# CSE 431/531: Algorithm Analysis and Design (Spring 2021)

# Graph Algorithms


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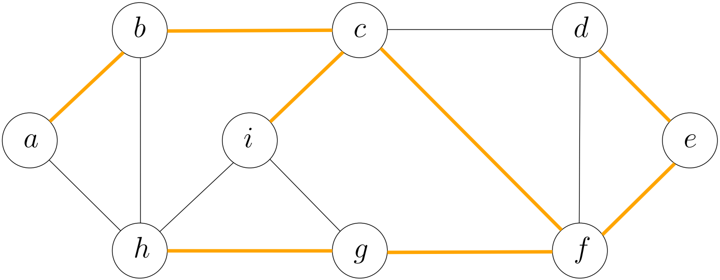
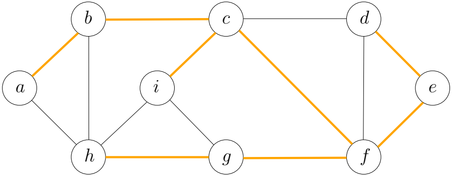
# Outline

# Spanning Tree

**Def.** Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of $G$ is a sub-graph of $G$ that is a tree including all vertices $V$.





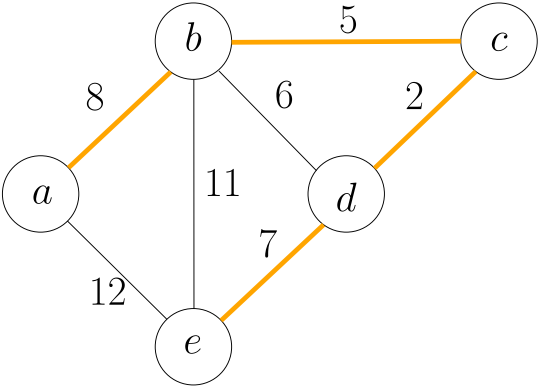

**Lemma** Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- $T$ is a spanning tree of $G$;
- $T$ is acyclic and connected;
- $T$ is connected and has $n - 1$ edges;
- $T$ is acyclic and has $n - 1$ edges;
- $T$ is minimally connected: removal of any edge disconnects it;
- $T$ is maximally acyclic: addition of any edge creates a cycle;
- $T$ has a unique simple path between every pair of nodes.

## Minimum Spanning Tree (MST) Problem

**Input:** Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

**Output:** the spanning tree $T$ of $G$ with the minimum total weight



## Recall: Steps of Designing A Greedy Algorithm

- Design a "reasonable" strategy
- Prove that the reasonable strategy is "safe" (key, usually done by "exchanging argument")
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

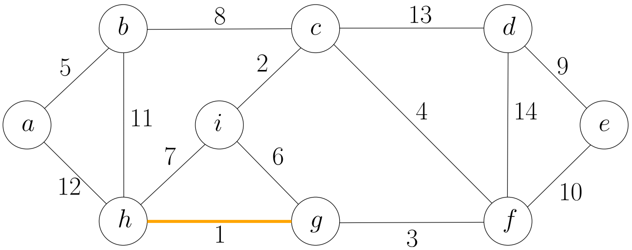
**Def.** A choice is "safe" if there is an optimum solution that is "consistent" with the choice

## Two Classic Greedy Algorithms for MST

- Kruskal's Algorithm
- Prim's Algorithm

# Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
   - Bellman-Ford Algorithm
4. All-Pair Shortest Paths and Floyd-Warshall



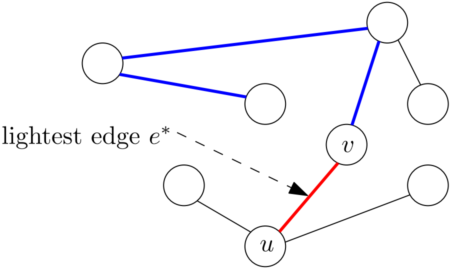

**Q:** Which edge can be safely included in the MST?

**A:** The edge with the smallest weight (lightest edge).

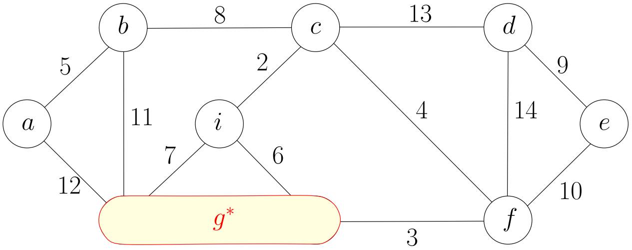
**Lemma** It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

## Proof.

- Take a minimum spanning tree $T$
- Assume the lightest edge $e^*$ is not in $T$
- There is a unique path in $T$ connecting $u$ and $v$
- Remove any edge $e$ in the path to obtain tree $T'$
- $w(e^*) \leq w(e) \implies w(T') \leq w(T)$: $T'$ is also a MST $\square$

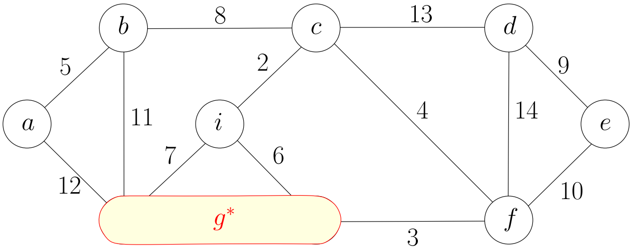


# Is the Residual Problem Still a MST Problem?




- Residual problem: find the minimum spanning tree that contains edge $(g, h)$
- Contract the edge $(g, h)$
- Residual problem: find the minimum spanning tree in the contracted graph

# Contraction of an Edge $(u, v)$




- Remove $u$ and $v$ from the graph, and add a new vertex $u^*$
- Remove all edges $(u, v)$ from $E$
- For every edge $(u, w) \in E, w \neq v$, change it to $(u^*, w)$
- For every edge $(v, w) \in E, w \neq u$, change it to $(u^*, w)$
- May create parallel edges! E.g. : two edges $(i, g^*)$

# Greedy Algorithm

Repeat the following step until $G$ contains only one vertex:

1. Choose the lightest edge $e^*$, add $e^*$ to the spanning tree
2. Contract $e^*$ and update $G$ be the contracted graph

