CHAPTER 10

Fourier Series

1. Introduction. 2. Euler's Formulae. 3. Conditions for a Fourier expansion. 4. Functions having points of discontinuity. 5. Change of interval. 6. Odd and even function—Expansions of odd or even periodic functions. 7. Half-range series. 8. Typical wave-forms. 9. Parseval's formula. 10. Complex form of F-series. 11. Practical Harmonic Analysis. 12. Objective Type of Questions.

10.1 INTRODUCTION

In many engineering problems, especially in the study of periodic phenomenae* in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. Most of the single-valued functions which occur in applied mathematics can be expressed in the form.

$$\tfrac{1}{2}a_0\dagger + a_1 \cos x + a_2 \cos 2x + \ldots\ldots\dagger$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots\ldots$$

within a desired range of values of the variable. Such a series is known as the **Fourier series$**.

10.2 EULER'S FORMULAE

The Fourier series for the function f(x) in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \quad \ldots(1)$$

These values of a_0, a_n, b_n are known as *Euler's formulae***.

***Periodic functions.** If at equal intervals of abscissa x, the value of each ordinate $f(x)$ repeats itself, *i.e.*, $f(x) = f(x + a)$, for all x, then $y = f(x)$ is called a *periodic function* having **period** α, *e.g.*, $\sin x$, $\cos x$ are periodic functions having a period 2π.

† To write $a_0/2$ instead of a_0 is a conventional device to be able to get more symmetric formulae for the coefficients.

$ Named after the French mathematician and physicist *Jacques Fourier* (1768–1830) who was first to use Fourier series in his memorable work *'Theorie Analytique de la Chaleur'* in which he developed the theory of heat conduction. These series had a deep influence in the further development of mathematics and mathematical physics.

**See footnote p. 205.

To establish these formulae, the following definite integrals will be required :

1. $\int_{\alpha}^{\alpha+2\pi} \cos nx \, dx = \left| \frac{\sin nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

2. $\int_{\alpha}^{\alpha+2\pi} \sin nx \, dx = -\left| \frac{\cos nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

3. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx \, dx$ $(n \neq 0)$

$$= \frac{1}{2} \int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x] \, dx$$

$$= \frac{1}{2} \left| \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right|_{\alpha}^{\alpha+2\pi} = 0 \quad (m \neq n)$$

4. $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx \, dx = \left| \frac{x}{2} + \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi$ $(n \neq 0)$

5. $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx \, dx = -\frac{1}{2} \left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n} \right] = 0$ $(m \neq n)$

6. $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx \, dx = \left| \frac{\sin^2 nx}{2n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

7. $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx \, dx = \frac{1}{2} \left| \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(m \neq n)$

8. $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx \, dx = \left| \frac{x}{2} - \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi.$ $(n \neq 0)$

Proof. Let $f(x)$ be represented in the interval $(\alpha, \alpha + 2\pi)$ by the Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(i)$$

To find the coefficients a_0, a_n, b_n, we assume that the series (i) can be integrated term by term from $x = \alpha$ to $x = \alpha + 2\pi$.

To find a_0, integrate both sides of (i) from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x) \, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) dx$$

$$= \frac{1}{2} a_0 (\alpha + 2\pi - \alpha) + 0 + 0 = a_0 \pi \quad \text{[By integrals (1) and (2) above]}$$

Hence $$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \, dx.$$

To find a_n, multiply each side of (i) by $\cos nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx \, dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \cos nx \, dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \cos nx \, dx$$

$$= 0 + \pi a_n + 0 \quad \text{[By integrals (1), (3), (4), (5) and (6)]}$$

Hence $$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx.$$

To find b_n, multiply each side of (i) by $\sin nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{1}{2}a_0\int_{\alpha}^{\alpha+2\pi}\sin nx\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\sin nx\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + \pi b_n \qquad \text{[By integrals (2), (5), (6), (7) and (8)]}$$

Hence $\qquad b_n = \dfrac{1}{\pi}\displaystyle\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx.$

Cor. 1. Making $\alpha = 0$, *the interval becomes* $0 < x < 2\pi$, and the formulae (I) reduce to

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \text{...(II)}$$

Cor. 2. Putting $\alpha = -\pi$, *the interval becomes* $-\pi < x < \pi$ and the formulae (I) take the form :

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \text{...(III)}$$

Example 10.1. *Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.* *(S.V.T.U., 2007)*

Solution. Let
$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \qquad \text{...}(i)$$

Then
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\Big| -e^{-x}\Big|_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$$

and
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\Big| e^{-x}(-\cos nx + n\sin nx)\Big|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{n^2+1}$$

$\therefore$
$$a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\ a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{5} \text{ etc.}$$

Finally,
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\ \sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\Big| e^{-x}(-\sin nx - n\cos nx)\Big|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{n}{n^2+1}$$

$\therefore$
$$b_1 = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{1}{2},\ b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{2}{5} \text{ etc.}$$

Substituting the values of a_0, a_n, b_n in (i), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\right)\right\}.$$

Example 10.2. *Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.*
(V.T.U., 2011 ; Madras, 2006)

Solution. Let $$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ...(i)$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx = \frac{1}{\pi}\left| \frac{x^2}{2} - \frac{x^3}{3} \right|_{-\pi}^{\pi} = -\frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos nx\, dx^*$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x) \times \left(-\frac{\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

$\therefore$ $a_1 = 4/1^2, a_2 = -4/2^2, a_3 = 4/3^2, a_4 = -4/4^2$ etc.

Finally, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1 - 2x) \times \left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi} = -2(-1)^n/n$$

$\therefore$ $b_1 = 2/1, b_2 = -2/2, b_3 = 2/3, b_4 = -2/4$ etc.

Substituting the values of a's and b's in (i), we get

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ...\right]$$
$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...\right]$$

Obs. Putting $x = 0$, we find another interesting series $0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\right)$

i.e., $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... = \frac{\pi^2}{12}.$$ *(V.T.U., 2011)*

Note. In the above example, we have used the results $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

Also $\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \frac{1}{2}\right)\pi = 0$. *The reader should remember these results.*

Example 10.3. *Expand $f(x) = x \sin x$ as a Fourier series in the interval $0 < x < 2\pi$.*
(S.V.T.U., 2009 ; Bhopal, 2009 ; Rohtak, 2006)

Solution. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ...(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} x \sin x\, dx = \frac{1}{\pi}\left| x(-\cos x) - 1.(-\sin x) \right|_0^{2\pi} = -2.$$

* Apply the *general rule of integration by parts* which states that if u, v be two functions of x and dashes denote differentiations and suffixes integrations w.r.t. x, then

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$$

In other words : *Integral of the product of two functions*

= 1st function × integral of 2nd – go on differentiating 1st, integrating 2nd signs alternately +ve and –ve.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\cos nx \sin x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,[\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right] = \frac{2}{n^2-1}.\ (n \neq 1)$$

When $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x\, dx$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = -\frac{1}{2}.$$

Finally, $b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(n-1)x - \cos(n+1)x]dx$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)}{n+1}\right\} - 1.\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0\ (n \neq 1)$$

When $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(1 - \cos 2x)\, dx$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi} = \pi$$

Substituting the values of a's and b's, in (i), we get

$$x \sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \ldots$$

Example 10.4. *Expand $f(x) = \sqrt{(1-\cos x)}$, $0 < x < 2\pi$ in a Fourier series. Hence evaluate*

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots$$ *(Mumbai, 2006 ; J.N.T.U., 2006)*

Solution. We have $f(x) = \sqrt{(1-\cos x)} = \sqrt{2}\sin x/2$.

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin x/2\, dx = \frac{\sqrt{2}}{\pi}\left| -2\cos\frac{\pi}{2}\right|_0^{2\pi} = \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\cos nx\, dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\cos nx \sin x/2\, dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi} \sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\Big]dx$$

$$= \frac{1}{\sqrt{2}\pi}\left| -\frac{2}{2n+1}\cos\left(\frac{2n+1}{2}\right) + \frac{2}{2n-1}\cos\frac{2n-1}{2}x\right|_0^{2\pi}$$

$$= \frac{2}{\sqrt{2}\pi}\left\{-\frac{1}{2n+1}[\cos(2n+1)\pi - 1] + \frac{1}{2n-1}[\cos(2n-1)\pi - 1]\right\}$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right) = -\frac{4\sqrt{2}}{\pi(4n^2-1)} \qquad [\because \cos(2n+1)\pi = \cos(2n-1)\pi = -1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx\, dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin nx \sin x/2\, dx$$

$$= \frac{1}{\sqrt{2}\,\pi}\int_0^{2\pi}\left[\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\,\pi}\left|\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{2n-1}\{\sin(2n-1)\pi - 0\} - \frac{1}{2n+1}\{\sin(2n+1)\pi - 0\}\right] = 0$$

Substituting the values of a's and b's in (i), we get

$$\sqrt{(1-\cos x)} = \frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{(4n^2-1)\pi}\cos nx$$

When $x = 0$, we have

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)} \qquad i.e., \quad \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots \infty = \frac{1}{2}.$$

PROBLEMS 10.1

1. Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\pi/\sinh \pi$.
2. Prove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$, $-\pi < x < \pi$. *(P.T.U., 2009 ; Bhopal, 2008 ; B.P.T.U., 2006)*

   Hence show that (i) $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$. *(Anna, 2009 ; P.T.U., 2009 ; Osmania, 2003)*

   (ii) $\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ (iii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$ *(S.V.T.U., 2008)*

   (iv) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$. *(Bhopal, 2008)*
3. If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the range 0 to 2π, show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$. *(Delhi, 2002 ; Madras, 2000)*
4. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \frac{2a^2}{\pi}\sinh a\pi\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nx\right]$.
5. $f(x) = x + x^2$ for $-\pi < x < \pi$ and $f(x) = \pi^2$ for $x = \pm\pi$. Expand $f(x)$ in Fourier series. *(Kurukshetra, 2005 ; U.P.T.U., 2003)*

   Hence show that $x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left\{\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right\}$

   and $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \frac{\pi^2}{8}$ *(V.T.U., 2008)*

10.3 CONDITIONS FOR A FOURIER EXPANSION

The reader must not be misled by the belief that the Fourier expansion of $f(x)$ in each case shall be valid. The above discussion has merely shown that if $f(x)$ has an expansion, then the coefficients are given by Euler's formulae. The problems concerning the possibility of expressing a function by Fourier series and convergence of this series are many and cumbersome. Such questions should be left to the curiosity of a pure-mathematician. However, almost all engineering applications are covered by the following well-known **Dirichlet's conditions***:

Any function f(x) can be developed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ *where* a_0, a_n, b_n *are constants, provided :*

(*i*) $f(x)$ *is periodic, single-valued and finite;*

(*ii*) $f(x)$ *has a finite number of discontinuties in any one period;*

(*iii*) $f(x)$ *has at the most a finite number of maxima and minima.* (*Anna, 2009 ; P.T.U., 2009*)

In fact the problem of expressing any function $f(x)$ as a Fourier series depends upon the evaluation of the integrals.

$$\frac{1}{\pi}\int f(x)\cos nx\,dx\,;\frac{1}{\pi}\int f(x)\sin nx\,dx$$

within the limits $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$ according as $f(x)$ is defined for every value of x in $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$.

PROBLEMS 10.2

State giving reasons whether the following functions can be expanded in Fourier series in the interval $-\pi \le x \le \pi$.

1. $\operatorname{cosec} x$
2. $\sin 1/x$ (*P.T.U., 2002*)
3. $f(x) = (m + 1)/m, \pi/(m + 1) < | x | \le \pi/m, m = 1, 2, 3, \ldots \infty$

10.4 FUNCTIONS HAVING POINTS OF DISCONTINUITY

In deriving the Euler's formulae for a_0, a_n, b_n, it was assumed that $f(x)$ was continuous. Instead a function may have a finite number of points of finite discontinuity *i.e.*, its graph may consist of a finite number of different curves given by different equations. Even then such a function is expressible as a Fourier series.

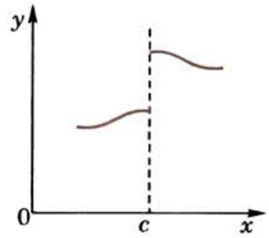


Fig. 10.1

For instance, if in the interval $(\alpha, \alpha + 2\pi)$, $f(x)$ is defined by

$$f(x) = \phi(x), \alpha < x < c.$$
$$= \psi(x), c < x < \alpha + 2\pi,$$ *i.e.*, c is the point of discontinuity, then

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\,dx + \int_{c}^{\alpha+2\pi} \psi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\cos nx\,dx + \int_{c}^{\alpha+2\pi} \psi(x)\cos nx\,dx\right]$$

and $$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} \phi(x)\sin nx\,dx + \int_{c}^{\alpha+2\pi} \psi(x)\sin nx\,dx\right]$$

At a point of finite discontinuity $x = c$, there is a finite jump in the graph of function (Fig. 10.1). Both the limit on the left [*i.e.*, $f(c - 0)$] and the limit on the right [*i.e.*, $f(c + 0)$] exist and are different. At such a point, Fourier series gives the value of $f(x)$ as the arithmetic mean of these two limits,

i.e., at $x = c$, $$f(x) = \frac{1}{2}[f(c-0) + f(c+0)].$$

Example 10.5. *Find the Fourier series expansion for* $f(x)$, *if*

$$f(x) = -\pi, -\pi < x < 0$$
$$x, 0 < x < \pi.$$ (*Bhopal, 2008 S*)

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. (*Kottayam, 2005*)

*See footnote p. 307.

Solution. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(*i*)

Then $$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\, dx + \int_0^{\pi} x\, dx\right] = \frac{1}{\pi}\left[-\pi \left| x \right|_{-\pi}^{0} + \left| x^2/2 \right|_0^{\pi}\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = -\frac{\pi}{2};$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \cos nx\, dx + \int_0^{\pi} x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi \left| \frac{\sin nx}{n} \right|_{-\pi}^{0} + \left| \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1)$$

$\therefore$ $$a_1 = \frac{-2}{\pi . 1^2},\ a_2 = 0,\ a_3 = -\frac{2}{\pi . 3^2},\ a_4 = 0,\ a_5 = -\frac{2}{\pi . 5^2} \text{ etc.}$$

Finally, $$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \sin nx\, dx + \int_0^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left| \frac{\pi \cos nx}{n} \right|_{-\pi}^{0} + \left| -x\frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$\therefore$ $$b_1 = 3,\ b_2 = -\frac{1}{2},\ b_3 = 1,\ b_4 = -\frac{1}{4}, \text{ etc.}$$

Hence substituting the values of a's and b's in (*i*), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3 \sin 3x}{3} - \frac{\sin 4x}{4} + \ldots \quad \text{...(ii)}$$

which is the required result.

Putting $x = 0$ in (*ii*), we obtain $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \infty\right)$ ...(*iii*)

Now $f(x)$ is discontinuous at $x = 0$. As a matter of fact

$$f(0-0) = -\pi \text{ and } f(0+0) = 0 \quad \therefore \quad f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\pi/2.$$

Hence (*iii*) takes the form $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$ whence follows the result.

Example 10.6. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, *prove that* $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$.

Hence show that $\frac{1}{1 . 3} - \frac{1}{3 . 5} + \frac{1}{5 . 7} - \ldots - \infty = \frac{1}{4}(\pi - 2)$ *(Bhopal, 2008 ; Mumbai, 2005 S ; Rohtak, 2005)*

Solution. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_0^{\pi} \sin x\, dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 . dx + \int_0^{\pi} \sin x \cos nx\, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\sin (n+1)x - \sin (n-1)x]\, dx = \frac{1}{2\pi}\left[-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \qquad (n \neq 1)$$

$$= \frac{1}{2\pi}\left\{\frac{1-(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right\} = 0, \text{ when } n \text{ is odd}$$

$$= -\frac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even.}$$

When $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{\pi} \sin 2x \, dx = \frac{1}{2\pi}\left| -\frac{\cos 2x}{2}\right|_0^{\pi} = 0$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \, . \, dx + \int_0^{\pi} \sin x \sin nx \, dx\right]$

$$= \frac{1}{2\pi}\int_0^{\pi} [\cos \overline{n-1}x - \cos \overline{n+1}x] \, dx = \frac{1}{2\pi}\left[\frac{\sin \overline{n-1}x}{n-1} - \frac{\sin \overline{n+1}x}{n+1}\right]_0^{\pi} = 0 \quad (n \neq 1)$$

When $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2x) \, dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}$

Hence $f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\right] + \frac{1}{2}\sin x$ ...(*i*)

Putting $x = \frac{\pi}{2}$ in (*i*), we get $1 = \frac{1}{\pi} - \frac{2}{\pi}\left(-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \ldots \infty\right) + \frac{1}{2}$

Whence $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots \infty = \frac{1}{4}(\pi - 2).$

Example 10.7. *Find the Fourier series for the function*

$$f(t) = \begin{cases} -1 & \text{for } -\pi < t < -\pi/2 \\ 0 & \text{for } -\pi/2 < t < \pi/2 \\ 1 & \text{for } \pi/2 < t < \pi \end{cases}$$

Solution. Let $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$ ...(*i*)

Then $a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1) dt + \int_{-\pi/2}^{\pi/2} (0) dt + \int_{\pi/2}^{\pi} (1) \, dt\right\}$

$$= \frac{1}{\pi}\left\{\left| -x \right|_{-\pi}^{-\pi/2} + \left| x \right|_{\pi/2}^{\pi}\right\} = \frac{1}{\pi}(\pi/2 - \pi + \pi - \pi/2) = 0$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1) \cos nt \, dt + \int_{-\pi/2}^{\pi/2} (0) \cos nt \, dt + \int_{\pi/2}^{\pi} (1) \cos nt \, dt\right\}$$

$$= \frac{1}{\pi}\left\{\left| -\frac{\sin nt}{n}\right|_{-\pi}^{-\pi/2} + \left|\frac{\sin nt}{n}\right|_{\pi/2}^{\pi}\right\} = \frac{1}{n\pi}\left(\frac{\sin n\pi}{2} - \frac{\sin n\pi}{2}\right) = 0$$

and $b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} (-1) \sin nt \, dt + \int_{-\pi/2}^{\pi/2} (0) \sin nt \, dt + \int_{\pi/2}^{\pi} (1) \sin nt \, dt\right\}$

$$= \frac{1}{\pi}\left\{\left|\frac{\cos nt}{n}\right|_{-\pi}^{-\pi/2} + \left| -\frac{\cos nt}{n}\right|_{\pi/2}^{\pi}\right\} = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

$\therefore \qquad b_1 = \frac{2}{\pi}, b_2 = -\frac{2}{\pi}, b_3 = \frac{2}{3\pi}$ etc.

Hence substituting the values of a's and b's in (i), we get $f(t) = \frac{2}{\pi}\left(\sin t - \sin 2t + \frac{1}{3}\sin 3t + \ldots\right)$.

PROBLEMS 10.3

1. Find the Fourier series to represent the function $f(x)$ given by
$f(x) = x$ for $0 \le x \le \pi$, and $= 2\pi - x$ for $\pi \le x \le 2\pi$. *(S.V.T.U., 2008 ; B.P.T.U., 2005 S)*
Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$. *(Madras 2000 S ; V.T.U., 2000 S)*

2. An alternating current after passing through a rectifier has the form
$$i = I_0 \sin x \quad \text{for } 0 \le x \le \pi$$
$$= 0 \quad \text{for } \pi \le x \le 2\pi$$
where I_0 is the maximum current and the period is 2π (Fig. 10.2). Express i as a Fourier series and evaluate
$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots \infty$$
(V.T.U., 2007 ; Calicut, 2005)

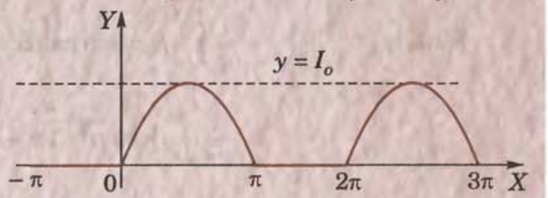


Fig. 10.2

3. Draw the graph of the function $f(x) = 0, -\pi < x < 0$
$= x^2, 0 < x < \pi$.
If $f(2\pi + x) = f(x)$, obtain Fourier series of $f(x)$.

4. Find the Fourier series of the following function :
$$f(x) = x^2, \quad 0 \le x \le \pi,$$
$$= -x^2, \quad -\pi \le x \le 0.$$
(Mumbai, 2009) *(Hissar, 2007)*

5. Find a Fourier series for the function defined by
$$f(x) = \begin{cases} -1, & \text{for } -\pi < x < 0 \\ 0, & \text{for } x = 0 \\ 1, & \text{for } 0 < x < \pi \end{cases}$$
Hence prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \infty = \frac{\pi}{4}$. *(U.P.T.U., 2005)*

10.5 CHANGE OF INTERVAL

In many engineering problems, the period of the function required to be expanded is not 2π but some other interval, say : $2c$. In order to apply the foregoing discussion to functions of period $2c$, this interval must be converted to the length 2π. This involves only a proportional change in the scale.

Consider the periodic function $f(x)$ defined in $(\alpha, \alpha + 2c)$. To change the problem to period 2π

put $\qquad z = \pi x/c \quad$ or $\quad x = cz/\pi \qquad$ ...(1)

so that when $\qquad x = \alpha, \qquad z = \alpha\pi/c = \beta$ (say)

when $\qquad x = \alpha + 2c, \qquad z = (\alpha + 2c)\pi/c = \beta + 2\pi$.

Thus the function $f(x)$ of period $2c$ in $(\alpha, \alpha + 2c)$ is transformed to the function $f(cz/\pi)$ [$= F(z)$ say] of period 2π in $(\beta, \beta + 2\pi)$. Hence $f(cz/\pi)$ can be expressed as the Fourier series

$$f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad \text{...(2)}$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) dz \\ a_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \cos nz\, dz \\ b_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \sin nz\, dz \end{aligned}\right\} \qquad \text{...(3)}$$

Making the inverse substitutions $z = \pi x/c$, $dz = (\pi/c)\,dx$ in (2) and (3) *the Fourier expansion of* $f(x)$ *in the interval* $(\alpha, \alpha + 2c)$ *is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx \\ a_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx \\ b_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx \end{aligned}\right\} \quad ...(4)$$

Cor. Putting $\alpha = 0$ in (4), we get the results for the interval $(0, 2c)$ and putting $\alpha = -c$ in (4), we get results for the interval $(-c, c)$.

Example 10.8. *Expand* $f(x) = e^{-x}$ *as a Fourier series in the interval* $(-l, l)$.
(Kerala, 2005 ; V.T.U., 2004)

Solution. The required series is of the form

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad ...(i)$$

Then $$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx = \frac{1}{l}\left| -e^{-x}\right|_{-l}^{l} = \frac{1}{l}(e^{l} - e^{-l}) = \frac{2\sinh l}{l}$$

and $$a_n = \frac{1}{l}\int_{-l}^{l} e^{x}\cos\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2l(-1)^n \sinh l}{l^2 + (n\pi)^2} \qquad [\because \cos n\pi = (-1)^n$$

$\therefore$ $$a_1 = \frac{-2l\sinh l}{l^2+\pi^2},\ a_2 = \frac{2l\sinh l}{l^2+2^2\pi^2},\ a_3 = \frac{2l\sinh l}{l^2+3^2\pi^2} \text{ etc.}$$

Finally, $$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2n\pi(-1)^n \sinh l}{l^2+(n\pi)^2}$$

$\therefore$ $$b_1 = \frac{-2\pi\sinh l}{l^2+\pi^2},\ b_2 = \frac{4\pi\sinh l}{l^2+2^2\pi^2},\ b_2 = \frac{-6\pi\sinh l}{l^2+3^2\pi^2} \text{ etc.}$$

Substituting the values of a's and b's in (i), we get

$$e^{-x} = \sinh l\left\{\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - \dots\right)\right.$$
$$\left. -2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{n\pi}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \dots\right)\right\}$$

Example 10.9. *Find the Fourier series expansion of* $f(x) = 2x - x^2$ *in* $(0, 3)$ *and hence deduce that* $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots - \infty = \frac{\pi}{12}$. *(Mumbai, 2005)*

Solution. The required series is of the form

$$2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \quad \text{where } l = 3/2. \qquad ...(i)$$

Then
$$a_0 = \frac{1}{l}\int_0^{2l} (2x - x^2)dx = \frac{2}{3}\left|x^2 - \frac{x^3}{3}\right|_0^3 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\cos\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{\sin 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\cos 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}[(2-6)\cos 2n\pi - 2] = -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\sin\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{-\cos 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{\cos 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\left\{-\frac{6}{n^2\pi^2}\cos 2n\pi - \frac{27}{4n^3\pi^3}(\cos 2n\pi - 1)\right\} = \frac{3}{n\pi}$$

Substituting the values of a_0, a_n, b_n in (*i*), we get

$$2x - x^2 = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

Putting $x = 3/2$, we get

$$3 - \frac{9}{4} = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos n\pi \quad \text{or} \quad -\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2} = \frac{\pi^2}{9}\cdot\frac{3}{4}$$

or
$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots\infty = \frac{\pi^2}{12}.$$

Example 10.10. *Obtain Fourier series for the function*

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$$ (*V.T.U., 2011 ; Bhopal, 2008 ; Mumbai, 2007*)

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\infty = \frac{\pi^2}{8}$.

Solution. The required series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Then
$$a_0 = \int_0^1 \pi x dx + \int_1^2 \pi(2 - x)dx = \pi\left|\frac{x^2}{2}\right|_0^1 + \pi\left|2x - \frac{x^2}{2}\right|_1^2 = \pi\left(\frac{1}{2}\right) + \pi\left\{(4-2) - \left(2 - \frac{1}{2}\right)\right\} = \pi$$

$$a_n = \int_0^1 \pi x\cos n\pi x dx + \int_1^2 \pi(2 - x)\cos n\pi x dx$$

$$= \left|\pi x\cdot\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|\pi(2 - x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi^2}\right) - \left(\frac{\cos 2n\pi}{n^2\pi} - \frac{\cos n\pi}{n^2\pi}\right) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$= 0 \text{ when } n \text{ is even ; } -\frac{4}{n^2\pi} \text{ when } n \text{ is odd.}$$

$$b_n = \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x) \sin n\pi x \, dx$$

$$= \left| \pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left| \pi(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(-\frac{\cos n\pi}{n}\right) + \left(\frac{\cos n\pi}{n}\right) = 0$$

Hence $\qquad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots \infty\right)$

Putting $x = 2$, $0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos 2\pi}{1^2} + \frac{\cos 6\pi}{3^2} + \frac{\cos 10\pi}{5^2} + \ldots \infty\right)$

Whence $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$.

Example 10.11. *Find the Fourier series for*

$$f(t) = 0, \; -2 < t < -1$$
$$= 1 + t, \; -1 < t < 0$$
$$= 1 - t, \; 0 < t < 1$$
$$= 0, \quad 1 < t < 2.$$

Solution. Let $\quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{2} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{2} \qquad \ldots(i)$

[$\because$ $2c = 2 - (-2)$ so that $c = 2$]

Then

$$a_0 = \frac{1}{2}\left\{\int_{-2}^{-1} (0)dt + \int_{-1}^{0} (1+t)dt + \int_0^1 (1-t)dt + \int_1^2 (0)dt\right\} = \frac{1}{2}\left\{\left| t + \frac{t^2}{2}\right|_{-1}^0 + \left| t - \frac{t^2}{2}\right|_0^1\right\}$$

$$= \frac{1}{2}\left\{-\left(-1+\frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)\right\} = \frac{1}{2}$$

$$a_n = \frac{1}{2}\left\{\int_{-1}^{0} (1+t) \cos \frac{n\pi t}{2} dt + \int_0^1 (1-t) \cos \frac{n\pi t}{2} dt\right\} \qquad \text{[Integrate by parts]}$$

$$= \frac{1}{2}\left\{\left| (1+t)\left(\sin \frac{n\pi t}{2}\right)\frac{2}{n\pi} - (1)\left(-\cos \frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_{-1}^0 \right.$$
$$\left. + \left| (1-t)\left(\sin \frac{n\pi t}{2}\right)\frac{2}{n\pi} - (-1)\left(-\cos \frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_0^1\right\}$$

$$= \frac{4}{n^2\pi^2}(1 - \cos n\pi/2)$$

$$b_n = \frac{1}{2}\left\{\int_{-1}^{0} (1+t) \sin \frac{n\pi t}{2} dt + \int_0^1 (1-t) \sin \frac{n\pi t}{2} dt\right\}$$

$$= \frac{1}{2}\left\{\left| (1+t)\left(-\cos \frac{n\pi t}{2}\right)\frac{2}{n\pi} - 1\left(-\sin \frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_{-1}^0 \right.$$
$$\left. + \left| (1-t)\left(-\cos \frac{n\pi t}{2}\right)\frac{2}{n\pi} - (-1)\left(-\sin \frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_0^1\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) - \left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right\} = 0$$

Substituting the values of a's and b's in (i), we get

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi t}{2}.$$

PROBLEMS 10.4

1. Obtain the Fourier series for $f(x) = \pi x$ in $0 \le x \le 2$.
2. (i) Find the Fourier series to represent x^2 in the interval $(0, a)$. *(Mumbai, 2009)*
   (ii) Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$. *(Mumbai, 2006)*
3. If $f(x) = 2x - x^2$ in $0 \le x \le 2$, show that $f(x) = \frac{2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x$. *(V.T.U., 2006)*
4. Find the Fourier series for $f(x) = \begin{cases} x & \text{in } 0 \le x \le 3 \\ 6 - x & \text{in } 3 \le x \le 6 \end{cases}$ *(Anna, 2008)*
5. A sinusoidal voltage $E \sin \omega t$ is passed through a half-wave rectifier which clips the negative portion of the wave. Develop the resulting periodic function

   $U(t) = 0$ when $-T/2 < t < 0$
   $= E \sin \omega t$ when $0 < t < T/2$,
   and $T = 2\pi/\omega$, in a Fourier series. *(Calicut, 1999)*
6. Find the Fourier series of the function $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & x = 1 \\ \pi(x-2), & 1 < x < 2 \end{cases}$

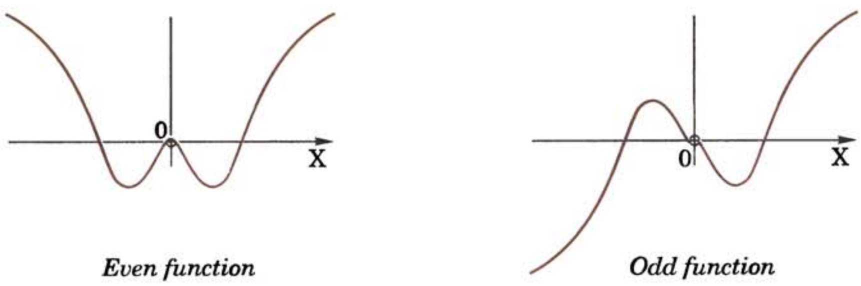
   Hence show that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ *(Mumbai, 2008)*

10.6 (1) EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be **even** *if $f(-x) = f(x)$,*

e.g., $\cos x$, $\sec x$, x^2 are all even functions. Graphically an even function is symmetrical about the y-axis.

A function $f(x)$ is said to be **odd** *if $f(-x) = -f(x)$,*



Even function *Odd function*

Fig. 10.3

e.g. $\sin x$, $\tan x$, x^3 are odd functions. Graphically, an odd function is symmetrical about the origin.

We shall be using the following property of definite integrals in the next paragraph :

$$\int_{-c}^{c} \mathbf{f(x)dx} = 2\int_0^c \mathbf{f(x)dx}, \textit{ when } f(x) \textit{ is an even function.}$$

$$= \mathbf{0}, \textit{ when } f(x) \textit{ is an odd function.}$$

(2) Expansions of even or odd periodic functions. We know that a periodic function $f(x)$ defined in $(-c, c)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c},$$

where $$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx,\ a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx,\ b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx.$$

Case I. *When f(x) is an even function* $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{2}{c}\int_0^c f(x)dx.$

Since $f(x)\cos\frac{n\pi x}{c}$ is also an even function,

$$\therefore \quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx$$

Again since $f(x)\sin\frac{n\pi x}{c}$ is an odd function, $\therefore\ b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = 0.$

Hence, *if a periodic function f(x) is even, its Fourier expansion contains only cosine terms, and*

$$\left.\begin{aligned} a_0 &= \frac{2}{c}\int_0^c f(x)dx \\ a_n &= \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx \end{aligned}\right\} \quad ...(1)$$

Case II. *When f(x) is an odd function,* $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = 0,$

Since $\cos\frac{n\pi x}{c}$ is an even function, therefore, $f(x)\cos\frac{n\pi x}{c}$ is an odd function.

$$\therefore \quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c} = 0$$

Again since $\sin\frac{n\pi x}{c}$ is an odd function, therefore, $f(x)\sin\frac{n\pi x}{c}$ is an even function.

$$\therefore \quad b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx$$

Thus, *if a periodic function f(x) is odd, its Fourier expansion contains only sine terms and*

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx \quad ...(2)$$

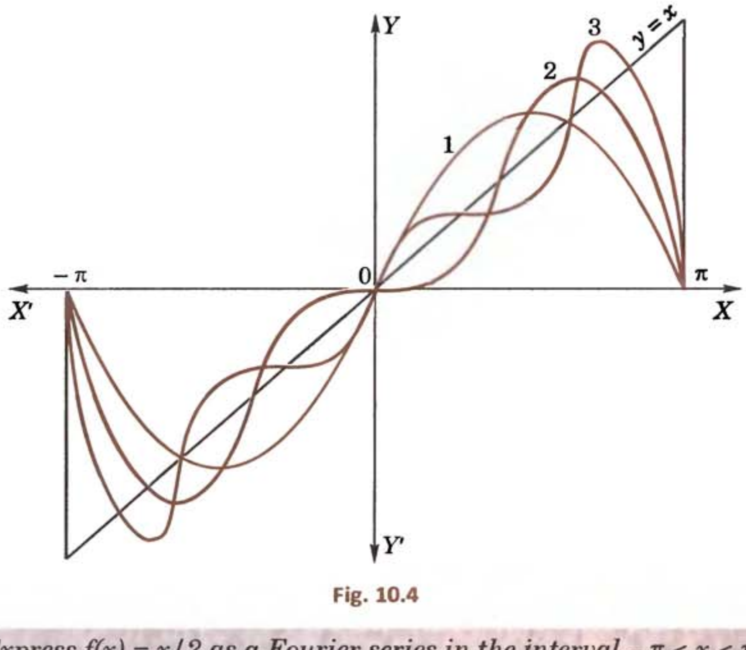


Fig. 10.4

Example 10.12. *Express f(x) = x / 2 as a Fourier series in the interval* $-\pi < x < \pi$. (*J.N.T.U., 2006*)

Solution. Since $f(-x) = -x/2 = -f(x).$

$\therefore$ $f(x)$ is an odd function and hence $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \frac{x}{2} \sin nx\, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = -\frac{\cos n\pi}{n}.$$

$\therefore$ $b_1 = 1/1, b_2 = -1/2, b_3 = 1/3, b_4 = -1/4$, etc.

Hence the series is $x/2 = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots$ ...(*i*)

Obs. The graphs of $y = 2 \sin x$, $y = 2(\sin x - \frac{1}{2}\sin 2x)$ and $y = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x)$ are shown in Fig. 10.4, by the curves 1, 2 and 3 respectively. These illustrate the manner in which the successive approximations to the series (*i*) approach more and more closely to $y = x$ for all values of x in $-\pi < x < \pi$, but not for $x = \pm \pi$.

As the series has a period 2π, it represents the discontinuous function, called *saw-toothed waveform*, shown in Fig. 10.5. It is important to note that the given function $y = x$ is continuous and each term of the series (*i*) is continuous, but the function represented by the series (*i*) has finite discontinuities at $x = \pm\pi, \pm 3\pi, \pm 5\pi$ etc.

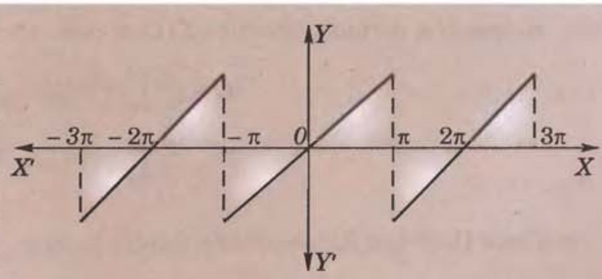


Fig. 10.5

Example 10.13. *Find a Fourier series to represent x^2 in the interval $(-l, l)$.* (*S.V.T.U., 2008*)

Solution. Since $f(x) = x^2$ is an even function in $(-l, l)$,

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad ...(i)$$

Then $$a_0 = \frac{2}{l}\int_0^l x^2\, dx = \frac{2}{l}\left|\frac{x^3}{3}\right|_0^l = \frac{2l^2}{3}$$

$$a_n = \int_0^l x^2 \cos\frac{n\pi x}{l}\, dx$$ [*See* footnote p. 398]

$$= \frac{2}{l}\left[x^2\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 2x\left(-\frac{\cos n\pi x/l}{n^2\pi^2/l^2}\right) + 2\left(-\frac{\sin n\pi x/l}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= 4l^2(-1)^n/n^2\pi^2$$ [$\because \cos n\pi = (-1)^n$]

$\therefore$ $a_1 = -4l^2/\pi^2, a_2 = 4l^2/2^2\pi^2, a_3 = -4l^2/3^2\pi^2, a_4 = 4l^2/4^2\pi^2$ etc.

Substituting these values in (*i*), we get

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\frac{\cos \pi x/l}{1^2} - \frac{\cos 2\pi x/l}{2^2} + \frac{\cos 3\pi x/l}{3^2} - \frac{\cos 4\pi x/l}{4^2} + \ldots\right)$$

which is the required Fourier series.

Example 10.14. *If $f(x) = |\cos x|$, expand $f(x)$ as a Fourier series in the interval $(-\pi, \pi)$.*

Solution. As $f(-x) = |\cos(-x)| = |\cos x| = f(x)$, $|\cos x|$ is an even function.

$$\therefore \quad f(x) = \frac{a_0}{2} + \Sigma a_n \cos nx$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x|\, dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)\, dx$$

[$\because$ $\cos x$ is $-$ve when $\pi/2 < x < \pi$]

$$= \frac{2}{\pi}\left\{\left| \sin x \right|_0^{\pi/2} - \left| \sin x \right|_{\pi/2}^{\pi}\right\} = \frac{2}{\pi}[(1-0)-(0-1)] = \frac{4}{\pi}$$

and
$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx \, dx + \int_{\pi/2}^{\pi} (-\cos x) \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2} [\cos (n+1)x + \cos (n-1)x] \, dx - \int_{\pi/2}^{\pi} [\cos (n+1)x + \cos (n-1)x] \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left| \frac{\sin (n+1)x}{n+1} + \frac{\sin (n-1)x}{n-1} \right|_0^{\pi/2} - \left| \frac{\sin (n+1)x}{n+1} + \frac{\sin (n-1)x}{n-1} \right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin (n+1)\pi/2}{n+1} + \frac{\sin (n-1)\pi/2}{n-1}\right\} + \left\{\frac{\sin (n+1)\pi/2}{n+1} + \frac{\sin (n-1)\pi/2}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left(\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right) = \frac{-4 \cos n\pi/2}{\pi(n^2-1)} \, (n \neq 1)$$

In particular $a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx\right] = 0$

Hence $\quad |\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left\{\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \ldots\right\}.$

Example 10.15. *Obtain Fourier series for the function f(x) given by*

$$f(x) = 1 + 2x/\pi, \qquad -\pi \le x \le 0,$$
$$= 1 - 2x/\pi, \qquad 0 \le x \le \pi.$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. *(V.T.U., 2010 ; Mumbai, 2007)*

Solution. Since $\quad f(-x) = 1 - \frac{2x}{\pi}$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$

and $\quad f(-x) = 1 + \frac{2x}{\pi}$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$

$\therefore$ $f(x)$ is an even function in $(-\pi, \pi)$. This is also clear from its graph $A'BA$ (Fig. 10.6) which is symmetrical about the y-axis.

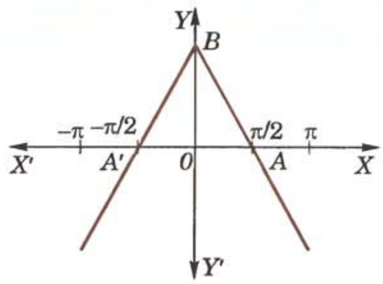


Fig. 10.6

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \ldots(i)$$

where
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left(x - \frac{x^2}{\pi}\right)_0^{\pi} = 0$$

and
$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx \, dx$$

$$= \frac{2}{\pi}\left|\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} = \frac{2}{\pi}\left(-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right) = \frac{4}{n^2\pi^2}[1-(-1)^n]$$

$\therefore \quad a_1 = 8/\pi^2, a^3 = 8/3^2\pi^2, a_5 = 8/5^2\pi^2, \ldots$

and $\quad a_2 = a_4 = a_6 = \ldots = 0.$

Thus substituting the values of a's in (i), we get

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \quad \ldots(ii)$$

as the required Fourier expansion

Putting $x = 0$ in (*ii*), we get $1 = f(0) = \dfrac{8}{\pi^2}\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\right)$

whence follows the desired result.

