frac{1}{3}\log(x+1) + \frac{5}{3}\log(x-2) + C$ 2. $\frac{1}{2}\log(x-1) - 2\log(x-2) + \frac{3}{2}\log(x-3) + C$

3. $\frac{11}{4}\log\left(\frac{x+1}{x+3}\right) + \frac{5}{2(x+1)} + C$

LEVEL II 1. $x - 11\log(x-1) + 16\log(x-2) + C$ 2. $\frac{1}{4}\log x - \frac{1}{2x} + \frac{3}{4}\log(x+2) + C$

3. $\frac{3}{8}\log(x-1) - \frac{1}{2(x-1)} + \frac{5}{8}\log(x+3) + C$

LEVEL III 1. $\log(x+2)-\frac{1}{2}\log\left(x^2+4\right)+\tan^{-1}\frac{x}{2}$ 2. $\frac{\log(1-\cos x)}{6}+\frac{\log(1+\cos x)}{2}-\frac{2\log(1+2\cos x)}{3}+C$

3. $\frac{1}{3}\log(1+x)-\frac{1}{6}\log\left(1-x+x^2\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$ + C [Hint: Partial fractions]

(v) Integration by Parts

LEVEL I 1.x.tanx + logcosx + C 2.xlogx – x + C 3.e^x.logsecx + C

LEVEL II 1. $x\sin^{-1}x+\sqrt{1-x^2}+C$ 2. $\frac{x^3}{3}\sin^{-1}x+\frac{\left(x^2+2\right)\sqrt{1-x^2}}{9}+C$

3. $-\sqrt{1-x^2}\sin^{-1}x+x+C$ 4. $2x\tan^{-1}x-\log\left(1+x^2\right)+C$

5. $\frac{1}{2}(\sec x.\tan x+\log(\sec x+\tan x))+C$

LEVEL III 1. $\frac{x}{2}[\cos(\log x)+\sin(\log x)]+C$ 2. $\frac{e^x}{2+x}+C$ [Hint: $\int\left[e^x f(x)+f'(x)\right]dx = e^x f(x)+c$

3. $\frac{x}{1+\log x}+C$ 4. e^x.tanx + C 5. $\frac{e^{2x}}{13}(3\sin 3x+2\cos 3x)+C$

(vi) Some Special Integrals

LEVEL I 1. $\frac{x\sqrt{4+x^2}}{2}+2\log\left|x+\sqrt{4+x^2}\right|+C$ 2. $\frac{x\sqrt{1-4x^2}}{2}+\frac{1}{4}\sin^{-1}2x+C$

LEVEL II 1. $\frac{(x+2)\sqrt{x^2+4x+6}}{2}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

2. $\frac{(x+2)\sqrt{1-4x-x^2}}{2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$

LEVEL III 1. $-\frac{1}{3}\left(1-x-x^2\right)^{3/2}+\frac{1}{8}(2x-1)\sqrt{1-x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+C$

2. $\frac{1}{3}\left(x^2+x\right)^{3/2}-\frac{11}{8}(2x+1)\sqrt{x^2+x}+\frac{11}{16}\log\left[(2x+1)+2\sqrt{x^2+x}\right]+C$

(vii) Miscellaneous Questions

LEVEL II 1. $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right)+C$

LEVEL III 1. $\frac{x}{2}+\frac{1}{2}$ log|cosx+sinx|+C

Detail of the concepts to be mastered by every student of class second PUC with exercises and examples of NCERT Text Book.

Indefinite Integrals	(i) Integration by substitution	*	Text book , Vol. II Examples 5&6 Page 300, 302,301,303
	(ii)) Application of trigonometric function in integrals	**	Text book , Vol. II Ex 7 Page 306, Exercise 7.3 Q13&Q24
	(iii) Integration of some particular function $\int \frac{dx}{x^2 \pm a^2}$, $\int \frac{dx}{\sqrt{x^2 \pm a^2}}$, $\int \frac{1}{\sqrt{a^2 - x^2}} dx$, $\int \frac{dx}{ax^2 + bx + c}$, $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$, $\int \frac{(px+q)dx}{ax^2 + bx + c}$, $\int \frac{(px+q)dx}{\sqrt{ax^2 + bx + c}}$	***	Text book , Vol. II Exp 8, 9, 10 Page 311,312,313, Exercise 7.4 Q 3,4,8,9,13&23
	(iv) Integration using Partial Fraction	***	Text book , Vol. II Exp 11&12 Page 318 Exp 13 319,Exp 14 & 15 Page320
	(v) Integration by Parts	**	Text book , Vol. II Exp 18,19&20 Page 325 Exs 7.6 QNO ,10,11, 17,18,20
	(vi)Some Special Integrals $\int \sqrt{a^2 \pm x^2}\, dx$, $\int \sqrt{x^2 - a^2}\, dx$	***	Text book , Vol. II Exp 23 &24 Page 329
	(vii) Miscellaneous Questions	**	Text book , Vol. II Solved Ex. 40, 41
	viii)Some special integrals		Text book Supplimentary material Page 614,615

SYMBOLS USED :

*** : Important Questions, ** :Very Important Questions, *** : Very-Very Important Questions**