**Q:** What edges are removed due to contractions?

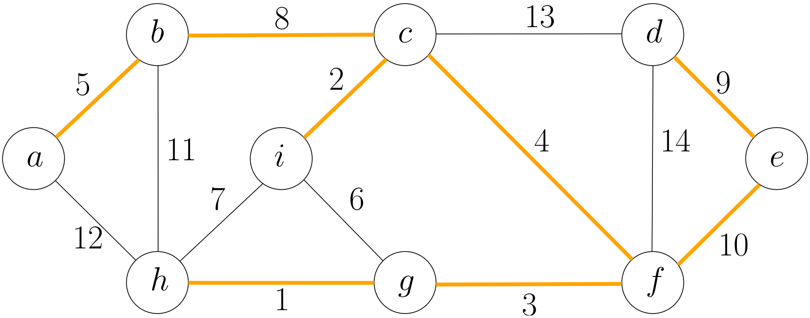
**A:** Edge $(u, v)$ is removed if and only if there is a path connecting $u$ and $v$ formed by edges we selected

# Greedy Algorithm

**MST-Greedy**$(G, w)$

```
F ← ∅
sort edges in E in non-decreasing order of weights w
for each edge (u, v) in the order do
    if u and v are not connected by a path of edges in F then
        F ← F ∪ {(u, v)}
return (V, F)
```

# Kruskal's Algorithm: Example




Sets: $\{a, b, c, i, f, g, h, d, e\}$

# Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

**MST-Kruskal($G$, $w$)**

```
F ← ∅
𝒮 ← {{v} : v ∈ V}
sort the edges of E in non-decreasing order of weights w
for each edge (u, v) ∈ E in the order do
    S_u ← the set in 𝒮 containing u
    S_v ← the set in 𝒮 containing v
    if S_u ≠ S_v then
        F ← F ∪ {(u, v)}
        𝒮 ← 𝒮 \ {S_u} \ {S_v} ∪ {S_u ∪ S_v}
return (V, F)
```

# Running Time of Kruskal's Algorithm

## MST-Kruskal($G$, $w$)

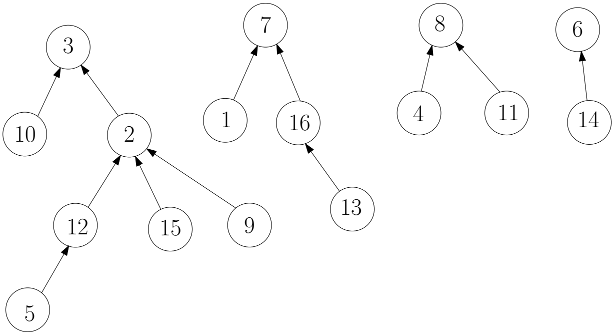
1: $F \leftarrow \emptyset$
2: $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
3: sort the edges of $E$ in non-decreasing order of weights $w$
4: **for** each edge $(u, v) \in E$ in the order **do**
5: $\quad S_u \leftarrow$ the set in $\mathcal{S}$ containing $u$
6: $\quad S_v \leftarrow$ the set in $\mathcal{S}$ containing $v$
7: $\quad$ **if** $S_u \neq S_v$ **then**
8: $\qquad F \leftarrow F \cup \{(u, v)\}$
9: $\qquad \mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10: **return** $(V, F)$

Use union-find data structure to support ❷, ❺, ❻, ❼, ❾.

# Union-Find Data Structure

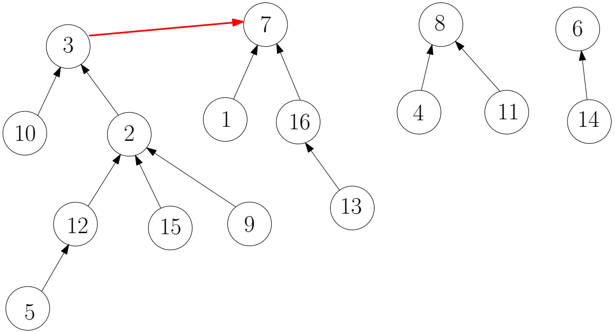
- $V$: ground set
- We need to maintain a partition of $V$ and support following operations:
  - Check if $u$ and $v$ are in the same set of the partition
  - Merge two sets in partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$




- $par[i]$: parent of $i$, ($par[i] = \perp$ if $i$ is a root).

# Union-Find Data Structure




- Q: how can we check if $u$ and $v$ are in the same set?
- A: Check if root$(u) =$ root$(v)$.
- root$(u)$: the root of the tree containing $u$
- Merge the trees with root $r$ and $r'$: $par[r] \leftarrow r'$.

# Union-Find Data Structure

**root($v$)**

**if** $par[v] = \perp$ **then**
    **return** $v$
**else**
    **return** root($par[v]$)

**root($v$)**

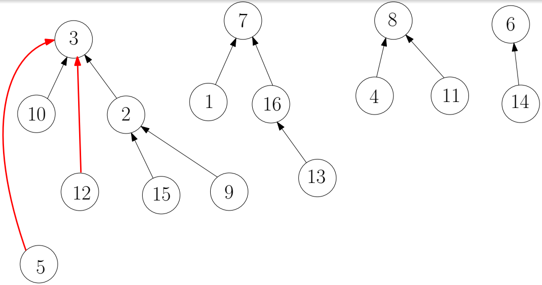
**if** par$[v] = \perp$ **then**
    **return** $v$
**else**
    $par[v] \leftarrow$ root($par[v]$)
**return** $par[v]$

- Problem: the tree might too deep; running time might be large
- Improvement: all vertices in the path directly point to the root, saving time in the future.