PROBLEMS 10.5

1. Obtain the Fourier series expansion of $f(x) = x^2$ in $(0, a)$. Hence show that
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$ (*Mumbai, 2009 ; S.V.T.U., 2008*)

2. Show that for $-\pi < x < \pi$, $\sin ax = \dfrac{2 \sin a\pi}{\pi}\left(\dfrac{\sin x}{1^2 - a^2} - \dfrac{2 \sin 2x}{2^2 - a^2} + \dfrac{3 \sin 3x}{3^2 - a^2} - \ldots\right)$

3. Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$. (*V.T.U., 2008 ; Anna, 2003*)
Deduce that $\dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \dfrac{1}{7.9} + \ldots = \dfrac{1}{4}(\pi - 2)$. (*U.P.T.U., 2005*)

4. Prove that in the interval $-\pi < x < \pi$, $x \cos x = -\dfrac{1}{2}\sin x + 2\displaystyle\sum_{n=2}^{\infty} \dfrac{n(-1)^n}{n^2 - 1} \sin nx$. (*S.V.T.U., 2009*)

5. For a function $f(x)$ defined by $f(x) = |\, x \,|, -\pi < x < \pi$, obtain a Fourier series. (*Bhopal, 2007 ; V.T.U., 2004*)
Deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ldots = \dfrac{\pi^2}{8}$. (*S.V.T.U., 2009 ; Kerala, 2005 ; P.T.U., 2005*)

6. Find the Fourier series to represent the function
(*i*) $f(x) = |\, \sin x \,|, -\pi < x < \pi$. (*Mumbai, 2008*)
(*ii*) $f(x) = |\, \cos (\pi x/l) \,|$ in the interval $(-1, 1)$. (*P.T.U., 2009 S*)

7. Given $f(x) = \begin{cases} -x + 1 \text{ for } -\pi \le x \le 0, \\ x + 1 \text{ for } 0 \le x \le \pi. \end{cases}$
Is the function even or odd? Find the Fourier series for $f(x)$ and deduce the value of
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

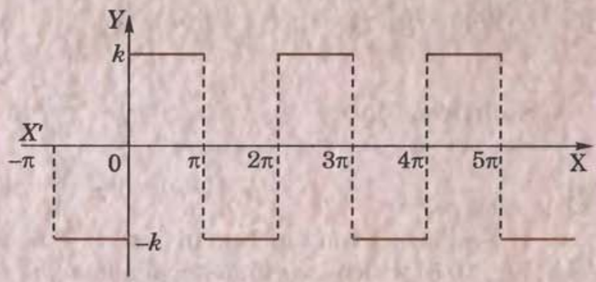


Fig. 10.7

8. Find the Fourier series of the periodic function $f(x)$: $f(x) = -k$ when $-\pi < x < 0$ and $f(x) = k$ when $0 < x < \pi$, and $f(x + 2\pi) = f(x)$. Sketch the graph of $f(x)$ and the two partial sums. (See Fig. 10.7)
Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n - 1} = \frac{\pi}{4}$. (*Rohtak, 2005*)

9. A function is defined as follows :
$$f(x) = -x \text{ when } -\pi < x \le 0 = x \quad \text{when } 0 < x < \pi.$$
Show that $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left[\dfrac{1}{1^2}\cos x + \dfrac{1}{3^2}\cos 3x + \dfrac{1}{5^2}\cos 5x + \ldots\right]$
Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n - 1)^2} = \frac{\pi^2}{8}$.

10.7 HALF RANGE SERIES

Many a time is is required to obtain a Fourier expansion of a function $f(x)$ for the range $(0, c)$ which is half the period of the Fourier series. As it is immaterial whatever the function may be outside the range $0 < x < c$, we extend the function to cover the range $-c < x < c$ so that the new function may be odd or even. The Fourier expansion of such a function of half the period, therefore, consists of sine or cosine terms only. In such cases the graphs for the values of x in $(0, c)$ are the same but outside $(0, c)$ are different for odd or even functions. That is why we get different forms of series for the same function as is clear from the examples 10.16 and 10.17.

Sine series. If it be required to expand $f(x)$ as a sine series in $0 < x < c$; then we extend the function reflecting it in the origin, so that $f(x) = -f(-x)$.

Then the extended function is odd in $(-c, c)$ and the expansion will give the desired Fourier sine series :

$$\left.\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \\ \text{where} \quad b_n &= \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c}\, dx \end{aligned}\right\} \quad ...(1)$$

Cosine series. If it be required to express $f(x)$ as a cosine series in $0 < x < c$, we extend the function reflecting it in the y-axis, so that $f(-x) = f(x)$.

Then the extended function is even in $(-c, c)$ and its expansion will give the required Fourier cosine series :

$$\left.\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} \\ \text{where} \quad a_0 &= \frac{2}{c}\int_0^c f(x)\, dx \\ \text{and} \quad a_n &= \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi x}{c}\, dx \end{aligned}\right\} \quad ...(2)$$

Example 10.16. *Express $f(x) = x$ as a half-range sine series in $0 < x < 2$.* *(U.P.T.U., 2004)*

Solution. The graph of $f(x) = x$ in $0 < x < 2$ is the line OA. Let us extend the function $f(x)$ in the interval $-2 < x < 0$ (shown by the line BO) so that the new function is symmetrical about the origin and, therefore, *represents an odd function in* $(-2, 2)$ (*Fig. 10.8*)

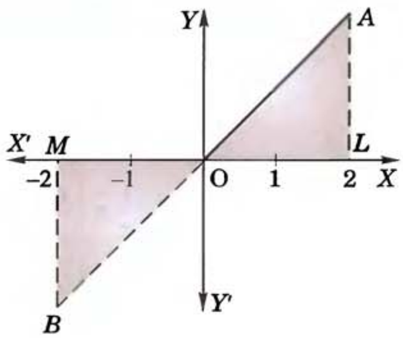


Fig. 10.8

Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

where

$$b_n = \frac{2}{2}\int_0^2 f(x) \sin \frac{n\pi x}{2}\, dx = \int_0^2 x \sin \frac{n\pi x}{2}\, dx$$

$$= \left| -\frac{2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2} \right|_0^2 = -\frac{4(-1)^n}{n\pi}$$

Thus $b_1 = 4/\pi$, $b_2 = -4/2\pi$, $b_3 = 4/3\pi$, $b_4 = -4/4\pi$ etc.

Hence the Fourier sine series for $f(x)$ over the half-range $(0, 2)$ is

$$f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \frac{1}{4}\sin\frac{4\pi x}{2} + ... \right).$$

Example 10.17. *Express $f(x) = x$ as a half-range cosine series in $0 < x < 2$.*
(S.V.T.U., 2009 ; Bhopal, 2007 ; Mumbai, 2006)

Solution. The graph of $f(x) = x$ in $(0, 2)$ is the line OA. Let us extend the function $f(x)$ in the interval $(-2, 0)$ shown by the line OB' so that the new function is symmetrical about the y-axis and, therefore, *represents an even function in $(-2, 2)$*. (Fig. 10.9)

Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

where $$a_0 = \frac{2}{2}\int_0^2 f(x)dx = \int_0^2 x\,dx = 2$$

and $$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^2 x\cos\frac{n\pi x}{2}dx$$

$$= \left| \frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right|_0^2 = \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

Thus $a_1 = -8/\pi^2, a_2 = 0, a_3 = -8/3^2\pi^2, a_4 = 0, a_5 = -8/5^2\pi^2$ etc.

Hence the desired Fourier series for $f(x)$ over the half-range (0, 2) is

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2} + \dots\right]$$

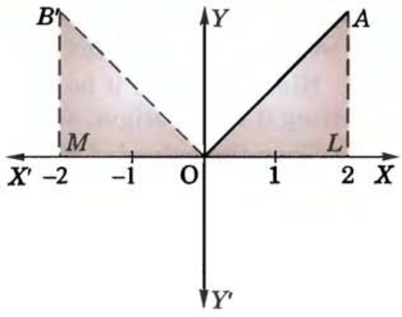


Fig. 10.9

Important Obs. It must be clearly understood that we expand a function in $0 < x < c$ as a series of sines or cosines, merely looking upon it as an odd or even function of period $2c$. It hardly matters whether the function is odd or even or neither.

Example 10.18. *Obtain the Fourier expansion of x sin x as a cosine series in (0, π).* *(V.T.U., 2003 ; U.P.T.U., 2002)*

Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots \infty = \frac{\pi - 2}{4}$. *(Anna, 2001)*

Solution. Let $x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $$a_0 = \frac{2}{\pi}\int_0^\pi x\sin x\,dx = \frac{2}{\pi}\left| x(-\cos x) - 1(-\sin x)\right|_0^\pi = 2$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^\pi x(\sin \overline{n+1}\,x - \sin\overline{n-1}\,x)\,dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{\frac{-\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^\pi$$

$$= \frac{1}{\pi}\pi\left\{\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right\} (n \neq 1).$$

When $n = 1$, $a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$

$$= \frac{1}{\pi}\left| x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{2}\right)\right|_0^\pi = \frac{1}{\pi}\left(-\frac{\pi\cos 2\pi}{2}\right) = -\frac{1}{2}.$$

Hence $x\sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{1.3} - \frac{\cos 3x}{3.5} + \frac{\cos 4x}{5.7} - \dots\infty\right\}$

Putting $x = \pi/2$, we obtain $\pi/2 = 1 + 2\left\{\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\infty\right\}$

Hence $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\infty = \frac{\pi - 2}{4}$.

Example 10.19. *Obtain a half range cosine series for*

$$f(x) = \begin{cases} kx, 0 \le x \le l/2 \\ k(l - x), l/2 \le x \le l. \end{cases}$$ *(Bhopal, 2008 ; V.T.U., 2008)*

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\infty$ *(Rohtak, 2006 ; U.P.T.U., 2003)*

Solution. Let the half-range cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n+1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Then $$a_0 = \frac{2}{l}\left\{\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\right\}dx = \frac{2k}{l}\left\{\left|\frac{x^2}{2}\right|_0^{l/2} - \left|\frac{(l-x)^2}{2}\right|_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\cdot\frac{1}{2}\left\{\frac{l^2}{4} - \left(0 - \frac{l^2}{4}\right)\right\} = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\left\{\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{2k}{l}\left| x\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 1\left\{-\cos\frac{n\pi x/l}{(n\pi/l)^2}\right\}\right|_0^{l/2}$$

$$+ \frac{2k}{l}\left|\left\{\frac{(l-x)\sin n\pi x/l}{n\pi/l}\right\} - (-1)\left(\frac{-\cos n\pi x/l}{(n\pi/l)^2}\right)\right|_{l/2}^{l}$$

$$= \frac{2k}{l}\left[\left(\frac{l^2}{2n\pi}\cdot\sin\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - \cos 0\right)\right] + \frac{2k}{l}\left[\left(\frac{l}{n\pi}\left(-\frac{l}{2}\sin\frac{n\pi}{2}\right)\right.\right.$$
$$\left.\left. - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right)\right]$$

$$= \frac{2k}{l}\cdot\frac{l^2}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right] = \frac{2kl}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}$$

Hence the required Fourier series is

$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \ldots\right]$$

Putting $x = l$, we get

$$0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \ldots\infty\right)$$

Thus $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty = \frac{\pi^2}{8}.$$

Example 10.20. *Expand* $f(x) = \frac{1}{4} - x$, *if* $0 < x < \frac{1}{2}$,

$$= x - \frac{3}{4},\ \text{if } \frac{1}{2} < x < 1,$$

as the Fourier series of sine terms. (V.T.U., 2011; Andhra, 2000)

Solution. Let $f(x)$ represent an odd function in $(-1, 1)$ so that $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

where $$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx$$

$$= 2\left[\int_0^{\frac{1}{2}}\left(\frac{1}{4} - x\right)\sin n\pi x\,dx + \int_{\frac{1}{2}}^{1}\left(x - \frac{3}{4}\right)\sin n\pi x\,dx\right]$$

$$= 2\left|-\left(\frac{1}{4} - x\right)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right|_0^{\frac{1}{2}} + 2\left|-\left(x - \frac{3}{4}\right)\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right|_{\frac{1}{2}}^{1}$$

$$= 2\left[\frac{1}{4n\pi}\cos\frac{n\pi}{2} + \frac{1}{4n\pi} - \frac{\sin n\pi/2}{n^2\pi^2}\right] + 2\left[-\frac{1}{4n\pi}\cos n\pi - \frac{1}{4n\pi}\cos\frac{n\pi}{2} - \frac{\sin n\pi/2}{n^2\pi^2}\right]$$

$$= \frac{1}{2n\pi}[1 - (-1)^n] - \frac{4\sin n\pi/2}{n^2\pi^2}$$

Thus $$b_1 = \frac{1}{\pi} - \frac{4}{\pi^2};\ b_2 = 0$$
$$b_3 = \frac{1}{3\pi} + \frac{4}{3^2\pi^2};\ b_4 = 0$$
$$b_5 = \frac{1}{5\pi} - \frac{4}{5^2\pi^2};\ b_6 = 0 \text{ etc.}$$

Hence $$f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \cdots$$

PROBLEMS 10.6

1. Show that a constant c can be expanded in an infinite series $\frac{4c}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right\}$ in the range $0 < x < \pi$. *(Marathwada, 2008 ; Kerala, 2005)*

2. Obtain cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$
(Osmania, 2003 S)

3. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$. *(B.P.T.U., 2005 ; Kottayam, 2005)*

4. Find the Fourier cosine series of the function $f(x) = \pi - x$ in $0 < x < \pi$. Hence show that
$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}$$
(West Bengal, 2004)

5. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. *(V.T.U., 2010 ; J.N.T.U., 2006)*

Hence show that $\pi^2 = 8\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$ *(Anna, 2003)*

6. Find the half-range sine series for the function $f(t) = t - t^2$, $0 < t < 1$.

7. Represent $f(x) = \sin(\sin(\pi x/l)$, $0 < x < l$ by a half-range cosine series. *(Mumbai, 2009)*

8. Find the half range sine series for $f(x) = x \cos x$ in $(0, \pi)$. *(Anna, 2008 S)*

9. Obtain the half-range sine series for e^x in $0 < x < 1$.

10. Find the half range Fourier sine series of $f(x) = x(\pi - x)$, $0 \le x \le \pi$ and hence deduce that

(*i*) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$ *(Anna, 2009)* (*ii*) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$ *(Mumbai, 2005)*

11. If $f(x) = x$, $0 < x < \pi/2$
$= \pi - x$, $\pi/2 < x < \pi$,

show that (*i*) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right]$ *(Mumbai, 2008 ; S.V.T.U., 2008 ; V.T.U., 2004)*

(*ii*) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{12}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right]$ *(V.T.U., 2011)*

12. Find the half-range cosine series expansion of the function $f(x) = \begin{cases} 0, & 0 \le x \le l/2 \\ l - x, & l/2 \le x \le l \end{cases}$. *(P.T.U., 2010)*

13. If $f(x) = \sin x$ for $0 \le x \le \pi/4$
$= \cos x$ for $\pi/4 \le x \le \pi/2$, expand $f(x)$ in a series of sines.

14. For the function defined by the graph OAB in Fig. 10.10, find the half-range Fourier sine series.

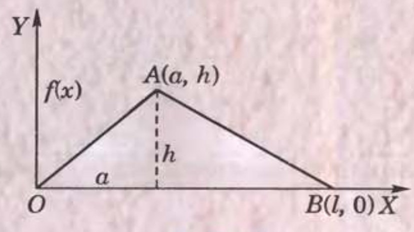


Fig. 10.10

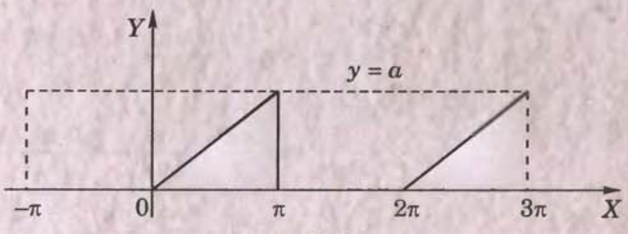


Fig 10.11

10.8 TYPICAL WAVEFORMS

We give below six typical waveforms usually met with in communication engineering :

(1) *Square waveform* (Fig. 10.7) is an extension of the function of Problem 8, page 412.

(2) *Saw-toothed waveform* (Fig. 10.5) is an extension of the function in Ex. 10.12, page 409.

(3) *Modified saw-toothed waveform* (Fig. 10.11) is extension of the function

$$f(x) = 0, \quad -\pi < x \le 0$$
$$= x, \quad 0 \le x < \pi,$$

Its Fourier expansion is

$$f(x) = \frac{a}{4} - \frac{2a}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right) + \frac{\alpha}{\pi}\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right)$$

(4) *Triangular waveform* (Fig. 10-6) is an extension of the function of Ex. 10.15, page 411.

(5) *Half-wave rectifier* (Fig. 10.2) is an extension of the function of Problem 2, page 412.

(6) *Full-wave rectifier* (Fig. 10.12) is an extension of the function $f(x) = a \sin x$, $0 \le x \le \pi$. Its Fourier expansion is

$$f(x) = \frac{4a}{\pi}\left\{\frac{1}{2} - \frac{1}{1.3}\cos 2x - \frac{1}{3.5}\cos 4x - \frac{1}{5.7}\cos 6x - \dots\right\}$$

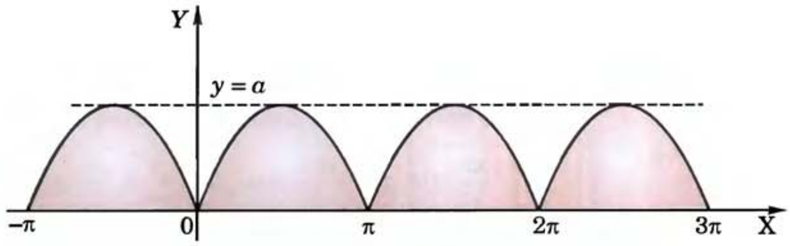


Fig. 10.12

10.9 (1) PARSEVAL'S FORMULA*

To prove that $\int_{-l}^{l} [f(x)]^2\,dx = l\left\{\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$,

provided the Fourier series for $f(x)$ converges uniformly in $(-l, l)$.

The Fourier series for $f(x)$ in $(-l, l)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$ ...(1)

Multiplying both sides of (1) by $f(x)$ and integrating term by term from $-l$ to l [which is justified as the series (1) is uniformly convergent –p. 389], we get

$$\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx + b_n\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right\} \quad \text{...(2)}$$

Now $\int_{-l}^{l} f(x)\,dx = la_0$,

$\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = la_n$ and $\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = lb_n$, by (4) of p. 405

$\therefore$ (2) takes the form $\int_{-l}^{l}[f(x)]^2\,dx = l\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$ ...(3)

which is the desired *Parseval's formula*. *(Mumbai, 2005 S)*

*Named after the French mathematician *Marc Antoine Parseval* (1755–1836).

Cor. 1. *If* $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$ *in* $(0, 2l)$, then

$$\int_0^{2l} [f(x)]^2 \, dx = l \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\} \qquad ...(4)$$

Cor. 2. *If the half-range cosine series is* $(0, l)$ *for* $f(x)$ *is*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \text{ then}$$

$$\int_0^{l} [f(x)]^2 \, dx = \frac{l}{2} \left(\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + ... \infty \right) \qquad ...(5)$$

Cor. 3. *If the half-range sine series in* $(0, l)$ *for* $f(x)$ *is* $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$, then

$$\int_0^{l} [f(x)]^2 = \frac{l}{2} (b_1^2 + b_2^2 + b_3^2 + ... \infty) \qquad ...(6)$$

(2) Root mean square (rms) value. *The root mean square value of the function* $f(x)$ *over an interval* (a, b) *is defined as*

$$[f(x)]_{\text{rms}} = \sqrt{\left\{ \frac{\int_a^b [f(x)]^2 \, dx}{b - a} \right\}} \qquad ...(7)$$

The use of root mean square value of a periodic function is frequently made in the theory of mechanical vibrations and in electric circuit theory. The r.m.s. value is also known as the effective value of the function.

Example 10.21. *Obtain the Fourier series for* $y = x^2$ *in* $-\pi < x < \pi$. *Using the two values of* y, *show that*

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ... = \frac{\pi^4}{90}$$

Solution. Let $\quad y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

We have $\quad a_0 = 2\frac{\pi^2}{3}, a_n = \frac{4}{n^2}(-1)^n, b_n = 0$ for all n (See problem 2, p. 400)

If $\bar{y}$ be the r.m.s. value of y in $(-\pi, \pi)$, then

$$(\bar{y})^2 = \frac{\pi}{2\pi} \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right] \qquad \text{[By (3) and (7) §10.9]}$$

$$= \frac{1}{4}\left(\frac{2\pi^2}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{16}{n^4}(-1)^{2n} + 0\right] = \frac{\pi^4}{9} + 8\sum_{1}^{\infty}\frac{1}{n^4}$$

Also by definition,

$$(\bar{y})^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} y^2 \, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 \, dx = \frac{\pi^4}{5}$$

Equating the two values of $(\bar{y})^2$, we get

$$\frac{\pi^4}{9} + 8\sum_{1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{5} \text{ i.e., } \sum_{1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

PROBLEMS 10.7

1. By using the sine series for $f(x) = 1$ in $0 < x < \pi$, show that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

2. Prove that in $0 < x < l$, $x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\right)$

   and deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}$.

3. If $\frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l}$ is the half-range cosine series of $f(x)$ of period $2l$ in $(0, l)$, then show that the mean square value of $f(x)$ in $(0, l)$ is $\frac{l}{2}\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right\}$

   Use this result to evaluate $1^{-4} + 3^{-4} + 5^{-4} + \ldots\ldots$ from the half-range cosine series of the function $f(x)$ of period 4 defined in (0, 2) by

$$f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2 - x), & 1 < x < 2 \end{cases}.$$

10.10 COMPLEX FORM OF FOURIER SERIES

The Fourier series of a periodic function $f(x)$ of period $2l$, is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \ldots(1)$$

Since $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$,

therefore, we can express (1) as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)\right\}$$

$$= c_0 + \sum_{n=1}^{\infty}\left\{c_n\, e^{in\pi x/l} + c_{-n}\, e^{-in\pi x/l}\right\} \quad \ldots(2)$$

where $c_0 = \tfrac{1}{2}a_0,\ c_n = \tfrac{1}{2}(a_n - ib_n),\ c_{-n} = \tfrac{1}{2}(a_n + ib_n)$

Now $c_n = \frac{1}{2l}\left\{\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - i\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right\}$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$$

and $c_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{in\pi x/l}\,dx$

Combining these, we have $c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$ ...(3)

Then the series (2) can be compactly written as :

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/l}$$

which is the *complex form of Fourier series* and its coefficients are given by (3).

Obs. The complex form of a Fourier series is especially useful in problems on electrical circuits having impressed periodic voltage.

CHAPTER 22

Fourier Transforms

22.1 INTRODUCTION

In the previous chapter, the reader has already been acquainted with the use of Laplace transforms in the solution of ordinary differential equations. In this chapter, the well-known Fourier transforms will be introduced and their properties will be studied which will be used in the solution of partial differential equations. The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ease with which the transform can be inverted. A Fourier transform when applied to a partial differential equation reduces the number of its independent variables by one.

The theory of integral transforms afford mathematical devices through which solutions of numerous boundary value problems of engineering can be obtained *e.g.*, conduction of heat, transverse vibrations of a string, transverse oscillations of an elastic beam, free and forced vibrations of a membrane, transmission lines etc. Some of these applications will be illustrated in the last section.

22.2 DEFINITION

The integral transform of a function $f(x)$ denoted by $I[f(x)]$, is defined by

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x) K(s, x)\, dx$$

where $K(s, x)$ is called the *kernel* of the transform and is a known function of s and x. The function $f(x)$ is called the *inverse transform* of $\bar{f}(s)$.

Three simple examples of a kernel are as follows :

(*i*) *When $K(s, x) = e^{-sx}$, it leads to the* **Laplace transform** *of $f(x)$, i.e.,*

$$\bar{f}(s) = \int_0^{\infty} f(x) e^{-sx}\, dx. \qquad \text{[Chap. 21]}$$

(*ii*) *When $K(s, x) = e^{isx}$, we have the* **Fourier transform** *of $f(x)$, i.e.,*

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx.$$

(*iii*) When $K(s, x) = x^{s-1}$, it gives the *Mellin transform* of $f(x)$ *i.e.*,

$$M(s) = \int_0^\infty f(x) x^{s-1}\, dx.$$

Other special transforms arise when the kernel is a sine or a cosine function or a Bessel's function. These lead to *Fourier sine* or *cosine transforms* and the *Hankel transform* respectively.

In order to introduce the *Fourier transforms*, we shall first derive the Fourier integral theorem.

22.3 (1) FOURIER INTEGRAL THEOREM

Consider a function $f(x)$ which satisfies the Dirichlet's conditions (Art. 10.3) in every interval $(-c, c)$ so that, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right) \qquad \text{...(1)}$$

where $$a_0 = \frac{1}{c}\int_{-c}^{c} f(t)\, dt,\ a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\, dt, \text{ and } b_n = \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\, dt.$$

Substituting the values of a_0, a_n and b_n in (1), it takes the form

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\, dt + \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c} f(t)\cos\frac{n\pi(t-x)}{c}\, dt \qquad \text{...(2)}$$

If we assume that $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, the first term on the right side of (2) approaches 0 as $c \to \infty$, since

$$\left|\frac{1}{2c}\int_{-c}^{c} f(t)\, dt\right| \le \frac{1}{2c}\int_{-\infty}^{\infty} |f(t)|\, dt$$

The second term on the right side of (2) tends to

$$\underset{c\to\infty}{\text{Lt}}\ \frac{1}{c}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi(t-x)}{c}\, dt$$

$$= \underset{\delta\lambda\to 0}{\text{Lt}}\ \frac{1}{\pi}\sum_{n=1}^{\infty}\delta\lambda\int_{-\infty}^{\infty} f(t)\cos n\delta\lambda\,(t-x)\, dt, \text{ on writing } \pi/c = \delta\lambda$$

This is of the form $\underset{\delta\lambda\to 0}{\text{Lt}} \sum_{n=1}^{\infty} F(n\delta\lambda)$, *i.e.*, $\int_0^\infty F(\lambda)\, d\lambda$

Thus as $c \to \infty$, (2) becomes $f(x) = \dfrac{1}{\pi}\displaystyle\int_0^\infty\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\, dt d\lambda$ ...(3)

which is known as the **Fourier integral** of $f(x)$.

Obs. We have given a heuristic demonstration of the Fourier integral theorem which simply helps in deriving the result (3). It cannot however, be taken as a rigorous proof for that would, involve a proof of the convergence of the Fourier integral which is beyond the scope of this book. When $f(x)$ satisfies the above-mentioned conditions, equation (3) holds good at a point of continuity. If however, x is point of discontinuity, we replace $f(x)$ by $\frac{1}{2}[f(x+0) + f(x-0)]$ as in the case of Fourier series.

(2) Fourier sine and cosine integrals. Expanding $\cos\lambda(t-x)$, (3) may be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos\lambda x\int_{-\infty}^{\infty} f(t)\cos\lambda t\, dt\, d\lambda + \frac{1}{\pi}\int_0^\infty \sin\lambda x\int_{-\infty}^{\infty} f(t)\sin\lambda t\, dt\, d\lambda \qquad \text{...(4)}$$

If $f(x)$ is an odd function, $f(t)\cos\lambda t$ is also an odd function while $f(t)\sin\lambda t$ is even. Then the first term on the right side of (4) vanishes and, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\int_0^\infty f(t)\sin\lambda t\, dt\, d\lambda \qquad \text{...(5)}$$

which is known as the *Fourier sine integral*.

Similarly, if $f(x)$ is an even function, (4) takes the form

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t \, dt \, d\lambda \qquad \text{...(6)}$$

which is known as the *Fourier cosine integral.*

Obs. A function $f(x)$ defined in the interval $(0, \infty)$ is expressed either as a Fourier sine integral or as a Fourier cosine integral, merely looking upon it as an odd or even function in $(-\infty, \infty)$ on the lines of half-range Fourier series.

(3) **Complex form of Fourier integrals.** Equation (3) can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda \qquad \text{..(7)}$$

because $\cos \lambda(t-x)$ is an even function of λ. Also since $\sin \lambda(t-x)$ is an odd function of λ, we have

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin \lambda(t-x) dt d\lambda \qquad \text{...(8)}$$

Now multiply (8) by i and add it to (7), so that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t-x)} dt d\lambda \qquad \text{...(9)}$$

which is the *complex* form of the *Fourier integral.*

(4) Fourier integral representation of a function

Using (4), *a function F(x) may be represented by a Fourier integral as*

$$F(x) = \frac{1}{\pi} \int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] \, d\lambda$$

where $$A(\lambda) = \int_{-\infty}^{\infty} f(t) \cos \lambda t \, dt \,; B(\lambda) = \int_{-\infty}^{\infty} f(t) \sin \lambda t \, dt \qquad \text{...(10)}$$

If f (x) is an odd function, then [By (5)]

$$f(x) = \frac{1}{\pi} \int_0^\infty B(\lambda) \sin \lambda x \, d\lambda \text{ where } B(\lambda) = 2 \int_0^\infty f(t) \sin \lambda t \, dt \qquad \text{...(11)}$$

If f (x) is an even function, then [By (6)]

$$f(x) = \frac{1}{\pi} \int_0^\infty A(\lambda) \cos \lambda x \, d\lambda \text{ where } A(\lambda) = 2 \int_0^\infty f(t) \cos \lambda t \, dt \qquad \text{...(12)}$$

Example 22.1. *Express* $f(x) = 1$ *for* $0 \le x \le \pi$,
$= 0$ *for* $x > \pi$,
as a Fourier sine integral and hence evaluate

$$\int_0^\infty \frac{1-\cos(\pi\lambda)}{\lambda} \sin(x\lambda) d\lambda.$$ *(Kottayam, 2005 ; J.N.T.U., 2004 S)*

Solution. The Fourier sine integral for $f(x) = \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \, d\lambda \int_0^\infty f(t) \sin(\lambda t) \, dt$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \, d\lambda \int_0^\pi \sin(\lambda t) \, dt$$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \, d\lambda \left| \frac{-\cos(\lambda t)}{\lambda} \right|_0^\pi = \frac{2}{\pi} \int_0^\infty \frac{1-\cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda$$

$$\therefore \quad \int_0^\infty \frac{1-\cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2 & \text{for } 0 \le x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2}\left[\frac{f(\pi-0)+f(\pi+0)}{2}\right] = \frac{\pi}{2} \cdot \frac{1+0}{2} = \frac{\pi}{4}.$$

22.4 (1) FOURIER TRANSFORMS

Rewriting (9) of § 22.3 as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, ds \int_{-\infty}^{\infty} f(t)e^{ist}\, dt,$$

it follows that if $\quad F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \quad$ ...(1)

then $\quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \quad$ ...(2)

The function F(s), defined by (1), is called the **Fourier transform** *of f(x). Also the function f(x), as given by* (2), *is called the* **inverse Fourier transform** *of F(s).* Sometimes, we call (2) as an *inversion formula* corresponding to (1).

(2) Fourier sine and cosine transforms. From (5) of § 22.3, it follows that if

$$F_s(s) = \int_0^{\infty} f(x) \sin sx\, dx \qquad ...(3)$$

then $\quad f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(s) \sin sx\, ds \quad$ ...(4)

The function $F_s(s)$, as defined by (3), *is known as the* **Fourier sine transform** *of f(x) in $0 < x < \infty$. Also the function f(x), as given by* (4) *is called the* **inverse Fourier sine transform** *of $F_s(s)$.*

Similarly, if follows from (6) of § 22.3 that if

$$F_c(s) = \int_0^{\infty} f(x) \cos sx\, dx \qquad ...(5)$$

then $\quad f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(s) \cos sx\, ds \quad$ ...(6)

The function $F_c(s)$ as defined by (5) *is known as the* **Fourier cosine transform** *of f(x) in $0 < x < \infty$. Also the function f(x), as given by* (6), *is called the* **inverse Fourier cosine transform** *of $F_c(s)$.*

(3) **Finite Fourier sine and cosine transforms.** These transforms are useful for such a boundary-value problem in which at least two of the boundaries are parallel and separated by a finite distance.

The **finite Fourier sine transform** *of f(x), in $0 < x < c$, is defined as*

$$F_s(n) = \int_0^{c} f(x) \sin \frac{n\pi x}{c}\, dx \qquad ...(7)$$

where n is an integer.

The function f(x) is then called the **inverse finite Fourier sine transform** *of $F_s(n)$ which is given by*

$$f(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{c} \qquad ...(8)$$

The **finite Fourier cosine transform** *of f(x), in $0 < x < c$, is defined as*

$$F_c(n) = \int_0^{c} f(x) \cos \frac{n\pi x}{c}\, dx \qquad ...(9)$$

where n is an integer.

The function f(x) is then called the **inverse finite Fourier cosine transform** *of $F_c(n)$ which is given by*

$$f(x) = \frac{1}{c} F_c(0) + \frac{2}{c}\sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{c} \qquad ...(10)$$

Obs. The finite Fourier sine transform is useful for problems involving boundary conditions of heat distribution on two parallel boundaries, while the finite cosine transform is useful for problems in which the velocities normal to two parallel boundaries are among the boundary conditions.

22.5 PROPERTIES OF FOURIER TRANSFORMS

(1) **Linear property.** *If F(s) and G(s) are Fourier transforms of f(x) and g(x) respectively, then*

$$F[a\, f(x) + bg(x)] = a\, F(s) + bG(s)$$

where a and b are constants.

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$ and $G(s) = \int_{-\infty}^{\infty} e^{isx} g(x)\,dx$

$$\therefore \quad F[af(x) + bg(x)] = \int_{-\infty}^{\infty} e^{isx} [af(x) + bg(x)]\,dx = a\int_{-\infty}^{\infty} e^{isx} f(x)\,dx + b\int_{-\infty}^{\infty} e^{isx} g(x)dx$$

$$= aF(s) + bG(s)$$

(2) Change of scale property. *If F(s) is the complex Fourier transform of f(x), then*

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right), a \neq 0$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$ ...(*i*)

$$\therefore \quad F\{f(ax)\} = \int_{-\infty}^{\infty} e^{isx} f(ax)dx \qquad \left|\begin{array}{l}\text{Put } ax = t \\ \text{so that } dx = dt/a\end{array}\right.$$

$$= \int_{-\infty}^{\infty} e^{ist/a} f(t)dt/a = \frac{1}{a}\int_{-\infty}^{\infty} e^{i(s/a)t} f(t)dt = \frac{1}{a} F\left(\frac{s}{a}\right) \qquad \text{[By } (i)]$$

Cor. *If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of f (x) respectively, then*

$$F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right) \quad \text{and} \quad F_c\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right).$$

(3) Shifting property. *If F(s) is the complex Fourier transform of f (x), then*

$$F\{f(x-a)\} = e^{isa} F(s)$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$ ...(*i*)

$$\therefore \quad F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{isx} f(x-a)dx \qquad \left|\begin{array}{l}\text{Put } x - a = t \\ \text{so that } dx = dt\end{array}\right.$$

$$= \int_{-\infty}^{\infty} e^{is(t+a)} f(t)dt = e^{isa}\int_{-\infty}^{\infty} e^{ist} f(t)dt = e^{isa} F(s) \qquad \text{[By } (i)]$$

(4) Modulation theorem. *If F(s) is the complex Fourier transform of f (x), then*

$$F\{f(x)\cos ax\} = \frac{1}{2}[F(s+a) + F(s-a)\}$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx$ ...(*i*)

$$\therefore \quad F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{isx} f(x)\cos ax\,dx = \int_{-\infty}^{\infty} e^{isx} . f(x) . \frac{e^{iax} + e^{-iax}}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x)dx\right] = \frac{1}{2}[F(s+a) + F(s-a)].$$

Cor. *If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of f (x) respectively, then*

(*i*) $F_s\{f(x)\cos ax\} = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$ (*Anna, 2008*)

(*ii*) $F_c\{f(x)\sin ax\} = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

(*iii*) $F_s\{f(x)\sin ax\} = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

Obs. This theorem is of great importance in radio and television where the harmonic carrier wave is modulated by an envelope.

Example 22.2. *Find the Fourier transform of*

$$f(x) = \begin{cases} 1 \text{ for } |x| < 1 \\ 0 \text{ for } |x| > 1 \end{cases}.$$

Hence evaluate $\int_0^{\infty} \frac{\sin x}{x}\,dx.$ (*V.T.U., 2010 ; S.V.T.U., 2009 ; U.P.T.U., 2008*)

Solution. The Fourier transform of $f(x)$, *i.e.*,

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{isx}\,dx = \int_{-1}^{1}(1)\,e^{isx}\,dx = \left|\frac{e^{isx}}{is}\right|_{-1}^{1} = \frac{e^{is}-e^{-is}}{is}$$

Thus $\quad F\{f(x)\} = F(s) = 2\,\dfrac{\sin s}{s}$, $s \neq 0$. For $s = 0$, we have $F(s) = 2$.

Now by the inversion formula, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds, \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin s}{s}\,e^{-isx}ds = \begin{cases}1 \text{ for } |x|<1\\ 0 \text{ for } |x|>1\end{cases}$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty}\frac{\sin s}{s}\,ds = \pi \;\therefore\; \int_{0}^{\infty}\frac{\sin s}{s}\,ds = \frac{\pi}{2}, \text{ since the integrand is even.}$$

Example 22.3. *Find the Fourier transform of :*

$$f(x) = \begin{cases}1-x^2, & |x| \le 1\\ 0 & , |x|>1\end{cases}$$

Hence evaluate $\displaystyle\int_0^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx.$ *(V.T.U., 2011 S ; Anna, 2005 S ; Mumbai, 2005 S)*

Solution. $F\{f(x)\} = \displaystyle\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = F(s)$, say

$$= \int_{-\infty}^{-1}(0)\,e^{isx}\,dx + \int_{-1}^{1}(1-x^2)\,e^{isx}\,dx + \int_{1}^{\infty}(0)\,e^{isx}\,dx = \left|(1-x^2)\frac{e^{isx}}{is} - (2x)\frac{e^{isx}}{(is)^2} + (-2)\frac{e^{isx}}{(is)^3}\right|_{-1}^{1}$$

$$= 2\left(\frac{e^{is}+e^{-is}}{-s^2}\right) - 2\left(\frac{e^{is}-e^{-is}}{-is^3}\right) = -\frac{4}{s^3}(s\cos s - \sin s)$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

or $$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)\,e^{-isx}\,ds = \begin{cases}1-x^2, & |x|\le 1\\ 0, & |x|>1\end{cases}$$

Putting $x = 1/2$, we obtain

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)e^{-is/2}ds = \frac{3}{4}$$

or $$\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right)ds = -\frac{3\pi}{8}$$

or $$\int_{-\infty}^{\infty}\frac{s\cos s - \sin s}{s^3}\,.\cos\frac{s}{2}\,ds = -\frac{3\pi}{8}$$

or $$\int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}\,.\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}, \text{ since the integral is even.}$$

Example 22.4. (*a*) *Find the Fourier transform of* $e^{-a^2x^2}$, $a < 0$. *Hence deduce that* $e^{-x^2/2}$ *is self reciprocal in respect of Fourier transform.* *(Madras, 2006 ; Kottayam, 2005)*

(*b*) *Find Fourier transform of* (*i*) $e^{-2(x-3)^2}$ (*ii*) $e^{-x^2}\cos 3x$.

Solution. (*a*) $\quad F(e^{-a^2x^2}) = \displaystyle\int_{-\infty}^{\infty} e^{-a^2x^2}\,.\,e^{isx}\,dx = \int_{-\infty}^{\infty} e^{-a^2(x^2 - isx/a^2)}dx$

$$= \int_{-\infty}^{\infty} e^{-a^2(x - is/2a^2)^2}\,.\,e^{-s^2/4a^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-s^2/4a^2}\, dt/a \qquad \text{[Putting } a(x - is/2a^2) = t,\ dx = dt/a$$

$$= \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2}\, dt = \frac{e^{-s^2/4a^2}}{a}\sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\right]$$

Hence $F(e^{-a^2x^2}) = \dfrac{\sqrt{\pi}}{a}\, e^{-s^2/4a^2}$

Taking $a^2 = 1/2$, we have

$$F(e^{-x^2/2}) = \frac{\sqrt{\pi}}{(1/\sqrt{2})}\, e^{-s^2/2} = \sqrt{2\pi} e^{-s^2/2}$$

i.e., Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-s^2/2}$. Also the functions $e^{-x^2/2}$ and $e^{-s^2/2}$ are the same. Hence it follows that $e^{-x^2/2}$ *is self-reciprocal under the Fourier transform.*

(*b*) Since $e^{-2x^2} = e^{-(2x)^2/2} = f(2x)$ where $f(x) = e^{-x^2/2}$

$\therefore$ by change of scale property, $F\{f(2x)\} = \dfrac{1}{2} F(s/2)$

i.e.,
$$F(e^{-2x^2}) = F[e^{-(2x)^2/2}] = \sqrt{2\pi} e^{-(s/2)^2/2} = \sqrt{2\pi} e^{-s^2/8}$$

By shifting property $F\, f(x-3) = e^{i3s} F(3)$

$$\therefore \quad F[e^{-2(x-3)^2}] = e^{3is}\sqrt{2\pi} e^{-s^2/8} = \sqrt{2\pi} e^{(3is - s^2/8)} \qquad ...(i)$$

Also by modulation theorem,

$$F[f(x)\cos 2x] = \frac{1}{2}[F(s+a) + F(s-a)]$$

$$F(e^{-x^2}\cos 3x) = \frac{1}{2}\sqrt{2\pi}[e^{-(s+3)^2/2} + e^{-(s-3)^2/2}]. \qquad ...(ii)$$

Example 22.5. *Find the Fourier cosine transform of* e^{-x^2}. (*V.T.U., 2010; Rajasthan, 2006*)

Solution. We have $F_c(e^{-x^2}) = \int_0^{\infty} e^{-x^2} \cos sx\, dx = I$ (say)

Differentiating under the integral sign w.r.t. s,

$$\frac{dI}{ds} = -\int_0^{\infty} xe^{-x^2} \sin sx\, dx = \frac{1}{2}\int_0^{\infty} (\sin sx)(-2xe^{-x^2})dx$$

$$= \frac{1}{2}\left\{\left|\sin sx \cdot e^{-x^2}\right|_0^{\infty} - s\int_0^{\infty} \cos sx \cdot e^{-x^2}\, dx\right\}$$

$$= -\frac{s}{2}\int_0^{\infty} e^{-x^2}\cos sx\, dx = -\frac{s}{2} I \quad \text{or} \quad \frac{dI}{I} = -\int \frac{s}{2} ds + \log c$$

or
$$\log I = -\frac{s^2}{4} + \log c = \log\,(ce^{-s^2/4})$$

$$\therefore \quad I = ce^{-s^2/4} \quad \text{or} \quad \int_0^{\infty} e^{-x^2}\cos sx\, dx = ce^{-s^2/4}$$

Putting $s = 0$,
$$c = \int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}. \ \ i.e.\ I = \frac{\sqrt{\pi}}{2} e^{-s^2/4}.$$

Hence
$$F_c(e^{-x^2}) = \frac{\sqrt{\pi}}{2} e^{-s^2/4}.$$

Example 22.6. *Find the Fourier sine transform of* $e^{-|x|}$.

