# Financial Mathematics

## Bonds

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Bonds (fixed income securities) are a fundamental class of assets related to debt

In this lecture we will introduce fundamental concepts such as

- the yield and the yield curve,
- futures and futures returns
- stochastic models for yields

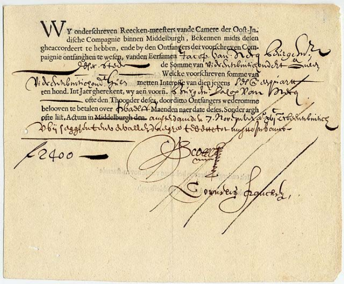
# What is a bond?

A bond is essentially a loan:

- The debtor issues (today) in the market contracts which guarantee future payments (either one off $F$ or in terms of coupons $C$) for a price
- The total sum collected today is the loan that the deptor has amassed (in return for the future payments)
- The interest on the loan can be internally calculated in terms of the connection of the price and the future payments (disounting)

Bonds are not personalized and can be traded at any time in well organized markets (bond markets)

Bonds are a rather old type of asset – essentially traced back to the 17th century and used to fund various causes

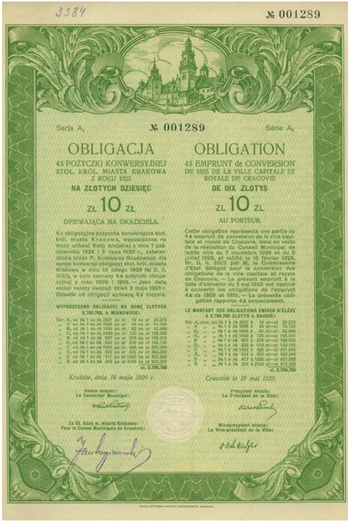


WY onderschreven Reecken-meesters vande Camere der Oost-Indische Compagnie binnen Middelburgh, Bekennen midts desen gheaccordeert te hebben, ende by den Ontfangers der voorschreven Compaignie ontfanghen te wesen, vanden Eersamen [illegible] de Somme van [illegible] Welcke voorschreven somme van [illegible] metten Interesse van dien jegens [illegible] ten hond. Int Jaer gherekent, wy aen voorn. [illegible] ofte den Thoonder deses, door ditto Ontfangers wederomme belooven te betalen over [illegible] Maenden naer date deses. Sonder argh ofte list. Actum in ~~Middelburgh den.~~ [illegible]

f 2400

Bond of the West Indies Company 1623


Bond of the State of South Carolina 1888




Bond of the Crakow Municipality (Poland) 1929

Characteristics of bonds are:

- The maturity $T$
- The principal or face value: The sum payable to the holds at maturity
- The coupons: Intermediate payments
- Its price $P$ at any time $t$ before maturity

The above characteristics can be used to calculate the yields of the bond and various risk measures (e.g. duration or convexity)

# Price and yield

Suppose that you buy a bond of maturity $T$ at $t = 0$ for price $P$ and you keep it till expiry.

Discounting would give

$$P = \sum_{i=1}^{n} \frac{c_i}{(1+y)^i} + \frac{P_p}{(1+y)^n} \tag{1}$$

where

- $P_p$ is the principal of the bond
- $c_i$ are the coupons p.a.
- $P$ is the price of the bond in the market at time $t = 0$

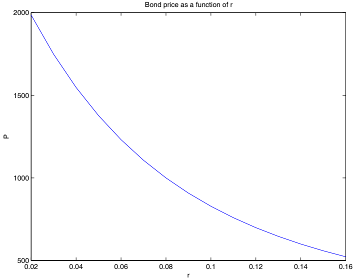
The discount factor $y$ that satisfies this equation is called the yield of the bond: This is implicitly defined by the relation between the payments of the bond and the price that the bond achieves in the market.

This equation can be only be solved numerically for $y$

## Example (How does price and yield vary)

Consider a bond of maturity $T = 20y$ paying coupons biannually of 8% of its principal value which is $P_p = 1000$.

Perform the following thought experiment: If you could affect the yield $y$ of the bond (which you can't!) how would this affect the price $P$ of the bond?




If yields increase prices will fall: This is the basic type of risk a bond portfolio will face

In actual practice what we know is the price and the payoff structure of the bond: From these we calculate the yield

# Calculating the yield

Suppose that you known the price $P$ of the bond (observed in the market today) and you know the characteristics of the bond.