# Union-Find Data Structure

## root($v$)

**if** $par[v] = \bot$ **then**
    **return** $v$
**else**
    $par[v] \leftarrow \mathrm{root}(par[v])$
    **return** $par[v]$



## MST-Kruskal($G$, $w$)

$F \leftarrow \emptyset$

$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$

sort the edges of $E$ in non-decreasing order of weights $w$

**for** each edge $(u, v) \in E$ in the order **do**

    $S_u \leftarrow$ the set in $\mathcal{S}$ containing $u$

    $S_v \leftarrow$ the set in $\mathcal{S}$ containing $v$

    **if** $S_u \neq S_v$ **then**

        $F \leftarrow F \cup \{(u, v)\}$

        $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$

**return** $(V, F)$

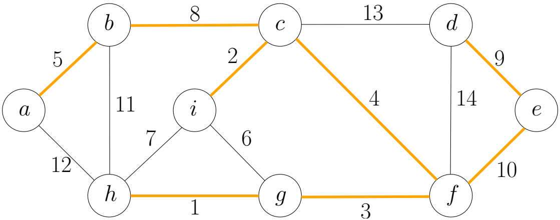
## MST-Kruskal($G$, $w$)

1: $F \leftarrow \emptyset$
2: **for** every $v \in V$ **do**: $par[v] \leftarrow \perp$
3: sort the edges of $E$ in non-decreasing order of weights $w$
4: **for** each edge $(u, v) \in E$ in the order **do**
5: $\quad u' \leftarrow \text{root}(u)$
6: $\quad v' \leftarrow \text{root}(v)$
7: $\quad$ **if** $u' \neq v'$ **then**
8: $\quad\quad F \leftarrow F \cup \{(u, v)\}$
9: $\quad\quad par[u'] \leftarrow v'$
10: **return** $(V, F)$

- 2,5,6,7,9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \le 4$ for $n \le 10^{80}$.
- Running time $=$ time for 3 $= O(m \lg n)$.

**Assumption** Assume all edge weights are different.

**Lemma** An edge $e \in E$ is not in the MST, if and only if there is cycle $C$ in $G$ in which $e$ is the heaviest edge.



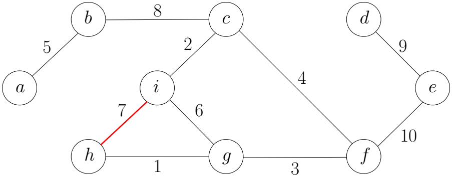

- $(i, g)$ is not in the MST because of cycle $(i, c, f, g)$
- $(e, f)$ is in the MST because no such cycle exists

# Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
   - Bellman-Ford Algorithm
4. All-Pair Shortest Paths and Floyd-Warshall

# Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to $F$ one by one until we obtain a spanning tree
2. Start from $F \leftarrow E$, and remove edges from $F$ one by one until we obtain a spanning tree




**Q:** Which edge can be safely excluded from the MST?

**A:** The heaviest non-bridge edge.

**Def.** A bridge is an edge whose removal disconnects the graph.

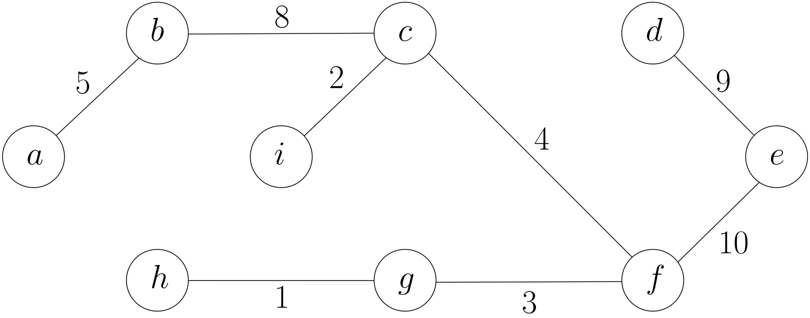
**Lemma** It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

# Reverse Kruskal's Algorithm

**MST-Greedy($G, w$)**

$F \leftarrow E$
sort $E$ in non-increasing order of weights
**for** every $e$ in this order **do**
    **if** $(V, F \setminus \{e\})$ is connected **then**
        $F \leftarrow F \setminus \{e\}$
**return** $(V, F)$

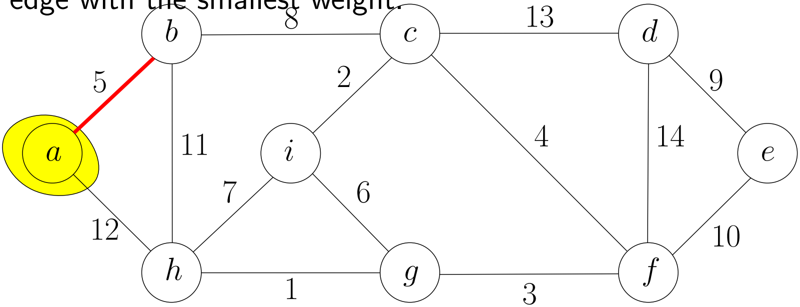
# Reverse Kruskal's Algorithm: Example



# Outline

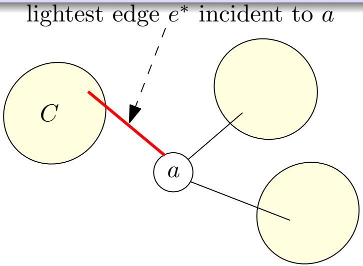
# Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.




- Greedy strategy for Prim's algorithm: choose the lightest edge incident to $a$.

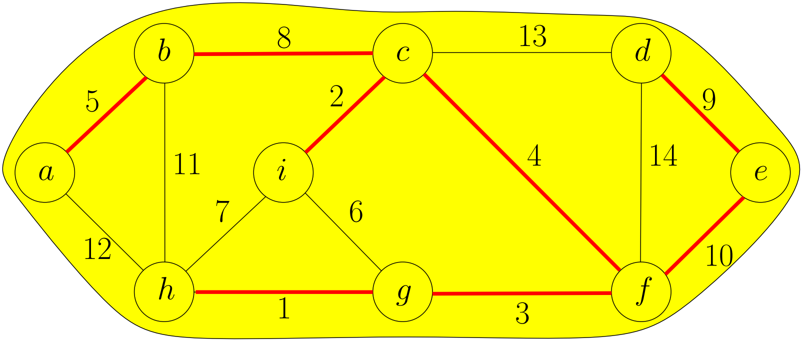
**Lemma** It is safe to include the lightest edge incident to $a$.