Hence show that $\int_0^{\infty} \dfrac{x \sin mx}{1+x^2} dx = \dfrac{\pi e^{-m}}{2}$, $m > 0$. (*V.T.U., 2010 ; S.V.T.U., 2008 ; Kottayam, 2005*)

Solution. x being positive in the interval $(0, \infty)$, $e^{-|x|} = e^{-x}$

$\therefore$ Fourier sine transform of $f(x) = e^{-|x|}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \left| \frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right|_0^\infty = \frac{s}{1+s^2}$$

Using Inversion formula for Fourier sine transforms, we get

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s\{f(x)\} \sin sx \, dx \quad \text{or} \quad e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s}{1+s^2} \sin sx \, ds$$

or changing x to m, $$e^{-m} = \frac{2}{\pi} \int_0^\infty \frac{s \sin ms}{1+s^2} ds = \frac{2}{\pi} \int_0^\infty \frac{x \sin mx}{1+m^2} dx$$

Hence $\displaystyle\int_0^\infty \frac{x \sin mx}{1+m^2} dx = \frac{\pi e^{-m}}{2}$.

Example 22.7. *Find the Fourier cosine transform of* $f(x) = \begin{cases} x, \text{ for } 0 < x < 11 \\ 2 - x, \text{ for } 1 < x < 2 \\ 0, \text{ for } x > 2 \end{cases}$ *(J.N.T.U., 2006)*

Solution. Fourier cosine transform of $f(x)$ *i.e.*, $F_c[f(x)]$

$$= \int_0^\infty f_c(x) \cos sx \, dx = \int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx + \int_2^\infty 0 \, . \, dx$$

$$= \left| x \frac{\sin sx}{s} - \left(\frac{-\cos sx}{s^2} \right) \right|_0^1 + \left| (2-x) \frac{\sin sx}{s} - (-1) \frac{-\cos sx}{s^2} \right|_1^2$$

$$= \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} \right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right)$$

$$= \frac{2 \cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}.$$

Example 22.8. *Find the Fourier sine transform of* e^{-ax}/x. *(V.T.U., 2010 S ; P.T.U., 2006 ; Rohtak, 2005)*

Solution. Let $f(x) = e^{-ax}/x$, then its Fourier sine transform

i.e. $$F_s\{f(x)\} = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = F(s), \text{ say}$$

Differentiating both sides w.r.t. s, we get

$$\frac{d}{ds}\{F(s)\} = \int_0^\infty \frac{xe^{-ax} \cos sx}{x} dx = \int_0^\infty e^{-ax} \cos sx \, dx = \frac{a}{s^2 + a^2}$$

Integrating w.r.t. s, we obtain $F(s) = \displaystyle\int_0^\infty \frac{a}{s^2+a^2} ds = \tan^{-1} \frac{s}{a} + c$

But $F(s) = 0$, when $s = 0$; $\therefore$ $c = 0$. Hence $F(s) = \tan^{-1}(s/a)$.

Example 22.9. *Find the Fourier cosine transform of* $f(x) = 1/(1 + x^2)$. *(V.T.U., 2011 S ; Anna, 2009)*
Hence derive Fourier sine transform of $\phi(x) = x/(1 + x^2)$. *(V.T.U., 2009 S)*

Solution. $$F_c[f(x)] = \int_0^\infty \frac{\cos sx}{1+x^2} dx = I, \text{ say} \quad ...(i)$$

$\therefore$ $$\frac{dI}{ds} = \int_0^\infty \frac{-x \sin sx}{1+x^2} dx = -\int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)} dx \quad ...(ii)$$

$$= - \int_0^\infty \frac{[(1+x^2)-1]\sin sx}{x(1+x^2)} dx = - \int_0^\infty \frac{\sin sx}{x} dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx$$

or
$$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} dx \qquad ...(iii)$$

$\therefore$
$$\frac{d^2 I}{ds^2} = \int_0^\infty \frac{x \cos sx}{x(1+x^2)} dx = I$$

or
$$\frac{d^2 I}{ds^2} - I = 0 \quad \text{or} \quad (D^2 - 1)I = 0, \text{ where } D = \frac{dI}{ds}$$

Its solution is $\quad I = c_1 e^s + c_2 e^{-s}$ ...(*iv*)

$\therefore \quad dI/ds = c_1 e^s - c_2 e^{-s}$ ...(*v*)

When $s = 0$, (*i*) and (*iv*) give $c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

Also when $s = 0$, (*iii*) and (*v*) give $c_1 - c_2 = -\pi/2$.

Solving these, $c_1 = 0, c_2 = \pi/2$.

Thus from (*i*) and (*iv*), we have $F_c[f(x)] = I = (\pi/2)e^{-s}$

Now
$$F_s[\phi(x)] = \int_0^\infty \frac{x \sin sx}{1+x^2} dx = -\frac{dI}{ds}, \text{ from } (ii)$$
$$= (\pi/2)e^{-s}, \text{ from } (v), \text{ with } c_1 = 0, c_2 = \pi/2.$$

Example 22.10. *Find the Fourier sine and cosine transform of x^{n-1}, $n > 0$.* (*Madras, 2006*)

Solution. We know that $F_s(x^{n-1}) = \int_0^\infty x^{n-1} \sin sx \, dx$ ...(*i*)

and
$$F_c(x^{n-1}) = \int_0^\infty x^{n-1} \cos sx \, dx \qquad ...(ii)$$

$\therefore$
$$F_c(x^{n-1}) + i F_s(x^{n-1}) = \int_0^\infty (\cos sx + i \sin sx) x^{n-1} dx$$
$$= \int_0^\infty e^{isx} x^{n-1} dx = \int_0^\infty e^{-t} \left(-\frac{t}{is}\right)^{n-1} \left(-\frac{dt}{is}\right) \qquad [\text{Where } isx = -t]$$
$$= \left(-\frac{1}{(i)}\right)^n \int_0^\infty e^{-t} t^{n-1} dt = \frac{(i)^{2n}}{(i)^n s^n} \Gamma(n) = \frac{(i)^n}{s^n} \Gamma(n)$$
$$= \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^n \Gamma(n)/s^n = \left(\cos\frac{n\pi}{2} + i \sin\frac{n\pi}{2}\right) \Gamma(n)/s^n$$

Equating real and imaginary parts, we get

$$F_c(x^{n-1}) = \frac{\Gamma(n)}{s^n} \cos\frac{n\pi}{2} \quad \text{and} \quad F_s(x^{n-1}) = \frac{\Gamma(n)}{s^n} \sin\frac{n\pi}{2}.$$

Example 22.11. *(a) Show that* $F_s[x f(x)] = -\frac{d}{ds}\{F_c(s)\}$; $F_c[x f(x)] = \frac{d}{ds}\{F_s(s)\}$.

(b) Find the Fourier sine and cosine transform of xe^{-ax} (*Madras, 2006*)

Solution. (*a*)
$$\frac{d}{ds}\{F_c(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x) \cos sx \, dx\right\} = \int_0^\infty f(x)(-x \sin sx) dx$$
$$= -\int_0^\infty \{x f(x)\} \sin sx dx = -F_s[x f(x)] \qquad ...(i)$$

$$\frac{d}{ds}\{F_s(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x) \sin sx \, dx\right\} = \int_0^\infty f(x)(x \cos sx) dx$$
$$= \int_0^\infty \{x f(x)\} \cos sx dx = F_c[x f(x)] \qquad ...(ii)$$

(*b*) We have $$F_s(e^{-ax}) = \int_0^\infty e^{-ax} \sin sx\, dx = \frac{e^{-ax}}{a^2+s^2} \left| -a \sin sx - s \cos sx \right|_0^\infty$$

$$= \frac{s}{a^2+s^2} \qquad ...(iii)$$

and $$F_c(e^{-ax}) = \int_0^\infty e^{-ax} \cos sx\, dx = \frac{e^{-ax}}{a^2+s^2} \left| -a \cos sx + s \sin sx \right|_0^\infty$$

$$= \frac{a}{a^2+s^2} \qquad ...(iv)$$

Now $$F_c(xe^{-ax}) = -\frac{d}{ds}\{F_c(e^{-ax})\} \qquad \text{[by } (i)$$

$$= -\frac{d}{ds}\left(\frac{a}{a^2+s^2}\right) = \frac{2as}{(a^2+s^2)^2} \qquad \text{[by } (iv)$$

$$F_c(xe^{-ax}) = \frac{d}{ds}\{F_s(e^{-ax})\} \qquad \text{[by } (ii)$$

$$= \frac{d}{ds}\left(\frac{s}{a^2+s^2}\right) = \frac{(a^2+s^2) - s(2s)}{(a^2+s^2)^2} = \frac{a^2-s^2}{(a^2+s^2)^2}. \qquad \text{[by } (iii)$$

Example 22.12. *If the Fourier sine transform of* $f(x) = \dfrac{1-\cos n\pi}{n^2\pi^2}$ $(0 \le x \le \pi)$, *find* $f(x)$. (*Delhi, 2002*)

Solution. We have $f(x)$ = inverse finite Fourier sine transform of $F_s(n)$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-\cos n\pi}{n^2\pi^2}\right\} \sin nx$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\left\{\frac{1-\cos n\pi}{n^2}\right\} \sin nx.$$

Example 22.13. *Solve the integral equation**

$$\int_0^\infty f(\theta) \cos \alpha\theta\, d\theta = \begin{bmatrix} 1-\alpha, & 0 \le a \le 1 \\ 0 & , \alpha > 1 \end{bmatrix}$$

Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt$. (*V.T.U., 2011 S ; Kurukshetra, 2005*)

Solution. We have $$\int_0^\infty f(\theta) \cos \alpha\theta\, d\theta = F_c(\alpha)$$

$$\therefore \quad F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le a \le 1 \\ 0 & , \alpha > 1 \end{cases} \qquad ...(i)$$

By the inversion formula, we have

$$f(\theta) = \frac{2}{\pi}\int_0^\infty F_c(\alpha) \cos \alpha\theta\, d\alpha = \frac{2}{\pi}\int_0^1 (1-\alpha) \cos \alpha\theta\, d\alpha \qquad \text{[Integrating by parts]}$$

$$= \frac{2}{\pi}\left[\left|(1-\alpha)\frac{\sin \alpha\theta}{\theta}\right|_0^1 - \int_0^1 (-1)\frac{\sin \alpha\theta}{\theta}\, d\alpha\right] = \frac{2}{\pi\theta}\left|-\frac{\cos \alpha\theta}{\theta}\right|_0^1 = \frac{2(1-\cos\theta)}{\pi\theta^2}$$

Now $$F_c(\alpha) = \int_0^\infty f(\theta) \cos \alpha\theta\, d\theta = \int_0^\infty \frac{2(1-\cos\theta)}{\pi\theta^2} \cos \alpha\theta\, d\theta \qquad ...(ii)$$

* Refer to Chapter 26.

$\therefore$ From (*i*) and (*ii*), we have

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\cos\alpha\theta\, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0 & , \alpha > 1 \end{cases}$$

Now letting $\alpha \to 0$, we get $\frac{2}{\pi}\int_0^\infty \frac{1-\cos\theta}{\theta^2}\, d\theta = 1$ (*V.T.U., 2008*)

or $$\int_0^\infty \frac{2\sin^2\theta/2}{\theta^2}\, d\theta = \pi/2$$ [Put $\theta/2 = t$, so that $d\theta = 2dt$]

$\therefore$ $$\int_0^\infty \frac{\sin^2 t}{t^2} = \pi/2.$$

PROBLEMS 22.1

1. Express the function $f(x) = \begin{cases} 1 \text{ for } |x| \le 1 \\ 0 \text{ for } |x| > 1 \end{cases}$ as a Fourier integral.

   Hence evaluate $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda.$ (*Kottayam, 2005*)

2. Find the Fourier integral representation for

   (*i*) $f(x) = \begin{cases} 1-x^2, \text{ for } |x| \le 1 \\ 0, \text{ for } |x| > 1 \end{cases}$ (*Mumbai, 2008*) (*ii*) $f(x) = \begin{cases} e^{ax}, \text{ for } x \le 0, a > 0 \\ e^{-ax}, \text{ for } x \ge 0\ a < 0 \end{cases}$

3. Using the Fourier integral representation, show that

   (*i*) $\int_0^\infty \frac{\omega\sin x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x}\ (x > 0)$ (*ii*) $\int_0^\infty \frac{\cos\omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2} e^{-x}\quad (x \ge 0)$ (*U.P.T.U., 2008*)

   (*iii*) $\int_0^\infty \frac{\sin\omega\cos x\omega}{\omega}\, d\omega = \frac{\pi}{2}$ when $0 \le x < 1.$ (*iv*) $\int_0^\infty \frac{\sin\pi\alpha\sin\alpha\theta}{1-\alpha^2}\, d\alpha = \begin{cases} \frac{1}{2}\pi\sin\theta, & 0 \le \theta \le \pi \\ 0, & \theta > \pi \end{cases}$

4. Find the Fourier transforms of

   (*i*) $f(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$ (*W.B.T.U., 2005 ; Madras, 2003 ; P.T.U., 2003*)

   Hence evaluate $\int_{-\infty}^\infty \frac{\sin ax}{x}\, dx$ (*Mumbai, 2009*)

   (*ii*) $f(x) = \begin{cases} x^2, |x| < a \\ 0, |x| > a \end{cases}$ (*S.V.T.U., 2008*)

5. Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2 \text{ for } |x| \le a \\ 0 \text{ for } |x| > a \end{cases}$ (*V.T.U., 2007*)

   Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}.$ (*Anna, 2009*)

6. Given $F(e^{-x^2}) = \sqrt{\pi}e^{-s^2/4}$, find the Fourier transform of

   (*i*) $e^{-x^2/3}$ (*ii*) $e^{-4(x-3)^2}$

7. Find the Fourier sine and cosine transforms of $f(x) = \begin{cases} 1, 0 \le x < 2 \\ 0, x \ge 2 \end{cases}$ (*V.T.U., 2008*)

8. Using the Fourier sine transform of e^{-ax} $(a > 0)$, show that $\int_0^\infty \frac{x\sin kx}{a^2+x^2}\, dx = \frac{\pi}{2}e^{-ak}\ (k > 0).$

   Hence obtain the Fourier sine transform of $x/(a^2 + x^2)$. (*Rohtak, 2006 ; Madras, 2003 S*)

9. Find the Fourier cosine transform of e^{-ax}. (*Anna, 2009*)

   Hence evaluate $\int_0^\infty \frac{\cos\lambda x}{x^2+\alpha^2}\, dx.$ (*V.T.U., 2003 S*)

10. If the Fourier sine transform of $f(x)$ is e^{-as}/s, find $f(x)$. Hence obtain the inverse Fourier sine transform of $1/s$. (*Mumbai, 2009*)

11. Find the Fourier cosine transform of e^{-x^2} and hence evaluate Fourier sine transform of xe^{-x^2}.

12. Find the Fourier cosine transform of $e^{-a^2x^2}$ for any $a > 0$ and hence prove that $e^{-x^2/2}$ is self-reciprocal under Fourier cosine transform. *(Anna, 2009)*

13. Find the Fourier sine transform of (*i*) $\dfrac{1}{x(x^2+a^2)}$. *(Rohtak, 2006)*

(*ii*) $[e^{-ax}/x]$, $a > 0$ *(U.P.T.U., 2008)*

14. Obtain Fourier sine transform of

(*i*) $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$ *(Madras, 2000)* (*ii*) $f(x) = \begin{cases} 4x, & \text{for } 0 < x < 1 \\ 4 - x, & \text{for } 1 < x < 4 \\ 0, & \text{for } x > 4 \end{cases}$ *(V.T.U., 2006)*

15. Find the Fourier cosine transform of $(1 - x/\pi)^2$. *(P.T.U., 2006)*

16. Find the finite Fourier sine and cosine transforms of $f(x) = 2x$, $0 < x < 4$. *(V.T.U., 2011)*

17. Find the finite sine transform of $f(x) = \begin{cases} -x, & x < c \\ \pi - x, & x > c \end{cases}$ where $0 \le c \le \pi$.

18. Show that the inverse finite Fourier sine transform of $F_s(n) = \dfrac{1}{\pi}\left\{1 + \cos n\pi - 2\cos\dfrac{n\pi}{2}\right\}$ is

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{cases}$$ *(V.T.U., 2008)*

19. Solve the integral equation $\int_0^\infty f(x) \sin tx\, dx = \begin{cases} 1, & 0 \le t < 1, \\ 2, & 1 \le t < 2, \\ 0, & t \ge 2 \end{cases}$ *(Kottayam, 2005)*

20. Solve the integral equation $\int_0^\infty f(x) \cos \alpha x\, dx = e^{-\alpha}$. *(S.V.T.U., 2009 ; Rohtak, 2004)*

22.6 (1) CONVOLUTION

The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as

$$f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u) du = h(x).$$

(2) **Convolution theorem for Fourier transforms.** *The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms, i.e.,*

$$F\{f(x) * g(x)\} = F\{f(x)\} \,.\, F\{g(x)\}$$

We have $$F\{f(x) * g(x)\} = F\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u) du\right\}$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)du\right\} e^{isx}\, dx = \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} g(x-u)\,.\, e^{isx}\, dx\right\}du$$

[Changing the order of integration]

$$= \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} e^{is(x-u)}\,.\, g(x-u)\, d(x-u)\right\} e^{isu}\, du$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u)\left\{\int_{-\infty}^{\infty} e^{ist}\, g(t)\, dt\right\} du \text{ where } x - u = t$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u)\, du \,.\, F\{g(t)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \,.\, F\{g(x)\} = F\{f(x)\} \,.\, F\{g(x)\}$$

22.7 PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If the Fourier transforms of $f(x)$ and $g(x)$ are $F(s)$ and $G(s)$ respectively, then

(*i*) $\dfrac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\overline{G}(s) ds = \int_{-\infty}^{\infty} f(x)\overline{g}(x) dx$ (*ii*) $\dfrac{1}{2\pi}\int_{-\infty}^{\infty} [F(s)]^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$

where bar implies the complex conjugate.

(*i*) $\int_{-\infty}^{\infty} f(x)\overline{g}(dx) \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G}(s)\, e^{isx}\, ds\right\}dx$ [Using the inversion formula for Fourier transform]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G}(s)\left\{\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx\right\}ds$$ [Changing the order of integration]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G}(s)F(s)ds,$$ by definition of F-transform.

(*ii*) Taking $g(x) = f(x)$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\overline{F}(s)ds = \int_{-\infty}^{\infty} f(x)\,\overline{f}(x)dx \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

Obs. The following Parseval's identities for Fourier cosine and sine transforms can be proved as above :

(*i*) $\frac{2}{\pi}\int_0^{\infty} F_c(s)\, G_c(s)\, ds = \int_0^{\infty} f(x)\, g(x) dx$ (*ii*) $\frac{2}{\pi}\int_0^{\infty} F_s(s)\, G_s(s)\, ds = \int_0^{\infty} f(x)\, g(x) dx$

(*iii*) $\frac{2}{\pi}\int_0^{\infty} [F_c(s)]^2\, ds = \int_0^{\infty} [f(x)]^2\, dx$ (*iv*) $\frac{2}{\pi}\int_0^{\infty} [F_s(s)]^2\, ds = \int_0^{\infty} [f(x)]^2\, dx.$

Example 22.14. *Using Parseval's identities, prove that*

(*i*) $\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$ *(S.V.T.U., 2009 ; [illegible] 2008)*

(*ii*) $\int_0^{\infty} \frac{t^2}{(t^2+1)^2}\, dt = \frac{\pi}{4}$ (*iii*) $\int_0^{\infty} \frac{\sin at}{t(a^2+t^2)}\, dt = \frac{\pi}{2}\cdot\frac{1-e^{-a^2}}{a^2}.$

Solution. (*i*) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. Then $F_c(s) = \frac{a}{a^2+s^2}, G_c(s) = \frac{b}{b^2+s^2}$

Now using Parseval's identity for Fourier cosine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} F_c(s) G_c(s) ds = \int_0^{\infty} f(x)\, g(x) dx \quad ...(1)$$

We have $$\frac{2}{\pi}\int_0^{\infty} \frac{ab}{(a^2+s^2)(b^2+s^2)}\, ds = \int_0^{\infty} e^{-(a+b)x} dx$$

or $$\frac{2ab}{\pi}\int_0^{\infty} \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left|\frac{e^{-(a+b)x}}{-(a+b)}\right|_0^{\infty} = \frac{1}{a+b}$$

Thus $$\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

(*ii*) Let $f(x) = \frac{x}{x^2+1}$ so that $F_s[f(x)] = \frac{\pi}{2} e^{-s}$

Now using Parseval's identity for sine transform, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} [F_s\{f(x)\}]^2\, ds = \int_0^{\infty} [f(x)]^2\, dx$$

or $$\int_0^{\infty} \left(\frac{x}{x^2+1}\right)^2 dx = \frac{2}{\pi}\int_0^{\infty} \left(\frac{\pi}{2} e^{-s}\right)^2 ds = \frac{\pi}{2}\left| e^{-2s}/-2\right|_0^{\infty} = \frac{\pi}{-4}(0-1) = \frac{\pi}{4}$$

Hence $$\int_0^{\infty} \frac{t^2}{(t^2+1)^2}\, dt = \frac{\pi}{4}$$

(*iii*) Let $f(x) = e^{-ax}$ and $g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$. Then $F_c(s) = \frac{a}{a^2+s^2}, G_c(s) = \frac{\sin as}{s}$

Now using (1) above, we have $\frac{2}{\pi}\int_0^\infty \frac{a \sin as}{s(a^2+s^2)} ds = \int_0^a e^{-ax} . 1\, dx = \frac{1-e^{-a^2}}{a}$

Thus $$\int_0^\infty \frac{\sin at}{t(a^2+t^2)} dt = \frac{\pi}{2a^2}(1-e^{-a^2}).$$

Example 22.15. *Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} a-|x|, & \text{for } |x|<a \\ 0 & , \text{for } |x|>a \end{cases}$*

Hence show that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ and $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \pi/3.$ (Anna, 2008)

Solution. Fourier transform of $f(x)$ *i.e.* $F[f(x)] = \int_{-\infty}^\infty f(x)\, e^{isx}\, dx = \int_{-a}^a [a-|x|]e^{isx}\, dx$

$$= \int_{-a}^a [a-|x|](\cos x + i \sin sx) dx$$

$$= 2\int_0^a (a-x) \cos sx\, dx + 0 \qquad \left\{\begin{array}{l} \because [a-|x|]\cos x \text{ in an even function} \\ [a-|x|]\sin x \text{ is an odd function}\end{array}\right.$$

$$= 2\left|(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right|_0^a = 2\,\frac{1-\cos as}{s^2} = 4\,\frac{\sin^2 as/2}{s^2}$$

(*i*) By inversion formula,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(s)\, e^{-isx}\, ds = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{4\sin^2 as/2}{s^2} e^{-isx}\, ds$$

To evaluate $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$, put $x = 0$ and $a = 2$ so that

$$f(0) = \frac{2}{\pi}\int_{-\infty}^\infty \frac{\sin^2 s}{s^2} ds = \frac{4}{\pi}\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds \qquad \left[\because \frac{\sin s}{s} \text{ is an even function}\right]$$

$$\therefore \int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{4} f(0) = \frac{\pi}{2}. \qquad [\because f(0) = a = 2$$

(*ii*) Using Parseval's identity

$$\frac{1}{2\pi}\int_{-\infty}^\infty [F(s)]^2 ds = \int_{-\infty}^\infty |f(x)|^2\, dx$$

$$\frac{1}{2\pi}\int_{-\infty}^\infty \left(\frac{4\sin^2 as/2}{s^2}\right)^2 dx = \int_{-a}^a |[a-|x|]|^2\, dx$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin as/2}{s}\right)^4 ds = 2\int_0^a (a-x)^2\, dx = 2\left|\frac{(a-x)^3}{-3}\right|_0^a = \frac{2}{3}a^3$$

Putting $t = as/2$ and $dt = ads/2$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin t}{2t/a}\right)\frac{2}{a} dt = \frac{2}{3}a^3 \quad \text{or} \quad \frac{2a^3}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{2}{3}a^3$$

Hence $$\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

PROBLEMS 22.2

1. Verify Convolution theorem for $f(x) = g(x) = e^{-x^2}$. (V.T.U., 2000 S)
2. Use Convolution theorem to find the inverse Fourier transform of $\frac{i}{(1+s^2)^2}$, given that $\frac{2}{(1+s^2)}$ is the Fourier transform of $e^{-|x|}$. (V.T.U., 2010 S)
3. Using Parseval's identity, show that

   (i) $\int_0^\infty \frac{dx}{(t^2+1)^2} = \frac{\pi}{4}$, (Hissar, 2007) (ii) $\int_0^\infty \frac{t^2}{(4+t^2)(9+t^2)} dt = \frac{\pi}{10}$. (Rohtak, 2003)
4. Find the Fourier transform of $f(x) = \begin{cases} 1 \text{ if } |x| < 1 \\ 0 \text{ if } |x| > 1 \end{cases}$.

   Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$. (Anna, 2009)
5. Evaluate $\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx$.

22.8 RELATION BETWEEN FOURIER AND LAPLACE TRANSFORMS

If $$f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0 , & t < 0 \end{cases} \qquad ...(i)$$

then $$F\{f(t)\} = L\{g(t)\}.$$

We have $$F\{f(t)\} = \int_{-\infty}^{\infty} e^{ist} f(t)dt = \int_{-\infty}^{0} e^{ist} . 0 . dt + \int_0^\infty e^{ist} . e^{-xt} g(t)dt$$

$$= \int_0^\infty e^{(is-x)t} g(t)\, dt = \int_0^\infty e^{-pt} g(t)dt \quad \text{where } p = x - is$$

Hence *the Fourier transform of* $f(t)$ [defined by (i)] is the *Laplace transform of* $g(t)$.

22.9 FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

The Fourier transform of the function $u(x, t)$ is given by

$$F[u(x,t)] = \int_{-\infty}^{\infty} ue^{isx}\, dx$$

Then the Fourier transform of $\partial^2 u/\partial x^2$, *i.e.*

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx}\, dx = \left| e^{isx} \frac{\partial u}{\partial x} - is\, e^{isx} . u \right|_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} ue^{isx}\, dx,$$

on applying the general rule of integration by parts (p. 398). If u and $\frac{\partial u}{\partial x}$ tend to zero as x tends to $\pm\infty$, then

$$\mathbf{F}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\mathbf{s}^2 \mathbf{F}[\mathbf{u}] \qquad ...(1)$$

Similarly in the case of Fourier sine and cosine transforms, we have

$$\mathbf{F_s}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = \mathbf{s}(\mathbf{u})_{\mathbf{x}=0} - \mathbf{s}^2\mathbf{F_s}[\mathbf{u}] \qquad ...(2)$$

and $$\mathbf{F_c}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)_{\mathbf{x}=0} - \mathbf{s}^2 \mathbf{F_c}[\mathbf{u}] \qquad ...(3)$$

In general, the *Fourier transform of the nth derivative of* $f(x)$ is given by

$$\mathbf{F}\left[\frac{\mathbf{d^n f}}{\mathbf{dx^n}}\right] = \mathbf{(-\,is)^n\, F[f(x)]} \qquad ...(4)$$

provided the first $n - 1$ derivatives vanish as $x \to \pm\,\infty$.

For $\quad F[f^n(x)] = \int_{-\infty}^{\infty} f^n(x)\, e^{isx}\, dx$

$$= \left| e^{isx} f^{n-1} - ise^{isx} f^{n-2} + (is)^2 e^{isx} f^{n-3} - \ldots \right|_{-\infty}^{\infty} + (-is)^n \int_{-\infty}^{\infty} f \,.\, e^{isx}\, dx$$

by the general rule of integration by parts, whence follows (4).

22.10 INVERSE LAPLACE TRANSFORMS BY METHOD OF RESIDUES

Let the Laplace transform of $f(x)$ be $\bar{f}(s)$ so that

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st}\, dt \qquad ...(1)$$

Multiply both sides by e^{xs} and integrate w.r.t. s within the limits $a - ir$ and $a + ir$. Then

$$\int_{a-ir}^{a+ix} e^{xs}\, \bar{f}(s) ds = \int_{a-ir}^{a+ir} e^{xs} \int_0^{\infty} f(t) e^{-st}\, dt\, ds \qquad \text{[Put } s = a - iu]$$

$$= \int_r^{-r} e^{x(a-iu)} \int_0^{\infty} f(t) e^{-(a-iu)t}\, dt\, (-idu) = ie^{ax} \int_{-r}^{r} e^{-ixu} \int_0^{\infty} [e^{-at} f(t)] e^{iut}\, dtdu$$

$$= ie^{ax} \int_{-r}^{r} e^{-ixu} \int_{-\infty}^{\infty} \phi(t)\, e^{iut}\, dt\, du$$

where $\phi(t) = \begin{cases} e^{-at} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$.

Proceeding to limits as $r \to \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} e^{xs}\, \bar{f}(s)\, ds = ie^{ax} \,.\, 2\pi\phi(x), \text{ by (2) of § 22.4} = 2\pi i e^{ax}.e^{-ax} f(x) \text{ for } x > 0.$$

Hence $\qquad f(x) = \int_{a-i\infty}^{a+i\infty} e^{xs}\, \bar{f}(s) ds \; (x > 0) \qquad ...(2)$

which is called the *complex inversion formula*. It provides a direct means for obtaining the inverse Laplace transform of a given function.

The integration in (2) is performed along a line LM parallel to the imaginary axis in the complex plane $z = x + iy$ such that all the singularities of $\bar{f}(s)$ lie to its left* (Fig. 22.1). Let us take a contour C which is composed of the line LM and the semi-circle C' (*i.e.*, MNL). Then from (2)

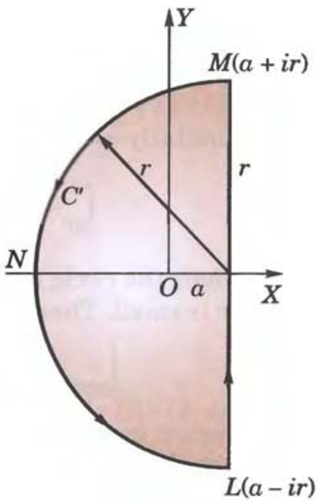


Fig. 22.1

$$\frac{1}{2\pi i} \int_{LM} e^{xs}\, \bar{f}(s)\, ds = \frac{1}{2\pi i} \int_C e^{xs}\, \bar{f}(s)\, ds - \frac{1}{2\pi i} \int_{C'} e^{xs}\, \bar{f}(s)\, dx$$

The integral over C' tends to zero as $r \to \infty$ (under certain conditions†). Therefore,

$$\mathbf{f(x)} = \underset{r \to \infty}{\mathbf{Lt}} \frac{\mathbf{1}}{\mathbf{2\pi i}} \int_C \mathbf{e^{xs}\, \bar{f}(s)\, dx}$$

$$= \textbf{sum of the residues of } e^{xs} \bar{f}(s) \textbf{ at the poles of } f(s) \qquad ...(3)$$

[By §20.18]

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* This has been so assumed simply to ensure the convergence of the integral (1).

† If positive constants A and k can be so found that $|\bar{f}(s)| < Ar^{-k}$ for every point on C', then

$$\underset{r \to \infty}{\text{Lt}} \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{f}(s)\, ds = 0. \qquad \textit{(Jordan's Lemma)}$$

Example 22.16. *Evaluate* $L^{-1}\left\{\dfrac{1}{(s-1)(s^2+1)}\right\}$ *by the method of residues.*

Solution. Since $\left|\dfrac{1}{(s-1)(s^2+1)}\right| \sim \left|\dfrac{1}{s^3}\right|$ for $|s| \to \infty$, therefore,

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \text{sum of Res}\left[\frac{e^{xs}}{(s-1)(s^2+1)}\right] \text{at the poles } s = 1, \pm i$$

Now $$(\text{Res})_{s=1} = \underset{s\to 1}{\text{Lt}}\left[\frac{(s-1)\,.\,e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^x}{2} \qquad \text{[By § 20.19 (1)]}$$

$$(\text{Res})_{s=i} = \underset{s\to i}{\text{Lt}}\left[\frac{(s-i)\,.\,e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^{ix}}{(i-1)(i-1)} = -\frac{1}{2}\cdot\frac{e^{ix}}{1+i}$$

Changing i to $-i$, we get $(\text{Res})_{s=-i} = -\dfrac{1}{2}\cdot\dfrac{e^{ix}}{1-i}$

$$\therefore \qquad L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{e^x}{2} - \frac{1}{2}\left(\frac{e^{ix}}{1+i} + \frac{e^{-ix}}{1-i}\right) = \frac{1}{2}(e^x - \sin x - \cos x).$$

Example 22.17. *Prove that* $L^{-1}\left(\dfrac{e^{-c\sqrt{s}}}{s}\right) = 1 - erf\left(\dfrac{c}{\sqrt{2x}}\right)$.

Solution. By the complex inversion formula,

$$L^{-1}\left(\frac{e^{-c\sqrt{s}}}{s}\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{xs}\,.\,\frac{e^{-c\sqrt{s}}}{s}\,ds.$$

Since $s = 0$ is a branch point of the integrand, we take a contour $LMNPQST$ as shown in Fig. 22.2, so that it doesn't include any singularity. Therefore, by Cauchy's theorem (§ 20.13), we have

$$\left\{\int_{LM} + \int_{MN} + \int_{NP} + \int_{PQS} + \int_{ST} + \int_{TL}\right\} \times e^{xs}\,\frac{e^{-c\sqrt{s}}}{s}\,ds = 0 \qquad ...(i)$$

If $ON = \rho$ and $OP = \varepsilon$, then along NP, $s = Re^{i\pi}$, therefore,

$$\int_{NP} = \int_{\rho}^{\varepsilon} e^{-xR}\,\frac{e^{-ic\sqrt{R}}}{R}\,dR$$

Similarly along ST, $s = Re^{-i\pi}$, therefore,

$$\int_{ST} = \int_{\varepsilon}^{\rho} e^{-xR}\,\frac{e^{c\sqrt{R}}}{R}\,dR$$

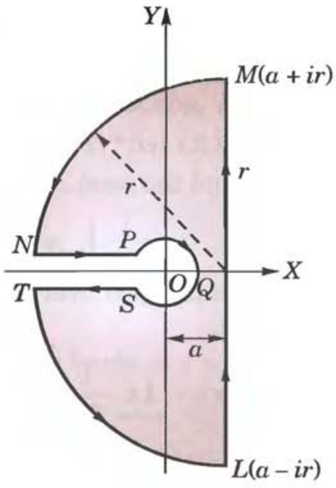

Fig. 22.2

Along the circle PQS, $s = \varepsilon\, e^{i\theta}$. Also e^{xs} and $e^{-c\sqrt{\sigma}}$ are both approximately 1 since ε is small. Therefore,

$$\int_{PQS} = \int_{\pi}^{-\pi} \frac{1}{\varepsilon e^{i\theta}}\,.\,\varepsilon e^{i\theta}\, i d\theta = -2\pi i \text{ approximately.}$$

For $c > 0$, $|\, e^{-c\sqrt{s}}/s \,| < |s|^{-1}$.

But $\int_{MN}$ and $\int_{TL}$ both tend to zero as $r \to \infty$

Thus (i) takes the form

$$\int_{a-ir}^{a+ir} \frac{e^{xs - c\sqrt{s}}}{s}\,ds + \int_{\varepsilon}^{\rho} e^{-xR}\,\frac{e^{ic\sqrt{R}} - e^{-ic\sqrt{R}}}{R}\,dR - 2\pi i = 0$$

Taking limits as $\varepsilon \to 0$ and $\rho \to \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} \frac{e^{xs - c\sqrt{s}}}{s}\,ds = 2\pi i - 2i\int_0^{\infty} e^{-xR}\,\frac{\sin c\sqrt{R}}{R}\,dR$$

or $$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{xs-c\sqrt{s}}}{s}\,ds = 1-\frac{2}{\pi}\int_0^\infty e^{-t^2}\,\frac{\sin(ct/\sqrt{x})}{t}\,dt^*, \text{ where } R = t^2/x$$

$$= 1-\frac{2}{\pi}\cdot\frac{\pi}{2}\,erf\left(\frac{C}{2\sqrt{x}}\right) \text{ whence follows the result.}$$

PROBLEMS 22.3

Using the method of residues, evaluate the inverse Laplace transform of each of the following :

1. $\dfrac{1}{(s+1)(s-2)^2}$.
2. $\dfrac{1}{(s-2)(s^2+1)}$.
3. $\dfrac{1}{s^2(s^2-a^2)}$.
4. $\dfrac{1}{(s-1)^2(s^2+1)}$.
5. $\dfrac{1}{(s^2+1)^2}$. *(V.T.U., 2008 S)*

22.11 APPLICATION OF TRANSFORMS TO BOUNDARY VALUE PROBLEMS

In one dimensional boundary value problems, the partial differential equation can easily be transformed into an ordinary differential equation by applying a suitable transform. The required solution is then obtained by solving this equation and inverting by means of the complex inversion formula or by any other method. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

(*i*) If in a problem $u(x, t)_{x=0}$ is given then we use infinite sine transform to remove $\partial u^2/\partial x^2$ from the differential equation.

In case $[\partial u(x, t)/\partial x]_{x=0}$ is given then we employ infinite cosine transform to remove $\partial^2 u/\partial x^2$.

(*ii*) If in a problem $u(0, t)$ and $u(l, t)$ are given, then we use finite sine transform to remove $\partial^2 u/\partial x^2$ from the differential equation.

In case $(\partial u/\partial x)_{x=0}$ and $(\partial u/\partial x)_{x=l}$ are given, then we employ finite cosine transform to remove $\partial^2 u/\partial x^2$.

The method of solution is best explained through the following examples.

Heat conduction

Example 22.18. *Determine the distribution of temperature in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.* *(Osmania, 2003)*

Solution. Let $u(x, t)$ be the temperature at any point x and at any time t. We have to solve the heat-flow equation (§ 18.5)

$$\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2}\ (x > 0, t > 0) \qquad ...(i)$$

subject to the initial condition $u(x, 0) = f(x)$ ...(*ii*)

and the boundary condition $u(0, t) = 0$ ...(*iii*)

Taking Fourier sine transform of (1) and denoting $F_s[u(x, t)]$ by $\bar{u}_s$, we have

$$\frac{d\bar{u}_s}{dt} = c^2\,[su(0, t) - s^2\bar{u}_s] \qquad \text{[By (2) of § 22.9]}$$

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* We know that $\int_0^\infty e^{-t^2}\cos 2mt\,dt = \frac{1}{2}\sqrt{\pi}e^{-m^2}$ [Example 20.44

Integrating both sides w.r.t. m from 0 to $c/2\sqrt{x}$.

$$\int_0^\infty e^{-t^2}\left|\frac{\sin 2mt}{2t}\right|_0^{c/2\sqrt{x}} dt = \frac{1}{2}\sqrt{\pi}\int_0^{c/2\sqrt{\pi}} e^{-m^2}\,dm$$

or $$\int_0^\infty e^{-t^2}\,\frac{\sin(ct/\sqrt{x})}{t}\,dt = \frac{\pi}{2}\,erf\left(\frac{c}{2\sqrt{x}}\right). \qquad \text{[By § 7.18(1)]}$$

or $$\frac{d\bar{u}_s}{dt} + c^2s^2\bar{u}_s = 0 \qquad \text{[By (iii)] ...(iv)]}$$

Also the Fourier sine transform of (*ii*) is $\bar{u}_s = \bar{f}(s)$ at $t = 0$. ...(*v*)

Solving (*iv*) and using (*v*), we get $\bar{u}_s = \bar{f}_s(s)e^{-c^2s^2t}$

Hence taking its inverse Fourier sine transform, we obtain

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(s)e^{-c^2s^2t} \sin xs\, ds.$$

Example 22.19. *Solve $\partial u/\partial t = 2\partial^2u/\partial x^2$, if $u(0, t) = 0$, $u(x, 0) = e^{-x}$ $(x > 0)$, $u(x, t)$ is bounded where $x > 0$, $t > 0$.* *(Rohtak, 2006)*

Solution. Given $\partial u/\partial t = 2\partial^2u/\partial x^2, x > 0, t > 0$ ...(*i*)

with boundary conditions : $u(0, t) = 0$, $u(x, t)$ is bounded ...(*ii*)

and initial condition $u(x, 0) = e^{-x}, x > 0$ ...(*iii*)

Since $u(0, t)$ is given, we take Fourier sine transform of both sides of (*i*) so that

$$\int_0^\infty \frac{\partial u}{\partial t}\sin px\, dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin px\, dx$$

or $$\frac{d}{dt}\int_0^\infty u(x, t)\sin px\, dx = 2\left[\left|\frac{\partial u}{\partial x}\sin px\right|_0^\infty - \int_0^\infty \frac{\partial u}{\partial x}. p\cos px\, dx\right] \qquad \text{(Integrating by parts)}$$

or $$\frac{d\bar{u}_s}{dt} = -2p\int_0^\infty \frac{\partial u}{\partial x}\cos px\, dx, \text{ if } \frac{\partial u}{\partial x} \to \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty \text{ where } \bar{u}_s(p, t) = \int_0^\infty u(x, t)\sin px\, dx$$

$$= -2p[|u(x, t)\cos px|_0^\infty - \int_0^\infty u(x, t) - (-p\sin px)\, dx] \qquad \text{[Again integrating by parts]}$$

$$= -2p\ [0 - u(0, t) + p\int_0^\infty u(x, t)\sin px\, dx] \qquad [\because\ u(x, t) \to 0 \text{ as } x \to \infty \text{ by } (ii)]$$

$$= 2pu(0, t) - 2p^2\bar{u}_s$$

or $$\frac{d\bar{u}_s}{dt} = -2p^2\bar{u}_s \qquad \text{[By } (ii)]$$

Integrating $\int \frac{\partial\bar{u}_s}{\bar{u}_s} - \log c = -2p^2\int dt$ or $\log\bar{u}_s - \log c = -2p^2t$

$\therefore$ $$\bar{u}_s(p, t) = ce^{-2p^2t} \qquad ...(iv)$$

Taking Fourier sine transform of both sides of (*iii*), we get

$$\int_0^\infty u(x, 0)\sin px\, dx = \int_0^\infty e^{-x}\sin px\, dx$$

or $$\bar{u}_s(p, 0) = \left|\frac{e^{-x}}{1 + p^2}(-\sin px - p\cos px)\right|_0^\infty = \frac{p}{1 + p^2} \qquad ...(v)$$

Putting $t = 0$ in (*iv*) and using (*v*), we obtain $p/(1 + p^2) = c$

Thus (*iv*) becomes $\bar{u}_s(p, t) = \dfrac{p}{1 + p^2}e^{-2p^2t}$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{pe^{-2p^2t}}{1 + p^2}\sin px\, dp.$$

Example 22.20. *Solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $(x > 0, t > 0)$ subject to the conditions*

(i) $u = 0$, when $x = 0, t > 0$ (ii) $u = \begin{cases} 1, 0 < x < 1 \\ 0, x \le 1, \text{when } t = 0\end{cases}$ (iii) $u(x, t)$ is bounded. *(U.P.T.U., 2003 S)*

Solution. Since $u(0, t) = 0$, we take Fourier sine transform of both sides of the given equation, we get

$$\int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

$$\frac{\partial}{\partial t} \int_0^\infty u \sin sx \, dx = -s^2 \, \bar{u}(s) + s \, u(0) \qquad [\because u = 0, \text{ when } x = 0]$$

or $$\frac{\partial \bar{u}}{\partial t} = -s^2 \bar{u} \quad \text{or} \quad \frac{\partial \bar{u}}{\partial t} + s^2 \bar{u} = 0 \quad \text{or} \quad (D^2 + s^2)\bar{u} = 0 \text{ i.e., } D = \pm is$$

$\therefore$ Its solution is $\bar{u}(s, t) = e^{-s^2 t}$ ...(1)

Since $$\bar{u}(s, t) = \int_0^\infty u(x, t) \sin sx \, dx$$

$\therefore$ $$\bar{u}(s, 0) = \int_0^\infty u(x, 0) \sin sx \, dx = \int_0^1 1 \,.\, \sin sx \, dx \qquad [\text{By } (ii)]$$

$$= \frac{1 - \cos s}{s} \qquad ...(2)$$

From (1) and (2), $$c = \bar{u}(s, 0) = \frac{1 - \cos s}{s}$$

Thus (1) gives $$\bar{u}(s, t) = \frac{1 - \cos s}{s} e^{-s^2 t}$$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \int_0^\infty \frac{1 - \cos s}{s} e^{-s^2 t} \, ds$$

which is the desired solution.