Define the function

$$f(x) = P - \sum_{i=1}^{T} \frac{c_i}{(1+x)^i} - \frac{P_p}{(1+x)^T}$$

To find the yield of the bond $y$ you must solve the equation $f(x) = 0$.

This can be done numerically, e.g. by the Newton-Raphson scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For zero coupon bonds $c_i = 0$ we can get an analytic expression for the yield

$$P(t,T) = \frac{P_p}{(1+y)^{T-t}} \iff y(t,T) = \left(\frac{1}{P}\right)^{\frac{1}{T-t}} - 1.$$

In the case of continuous discounting

$$P(t,T) = P_p \exp(-(T-t)y) \iff y(t,T) = -\frac{\ln(P/P_p)}{T-t}.$$

Often we simply use as a variable the time to maturity $\tau = T - t$.

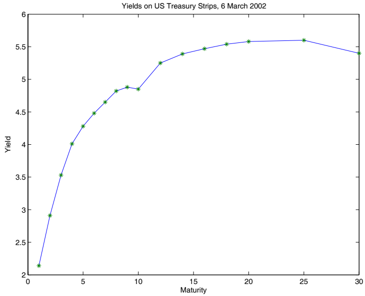
The short rate

$$r(t) = \lim_{T \to t} y(t,T) = y(t,t) \iff P(t,T) = \exp\left(-\int_t^T r(s)\,ds\right)$$

# The yield curve: Term structure

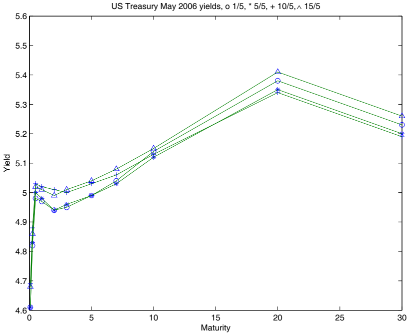
In a market at each time instance $t$ more than one bonds from the same issuer, with different maturities $T$ are in circulation: Each of them has a different price $P(t;T)$

The curve that plots $y(t,T)$ versus $T$ for a given time $t$, is called the yield curve for the market at time $t$



These prices change daily!

Yield curves change daily and in a stochastic fashion



# Futures and futures returns

Futures is a very popular type of contracts:

A typical type of futures contract is one where at time $t$ an investor pledges to invest 1 euro at time $T$ in return to the sum of $e^{(S-T)F(t,T,S)}$ at time $S$, where $t < T < S$.

A futures contract pre-determines at time $t$ the (future returns) between times $T$ and $S$ at level $F(t,T,S)$.

$F(t,T,S)$ is called the futures returns and must be related to the bond prices

$$F(t,T,S) = \frac{1}{S-T} \ln\left(\frac{P(t,T)}{P(t,S)}\right)$$

Why?

- Two different ways to guarantee $e^{(S-T)F(t,T,S)}$ at time $S$:
  1. Buy at time $t$ a quantity of $e^{(S-T)F(t,T,S)}$ zero coupon bonds of maturity $S$, each of price $P(t,S)$
  2. Buy at time $t$ a zero coupon bond of maturity $T$ of price $P(t,T)$ and use the payoff of 1 euro at $T$ from the bond to enter the futures contract

  No arbitrage guarantees that the two strategies should be equally valued!

# Related quantities

Instantaneous forward returns at time $t$ for investments at time $T$:

$$f(t,T) = \lim_{S \to T} F(t,T,S) = -\frac{\partial}{\partial T} \ln P(t,T) \iff \tag{2}$$

$$P(t,T) = \exp\left(-\int_t^T f(t,s)\,ds\right)$$

The short rate is related to the instantaneous forward returns by

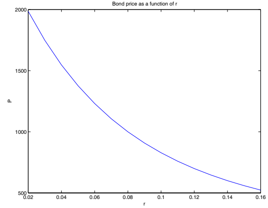
$$r(t) = f(t,t)$$

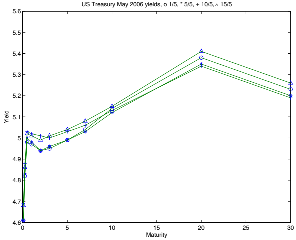
Since forward prices and returns are related to the bonds prices and returns, and the forwards market is a more liquid market, we often model the forwards market and use the above formulae for obtaining results for the bonds market.

