Optimization

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```
library(latex2exp)
library(lpSolve)
```

Exercise 1:

Assume that :

$$A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

Is **A** positively definite? Are there values of b that matrix **B** is positively definite? If yes what are those values?

Solution:

We know that a symmetrical matrix $M_{k \times k} \quad k = 1, \dots, n$ is positively definite if

1. All the eigen values are positive.
2. All the n upper left determinants of a $M_{k \times k}$ are positive
3. All the drivers $d_i, \quad \forall i = 1, \dots, n$ in the Gaussian Elimination without interchanging of lines are positive.

In order to check whether A is positive definite, we are going to calculate all the upper left determinants. As a result we have the following observations. The first term $m_{1 \times 1}$ is positive.

$$3 > 0$$

Furthermore, we can see that the next *left-upper* determinant is the determinant of the 2×2 submatrix, as shown below, which is positive as well.

$$\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3 > 0$$

Finally what is left is the determinant of the full 3×3 matrix A which as shown below, positive.

$$\begin{vmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \\ 3 & -2 & 8 \end{vmatrix} =$$

$$3 \cdot 1 \cdot 8 + 0 \cdot (-2) \cdot 3 + 3 \cdot 0 \cdot (-2) - 3 \cdot 1 \cdot 3 - (-2) \cdot (-2) \cdot 3 - 8 \cdot 0 \cdot 0 =$$

$$24 + 0 + 0 - 9 - 12 - 0 = 3 > 0$$

In conclusion, we can see that matrix A is **positive define**. We are going to follow the exact same methodology in order to answer to the second part of this exercise. Initially we observe that $2 > 0$ and as a result we move forward and calculate the next determinant of the upper-left submatrix $M_{2\times2}$.

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2 \cdot 2 - (-1) \cdot (-1) = 4 - 1 = 3 > 0$$

The determinant of the submatrix is a positive number. What is left is to calculate the determinant of the full initial matrix, containing the unknown b. What we need is B to be a positive definite. This means that:

$$\begin{vmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{vmatrix} =$$

$$2 \cdot 2 \cdot 2 + (-1) \cdot (-1) \cdot b + b \cdot (-1) \cdot (-1) - b \cdot 2 \cdot b - (-1) \cdot (-1) \cdot 2 - 2 \cdot (-1) \cdot (-1) =$$

$$8 + b + b - 2b^2 - 2 - 2 = -2b^2 + 2b + 4$$

As a result we have to find all those b that the equation we ended up. If we follow the usual steps of solving a second order equation we have that:
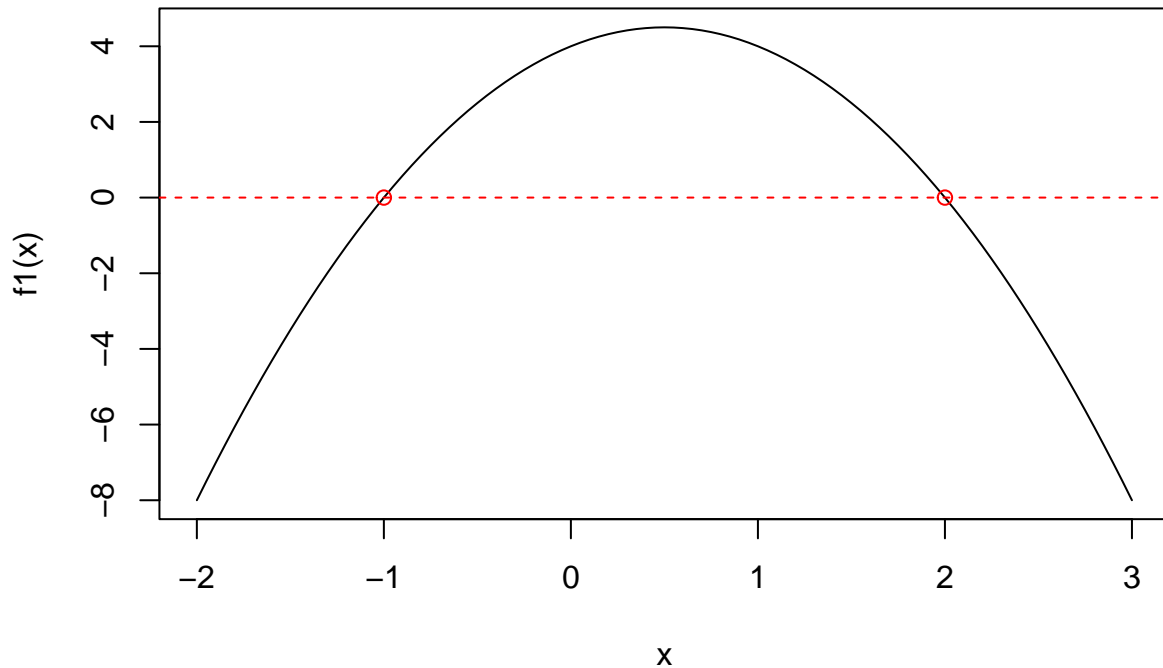
$$-2b^2 + 2b + 4 = 2(-b - 1)(b - 2)$$

which means that the b that make this equation equal to zero are $b = -1$ and $b = 2$. We know that the equation is going to be positive $\forall b \in (-1, 2)$ and negative $\forall b \in (-\infty, -1) \cup (2, \infty)$. This is something one can observe in the following graph.