Example 22.21. *Using finite Fourier transform, solve*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

given $u(0, t) = 0$, $u(4, t) = 0$ and $u(x, 0) = 2x$ where $0 < x < 4$, $t > 0$. *(Rajasthan, 2006)*

Solution. Since $u(0, t) = 0$, we take finite Fourier sine transform of both sides of the given equation

$$\int_0^4 \frac{\partial u}{\partial t} \sin \frac{n\pi}{4} x \, dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi}{4} x \, dx$$

or $$\frac{d}{dt}(\bar{u}s) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$= -\frac{n^2\pi^2}{16} \bar{u}_s + \frac{n\pi}{4} [u(0, t) - (-1)^n u(4, t)]$$

$$= -\frac{n^2\pi^2}{16} \bar{u}_s \qquad [\because u(0, t) = 0, u(4, t) = 0.]$$

or $$\frac{d\bar{u}_s}{\bar{u}_s} = -\frac{n^2\pi^2}{16} dt$$

Integrating both sides, $$\log \bar{u}_s = -\frac{n^2\pi^2}{16} t + c$$

or $$\bar{u}_s(x, 0) = \alpha e^{\frac{-n^2\pi^2 t}{16}} \qquad ...(i)$$

Putting $t = 0$, $$a = \bar{u}_s(x, 0) = \int_0^4 u(x, 0) \sin \frac{n\pi x}{4} dx \qquad [\because \; u(x, 0) = 2x]$$

$$= \int_0^4 2x \sin \frac{n\pi x}{4} dx = -\frac{32}{n\pi} \cos n\pi$$

Thus (*i*) gives, $\bar{u}_s(x, 0) = -\dfrac{32}{n\pi}\cos n\pi e^{-n^2\pi^2 t/16} = -\dfrac{32}{n\pi}(-1)^n\, e^{-n^2\pi^2 t/16}$

Now taking inverse Fourier sine transform, we get

$$u(x, 0) = \frac{2}{4}\sum_{n=1}^{\infty}\frac{32}{n\pi}(-1)^{n+1}\, e^{-n^2\pi^2 t/16}\sin\left(\frac{n\pi x}{4}\right)$$

$$= 16\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}\, e^{-n^2\pi^2 t/16}\sin\left(\frac{n\pi x}{4}\right).$$

Example 22.22. *If the initial temperature of an infinite bar is given by*

$$\theta(x) = \begin{cases}\theta_0 \text{ for } |x| < a\\ 0 \text{ for } |x| > a,\end{cases}$$

determine the temperature at any point x and at any instant t. (*S.V.T.U., 2008 ; Rohtak, 2004*)

Solution. To determine the temperature $\theta(x, t)$ at any point at any time, we have to solve the equation

$$\frac{\partial\theta}{\partial t} = c^2\frac{\partial^2\theta}{\partial x^2}\quad (t > 0) \qquad ...(i)$$

subject to the initial condition $\theta(x, 0) = \begin{cases}\theta_0 \text{ for } |x| < a\\ 0 \text{ for } |x| > a\end{cases}$ ...(*ii*)

Taking Fourier transform of (*i*) and denoting $F[\theta(x, t)]$ by $\bar{\theta}$, we find

$$\frac{d\bar{\theta}}{dt} = -c^2 s^2\bar{\theta} \qquad \text{[by (1) of § 22.9] ...(iii)}$$

Also the Fourier transform of (2) is

$$\bar{\theta}(s, 0) = \int_{-\infty}^{\infty}\theta(x, 0)\, e^{isx}\, dx = \int_{-a}^{a}\theta_0\, e^{isx}\, dx = \theta_0\frac{e^{isa} - e^{-isa}}{is} = 2\theta_0\frac{\sin as}{s} \qquad ...(iv)$$

Solving (*iii*) and using (*iv*), we get $\bar{\theta} = \dfrac{2\theta_0 \sin as}{s}\, e^{-c^2 s^2 t}$

Hence taking its inverse Fourier transform, we get

$$\theta(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\theta_0\sin as}{s}\, e^{-c^2s^2t}\, e^{-isx}\, ds = \frac{\theta_0}{\pi}\int_{-\infty}^{\infty}\frac{\sin as}{s}\, e^{-c^2s^2t}\,(\cos xs - i\sin xs)\, ds$$

$$= \frac{2\theta_0}{\pi}\int_0^{\infty}\frac{\sin as}{s}\, e^{-c^2s^2t}\cos xs\, ds \qquad \left\{\begin{array}{l}\text{The second integral vanishes as}\\ \text{its integrand is an odd function}\end{array}\right.$$

$$= \frac{\theta_0}{\pi}\int_0^{\infty} e^{-c^2s^2t}\,\frac{\sin(a + x)s + \sin(a - x)s}{s}\, ds$$

$$= \frac{\theta_0}{\pi}\int_0^{\infty} e^{-v^2}\left\{\sin\frac{(a + x)v}{c\sqrt{t}} + \sin\frac{(a - x)v}{c\sqrt{t}}\right\}\frac{dv}{v} \quad \text{where } v^2 = c^2s^2t$$

$$= \frac{\theta_0}{\pi}\left\{erf\,\frac{(a + x)}{2c\sqrt{t}} + erf\,\frac{(a - x)}{2c\sqrt{t}}\right\}. \qquad \text{[See footnote on p. 783]}$$

Example 22.23. *A bar of length a is at zero temperature. At t = 0, the end x = a is suddenly raised to temperature u_0 and the end x = 0 is insulated. Find the temperature at any point x of the bar at any time t > 0, assuming that the surface of the bar is insulated.*

Solution. Here we have to solve the differential equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}\quad (0 < x < a,\ t > 0) \qquad ...(i)$$

subject to the conditions

$u(x, 0) = 0$ ...(*ii*) ; $u_x(0, t) = 0$ ...(*iii*) and $u(a, t) = u_0$ (*Rohtak, 2005*) ...(*iv*)

The Laplace transform of (i), if $L[u(x, t)] = \bar{u}(x, s)$, is

$$s\bar{u} - u(x, 0) = c^2 \frac{d^2\bar{u}}{dx^2}$$

Using (ii), we get $$\frac{d^2\bar{u}}{dx^2} - \frac{s}{c^2}\bar{u} = 0 \qquad ...(v)$$

Similarly the Laplace transform of (iii) and (iv) are

$$\bar{u}_x(0, s) = 0 \qquad ...(vi)\,; \qquad \bar{u}(a, s) = \frac{u_0}{s} \qquad ...(vii)$$

Solving (v), we have $$\bar{u} = C_1 e^{x\sqrt{s}/c} + C_2 e^{-x\sqrt{s}/c}$$

Using (vi), we find $C_1 = C_2$ so that

$$\bar{u} = C_1 (e^{\sqrt{s}x/c} + e^{-\sqrt{s}x/c}) = 2C_1 \cosh(\sqrt{s}x/c)$$

Now using (vii), we have $$\bar{u} = \frac{u_0 \cosh(\sqrt{s}x/c)}{s\cosh(\sqrt{s}a/c)}$$

By the inversion formula (3) § 22.10, we get

$$u(x, t) = \text{sum of the residues of } \left(\frac{e^{st} . u_0 \cosh(\sqrt{s}x/c)}{s\cosh(\sqrt{s}a/c)}\right) \text{ at all the poles which occur at } s = 0$$

and $$\cosh(\sqrt{s}a/c) = 0 \text{ i.e., at } s = 0,\ \sqrt{s}a/c = \left(n - \frac{1}{2}\right)\pi i,\ n = 0, \pm 1, \pm 2, ...$$

or at $$s = 0,\ s\ (= s_n) = -\frac{(2n-1)^2 c^2 \pi^2}{4a^2} = 0, 1, 2, ...$$

Now $$(\text{Res})_{s=0} = \underset{s\to 0}{\text{Lt}} \left\{s . \frac{u_0 e^{st} \cosh(\sqrt{s}x/c)}{s\cosh(\sqrt{s}a/c)}\right\} = u_0$$

$$(\text{Res})_{s=s_n} = u_0 \underset{s\to s_n}{\text{Lt}} \left\{(s - s_n) . \frac{u_0 e^{st} \cosh(\sqrt{s}x/c)}{s\cosh(\sqrt{s}a/c)}\right\}$$

$$= u_0 \underset{s\to s_n}{\text{Lt}} \left\{\frac{s - s_n}{\cosh(\sqrt{s}a/c)}\right\} . \underset{s\to s_n}{\text{Lt}} \left\{\frac{e^{st}\cosh(\sqrt{s}x/c)}{s}\right\} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= u_0 \underset{s\to s_n}{\text{Lt}} \frac{1}{\sinh(\sqrt{s}a/c) . (a/2\sqrt{s}/c)} . \underset{s\to s_n}{\text{Lt}} \left\{\frac{e^{st}\cosh(\sqrt{s}x/c)}{s}\right\}$$

$$= \frac{4u_0(-1)^n}{(2n-1)\pi} e^{-(2n-1)^2\pi^2c^2t/4a^2} \cos\frac{(2n-1)\pi x}{2a}$$

Thus we get $$u(x, t) = u_0 + \frac{4u_0}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2\pi^2c^2t/4a^2} \cos\frac{(2n-1)\pi x}{2a}.$$

Vibrations of a string

Example 22.24. *An infinite string is initially at rest and that the initial displacement is $f(x)$, $(-\infty < x < \infty)$. Determine the displacement $y(x, t)$ of the string.* (*Rohtak, 2000*)

Solution. The equation for the vibration of the string is

$$\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2 \qquad ...(i)$$

and the initial conditions are

$$(\partial y/\partial t)_{t=0} = 0\,;\ y(x, 0) = f(x) \qquad ...(ii)$$

Multiplying (i) by e^{isx} and integrating w.r.t. x from $-\infty$ to ∞, we get

$$\partial^2Y/\partial t^2 = c^2(-s^2Y) \qquad \text{provided } y \text{ and } \partial y/\partial x \to 0 \text{ as } x \to \infty$$

$\therefore$ a solution of $d^2Y/dt^2 + c^2s^2Y = 0$ is $Y = A_1 \cos cst + A_2 \sin cst$ ...(*iii*)

Also Fourier transforms of (*ii*) are

$$\partial y/\partial t = 0 \quad \text{and} \quad Y = F(s) \text{ when } t = 0$$

Applying these to (*iii*), we get

$$A_2 = 0 \quad \text{and} \quad A_1 = F(s)$$

Thus $Y = F(s) \cos cst$

Now taking inverse Fourier transforms, we get

$$y(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \cos cst \,.\, e^{-isx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) \frac{e^{icst} + e^{-icst}}{2} \,.\, e^{-isx}\, dx$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} [F(s)\, e^{-is(x-ct)} + F(s) e^{-is(x+ct)}]\, ds$$

$$= \frac{1}{2} [f(x - ct) + f(x + ct)] \qquad \left[\because\ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds\right.$$

Example 22.25. *An infinitely long string having one end at $x = 0$, is initially at rest along the x-axis. The end $x = 0$ is given a transverse displacement $f(t)$, $t > 0$. Find the displacement of any point of the string at any time.*

Solution. Let $y(x, t)$ be the transverse displacement of any point x of the string at any time t. Then we have to solve the wave equation (§ 18.4)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad (x > 0, t > 0) \qquad ...(i)$$

subject to the conditions $y(x, 0) = 0$, $y_t(x, 0) = 0$, $y(0, t) = f(t)$ and the displacement $y(x, t)$ is bounded.

The Laplace transform of (*i*), writing $L[y(x, t)] = \bar{y}(x, s)$ is

$$s^2\bar{y} - sy(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2}$$

Using the first two conditions, we have

$$\frac{\partial^2 \bar{y}}{\partial x^2} = \left[\frac{s}{c}\right]^2 \bar{y} \qquad ...(ii)$$

Similarly the Laplace transforms of the third and fourth conditions are

$$\bar{y}(0, s) = \bar{f}(s) \text{ at } x = 0 \qquad ...(iii) \qquad \text{and} \quad \bar{y}(x, s) \text{ is bounded.} \qquad ...(iv)$$

Solving (*ii*), we get

$$\bar{y}(x, s) = C_1 e^{sx/c} + C_2 e^{-sx/c}$$

To satisfy condition (*iv*), we must have $C_1 = 0$

Using the condition (*iii*), we get $C_2 = \bar{f}(s)$.

$\therefore$ $\bar{y}(x, s) = \bar{f}(s) e^{-sx/c}$

Using the complex inversion formula, we obtain

$$y = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} e^{(t - x/c)s}\, \bar{f}(s)\, ds = f(t - x/c).$$

Example 22.26. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where μ is a constant and then released. Find the displacement of any point x of the string at any time $t > 0$.* (V.T.U., M.E., 2006)

Solution. We have to solve the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ $\qquad (x > 0, t > 0)$

subject to the conditions $y(0, t) = 0, y(l, t) = 0$

and $y(x, 0) = \mu x(l - x), y_t(x, 0) = 0$

Now taking Laplace transform, writing $L[y(x, t)] = \bar{y}(x, s)$, we get

$$s^2\bar{y} - s\bar{y}(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2} \quad ...(i)$$

where $\bar{y}(0, s) = 0, \ \bar{y}(l, s) = 0$ ...(ii)

$\therefore$ (i) reduces to $\dfrac{\partial^2 \bar{y}}{\partial x^2} - \left(\dfrac{s}{c}\right)^2 \bar{y} = -\dfrac{\mu s x(l - x)}{c^2}$

Its solution is $\bar{y}(x, s) = c_1 \cosh(sx/c) + c_2 \sinh(sx/c) + \dfrac{\mu x(l - x)}{s} - \dfrac{2c^2\mu}{s^3}$

Applying the conditions (ii), we get

$$c_1 = 2c^2\mu/s^2 \quad \text{and} \quad c_2 = \frac{2c^2\mu}{s^3}\left[\frac{1 - \cosh(sl/c)}{\sinh(sl/c)}\right] - \frac{2c^2\mu}{s^3} \tanh(s/2c)$$

Thus $\bar{y}(x, s) = \dfrac{2c^2\mu}{s^3}\left[\dfrac{\cosh\{s(2x - l)/2c\}}{\cosh(sl/2c)}\right] + \dfrac{\mu x(l - x)}{s} - \dfrac{2c^2\mu}{s^3}$

Now using the inversion formula (3) § 22.10, we get

$y(x, t) =$ sum of the residues of

$$2c^2\mu\left[e^{st}\frac{\cosh\{s(2x - l)/2c\}}{s^3\cosh(sl/2c)}\right] \text{ at all the poles } + \mu x(l - x) - c^2\mu t^2$$

Proceeding exactly as in Example 22.23, we have,

sum of the residues of $2c^2\mu\left[\dfrac{e^{st}\cosh\{s(2x - l)/2c\}}{s^3\cosh sl/2c}\right]$ at all the poles

$$= c^2\mu\left[t^2 + \left(\frac{2x - l}{2c}\right)^2 - \left(\frac{l}{2c}\right)^2\right]$$

$$- \frac{32c^2\mu}{\pi^3}\left(\frac{l}{2c}\right)^2 \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(2n - 1)^3}\cos\left\{\frac{(2n - 1)\pi(2x - l)}{2l}\right\}\cos\left\{\frac{(2n - 1)\pi ct}{l}\right\}\right]$$

$$= c^2\mu t^2 - \mu x(l - x) + \frac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n - 1)^3}\sin\frac{(2n - 1)\pi x}{l}\cos\frac{(2n - 1)\pi ct}{l}\right]$$

Hence $y(x, t) = \dfrac{8\mu l^2}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\left[\dfrac{1}{(2n - 1)^3}\sin\dfrac{(2n - 1)\pi x}{l}\cos\dfrac{(2n - 1)\pi ct}{l}\right].$

Transmission lines

Example 22.27. *A semi-infinite transmission line of negligible inductance and leakance per unit length has its voltage and current equal to zero. A constant voltage v_0 is applied at the sending end ($x = 0$) at $t = 0$. Find the voltage and current at any point ($x > 0$) and at any instant.*

Solution. Let $v(x, t)$ and $i(x, t)$ be the voltage and current at any point x and at any time t. If $L = 0$ and $G = 0$, then the transmission line equations [(1) and (2) of § 18.10] become

$$\frac{\partial v}{\partial x} = -Ri, \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \quad i.e., \quad \frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad ...(i)$$

The boundary conditions are $v(0, t) = v_0$ and $i(x, t)$ is finite for all x and t.

The initial conditions are $v(x, 0) = 0, i(x, 0) = 0.$ ...(ii)

Laplace transforms of (*i*), are

$$\frac{d^2\bar{v}}{dx^2} = RC(s\bar{v} - 0) \quad \text{or} \quad \frac{d^2\bar{v}}{dx^2} - RCs\bar{v} = 0 \qquad ...(iii)$$

Laplace transforms of the conditions in (*ii*), are

$$\bar{v}(0, s) = \frac{v_0}{s} \text{ at } x = 0 \qquad ...(iv)$$

and $$\bar{v}(x, s) \text{ remains finite as } x \to \infty \qquad ...(v)$$

$\therefore$ the solution of (*iii*) is

$$\bar{v}(x, s) = C_1 e^{\sqrt{RCs}\,x} + C_2 e^{-\sqrt{RCs}\,x}$$

To satisfy condition (*v*), we must have $C_1 = 0$.

Using the condition (*iv*), we get $C_2 = v_0/s$

Thus $$\bar{v}(x, s) = \frac{v_0}{s} e^{-\sqrt{RCs}\,x}$$

Using the inversion formula, we obtain

$$v(x, t) = v_0 L^{-1}\left\{\frac{e^{-\sqrt{RC}\,x\,.\sqrt{s}}}{s}\right\} = v_0\, erfc\left(x\,\frac{\sqrt{RC}}{2\sqrt{t}}\right) \qquad \text{[By Ex. 22.17]}$$

$$= v_0\,\frac{x\sqrt{RC}}{2\sqrt{\pi}}\int_0^t u^{-3/2}\, e^{-(RC\,x^2/4u)}\, du$$

$\therefore$ since $i = -\dfrac{1}{R}\dfrac{\partial v}{\partial x}$, we obtain by differentiation,

$$i(x, t) = \frac{v_0 x}{2\sqrt{x}}\sqrt{\frac{C}{R}}\, t^{-3/2}\, e^{(-RC\,x^2/4t)}.$$

Example 22.28. *A transmission line of length l has negligible inductance and leakance. A constant voltage v_0 is applied at the sending end ($x = 0$) and is open circuited at the far end. Assuming the initial voltage and current to be zero, determine the voltage and current.*

Solution. For a transmission line with $L = G = 0$, the voltage v and current i are given by the equations

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \text{ and } \frac{\partial v}{\partial x} + Ri = 0 \qquad ...(i)$$

The boundary conditions are (for $t > 0$)

$$v = v_0 \text{ at } x = 0 \text{ and } i = \frac{\partial v}{\partial x} = 0 \quad \text{at } x = l \qquad ...(ii)$$

The initial condition is $v = 0$ at $t = 0$ ($x > 0$)

Laplace transforms of (*i*) and (*ii*) are

$$\frac{\partial^2\bar{v}}{\partial x^2} = RC(s\bar{v} - 0) \qquad ...(iii)$$

and $$\bar{v} = v_0/s \text{ at } x = 0,\ \frac{\partial\bar{v}}{\partial x} = 0 \text{ at } x = l \qquad ...(iv)$$

$\therefore$ the solution of (*iii*) is

$$\bar{v} = c_1 \cosh\sqrt{(RCs)}\,x + c_2 \sinh\sqrt{RCs}\,x$$

Applying conditions (*iv*), it gives

$$v_0/s = c_1,\ 0 = c_1 \sinh\sqrt{(RCs)}\,l + c_2 \cosh\sqrt{(RCs)}\,l$$

$$\therefore \quad \bar{v} = \frac{v_0}{s}\left[\cosh\sqrt{(RCs)}x - \frac{\sinh\sqrt{(RCs)}l}{\cosh\sqrt{(RCs)}l}\sinh\sqrt{(RCs)}x\right]$$

$$= \frac{v_0}{s} \frac{\cosh pq\sqrt{s}}{\cosh p\sqrt{s}}$$ where $p = \sqrt{(RC)}\,l$ and $q = (l - x)/l$

By the inversion formula (3) § 22.10, we get

$$v(x, t) = \text{sum of the residues of } (e^{st}\,\bar{v}) \text{ at all poles of } e^{st}\,\bar{v}. \qquad ...(iv)$$

These poles are at $s = 0$ and $p\sqrt{s} = \pm\, i(2n - 1)\pi/2 = \pm\, ipk$ (say)

Now $$Res\,(e^{st}\,\bar{v})_{s=0} = \underset{s \to 0}{\text{Lt}} \frac{se^{st}\, v_0 \cosh pq\sqrt{s}}{s \cosh p\sqrt{s}} = v_0$$

and $$Res(e^{st}\bar{v})_{s=-k^2} = \underset{s \to -k^2}{\text{Lt}} \frac{(s + k^2)e^{st}\, v_0 \cosh pq\sqrt{s}}{s \cosh p\sqrt{s}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{s \to -k^2}{\text{Lt}} \frac{v_0 . e^{st} \cosh pq\sqrt{s} + (s + k^2)(...)}{\cosh p\sqrt{s} + s \sinh p\sqrt{s} . \frac{1}{2} ps^{-1/2}}$$

$$= \frac{v_0\, e^{-k^2 t} \cosh (ipqk) + 0}{0 + 1/2\,(ipk) \sinh (ipk)} = \frac{2v_0 e^{-k^2 t} \cos (pqk)}{-\, pk \sin pk}$$

Adding up all the residues, (*iv*) gives

$$v(x, t) = v_0 + \frac{4v_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n - 1} e^{-[(2n-1)^2 \pi^2 t/4RCl^2]} \cos\, [(2n - 1)\, \pi(l - x)/2l]$$

$[\because\ pk = (2n - 1)\, \pi/2,\ -\sin pk = (-1)^n,\ pqk = \frac{1}{2}(2n - 1)\, \pi(l - x)/l,$
$k^2 = (2n - 1)^2\pi^2/4RCl^2]$

Also $$i = -\frac{1}{R}\frac{\partial v}{\partial x}. \qquad \text{[By } (i)]$$

PROBLEMS 22.4

1. Solve the differential equation using Laplace transform method, $\frac{\partial y}{\partial t} = 3\frac{\partial^2 y}{\partial x^2}$ where $y(\pi/2, t) = 0$, $(\partial y/\partial x)_{x=0} = 0$ and $y(x, 0) = 30 \cos 5x$. (*U.P.T.U., 2005*)

2. Using suitable transforms, solve the differential equation $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t}$, $0 \le x \le \pi$, $t \ge 0$. where $V(0, t) = 0 = V(\pi, t)$ and $V(x, 0) = V_0$ constant.

3. The initial temperature along the length of an infinite bar is given by $u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$. If the temperature $u(x, t)$ satisfies the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$, find the temperature at any point of the bar at any point t. (*Rohtak, 2006*)

4. Use the complex form of the Fourier transform to show that
$$V = \frac{1}{2\sqrt{(\pi t)}} \int_{-\infty}^{\infty} \bar{f}(u) e^{[-(x-u)^2/4t]}\, du$$
is the solution of the boundary value problem
$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2},\ -\infty < x < \infty,\ t > 0\ ;\ V = f(x) \text{ when } t = 0.$$ (*U.P.T.U., 2008*)

5. A semi-infinite solid ($x > 0$) is initially at temperature zero. At time $t = 0$, a constant temperature $\theta_0 > 0$ is applied and maintained at the face $x = 0$. Show that the temperature at any point x and at any time t, is given by $\theta(x, t) = \theta_0\, erfc(x/2c\sqrt{t})$.

6. A solid is initially at constant temperature θ_0, while the ends $x = 0$ and $x = a$ are maintained at temperature zero. Determine the temperature at any point of the solid at any later time $t > 0$.
7. An infinite string is initially at rest along the x-axis. Its one end which is at $x = 0$, is given a periodic transverse displacement $a_0 \sin \omega t$, $t > 0$. Show that the displacement of any point of the string at any time is given by

$$y(x, t) = \begin{cases} a_0 \sin \omega(t - x/c), & t > x/c \\ 0, & t < x/c, \end{cases}$$

where c is the wave velocity.
8. An infinite string has an initial transverse displacement $y(x, 0) = f(x), -\infty < x < \infty$, and is initially at rest. Show that

$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)].$$

9. A semi-infinite transmission line has negligible inductance and leakance per unit length. A voltage v is applied at the sending end ($x = 0$) which is given by

$$v(0, t) = \begin{cases} v_0, & 0 < t < \tau \\ 0, & t > \tau \end{cases}$$

Show that the voltage at any point $x > 0$ at any time $t > 0$ is given by

$$v(x, t) = v_0 \, erfc\left[\frac{x}{2}\sqrt{(RC/t)}\right].$$

22.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 22.5

Fill in the blanks or choose the correct answer in each of the following problems :

1. Fourier cosine transform of $f(t)$ is ....
2. Fourier sine transform of $1/x$ is ....
3. Convolution theorem for Fourier transforms states that ....
4. If Fourier transform of $f(x)$ is $F(s)$, then the inversion formula is ....
5. $F[x^n f(x)] =$ ....
6. If $F\{f(x)\} = F(s)$, then $F\{f(x - a)\} =$ ....
7. Fourier sine integral representation of a function $f(x)$ is given by ....
8. If $F_c[f(ax)] = k\, F_c(s/a)$, then $k =$ ....
9. Fourier transform of second derivative of $u(x, t)$ is ....
10. If $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$, then Fourier sine integral of $f(x)$ is ....
11. Fourier sine transform of $f'(x)$ in the interval $(0, l)$ is ....
12. If $F(\lambda)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(ax)$ is ....
13. Inverse finite Fourier sine transform of $F_s(p) = \dfrac{1 - \cos p\pi}{(p\pi)^2}$ for $p = 1, 2, 3, \ldots$ and $0 < x < \pi$ is ....
14. If Fourier transform of $f(x) = F(s)$, then Fourier Transform of $f(2x)$ is ....
15. Fourier cosine transform of e^{-x} is ....
16. $f(x) = 1, 0 < x < \infty$ cannot be represented by a Fourier integral. (True or False)
17. $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 dx.$ (True or False)
18. Fourier transform is a linear operation. (True or False)
19. $F_s[x f(x)] = -\dfrac{d}{ds} F_c(s).$ (True or False)
20. Kernel of Fourier transform is e^{sx}. (True or False)
21. Finite Fourier cosine transform of $f(x) = 1$ in $(0, \pi)$ is zero. (True or False)

(2) Continuity of f(z). *A function $w = f(z)$ is said to be* **continuous** *at $z = z_0$, if*

$$\underset{z \to z_0}{Lt} f(z) = f(z_0).$$

Further $f(z)$ is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

Also if $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0$, then $u(x, y)$ and $v(x, y)$ are also continuous at $z = z_0$, *i.e.*, at $x = x_0$ and $y = y_0$. Conversely if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), then $f(z)$ will be continuous at $z = z_0$. [cf. § 5.1 (3)].

20.3 (1) DERIVATIVE OF f(z)

Let $w = f(z)$ be a single-valued function of the variable $z = x + iy$. Then *the derivative of $w = f(z)$ is defined to be*

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z + \delta z) - f(z)}{\delta z},$$

provided the limit exists and has the same value for all the different ways in which δz approaches zero.

Suppose $P(z)$ is fixed and $Q(z + \delta z)$ is a neighbouring point (Fig. 20.2). The point Q may approach P along any straight or curved path in the given region, *i.e.*, δz may tend to zero in any manner and dw/dz may not exist. It, therefore, becomes a fundamental problem to determine the necessary and sufficient conditions for dw/dz to exist. The fact is settled by the following theorem.

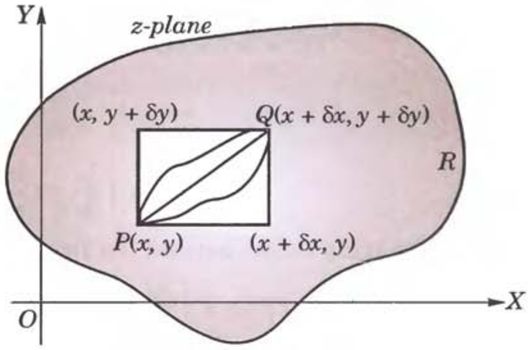


Fig. 20.2

(2) Theorem. *The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y)$ $= f(z)$ to exist for all values of z in a region R, are*

(*i*) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ *are continuous functions of x and y in R;*

(*ii*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

The relations (ii) are known as **Cauchy-Riemann* equations** or briefly *C-R* equations.

(*a*) *Condition is necessary.*

If $f(z)$ possesses a unique derivative at $P(z)$, then

$$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z}$$

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \frac{\{u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)\} - \{u(x, y) + iv(x, y)\}}{\delta x + i\delta y}$$

Since δz can approach zero in any manner, we can first assume δz to be wholly real and then wholly imaginary. When δz is wholly real, then $\delta y = 0$ and $\delta z = \delta x$.

$$\therefore \quad f'(z) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{u(x + \delta x, y) - u(x, y)}{\delta x} + i \frac{v(x + \delta x, y) - v(x, y)}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{...(1)}$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$.

$$\therefore \quad f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\frac{u(x, y + \delta y) - u(x, y)}{i\delta y} + i \frac{v(x, y + \delta y) - v(x, y)}{i\delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\delta y} - i \frac{\partial u}{\partial y} \quad \text{...(2)}$$

Now the existence of $f'(z)$ requires the equality of (1) and (2).

* Named after *Cauchy* (p. 144) and the German mathematician *Bernhard Riemann* (1826—1866) who along with *Weierstrass* (p. 390) laid the foundations of complex analysis. Riemann introduced the concept of integration and made basic contributions to number theory and mathematical analysis. He developed the Riemannian geometry which formed the mathematical base for Einstein's relativity theory.

$$\therefore \qquad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

On equating the real and imaginary parts from both sides, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(3)$$

Thus the necessary conditions for the existence of the derivative of $f(z)$ is that the C-R equations should be satisfied. (*V.T.U., 2011 S*)

(*b*) *Condition is sufficient.* Suppose $f(z)$ is a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region and the *C-R* equations (3) are satisfied.

Then by Taylor's theorem for functions of two variables (p. 220)

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= u(x, y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + ... + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + ...\right]$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\partial x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

[Omitting terms beyond the first powers of δx and δy]

or
$$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\partial x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y.$$

Now using the *C-R* equation (3), replace $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively.

Then
$$f(z + \delta z) - f(z) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\delta y = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]i\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta z$$

$$\therefore \qquad f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or} \quad \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

which by (1) or (2) proves the sufficiency of conditions.

20.4 ANALYTIC FUNCTIONS

A function $f(z)$ which is single-valued and possesses a unique derivative with respect to z at all points of a region R, is called an **analytic function** *of z in that region. An analytic function is also called a regular function* or an *holomorphic function.*

A function which is analytic everywhere in the complex plane, is known as an ***entire function.*** As derivative of a polynomial exists at every point, a polynomial of any degree is an *entire function.*

A point at which an analytic function ceases to possess a derivative is called a **singular point** *of the function.*

Thus if u and v are real single-valued functions of x and y such that $\delta u/\delta x$, $\delta u/\delta y$, $\delta v/\delta x$, $\delta v/\delta y$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}} \text{ and } \frac{\partial \mathbf{u}}{\partial \mathbf{y}} = -\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \qquad ...(1)$$

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in R. The derivative of $f(z)$ is then, given by (1) of p. 664 or (2) of p. 665.

The real and imaginary parts of an analytic function are called *conjugate functions.* The relation between two conjugate functions is given by *C-R* equation (1).

Example 20.1. *If $w = \log z$, find dw/dz and determine where w is non-analytic.*
(*U.P.T.U., 2005 ; J.N.T.U., 2005*)

Solution. We have $w = u + iv = \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1} y/x$ [By (**2**), p. 665]

so that $u = \frac{1}{2}\log(x^2 + y^2),\ v = \tan^{-1} y/x.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x}.$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous except at (0, 0). Hence w is analytic everywhere except at $z = 0$.

$$\therefore \quad \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}\ (z \neq 0).$$

Obs. The definition of the derivative of a function of complex variable is identical in form to that of the derivative of a function of real variable. Hence the rules of differentiation for complex functions are the same as those of real calculus. *Thus* **if, a complex function is once known to be analytic, it can be differentiated just in the ordinary way.**

Example 20.2. *If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant.*
(U.P.T.U., 2008 ; Mumbai, 2005 S ; Madras 2003 ; Bhopal, 2002 S)

Solution. If $f(z) = u + iv$ is an analytic function, then

$$|f(z)| = \sqrt{(u^2 + v^2)} \text{ is constant} = c \text{ (say) or } u^2 + v^2 = c^2 \quad ...(i)$$

Differentiating (*i*) partially w.r.t. x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0; \qquad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

or $$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad ...(ii) \qquad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad ...(iii)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ by C-R equations,

$\therefore$ (*iii*) becomes $-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$ ...(*iv*)

Squaring and adding (*ii*) and (*iv*), we obtain

$$u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

or $$(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0 \text{ or } \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad [\because\ u^2 + v^2 = c^2 \neq 0] \quad ...(v)$$

Now $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad \text{[By } (v)]$$

or $f'(z) = 0.$ or $f(z) = $ constant.

Example 20.3. *Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though C.R. equations are satisfied thereof.* *(A.M.I.E.T.E., 2005 S ; Osmania, 2003)*

Solution. If $f(z) = \sqrt{|xy|} = u(x, y) + iv(x, y)$, then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin, we have

$$\frac{\partial u}{\partial x} = \underset{x\to 0}{\text{Lt}} \frac{u(x,0) - u(0,0)}{x} = \underset{x\to 0}{\text{Lt}} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{\text{Lt}} \frac{u(0,y) - u(0,0)}{y} = \underset{x\to 0}{\text{Lt}} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x\to 0}{\text{Lt}} \frac{v(x,0) - v(0,0)}{x} = \underset{x\to 0}{\text{Lt}} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., C.R. equations are satisfied at the origin.

However $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z - 0} = \underset{z \to 0}{\text{Lt}} \frac{\sqrt{|xy|} - 0}{x + iy}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|mx^2|} - 0}{x(1 + im)}, \text{ when } z \to 0 \text{ along the line } y = mx$$

$$= \frac{\sqrt{|m|}}{1 + im} \text{ which is not unique.}$$

$\therefore$ $f'(0)$ does not exist. Hence $f(z)$ is not analytic at the origin.

Example 20.4. *Prove that the function $f(z)$ defined by*

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \, (z \neq 0), f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

(S.V.T.U., 2009 ; V.T.U., 2001)

Solution. $$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \underset{y \to 0}{\text{Lt}} \frac{-y^3(1-i)}{y^2} = \underset{y \to 0}{\text{Lt}} [-y(1-i)] = 0$$

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to 0 \\ x \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i)}{x^2} = \underset{x \to 0}{\text{Lt}} [x(1+i)] = 0$$

Also $f(0) = 0$ (given).

Thus $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$ when $x \to 0$ first and then $y \to 0$ and also vice-versa. Now let both x and y tend to zero simultaneously along the path $y = mx$. Then

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to mx \\ x \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i) - m^3x^3(1-i)}{(1+m^2)x^2} = \underset{x \to 0}{\text{Lt}} \frac{x[1 + i - m^3(1-i)]}{1 + m^2} = 0$$

Hence $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$, in whatever manner $z \to 0$. $\therefore$ $f(z)$ is continuous at the origin.

Now $$f(z) = \frac{x^3 - y^3}{x^2 + y^2} + i \frac{x^3 + y^3}{x^2 + y^2} = u(x, y) + iv(x, y).$$

Also $u(0, 0) = 0$, and $v(0, 0) = 0$ $\qquad [\because \; f(0) = 0]$

$$\therefore \qquad \left(\frac{\partial u}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{-y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

and $$\left(\frac{\partial v}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{y}{y} = 1.$$

Hence at (0, 0), $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Thus the *C-R* equations are satisfied at the origin.

But $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z} = \underset{z \to 0}{\text{Lt}} \frac{f(z)}{z} = \underset{z \to 0}{\text{Lt}} \frac{x^3 - y^3 + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}.$$

If $z \to 0$ along the path $y = mx$, then $f'(0) = \dfrac{1 - m^3 + i(1 + m^3)}{(1 + m^2)(1 + im)}$

which assumes different values as m varies. So $f'(z)$ is not unique at (0, 0) *i.e.*, $f'(0)$ does not exist. Thus $f(z)$ is not analytic at the origin even though it is continuous and satisfies the *C-R* equations thereat.

Example 20.5. *Show that polar form of Cauchy-Riemann equations are*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$ *(U.P.T.U., 2008 ; V.T.U., 2006)*

Deduce that $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0.$ *(Bhopal, 2009 ; Kurukshetra, 2005)*

Solution. If (r, θ) be the coordinates of a point whose cartesian coordinates are (x, y), then $z = x + iy = re^{i\theta}$.

$$\therefore \quad u + iv = f(z) = f(re^{i\theta})$$

where u and v are now expressed in terms of r and θ.

Differentiating it partially w.r.t. r and θ, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta}$$

and $$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ...(i) \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad ...(ii)$$

Differentiating (*i*) partially w.r.t. r, we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \quad ...(iii)$$

Differentiating (*ii*) partially w.r.t. θ, we have

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r \partial \theta} \quad ...(iv)$$

Thus using (*i*), (*ii*) and (*iv*)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r \partial \theta}\right) = 0 \quad \left[\because \frac{\partial^2 v}{\partial \theta \partial r} = \frac{\partial^2 v}{\partial r \partial \theta}\right]$$

20.5 (1) HARMONIC FUNCTIONS

If $f(z) = u + iv$ be an analytic function in some region of the z-plane, then the Cauchy-Riemann equations are satisfied.

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ...(1) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad ...(2)$$

Differentiating (1) with respect to x and (2) with respect to y, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad ...(3) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \quad ...(4)$$

Adding (3) and (4) and assuming that $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad ...(5)$$

Similarly, by differentiating (1) with respect to y and (2) with respect to x and subtracting, we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \qquad ...(6)$$

Thus both the functions u and v satisfy the Laplace's equation in two variables. For this reason, they are known as **harmonic functions** and their theory is called **potential theory.** *(Rohtak, 2005)*

(2) Orthogonal system. Consider the two families of curves

$$u(x, y) = c_1 \quad ...(7) \qquad \text{and} \qquad v(x, y) = c_2 \qquad ...(8)$$

Differentiating (7), we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$

or $$\frac{dy}{dx} = -\frac{\partial u / \partial x}{\partial u / \partial y} = \frac{\partial v / \partial y}{\partial v / \partial x} = m_1 \text{ (say)} \qquad \text{[By (1) and (2)]}$$

Similarly (8) gives $$\frac{dy}{dx} = -\frac{\partial v / \partial x}{\partial v / \partial y} = m_2 \text{ (say)}$$

$\therefore$ $m_1 m_2 = -1$, *i.e.*, (7) and (8) form an orthogonal system.

Hence *every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.* *(U.P.T.U., 2009)*

20.6 APPLICATIONS TO FLOW PROBLEMS

As the real and imaginary parts of an analytic function are the solutions of the Laplace's equation in two variables, the conjugate functions provide solutions to a number of field and flow problems.

As an illustration, consider the irrotational motion of an incompressible fluid in two dimensions. Assuming the flow to be in planes parallel to the xy-plane, the velocity $\mathbf{V}$ of a fluid particle can be expressed as

$$\mathbf{V} = v_x \mathbf{I} + v_y \mathbf{J} \qquad ...(1)$$

Since the motion is irrotational, therefore, by § 6.18 (1), there exist a scalar function $\phi(x, y)$ such that

$$\mathbf{V} = \nabla\phi(x, y) = \frac{\partial \phi}{\partial x}\mathbf{I} + \frac{\partial \phi}{\partial y}\mathbf{J} \qquad ...(2)$$

[The function $\phi(x, y)$ is called the *velocity potential* and the curves $\phi(x, y) = c$ are known as *equipotential lines*.]