## Proof.

- Let $T$ be a MST
- Consider all components obtained by removing $a$ from $T$
- Let $e^*$ be the lightest edge incident to $a$ and $e^*$ connects $a$ to component $C$
- Let $e$ be the edge in $T$ connecting $a$ to $C$
- $T' = T \setminus e \cup \{e^*\}$ is a spanning tree with $w(T') \le w(T)$ $\square$

# Prim's Algorithm: Example



# Greedy Algorithm

**MST-Greedy1($G, w$)**

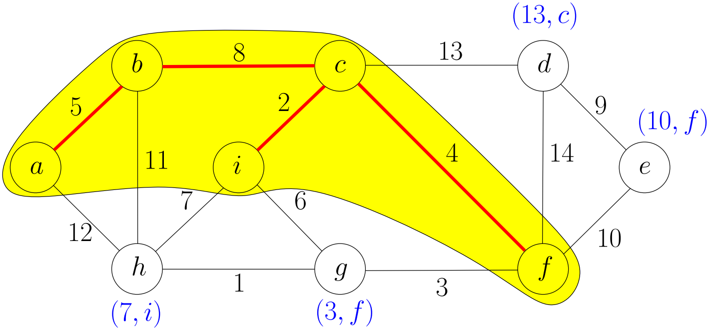
$S \leftarrow \{s\}$, where $s$ is arbitrary vertex in $V$
$F \leftarrow \emptyset$
**while** $S \neq V$ **do**
    $(u, v) \leftarrow$ lightest edge between $S$ and $V \setminus S$,
        where $u \in S$ and $v \in V \setminus S$
    $S \leftarrow S \cup \{v\}$
    $F \leftarrow F \cup \{(u, v)\}$
**return** $(V, F)$

- Running time of naive implementation: $O(nm)$

# Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between $v$ and $S$
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between $v$ and $S$



# Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between $v$ and $S$
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between $v$ and $S$

In every iteration

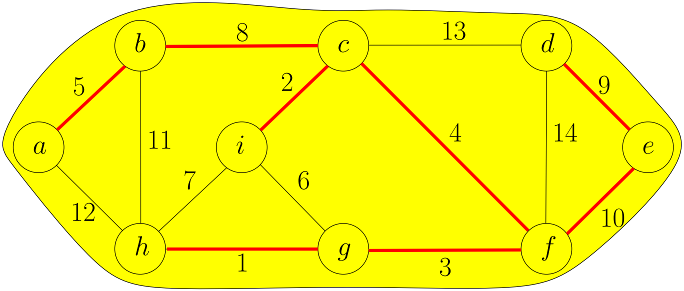
- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to $F$
- Add $u$ to $S$, update $d$ and $\pi$ values.

# Prim's Algorithm

## MST-Prim$(G, w)$

```
s ← arbitrary vertex in G
S ← ∅, d(s) ← 0 and d(v) ← ∞ for every v ∈ V \ {s}
while S ≠ V do
    u ← vertex in V \ S with the minimum d(u)
    S ← S ∪ {u}
    for each v ∈ V \ S such that (u, v) ∈ E do
        if w(u, v) < d(v) then
            d(v) ← w(u, v)
            π(v) ← u
return {(u, π(u)) | u ∈ V \ {s}}
```

# Example



# Prim's Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between $v$ and $S$
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between $v$ and $S$

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value — extract_min
- Add $(\pi(u), u)$ to $F$
- Add $u$ to $S$, update $d$ and $\pi$ values. — decrease_key

Use a priority queue to support the operations

**Def.** A priority queue is an abstract data structure that maintains a set $U$ of elements, each with an associated key value, and supports the following operations:

- insert$(v, key_value)$: insert an element $v$, whose associated key value is $key\_value$.
- decrease_key$(v, new_key_value)$: decrease the key value of an element $v$ in queue to $new\_key\_value$
- extract_min$()$: return and remove the element in queue with the smallest key value
- $\cdots$

# Prim's Algorithm

## MST-Prim$(G, w)$

$s \leftarrow$ arbitrary vertex in $G$
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$

**while** $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
    $S \leftarrow S \cup \{u\}$
    **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        **if** $w(u, v) < d(v)$ **then**
            $d(v) \leftarrow w(u, v)$
            $\pi(v) \leftarrow u$
**return** $\big\{(u, \pi(u)) | u \in V \setminus \{s\}\big\}$

# Prim's Algorithm Using Priority Queue

## MST-Prim$(G, w)$

$s \leftarrow$ arbitrary vertex in $G$
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: $Q$.insert$(v, d(v))$
**while** $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        **if** $w(u, v) < d(v)$ **then**
            $d(v) \leftarrow w(u, v)$, $Q$.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
**return** $\big\{(u, \pi(u)) | u \in V \setminus \{s\}\big\}$

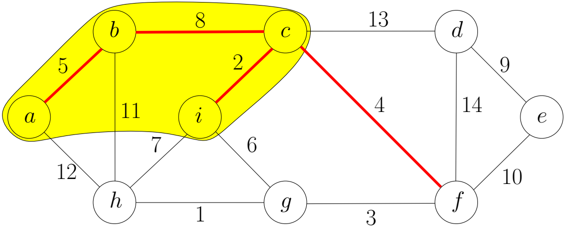
# Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+$ $O(m)\times$ (time for decrease_key)

| concrete DS | extract_min | decrease_key | overall time |
|---|---|---|---|
| heap | $O(\log n)$ | $O(\log n)$ | $O(m \log n)$ |
| Fibonacci heap | $O(\log n)$ | $O(1)$ | $O(n \log n + m)$ |

**Assumption** Assume all edge weights are different.

**Lemma** $(u, v)$ is in MST, if and only if there exists a cut $(U, V \setminus U)$, such that $(u, v)$ is the lightest edge between $U$ and $V \setminus U$.




- $(c, f)$ is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- $(i, g)$ is not in MST because no such cut exists

# "Evidence" for $e \in$ MST or $e \notin$ MST

**Assumption** Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which $e$ is the lightest edge
- $e \notin$ MST $\leftrightarrow$ there is a cycle in which $e$ is the heaviest edge

Exactly one of the following is true:

- There is a cut in which $e$ is the lightest edge
- There is a cycle in which $e$ is the heaviest edge

Thus, the minimum spanning tree is unique with assumption.