```r
f1 <- function(x){
        return(-2*x^2+2*x+4)
}

curve(expr = f1, from = -2, to = 3)
points(x = c(-1,2), y = c(0,0), col = 'red')
abline(h = 0, lty = 2, col = 'red')
title(main = TeX(r'(\textbf{Equation} : $-2x^2+2x+4$)'))
```

Equation : $-2x^2 + 2x + 4$

As a result the solution is the following

$$B \text{ positive-definite} \Leftrightarrow \forall b \in (-1, 2)$$

Exercise 2:

Find all the critical points and say whether they are local minimum, local maximum or saddle points, of the following functions:

$$f(x, y) = x^2 + 2y^2 - x$$
$$g(x, y) = x^3 + y^3 - 9xy + 27$$
$$h(x, y) = 2x^3 + y^3 - 3x^2y - 6y$$

Solution:

We know that a point (x^*, y^*) is critical if $\bigtriangledown f(x^*, y^*) = (0, 0)$. After finding a critical point, we can check whether the Hessian Matrix at this point is positive definite or negative definite. If the matrix is positive definite then the point is local minimum. On the other hand, if the Hessian Matrix is negative definite, then it is local maximum. Now we can start by finging the partial derivative in order to find out the critical point(s)

$$\partial_x f(x, y) = 2x - 1 \quad \text{and} \quad \partial_y f(x, y) = 4y$$

One can easily check that

$$\partial_x f(x, y) = 0 \Leftrightarrow x = \frac{1}{2}$$

$$\partial_y f(x, y) = 0 \Leftrightarrow y = 0$$

So function f has one critical point $(x^*, y^*) = (\frac{1}{2}, 0)$. Moving forward we have to calculate the Hessian matrix in order to check whether the point is local minimum, local maximum or saddle point. We know that

$$\begin{bmatrix} \partial_{x^2}^2 f(x, y) & \partial_{xy}^2 f(x, y) \\ \partial_{yx}^2 f(x, y) & \partial_{y^2}^2 f(x, y) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

We can easily see that this is a positive definite matrix and as a result the critical point is local minimum. Moving on we have function $g(x, y)$.

$$\partial_x g(x, y) = 3x^2 - 9y \quad \text{and} \quad \partial_y f(x, y) = 3y^2 - 9x$$

We can solve the equations above like a system, meaning:

$$\begin{cases} 3x^2 - 9y = 0 \\ 3y^2 - 9x = 0 \end{cases} \Rightarrow \begin{cases} x^2 - 3y = 0 \\ y^2 - 3x = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{x^2}{3} \\ y^2 - 3x = 0 \end{cases} \Rightarrow$$

$$(\frac{x^2}{3})^2 - 3x = 0 \Rightarrow x^4 - 27x = 0 \Rightarrow x(x^3 - 27) = 0 \Rightarrow$$

$$x = 0 \quad \text{or} \quad x^3 = 27 \Rightarrow$$

$$x = 0 \quad \text{or} \quad x = \sqrt[3]{27} \Rightarrow$$

$$x = 0 \quad \text{or} \quad x = 3$$

Let's assume that $x = 0$ then by inserting this in the next equation we have that $y = 0$. So we have the first critical point which is $(x_1^*, y_2^*) = (0, 0)$.
On the other hand, let's assume that $x = 3$, then we can see that $y = 3$. By those steps, we ended up in a second critical point which is $(x_2^*, y_2^*) = (3, 3)$. In order to check and identify what kind of critical points those are, we need to calculate the Hessian determinant in the points we just found. The Hessian matrix can be calculated as follows:

$$H(x^*, y^*) = \begin{vmatrix} \partial_{x^2}^2 g(x^*, y^*) & \partial_{xy}^2 g(x^*, y^*) \\ \partial_{yx}^2 g(x^*, y^*) & \partial_{y^2}^2 g(x^*, y^*) \end{vmatrix} = \begin{vmatrix} 6x^* & -9 \\ -9 & 6y^* \end{vmatrix}$$

So for the first critical point, we have the following result

$$H(0, 0) = \begin{vmatrix} 0 & -9 \\ -9 & 0 \end{vmatrix} = 0 \cdot 0 - (-9) \cdot (-9) = -81 < 0$$

We ended up that $H(0, 0) < 0$ meaning that this is a local maximum point. Now let us move forward and check the secont critical point.

$$H(3, 3) = \begin{vmatrix} 6 \cdot 3 & -9 \\ -9 & 6 \cdot 3 \end{vmatrix} = \begin{vmatrix} 18 & -9 \\ -9 & 18 \end{vmatrix} = 324 - 81 = 243 > 0$$

As we can see, $H(3,3) = 243$ is greater than zero. This implies that the critical point is local minimum. The last but not least function is

$$h(x, y) = 2x^3 + y^3 - 3x^2y - 6y$$

As you may have already thought, we are going to follow the exact same methodology.

$$\partial_x h(x, y) = 6x^2 - 6xy \quad \text{and} \quad \partial_y h(x, y) = 3y^2 - 3x^2 - 6$$

In order to find the critical points, we solve the following system.

$$\begin{cases} 6x^2 - 6xy = 0 \\ 3y^2 - 3x^2 - 6 = 0 \end{cases} \Rightarrow \begin{cases} 6x(x - y) = 0 \\ 3y^2 - 3x^2 - 6 = 0 \end{cases} \Rightarrow$$

From the first equation we have that $6x = 0 \Rightarrow x = 0$ or $x = y$.

Initially, let us assume that $x = 0$ from the second equation we have that

$$3y^2 - 6 = 0 \Leftrightarrow y^2 - 2 = 0 \Leftrightarrow y = \pm\sqrt{2}$$

Which implies that the critical points obtained are $(x_1^*, y_1^*) = (0, \sqrt{2})$ and $(x_2^*, y_2^*) = (0, -\sqrt{2})$. Now let us assume that $y = x$ and use this information in the second equation, we have that

$$3x^2 - 3x^2 - 6 = 0 \Leftrightarrow -6 = 0$$

Which is a not possible result. In order to continue and describe the kind of the critical points we have found $(x^*, y^*) = (0, \pm\sqrt{2})$ we have to define the Hessian matrix and calculate its determinant for each of the critical point.

$$H(x^*, y^*) = \begin{vmatrix} \partial_{x^2}^2 h(x^*, y^*) & \partial_{xy}^2 h(x^*, y^*) \\ \partial_{yx}^2 h(x^*, y^*) & \partial_{y^2}^2 h(x^*, y^*) \end{vmatrix} = \begin{vmatrix} 12x - 6y & -6x \\ -6x & 6y \end{vmatrix}$$

So moving forward to calculating the Hessian matrix of the critical points we have the following calculations

$$H(0, \sqrt{2}) = \begin{vmatrix} 12 \cdot 0 - 6 \cdot \sqrt{2} & -6 \cdot 0 \\ -6 \cdot 0 & 6 \cdot \sqrt{2} \end{vmatrix} = \begin{vmatrix} -6\sqrt{2} & 0 \\ 0 & 6\sqrt{2} \end{vmatrix} = -72 < 0$$

We ended up that $H(0, \sqrt{2}) = -72 < 0$ which implies that the critical point is local maximum. On the other hand we have that

$$H(0, -\sqrt{2}) = \begin{vmatrix} 12 \cdot 0 - 6 \cdot -\sqrt{2} & -6 \cdot 0 \\ -6 \cdot 0 & 6 \cdot -\sqrt{2} \end{vmatrix} = \begin{vmatrix} 6\sqrt{2} & 0 \\ 0 & -6\sqrt{2} \end{vmatrix} = -72 < 0$$

Again as the previous critical point we see that the determinant of $H(x_2^*, y_2^*)$ results to -72 which less than zero and this implies that the critical point is a local maximum as well.

Exercise 3:

Find all the extreme values of the functions

- a. $f(x, y, z) = x^2 + y^2 + z^2 + 4yz$ under the constraint that $x + y + z = 0$ and $x + y = 1$ with the method of Lagrange multipliers

- b. $f(x_1, x_2, x_3) = 2x_1^2 - 3x_1x_2 + x_2^2 + x_1x_3 - 2x_2x_3 + 3x_3^2$ under the constraint that $x_1 + x_2 - x_3 = 0$

Figure out if the point is minimal, maximal or a saddle point.

Solution:

Lagrange multipliers are used to help us solve constrained optimization problems, as we have to face in this exercise. The basic structure of a Lagrange multiplier problem with one constraint and two variables x, y is the following:

$$\mathcal{L}(x, y; \lambda) = f(x, y) + \lambda g(x, y)$$

where $f(x, y)$ is the function to optimize under the constraint $g(x, y)$, and λ is the Lagrange multiplier. In the next step, we set $\nabla \mathcal{L} = \nabla f + \lambda \nabla g = 0$ to solve the resulting system of equations. In our case we have the following:

$$\mathcal{L}(x, y, z; \lambda_1, \lambda_2) = f(x, y, z) + \lambda_1 g_1(x, y, z) + \lambda_2 g_2(x, y, z) \Leftrightarrow$$
$$\mathcal{L}(x, y, z; \lambda_1, \lambda_2) = x^2 + y^2 + z^2 + 4yz + \lambda_1(x + y + z) + \lambda_2(x + y - 1)$$

The partial derivatives of the Lagrange

$$\nabla \mathcal{L}(x, y, z; \lambda_1, \lambda_2) = \nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z)$$

results in the following system of equations

$$\begin{cases} \nabla_x \mathcal{L} = 0 \\ \nabla_y \mathcal{L} = 0 \\ \nabla_z \mathcal{L} = 0 \\ g_1(x, y, z) = 0 \\ g_2(x, y, z) = 0 \end{cases} \Rightarrow \begin{cases} 2x + \lambda_1 + \lambda_2 = 0 \\ 2y + 4z + \lambda_1 + \lambda_2 = 0 \\ 2z + 4y + \lambda_1 = 0 \\ x + y + z = 0 \\ x + y - 1 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} \lambda_2 = -2x - 2z - 4y \\ 2y + 4z + \lambda_1 + \lambda_2 = 0 \\ \lambda_1 = -2z - 4y \\ x + y + z = 0 \\ x + y = 1 \end{cases}$$

combining the first three equations, we end up

$$\begin{cases} -6y - 2x = 0 \\ x + y + z = 0 \\ x + y = 1 \end{cases}$$

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By solving the last system of equations we end up in the solution

$$(x^*, y^*, z^*) = (\frac{3}{2}, -\frac{1}{2}, -1)$$

In order to figure out whether this point is minimum, maximum or a saddle point, we can use the determinant of the Hessian matrix.

$$H(x^*, y^*, z^*) = \begin{bmatrix} \partial_{x^2}^2 f & \partial_{xy}^2 f & \partial_{xz}^2 f \\ \partial_{yx}^2 f & \partial_{y^2}^2 f & \partial_{yz}^2 f \\ \partial_{zx}^2 f & \partial_{zy}^2 f & \partial_{z^2}^2 f \end{bmatrix}$$

We already, have calculated that:

$$\partial_x(f)(x, y, z) = 2x, \quad \partial\partial_y(f)(x, y, z) = 2y + 4z, \quad \partial\partial_z(f)(x, y, z) = 2z + 4y$$

Moving forward we have that:

$$|H(x^*, y^*, z^*)| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{vmatrix}$$

All we now to do is to calculate all the upper-left determinants. We can see that the first argument at posision 1×1 is $2 > 0$. Then as a second step, we need to check the determinant of the submatrix $H_{2 \times 2} \subset H_{3 \times 3}$. Hence we see that