Thus from (1) and (2), $v_x = \frac{\partial \phi}{\partial x}$ and $v_y = \frac{\partial \phi}{\partial y}$ ...(3)

Also the fluid being incompressible div $\mathbf{V} = 0$ [by § 8.7 (1)] *i.e.*, $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.$

Substituting the values of v_x and v_y from (3), we get $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

which shows that the velocity potential ϕ is *harmonic*. It follows that there must exist a conjugate harmonic function $\psi(x, y)$ such that $w(z) = \phi(x, y) + i\psi(x, y)$ ...(4)
is analytic.

Also the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\frac{dy}{dx} = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y} = \frac{\partial\phi/\partial y}{\partial\phi/\partial x} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$= v_y/v_x \qquad \text{[By (3)]}$$

This shows that the velocity of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$, *i.e.* the particle moves along this curve. Such curves are known as *stream lines* and $\psi(x, y)$ is called the *stream function.* Also the equipotential lines $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$ cut orthogonally.

From (4), $\dfrac{\partial w}{dz} = \dfrac{\partial \phi}{\partial x} + i\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \phi}{\partial x} - i\dfrac{\partial \phi}{\partial y}$ [By *C-R* equations]

$= v_x - iv_y$ [By (3)]

∴ The magnitude of the fluid velocity $= \sqrt{(v_x^2 + v_y^2)} = | dw/dz |$.

Thus the flow pattern is fully represented by the function $w(z)$ which is known as the **complex potential.**

Similarly the complex potential $w(z)$ can be taken to represent any other type of 2-dimensional steady flow. In electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are *equipotential lines* and *lines of force*. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as *isothermals* and *heat flow lines* respectively.

Given $\phi(x, y)$, we can find $\psi(x, y)$ and *vice-versa*.

Example 20.6. *If $w = \phi + i\psi$ respresents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, determine the function ϕ.* (V.T.U., 2011 ; Mumbai, 2008 ; Bhopal, 2002 S)

Solution. It is readily verified that ψ satisfies the Laplace's equation.

∴ ϕ and ψ must satisfy the Cauchy-Riemann equations :

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad ...(i) \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad ...(ii)$$

∴ by (*i*), $$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y}\left[x^2 - y^2 + \frac{x}{x^2 + y^2}\right] = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

Integrating w.r.t. x, we get $\phi = -2xy + \dfrac{y}{x^2 + y^2} + \eta(y)$ where $\eta(y)$ is an arbitrary function of y.

∴ (*ii*) gives $-2x + \dfrac{x^2 - y^2}{(x^2 + y^2)^2} + \eta'(y) = -2x + \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

whence $\eta'(y) = 0$, *i.e.*, $\eta(y) = c$, an arbitrary constant.

Thus $$\phi = -2xy + \frac{y}{x^2 + y^2} + c$$

Otherwise (Milne-Thomson's method*) :

We have $$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x} = \left[-2y - \frac{2xy}{(x^2 + y^2)^2}\right] + i\left[2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right]$$

By Milne-Thomson's method, we express dw/dz in terms of z, on replacing x by z and y by 0.

∴ $$\frac{dw}{dz} = i\left(2z - \frac{1}{z^2}\right)$$

Integrating w.r.t. z, we get $w = i\,(z^2 + 1/z) + A$ where A is a complex constant.

---

* Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z + \bar{z})$$

∴ $$f(z) = \phi(x, y) + i\psi(x, y) \quad ...(1)$$

$$= \phi\left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right] + i\psi\left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right]$$

Now considering this as a formal identity in the two independent variables z, $\bar{z}$ and putting $\bar{z} = z$, we get

$$f(z) = \phi(z, 0) + i\psi(z, 0) \quad ...(2)$$

∴ (2) is the same as (1), if we replace x by z and y by 0.

Thus **to express any function in terms of z, replace x by z and y by 0.** *This provides an elegant method of finding $f(z)$ when its real part or the imaginary part is given. It is due to Milne-Thomson.*

Hence $$\phi = R\left[i\left(z^2 + \frac{1}{z}\right) + A\right] = -2xy + \frac{y}{x^2 + y^2} + c.$$

Example 20.7. *Find the analytic function, whose real part is sin 2x/(cosh 2y – cos 2x).*
(J.N.T.U., 2005 ; Anna, 2003)

Solution. Let $f(z) = u + iv$, where $u = \sin 2x/(\cosh 2y - \cos 2x)$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$= \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} - i\,\frac{\sin 2x(-2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x\cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2\sin 2x\sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

By Milne-Thomson's method, we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore \quad f'(z) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} + i(0) = \frac{-2}{1 - \cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

Integrating w.r.t. z, we get $f(z) = \cot z + ic$, taking the constant of integration as imaginary since u does not contain any constant.

Example 20.8. *Determine the analytic function* $f(z) = u + iv$, *if* $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ *and* $f(\pi/2) = 0$.
(A.M.I.E.T.E., 2005 ; Osmania, 2003)

Solution. We have $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

$$\therefore \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y}\sin x}{2(\cos x - \cosh y)^2} \qquad ...(i)$$

and $$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\cos x - \cosh y)\, e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

or $$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = \frac{(\sin x + \cos x)\sinh y + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2} \qquad ...(ii)$$

Subtracting (*ii*) from (*i*), we get

$$2\frac{\partial u}{\partial x} = \frac{(\sin x - \cos x)\cosh y - (\sin x + \cos x)\sinh y + 1 - e^{-y}(\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Adding (*i*) and (*ii*), we have

$$-2\frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + (\sin x + \cos x)\sinh y + 1 + e^{-y}(-\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Thus $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1 - \cos z}{2(1 - \cos z)^2} \qquad \text{[Putting } x = z \text{ and } y = 0]$$

$$= \frac{1}{2(1 - \cos z)} = \frac{1}{4\sin^2 z/2} = \frac{1}{4}\operatorname{cosec}^2\frac{z}{2} \quad \text{or} \quad f(z) = -\frac{1}{2}\cot\frac{z}{2} + c$$

Since $f(\pi/2) = 0$, $\quad 0 = -\dfrac{1}{2}\cot \pi/4 + c$, whence $c = \dfrac{1}{2}$

Hence $$f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right).$$

Example 20.9. *Find the conjugate harmonic of $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$. Show that v is harmonic.* *(Marathwada, 2008)*

Solution. Let $f(z) = u + v$. Using C-R equations in polar coordinates (Ex. 20.5),

$$r \frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r \sin \theta \quad ...(i)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r \cos 2\theta - \cos \theta \quad ...(ii)$$

$\therefore$ (i) gives, $\quad \dfrac{\partial u}{\partial r} = -2r \sin 2\theta + \sin \theta$

Integrating w.r.t., r

$$u = -r^2 \sin 2\theta + r \sin \theta + \phi(\theta) \qquad \text{where } \phi(\theta) \text{ is an arbitrary function.}$$

$$\therefore \quad \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta) \quad ...(iii)$$

From (ii) and (iii), we get

$$-2r^2 \cos 2\theta + r \cos \theta = \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta)$$

$$\therefore \quad \phi'(\theta) = 0 \quad \text{or} \quad \phi(\theta) = c$$

Thus $\quad u = -r^2 \sin 2\theta + r \sin \theta + c$ is the conjugate harmonic of v.

Now v will be harmonic if it satisfies the Laplace equation $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r} + \dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$

From (i), $\dfrac{\partial^2 v}{\partial \theta^2} = -4r^2 \cos 2\theta + r \cos \theta$. From (ii), $\dfrac{\partial^2 v}{\partial r^2} = 2 \cos 2\theta$

$$\therefore \quad \frac{\partial t^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 2 \cos 2\theta + \frac{1}{r}(2r \cos 2\theta - \cos \theta) + \frac{1}{r^2}(-4r^2 \cos 2\theta + r \cos \theta)$$

$$= 4 \cos 2\theta - \frac{1}{r} \cos \theta - 4 \cos 2\theta + \frac{1}{r} \cos \theta = 0$$

Hence v is harmonic.

Example 20.10. (a) *Find the orthogonal trajectories of the family of curves*

$$x^4 + y^4 - 6x^2y^2 = \textit{constant}$$

(b) Show that the curves $r^n = \alpha \sec n\theta$ and $r^n = \beta \operatorname{cosec} n\theta$ cut orthogonally. *(Mumbai, 2005 ; J.N.T.U., 2003)*

Solution. (a) Take $u(x, y) = x^4 + y^4 - 6x^2y^2$. Then the family of curves $v(x, y) =$ constant will be the required trajectories if $f(z) = u + iv$ is analytic.

Now $\quad \dfrac{\partial u}{\partial x} = 4x^3 - 12xy^2, \ \dfrac{\partial u}{\partial y} = 4y^3 - 12x^2y$

$$\therefore \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

Integrating, $\quad v = 4x^3y - 4xy^3 + c(x)$

Differentiating partially w.r.t. x

$$12x^2y - 4y^3 + \frac{dc(x)}{dx} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -4y^3 + 12x^2y$$

$$\therefore \quad \frac{dc(x)}{dx} = 0 \quad \text{or} \quad c = \text{constant}$$

Thus the required orthogonal trajectories are $v =$ constant or $x^3y - xy^3 =$ constant.

(b) Writing $u(r, \theta) = r^n \cos n\theta = \alpha$ and $v(r, \theta) = r^n \sin n\theta = \beta$,
we have $u(r, \theta) + iv(r, \theta) = \alpha + i\beta = r^n (\cos n\theta + i \sin n\theta) = r^n \cdot e^{in\theta} = (re^{i\theta})^n = z^n$

This is an analytic function.

Thus $f(z) = u + iv$, gives the curves $u = \alpha$ and $v = \beta$ which cut orthogonally.

Example 20.11. *Two concentric circular cylinders of radii r_1, r_2 ($r_1 < r_2$) are kept at potentials ϕ_1 and ϕ_2 respectively. Using complex function $w = a \log z + c$, prove that the capacitance per unit length of the capacitor formed by them is $2\pi\lambda / \log (r_2 / r_1)$ where λ is the dielectric constant of the medium.*

Solution. We have $\quad \phi + i\psi = a \log (re^{i\theta}) + c \quad$ where $z = x + iy = re^{i\theta}$

$$\therefore \qquad \phi = a \log r + c, \quad \text{and} \quad \psi = a\theta$$

so that $\quad \phi_1 = a \log r_1 + c, \quad \phi_2 = a \log r_2 + c$

Thus the potential difference $= \phi_2 - \phi_1 = a (\log r_2 - \log r_1)$

Also the total charge (or flux) $= \int_0^{2\pi} d\psi = \int_0^{2\pi} a \, d\theta = 2\pi a.$

The capacitance being the charge required to maintain a unit potential difference ; the capacitance without dielectric

$$= \frac{\text{charge}}{\text{potential difference}} = \frac{2\pi a}{a(\log r_2 - \log r_1)} = \frac{2\pi}{\log (r_2/r_1)}.$$

A medium of dielectric constant λ increases the potential difference to λ times that in vacuum for the same charge. Thus the capacitance with dielectric $= 2\pi\lambda/\log (r_2/r_1)$.

Example 20.12. *If $f(z)$ is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) | f(z) |^2 = 4 \,| f'(z) |^2. \qquad \textit{(J.N.T.U., 2006 ; Kottayam, 2005)}$$

or

$$\nabla^2 \,| f(z) |^2 = 4 \,| f'(z) |^2 \qquad \textit{(Madras, 2006)}$$

Solution. Let $f(z) = u(x, y) + iv(x, y)$ so that $| f(z) |^2 = u^2 + v^2 = \phi(x, y)$, (say).

$$\therefore \qquad \frac{\partial \phi}{\partial x} = 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2} = 2 \left\{ u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v \frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2 \right\}$$

Similarly, $$\frac{\partial^2 \phi}{\partial y^2} = 2 \left\{ u \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v \frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2 \right\}$$

Adding, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 \left\{ u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \right\} + 2 \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right\} \qquad ...(i)$$

Since u, v have to satisfy Cauchy-Riemann equations and the Laplace's equation.

$$\therefore \qquad \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 ; \left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

Thus (*i*) takes the form $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 4 \left\{ \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2 \right\}$

Hence $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) | f(z) |^2 = 4 \,| f'(z) |^2 \quad \text{or} \quad \nabla^2 \,| f(z) |^2 = 4 \,| f'(z) |^2.$$

PROBLEMS 20.1

1. If $f(z) = \begin{cases} x^3 y(y - ix)/(x^6 + y^2), & z \neq 0 \\ 0 & , z = 0 \end{cases}$ prove that $[f(z) - f(0)]/z \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ along the curve $y = ax^3$.

2. Show that (a) $f(z) = xy + iy$ is everywhere continuous but is not analytic. (Osmania, 2003 S)
(b) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane. (J.N.T.U., 2003)

3. If $f(z) = u + iv$ is analytic, then show that $|f'(z)|^2 = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$. (Mumbai, 2007)

4. Find the constants a, b, c, d and e if $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic. (Mumbai, 2008)

5. Show that z^n is analytic. Hence find its derivative. (V.T.U., 2010 S)

6. Determine which of the following functions are analytic :
(i) $2xy + i(x^2 - y^2)$ (ii) $(x - iy)/(x^2 + y^2)$ (iii) $\cosh z$.

7. (a) Determine p such that the function $f(z) = \dfrac{1}{2}\log_e(x^2 + y^2) + i\tan^{-1}(px/y)$ be an analytic function. (Mumbai, 2007 ; J.N.T.U., 2003)
(b) Show that $u = \dfrac{1}{2}\log(x^2 + y^2)$ is harmonic and find its harmonic conjugate function. (U.P.T.U., 2010)

8. Show that each of the following functions is not analytic at any point :
(i) $\bar{z}$ (J.N.T.U., 2003) (ii) $|z|^2$.

9. Show that $u + iv = (x - iy)/(x - iy + a)$ where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.

10. Show that $f(z) = \begin{cases} xy^2(x + iy) \div (x^2 + y^4), & z \neq 0 \\ 0, & z = 0 \end{cases}$
is not analytic at $z = 0$, although C-R equations are satisfied at the origin. (J.N.T.U., 2003)

11. Verify if $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}, z \neq 0; f(0) = 0$ is analytic or not. (U.P.T.U., 2008)

12. Examine the nature of the function $f(z) = \dfrac{x^2y^5(x + iy)}{x^4 + y^{10}}, z \neq 0 ; f(0) = 0$. (Rohtak, 2004)

13. For the function $f(z)$ defined by $f(z)^2 = (\bar{z})^2/z, z \neq 0; f(0) = 0$, show that the C-R equations are satisfied at (0, 0), but $f(z)$ is not differentiable at (0, 0). (P.T.U., 2010)

14. Determine the analytic function whose real part is
(i) $x^3 - 3xy^2 + 3x^2 - 3y^2$ (Bhopal, 2009) (ii) $\cos x \cosh y$ (Rohtak, 2004)
(iii) $y/(x^2 + y^2)$ (iv) $y + e^x \cos y$ (S.V.T.U., 2008 ; V.T.U., 2006)
(v) $e^{-x}(x \sin y - y \cos y)$ (U.P.T.U., 2008)
(vi) $e^{2x}(x \cos 2y - y \sin 2y)$ (V.T.U., 2008 S ; Mumbai, 2005 ; Kottayam, 2005)
(vii) $x \sin x \cosh y - y \cos x \sinh y$ (V.T.U., 2006)
(viii) $e^x[(x^2 - y^2)\cos y - 2xy \sin y]$. (V.T.U., 2010 S ; Rohtak, 2005)

15. Find the regular function whose imaginary part is
(i) $(x - y)/(x^2 + y^2)$ (ii) $-\sin x \sinh y$ (iii) $e^x \sin y$
(iv) $e^{-x}(x \sin y - y \cos y)$ (v) $e^{-x}(x \cos y + y \sin y)$ (U.P.T.U., 2009) (vi) $\dfrac{2\sin x \sin y}{\cos 2x + \cosh 2y}$. (Mumbai, 2006)

16. Find the analytic function $z = u + iv$, if
(i) $u - v = (x - y)(x^2 + 4xy + y^2)$ (Mumbai, 2008 ; V.T.U., 2007 ; W.B.T.U., 2005)
(ii) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}}$ when $f\left(\dfrac{\pi}{2}\right) = 0$ (Mumbai, 2007)
(iii) $u + v = \dfrac{2\sin 2x}{e^{2y} - e^{-2y} - 2\cos 2x}$. (P.T.U., 2002)

17. An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function.

18. If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function.

19. Prove that $u = x^2 - y^2$ and $v = \dfrac{y}{x^2 + y^2}$ are harmonic functions of (x, y) but are not harmonic conjugates. (U.P.T.U., 2004 S)

20. Show that the function $u = e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z. *(Bhopal, 2007)*
21. For $w = \exp(z^2)$, find u and v, and prove that the curves $u(x, y) = c_1$ and $v(x, y) = c_2$ where c_1 and c_2 are constants, cut orthogonally. *(J.N.T.U., 2003)*
22. Find the orthogonal trajectories of the family of curves
    (*i*) $x^3y - xy^3 = c$ *(Mumbai, 2007)* (*ii*) $e^x \cos y - xy = c$ *(Mumbai, 2008)* (*iii*) $r^2 \cos 2\theta = c$.
23. In a two dimensional fluid flow, the stream function ψ is given, find the velocity potential ϕ :
    (*i*) $\psi = -y/(x^2 + y^2)$ (*ii*) $\psi = \tan^{-1}(y/x)$.
24. Find the analytic function $f(z) = u + iv$, given
    (*i*) $u = a(1 + \cos\theta)$ (*ii*) $v = (r - 1/r)\sin\theta, r \neq 0$.
25. If $f(z)$ is an analytic function of z, show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$ *(U.P.T.U., 2009 ; V.T.U., 2008 S ; P.T.U., 2005)*

26. If $f(z)$ is an analytic function of z, prove that

    (*i*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$ *(Madras, 2000 S)* (*ii*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|R\,f(z)|^2 = 2\,|f'(z)|^2$

    (*iii*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f'(z)|^2\,|f(z)|^{p-2}$. *(Kerala, 2005)*
27. Prove that $\psi = \log[(x - 1)^2 + (y - 2)^2]$ is harmonic in every region which does not include the point (1, 2). Find a function ϕ such that $\phi + i\psi$ is an analytic function of the complex variable $z = x + iy$. Express $\phi + i\psi$ as a function of z.

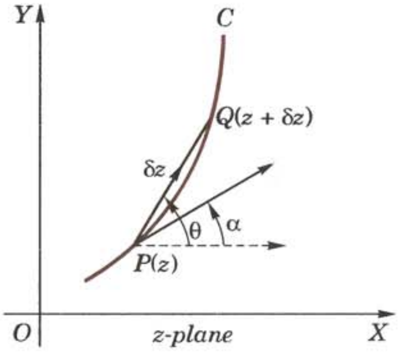
20.7 GEOMETRICAL REPRESENTATION OF w = f(z)

To find the geometrical representation of a function of a complex variable, it requires a departure from the usual practice of cartesian plotting, where we associate a curve to a real function $y = f(x)$.

In the complex domain, the function $w = f(z)$

i.e., $$u + iv = f(x + iy) \quad ...(1)$$

involves four real variables x, y, u, v. Hence a four dimensional region is required to plot (1) in the cartesian fashion. As it is not possible to have 4-dimensional graph papers, we make use of two complex planes, one for the variable $z = x + iy$, and the other for the variable $w = u + iv$. If the point z describes some curve C in the z-plane, the point w will move along a corresponding curve C' in the w-plane, since to each point (x, y), there corresponds a point (u, v) (Fig. 20.3). We then, say that *a curve C in the z-plane is mapped into the corresponding curve C′ in the w-plane by the function w = f(z) which defines a* **mapping** *or* **transformation** *of the z-plane into the w-plane.*



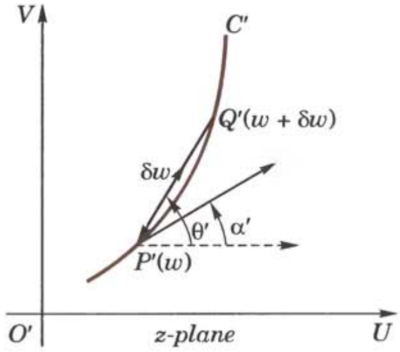


Fig. 20.3

20.8 SOME STANDARD TRANSFORMATIONS

(1) Translation. $w = z + c$, where c is a complex constant.

If $z = x + iy$, $c = c_1 + ic_2$ and $w = u + iv$, then the transformation becomes $u + iv = x + iy + c_1 + ic_2$ whence $u = x + c_1$ and $v = y + c_2$, *i.e.* the point $P(x, y)$ in the z-plane is mapped onto the point $P'(x + c_1, y + c_2)$ in the

w-plane. Every point in the z-plane is mapped onto w-plane in the same way. Thus if the w-plane is superposed on the z-plane, figure is shifted through a distance given by the vector c. Accordingly, this transformation maps a figure in the z-plane into a figure in the w-plane of the same shape and size.

In particular, this transformation changes circles into circles.

(2) Magnification and rotation. w = cz, where c is a complex constant.

If $\quad c = \rho e^{i\alpha}, z = re^{i\theta}$ and $w = Re^{i\phi}$, then

$$Re^{i\phi} = \rho e^{i\alpha} . re^{i\theta} = \rho re^{i(\theta + \alpha)}$$

whence $R = \rho r$ and $\phi = \theta + \alpha$, *i.e.* the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'(\rho r, \theta + \alpha)$ in the w-plane. Hence the transformation consists of magnification (or contraction) of the radius vector of P by $\rho = | c |$ and its rotation through an $\angle\alpha = \text{amp}(c)$. Accordingly any figure in the z-plane is transformed into a geometrically similar figure in the w-plane. In particular, this transformation maps circles into circles.

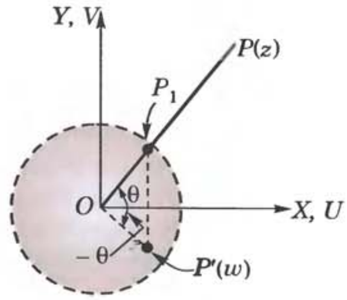


Fig. 20.4

(3) Inversion and reflection. w = 1/z.

Here it is convenient to think the w-plane as superposed on z-plane (Fig. 20.4).

If $z = re^{i\theta}$ and $w = Re^{i\phi}$, then $Re^{i\phi} = \frac{1}{r} e^{-i\theta}$

whence $R = 1/r$ and $\phi = -\theta$. Thus, if P be (r, θ) and P_1 be $(1/r, \theta)$, *i.e.* P_1 is the inverse* of P w.r.t. the unit circle with centre O, then the reflection P' of P_1 in the real axis represents $w = 1/z$.

Hence this transformation is an inversion of z w.r.t. the unit circle $| z | = 1$ followed by reflection of the inverse into the real axis.

Obs. 1. Clearly the function $w = 1/z$ maps the interior of the unit circle $| z | = 1$ onto the exterior of the unit circle $| w | = 1$ and the exterior of $| z | = 1$ onto the interior of $| w | = 1$. In particular, the origin $z = 0$ corresponds to the improper point $w = \infty$, called the *point at infinity* and the image of the improper point $z = \infty$ is the origin $w = 0$.

2. *This transformation maps a circle onto a circle or to a straight line if the former goes through the origin.*

To prove this, we write $z = 1/w$ as $x + iy = \dfrac{1}{u + iv} = \dfrac{u - iv}{u^2 + v^2}$

so that $$x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}. \qquad ...(1)$$

Now the general equation of any circle in the z-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(2)$$

which on substituting from (1), becomes $\dfrac{u^2}{(u^2 + v^2)^2} + \dfrac{v^2}{(u^2 + v^2)^2} + 2g\dfrac{u}{u^2 + v^2} + 2f\dfrac{-v}{u^2 + v^2} + c = 0$

or $$c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \qquad ...(3)$$

This is the equation of a circle in the w-plane. If $c = 0$, the circle (2) passes through the origin and its image, *i.e.*, (3) reduces to a straight line. Hence the result.

Regarding a straight line as the limiting form of a circle with infinite radius, we conclude that the *transformation $w = 1/z$ always maps a circle into a circle.*

(4) Bilinear transformation. The transformation

$$w = \frac{az + b}{cz + d} \qquad ...(1)$$

where a, b, c and d are complex constants and $ad - bc \neq 0$ is known as the **bilinear transformation.**** The condition $ad - bc \neq 0$ ensures that $dw/dz \neq 0$, *i.e.*, the transformation is conformal. If $ad - bc = 0$ every point of the z-plane is a *critical point.*

The inverse mapping of (1) is

$$z = \frac{-dw + b}{cw - a} \qquad ...(2)$$

which is also a bilinear transformation.

* The inverse of a point A w.r.t. a circle with centre O and radius k is defined as the point B on the line OA such that $OA.OB = k^2$.

** First studied by *Möbius* (p. 337). Hence, sometimes called *Möbius transformation*

Obs. 1. From (1), we see that each point in the z-plane except $z = -d/c$, corresponds a unique point in the w-plane. Similarly, (2) shows that each point in the w-plane except $w = a/c$, maps into a unique point in the z-plane. Including the images of the two exceptional points as the infinite points in the two planes, it follows that *there is one to one correspondence between all points in the two planes.*

Obs. 2. Invariant points of bilinear transformation. If z maps into itself in the w-plane (*i.e.*, $w = z$), then (1) gives

$$z = \frac{az+b}{cz+d} \quad \text{or} \quad cz^2 + (d-a)z - b = 0$$

The roots of this equation (say : z_1, z_2) *are defined as the invariant or fixed points of the bilinear transformation (1).* If however, the two roots are equal, the bilinear transformation is said to be *parabolic.*

Obs. 3. Dividing the numerator and denominator of the right side of (1) by one of the four constants, it is clear that (1) has only three essential arbitrary constants. Hence *three conditions are required to determine a bilinear transformation.* For instance, three distinct points z_1, z_2, z_3 can be mapped into any three specified points w_1, w_2, w_3.

Two important properties :

I. A bilinear transformation maps circles into circles.

By actual division, (1) can be written as $w = \dfrac{a}{c} + \dfrac{bc-ad}{c^2} \cdot \dfrac{1}{z+d/c}$

which is a combination of the transformations

$$w_1 = z + d/c,\ w_2 = 1/w_1,\ w_3 = \frac{bc-ad}{c^2} w_2,\ w = \frac{a}{c} + w_3 .$$

By these transformations, we successively pass from z-plane to w_1-plane, from w_1-plane to w_2-plane, from w_2-plane to w_3-plane and finally from w_3-plane to w-plane. Now each of these transformations is one or other of the standard transformations $w = z + c$, $w = cz$, $w = 1/z$ and under each of these a circle always maps onto a circle. Hence the bilinear transformation maps circles into circles.

II. A bilinear transformation preserves cross-ratio† *of four points.*

Let the points z_1, z_2, z_3, z_4 of the z-plane map onto the points w_1, w_2, w_3, w_4 of the w-plane respectively under the bilinear transformation (1). If these points are finite, then from (1), we have

$$w_j - w_k = \frac{az_j+b}{cz_j+d} - \frac{az_k+b}{cz_k+d} = \frac{ad-bc}{(cz_j+d)(cz_k+d)}(z_j - z_k).$$

Using this relation for $j, k = 1, 2, 3, 4$, we get $\dfrac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$

Thus the cross-ratio of four points is invariant under bilinear transformation.

This property is very useful in finding a bilinear transformation. If one of the points, say : $z_1 \to \infty$, the quotient of those two differences which contain z_1, is replaced by 1 *i.e.*,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{z_3-z_4}{z_3-z_2}.$$

Example 20.13. *Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$.*

Hence find (a) the image of $|z| < 1$, *(Mumbai, 2006 ; Delhi, 2002)*

(b) the invariant points of this transformation. *(U.P.T.U., 2008 ; V.T.U., 2000)*

Solution. Let the points $z_1 = 1, z_2 = i, z_3 = -1$ and $z_4 = z$ map onto the points $w_1 = i, w_2 = 0, w_3 = -i$ and $w_4 = w$. Since the cross-ratio remains unchanged under a bilinear transformation.

$$\therefore \quad \frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(i-0)(-i-w)}{(i-w)(-i-0)} \quad \text{or} \quad \frac{w+i}{w-i} = \frac{(z+1)(1-i)}{(z-1)(1+i)}$$

By componendo dividendo, we get $\dfrac{2w}{2i} = \dfrac{(z+1)(1-i)+(z-1)(1+i)}{(z+1)(1-i)-(z-1)(1+i)}$

† **Def.** *If t_1, t_2, t_3, t_4 be any four numbers, then* $\dfrac{(t_1-t_2)(t_3-t_4)}{(t_1-t_4)(t_3-t_2)}$ *is said to be their cross-ratio and is denoted* (t_1, t_2, t_3, t_4).

$\therefore \qquad w = \dfrac{1+iz}{1-iz}$ ...(i)

which is the required bilinear transformation.

(a) Rewriting (i) as $\quad z = i\dfrac{1-w}{1+w}$

$\therefore \qquad \left|\dfrac{i(1-w)}{1+w}\right| = |z| < 1 \quad \text{or} \quad |i|\,|1-w| < |1+w|$

or $\qquad |1-u-iv| < |1+u+iv|$ $\qquad [\because \ |i| = 1]$

or $\qquad (1-u)^2 + v^2 < (1+u)^2 + v^2$ which reduces to $u > 0$.

Hence the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped onto the entire half of the w-plane to the right of the imaginary axis.

(b) To find the invariant points of the transformation, we put $w = z$ in (i).

$\therefore \qquad z = \dfrac{1+iz}{1-iz} \quad \text{or} \quad iz^2 + (i-1)z + 1 = 0$

or $\qquad z = \dfrac{1-i \pm \sqrt{[(i-1)^2 - 4i]}}{2i} = -\tfrac{1}{2}\{1 + i \pm \sqrt{(6i)}\}$

which are the required invariant points.

Example 20.14. *Show that $w = \dfrac{i-z}{i+z}$ maps the real axis of z-plane into the circle $|w| = 1$ and the half plane $y > 0$ into the interior of the unit circle $|w| = 1$ in the w-plane.* *(Mumbai, 2007)*

Solution. Since $\qquad w = (i-z)/(i+z)$,

$\therefore \qquad |w| = 1$ becomes $|(i-z)/(i+z)| = 1 \quad \text{or} \quad |i-z| = |i+z|$

i.e., $\qquad |i - x - iy| = |i + x + iy| \quad \text{or} \quad |-x + i(1-y)| = |x + i(1+y)|$

$\therefore \qquad \sqrt{[x^2 + (1-y)^2} = \sqrt{(x^2 + (1+y)^2]} \text{ or } (1-y)^2 = (1+y)^2$

$\therefore \qquad 4y = 0 \quad \text{or} \quad y = 0$ which is the real axis.

Hence the real axis of the z-plane is mapped to the circle $|w| = 1$

Now for the interior of the circle $|w| = 1$

$\qquad |w| < 1 \quad i.e., \quad |i-z| < |i+z| \quad \text{or} \quad (1-y)^2 < (1+y)^2$

$\therefore \qquad -4y < 0 \quad i.e., \quad y > 0$

Hence the half plane $y > 0$ is mapped into the interior of the circle $|w| = 1$.

PROBLEMS 20.2

1. Find the invariant points of the transformation $w = (z-1)/(z+1)$. *(Madras, 2003)*
2. Find the transformation which maps the points $-1, i, 1$ of the z-plane onto $1, i, -1$ of the w-plane respectively. Also find its *invariant points*. *(V.T.U., 2011)*
3. Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation. *(S.V.T.U., 2008 ; Mumbai, 2007 ; V.T.U., 2006)*
4. Determine the bilinear transformation that maps the points $1-2i$, $2+i$, $2+3i$ respectively into $2+2i$, $1+3i$, 4. *(J.N.T.U., 2003 ; Coimbatore, 1999)*
5. Find the bilinear transformation which maps
   (i) the points $z = 1, i, -1$ into the points $w = 0, 1, \infty$ *(V.T.U., 2008 ; Mumbai, 2007)*
   (ii) the points $z = 0, 1, i$ into the points $w = 1+i, -i, 2-i$ *(V.T.U., 2010 S)*
   (iii) $R(z) > 0$ into interior of unit circle so that $z = \infty, i, 0$ map into $w = -1, -i, 1$.
6. Under the transformation $w = \dfrac{z-1}{z+1}$, show that the map of the straight line $x = y$ is a circle and find its centre and radius. *(Marathwada, 2008)*

$\therefore \qquad w = A \int \frac{dz}{\sqrt{(z^2-1)}} + B = A \cosh^{-1} z + B$

When $z = 1, w = 0. \quad \therefore \quad 0 = A \cosh^{-1}(1) + B$, *i.e.*, $B = 0$.

When $z = -1, w = ib. \quad \therefore \quad ib = A \cosh^{-1}(-1) + 0$, *i.e.*, $\cosh(ib/A) = -1$

or $\qquad \cos \frac{b}{A} = -1 = \cos \pi$. Thus $A = \frac{b}{\pi}$.

Hence $\qquad w = \frac{b}{\pi} \cosh^{-1} z$ or $z = \cosh \frac{\pi w}{b}$.

PROBLEMS 20.4

1. Find the transformation which maps the semi-infinite strip of width π bounded by the lines $v = 0, v = \pi$ and $u = 0$ into the upper half of the z plane.
2. Show how you will use Schwarz-Christoffel transformation to map the semi-infinite strip enclosed by the real axis and the lines $u = \pm 1$ of the w-plane into the upper half of the z-plane.
3. Find the mapping function which maps semi-infinite strip in the z-plane $-\pi/2 \le x \le \pi/2, y \ge 0$ into half w-plane for which $v \ge 0$, such that the points $(-\pi/2, 0), (\pi/2, 0)$ in the π-plane are mapped into the points $(-1, 0), (1, 0)$ respectively in w-plane.
4. Find the transformation which will map the interior of the infinite strip bounded by the lines $v = 0, v = \pi$ onto the upper half of the z-plane.

20.12 COMPLEX INTEGRATION

We have already discussed the concept of the line integral as applied to vector fields in § 8.11. Now we shall consider the line integral of a complex function.

Consider a continuous function $f(z)$ of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide C into n parts at the points

$$A = P_0(z_0), P_1(z_1), \ldots, P_i(z_i), \ldots P_n(z_n) = B.$$

Let $\delta z_i = z_i - z_{i-1}$ and ζ_i be any point on the arc $P_{i-1} P_i$. The limit of the sum $\sum_{i=1}^{n} f(\zeta_i)\,\delta z_i$ as $n \to \infty$ *in such a way that the length of the chord* δz_i *approaches zero, is called the* **line integral of f (z)** *taken along the path C, i.e.,*

$$\int_C f(z)\, dz.$$

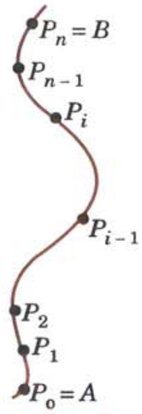


Fig. 20.13

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$,

$$\int_C f(z)\, dz = \int_C (udx - vdy) + i \int_C (vdx + udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Obs. *The value of the integral is independent of the path of integration when the integrand is analytic.*

Example 20.17. *Prove that*

(*i*) $\int_C \frac{dz}{z-a} = 2\pi i.$ $\qquad$ (*ii*) $\int_C (z-a)^n\, dz = 0$ [*n, any integer* $\neq -1$]

where C is the circle $|z - a| = r$. $\qquad$ (*U.P.T.U., 2003*)

Solution. The parametric equation of C is $z - a = re^{i\theta}$, where θ varies from 0 to 2π as z describes C once in the positive (anti-clockwise) sense. (Fig. 20.14)

(*i*) $$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}} \cdot ire^{i\theta}\, d\theta \qquad [\because dz = ire^{i\theta}\, d\theta]$$

$$= i \int_0^{2\pi} d\theta = 2\pi i$$

(ii) $\int_C (z-a)^n\, dz = \int_0^{2\pi} r^n e^{ni\theta} \,.\, ire^{i\theta}\, d\theta$

$$= ir^{n+1} \int_0^{2\pi} e^{(n+1)\theta i}\, d\theta = \frac{r^{n+1}}{n+1} \left| e^{(n+1)\theta i} \right|_0^{2\pi}, \text{ provided } n \neq -1$$

$$= \frac{r^{n+1}}{n+1} [e^{2(n+1)\pi i} - 1] = 0 \qquad [\because \; e^{2(n+1)\pi i} = 1]$$

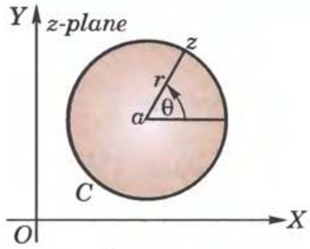


Fig. 20.14

Example 20.18. *Evaluate* $\int_0^{2+i} (\bar{z})^2\, dz$, *along (i) the line* $y = x/2$, *(Bhopal, 2007 ; U.P.T.U., 2002)*
(ii) the real axis to 2 and then vertically to 2 + i. *(S.V.T.U., 2009 ; P.T.U., 2008 S ; Mumbai, 2006)*

Solution. (*i*) Along the line OA, $x = 2y$, $z = (2+i)y$, $\bar{z} = (2-i)y$ and $dz = (2+i)\, dy$ (Fig. 20.15)

$$\therefore \qquad I = \int_0^{2+i} (\bar{z})^2\, dz = \int_0^1 (2-i)^2 y^2 \,.\, (2+i)\, dy$$

$$= 5(2-i) \left| \frac{y^3}{3} \right|_0^1 = \frac{5}{3}(2-i)$$

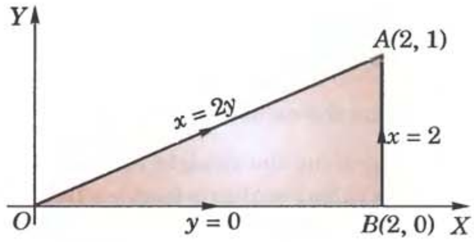


Fig. 20.15

(*ii*) $$I = \int_{OB} (\bar{z})^2\, dz + \int_{BA} (\bar{z})^2\, dz.$$

Now along OB, $z = x$, $\bar{z} = x$, $dz = dx$;

and along BA, $z = 2 + iy$, $\bar{z} = 2 - iy$, $dz = idy$

$$\therefore \qquad I = \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 \,.\, idy = \left| \frac{x^3}{3} \right|_0^2 + \int_0^1 [\, 4y + (4 - y^2) i \,]\, dy$$

$$= \frac{8}{3} + 4 \,.\, \frac{1}{2} + \left(4.1 - \frac{1}{3}\right) i = \frac{1}{3}(14 + 11i).$$

Example 20.19. *Evaluate* $\int_C (z^2 + 3z + 2)\, dz$ *where C is the arc of the cycloid* $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$ *between the points* $(0, 0)$ *and* $(\pi a, 2a)$. *(Rohtak, 2004)*

Solution. $f(z) = z^2 + 3z + 2$ is analytic in the z-plane being a polynomial. As such, the line integral of $f(z)$ between O and A is independent of the path (Fig. 20.16). We therefore, take the path from O to L and L to A so that

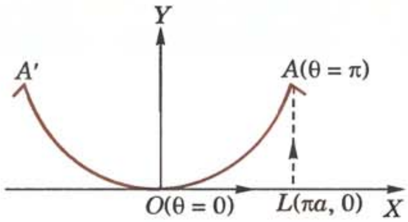


Fig. 20.16

$$\int_C f(z)\, dz = \int_{OL} f(z)\, dz + \int_{LA} f(z)\, dz \qquad \text{...}(i)$$

$$\therefore \quad \int_{OL} f(z)\, dz = \int_0^{\pi a} (x^2 + 3x + 2)\, dx$$

[$\because$ along OL, $y = 0$, $x = 0$ at O, $x = \pi a$ at L

$$= \left| \frac{x^3}{3} + \frac{3x^2}{2} + 2x \right|_0^{\pi a} = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) \qquad \text{...}(ii)$$

and $$\int_{LA} f(z)\, dz = \int_0^{2a} [(\pi a + iy)^2 + 3\,(\pi a + iy) + 2]\; idy$$

[$\because$ along LA, $x = \pi a$, $z = \pi a + iy$, $dz = idy$ and y varies from 0 (at L) to $2a$ (at A)

$$= L \left| \frac{(\pi a + iy)^3}{3i} + 3\,\frac{(\pi a + iy)^2}{2i} + 2y \right|_0^{2\pi} = \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia \qquad \text{...}(iii)$$

$\therefore$ substituting from (*ii*) and (*iii*) in (*i*), we get

$$\int_C f(z)\, dz = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) + \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia$$

PROBLEMS 20.5

1. Evaluate $\int_0^{1+i}(x^2 - iy)\,dz$ along the paths (*a*) $y = x$ and (*b*) $y = x^2$. (*U.P.T.U., 2010*)

2. Evaluate $\int_{1-i}^{2+i}(2x + iy + 1)\,dz$, along the two paths: (*U.P.T.U., 2010*)
   (*i*) $x = t + 1, y = 2t^2 - 1$ (*ii*) the straight line joining $1 - i$ and $2 + i$. (*U.P.T.U., 2006*)

3. Evaluate $\int_{1-i}^{2+3i}(z^2 + z)\,dz$ along the line joining the points $(1, -1)$ and $(2, 3)$. (*V.T.U., 2004*)

4. Show that for every path between the limits, $\int_{-2}^{-2+i}(2 + z)^2\,dz = -i/3$. (*Delhi, 2002*)

5. Show that $\%\,(z + 1)\,dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0, z = 1$, $z = 1 + i$ and $z = i$. (*Rohtak, 2006*)

6. Evaluate $\int_C |z|\,dz$, where C is the contour
   (*i*) straight line from $z = -i$ to $z = i$.
   (*ii*) left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.
   (*iii*) circle given by $|z + 1| = 1$ described in the clockwise sense.

7. Find the value of $\int_0^{1+i}(x - y + ix^2)\,dz$
   (*i*) along the straight line from $z = 0$ to $z = 1 + i$
   (*ii*) along real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$. (*U.P.T.U., 2003*)

8. Prove that $\int dz/z = -\pi i$ or πi, according as C is the semi-circular arc $|z| = 1$ above or below the real axis. (*Rohtak, 2005*)

9. Evaluate $\int_C (z - z^2)\,dz$, where C is the upper half of the circle $|z| = 1$.
   What is the value of this integral if C is the lower half of the above circle ?

20.13 CAUCHY'S THEOREM

If f (z) is an analytic function and f′(z) is continuous at each point within and on a closed curve C, then $\underset{C}{\%}$ $f(z)\,dz = 0$.