# Risks associated to bonds

There are various risks associated with bonds (e.g. default, inflation etc).

However, the most important risk is associated to the fluctuations of the yields




- Increased yields → Drop of prices
- Decreased yields → Increased prices

# Duration

A common way to measure the sensitivity of the price of a bond with respect to the yields is duration:

Definition (Macaulay duration and modified duration)

For a coupon paying bond of maturity $T$

(a) The Macaulay duration $D$ is

$$D = \frac{1}{P}\left(\frac{1 \times c}{(1+y)} + \frac{2 \times c}{(1+y)^2} + \frac{3 \times c}{(1+y)^3} + \cdots + \frac{n \times (c + P_p)}{(1+y)^T}\right) \tag{3}$$

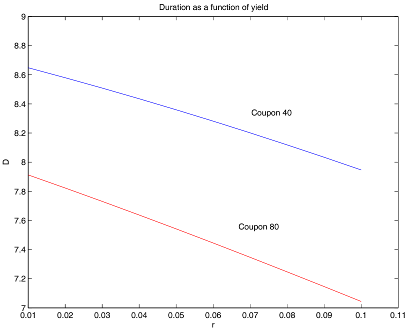
or equivalently

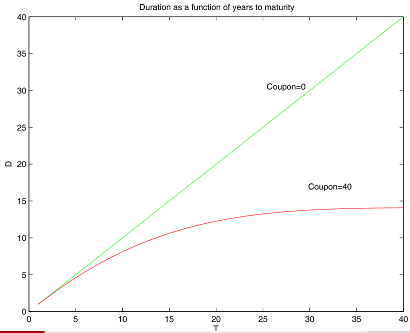
$$\frac{\partial P}{\partial y}\frac{1}{P} = -\frac{D}{1+y}. \tag{4}$$

(b) The modified duration is $D_m = \frac{D}{1+y}$

Qualitatively, the duration is a measure of the time required for a bond to "repay" the initial investment – the smaller the duration the safer the bond

- For $y$ constant: Increased coupons $\rightarrow$ Lower duration
- For zero coupon bonds $D = T$
- Increased yields $\rightarrow$ lower $D$.


Duration as a function of yield
Coupon 40
Coupon 80
D
r
9
8.8
8.6
8.4
8.2
8
7.8
7.6
7.4
7.2
7
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
0.11


Duration as a function of years to maturity
Coupon=0
Coupon=40
D
T
40
35
30
25
20
15
10
5
0
0
5
10
15
20
25
30
35
40

# Various interpretations of duration

A. Duration as an expected repayment time:

$$
\begin{aligned}
D &= \sum_{t=1}^{T} w_t t \\
w_t &= \frac{1}{P} \frac{c}{(1+y)^t} \\
w - T &= \frac{1}{P} \frac{c + P_p}{(1+y)^T},
\end{aligned}
$$

where the $w_t$ have the property

$$
\sum_{t=1}^{n} w_t = 1, \quad w_t > 0,
$$

i.e. can be interpreted as probabilities.

Hence, $D$ can be interpreted as the average time at which the bond repays the investment in terms of the intermediate coupon payments and the principal payment

B. Bonds as investment instruments:

We start from

$$P = \sum_{i=1}^{T} \frac{c_i}{(1+y)^i} + \frac{P_p}{(1+y)^T} \iff P(1+y)^T = \sum_{i=1}^{T} c_i (1+y)^{T-i} + P_p =: F_T(y)$$

- LHS: Future value of the price of the bond
- RHS: Payoffs from the coupon reinvestment in the market – $F_T(y)$ future value of all payments at yield $y$.

$$P(1+y)^T = F_T(y)$$

Horizon rate of return $r_H$:

- Assume that the investor buys the bond but does not intend to keep it until maturity $T$ but rather to sell it at some time $H < T$
- Up to time $H$ the investor will obtain payments from the coupons of total future value $F_H$
- The horizon rate of return $r_H$ is defined by

$$P(1+r_H)^H = F_H \iff r_H := \left(\frac{F_H}{P}\right)^{1/H} - 1$$

$F_H$ depends on the yields:

- If yields fall that future value of reinvestment will fall
- If yields rise then future value of reinvestment will rise

$P$ also depend on the yields:

- If yields fall then $P$ will rise
- If yields rise then $P$ will fall

Hence $r_H$ depends on $y$: Can we choose $H$ such that $r_H$ is insensitive to small changes in $y$?