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2 \cdot 2 - 0 \cdot 0 = 4 > 0$$

We see that the determinant is positive as well. Finally, we have that

$$|H(x^*, y^*, z^*)| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{vmatrix} =$$

$$2 \cdot 2 \cdot 2 + 0 \cdot 4 \cdot 0 + 0 \cdot 0 \cdot 4 - 0 \cdot 2 \cdot 0 - 4 \cdot 4 \cdot 2 - 2 \cdot 0 \cdot 0 = 8 - 32 = -24 < 0$$

As we see, the determinant of the full Hessian matrix is negative. This implies that the Hessian matrix is negative definite and as a result, the point is maximum.

Moving forward to the second part of the exercise we have the following constrained problem.

$$2x_1^2 - 3x_1 x_2 + x_2^2 + x_1 x_3 - 2x_2 x_3 + 3x_3^2$$

under the constraint

$$x_1 + x_2 - x_3 = 0$$

Again we need to solve the system of equations created by the partial derivated using the lgrange multipliers.

$$\mathcal{L}(x_1, x_2, x_3) = \nabla f(x_1, x_2, x_3) + \lambda \nabla g(x_1, x_2, x_3)$$

$$\begin{cases} 4x_1 - 3x_2 + x_3 + \lambda = 0 \\ -3x_1 + 2x_2 - 2x_3 + \lambda = 0 \\ x_1 - 2x_2 + 6x_3 - \lambda = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$$

The steps toward the solution of this system of equations is the following. From the third equation we have that $\lambda = x_1 - 2x_2 + 6x_3$ and by place this in the first two equations we have the following system of equations:

$$\begin{cases} 5x_1 - 5x_2 + 7x_3 = 0 \\ -2x_1 + 4x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$$

With the help of the last equation we have that $x_3 = x_1 + x_2$ and by placing this in the other two equations we have that $x_1 = -2x_2$ and as a result, $x_3 = -x_2$. Show by combining all those findings with the first equation we have that

$$(x^*, x_2^*, x_3^*) = (0, 0, 0)$$

Once again we are going to use the Hessian matrix to decide if this point is maximum, minimum or saddle point. The Hessian Matrix is the

$$H(x_1, x_2, x_3) = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 2 & -2 \\ 1 & -2 & 6 \end{bmatrix}$$

We observe that

$$4 > 0$$

$$\begin{vmatrix} 4 & -3 \\ -3 & 2 \end{vmatrix} = 4 \cdot 2 - (-3) \cdot (-3) = 8 - 9 = -1 < 0$$

and finally

$$\begin{vmatrix} 4 & -3 & 1 \\ -3 & 2 & -2 \\ 1 & -2 & 6 \end{vmatrix} =$$

$$4 \cdot 2 \cdot 6 + (-3) \cdot (-2) \cdot 1 + 1(-3) \cdot (-2) - 1 \cdot 2 \cdot 1 - (-2) \cdot (-2) \cdot 4 - 6 \cdot (-3) \cdot (-3)$$

$$48 + 6 + 6 - 2 - 16 - 54 = -12 < 0$$

As we have seen, the Hessian matrix is negative definite and as a result, we can say that the point

$$(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$$

is a maximum.

_____

Exercise 4:

Assume you have the following constrained problem:

$$\begin{aligned}
\text{maximize} \quad & 6x_1 + 4x_2 + 7x_3 + 5x_4 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 + 2x_4 \le 20 \\
& 6x_1 + 5x_2 + 3x_3 + 2x_4 \le 100 \\
& 3x_1 + 4x_2 + 9x_3 + 12x_4 \le 75 \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}$$

    a. Solve the problem (by using R-Programming Language)
    b. Write the dual problem and solve it as well.

Solution:

```
f.obj <- c(6,4,7,5)
f.con <- matrix(c(1,2,1,2,6,5,3,2,3,4,9,12,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1), nrow = 7, byrow = TRUE)
f.dir <- c("<=","<=","<=",">=",">=",">=",">=")
f.rhs <- c(20,100,75,0,0,0,0)

result <- lp("max",f.obj,f.con, f.dir,f.rhs)
```

The optimal solution $(x_1^*, x_2^*, x_3^*, x_4^*)$:

```
result$solution
```

```
## [1] 15.000000  0.000000  3.333333  0.000000
```

and the optimal value for the objective function is :

```
result$objval
```

```
## [1] 113.3333
```

Moving forward, the dual problem of the

$$\begin{aligned}
\text{maximize} \quad & 6x_1 + 4x_2 + 7x_3 + 5x_4 \\
\text{subject to} \quad & x_1 + 2x_2 + x_3 + 2x_4 \le 20 \\
& 6x_1 + 5x_2 + 3x_3 + 2x_4 \le 100 \\
& 3x_1 + 4x_2 + 9x_3 + 12x_4 \le 75 \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}$$

is the following

$$\begin{aligned}
\text{minimize} \quad & 20y_1 + 100y_2 + 75y_3 \\
\text{subject to} \quad & y_1 + 6y_2 + 3y_3 \ge 6 \\
& 2y_1 + 5y_2 + 4y_3 \ge 4 \\
& y_1 + 3y_2 + 9y_3 \ge 7 \\
& 2y_1 + 2y_2 + 12y_3 \ge 5 \\
& y_1, y_2, y_3 \ge 0
\end{aligned}$$

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In order to solve the dual problem we are going to follow the exact same steps as we did for the primal. Meaning,

```
f.obj <- c(20,100,75)
f.con <- matrix(c(1,6,3,2,5,4,1,3,9,2,2,12,1,0,0,0,1,0,0,0,1), nrow = 7, byrow = TRUE)
f.dir <- c(">=",">=",">=",">=",">=",">=",">=")
f.rhs <- c(6,4,7,5,0,0,0)

result <- lp("min",f.obj,f.con, f.dir,f.rhs)
```

The optimal solution of the dual is the (y_1^*, y_2^*, y_3^*) :

```
result$solution
```

```
## [1] 0.0000000 0.7333333 0.5333333
```

and the optimal value for the objective function is :