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$

$$\underset{C}{\%} f(z)dz = \underset{C}{\%}(udx - vdy) = i\,\underset{C}{\%}(vdx + udy) \qquad ...(1)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. Hence the Green's theorem (p. 376) can be applied to (1), giving

$$\underset{C}{\%} f(z)dz = -\iint_D \left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right] dxdy + i \iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy \qquad ...(2)$$

Now $f(z)$ being analytic, u and v necessarily satisfy the Cauchy-Riemann equations and thus the integrands of the two double integrals in (2) vanish identically.

Hence $\oint_C f(z)dz = 0.$

Obs. 1. The Cauchy-Riemann equations are precisely the conditions for the two real integrals in (1) to be independent of the path. *Hence the line integral of a function f (z) which is analytic in the region D, is independent of the path joining any two points of D.*

Obs. 2. Extension of Cauchy's theorem. *If f (z) is analytic in the region D between two simple closed curves C and* C_1, then $\oint_C f(z)dz = \oint_{C_1} f(z)dz$.

To prove this, we need to introduce the cross-cut AB. Then $\oint f(z)dz = 0$ where the path is as indicated by arrows in Fig. 20.17, *i.e.*, along AB—along C_1 in clockwise sense and along BA—along C in anti-clockwise sense.

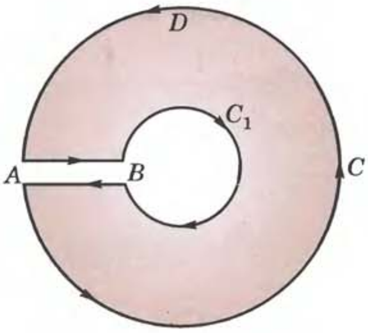


Fig. 20.17(a)

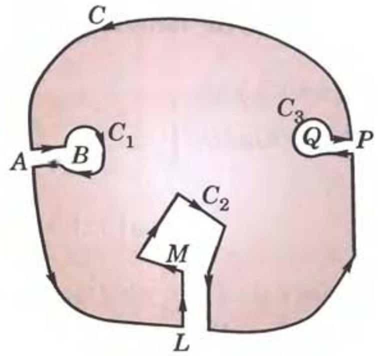


Fig. 20.17(b)

i.e., $$\int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{AB} f(z)dz + \int_C f(z)dz = 0$$

But, since the integrals along AB and along BA cancel, it follows that

$$\int_C f(z)dz + \int_{C_1} f(z)dz = 0$$

Reversing the direction of the integral around C_1 and transposing, we get

$\int_C f(z)dz + \int_{C_1} f(z)dz$ each integration being taken in the anti-clockwise sense.

If C_1, C_2, C_3, ... *be any number of closed curves within* C (*Fig. 20.17(b)*), *then*

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \ldots$$

20.14 CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic within and on a closed curve and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i} = \oint_C \frac{f(z)dz}{z-a}$$

Consider the function $f(z)/(z-a)$ which is analytic at all points within C except at $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $f(z)/(z-a)$ being analytic in the region enclosed by C and C_1, we have by Cauchy's theorem,

$$\oint_C \frac{f(z)}{z-a}dz = \oint_{C_1} \frac{f(z)}{z-a}dz \qquad \left\{\begin{array}{l}\text{For any point on } C_1,\\ z-a = re^{i\theta} \text{ and } dz = ire^{i\theta}\, d\theta\end{array}\right.$$

$$= \oint_C \frac{f(a+re^{i\theta})}{re^{i\theta}} . ire^{i\theta}\, d\theta = i\oint_{C_1} f(a+re^{i\theta})d\theta \qquad \ldots(1)$$

In the limiting form, as the circle C_1 shrinks to the point a, *i.e.*, as $r \to 0$, the integral (1) will approach to

$$\oint_C f(a)d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a) \oint f(z)dz. \text{ Thus } \oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

i.e., $$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \qquad \ldots(2)$$

which is the desired *Cauchy's integral formula*. (*V.T.U., 2011 S*)

Cor. Differentiating both sides of (2) w.r.t. a,

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz \qquad \ldots(3)$$

Similarly, $$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\oint_C dz \qquad \ldots(4)$$

and in general, $$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz. \quad \text{...(5)}$$

Thus it follows from the results (2) to (5) that *if a function $f(z)$ is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C.* Incidently, we have established a remarkable fact that **an analytic function possesses derivatives of all orders and these are themselves all analytic.**

Example 20.20. *Evaluate* $\int_C \frac{z^2 - z + 1}{z - 1}\,dx$, *where C is the circle*

(*i*) $|z| = 1$, (*ii*) $|z| = \frac{1}{2}$. (*S.V.T.U., 2007*)

Solution. (*i*) Here $f(z) = z^2 - z + 1$ and $a = 1$.

Since $f(z)$ is analytic within and on circle $C : |z| = 1$ and $a = 1$ lies on C.

$\therefore$ by Cauchy's integral formula $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} = f(a) = 1$ *i.e.*, $\int_C \frac{z^2 - z + 1}{z - 1}\,dz = 2\pi i$.

(*ii*) In this case, $a = 1$ lies outside the circle $C : |z| = 1/2$. So $(z^2 - z + 1)/(z - 1)$ is analytic everywhere within C.

$\therefore$ by Cauchy's theorem $\int_C \frac{z^2 - z + 1}{z - 1}\,dz = 0$.

Example 20.21. *Evaluate, using Cauchy's integral formula:*

(*i*) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ *where C is the circle* $|z| = 3$ (*U.P.T.U., 2010*)

(*ii*) $\oint_C \frac{\cos \pi z}{z^2 - 1}\,dz$ *around a rectangle with vertices* $2 \pm i, -2 \pm 1$

(*iii*) $\oint_C \frac{e^{tz}}{z^2 + 1}\,dz$ *where C is the circle* $|z| = 3$. (*U.P.T.U., 2009*)

Solution. (*i*) $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within the circle $|z| = 3$ and the two singular points $z = 1$ and $z = 2$ lie inside this circle.

$$\therefore \quad \oint_C \frac{f(z)dz}{(z-1)(z-2)} = \oint_C (\sin \pi z^2 + \cos \pi z^2) = \left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$= \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2}\,dz - \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-1}\,dz$$

$$= 2\pi i\,[\sin \pi(2)^2 + \cos \pi(2)^2] - 2\pi i[\sin \pi(1)^2 + \cos \pi(1)^2]$$

[By Cauchy's integral formula]

$$= 2\pi i\,(0 + 1) - 2\pi i\,(0 - 1) = 4\pi i$$

(*ii*) $f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and the two singular points $z = 1$ and $z = -1$ lie inside this rectangle. (Fig. 20.18)

$$\therefore \quad \oint_C \frac{\cos \pi z}{z^2 - 1}\,dz = \frac{1}{2}\oint_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right)\cos \pi z\,dz$$

$$= \frac{1}{2}\oint_C \frac{\cos \pi z}{z-1}\,dz - \oint_C \frac{\cos \pi z}{z+1}\,dz$$

$$= \frac{1}{2}\{2\pi i \cos \pi\,(1)\} - \frac{1}{2}\{2\pi i \cos \pi\,(-1)\} = 0.$$

[By Cauchy's integral formula]

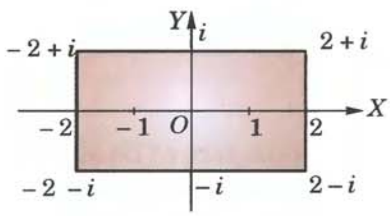


Fig. 20.18

(*iii*) $f(z) = e^{tz}$ is analytic within the circle $|z| = 3$.

The singular points are given by $z^2 + 1 = 0$ *i.e.*, $z = i$ and $z = -i$ which lie within this circle.

$$\therefore \quad \oint_C \frac{e^{tz}}{z^2+1}dz = \oint_C \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)e^{tz}dz = \frac{1}{2i}\left\{\oint_C \frac{e^{tz}}{z-i}dz - \oint_C \frac{e^{tz}}{z+i}dz\right\}$$

$$= \frac{1}{2i}\{2\pi i e^{t(i)} - 2\pi i e^{t(-i)}\} \quad \text{[By Cauchy's integral formula]}$$

$$= 2\pi i\left(\frac{e^{it} - e^{-it}}{2i}\right) = 2\pi i \sin t.$$

Example 20.22. *Evaluate*

(*i*) $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}dz$, *where C is the circle* $|z| = 1$ (*Rohtak, 2005*)

(*ii*) $\oint_C \frac{e^{2z}}{(z+i)^4}dz$, *where C is the circle* $|z| = 3$ (*U.P.T.U., 2008*)

(*iii*) $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz$, *where C is* $|z| = 4$. (*U.P.T.U., 2008; J.N.T.U., 2000*)

Solution. (*i*) $f(z) = \sin^2 z$ is analytic inside the circle C: $|z| = 1$ and the point $a = \pi/6$ (= 0.5 approx.) lies within C.

$\therefore$ by Cauchy's integral formula $f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}dz$,

we get $$\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}dz = \pi i\left[\frac{d^2}{dz^2}(\sin^2 z)\right]_{z=\pi/6}$$

$$= \pi i(2\cos 2z)_{z=\pi/6} = 2\pi i \cos \pi/3 = \pi i.$$

(*ii*) $f(z) = e^{2z}$ is analytic within the circle C: $|z| = 3$. Also $z = -1$ lies inside C.

$\therefore$ By Cauchy's integral formula: $f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^4}$

we get $$\int_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{6}\left|\frac{d^3(e^{2z})}{dz^3}\right|_{z=-1} = \frac{\pi i}{3}[8e^{2z}]_{z=-1} = \frac{8\pi i}{3}e^{-2}$$

(*iii*) $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$ is not analytic at $z = \pm \pi i$.

However both $z = \pm \pi i$ lie within the circle $|z| = 4$.

Now $$\frac{1}{(z+\pi i)^2(z-\pi i)^2} = \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$$

where $A = 7/2\pi^3 i$, $C = -7/2\pi^3 i$, $B = D = -1/4\pi^2$

$$\therefore \quad \int_C \frac{e^z}{(z^2+\pi^2)^2}dz = \frac{7}{2\pi^3 i}\left\{\int_C \frac{e^z}{z+\pi i}dz - \int_C \frac{e^z}{z-\pi i}dz\right\} - \frac{1}{4\pi^2}\left\{\int_C \frac{e^z}{(z+\pi i)^2}dz + \int \frac{e^z}{(z-\pi i)^2}dz\right\}$$

$$= \frac{7}{2\pi^3 i}[2\pi i f(-\pi i) - 2\pi i f(\pi i)] - \frac{1}{4\pi^2}[2\pi i f'(-\pi i) + 2\pi i f'(\pi i)] \quad \text{where } f(z) = e^z.$$

$$= \frac{7}{\pi^2}(e^{-\pi i} - e^{\pi i}) - \frac{i}{2\pi}(e^{-\pi i} + e^{\pi i}) = -\frac{14i}{\pi^2}\left(\frac{e^{\pi i} - e^{-\pi i}}{2i}\right) - \frac{i}{\pi}\left(\frac{e^{\pi i} + e^{-\pi i}}{2}\right) \quad [\S\ 19.9]$$

$$= -\frac{14i}{\pi^2}\sin \pi - \frac{i}{\pi}\cos \pi = \frac{i}{\pi}.$$

Example 20.23. *Verify Cauchy's theorem by integrating e^{iz} along the boundary of the triangle with the verticles at the points $1 + i, -1 + i$ and $-1 - i$.* (*U.P.T.U., 2006*)

Solution. The boundary of the given triangle consists of three lines AB, BC, CA. (Fig. 29.19).

$$\oint_{ABC} e^{iz} dz = \int_{AB} e^{iz} dz + \int_{BC} e^{iz} dz + \int_{CA} e^{iz} dz$$

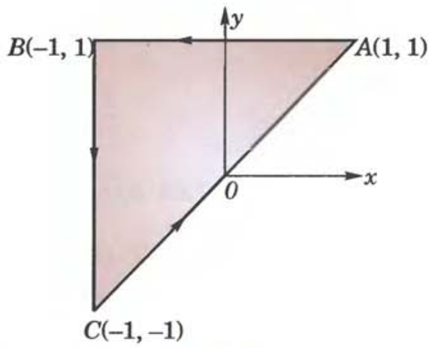


Fig. 20.19

Now $\displaystyle\int_{AB} e^{iz} dz = \int_1^{-1} e^{i(x+i)} dx$ $\left| \begin{array}{l} \because \text{Along } AB : y = 1 \\ \therefore z = x + i \text{ and } dz = dx \end{array} \right.$

$$= \int_1^{-1} e^{(ix-1)} dx = \left| \frac{e^{ix-1}}{i} \right|_1^{-1} = \frac{e^{-i-1} - e^{i-1}}{i}$$

$$\int_{BC} e^{iz} dz = \int_1^{-1} e^{i(-1+1iy)} idy$$

$\left| \begin{array}{l} \because \text{Along } BC : x = -1 \\ \therefore z = -1 + iy, dz = idy \end{array} \right.$

$$= i \int_1^{-1} e^{-i-y} dy = i \left| \frac{e^{-i-y}}{-1} \right|_1^{-1} = \frac{e^{-i+1} - e^{-i-1}}{i}$$

$$\int_{CA} e^{iz} dz = \int_{-1}^{1} e^{i(1+i)x} (1+i) dx$$ $\left| \begin{array}{l} \because \text{Along } CA : y = x \\ \therefore z = (1 + i, x, dz = (1 + i) dx \end{array} \right.$

$$= (1+i) \frac{e^{(i-1)} - e^{-(i-1)}}{i(1+i)} = \frac{e^{i-1} - e^{-i+1}}{i}$$

Thus from (*i*) $\displaystyle\oint_{ABC} e^{iz} dz = \frac{e^{-i-1} - e^{i-1}}{i} + \frac{e^{-i+1} - e^{-i-1}}{i} + \frac{e^{i-1} - e^{-i+1}}{i} = 0$ ...(*ii*)

Also since $f(x) = e^{iz}$ is analytic everywhere,

$\therefore$ by Cauchy's theorem $\oint f(z) = 0$ ...(*iii*)

Hence from (*ii*) and (*iii*), $\oint_{ABC}$ Cauchy's theorem is verified.

Example 20.24. *If* $F(\zeta) = \oint_C \dfrac{4z^2 + z + 5}{z - \zeta} dz$, *where C is the ellipse* $(x/2)^2 + (y/3)^2 = 1$, *find the value of* (*a*) $F(3.5)$; (*b*) $F(i)$, $F'(-1)$ *and* $F''(-i)$. (*Bhopal, 2009 ; Marathwada, 2008 ; Mumbai, 2006*)

Solution. (*a*) $$F(3.5) = \oint_C \frac{z^3 + z + 1}{z^2 - 7z + 2} dz$$

Since $\zeta = 3.5$ is the only singular point of $(4z^2 + z + 5)/(z - 3.5)$ and it lies outside the ellipse C, therefore, $(4z^2 + z + 5)/(z - 3.5)$ is analytic everywhere within C.

Hence by Cauchy's theorem,

$$\oint_C \frac{4z^2 + z + 5}{z - 3.5} dz = 0, \text{ i.e., } F(3.5) = 0.$$

(*b*) Since $f(z) = 4z^2 + z + 5$ is analytic within C and $\zeta = i, -1$ and $-i$ all lie within C, therefore, by Cauchy's integral formula

$$f(\zeta) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - \zeta} dz$$

i.e., $$\oint_C \frac{4z^2+z+5}{z-\zeta}dz = 2\pi i(4\zeta^2+\zeta+5)$$

i.e., $$F(\zeta) = 2\pi i(4\zeta^2+\zeta+5)$$

$\therefore$ $$F'(\zeta) = 2\pi i\,(8\zeta+1) \text{ and } F''(\zeta) = 16\pi i$$

Thus $$F(i) = 2\pi i(-4+i+5) = 2\pi(i-1)$$

$$F'(-1) = 2\pi i[8(-1)+1] = -14\pi i \text{ and } F''(-i) = 16\pi i.$$

20.15 (1) CONVERSE OF CAUCHY'S THEOREM: MORERA'S THEOREM*

If $f(z)$ is continuous in a region D and $\oint_C f(z)dz = 0$ around every simple closed curve C in D, then $f(z)$ is analytic in D.

Since $\oint f(z)dz = 0$, then the line integral of $f(z)$ from a fixed point z_0 to a variable point z must be independent of the path and hence must be a function of z only. Thus

$$\int_{z_0}^{z} f(z)dz = \phi(z), \text{ (say)},$$

Let $$\phi(z) = U + iV \text{ and } f(z) = u + iv$$

Then $$U + iV = \int_{(x_0, y_0)}^{(x,y)} (u+iv)(dx+idy) = \int_{(x_0, y_0)}^{(x,y)} (udx - vdy) + i\int_{(x_0, y_0)}^{(x,y)} (vdx + udy)$$

$\therefore$ $$U = \int_{(x_0, y_0)}^{(x,y)} (udx - vdy),\ V = \int_{(x_0, y_0)}^{(x,y)} (vdx + udy)$$

Differentiating under the integral sign,

$$\frac{\partial U}{\partial x} = u, \frac{\partial U}{\partial y} = -v, \frac{\partial V}{\partial x} = v, \frac{\partial V}{\partial y} = u \quad \therefore \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Thus U and V satisfy C-R equations.

Also, since $f(z)$ is given to be continuous, u and v and therefore, $\partial U/\partial x$, $\partial U/\partial y$, $\partial V/\partial x$, $\partial V/\partial y$, are also continuous.

$\therefore$ $\phi(z)$ is an analytic function and

$$\phi'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z).$$

Thus, $f(z)$ is the derivative of an analytic function $\phi(z)$. Hence $f(z)$ is analytic by § 20.14 Cor.

(2) Cauchy's inequality†. *If $f(z)$ is analytic within and on the circle C: $|z-a| = r$, then*

$$|f^n(a)| \le \frac{Mn!}{r^n} \quad \text{...(I)}$$

where M is the maximum value of $|f(z)|$ on C.

From (5) of § 20.14, we get

$$|f^n(a)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)dz}{(z-a)^{n+1}}\right|$$

$$\le \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\oint_C |z| \qquad [\because |f(z)| < M]$$

$$= \frac{n!M}{2\pi r^{n+1}}\oint_C ds = \frac{Mn!}{2\pi r^{n+1}}2\pi r = \frac{Mn!}{r^n} \qquad (U.P.T.U., 2005)$$

(3) Liouville's theorem$. *If $f(z)$ is analytic and bounded for all z in the entire complex plane, then $f(z)$ is a constant.* (*U.P.T.U., 2008*)

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* Named after the Italian mathematician, *Giacinto Morera* (1856–1909) who worked in Turin.

† See footnote p. 144

$ See footnote p. 573.

Taking $n = 1$ and replacing a by z, (I) gives

$$| f'(z) | \le M/r$$

As $r \to \infty$, it gives $f'(z) = 0$ *i.e.*, $f(z)$ is constant for all z.

(4) Poisson's integral formulae. *If $f(z)$ is analytic within and on the circle C: $|z| = \rho$ and $z = re^{i\theta}$ is any point within C, then*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} f(re^{i\phi})d\phi$$

By Cauchy's integral formula, $f(z) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(w)}{w - z}\, dw$ ...(1)

As the inverse of the point x w.r.t. C lies outside C and is given by $\rho^2/\bar{z}$. [See footnote p. 685]

$\therefore$ by Cauchy's theorem,

$$0 = \frac{1}{2\pi i}\int \frac{f(w)}{w - \rho^2/\bar{z}}\, dw \qquad ...(2)$$

Subtracting (2) from (1), $f(z) = \dfrac{1}{2\pi i}\displaystyle\int\left(\frac{1}{w - z} - \frac{1}{w - \rho^2/\bar{z}}\right) f(w)\, dw$

$$= \frac{1}{2\pi i}\oint_C \frac{z\bar{z} - \rho^2}{\bar{z}w^2 - (z\bar{z} + \rho^2)\, w + z\rho^2)} f(w)\, dw \qquad ...(3)$$

Taking $w = \rho e^{i\phi}$ and noting that $\bar{z} = re^{-i\theta}$, (3) gives

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(r^2 - \rho^2)\, f(\rho e^{i\phi})\, .\, \rho i e^{i\phi}\, d\phi}{re^{-i\theta}\, .\, \rho^2 e^{2i\phi} - (r^2 + \rho^2)\, \rho e^{i\phi} + re^{i\theta}\rho^2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - r^2)\, f(\rho e^{i\phi})d\phi}{\rho^2 + r^2 - r\rho[e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}]} = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - r^2)\, f(\rho e^{i\phi})d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \qquad ...(4)$$

This is called *Poisson's integral formula** for a circle. It expresses the values of a harmonic function within a circle in terms of its values on the boundary.

Writing $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ and $f(\rho e^{i\phi}) = u(\rho, \phi) + iv(\rho, \phi)$ in (4) and equating real and imaginary parts from both sides, we get the formulae:

$$u(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2) u(\rho, \phi)d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \qquad ...(5)$$

and

$$v(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2) v(\rho, \phi)d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \qquad ...(6)$$

PROBLEMS 20.6

1. Evaluate $\oint_C (z - a)^{-1} dz$, where C is a simple closed curve and the point $z = a$ is (i) outside C, (ii) inside C.
2. Evaluate $\oint_C \dfrac{dz}{(z - a)^n}$, $n = 2, 3, 4, \ldots$, where C is a closed curve containing the point $z = a$.
3. Evaluate (i) $\oint_C \dfrac{e^z}{z^2 + 1} dz$, where C is the circle $|z| = 1/2$. (*P.T.U., 2010*)

   (ii) $\oint_C \dfrac{e^{3iz}}{(z + \pi)^3} dz$, where C is the circle $|z - \pi| = 3$. (*U.P.T.U., 2007*)

* Named after the French mathematician *Simeon Denis Poisson* (1781–1840) who was a professor in Paris and made contributions to partial differential equations, potential theory and probability.

4. Use Cauchy's integral formula to calculate:

(i) $\oint_C \frac{3z\ \ 5}{z^2 + 2z} dz$, where C is $|z| = 1$. (*P.T.U., 2005 S*) (ii) $\oint_C \frac{z^2 + 1}{z(2z + 1)} dz$, where C is $|z| = 1$.

(iii) $\oint_C \frac{\sin \pi z + \cos \pi z}{(z - 1)(z - 2)} dz$ where C is $|z| = 4$. (*U.P.T.U., 2008*)

5. Evaluate (a) $\oint_C \frac{z^3 - 2z + 1}{(z - i)^2}$ where C is $|z| = 2$.

(b) $\oint_C \frac{e^{-z}}{(z - 1)(z - 2)^2} dz$ where C is $|z| = 3$. (*Rohtak, 2003*)

6. Evaluate, using Cauchy's integral formulae:

(i) $\oint_C \frac{z}{z^2 - 3z + 2} dz$, where C is $|z - 2| = \frac{1}{2}$. (*U.P.T.U., 2009 ; Hissar, 2007 ; Madras, 2000*)

(ii) $\oint_C \frac{e^z dz}{(z + 1)^2}$, where C is $|z - 1| = 3$. (*Bhopal, 2009*)

(iii) $\oint_C \frac{\log z}{(z - 1)^3} dz$ where C is $|z - 1| = \frac{1}{2}$. (*J.N.T.U., 2003*)

7. Evaluate $f(2)$ and $f(3)$ where $f(a) = \oint_C \frac{2z^2 - z - 2}{z - a} dz$ and C is the circle $|z| = 2.5$.

8. If $\phi(\zeta) = \oint_C \frac{3z^2 + 7z + 1}{z - \zeta} dz$, where C is the circle $|z| = 2$ find the values of

(i) $\phi(3)$, (ii) $\phi'(1 - i)$ (iii) $\phi''(1 - i)$. (*Mumbai, 2006*)

9. Evaluate $\oint_C \frac{z^3 + z + 1}{z^2 - 7z + 2} dz$, where C is the ellipse $4x^2 + 9y^2 = 1$. (*Rohtak, 2006*)

10. Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the (i) rectangle with vertices $-1, 1, 1 + i, -1 + i$; (ii) triangle with vertices $(1, 2), (1, 4), (3, 2)$. (*V.T.U., 2003*)

20.16 (1) SERIES OF COMPLEX TERMS

Let
$$(a_1 + ib_1) + (a_2 + ib_2) + \dots + (a_n + ib_n) + \dots \qquad \dots(1)$$
be an infinite series of complex terms ; a's and b's being real numbers. *If the series Σa_n and Σb_n converge to the sums A and B, then series (1) is said to* **converge** *to the sum $A + iB$.* Also if (1) is a convergent series, then $\underset{n \to \infty}{\text{Lt}} (a_n + ib_n) = 0$.

The series (1) is said to be **absolutely convergent** *if the series*
$$|a_1 + ib_1| + |a_2 + ib_2| + \dots + |a_n + ib_n| + \dots$$
is convergent. Since $|a_n|$ and $|b_n|$ are both $\le |a_n + ib_n|$, it follows *that an absolutely convergent series is convergent.*

Next let the series of functions $u_1(z) + u_2(z) + \dots + u_n(z) + \dots\dots \qquad \dots(2)$

converge to the sum $S(z)$ and $S_n(z)$ be the sum of its first n terms. Then *the series (2) is said to be* **uniformly convergent** *in a region R, if corresponding to any positive number ε, there exists a positive number N, depending on ε, but not on z, such that for every z in R.*
$$|S(z) - S_n(z)| < \varepsilon \text{ for } n > N. \qquad [\textit{cf. Def. p. 389}]$$

As in the case of real series (p. 390) **Weirstrass's M-test** holds for series of complex terms. So *the series (2) is uniformly convergent in a region R if there is a convergent series of positive constants ΣM_n such that $|u_n(z)| \le M_n$ for all z in R.*

Also a *uniformly convergent series of continuous complex functions is itself continuous and can be integrated term by term.*

Obs. *If a power series $\Sigma a_n z^n$ converges for $z = z_1$, then it converges absolutely for $|z| < |z_1|$.*

Since $\Sigma a_n z_1^n$ converges, therefore, $\underset{n \to \infty}{\text{Lt}}\ a_1 z_1^n = 0$ and so we can find a number k such that $|a_n z_1^n| < k$ for all n. Then $\Sigma a_n z^n = \Sigma |a_n z_1^n| \cdot |z/z_1|^n < \Sigma k t^n$ where $t = |z/z_1|$.

But the series Σt^n converges for $t < 1$. Hence the series $\Sigma a_n z^n$ converges absolutely for $|z| < |z_1|$, *i.e.*, if a circle with centre at the origin and radius $|z|$ be drawn, then the given series converges absolutely at all points inside the circle.

Such a circle $|z| = R$ within which series $\Sigma a_n z^n$ converges, is called the *circle of convergence and R is called the radius of convergence*.

A power series is uniformly convergent in any region which lies entirely within its circle of convergence.

(2) Taylor's series*. *If $f(z)$ is analytic inside a circle C with centre at a, then for z inside C,*

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^n(a)}{n!}(z-a)^n + \dots \qquad \dots(i)$$

Proof. Let z be any point inside C. Draw a circle C_1 with centre at a enclosing z (Fig. 20.20). Let t be a point on C_1. We have

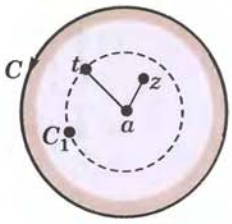


Fig. 20.20

$$\frac{1}{t-z} = \frac{1}{t-a-(z-a)} = \frac{1}{t-a}\left(1 - \frac{z-a}{t-a}\right)^{-1}$$

$$= \frac{1}{t-a}\left[1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right) + \dots + \left(\frac{z-a}{t-a}\right)^n + \dots\right] \qquad \dots(ii)$$

As $|z-a| < |t-a|$, *i.e.* $|(z-a)/(t-a)| < 1$, this series converges uniformly. So, multiplying both sides of (*ii*) by $f(t)$, we can integrate over C_1.

$$\therefore \quad \oint_{C_1} \frac{f(t)}{t-z} dz = \oint_{C_1} \frac{f(t)}{t-a} dz + (z-a)\oint_{C_1} \frac{f(t)}{(t-a)^2} dt + \dots + (z-a)^n \cdot \oint_{C_1} \frac{f(t)}{(t-a)^{n+1}} dt + \dots \qquad \dots(iii)$$

Since $f(t)$ is analytic on and inside C_1, therefore, applying the formulae (2) to (5) of p. 697-698 (*iii*), we get (*i*) which is known as *Taylor's series*.

Obs. Another remarkable fact is that *complex analytic functions can always be represented by power series of the form* (*i*).

(3) Laurent's series†. *If f(z) is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 ($r > r_1$) and with centre at a, then for all z in R*

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\oint \frac{f(t)}{(t-a)^{n+1}}\, dt,$

Γ being any curve in R, encircling C_1 (as in Fig. 20.21).

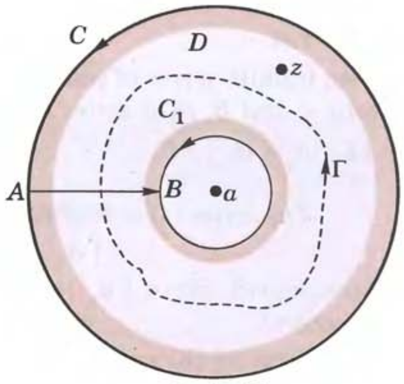


Fig. 20.21

Proof. Introduce cross-out AB, then $f(z)$ is analytic in the region D bounded by AB, C_1 described clockwise, BA and C described anti-clockwise (see Fig. 20.17). Then if z be any point in D, we have

$$f(z) = \frac{1}{2\pi i}\left[\int_{AB} \frac{f(t)}{t-z} dt + \oint_{C_1} \frac{f(t)}{t-z} dt + \int_{BA} \frac{f(t)}{t-z} dt + \oint_{C_1} \frac{f(t)}{t-z} dt\right]$$

$$= \frac{1}{2\pi i}\left[\oint_{C} \frac{f(t)}{t-z} dt - \oint_{C_1} \frac{f(t)}{t-z} dt\right] \qquad \dots(i)$$

where both C and C_1 are described anti-clockwise in (*i*) and integrals along AB and BA cancel (Fig. 20.21).

For the first integral in (*i*), expanding $1/(t-z)$ as in § 20.16 (2), we get

$$\frac{1}{2\pi i}\oint_C \frac{f(t)}{t-z} dt = \sum_{n=1}^{\infty} \frac{(z-a)^n}{2\pi i} \oint_C \frac{f(t)}{(t-a)^{n+1}} dt$$

* See footnote p. 145.

† Named after the French engineer and mathematician *Pierre Alphonse Laurent* (1813–1854) who published this theorem in 1843.

$$= \Sigma a_n (z-a)^n \text{ where } \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t-a)^{n+1}} dt \qquad ...(ii)$$

For the second integral in (i), let t lie on C_1. Then we write

$$\frac{1}{t-z} = \frac{1}{(t-a)-(z-a)} = -\frac{1}{z-a}\left(1-\frac{t-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{z-a}\left[1+\frac{t-a}{z-a}+\left(\frac{t-a}{z-a}\right)^2+\ldots+\left(\frac{t-a}{z-a}\right)^{n-1}+\ldots\right]$$

As $|t-a| < |z-a|$, *i.e.*, $|(t-a)/(z-a)| < 1$, this series converges uniformly. So multiplying both sides by $f(t)$ and integrating over C_1, we get

$$-\frac{1}{2\pi i}\oint_C \frac{f(t)}{t-z} dt = \sum_{n=1}^{\infty} \frac{1}{(z-a)^n} \cdot \frac{1}{2\pi i}\oint_C (t-a)^{n-1} f(t) dt = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \qquad ...(iii)$$

where $$a_{-n} = \frac{1}{2\pi i}\oint_{C_1} \frac{f(t)}{(t-a)^{-n+1}} dt$$

Substituting from (ii) and (iii) in (i), we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}. \qquad ...(iv)$$

Now $f(t)/(t-a)^{n+1}$ being analytic in the region between C and Γ, we can take the integral giving a_n over Γ. Similarly we can take the integral giving a_{-n} over Γ. Hence (iv) can be written as

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}} dt$$

which is known as *Laurent's series.*

Obs. 1. *As $f(z)$ is not given to be analytic inside Γ,* $a_n = \dfrac{1}{2\pi i}\oint_\Gamma \dfrac{f(t)}{(t-a)^{n+1}} dt \neq \dfrac{f^n(a)}{n!}$

However, *if $f(z)$ is analytic inside Γ,* then $a_{-n} = 0$; $a_n = \dfrac{1}{2\pi i}\oint_\Gamma \dfrac{f(t)}{(t-a)^{n+1}} dt = \dfrac{f^n(a)}{n!}$

and *Laurent's series reduces to Taylor's series.*

Obs. 2. *To obtain Taylor's or Laurent's series, simply expand f(z) by binomial theorem* instead of finding a_n by complex integration which is quite complicated.

Obs. 3. *Laurent series of a given analytic function f(z) in its annulus of convergence is unique.* There may be different Laurent series of $f(z)$ in two annuli with the same centre.

Example 20.25. *Show that the series $z(1-z) + z^2(1-z) + z^3(1-z) + \ldots\infty$ converges for $|z| < 1$. Determine whether it converges absolutely or not.*

Solution. Let the sum of the first n terms of the series be s_n, so that

$$s_n = z - z^2 + z^2 - z^3 + z^3 - z^4 + \ldots + z^n - z^{n+1} = z - z^{n+1}$$

For $|z| < 1$, $z^{n+1} \to 0$ as $n \to \infty$.

$\therefore$ $\quad \underset{n\to\infty}{\text{Lt}} (s_n) = z$, *i.e.*, the given series converges for $|z| < 1$.

$$|s_n(z)| = |z(1-z)| + |z^2(1-z)| + \ldots + |z^n(1-z)|$$
$$= |1-z| \{|z| + |z|^2 + |z|^3 + \ldots + |z|^n\}$$

For $|z| < 1$, $\quad \underset{n\to\infty}{\text{Lt}} |s_n(z)| = |1-z| \dfrac{|z|}{1-|z|}$ [G.P.]

Hence the given series converges absolutely.

Example 20.26. *Expand sin z in a Taylor's series about z = 0 and determine the region of convergence.* *(P.T.U., 2009 S)*

Solution. Given $f(z) = \sin z, f'(z) = \cos \theta, f''(z) = -\sin z, f''(z) = -\cos z, \ldots$

$\therefore$ $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 1$

By Taylor's series about $z = 0$, we have

$$f(z) = f(0) + \frac{(z-0)}{1!} f'(0) + \frac{(z-0)^2}{2!} f''(0) + \frac{(z-0)^3}{3!} f'''(0) + \ldots$$

i.e.,
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + (-1)^{n-1} \frac{z^{2n}-1}{(2n-1)!} + \ldots$$

Hence
$$\sin z = \sum_{n=1}^{\infty} a_n (z-0)^{2n-1} \text{ where } a_n = \frac{(-1)^{n-1}}{(2n-1)!}$$

Since
$$\rho = \underset{n\to\infty}{\text{Lt}} \left| \frac{a_{n+1}}{a_n} \right| = \underset{n\to\infty}{\text{Lt}} \left| \frac{(2n-1)!}{(2n+1)!} \right| = 0$$

Thus the radius of convergence of $f(z) = 1/\rho = \infty$

i.e., the region of convergence of $f(z)$ is all reals.

Example 20.27. *Find Taylor's expansion of*

(i) $f(z) = \dfrac{1}{(z+1)^2}$ *about the point* $z = -i$. *(V.T.U., 2009 S)*

(ii) $f(z) = \dfrac{2z^3+1}{z^2+z}$ *about the point* $z = i$. *(P.T.U., 2003)*

Solution. *(i)* To expand $f(z)$ about $z = -i$, *i.e.*, in powers of $z + i$, put $z + i = t$. Then

$$f(z) = \frac{1}{(t-i+1)^2} = (1-i)^{-2}\,[1 + t/(1-i)]^{-2} = \frac{i}{2}\left[1 - \frac{2t}{1-i} + \frac{3t^2}{(1-i)^2} - \frac{4t^3}{(1-i)^3} + \ldots\right]$$

[Expanding by Binomial theorem]

$$= \frac{i}{2}\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(z+i)^n}{(1-i)^n}\right]$$

(ii)
$$f(z) = \frac{2z^3+1}{z(z+1)} = 2z - 2 + \frac{2z+1}{z(z+1)} = (2i-2) + 2(z-i) + \frac{1}{z} + \frac{1}{z+1} \qquad \ldots(i)$$

[By partial fractions]

To expand $1/z$ and $1/(z+1)$ about $z = i$, put $z - i = t$, so that

$$\frac{1}{z} = \frac{1}{(t+i)} = \frac{1}{i}\left(1 + \frac{t}{i}\right)^{-1}$$ [Expanding by Binomial theorem]

$$= \frac{1}{i}\left[1 - \frac{t}{i} + \frac{t^2}{i^2} - \frac{t^3}{i^3} + \frac{t^4}{i^4} - \ldots \infty\right] = \frac{1}{i} + \frac{t}{1} + \frac{t^2}{i^3} - \frac{t^3}{i^4} + \frac{t^4}{i^5} - \ldots \infty$$

$$= -i + (z-i) + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}} \qquad \ldots(ii)$$

and
$$\frac{1}{z+1} = \frac{1}{t+i+1} = \frac{1}{1+i}\left(1 + \frac{t}{1+i}\right)^{-1}$$ [Expanding by Binomial theorem]

$$= \frac{1}{1+i}\left[1 - \frac{t}{1+i} + \frac{t^2}{(1+t)^2} - \frac{t^3}{(1+i)^3} + \frac{t^4}{(1+i)^4} - \ldots \infty\right]$$

$$= \frac{1-i}{2} - \frac{t}{2i} + \left[\frac{t^2}{(1+i)^3} - \frac{t^3}{(1+i)^4} + \frac{t^4}{(1+i)^5} - \ldots \infty\right] = \frac{1}{2} - \frac{i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}} \qquad \ldots(iii)$$

Substituting from (*ii*) and (*iii*) in (*i*), we get

$$f(z) = \left(2i - 2 - i + \frac{1}{2} - \frac{i}{2}\right) + \left(2 + 1 - \frac{1}{2i}\right)(z - i) + \sum_{n=2}^{\infty} (-1)^n \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z - i)^n$$

$$= \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z - i) + \sum_{n=2}^{\infty} (-1)^n \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z - i)^n.$$

Example 20.28. *Expand* $f(z) = 1/[(z-1)(z-2)]$ *in the region:*
(*a*) $|z| < 1$, (*b*) $1 < |z| < 2$, (*c*) $|z| > 2$, (*d*) $0 < |z-1| < 1$.
(*U.P.T.U., 2010 ; V.T.U., 2010 ; Bhopal, 2009*)

Solution. (*a*) By partial fractions $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$ ...(*i*)

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + (1 - z)^{-1} \quad ...(ii)$$

For $|z| < 1$, both $|z/2|$ and $|z|$ are less than 1. Hence (*ii*) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + ...\right) + (1 + z + z^2 + z^3 + ...)$$

$$= \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + ...$$ which is a *Taylor's series*.

(*b*) *For* $1 < |z| < 2$, we write (*i*) as

$$f(z) = -\frac{1}{2}\frac{1}{(1 - z/2)} - \frac{1}{z(1 - z^{-1})} \quad ...(iii)$$

and notice that both $|z/2|$ and $|z^{-1}|$ are less than 1. Hence (*iii*) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + ...\right) - \frac{1}{z}(1 + z^{-1} + z^{-2} + z^{-3} + ...)$$

$$= ... - z^{-4} - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - ...$$

which is a *Laurent's series*.

(*c*) *For* $|z| > 2$, we write (*i*) as

$$f(z) = \frac{1}{z(1 - 2z^{-1})} - \frac{1}{z(1 - z^{-1})}$$

$$= z^{-1}(1 + 2z^{-1} + 4z^{-2} + 8z^{-3} + ...) - z^{-1}(1 + z^{-1} + z^{-2} + z^{-3} + ...)$$

$$= ... + 7z^{-4} + 3z^{-3} + z^{-2} + ...$$

(*d*) *For* $0 < |z - 1| < 1$, we write (*i*) as

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= -(z-1)^{-1} - [1 - (z-1)]^{-1}$$

$$= -(z-1)^{-1} - [1 + (z-1) + (z-1)^2 + (z-1)^3 + ...].$$

Example 20.29. *Find the Laurents' expansion of* $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$ *in the region* $1 < z + 1 < 3$.
(*S.V.T.U., 2009 ; Anna, 2003 ; V.T.U., 2003*)

Solution. Writing $z + 1 = u$, we have

$$f(z) = \frac{7(u-1)-2}{u(u-1)(u-1-2)} = \frac{7u-9}{u(u-1)(u-3)}$$

$$= -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3} \quad \text{(splitting into partial fraction)}$$

$$= -\frac{3}{u} + \frac{1}{u(1-1/u)} - \frac{2}{3(1-u/3)} = -\frac{3}{u} + \frac{1}{u}\left(1 - \frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1 - \frac{u}{3}\right)^{-1}$$

Since $1 < u < 3$ or $1/u < 1$ and $u/3 < 1$, expanding by Binomial theorem,

$$f(z) = \frac{-3}{u} + \frac{1}{u}\left(1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots \infty\right) - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots \infty\right)$$

$$= -\frac{2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots \infty - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots \infty\right)$$

Hence $f(z) = -\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \dots \infty - \frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \frac{(z+1)^3}{3^3} + \dots \infty\right]$

which is valid in the region $1 < z + 1 < 3$.

20.17 (1) ZEROS OF AN ANALYTIC FUNCTION

Def. *A zero of an analytic function f(z) is that value of z for which f(z) = 0.*

If $f(z)$ is analytic in the neighbourhood of a point $z = a$, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + \dots \qquad \text{where } a_n = \frac{f^n(a)}{n!}.$$

If $a_0 = a_1 = a_2 = \dots = a_{m-1} = 0$ but $a_m \neq 0$, then $f(z)$ is said to have a *zero of order* m at $z = a$.

When $m = 1$, the zero is said to be *simple*. In the neighbourhood of zero ($z = a$) of order m,

$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \dots \infty$$
$$= (z-a)^m \phi(z) \text{ where } \phi(z) = a_m + a_{m+1}(z-a) + \dots$$

Then $\phi(z)$ is analytic and non-zero in the neighbourhood of $z = a$.