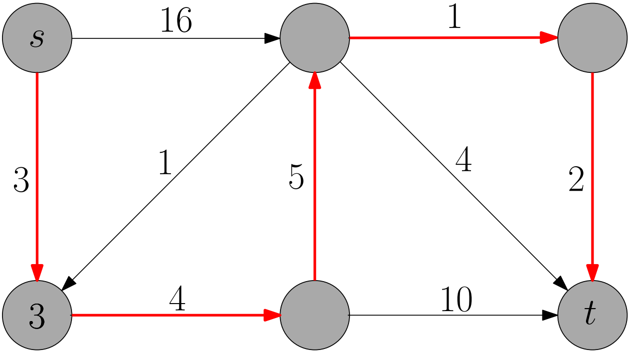
# Outline

## $s$-$t$ Shortest Paths

**Input:** (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \to \mathbb{R}_{\geq 0}$

**Output:** shortest path from $s$ to $t$



## Single Source Shortest Paths

**Input:** directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$

## Reason for Considering Single Source Shortest Paths Problem

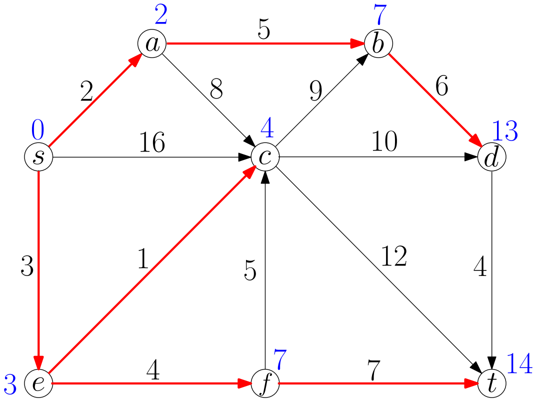
- We do not know how to solve $s$-$t$ shortest path problem more efficiently than solving single source shortest path problem

- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight

- Shortest path from $s$ to $v$ may contain $\Omega(n)$ edges
- There are $\Omega(n)$ different vertices $v$
- Thus, printing out all shortest paths may take time $\Omega(n^2)$
- Not acceptable if graph is sparse

# Shortest Path Tree

- $O(n)$-size data structure to represent all shortest paths
- For every vertex $v$, we only need to remember the parent of $v$: second-to-last vertex in the shortest path from $s$ to $v$ (why?)



## Single Source Shortest Paths

**Input:** directed graph $G = (V, E)$, $s \in V$

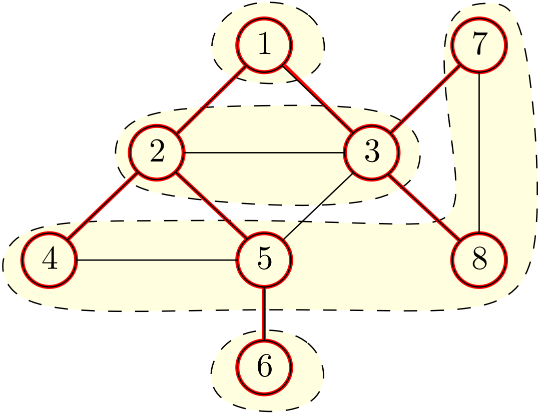
$w : E \to \mathbb{R}_{\geq 0}$

**Output:** $\pi(v), v \in V \setminus s$: the parent of $v$

$d(v), v \in V \setminus s$: the length of shortest path from $s$ to $v$

**Q:** How to compute shortest paths from $s$ when all edges have weight 1?

**A:** Breadth first search (BFS) from source $s$



**Assumption** Weights $w(u,v)$ are integers (w.l.o.g).

- An edge of weight $w(u,v)$ is equivalent to a pah of $w(u,v)$ unit-weight edges

## Shortest Path Algorithm by Running BFS

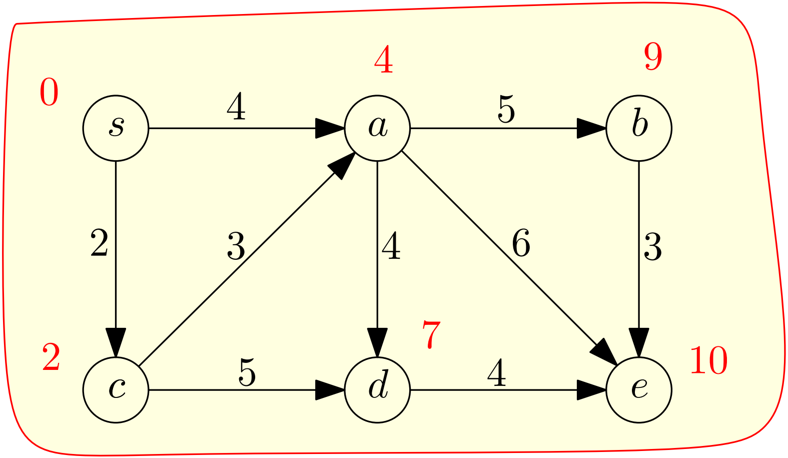
1: replace $(u,v)$ of length $w(u,v)$ with a path of $w(u,v)$ unit-weight edges, for every $(u,v) \in E$
2: run BFS virtually
3: $\pi(v) \leftarrow$ vertex from which $v$ is visited
4: $d(v) \leftarrow$ index of the level containing $v$

- Problem: $w(u,v)$ may be too large!

## Shortest Path Algorithm by Running BFS Virtually

$S \leftarrow \{s\}, d(s) \leftarrow 0$
**while** $|S| \leq n$ **do**
    find a $v \notin S$ that minimizes $\displaystyle\min_{u \in S:(u,v) \in E} \{d(u) + w(u,v)\}$
    $S \leftarrow S \cup \{v\}$
    $d(v) \leftarrow \min_{u \in S:(u,v) \in E} \{d(u) + w(u,v)\}$