Such a choice would be related to the duration

Example

Consider a bond of maturity 10 ye with principal 1000 euros paying annual coupons of 40 euros. Suppose that the yield curve shows a decrease from $y = 8\%$ to 7.8%.
What will be the change in the value of the bond?

A simple application of the formulae gives $D = 8.1184$ and $P = 731.5967$.

Hence

$$\Delta P = -D \, \frac{1}{1+y} \, P \, \Delta y = -8.1184 \times \frac{1}{1.08} \times 731.5967 \, (-0.002) = 10.9989,$$

i.e. we expect an increase in the price of the bond by roughly 11 euros.

Duration can be used to quantify the changes in bond prices due to small changes in the yields (linearized approximation).

This approximation is not sufficient if the changes in the yields are large – Convexity

**Definition (Convexity)**

Convexity for a bond is defined as

$$\Delta P = -D_m \, P \, \Delta y \, + \, \frac{1}{2} \, C \, P \, (\Delta y)^2.$$

The change of the price of a bond $\Delta P$ due to the yield change $\Delta y$ is given by


$$\Delta P = -D_m \, P \, \Delta y \, + \, \frac{1}{2} \, C \, P \, (\Delta y)^2.$$

Example

Calculate the convexity for a bond of maturity 10 years with principal at 100 euros paying coupons of 40 euros annually when the yields are at 8%
What will be the change in the bond price if the yields fall to 7/5%?

By direct calculation $P = 731.5967$, $D_m = 7.5171$, $C = 71.2235$.

Hence,

$$\begin{aligned} \Delta P &= -D_m \, P \, \Delta y + \frac{1}{2} \, C \, P \, (\Delta y)^2 \\ &= -7.5171 \times 731.5967 \, (-0.005) + \frac{1}{2} \times 71.2235 \times 731.5967 \times (-0.005)^2 = 28.1488 \end{aligned}$$

The exact calculation would give

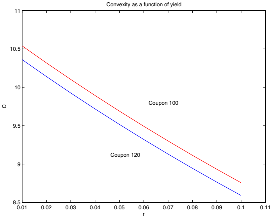
$$\Delta P = P(0.075) - P(0.8) = 28.1604$$

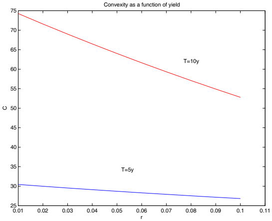
while the linear approximation would underestimate the change

$$(\Delta P)_{lin} = -D_m \, P \, \Delta y = 27.4974$$

The following hold:

- Low coupons $\rightarrow$ high convexity
- High maturities $\rightarrow$ high convexity
- High yields $\rightarrow$ low convexity



# Models for bond yields

When modelling bonds we may either model the yields, the short rate or the forward rates.

In the case of yields or forwards we must model continuous curves: e.g. $f(t, T)$ as a function of maturity $T$

Function space valued random variables and processes – Infinite dimensional stochastic processes

For the short rate we can just model the rate which is a real number – Real valued stochastic processes

# Parametric models: The Nelson-Siegel family

$$f_{NS}(t,\tau;\mathbf{z}) = z_1 + (z_2 + z_3\,\tau)\,e^{-z_4\,\tau}, \quad \tau = T - t$$

where $\theta = (z_1, z_2, z_3, z_4)$ are time dependent parameters

This model allows to reproduce various types of yield curves – convex as well as concave

The parameters $\theta$ can be estimated using various methodologies e.g. maximum likelihood.

This family of curves is used by various banks e.g. in Italy or Finland.

# Stochastic models

An alternative is to use stochastic models that will generate as random variables either the yields $y(t, T, \omega)$ or the short rate $r(t, \omega)$.

From the yields or the short rate we may generate the bond prices.

We may either used discrete or continuous time models e.g.