(2) Singularities of an analytic function

We have already defined a *singular point of a function as the point at which the function ceases to be analytic.*

(*i*) **Isolated singularity.** *If z = a is a singularity of f (z) such that f (z) is analytic at each point in its neighourhood* (*i.e.*, there exists a circle with centre a which has no other singularity), *then z = a is called an* **isolated singularity.**

In such a case, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots \qquad \dots(1)$$

For example, $f(z) = \cot(\pi/z)$ is not analytic where $\tan(\pi/z) = 0$ *i.e.* at the points $\pi/z = 4\pi$ or $z = 1/n$ ($n = 1, 2, 3, \dots$).

Thus $z = 1, 1/2, 1/3, \dots$ are all *isolated singularities* as there is no other singularity in their neighbourhood.

But when n is large, $z = 0$ is such a singularity that there are infinite number of other singularities in its neighbourhood. Thus $z = 0$ is the *non-isolated singularity* of $f(z)$.

(*ii*) **Removable singularity.** If all the negative powers of $(z-a)$ in (1) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$.

Here the singularity can be removed by defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$. Such a singularity is called a *removable singularity*.

Thus *if* $\underset{z \to a}{Lt} f(z)$ *exists finitely, then z = a is a removable singularity.*

(*iii*) **Poles.** *If all the negative powers of (z – a) in (i) after the nth are missing, then the singularity at z = a is called a* **pole of order n.**

A pole of first order is called a **simple pole.**

(*iv*) **Essential singularity.** *If the number of negative powers of (z – a) in (1) is infinite, then z = a is called an essential singularity.* In this case, $\underset{z \to a}{\text{Lt}} f(z)$ does not exist.

Example 20.30. *Find the nature and location of singularities of the following functions:*

(*i*) $\frac{z - \sin z}{z^2}$ (*ii*) $(z + 1) \sin \frac{1}{z-2}$ (*iii*) $\frac{1}{\cos z - \sin z}$.

Solution. (*i*) Here $z = 0$ is a singularity.

Also $$\frac{z-\sin z}{z^2} = \frac{1}{z^2}\left\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots\right)\right\} = \frac{z}{3!}-\frac{z^3}{5!}+\frac{z^5}{7!}-\ldots$$

Since there are no negative powers of z in the expansion, $z = 0$ *is a removable singularity.*

(*ii*) $(z+1)\sin\frac{1}{z-2} = (t+2+1)\sin\frac{1}{t}$ where $t = z-2$

$$= (t+3)\left\{\frac{1}{t}-\frac{1}{3!t^3}+\frac{1}{5!t^5}-\ldots\right\} = \left(1-\frac{1}{3!t^2}+\frac{1}{5!t^4}-\ldots\right)+\left(\frac{3}{t}-\frac{1}{2t^3}+\frac{3}{5!t^5}-\ldots\right)$$

$$= 1+\frac{3}{t}-\frac{1}{6t^2}-\frac{1}{2t^3}+\frac{1}{120t^4}-\ldots = 1+\frac{3}{z-2}-\frac{1}{6(z-2)^2}-\frac{1}{2(z-2)^3}+\ldots$$

Since there are infinite number of terms in the negative powers of $(z-2)$, $z = 2$ *is an essential singularity.*

(*iii*) Poles of $f(z) = \frac{1}{\cos z-\sin z}$ are given by equating the denominator to zero, *i.e.*, by $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \pi/4$ is a *simple pole of* $f(z)$.

Example 20.31. *What type of singularity have the following functions :*

(*i*) $\frac{1}{1-e^z}$ (*ii*) $\frac{e^{2z}}{(z-1)^4}$ (*iii*) $\frac{e^{1/z}}{z^2}$. (*U.P.T.U., 2009*)

Solution. (*i*) Poles of $f(z) = 1/(1-e^z)$ are found by equating to zero $1-e^z = 0$ or $e^z = 1 = e^{2n\pi i}$

$\therefore$ $z = 2n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$

Clearly $f(z)$ *has a simple pole at* $z = 2\pi i$.

(*ii*) $$\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4} = \frac{e^2}{t^4}\cdot e^{2t} \quad \text{where } t = z-1$$

$$= \frac{e^2}{t^4}\left\{1+\frac{2t}{1!}+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\frac{(2t)^5}{5!}+\ldots\right\} = e^2\left\{\frac{1}{t^4}+\frac{2}{t^3}+\frac{2}{t^2}+\frac{4}{3t}+\frac{2}{3}+\frac{4t}{15}+\ldots\right\}$$

$$= e^2\left\{\frac{1}{(z-1)^4}+\frac{2}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{4}{3(z-1)}+\frac{2}{3}+\frac{4}{15}(z-1)+\ldots\right\}$$

Since there are finite (4) number of terms containing negative powers of $(z-1)$,

$\therefore$ $z = 1$ *is a pole of 4th order.*

(*iii*) $$f(z) = \frac{e^{1/z}}{z^2} = \frac{1}{z^2}\left\{1+\frac{1}{1!z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\ldots\right\} = z^{-2}+z^{-3}+\frac{z^{-4}}{2}+\ldots\infty$$

Since there are infinite number of terms in the negative powers of z, therefore $f(z)$ has *an essential singularity* at $z = 0$.

PROBLEMS 20.7

1. Obtain the expansion of $(z-1)/z^2$ in a Taylor's series in powers of $(z-1)$ and determine the region of convergence.
2. Find the first three times of the Taylor's series expansion of $f(z) = 1/(z^2+4)$ about $z = -i$. Also find the region of convergence. (*U.P.T.U., 2006*)
3. Expand in Taylor's series (*i*) $(z-1)/(z+1)$ about the point $z = 1$. (*Andhra, 2000*)

   (*ii*) $\cos z$ about the point $z = \pi/2$. (*Marathwada, 2008*) (*iii*) $\frac{1}{z^2-z-6}$ about (*a*) $z = -1$ (*b*) $z = 1$ (*P.T.U., 2009*)
4. Expand the following functions in Laurent's series :

   (*i*) $f(z) = \frac{1}{z-z^2}$ for $1 < |z+1| < 2$. (*Madras, 2006*) (*ii*) $f(z) = \dfrac{1}{(z-1)(z+3)}$ for $1 < |z| < 3$. (*J.N.T.U., 2006*)

(*iii*) $f(z) = z/[(z-1)(z-3)]$ for $|z-1| < 2$. (*V.T.U., 2007*)

5. Find the Laurent's expansion of (*i*) $\dfrac{e^z}{(z-1)^2}$, about $z = 1$. (*Rohtak, 2006*)

(*ii*) $e^{2z}/(z-1)^3$ about the singularity $z = 1$.

6. Expand the following functions in Laurent series.

(*i*) $(z-1)/z^2$ for $|z-1| > 1$ (*ii*) $\dfrac{1-\cos z}{z^3}$, about $z = 0$. (*Rohtak, 2004*)

7. Find the Laurent's series expansion of

(*i*) $\dfrac{z^2-1}{z^2+5z+6}$ about $z = 0$ in the region $2 < |z| < 3$ (*V.T.U., 2011 S ; Osmania, 2003*)

(*ii*) $\dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$

(*iii*) $\dfrac{7z^2-9z-18}{z^3-9z}$ in the region (*a*) $|z| > 3$ (*b*) $0 < |z-3| < 3$. (*V.T.U., 2010 S*)

8. Find the Laurent's expansion of $1/[(z^2+1)(z^2+2)]$ for (*a*) $0 < |z| < 1$; (*b*) $1 < |z| < \sqrt{2}$; (*c*) $|z| > 2$.

Find the nature and location of the singularities of the following functions : (*P.T.U., 2005*)

9. $\dfrac{1}{z(2-z)}$. 10. $\sin(1/z)$. (*U.P.T.U., 2009*) 11. $\tan\left(\dfrac{1}{z}\right)$. (*P.T.U., 2006*)

12. $\dfrac{z^2-1}{(z-1)^3}$. (*Osmania, 2003*) 13. $\dfrac{e^z}{(z-1)^4}$. 14. $\dfrac{\cot \pi z}{(z-a)^2}$. (*U.P.T.U., 2008*)

20.18 (1) RESIDUES

The coefficient of $(z-a)^{-1}$ *in the expansion of* $f(z)$ *around an isolated singularity is called the* **residue** *of* $f(z)$ *at that point.* Thus in the Laurent's series expansion of $f(z)$ around $z = a$ *i.e.*, $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$, the residue of $f(z)$ at $z = a$ is a_{-1}.

$$\therefore \qquad Res\, f(a) = \frac{1}{2\pi i}\oint_C f(z)$$

i.e.,
$$\oint_C f(z)\,dz = 2\pi i \text{ Res } f(a). \qquad \ldots(1)$$

(2) Residue Theorem

If $f(z)$ *is analytic in a closed curve C except at a finite number of singular points within C, then* $\oint_C f(z)\,dz = 2\pi i \times$ (*sum of the residues at the singular points within C*).

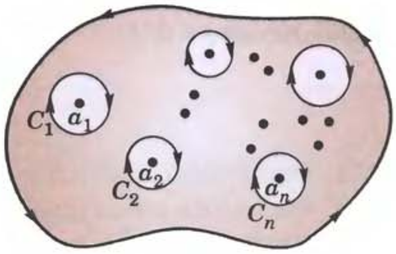


Fig. 20.22

Let us surround each of the singular points $a_1, a_2, \ldots, a_n$ by a small circle such that it encloses no other singular point (Fig. 20.22). Then these circles $C_1, C_2, \ldots, C_n$ together with C, form a multiply connected region in which $f(z)$ is analytic.

$\therefore$ applying Cauchy's theorem, we have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \ldots + \oint_{C_n} f(z)dz \qquad \text{[by (1)}$$

$$= 2\pi i\,[\text{Res } f(a_1) + \text{Res } f(a_2) + \ldots + \text{Res } f(a_n)] \text{ which is the desired result.}$$

20.19 CALCULATION OF RESIDUES

(1) *If $f(z)$ has a simple pole at $z = a$, then*

$$\textbf{Res f(a)} = \underset{z \to a}{\textbf{Lt}} \, [(\mathbf{z - a})\mathbf{f(z)}]. \qquad ...(1)$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 \ldots + c_{-1}(z-a)^{-1}$$

Multiplying throughout by $z - a$, we have

$$(z-a)f(z) = c_0(z-a) + c_1(z-a)^2 + \ldots + c_{-1}.$$

Taking limits as $z \to a$, we get

$$\underset{z \to a}{\text{Lt}} \, [(z-a)f(z)] = c_{-1} = \text{Res}\, f(a).$$

(2) *Another formula for Res $f(a)$:*

Let $f(z) = \phi(z)/\psi(z)$, where $\psi(z) = (z-a)\,F(z)$, $F(a) \neq 0$.

Then $\quad \underset{z \to a}{\text{Lt}} \, [(z-a)\phi(z)/\psi(z)]$

$$= \underset{z \to a}{\text{Lt}} \, \frac{(z-a)[\phi(a) + (z-a)\phi'(a) + \ldots]}{\psi(a) + (z-a)\psi'(a) + \ldots}$$

$$= \underset{z \to a}{\text{Lt}} \, \frac{\phi(a) + (z-a)\phi'(a) + \ldots}{\psi'(a) + (z-a)\psi''(a) + \ldots}, \quad \text{since } \psi(a) = 0$$

Thus $\quad \textbf{Res f(a)} = \dfrac{\boldsymbol{\phi}(\mathbf{a})}{\boldsymbol{\psi}'(\mathbf{a})}.$

(3) *If $f(z)$ has a pole of order n at $z = a$, then*

$$\textbf{Res f(a)} = \frac{\mathbf{1}}{(\mathbf{n-1})\,!}\left\{\frac{\mathbf{d^{n-1}}}{\mathbf{dz^{n-1}}}[(\mathbf{z-a})^{\mathbf{n}}\mathbf{f(z)}]\right\}_{\mathbf{z=a}}$$

Here $\quad f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_{-1}(z-a)^{-1} + \ldots + c_{-n}(z-a)^{-n}.$

Multiplying throughout by $(z-a)^n$, we get

$$(z-a)^n f(z) = c_0(z-a)^n + c_1(z-a)^{n+1} + c_2(z-a)^{n+2} + \ldots + c_{-1}(z-a)^{n-1} + c_{-2}(z-a)^{n-2} + \ldots + c_{-n}.$$

Differentiating both sides w.r.t. z, $n - 1$ times and putting $z = a$, we get

$$\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a} = (n-1)\,!\, c_{-1} \text{ whence follows the result.}$$

Obs. In many cases, the residue of a pole $(z = a)$ can be found, by putting $z = a + t$ in $f(z)$ and expanding it in powers of t where $|\, t \,|$ is quite small.

Example 20.32. *Find the sum of the residues of $f(z) = \dfrac{\sin z}{z \cos z}$ at its poles inside the circle $|z| = 2$.*

(Rohtak, 2004)

Solution. $f(z)$ has simple poles at $z = 0, \pm \pi/2, \pm 3\pi/2, - \ldots$

Only the poles $z = 0$ and $z = \pm \pi/2$ lies inside $|z| = 2$.

$$\therefore \quad \text{Res}\, f(0) = \underset{z \to 0}{\text{Lt}} \, [z \,.\, f(z)] = \underset{z \to 0}{\text{Lt}} \left(\frac{\sin z}{\cos z}\right) = 0.$$

$$\text{Res}\, f(\pi/2) = \underset{z \to \pi/2}{\text{Lt}} \left[\left(z - \frac{\pi}{2}\right) f(z)\right] = \underset{z \to \pi/2}{\text{Lt}} \left\{\frac{(z - \pi/2)\sin z}{z \cos z}\right\}$$

$$= \underset{z \to \pi/2}{\text{Lt}} \, \frac{(z - \pi/2)\cos z + \sin z}{\cos z - z \sin z} \qquad \left[\text{Being } \frac{0}{0} \text{ form}\right]$$

$$= \frac{1}{-\pi/2} = -\frac{2}{\pi}$$

and $\quad \text{Res} f(-\pi/2) = \underset{z\to-\pi/2}{\text{Lt}} \left\{\frac{(z+\pi/2)\sin z}{z\cos z}\right\} = \underset{z\to-\pi/2}{\text{Lt}} \frac{(z+\pi/2)\cos z + \sin z}{\cos z - z\sin z} = \frac{-1}{-\pi/2} = \frac{2}{\pi}$

Hence sum of residues $= 0 - \frac{2}{\pi} + \frac{2}{\pi} = 0$.

Example 20.33. *Determine the poles of the function*
$f(z) = z^2/(z-1)^2(z+2)$ and the residue at each pole. (S.V.T.U., 2008 ; J.N.T.U., 2005)
Hence evaluate $\oint_C f(z)dz$, where C is the circle $|z| = 2.5$.

Solution. Since $\underset{z\to-2}{\text{Lt}} \{(z+2)f(z)\} = \underset{z\to-2}{\text{Lt}} \frac{z^2}{(z-1)^2} = \frac{4}{9}$.
which is finite and non-zero, the function has a simple pole at $z = -2$ and Res $f(-2) = 4/9$.

Also since $\underset{z\to1}{\text{Lt}} [(z-1)^2 f(z)]$ is finite and non-zero, $f(z)$ has a pole of order two at $z = 1$.

$$\therefore \quad \text{Res} f(1) = \frac{1}{1!}\left[\frac{d}{dz}\{(z-1)^2 f(z)\}\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{z^2}{z+2}\right)\right]_{z=1} = \left[\frac{z^2+4z}{(z+2)^2}\right]_{z=1} = \frac{5}{9}.$$

[*Otherwise writing* $z = 1 + t$,

$$f(z) = \frac{(1+t)^2}{t^2(3+t)} = \frac{1}{3t^2}(1+t)^2(1+t/3)^{-1} = \frac{1}{3t^2}(1+t)^2\left(1 - \frac{t}{3} + \frac{t^2}{9} - \ldots\right)$$

$$= \frac{1}{3t^2}\left(1 + \frac{5}{3}t + \frac{4}{9}t^2 - \ldots\right) = \frac{1}{3t^2} + \frac{5}{9t} + \frac{4}{27} - \ldots \qquad \ldots(i)$$

$$\therefore \quad \text{Res} f(1) = \text{coefficient of } \frac{1}{t} \text{ in } (i) = \frac{5}{9}.]$$

Clearly $f(z)$ is analytic on $|z| = 2.5$ and at all points inside except the poles $z = -2$ and $z = 1$. Hence by residue theorem

$$\oint_C f(z)dz = 2\pi i\,[\text{Res} f(-2) + \text{Res} f(1)] = 2\pi i\left[\frac{4}{9} + \frac{5}{9}\right] = 2\pi i.$$

Example 20.34. *Find the residue of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its poles and hence evaluate $\oint_C f(z)dz$ where C is the circle $|z| = 2.5$.* (U.P.T.U., 2003)

Solution. The poles of $f(z)$ are given by $(z-1)^4(z-2)(z-3) = 0$.

$\therefore$ $z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res} f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1} = \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$\therefore$ $z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res} f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1} = \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$$= \frac{1}{6}\frac{d^3}{dz^3}\left[z + 5 - \frac{8}{z-2} + \frac{27}{z-3}\right] = \frac{1}{6}\left[-8 \cdot \frac{(-1)^3\,3!}{(z-2)^4} + \frac{27 \cdot (-1)^3\,3!}{(z-2)^4}\right]_{z=1}$$

$$= -\left[-8 + \frac{27}{16}\right] = \frac{101}{16}.$$

$$\text{Res}\, f(2) = \underset{z \to 2}{\text{Lt}} \left\{(z-2) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)}\right\} = \underset{z \to 2}{\text{Lt}} \left\{\frac{z^3}{(z-1)^4 (z-3)}\right\} = \frac{8}{(1)^4(-1)} = -8$$

$$\text{Res}\, f(3) = \underset{z \to 3}{\text{Lt}} \left\{(z-3) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)}\right\} = \frac{27}{(2)^4 \cdot 1} = \frac{27}{16}$$

Now $\oint_C f(z)dz = 2\pi i\,[\text{Res}\, f(1) + \text{Res}\, f(2)]$ [$\because$ Pole $z = 3$ is outside C]

$$= 2\pi i \left(\frac{101}{16} - 8\right) = \frac{-27\pi i}{8}.$$

Example 20.35. *Evaluate*

$$\oint_C \frac{z-3}{z^2+2z+5} dz, \text{ where } C \text{ is the circle}$$

(i) $|z| = 1$, *(ii)* $|z + 1 - i| = 2$, *(iii)* $|z + 1 + i| = 2$. *(J.N.T.U., 2003)*

Solution. The poles of $f(z) = \dfrac{z-3}{z^2+2z+5}$ are given by $z^2 + 2z + 5 = 0$

i.e., by $$z = \frac{-2 \pm \sqrt{(4-20)}}{2} = -1 \pm 2i.$$

(*i*) Both the poles $z = -1 + 2i$ and $z = -1 - 2i$ lie outside the circle $|z| = 1$. Therefore, $f(z)$ is analytic everywhere within C.

Hence by Cauchy's theorem, $\oint_C \dfrac{z-3}{z^2+2z+5} dz = 0$.

(*ii*) Here only one pole $z = -1 + 2i$ lies inside the circle $C : |z + 1 - i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\therefore \quad \text{Res}\, f(-1+2i) = \underset{z \to -1+2i}{\text{Lt}} [\{z - (-1+2i)\} f(z)] = \underset{z \to -1+2i}{\text{Lt}} \frac{(z+1-2i)(z-3)}{z^2+2z+5}$$

$$= \underset{z \to -1+2i}{\text{Lt}} \frac{z-3}{z+1+2i} = \frac{-4+2i}{4i} = i + 1/2.$$

Hence by residue theorem $\oint_C f(z)dz = 2\pi i \,\text{Res}\, f(-1+2i) = 2\pi i(i + 1/2) = \pi(i-2)$.

(*iii*) Here only the pole $z = -1 - 2i$ lies inside the circle $C : |z + 1 + i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\therefore \quad \text{Res}\, f(-1-2i) = \underset{z \to -1-2i}{\text{Lt}} \frac{(z+1+2i)(z-3)}{z^2+2z+5}$$

$$= \underset{z \to -1-2i}{\text{Lt}} \frac{z-3}{z+1-2i} = \frac{-4-2i}{-4i} = \tfrac{1}{2} - i$$

Hence by residue theorem, $\oint_C f(z)dz = 2\pi i \,\text{Res}\, f(-1-2i) = 2\pi i\,(\tfrac{1}{2} - i) = \pi(2+i)$.

Example 20.36. *Evaluate* $\oint_C \dfrac{e^z}{\cos \pi z} dz$, *where C is the unit circle* $|z| = 1$. *(Rohtak, 2006)*

Solution. $f(z) = e^x/\cos \pi z$ has simple poles at $z = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$

Out of these only the poles at $z = 1/2$ and $z = -1/2$ lie inside the given circle $|z| = 1$.

$$\therefore \quad \text{Res}\, f(1/2) = \underset{z \to 1/2}{\text{Lt}} \left[\left(z - \frac{1}{2}\right) f(z)\right] = \underset{z \to 1/2}{\text{Lt}} \left\{\frac{\left(z - \frac{1}{2}\right) e^z}{\cos \pi z}\right\} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{z \to 1/2}{\text{Lt}} \frac{e^z + \left(z - \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{1/2}}{-\pi}$$

and $$\text{Res} f(-1/2) = \underset{z \to -1/2}{\text{Lt}} \left\{ \frac{\left(z + \frac{1}{2}\right)e^z}{\cos \pi z} \right\} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{z \to -1/2}{\text{Lt}} \frac{e^z + \left(z + \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{-1/2}}{\pi}$$

Hence $$\oint_C \frac{e^z}{\cos \pi z} dz = 2\pi i \left\{ \text{Res } f\left(\frac{1}{2}\right) + \text{Res } f\left(-\frac{1}{2}\right) \right\}$$

$$= 2\pi i \left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi} \right) = -4i \left(\frac{e^{1/2} - e^{-1/2}}{2} \right) = -4i \sinh \frac{1}{2}.$$

Example 20.37. *Evaluate* $\oint_C \tan z \, dz$ *where C is the circle* $|z| = 2$. *(V.T.U., 2010 S)*

Solution. The poles of $f(z) = \sin z/\cos z$ are given by $\cos z = 0$ *i.e.* $z = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$
Of these poles, $z = \pi/2$, and $-\pi/2$ only are within the given circle.

$\therefore$ $$\text{Res } f(\pi/2) = \underset{z \to \pi/2}{\text{Lt}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \underset{z \to \pi/2}{\text{Lt}} \left(\frac{\sin z}{-\sin z} \right) = -1 \qquad \text{[By § 20.19 (2)]}$$

Similarly $$\text{Res } f(-\pi/2) = \underset{z \to -\pi/2}{\text{Lt}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = -1.$$

Hence by residue theorem,

$$\oint_C f(z)dz = 2\pi i \, \{\text{Res } f(\pi/2) + \text{Res } f(-\pi/2)\} = 2\pi i \, (-1 - 1) = -4\pi i.$$

Example 20.38. *Evaluate* $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, *where C is the circle* $|z| = 3$.
(V.T.U., 2010 ; Anna, 2003 S ; U.P.T.U., 2002)

Solution. $$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$$

is analytic within the circle $|z| = 3$ excepting the poles $z = 1$ and $z = 2$.
Since $z = 1$ is a pole of order 2.

$\therefore$ $$\text{Res } f(1) = \frac{1}{1!}\left[\frac{d}{dz}\{(z-1)^2 f(z)\}\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}\right)\right]_{z=1}$$

$$= \left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2}\right]_{z=1}$$

$$= (-1)(-2\pi) - (-1) = 2\pi + 1$$

Also $$\text{Res } f(2) = \underset{z \to 2}{\text{Lt}} [(z-2) f(z)] = \underset{z \to 2}{\text{Lt}} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

Hence by residue theorem,

$$\oint_C f(z) \, dz = 2\pi i \, [\text{Res } f(1) + \text{Res } f(2)] = 2\pi i \, (2\pi + 1 + 1) = 4\pi(\pi + 1)i.$$

PROBLEMS 20.8

1. Expand $f(z) = 1/[z^2 (z - i)]$ as a Laurent's series about i and hence find the residue thereat.
2. Find the residue of (*i*) $ze^z/(z-1)^3$ at its pole. (*J.N.T.U., 2003*)
   (*ii*) $z^2/(z^2 + a^2)$ at $z = ai$. (*P.T.U., 2009 S*)
3. Determine the poles of the following functions and the residue at each pole :
   (*i*) $\dfrac{z^2+1}{z^2-2z}$ (*ii*) $\dfrac{z^2-2z}{(z+1)^2(z^2+1)}$ (*J.N.T.U., 2005*) (*iii*) $\dfrac{2z+4}{(z+1)(z^2+1)}$ (*J.N.T.U., 2006*)
4. Find the residues of the following functions at each pole.
   (*i*) $(1 - e^{2z})/z^4$ (*ii*) $ze^{iz}/(z^2+1)$ (*P.T.U., 2010*) (*iii*) $\cot z$.
5. $\displaystyle\oint_C \frac{z^2+4}{(z-2)(z+3)}\, dz$, where C is (*i*) $|z+1| = 2$ (*ii*) $|z-2| = 2$. (*Mumbai, 2006*)
6. Evaluate the following integrals :
   (*i*) $\displaystyle\oint_C \frac{e^{2z}\, dz}{(z+2)(z+4)(z+7)}$ for C as circle $|z| = 3$. (*V.T.U., 2009*)
   (*ii*) $\displaystyle\oint_C \frac{4z^2-4z+1}{(z-2)(4+z^2)}\, dz,\ C : |z| = 1$
   (*iii*) $\displaystyle\oint_C \frac{3z^2+z+1}{(z^2-1)(z+3)}\, dz,\ C : |z| = 2$. (*U.P.T.U., 2004*)
7. Evaluate
   (*i*) $\displaystyle\int_C \frac{2z+1}{(2z-1)^2}\, dz$, where C is $|z| = 1$ (*ii*) $\displaystyle\oint_C \frac{z+4}{z^2+2z+5}\, dz$, where C is $|z+1-i| = 2$
   (*iii*) $\displaystyle\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\, dz$, where C is the circle $|z| = 10$. (*U.P.T.U., 2009*)
8. Evaluate :
   (*i*) $\displaystyle\oint_C \frac{z\, dz}{(z-1)(z-2)^2},\ C : |z-2| = \tfrac{1}{2}$. (*Madras, 2006*)
   (*ii*) $\displaystyle\oint_C \frac{3z^2+2}{(z-1)(z^2+9)}\, dz,\ C : |z-2| = 2$. (*Rohtak, 2005*)
   (*iii*) $\displaystyle\oint_C \frac{dz}{(z^2+4)^2},\ C : |z-i| = 2$. (*Hissar, 2007 ; Anna, 2003 S ; Osmania, 2003*)
9. Evaluate :
   (*i*) $\displaystyle\oint_C \frac{e^{-z}}{z^2} dz,\ C : |z| = 1$. (*ii*) $\displaystyle\oint_C z^2 e^{1/z},\ C : |z| = 1$.
   (*iii*) $\displaystyle\oint_C \frac{e^z\, dz}{z^2+4},\ C : |z-i| = 2$. (*V.T.U., 2006*) (*iv*) $\displaystyle\oint_C \frac{e^{2z}\, dz}{(z+1)^4},\ C : |z| = 2$.
10. Evaluate the following integrals : (*i*) $\displaystyle\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\, dz,\ C : |z| = 1$
   (*ii*) $\displaystyle\oint_C \frac{z \sec z}{(1-z)^2}\, dz,\ C : |z| = 3$ (*iii*) $\displaystyle\oint_C \frac{z \cos z}{(z-\pi/2)^3}\, dz,\ C : |z-1| = 1$. (*V.T.U., 2007*)
11. Evaluate $\displaystyle\oint_C \frac{dz}{\sinh 2z}$ where C is the circle $|z| = 2$. (*Marathwada, 2008*)
12. Obtain Laurent's expansion for the function $f(z) = 1/z^2 \sinh z$ and evaluate
   $\displaystyle\oint_C \frac{z}{z^2 \sinh z}$, where C is the circle $|z-1| = 2$. (*J.N.T.U., 2005*)

20.20 EVALUATION OF REAL DEFINITE INTEGRALS

Many important definite integrals can be evaluated by applying the Residue theorem to properly chosen integrals. The contours chosen will consist of straight lines and circular arcs.

(*a*) **Integration around the unit circle.** An integral of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$, where the integrand is a rational function of $\sin\theta$ and $\cos\theta$ can be evaluated by writing $e^{i\theta} = z$.

Since $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, then integral takes the form $\int_C f(z)\, dz$, where $f(z)$ is a rational function of z and C is a unit circle $|z| = 1$.

Hence *the integral is equal to $2\pi i$ times the sum of the residues at those poles of $f(z)$ which are within C.*

Example 20.39. *Show that*

$$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a^2}{1 - a^2}, \ (a^2 < 1).$$ *(Bhopal, 2009 ; Rohtak, 2003)*

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \frac{1}{2}(z + 1/z)$ and $\cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) = \frac{1}{2}(z^2 + 1/z^2)$

$\therefore$ the given integral

$$I = \int_C \frac{\frac{1}{2}(z^2 + 1/z^2)}{1 - a(z + 1/z) + a^2} \cdot \frac{dz}{iz} = \frac{1}{2i}\int_C \frac{(z^4 + 1)\, dz}{z^2(z - az^2 - a + a^2 z)}$$

$$= \frac{1}{2i}\int_C \frac{(z^4 + 1)\, dz}{z^2(z - a)(1 - az)} = \int_C f(z)\, dz$$ where C is the unit circle $|z| = 1$.

Now $f(z)$ has simple poles at $z = a$, $1/a$ and the second order pole at $z = 0$, of which the poles at $z = 0$ and $z = a$ lie within the unit circle.

$$\therefore \quad \text{Res}\, f(a) = \underset{z \to a}{\text{Lt}}\, [(z - a) f(z)] = \frac{1}{2i}\underset{z \to a}{\text{Lt}} \left[\frac{z^4 + 1}{z^2(1 - az)}\right] = \frac{a^4 + 1}{2ia^2(1 - a^2)}$$

and

$$\text{Res}\, f(0) = \underset{z \to 0}{\text{Lt}}\, \frac{d}{dz}[z^2 f(z)] = \frac{1}{2i}\underset{z \to 0}{\text{Lt}}\, \frac{d}{dz}\left[\frac{z^4 + 1}{z - az^2 - a + a^2 z}\right]$$

$$= \frac{1}{2i}\underset{z \to 0}{\text{Lt}}\, \frac{(z - az^2 - a + a^2 z)(4z^3) - (z^4 + 1)(1 - 2az + a^2)}{(z - az^2 - a + a^2 z)^2} = -\frac{1 + a^2}{2ia^2}$$

Hence

$$I = 2\pi i\, [\text{Res}\, f(a) + \text{Res}\, f(0)] = 2\pi i \left[\frac{a^4 + 1}{2ia^2(1 - a^2)} - \frac{1 + a^2}{2ia^2}\right] = \frac{2\pi a^2}{1 - a^2}.$$

Example 20.40. *By integrating around a unit circle, evaluate* $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta$.

(S.V.T.U, 2009 ; U.P.T.U., 2009 ; Madras, 2003)

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \frac{1}{2}(z + 1/z)$

and

$$\cos 3\theta = \frac{1}{2}(e^{3i\theta} + e^{-3i\theta}) = \frac{1}{2}(z^3 + 1/z^3).$$

$\therefore$ the given integral

$$I = \int_C \frac{\frac{1}{2}(z^3 + 1/z^3)}{5 - 2(z + 1/z)} \cdot \frac{dz}{iz}$$

$$= -\frac{1}{2i}\int_C \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)}\, dz = -\frac{1}{2i}\int_C \frac{(z^6 + 1)\, dz}{z^3(2z - 1)(z - 2)}$$

$$= -\frac{1}{2i}\int f(z)\,dz, \quad \text{where } C \text{ is the unit circle } |\,z\,| = 1.$$

Now $f(z)$ has a pole of order 3 at $z = 0$ and simple poles at $z = \frac{1}{2}$ and $z = 2$. Of these only $z = 0$ and $z = 1/2$ lie within the unit circle.

$$\therefore \quad \text{Res}\, f(1/2) = \underset{z\to 1/2}{\text{Lt}} \frac{(z-1/2)(z^6+1)}{(2z-1)(z-2)} = \underset{z\to 1/2}{\text{Lt}}\left\{\frac{z^6+1}{2z^3(z-2)}\right\} = -\frac{65}{24}$$

$$\text{Res}\, f(0) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-0)^n f(z)]\right\}_{z=0} \quad \text{where } n = 3$$

$$= \frac{1}{2}\left\{\frac{d^2}{dz^2}\left(\frac{z^6+1}{2z^2-5z+2}\right)\right\}_{z=0} = \frac{d}{dz}\left[\frac{(2z^2-5z+2)\,6z^5-(z^6+1)(4z-5)}{2(2z^2-5z+2)^2}\right] \text{ at } z=0$$

$$= \left\{\frac{d}{dz}\left[\frac{8z^7-25z^6+12z^5-4z+5}{2(2z^2-5z+2)^2}\right]\right\}_{z=0}$$

$$= \left[\frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4)-(8z^7-25z^6+12z^5-4z+5)2(2z^2-5z+2)(4z-5)}{2(2z^2-5z+2)^4}\right]_{z=0}$$

$$= \frac{4(-4)-5(-20)}{2\times 16} = \frac{84}{32} = \frac{21}{8}$$

Hence $\quad I = \dfrac{-1}{2i}\{2\pi i\,[\text{Res}\, f(1/2) + \text{Res}\, f(0)]\} = -\pi\left[-\dfrac{65}{24}+\dfrac{21}{8}\right] = -\pi\left(-\dfrac{1}{12}\right) = \dfrac{\pi}{12}.$

(b) **Integration around a small semi-circle.** To evaluate $\int_{-\infty}^{\infty} f(x)\,dx$, we consider $\int_C f(z)$, where C is the contour consisting of the semi-circle $C_R : |\,z\,| = R$, together with the diameter that closes it.

Supposing that $f(z)$ has no singular point on the real axis, we have, by the Residue theorem,

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i\,\Sigma\,\text{Res}\, f(a).$$

Finally making R tend to ∞, we find the value of $\int_{-\infty}^{\infty} f(x)\,dx$, provided $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

Example 20.41. *Evaluate* $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$. *(U.P.T.U., 2008)*

Solution. Consider $\displaystyle\int_C \frac{z^2dz}{(z^2+1)(z^2+4)} = \int_C f(z)\,dz$ ...(i)

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

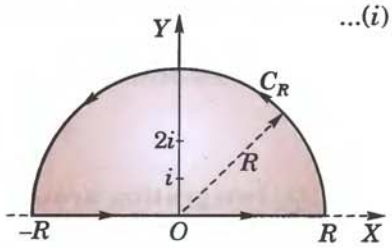


Fig. 20.23

The integrand has simple poles at $z = \pm i$, $z = \pm 2i$ of which $z = i, 2i$ only lie inside C.

$\therefore$ by the Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Res}\, f(i) + \text{Res}\, f(2i)]$$

$$= 2\pi i\,[\underset{z\to i}{\text{Lt}}\,(z-i)\,f(z) + \underset{z\to 2i}{\text{Lt}}\,(z-2i)\,f(z)]$$

$$= 2\pi i\left[\frac{i^2}{2i(i^2+4)} + \frac{4i^2}{(4i^2+1)(4i)}\right] = 2\pi i\left(\frac{i}{6}-\frac{i}{3}\right) = \frac{\pi}{3} \quad \text{...(ii)}$$

Also $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \quad ...(iii)$$

Now let $R \to \infty$, so as to show that the second integral in (*iii*) vanishes. For any point on C_R as $|z| \to \infty$

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{(1+z^{-2})(1+4z^{-2})}$$

decreases as $1/z^2$ and tends to zero whereas the length of C_R increases with z.

Consequently, $\underset{|z|\to\infty}{\text{Lt}} \int_{C_R} f(z)\,dz = 0.$

Hence from (*i*), (*ii*) and (*iii*), we get $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$.

Example 20.42. *Evaluate* $\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx$. *(U.P.T.U., 2006 ; Delhi, 2002)*

Solution. Consider $\int_C \frac{e^{iaz}}{z^2+1}\,dz = \int_C f(z)\,dz$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

The integrand has simple poles at $z = i$ amd $z = -i$, of which $z = i$ only lies inside C.

$\therefore$ by Residue theorem, $\int_C f(z)\,dz = 2\pi i \operatorname{Res} f(i) = 2\pi i \underset{z\to i}{\text{Lt}} [(z-i) f(z)]$

$$= 2\pi i \underset{z\to i}{\text{Lt}} \frac{(z-i)\,e^{iaz}}{z^2+1} = 2\pi i \underset{z\to i}{\text{Lt}} \frac{e^{iaz}}{z+i} = 2\pi i \frac{e^{-a}}{2i} = \pi e^{-a} \quad ...(i)$$

Also $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \quad ...(ii)$$

Now $|z| = R$ on C_R and $|z^2+1| \geq R^2 - 1$.

Also $|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax} \cdot e^{-ay}| = e^{-ay} < 1$ $\quad [\because \ y > 0]$

$$\therefore \quad \left|\frac{e^{iaz}}{z^2+1}\right| = |e^{iaz}| \cdot \frac{1}{|z^2+1|} < 1 \cdot \frac{1}{R^2-1}$$

Thus $\int_{C_R} f(z)\,dz = \left|\int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz\right| < \int_{C_R} \frac{1}{R^2-1}|dz| < \frac{\pi R}{R^2-1}$ which $\to$ to 0 as $R \to \infty$. $\quad ...(iii)$

Hence from (*i*), (*ii*) and (*iii*), we get

$$\pi e^{-a} = \int_{-\infty}^{\infty} f(x)\,dx + 0 \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a}$$

Equating real parts from both sides, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

Since $\cos ax/(x^2+1)$ is an even function of x, we have

$$2\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a} \quad \text{or} \quad \int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2} e^{-a}.$$

(*c*) Integration around rectangular contours

Example 20.43. *Evaluate* $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx$.

Solution. Consider $\int_C \frac{e^{az}}{e^z+1}\,dz = \int_C f(z)\,dz$ where C is the rectangle $ABCD$ with vertices at $(R, 0)$, $(R, 2\pi)$, $(-R, 2\pi)$ and $(-R, 0)$, R being positive (Fig. 20.24).

$f(z)$ has finite poles given by

$$e^z = -1 = e^{(2n+1)\pi i}$$

or $$z = (2n+1)\pi i, \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The only pole inside the rectangle is $z = \pi i$.

$\therefore$ by Residue theorem,

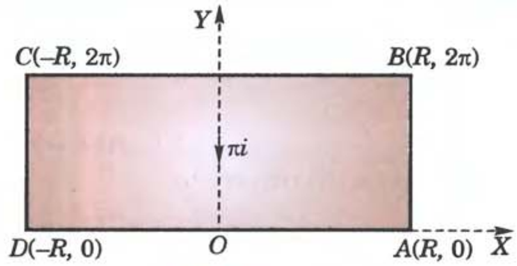


Fig. 20.24

$$\int_C f(z)dz = 2\pi i \text{ Res } f(\pi i)$$

$$= 2\pi i \left[e^{az} \Big/ \frac{d}{dz}(e^z + 1) \right]_{z=\pi i}$$

$$= 2\pi i e^{a\pi i}/e^{\pi i} = -2\pi i e^{a\pi i} \qquad [\because \; e^{\pi i} = -1] \qquad \ldots(i)$$

Also $$\int_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz$$

$$= \int_0^{2\pi} f(R+iy)idy + \int_R^{-R} f(x+2\pi i)dx + \int_{2\pi}^0 f(-R+iy)idy + \int_{-R}^R f(x)dx$$

[$\because$ $z = R + iy$ along AB, $z = x + 2\pi i$ along BC, $z = -R + iy$ along CD and $z = x$ along DA.]

or $$\int_C f(z)dz = i\int_0^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy}+1}dy - \int_{-R}^R \frac{e^{a(x+2\pi i)}}{e^{x+2\pi i}+1}dx - i\int_0^{2\pi} \frac{e^{a(-R+iy)}}{e^{-R+iy}+1}dy + \int_{-R}^R \frac{e^{ax}}{e^x+1}dx \qquad \ldots(ii)$$

Now for any two complex numbers z_1, z_2

$$|z_1| \geq |z_2|, \text{ we have } |z_1 + z_2| \geq |z_1| - |z_2|$$

so that $|e^{R+iy} + 1| \geq e^R - 1$. Also $|e^{a(R+iy)}| = e^{aR}$

$\therefore$ for the integrand of first integral in (ii), we have

$$\left| \frac{e^{a(R+iy)}}{e^{R+iy}+1} \right| \leq \frac{e^{aR}}{e^R - 1} \text{ which} \to 0 \text{ as } R \to \infty. \qquad [\because \; a > 1]$$

Similarly, for the integrand of the third integral in (ii), we get

$$\left| \frac{e^{a(-R+iy)}}{e^{-R+iy}+1} \right| \leq \frac{e^{-aR}}{1 - e^{-R}} \text{ which also} \to 0 \text{ as } R \to \infty. \qquad [\because \; a < 0]$$

Hence as $R \to \infty$, since the first and third integrals in (ii) approach zero, we get

$$\int_C f(z)dz = -e^{2a\pi i}\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = \left(1 - e^{2a\pi i}\right)\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx \qquad \ldots(iii)$$

Thus from (i) and (iii), we obtain $(1 - e^{2a\pi i}) \displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = -2\pi i e^{a\pi i}$

$\therefore$ equating real parts, we get $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = \frac{\pi}{\sin a\pi}$.

Example 20.44. *Show that* $\displaystyle\int_0^{\infty} e^{-x^2} \cos 2mx \, dx = \frac{1}{2}\sqrt{\pi e^{-m^2}}$.

Solution. Integrate $f(z) = e^{-z^2}$ along the rectangle ABCDA having vertices $A(-l)$, $B(l)$, $C(l + im)$, $D(-l + im)$ (Fig. 20.25). $f(z)$ has no poles inside this contour. As such

$$\int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = 0 \qquad \ldots(i)$$

On AB : $z = x$, on BC : $z = l + iy$, on CD : $z = x + im$ and on DA : $z = -l + iy$.

