Financial Mathematics

Introduction

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Financial Markets

- An economy consists of various interdependent markets, the most important of which are financial markets, as they are interconnected with any other market or individual.
- Financial markets offer firms and individuals the opportunity to materialize their investment plans and transfer value to the future.
- Financial markets are subject to risk and uncertainty which must be modelled in terms of stochastics.
- Stochastic finance is an exciting and rich academic field with a very strong two-way interaction with the market.

Definition

A financial asset is a contract allowing the bearer to have claims to a future (often uncertain) return or payoff which is (often) traded in organized markets called financial markets.
The value of an asset is the price for which this asset is traded in a market.

Examples of assets:

- Bonds (fixed income securities) – Trading debt
- Equity (stocks) – Trading future returns of a firm
- Derivatives – Trading future returns of other assets

Time and Value : Discounting

- An asset is a claim to a future payoff which must be traded now: Need to transfer value from the future to the present
- Discounting: The value of a yuan claimed tomorrow is lower today (Impatience?)
- Present value: The value today P of a future payoff F

$$P = \frac{F}{1+r} \iff F = (1+r)P$$

- r is called the discount factor
- Given any two assets: You can only compare like quantities
    - Present values with present values
    - Future values with future values

  Never mix them! (oranges and apples)

Variations on a theme:

- Present value: The value today P of a future payoff F (to be paid in N periods)

$$P = \frac{F}{(1+r)^N} \iff F = (1+r)^N P$$

- Present value in continuous discounting (idealization): The value today P of a future payoff F to be paid at time T:

$$P = e^{-rT} F \iff F = e^{rT} P$$

- r is the rate of discounting
- Recall that $\lim_{N\to\infty}(1+\frac{rT}{N})^N = e^{rT}$.

Example

Future value of 1 yuan in 10 years with discount factor 10% (fiction!)

$$FV = 1(1+r)^N = (1+10/100)^{10} = 2.59$$

Example

Present value of 1 yuan payable in 10 years with discount factor 10% (fiction!)

$$PV = 1(1+r)^{-N} = (1+10/100)^{-10} = 0.385$$

Example

Future value of 1 yuan payable each year for 10 years with discount factor 10% (fiction!) by the end of the 10th period

$$F = 1(1 + r) + 1(1 + 10/100)^2 + \cdots + 1(1 + 10/100)^{10} = 17.53$$

Example

Present value of 1 yuan payable each year (at the end of the year) for 10 years with discount factor 10% (fiction!) by the end of the 10th period

$$P = 1(1 + 10/100)^{-1} + \cdots + 1(1 + 10/100)^{-10} = 6.1446$$

Example

A firm contemplates entering an investment which is to be paid one off today, at cost $30,000$ yuan which is going to provide to the firm an annual turnover of $10,000$ yuan (payable at the end of year) for the next 10 years. Is this investment worthwhile?

The present value of the investment is

$$P = \sum_{t=1}^{10} \frac{10,000}{1.12^t} = 56,502$$

The net present value is

$$NP = P - C = 56,502 - 30,000 = 26,502 \;\; > 0!$$

Uncertainty

- In principle using discounting and present values we could price any asset from its future payoff.
- However, in practice future payoffs F are subject to uncertainty and hence not known today: We must use predictions for the future payoffs in terms of probabilistic models
- If the payoff of an asset is the random variable F, then the present value P is a candidate for the price of the asset today:

$$P = \frac{1}{1+r}\mathbb{E}[F]$$

- But the expectation (best estimate) comes with a probability measure concerning the distribution of F: Which is the proper probability measure to use?
  - Spoiler: Certainly not the statistical measure – Risk neutral measure (To be introduced shortly)

Arbitrage

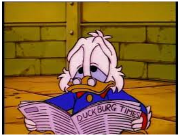
- Arbitrage is a very important concept in financial mathematics: It is an investment strategy (based on exploiting the fluctuations of the asset prices from their equilibrium values) that allows the clever investor to have returns without risk
- This is the dream of any investor, but will create disruptions in the market, hence financial mathematics place in important role in the absense of arbitrage as a pricing mechanism (martingale properties of asset prices, market efficiency etc)

Example (Arbitrage in the FX market)

Assume the following exchange rates in 3 markets:

$$
\begin{aligned}
1\,\textit{pound} &= 1.50\,\textit{dollars} \\
150\,\textit{yen} &= 1\,\textit{pound} \\
1\,\textit{dollar} &= 120\,\textit{yen}
\end{aligned}
$$

- Sell 1 pound for 1.50 dollars, use that to buy 180 *yen*, then spend 150 *yen* to buy 1 pound.
- Hence, you get back your 1 pound and have 30 yen as profit.
  - Repeating that we get a certain profit of 30 yen each time.
  - Constant disruption in the demand and supply of the different currencies will lead to change in the prices and hence loss of this arbitrage opportunity.



Can you find the right rates of exchange so that this market is in equilibrium (no arbitrage as a pricing scheme)

Example (Pricing a bond)

Assume a market consisting of a

- Bond (debt) with a future payoff of 1.
- A bank account with certain interest rate r

Use the above and a no arbitrage argument to find the price P of the bond today.

- Two assets in the market
  - Bond: Return $y = \frac{F-P}{P}$
  - Bank account: Return r.
- If $y \neq r$ then we have an arbitrage opportunity (why?)
- Absense of arbitrage

$$r = y = \frac{F-P}{P} \iff P = \frac{1}{1+r} F$$

Same as the present value approach but we arrived to that from a different route (generalizable under uncertainty!)