- The binomial model for bond yields (discrete)
- The Dothan model $r(t, \omega) \sim N(\alpha t, \beta t)$ for fixed $T$.
- The Vasiseck model
- etc

# The Vasicek model for the short rate

This is based on the Ornstein-Uhlenbeck process

$$dr(t) = a(\beta - r(t))dt + \sigma dW(t), \;\; a, b, \sigma > 0.$$

It can be shown that

$$\begin{aligned} P(t, T) &= \exp(A(T-t) - B(T-t)r(t)), \\ B(\tau) &= \frac{1 - e^{-a\tau}}{a}, \\ A(\tau) = (B(\tau) - \tau)\left(b - \frac{\sigma^2}{2a^2}\right) &= \frac{\sigma^2}{4a} B^2(\tau) \end{aligned}$$

Moreover,

$$R(t, T) = -\frac{1}{T-t} \ln P(t, T) = -\frac{A(T-t)}{T-t} + \frac{B(T-t)}{T-t} r(t)$$

The calculation of $P(t, T)$ is based on the relation (using the Markov property)

$$P(t,T) = \mathbb{E}[\exp(-\int_t^T r(s)ds) \mid \mathcal{F}_t] = \mathbb{E}[\exp(-\int_t^T r(s)ds) \mid r(t)],$$

Using Itō's lemma we can calculate this exponential moment in closed form, obtaining the above result.

Alternatively, assuming that $P(t,T) = V(T-t, r(t))$ for some function $V$ (Markov property) and applying Itō's lemma for the process $Y(t) = V(T-t, r(t))$ we see that

$$dY(t) = -\frac{\partial V(\tau, r)}{\partial \tau}dt + \frac{\partial V(\tau, r(t))}{\partial r}dr(t) + \frac{1}{2}\frac{\partial^2 V(\tau, r(t)}{\partial r^2}dr(t)^2$$
$$= \left(-\frac{\partial V}{\partial \tau} + (a(b-r))\frac{\partial V}{\partial r} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma^2\right)(\tau, r(t))dt + \frac{\partial V}{\partial r}(\tau, r(t))\sigma dW(t)$$

The expected change of $P(t, T, r(t))$ should satisfy $dP(t, T, r(T)) = r(t)P(t, T, r(t))$ hence $dE[Y(t)] = r(t)Y(t)$ and this implies

$$\left(-\frac{\partial V}{\partial \tau} + (a(b-r))\frac{\partial V}{\partial r} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\sigma^2\right) = rV(\tau, r)$$

The function $V$ solves the above PDE, and since $P(T, T) = 1$ it must hold that $V(0, r(T)) = 1$ for any $r(T)$.

Looking for solutions of the form

$$V(\tau, r) = \exp(A(\tau) - B(\tau)r).$$

where $A(\tau), B(\tau)$ are suitably chosen functions such that $B(0) = 0$ and $A(0) = 0$

Substituting this ansatz into the PDE we obtain 2 ODEs for A and B the solution of which provides the above results.

# The Cox-Ingersol-Ross model

According to this model

$$dr(r) = a(b - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

Then,

$$f(r(T) \mid r(t)) = c\,\chi^2_{2q+2,2u}(2c\,r(t)),$$

$$q = \frac{2ab}{\sigma^2} - 1,\ \ u = cr(t)e^{-a(T-t)},\ \ c = \frac{2a}{\sigma^2(1 - e^{-a(T-t)})},$$

where $\chi^2_{n,m}$ is the $\chi^2$ distribution.

It can be shown (using the properties of this distribution) that

$$\mathbb{E}[r(T) \mid r(t)] = r(t)e^{-a(T-t)} + b(1 - e^{-a(T-t)},$$

$$Var(r(T) \mid r(t)) = \frac{\sigma^2 r(t)}{a}(e^{-a(T-t)} - e^{-2a(T-t)}) + \frac{\sigma^2 b}{2a}(1 - e^{-a(T-t)})^2.$$

We can then compute the bonds prices as

$$
\begin{aligned}
P(t,T) &= \exp(A(T-t) - B(T-t)\,r(t)), \\
B(\tau) &= \frac{2(e^{\gamma\tau}-1)}{(\gamma+a)(e^{\gamma\tau}-1)+2\gamma}, \\
A(\tau) &= \frac{2ab}{\sigma^2}\ln\left(\frac{2\gamma e^{\frac{1}{2}(\gamma+a)\tau}}{(\gamma+a)(e^{\gamma\tau}-1)+2\gamma}\right), \\
\gamma &= \sqrt{a^2+2\sigma^2}
\end{aligned}
$$