```
result$objval
```

```
## [1] 113.3333
```

Exercise 5:

In order for Company AB to start production of two products, P_1 and P_2, for the next quarter, is considering obtaining a loan from the bank. Specifically, it deals with the problem of the amount of production of the two products as well as the amount of the loan. The purpose of the company is to maximize its total profits under the given production, financing and market conditions. Production and sales figures are given in the following table. The company also considers that the market can contain two practically unlimited quantities of the products at the selling prices shown in the table.

Product	Price	Production Cost	Wokring Hours per Sector		
			1	2	3
P_1	140	100	0.5	0.3	0.3
P_2	110	80	0.3	0.4	0.1
Available Working Hours			1800	1000	700

Production Plan's Data

In addition, the company must take into account the following data

1. The company's available funds for the next quarter are €300,000.
2. The bank grants a loan of up to €200,000 for three months with an interest rate of 5% per quarter, as long as the degree of immediate liquidity of the company, i.e. the ratio

$$\frac{\text{Available Funds} + \text{Accounts Receivable}}{\text{Accounts Payable}}$$

is at least 3 to 1 during the entire period in which the loan remains outstanding. 3. Payments for wages and materials are made at the end of the production period (the quarter) and therefore any credit needed is secured at that point in time. The products are shipped on credit at the end of the production period and finally the settlement of the accounts, i.e. the revenue after the sale and the payments for the outstanding liabilities are made at the end of the next period.

Formulate a linear programming model for the situation described above and solve it computationally using R-programming language.

Solution:

Assume that

- n_1 is the amount of product P_1 produced.
- n_2 is the amount of product P_2 produced.
- x is the amount of loan that company will acquire from the bank.

The objective function that company AB is trying to maximize is the following,

$$max \quad 140n_1 + 110n_2 - 100n_1 - 80n_2 - \frac{105}{100}x$$

What the function above tells us is that the company tries to maximize their Profit which is the amount of money obtained by their sells - the cost that they had to pay - the loan that they have to repay (plys the 5% interest). After making some adjastments, we can see that

$$max \quad 40n_1 + 30n_2 - \frac{105}{100}x$$

But as you may have guessed, this can not be left with out some constraints. After, carefully reading the data given in the exercise, we end up in the following constrained linear problem.

$$
\begin{aligned}
\text{maximize} \quad & 40n_1 + 30n_2 + \frac{105}{100}x \\
\text{subject to} \quad & 0.5n_1 + 0.3n_2 \leq 1800 \\
& 0.3n_1 + 0.4n_2 \leq 1000 \\
& 0.3n_1 + 0.1n_2 \leq 700 \\
& 100n_1 + 80n_2 - x \leq 300000 \\
& 160n_1 + 130n_2 - x \leq 300000 \\
& x \leq 200000 \\
& n_1, n_2, x \geq 0
\end{aligned}
$$

Let us see, one by one, all the constraints. The first three inequalities are ones obtained with the help of the table. To be more specific those are the sector's available working hours. The fourth one is the cost constraint. Meaning that the company's, concerning the production of P_1 and P_2 can not overcome this amount. The fifth one is the constraint is one set by the bank in order to be able to obtain the loan, in case this is wise. Lastly, x is the constraint, the maximum amount of money the company can obtain by the bank. In the last line, we set the constraints that the minimum amount of n_1, n_1, x can be 0 (non-negative). So in order to use R-Programming language in order to solve this problem, all we have to do is use the packages *lpSolve* and its functions in order to solve this constrained linear problem.

```
f.obj <- c(40,30,-105/100)
f.con <- matrix(c(0.5,0.3,0,0.3,0.4,0,0.3,0.1,0,100,80,-1,160,130,-1,0,0,1,1,0,0,0,1,0,0,0,1),
                nrow = 9, byrow = TRUE)
f.dir <- c("<=","<=","<=","<=","<=","<=",">=",">=",">=")
f.rhs <- c(1800,1000,700,300000,300000,200000,0,0,0)

result <- lp("max",f.obj,f.con,f.dir,f.rhs, all.int =TRUE)
```

As a result, the solution is

```
result$solution
```

```
## [1] 1875    0    0
```

As we have seen, the optimal solution is obtained when no loan is taken from the bank, according to this linear problem. We have obtained $(n_1^*, n_2^*, x^*) = (1875, 0, 0)$ as optimal solution. In order to interpret this result, we can say that, company AB will produce 1875 units of product P_1, 0 units of P_2 by taking 0 amount of loan from bank.