# Virtual BFS: Example




Time 10

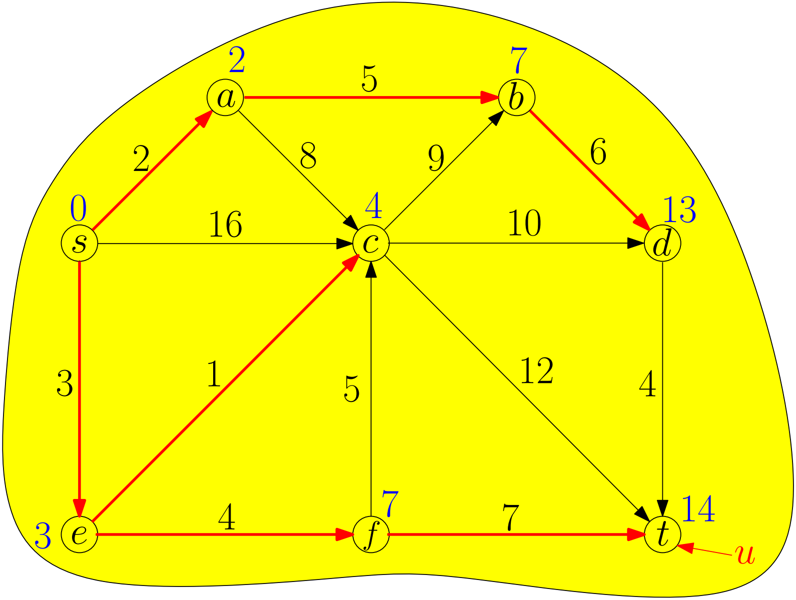
# Outline

# Dijkstra's Algorithm

**Dijkstra**$(G, w, s)$

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
**while** $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
    add $u$ to $S$
    **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        **if** $d(u) + w(u, v) < d(v)$ **then**
            $d(v) \leftarrow d(u) + w(u, v)$
            $\pi(v) \leftarrow u$
**return** $(d, \pi)$

- Running time $= O(n^2)$



$s$ 0
$a$ 2
$b$ 7
$c$ 4
$d$ 13
$e$ 3
$f$ 7
$t$ 14
$u$
2
5
8
9
6
16
10
3
1
5
12
4
4
7

# Improved Running Time using Priority Queue

**Dijkstra$(G, w, s)$**

$s \leftarrow$ arbitrary vertex in $G$
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: $Q$.insert$(v, d(v))$
**while** $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        **if** $d(u) + w(u, v) < d(v)$ **then**
            $d(v) \leftarrow d(u) + w(u, v)$, $Q$.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
**return** $(\pi, d)$

# Recall: Prim's Algorithm for MST

## MST-Prim$(G, w)$

$s \leftarrow$ arbitrary vertex in $G$
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: $Q$.insert$(v, d(v))$
**while** $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        **if** $w(u, v) < d(v)$ **then**
            $d(v) \leftarrow w(u, v)$, $Q$.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
**return** $\big\{(u, \pi(u)) | u \in V \setminus \{s\}\big\}$

# Improved Running Time

Running time:
$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

| Priority-Queue | extract_min | decrease_key | Time |
|---|---|---|---|
| Heap | $O(\log n)$ | $O(\log n)$ | $O(m \log n)$ |
| Fibonacci Heap | $O(\log n)$ | $O(1)$ | $O(n \log n + m)$ |

# Outline

# Recall: Single Source Shortest Path Problem

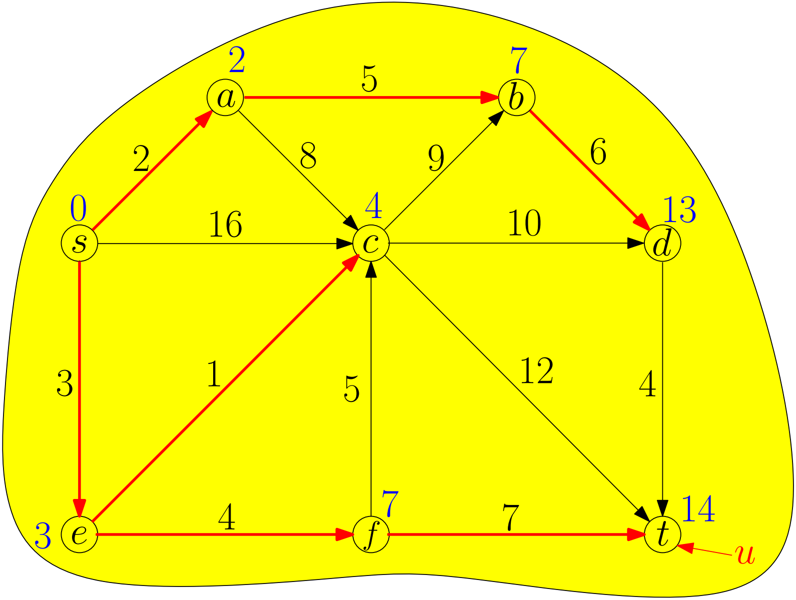
**Single Source Shortest Paths**

**Input:** directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$

- Algorithm for the problem: Dijkstra's algorithm



2
7
5
a
b
2
8
9
6
0
4
13
s
16
c
10
d
3
1
5
12
4
7
4
7
14
3
e
f
t
u

# Dijkstra's Algorithm Using Priorty Queue

**Dijkstra$(G, w, s)$**

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: $Q$.insert$(v, d(v))$
**while** $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        **if** $d(u) + w(u, v) < d(v)$ **then**
            $d(v) \leftarrow d(u) + w(u, v)$, $Q$.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
**return** $(\pi, d)$

- Running time $= O(m + n \lg n)$.

## Single Source Shortest Paths, Weights May be Negative

**Input:** directed graph $G = (V, E)$, $s \in V$

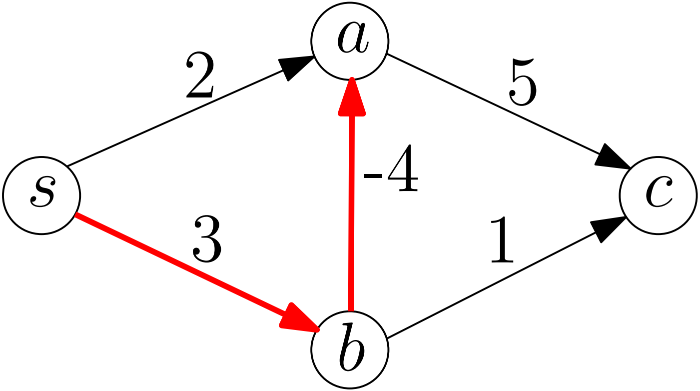
assume all vertices are reachable from $s$

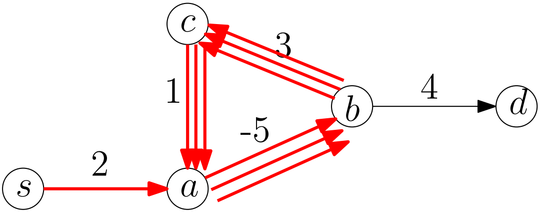
$w : E \to \mathbb{R}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$

- In transition graphs, negative weights make sense
- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)
- Dijkstra's algorithm does not work any more!

# Dijkstra's Algorithm Fails if We Have Negative Weights





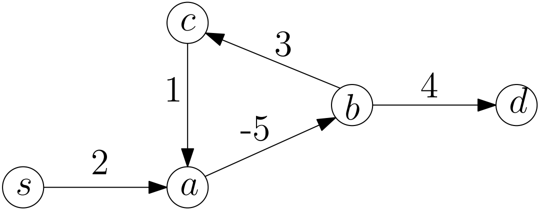

**Q:** What is the length of the shortest path from $s$ to $d$?

**A:** $-\infty$

**Def.** A negative cycle is a cycle in which the total weight of edges is negative.

### Dealing with Negative Cycles

- assume the input graph does not contain negative cycles, or
- allow algorithm to report "negative cycle exists"




**Q:** What is the length of the shortest simple path from $s$ to $d$?

**A:** 1

- Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.

# Outline

# Defining Cells of Table

**Single Source Shortest Paths, Weights May be Negative**

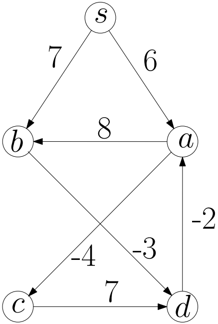
**Input:** directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from $s$

$w : E \to \mathbb{R}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$

- first try: $f[v]$: length of shortest path from $s$ to $v$
- issue: do not know in which order we compute $f[v]$'s
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n-1\}$, $v \in V$ : length of shortest path from $s$ to $v$ that uses at most $\ell$ edges




- $f^{\ell}[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n-1\}$, $v \in V$ : length of shortest path from $s$ to $v$ that uses at most $\ell$ edges
- $f^2[a] = 6$
- $f^3[a] = 2$

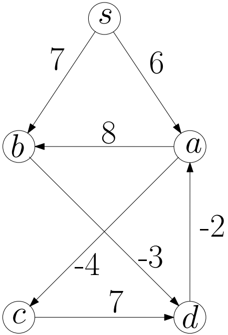
$$f^{\ell}[v] = \begin{cases} 0 & \ell = 0, v = s \\ \infty & \ell = 0, v \neq s \\ \min \begin{cases} f^{\ell-1}[v] \\ \min_{u:(u,v)\in E} \left(f^{\ell-1}[u] + w(u,v)\right) \end{cases} & \ell > 0 \end{cases}$$

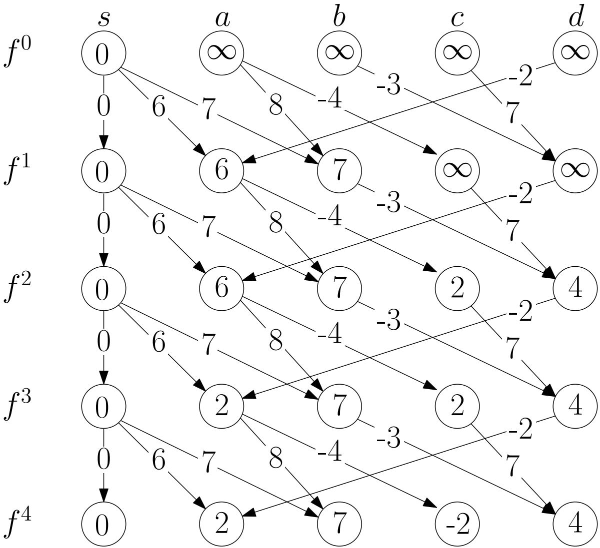
**dynamic-programming$(G, w, s)$**

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
**for** $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\ell-1} \to f^{\ell}$
    **for** each $(u, v) \in E$ **do**
        **if** $f^{\ell-1}[u] + w(u, v) < f^{\ell}[v]$ **then**
            $f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
**return** $(f^{n-1}[v])_{v \in V}$

**Obs.** Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

# Dynamic Programming: Example



**dynamic-programming**$(G, w, s)$

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
**for** $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\ell-1} \rightarrow f^{\ell}$
    **for** each $(u, v) \in E$ **do**
        **if** $f^{\ell-1}[u] + w(u,v) < f^{\ell}[v]$ **then**
            $f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u,v)$
**return** $(f^{n-1}[v])_{v \in V}$

**Obs.** Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

**Q:** What if there are negative cycles?

# Dynamic Programming With Negative Cycle Detection

**dynamic-programming$(G, w, s)$**

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
**for** $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\ell-1} \rightarrow f^{\ell}$
    **for** each $(u, v) \in E$ **do**
        **if** $f^{\ell-1}[u] + w(u, v) < f^{\ell}[v]$ **then**
            $f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
**for** each $(u, v) \in E$ **do**
    **if** $f^{n-1}[u] + w(u, v) < f^{n-1}[v]$ **then**
        report "negative cycle exists" and exit
**return** $(f^{n-1}[v])_{v \in V}$

# Bellman-Ford Algorithm

**Bellman-Ford$(G, w, s)$**

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$<br>
**for** $\ell \leftarrow 1$ to $n-1$ **do**<br>
    **for** each $(u, v) \in E$ **do**<br>
        **if** $f[u] + w(u, v) < f[v]$ **then**<br>
            $f[v] \leftarrow f[u] + w(u, v)$<br>
**return** $f$

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only "accelerate" the process!
- After iteration $\ell$, $f[v]$ is at most the length of the shortest path from $s$ to $v$ that uses at most $\ell$ edges
- $f[v]$ is always the length of some path from $s$ to $v$

# Bellman-Ford Algorithm

**Bellman-Ford$(G, w, s)$**

1. $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. **for** $\ell \leftarrow 1$ to $n-1$ **do**
3.     **for** each $(u, v) \in E$ **do**
4.         **if** $f[u] + w(u, v) < f[v]$ **then**
5.             $f[v] \leftarrow f[u] + w(u, v)$
6. **return** $f$

- After iteration $\ell$, $f[v]$ is at most the length of the shortest path from $s$ to $v$ that uses at most $\ell$ edges
- $f[v]$ is always the length of some path from $s$ to $v$
- Assuming there are no negative cycles, after iteration $n-1$, $f[v]$ = length of shortest path from $s$ to $v$