```
result$objval
```

```
## [1] 75000
```

As we can see, the profit of the company by following the strategy proposed above, is going to be 75000.

Exercise 6:

Suppose we have €250,000 which we intend to invest in some of the additional investments. The following table describes the cash flows resulting from the investments.

Table 1: Cash Flows

	Year 1	Year 2	Year 3	Year 4
Investment 1	-1.00		1.18	
Investment 2		-1.00		1.22
Investment 3			-1.00	1.10
Investment 4	-1.00	0.14	0.14	1.00
Investment 5		-1.00	0.20	1.00

For example, if we invest €1 in Investment 1 at the beginning of the first year, we will get €1.18 at the beginning of the third year. Investment 2 is available at the beginning of second year and a return of 1.22 in the fourth year. Available funds that have not been placed in one of the investments at the beginning of a year can be placed in a term bank account with a return of 3% per annum. Any of 5 investment schemes requires an investment amount of at least €100,000.

Formulate the problem of optimal placement of the amount of money for the 4-year horizon as a Linear-Integer programming problem and solve it computationally (e.g. R).

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Solution:

In order to solve the problem, we need to define the following:

- x_1 : is the amount of money we give in the *first* investment plan in the *first* year.
- x_2 : is the amount of money we give in the *second* investment plan in the *second* year.
- x_3 : is the amount of money we give in the *third* investment plan in the *third* year.
- x_4 : is the amount of money we give in the *fourth* investment plan in the *first* year.
- x_5 : is the amount of money we give in the *fifth* investment plan in the *second* year.
- x_6 : is the amount of money we give in the bank in the *first* year.
- x_7 : is the amount of money we give in the bank in the *second* year.
- x_8 : is the amount of money we give in the bank in the *third* year.

The objective of this problem is to obtain at the end, the largest possible return of our initial funds. As a result, we need to

$$\text{maximize} \quad 1.18x_1 + 1.22x_2 + 1.10x_3 + 0.14^2 x_4 + 0.2x_5 + 1.03x_8$$

But this has to be completed according to specific constraints. According to the data provided in the exercise, the full problem is going to be

$$
\begin{aligned}
\text{maximize} \quad & 1.18x_1 + 1.22x_2 + 1.10x_3 + 0.14^2 x_4 + 0.2x_5 + 1.03x_8 \\
\text{subject to} \quad & x_1 + x_4 + x_6 \leq 250000 \\
& x_1 + x_2 + 0.86x_4 + x_5 - 0.03x_6 + x_7 \leq 250000 \\
& -0.18x_1 + x_2 + x_3 + 0.9804x_4 + 0.8x_5 + x_6 + -0.03x_7 + x_8 \leq 250000 \\
& x_1, ..., x_8 \geq 0
\end{aligned}
$$

The solution to the problem provided above, with the help of R-programming (and the package *lpSolve*) the following:

```
f.obj <- c(1.18,1.22,1.10,0.14^2,0.2,0,0,1.03)
f.con <- matrix(c(1,0,0,0,0,0,0,0,
                  0,1,0,0,0,0,0,0,
                  0,0,1,0,0,0,0,0,
                  0,0,0,1,0,0,0,0,
                  0,0,0,0,1,0,0,0,
                  0,0,0,0,0,1,0,0,
                  0,0,0,0,0,0,1,0,
                  0,0,0,0,0,0,0,1,
                  1,0,0,1,0,1,0,0,
                  1,1,0,0.86,1,-0.03,1,0,
                  -0.18,1,1,0.9804,0.8,1,-0.03,1),
                nrow = 11, byrow = TRUE)
f.dir <- c(">=",">=",">=",">=",">=",">=",">=",">=","<=","<=","<=")
f.rhs <- c(0,0,0,0,0,0,0,0,250000,250000,250000)

result <- lp("max",f.obj,f.con,f.dir,f.rhs)
```

The optimal solution $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*)$

```
result$solution
```

```
## [1] 250000      0 295000      0      0      0      0      0
```

And the optimal amount of money that one should obtain by following the strategy provided above is

```
result$objval
```

```
## [1] 619500
```