# Bellman-Ford Algorithm

**Bellman-Ford$(G, w, s)$**

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
**for** $\ell \leftarrow 1$ to $n$ **do**
  $updated \leftarrow$ false
  **for** each $(u, v) \in E$ **do**
    **if** $f[u] + w(u, v) < f[v]$ **then**
      $f[v] \leftarrow f[u] + w(u, v)$, $\pi[v] \leftarrow u$
      $updated \leftarrow$ true
  if not $updated$, then return $f$
output "negative cycle exists"

- $\pi[v]$: the parent of $v$ in the shortest path tree
- Running time $= O(nm)$

# Outline

# Summary of Shortest Path Algorithms we learned

| algorithm | graph | weights | SS? | running time |
|---|---|---|---|---|
| Simple DP | DAG | $\mathbb{R}$ | SS | $O(n+m)$ |
| Dijkstra | U/D | $\mathbb{R}_{\geq 0}$ | SS | $O(n \log n + m)$ |
| Bellman-Ford | U/D | $\mathbb{R}$ | SS | $O(nm)$ |
| Floyd-Warshall | U/D | $\mathbb{R}$ | AP | $O(n^3)$ |

- DAG = directed acyclic graph    U = undirected    D = directed
- SS = single source    AP = all pairs

# All-Pair Shortest Paths

**All Pair Shortest Paths**

**Input:** directed graph $G = (V, E)$,
$w : E \to \mathbb{R}$ (can be negative)

**Output:** shortest path from $u$ to $v$ for every $u, v \in V$

**for** every starting point $s \in V$ **do**
    run Bellman-Ford$(G, w, s)$

- Running time $= O(n^2 m)$

# Design a Dynamic Programming Algorithm

- It is convenient to assume $V = \{1, 2, 3, \cdots, n\}$
- For simplicity, extend the $w$ values to non-edges:

$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

- For now assume there are no negative cycles

## Cells for Floyd-Warshall Algorithm

- First try: $f[i,j]$ is length of shortest path from $i$ to $j$
- Issue: do not know in which order we compute $f[i,j]$'s
- $f^k[i,j]$: length of shortest path from $i$ to $j$ that only uses vertices $\{1, 2, 3, \cdots, k\}$ as intermediate vertices

$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

- $f^k[i,j]$: length of shortest path from $i$ to $j$ that only uses vertices $\{1, 2, 3, \cdots, k\}$ as intermediate vertices

$$f^k[i,j] = \begin{cases} w(i,j) & k = 0 \\ \min \begin{cases} f^{k-1}[i,j] \\ f^{k-1}[i,k] + f^{k-1}[k,j] \end{cases} & k = 1, 2, \cdots, n \end{cases}$$

**Floyd-Warshall$(G, w)$**

$f^0 \leftarrow w$
**for** $k \leftarrow 1$ to $n$ **do**
  copy $f^{k-1} \rightarrow f^k$
  **for** $i \leftarrow 1$ to $n$ **do**
    **for** $j \leftarrow 1$ to $n$ **do**
      **if** $f^{k-1}[i,k] + f^{k-1}[k,j] < f^k[i,j]$ **then**
        $f^k[i,j] \leftarrow f^{k-1}[i,k] + f^{k-1}[k,j]$

**Floyd-Warshall$(G, w)$**

$f^{\text{old}} \leftarrow w$
**for** $k \leftarrow 1$ to $n$ **do**
  copy $f^{\text{old}} \rightarrow f^{\text{new}}$
  **for** $i \leftarrow 1$ to $n$ **do**
    **for** $j \leftarrow 1$ to $n$ **do**
      **if** $f^{\text{old}}[i,k] + f^{\text{old}}[k,j] < f^{\text{new}}[i,j]$ **then**
        $f^{\text{new}}[i,j] \leftarrow f^{\text{old}}[i,k] + f^{\text{old}}[k,j]$

**Lemma** Assume there are no negative cycles in $G$. After iteration $k$, for $i, j \in V$, $f[i,j]$ is exactly the length of shortest path from $i$ to $j$ that only uses vertices in $\{1, 2, 3, \cdots, k\}$ as intermediate vertices.

- Running time $= O(n^3)$.

# Recovering Shortest Paths

**Floyd-Warshall$(G, w)$**

$f \leftarrow w$, $\pi[i,j] \leftarrow \perp$ for every $i, j \in V$
**for** $k \leftarrow 1$ to $n$ **do**
  **for** $i \leftarrow 1$ to $n$ **do**
    **for** $j \leftarrow 1$ to $n$ **do**
      **if** $f[i,k] + f[k,j] < f[i,j]$ **then**
        $f[i,j] \leftarrow f[i,k] + f[k,j]$, $\pi[i,j] \leftarrow k$

**print-path$(i, j)$**

**if** $\pi[i,j] = \perp$ **then** then
  **if** $i \neq j$ **then** print$(i, \text{“,”})$
**else**
  print-path$(i, \pi[i,j])$, print-path$(\pi[i,j], j)$

# Detecting Negative Cycles

**Floyd-Warshall$(G, w)$**

$f \leftarrow w$, $\pi[i,j] \leftarrow \perp$ for every $i, j \in V$
**for** $k \leftarrow 1$ to $n$ **do**
  **for** $i \leftarrow 1$ to $n$ **do**
    **for** $j \leftarrow 1$ to $n$ **do**
      **if** $f[i,k] + f[k,j] < f[i,j]$ **then**
        $f[i,j] \leftarrow f[i,k] + f[k,j]$, $\pi[i,j] \leftarrow k$
**for** $k \leftarrow 1$ to $n$ **do**
  **for** $i \leftarrow 1$ to $n$ **do**
    **for** $j \leftarrow 1$ to $n$ **do**
      **if** $f[i,k] + f[k,j] < f[i,j]$ **then**
        report "negative cycle exists" and exit

# Summary of Shortest Path Algorithms

| algorithm | graph | weights | SS? | running time |
|---|---|---|---|---|
| Simple DP | DAG | $\mathbb{R}$ | SS | $O(n+m)$ |
| Dijkstra | U/D | $\mathbb{R}_{\geq 0}$ | SS | $O(n \log n + m)$ |
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- DAG $=$ directed acyclic graph U $=$ undirected D $=$ directed
- SS $=$ single source AP $=$ all pairs