Therefore, (i) becomes

$$\int_{-l}^{l} e^{-x^2} dx + \int_0^m e^{-(l+iy)^2} idy + \int_l^{-l} e^{-(x+im)^2} dx + \int_m^0 e^{-(-l+iy)^2} dy = 0$$

or $$\int_{-l}^{l} e^{-x^2} dx - \int_{-l}^{l} e^{-x^2 - 2imx + m^2} dx + \int_0^m e^{-l^2 - 2ily + y^2} . idy$$

$$- \int_0^m e^{-l^2 + 2ily + y^2} . idy = 0 \qquad \ldots(ii)$$

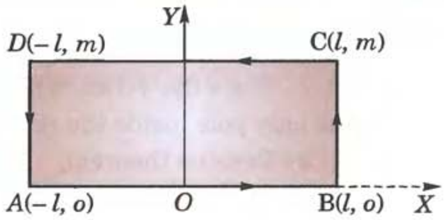


Fig. 20.25

Now let $l \to \infty$. Then the last two integrals

$$= ie^{-l^2} \int_0^m e^{y^2} (e^{-2ily} - e^{2ily})\, dy = 2e^{-l^2} \int_0^m e^{y^2} \sin 2ly\, dy \to 0$$

[$\because$ As $l \to \infty$, $e^{-l^2} \to 0$ and $\sin 2ly$ is finite]

Hence (*ii*) reduces to

$$\int_{-\infty}^{\infty} e^{-x^2} dx - e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} (\cos 2mx - i \sin 2mx) dx = 0$$

Equating real parts, we get

$$e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} \cos 2mx\, dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$ [See p. 289]

or $$\int_0^{\infty} e^{-x^2} \cos 2mx\, dx = \frac{1}{2}\sqrt{\pi e^{-m^2}}$$

(*d*) **Indenting the contours having poles on the real axis.** So far we have considered such cases in which there is no pole on the real axis. When the integrand has a simple pole on the real axis, we delete it from the region by indenting the contour (*i.e.*, by drawing a small semi-circle having the pole for the centre). The method will be clear from the following example.

Example 20.45. *Evaluate* $\int_0^{\infty} \frac{\sin mx}{x} dx$, *when* $m > 0$. *(U.P.T.U., 2007)*

Solution. Consider the integral $\int_C \frac{e^{miz}}{z} dz = \int_C f(z) dz$ where C consists of

(*i*) the real axis from r to R,

(*ii*) the upper half of the circle $C_R : |z| = R$,

(*iii*) the real axis $-R$ to $-r$,

(*iv*) the upper half of the circle $C_r : |z| = r$ (Fig. 20.26).

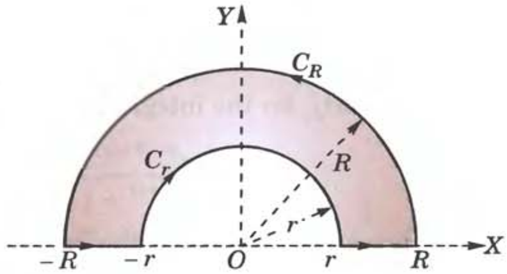


Fig. 20.26

Since $f(z)$ has no singularity inside C (its only singular point being a simple pole at $z = 0$ which has been deleted by drawing C_r), we have by Cauchy's theorem :

$$\int_r^R f(x) dx + \int_{C_R} f(z) dz + \int_{-R}^{-r} f(x) dx + \int_{C_r} f(z) dz = 0 \quad ...(i)$$

Now $$\int_{C_R} f(z) dz = \int_0^{\pi} \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} . Rie^{i\theta} d\theta$$ [$\because$ $z = Re^{i\theta}$]

$$= i \int_0^{\pi} e^{imR(\cos\theta + i\sin\theta)} d\theta$$

Since $| e^{imR(\cos\theta + i\sin\theta)} | = | e^{-mR\sin\theta + imR\cos\theta} | = e^{-mR\sin\theta}$

$\therefore$ $$\left| \int_{C_R} f(z) dz \right| \le \int_0^{\pi} e^{-mR\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta} d\theta$$

$$= 2\int_0^{\pi/2} e^{-2mR\theta/\pi} d\theta$$ [$\because$ for $0 \le \theta \le \pi/2$, $\sin\theta/\theta \ge 2/\pi$]

$$= \frac{\pi}{mR}\left(1 - e^{-mR}\right) \text{ which } \to 0 \text{ as } R \to \infty,$$

Also $$\int_{C_r} f(z) dz = i\int_{\pi}^{0} e^{imr(\cos\theta + i\sin\theta)} d\theta \to i\int_{\pi}^{0} d\theta \;\; i.e., -i\pi \text{ as } r \to 0.$$

Hence as $r \to 0$ and $R \to \infty$, we get from (*i*) $\int_0^{\infty} f(x) dx + 0 + \int_{-\infty}^{0} f(x) dx - i\pi = 0$

or $$\int_{-\infty}^{\infty} f(x) dx = i\pi \;\; i.e., \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi \quad ...(ii)$$

Equating imaginary parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi. \text{ Hence } \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}.$$

Obs. Equating real parts from both sides of (*ii*), we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x} dx = 0.$$

Example 20.46. *Show that* $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin px}$, $0 < p < 1$.

Solution. Integrate $f(z) = \frac{z^{p-1}}{1+z}$ along the contour consisting of the circles α and γ of radii a and R and the lines AB and FG along x-axis (Fig. 20.27). There is a simple pole at $z = -1$ which is within the contour.

$$\therefore \qquad \text{Res}\, f(-1) = \underset{z \to -1}{\text{Lt}} (1+z) \cdot \frac{z^{p-1}}{1+z} = \underset{z \to -1}{\text{Lt}} z^{p-1} = (-1)^{p-1} = e^{i\pi(p-1)}$$

Thus $$\int_{AB} f(z)dz + \int_\gamma f(z)dz + \int_{FG} f(z)dz + \int_\alpha f(z)dz = 2\pi i\, e^{i\pi(p-1)} \qquad ...(i)$$

On $AB : z = x$ and on $FG : z = xe^{2\pi i}$

$$\therefore \qquad \int_{AB} f(z)dz + \int_{FG} f(z)dz = \int_a^R \frac{x^{p-1}}{1+x} dx + \int_R^a \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}} dx\, e^{2\pi i}$$

$$= \int_a^R \frac{x^{p-1}}{1+x} [1 - e^{2\pi i(p-1)}]\, dx$$

On the circle $\gamma : z = Re^{i\theta}$. So

$$\int_\gamma f(z)dz = \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}}{1+Re^{i\theta}} Re^{i\theta}\, i\, d\theta$$

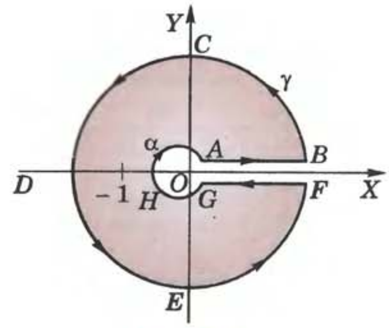


Fig. 20.27

For large R, the integrand is of the order $\frac{R^{p-1} \cdot R}{1+R}$ *i.e.* R^{p-1} which tends to zero as $R \to \infty$. $(\because \; p < 1)$

Hence $$\int_\gamma f(z)dz \to 0 \text{ as } R \to \infty$$

On the circle $\alpha : z = ae^{i\theta}$. So

$$\int_\alpha f(z)dz = \int_{2\pi}^0 \frac{(ae^{i\theta})^{p-1}}{1+ae^{i\theta}} ae^{i\theta}\, id\theta$$

For small α, the integrand is of the order a^p which tends to zero as $a \to 0$. $(\because \; p > 0)$

Thus on taking limits as $a \to 0$ and $R \to \infty$, (*i*) gives

$$\int_0^\infty \frac{x^{p-1}}{1+x} \{1 - e^{2\pi i(p-1)}\} dx = 2\pi i\, e^{i\pi(p-1)}$$

or $$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{2\pi i\, e^{i\pi(p-1)}}{1 - e^{2\pi i(p-1)}} = \frac{2\pi i\, e^{ip\pi}(-1)}{1 - e^{2ip\pi}\,(1)} = \frac{2i \cdot \pi}{e^{ip\pi} - e^{-ip\pi}} = \frac{\pi}{\sin p\pi}.$$

Example 20.47. *Prove that* $\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}$. (*Osmania, 2003*)

Solution. Consider $\int_C e^{-z^2} dz$ where C consists of the real axis from O to A, part of circle AB of radius R and the line $\theta = \frac{\pi}{4}$. (Fig. 20.28).

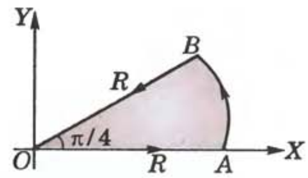


Fig. 20.28

e^{-z^2} has no singularity within C.

$$\therefore \qquad \int_{OA} e^{-z^2} dz + \int_{AB} e^{-z^2} dz + \int_{BO} e^{-z^2} dz = 0 \qquad ...(i)$$

On $OA : z = x$, $\therefore$ $\int_{OA} e^{-z^2} dz = \int_0^R e^{-x^2} dx \to \sqrt{\pi/2}$ as $R \to \infty$ [See p. 289]

On $AB : z = Re^{i\theta}$,

$\therefore$
$$\int_{AB} e^{-z^2} dz = \int_0^{\pi/4} e^{-R^2(\cos 2\theta + i \sin 2\theta)} \cdot Re^{i\theta} \cdot id\theta \to 0 \quad \text{as} \quad R \to \infty.$$

[$\because$ integrand $\to 0$ as $R \to \infty$]

On $BO : z = re^{i\pi/4}$ and $z^2 = r^2 e^{i\pi/2} = ir^2$

$\therefore$
$$\int_{BO} e^{-z^2} dz = \int_R^0 e^{-ir^2} \cdot e^{i\pi/4} dr = -\int_0^R e^{-ix^2} \frac{1+i}{\sqrt{2}} dx$$

$$\to -\int_0^\infty (\cos x^2 - i \sin x^2) \frac{1+i}{\sqrt{2}} dx \qquad \text{when } R \to \infty$$

Substituting these in (i), we get

$$\frac{1}{2}\sqrt{\pi} + 0 - \int_0^\infty (\cos x^2 - i \sin x^2)\left(\frac{1+i}{\sqrt{2}}\right) dx = 0$$

Equating real and imaginary parts, we obtain

$$\int_0^\infty (\cos x^2 + \sin x^2) dx = \frac{1}{2}\sqrt{(2\pi)} \quad \text{and} \quad \int_0^\infty (\cos x^2 - \sin x^2) dx = 0$$

Hence
$$\int_0^\infty \sin x^2 \, dx = \int_0^\infty \cos x^2 \, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}.$$

PROBLEMS 20.9

Apply the calculus of residues, to prove that

1. $\int_0^{2\pi} \frac{d\theta}{1 - 2p \sin\theta + p^2} = \frac{2\pi}{1-p^2} (0 < p < 1)$. *(Hissar, 2007 ; Mumbai, 2006 ; Kerala, 2005)*

2. $\int_0^{2\pi} \frac{d\theta}{1 - 2r \cos\theta + r^2} = \frac{\pi}{1-r^2}$. *(J.N.T.U., 2006 ; Madras, 2006 ; Anna, 2003)*

3. $\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{(a^2-1)}} (a > 1)$. *(P.T.U., 2010)*

4. $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \frac{\pi}{6}$. *(U.P.T.U., 2010)*

5. $\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}[a - \sqrt{(a^2 - b^2)}], (0 < b < a)$. *(J.N.T.U., 2003)*

6. $\int_0^{2\pi} \frac{a d\theta}{a^2 + \sin^2\theta} = \frac{2\pi}{\sqrt{(1+a^2)}}, (a > 0)$. *(S.V.T.U., 2009)*

7. $\int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2} = \frac{5\pi}{32}$.

8. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a+b} (a, b > 0)$. *(P.T.U., 2007 ; Mumbai, 2006 ; Anna, 2003)*

9. $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$. *(A.M.I.E.T.E., 2003 ; Delhi, 2002)*

10. $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{\sqrt{2}}$. *(J.N.T.U., 2006)*

11. $\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$. *(Madras, 2006 ; Kerala, 2005)*

12. $\int_0^{\infty} \frac{dx}{x^6 + 1} = \frac{\pi}{3}$. *(Kerala, 2005)*

13. $\int_0^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{\pi}{6}$. *(Rohtak, 2006)*

14. $\int_{-\infty}^{\infty} \frac{\cos mx}{e^x + e^{-x}} dx = \frac{\pi}{2} \operatorname{sech} \frac{m\pi}{2}$.

15. $\int_0^{\infty} \frac{1 - \cos x}{x^2} dx = \pi/2$. *(P.T.U., 2005)*

16. $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$. *(Kerala, 2005)*

17. $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx = \frac{-\pi \sin 2}{e}$.

18. $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) (a > b > 0)$.

19. $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$.

20.12. OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 20.10

Select the correct answer or fill up the blanks in each of the following questions :

1. The only function that is analytic from the following is
   (*i*) $f(z) = \sin z$ (*ii*) $f(z) = \bar{z}$ (*iii*) $f(z) = \text{Im}(z)$ (*iv*) $R(iz)$.
2. If $f(z) = u(x, y) + iv(x, y)$ is analytic, then $f'(z) =$
   (*i*) $\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$ (*ii*) $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (*iii*) $\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$.
3. If $2x - x^2 + ay^2$ is to be harmonic, then a should be
   (*a*) 1 (*b*) 2 (*c*) 3 (*d*) 0.
4. The analytic function which maps the angular region $0 \le \theta \le \pi/4$ onto the upper half plane is
   (*i*) z^2 (*ii*) $4z$ (*iii*) z^4 (*iv*) 2θ.
5. An angular domain in the complex plane is defined by $0 < \text{amp}(z) < \pi/4$. The mapping which maps this region onto the left half plane is
   (*i*) $w = z^4$ (*ii*) $w = iz^4$ (*iii*) $w = -z^4$ (*iv*) $w = -iz^4$.
6. The mapping $w = z^2 - 2z - 3$ is
   (*i*) conformal within $|z| = 1$ (*ii*) not conformal at $z = 1$
   (*iii*) not conformal at $z = -1$ and $z = 3$ (*iv*) conformal everywhere.
7. If $z = re^{i\theta}$, then the image of θ = constant under the mapping $w(z) = Re^{i\phi} = iz^3$ is
   (*i*) $\phi = 3\theta$ (*ii*) $\phi = 3\theta + \pi/2$ (*iii*) $\phi = 3\theta - \pi/2$ (*iv*) $\phi = \theta^3$.
8. The fixed points of the mapping $w = (5z + 4)/(z + 5)$ are
   (*i*) 2, 2 (*ii*) 2, – 2 (*iii*) – 2, – 2 (*iv*) – 4/5, 5.
9. The value of $\int_C (4x^3\,dx + 3y^2z^2\,dy + 2y^3\,z\,dz)$ where C is any path joining $A(-1, 1, 0)$ to $B(1, 2, 1)$ is
   (*i*) 0 (*ii*) 1 (*iii*) 8 (*iv*) – 8.
10. The value of $\int_C \frac{3z^2 + 7z + 1}{z + 1}\,dz$ where C is $|z| = 1/2$ is
    (*i*) $2\pi i$ (*ii*) 0 (*iii*) πi (*iv*) $\pi i/2$.
11. The value of $\int_C \frac{3z + 4}{z(2z + 1)}\,dz$ where C is the circle $|z| = 1$ is
    (*i*) $2\pi i$ (*ii*) $3\pi i$ (*iii*) 4 (*iv*) – 4.
12. The residue of a function can be found if the pole is an isolated singularity :
    (*i*) True (*ii*) False (*iii*) Partially false (*iv*) none of these.
13. The value of $\int_C \frac{z\,dz}{\sin z}$ where $C : |z| = 4$ is
    (*i*) $2\pi i$ (*ii*) 0 (*iii*) $-2\pi i$ (*iv*) $4\pi i$.
14. The value of $\int_C \tanh z\,dz$, where $C : |z| = 3$, is
    (*i*) 0 (*ii*) πi (*iii*) $2\pi i$ (*iv*) $4\pi i$.
15. The harmonic conjugate of the function $u(x, y) = 2x(1 - y)$ is ...... (*U.P.T.U., 2009*)
16. Harmonic conjugate of $x^3 - 3xy^2$ is ....
17. The curves $u(x, y) = c$ and $v(x, y) = c'$ are orthogonal if ......
18. The value of $\int_0^{1+i} z^2\,dz$ along the line $x = y$ is ......
19. Residue of $\frac{\cos z}{z}$ at $z = 0$ is ......
20. The critical point of the transformation $w^2 = (z - a)(z - b)$ is ......
21. Image of $|z + 1| = 1$ under the mapping $w = 1/z$ is......
22. The poles of $f(z) = (z^3 - 1)/(z^3 + 1)$ are $z =$ ......
23. $w = \log z$ is analytic everywhere except at $z =$ ......
24. If $f(z) = -\frac{1}{z - 1} - 2[1 + (z - 1) + (z - 1)^2 + ...]$, then the residue of $f(z)$ at $z = 1$ is ......
25. If $|z| < 1$ then Taylor's series expansion of $\log(1 + z)$ about $z = 0$ is ......

26. The value of $\int_C \frac{4z^2+z+5}{z-4}dz$ where C is $9x^2+4y^2=36$, is ......

27. The value of $\int z^4 e^{1/z}\,dz$, where C is $|z|=1$, is
(i) πi (ii) $\pi i/12$ (iii) $\pi i/60$ (iv) $-\pi i/60$.

28. If $f(z)$ has a pole of order three at $z=a$ Res $[f(a)]=$ ......

29. The value of $\int_C \frac{e^z\,dz}{(z-3)^2}$, C being $|z|=2$, is ......

30. The CR equations for $f(z)=u(x,y)+iv(x,y)$ to be analytic are ......

31. If $f(z)$ is analytic in a simply connected domain D and C is any simple closed path then $\int_C f(z)dz=$ ......

32. The harmonic conjugate of $e^x \cos y$ is ......

33. The value of $\oint_C \cos z\,dz$ where C is the circle $|z|=1$, is ......

34. The singularity of $f(z)=z/(z-2)^3$ is ......

35. The function $f(z)=\bar{z}$ is analytic at ......

36. C-R equations for a function to be analytic, in polar form, are ......

37. If C is the circle $|z-a|=r$, $\int_C (z-a)^n\,dz$ [n, any integer $\neq -1$] $=$ ......

38. A simply connected region is that ......

39. A holomorphic function is that ......

40. The poles of the function $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ are at $z=$ ......

41. The cross-ratio of four points z_1, z_2, z_3, z_4 is ......

42. The value of $\int_C |z|\,dz$, where C is the contour represented by the straight line from $z=-i$ to $z=i$, is ......

43. Taylor's series expansion of $\left(\frac{1}{z-2}-\frac{1}{z-1}\right)$ in the region $|z|<1$, is ......

44. The invariant points of the transformation $w=(1+z)/(1-z)$ are $z=$ ......

45. The residue at $z=0$ of $\frac{1+e^z}{z\cos z+\sin z}$ is ......

46. The transformation $w=Cz$ consists of ......

47. The residue of $f(z)$ at a pole is ......

48. The value of $\int_C \frac{1}{z-1}dz$, C being $|z|=2$, is ......

49. If C is $|z|=1/2$, $\int_C \frac{z^2-z+1}{z-1}dz=$ ......

50. Singular points of $\frac{\cos \pi z}{(z-1)(z-2)}$ are ......

51. Taylor series expansion of $\frac{1}{z-2}$ in $|z|<1$ is ......

52. $\underset{z\to\infty}{\text{Lt}} \frac{iz^3+iz-1}{(2z+3)(z-1)^2}=$ .... (*P.T.U., 2007*)

53. The poles of $\frac{(z-1)^2}{z(z-2)^2}$ are at $z=$ ......

54. Cauchy's integral theorem states that ......

55. The critical points of the transformation $w=z+1/z$ are ......

56. $\int_c \frac{dz}{2z-3}$, where $|z|=1$, is ....

57. The zeroes and singularities of $\frac{z^2+1}{1-z^2}$ are ......

58. Residue of $\tan z$ at $z=\pi/2$ is ......

59. Singularity of $e^{z^{-1}}$ at $z=0$ is of the type ....

60. Res $(e^{1/z})_{z=0}=$ ......

61. Taylor's series expansion of $\sin z$ about $z=\pi/4$ is ......

62. Image of $|z|=2$ under $w=z+3+2i$ is ....

63. The poles of $\cot z$ are ....

64. If a is simple pole, then Res $[\phi(z)/\psi(z)]_{z=a}=$ ....

65. Bilinear transformation always transforms circles into ....

66. If $f(z)$ and $\overline{f(z)}$ are analytic functions, then $f(z)$ is constant. (True or False) (*Mumbai, 2006*)

67. The function $u(x,y)=2xy+3xy^2-2y^3$ is a harmonic functions. (True or False) (*P.T.U., 2009 S*)

68. The function $e^x \cos y$ is harmonic. (True or False)

69. $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$, if $z = a$ is a point within C. (True or False)

70. The transformation affected by an analytic function $w = f(z)$ is conformal at every point of the z-plane where $f'(z) = 0$. (True or False)

71. The function $\bar{z}$ is not analytic at any point. (True or False)

72. Under the transformation $w = 1/z$, circle $x^2 + y^2 - 6x = 0$ transforms into a straight line in the w-plane. (True or False)

73. If $w = f(z)$ is analytic, then $\frac{dw}{dz} = -i\frac{\partial w}{\partial y}$. (True or False)

74. An analytic function with constant imaginary part is constant. (True or False)

75. If $u + iv$ is analytic, then $v - iu$ is also analytic. (True or False)

76. $f(z) = I_m(z)$ is not analytic. (True or False)

77. The cross-ratio of four points is not invariant under bilinear transformation. (True or False)

78. $z = 0$ is not a critical point of the mapping $w = z^2$. (True or False)

79. $f(z) = \text{Re}(z^2)$ is analytic. (True or False)

80. An analytic function with constant modulus is constant. (True or False)

81. The function $|\bar{z}|^2$ is not analytic at any point. (True or False)

82. If $f(z) = z^2$, then the family of curves $x^2 - y^2 = C_1$, and $xy = C_2$ are orthogonal. (True or False)

CHAPTER 23

Z-Transforms

23.1 INTRODUCTION

The development of communication branch is based on discrete analysis. Z-transform plays the same role in discrete analysis as Laplace transform in continuous systems. As such, Z-transform has many properties similar to those of the Laplace transform (§ 21.2). The main difference is that the Z-transform operates not on functions of continuous arguments but on sequences of the discrete integer-valued arguments, *i.e.* $n = 0, \pm 1, \pm 2,$ ...... The analogy of Laplace transform to Z-transform can be carried further. For every operational rule of Laplace transforms, there is a corresponding operational rule of Z-transforms and for every application of the Laplace transform, there is a corresponding application of Z-transform. A discrete system is expressible as a difference equation (§ 30.2) and its solutions are found using Z-transforms.

23.2 DEFINITION

If the function u_n is defined for discrete values ($n = 0, 1, 2, \ldots$) and $u_n = 0$ for $n < 0$, then its Z-transform is defined to be

$$Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n} \text{ whenever the infinite series converges.} \quad \ldots(i)$$

The inverse Z-transform is written as $Z^{-1}[U(z)] = u_n$.

If we insert a particular complex number z into the power series (i), the resulting value of $Z(u_n)$ will be a complex number. Thus the Z-transform $U(z)$ is a complex valued function of a complex variable z.

23.3 SOME STANDARD Z-TRANSFORMS

The direct application of the definition gives the following results :

(1) $Z(a^n) = \dfrac{z}{z-a}$ **(2)** $Z(n^p) = -z \dfrac{d}{dz} Z(n^{p-1})$, *p being a +ve integer.*

Proof. **(1)** By definition, $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= 1 + (a/z) + (a/z)^2 + (a/z)^3 + \ldots = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$ (*Kottayam, 2005*)

(2) $$Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z \sum_{n=0}^{\infty} n^{p-1} . n . z^{-(n+1)} \quad \ldots(i)$$

Changing p to $p - 1$, we get $Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} . z^{-n}$

Differentiating it w.r.t. z,

$$\frac{d}{dz}[Z(n^{p-1})] = \sum_{n=0}^{\infty} n^{p-1} . (-n) z^{-(n+1)} \quad \ldots(ii)$$

Substituting (*ii*) in (*i*), we obtain $Z(n^p) = -z\frac{d}{dz}[Z(n^{p-1})]$

which is the desired *recurrence formula.*

In particular, we have the following formulae :

(3) $Z(1) = \frac{z}{z-1}$ [Taking $a = 1$ in (1) **(4)** $Z(n) = \frac{z}{(z-1)^2}$ [Taking $p = 1$ in (2)]

(5) $Z(n^2) = \frac{z^2+z}{(z-1)^3}$ (*V.T.U., 2006*) **(6)** $Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$

(7) $Z(n^4) = \frac{z^4+11z^3+11z^2+z}{(z-1)^5}$.

23.4 LINEARITY PROPERTY

If a, b, c be any constants and u_n, v_n, w_n be any discrete functions, then

$$Z(au_n + bv_n - cw_n) = aZ(u_n) + bZ(v_n) - cZ(w_n)$$

Proof. By definition, $Z(au_n + bv_n - cw_n) = \sum_{n=0}^{\infty} (au_n + bv_n - cw_n)z^{-n}$

$$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} v_n z^{-n} - c\sum_{n=0}^{\infty} w_n z^{-n}$$

$$= aZ(u_n) + bZ(v_n) - cZ(w_n).$$

23.5 DAMPING RULE

If $Z(u_n) = U(z)$, then $Z(a^{-n} u_n) = U(az)$

Proof. By definition, $Z(a^{-n} u_n) = \sum_{n=0}^{\infty} a^{-n} u_n . z^{-n} = \sum_{n=0}^{\infty} u_n . (az)^{-n} = U(az)$. (*Madras, 2006*)

Cor. $\boldsymbol{Z(a^n u_n) = U(z/a)}$

Obs. The geometric factor a^{-n} when $|a| < 1$, damps the function u_n, hence the name *damping rule.*

23.6 SOME STANDARD RESULTS

The application of the damping rule leads to the following standard results :

(1) $Z(na^n) = \frac{az}{(z-a)^2}$ **(2)** $Z(n^2a^n) = \frac{az^2+a^2z}{(z-a)^3}$

(3) $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$ (4) $Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

(5) $Z(a^n \cos n\theta) = \dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$ (6) $Z(a^n \sin n\theta) = \dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$.

Proofs. **(1)** We know that $Z(n) = \dfrac{z}{(z-1)^2}$. Applying damping rule, we have

$$Z(na^n) = U(a^{-1}z) = \frac{a^{-1}z}{(a^{-1}z-1)^2} = \frac{az}{(z-a)^2}.$$ (*Madras, 2000 S*)

(2) We know that $Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$. Applying damping rule, we have

$$Z(n^2a^n) = U(a^{-1}z) = \frac{(a^{-1}z)^2 + a^{-1}z}{(a^{-1}z-1)^3} = \frac{a(z^2+az)}{(z-a)^3}.$$

(3) and **(4)** We know that $Z(1) = \dfrac{z}{z-1}$. Applying damping rule, we have

$$Z(e^{-in\theta}) = Z(e^{-i\theta})^n \cdot 1 = \frac{ze^{i\theta}}{ze^{i\theta}-1} = \frac{z}{z-e^{-i\theta}} = \frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})}$$
$$= \frac{z(z-\cos\theta) - iz\sin\theta}{z^2 - z(e^{i\theta}+e^{-i\theta})+1} = \frac{z(z-\cos\theta)-iz\sin\theta}{z^2-2z\cos\theta+1}$$

Equating real and imaginary parts, we get (3) and (4). (*V.T.U., 2010 S; Anna, 2009*)

(5) We know that $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$. By damping rule, we have

$$Z(a^n\cos n\theta) = \frac{a^{-1}z(a^{-1}z-\cos\theta)}{(a^{-1}z)^2 - 2(a^{-1}z)\cos\theta+1} = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$$ (*V.T.U., 2006*)

Similarly using (4) above, we get (6).

Example 23.1. *Find the Z-transform of the following :*
(i) $3n - 4\sin n\pi/4 + 5a$ *(ii)* $(n+1)^2$ (*V.T.U., 2010*)
(iii) $\sin(3n+5)$. (*V.T.U., 2009 S ; Kottayam, 2005*)

Solution. *(i)* $Z\left(3n - 4\sin\dfrac{n\pi}{4} + 5a\right) = 3Z(n) - 4Z\left(\sin\dfrac{n\pi}{4}\right) + 5a\,Z(1)$ [By Linearty property[

$$= 3\cdot\frac{z}{(z-1)^2} - 4\cdot\frac{z\sin n\pi/4}{z^2-2z\cos\pi/4+1} + 5a\cdot\frac{z}{z-1}$$ [Using formulae for $Z(1)$, $Z(n)$, $Z(\sin n\theta)$]

$$= \frac{(3-5a)z+5az^2}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2-\sqrt{2}z+1}$$

(ii) $Z(n+1)^2 = Z(n^2+2n+1) = Z(n^2) + 2Z(n) + Z(1)$

$$= \frac{z^2+z}{(z-1)^3} + 2\frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z^2(2z+1)}{(z-1)^3}$$

(iii) $Z[\sin(3n+5)] = Z(\sin 3n\cos 5 + \cos 3n\sin 5)$
$= \cos 5.Z(\sin 3n) + \sin 5.Z(\cos 3n)$ (using formulae for $Z(\sin n\theta)$, $Z(\cos n\theta)$

$$= \cos 5\cdot\frac{z\sin 3}{z^2-2z\cos 3+1} + \sin 5\cdot\frac{z(z-\cos 3)}{z^2-2z\cos 3+1} = z\cdot\frac{(z\sin 5-\sin 2)}{z^2-2z\cos 3+1}.$$

Example 23.2. *Find the Z-transforms of the following*
(i) e^{an} *(ii)* ne^{an} *(iii)* n^2e^{an}.

Solution. (*i*) Let $u_n = 1$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$. By damping rule $Z(k^{-n}\, u_n) = U(kz)$,

$$\therefore \quad Z(e^{an}) = Z(k^{-n}\,.\,1) = U(kz) = \frac{kz}{kz-1} \qquad \left[\because \; U(z) = Z(1) = \frac{z}{z-1}\right]$$

$$= \frac{z}{z-1/k} = \frac{z}{z-e^a}$$

(*ii*) Let $u_n = n$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule, $Z\,(e^{an}\,.\,n) = Z(k^{-n}\,.\,n) = U(k\,z)$ where $U(z) = Z(n) = \dfrac{z}{(z-1)^2}$

$$\frac{kz}{(kz-1)^2} = \frac{z}{k(z-1/k)^2} = \frac{e^a z}{(z-e^a)^2}$$

(*iii*) Let $u_n = n^2$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$
By damping rule,

$$Z(e^{an}\,.\,n^2) = Z(k^{-n}\,.\,n^2) = U(kz) \quad \text{where} \quad U(z) = Z(n^2) = \frac{z^2+z}{(z-1)^3}$$

$$= \frac{(kz)^2 + kz}{(kz-1)^3} = \frac{z(z+1/k)}{(z-1/(k))^3} = \frac{ze^a(z+e^a)}{(z-e^a)^3}.$$

Example 23.3. *Find the Z-transform of (i) cosh nθ.* (*V.T.U., 2011*) *(ii)* a^n *cosh nθ.*

Solution. (*i*)

$$Z\,(\cosh n\theta) = Z\left(\frac{e^{n\theta} + e^{-n\theta}}{2}\right)$$

$$= \frac{1}{2}\left[Z\left\{(e^{-\theta})^{-n}\,.\,1\right\} + Z\left\{(e^{\theta})^{-n}\,.\,1\right\}\right]$$

Apply damping rule to both terms, taking $u_n = 1$.

$$Z(\cosh n\theta) = \frac{1}{2}\left[\frac{ze^{-\theta}}{ze^{-\theta}-1} + \frac{ze^{\theta}}{ze^{\theta}-1}\right] \qquad \left[\because \; z(1) = \frac{z}{z-1}\right]$$

$$= \frac{1}{2}\left\{\frac{2z^2 - z(e^{\theta}+e^{-\theta})}{z^2 - z(e^{\theta}+e^{-\theta})+1}\right\} = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cosh\theta + 1}$$

(*ii*) $Z(a^n \cosh n\theta) = Z[(a^{-1})^{-n}\,.\,\cosh n\theta]$ [Apply damping rule using (*i*)]

$$= \frac{(a^{-1}z)^2 - (a^{-1}z)\cosh\theta}{(a^{-1}z)^2 - 2(a^{-1}z)\cosh\theta + 1} = \frac{z(z - a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}.$$

Example 23.4. *Find the Z-transforms of*
(i) $e^t \sin 2t$ (*Madras, 2003*) *(ii)* $c^k \cos k\alpha$. $(k \geq 0)$ (*U.P.T.U., 2004 S*)

Solution. (*i*) We know that $Z\,(\sin 2t) = \dfrac{z\sin 2}{z^2 - 2z\cos 2 + 1}$ ...(A)

$\therefore$ $Z\,(e^t \sin 2t) = Z\,[(e^{-1})^{-t}\,.\,\sin 2t]$ [Apply damping rule, using (A)]

$$= \frac{(e^{-1}z)\sin 2}{(e^{-1}z)^2 - 2(e^{-1}z)\cos 2 + 1} = \frac{ez\sin 2}{z^2 - 2ez\cos 2 + e^2}.$$

(*ii*) We know that $Z\,(\cos k\alpha) = \dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$ ...(B)

$\therefore$ $Z(c^k \cos k\alpha) = Z\,[(c^{-1})^{-k}\,.\,\cos k\alpha]$ [Apply damping rule, using (B)]

$$= \frac{(c^{-1}z)[c^{-1}z - \cos\alpha]}{(c^{-1}z)^2 - 2(c^{-1}z)\cos\alpha + 1} = \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}.$$

Example 23.5. *Find the Z-transforms of*

(i) $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$ *(V.T.U., 2011 S)* *(ii)* $\cosh\left(\frac{n\pi}{2}+\theta\right)$. *(U.P.T.U., 2008)*

Solution. *(i)* $Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right]=Z\left(\cos\frac{n\pi}{2}\cos\frac{\pi}{4}-\sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right)$

$$=\cos\frac{\pi}{4}.Z\left(\cos\frac{n\pi}{2}\right)-\sin\frac{\pi}{2}.Z\left(\sin\frac{n\pi}{2}\right)\quad\text{[Using formulae for } Z(\sin n\alpha) \text{ and } Z(\cos n\alpha)\text{]}$$

$$=\frac{1}{\sqrt{2}}\left\{\frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1}-\frac{z\sin\pi/2}{z^2-2z\cos\pi/2+1}\right\}=\frac{1}{\sqrt{2}}\left(\frac{z^2}{z^2+1}-\frac{z}{z^2+1}\right)=\frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

(ii) $$Z\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right]=Z\left[\frac{e^{n\pi/2+\theta}+e^{-(n\pi/2+\theta)}}{2}\right]=\frac{1}{2}\left[e^{\theta}\,Z(e^{n\pi/2})+e^{-\theta}\,Z(e^{-n\pi/2})\right]$$

Since, $Z(a^n)=\frac{z}{z-a}$, $\therefore\ Z(e^{n\pi/2})=Z(e^{\pi/2})^n=\frac{z}{z-e^{\pi/2}}$, $Z(e^{-n\pi/2})=\frac{z}{z-e^{-\pi/2}}$

Thus $$Z\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right]=\frac{1}{2}\left\{e^{\theta}.\frac{z}{z-e^{\pi/2}}+e^{-\theta}.\frac{z}{z-e^{-\pi/2}}\right\}$$

$$=\frac{z}{2}\left\{\frac{z(e^{\theta}+e^{-\theta})-[e^{(\pi/2-\theta)}+e^{-(\pi/2-\theta)}]}{z^2-z(e^{\pi/2}+e^{-\pi/2})+1}\right\}=\frac{z^2\cosh\theta-z\cosh\left(\frac{\pi}{2}-\theta\right)}{z^2-2z\cosh\left(\frac{\pi}{2}\right)+1}.$$

Example 23.6. *Find the Z-transform of*

(i) ${}^nC_p\ (0\le p\le n)$ *(ii)* ${}^{n+p}C_p$.

Solution. *(i)* $$Z({}^nC_p)=\sum_{p=0}^{n}\left({}^nC_pz^{-p}\right)=1+{}^nC_1z^{-1}+{}^nC_2z^{-2}+\ldots+{}^nC_nz^{-n}=(1+z^{-1})^n$$

(ii) $$Z({}^{n+p}C_n)=\sum_{p=0}^{n}{}^{n+p}C_pz^{-p}$$

$$=1+{}^{n+1}C_1z^{-1}+{}^{n+2}C_2z^{-2}+{}^{n+3}C_3z^{-3}+\ldots\infty$$

$$=1+(n+1)z^{-1}+\frac{(n+2)(n+1)}{2!}z^{-2}+\frac{(n+3)(n+2)(n+1)}{3!}z^{-3}+\ldots\infty$$

$$=1+(-n-1)(-z^{-1})+\frac{(-n-1)(-n-2)}{2!}(-z^{-1})^2+\frac{(-n-1)(-n-2)(-n-3)}{3!}(-z^{-1})^3+\ldots\infty$$

$$=(1-z^{-1})^{-n-1}.$$

Example 23.7. *Find the Z-transform of*

(i) unit impulse sequence $\delta(n)=\begin{cases}1 & n=0\\ 0 & n\ne 0\end{cases}$ *(ii) unit step sequence* $u(n)=\begin{cases}0 & n<0\\ 1 & n\ge 0\end{cases}$.

Solution. *(i)* $Z[\delta(n)]=\sum_{n=0}^{\infty}\delta(n)z^{-n}=1+0+0+\ldots=1$

(ii) $Z[u(n)]=\sum_{n=0}^{\infty}u(n)z^{-n}=1+z^{-1}+z^{-2}+z^{-3}+\ldots=\frac{1}{1-z^{-1}}=\frac{z}{z-1}.$

23.7 (1) SHIFTING U_N TO THE RIGHT

If $Z(u_n) = U(z)$, *then* $Z(u_{n-k}) = z^{-k}\, U(z)$ $\quad (k > 0)$

Proof. By definition,

$$Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k} z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + \ldots = z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} = z^{-k}\, U(z)$$

Obs. This rule will be very useful in applications to difference equations.

(2) **Shifting u_n to the left.** If $Z(u_n) = U(z)$, then

$$Z(u_{n+k}) = z^k[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \ldots - u_{k-1} z^{-(k-1)}]$$

Proof. $Z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)}$

$$= z^k \left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n} \right]$$

Hence $Z(u_{n+k}) = z^k[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \ldots - u_{k-1} z^{-(k-1)}]$ (*J.N.T.U., 2002*)

In particular, we have the following standard results :

(1) $Z(u_{n+1}) = z[U(z) - u_0]$; (2) $Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$

(3) $Z(u_{n+3}) = z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$.

Example 23.8. *Show that* $Z\left(\frac{1}{n!}\right) = e^{1/z}$.

Hence evaluate $Z[1/(n+1)!]$ *and* $Z[1/(n+2)!]$. (*Madras, 2006*)

Solution. We have $Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \ldots = e^{1/z}$.

Shifting $(1/n!)$ one unit to the left gives

$$Z\left[\frac{1}{(n+1)!}\right] = z\left[Z\left(\frac{1}{n!}\right) - 1\right] = z(e^{-1/z} - 1)$$

Similarly shifting $(1/n!)$ two units to the left gives

$$Z\left[\frac{1}{(n+2)!}\right] = z^2(e^{1/z} - 1 - z^{-1}).$$

23.8 MULTIPLICATION BY n

If $Z(u_n) = u(z)$, then $Z(nu_n) = -z\frac{dU(z)}{dz}$

Proof. $Z(nu_n) = \sum_{n=0}^{\infty} n \,.\, u_n z^{-n} = -z \sum_{n=0}^{\infty} u_n(-n) z^{-n-1} = -z \sum_{n=0}^{\infty} u_n \frac{d}{dz}(z^{-n})$.

$$= -z \sum_{n=0}^{\infty} \frac{d}{dz}(u_n z^{-n}) = -z\frac{d}{dz}\left(\sum_{n=0}^{\infty} u_n z^{-n}\right) = -z\frac{d}{dz}U(z).$$

Obs. We have, $Z(n^2 u_n) = \left(-z\frac{d}{dz}\right)^2 u(z)$ (*Madras, 2006*)

In general, $Z(n^m u_n) = \left(-z\frac{d}{dz}\right)^m u(z)$.

Example 23.9. *Find the Z-transform of (i) $n \sin n\theta$ (ii) $n^2 e^{n\theta}$.*

Solution. (*i*) We know that $Z(nu_n) = -z \dfrac{dU(z)}{dz}$ and $Z(\sin n\theta) = \dfrac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$

$$\therefore \quad Z(n \sin n\theta) = -z \frac{d}{dz} [Z(\sin n\theta)] = -z \frac{d}{dz}\left(\frac{z \sin \theta}{z^2 - 2z\cos\theta + 1}\right)$$

$$= -z \frac{\sin\theta - z^2 \sin\theta}{(z^2 - 2z\cos\theta + 1)^2} = \frac{z(z^2 - 1)\sin\theta}{(z^2 - 2z\cos\theta + 1)^2}$$

(*ii*) We know that $\quad Z(e^{n\theta}) = \dfrac{z}{z - e^\theta}$

$$\therefore \quad Z(n^2 e^{n\theta}) = \left(-z\frac{d}{dz}\right)^2 (Ze^{n\theta}) = \left(-z\frac{d}{dz}\right)\left[-z\frac{d}{dz}\left(\frac{z}{z - e^\theta}\right)\right]$$

$$= \left(-z\frac{d}{dz}\right)\left\{-z\frac{(z - e^\theta)(1) - z(1)}{(z - e^\theta)^2}\right\} = -z\frac{d}{dz}\left\{\frac{ze^\theta}{(z - e^\theta)^2}\right\}$$

$$= -ze^\theta \left\{\frac{(z - e^\theta)^2 (1) - z[2(z - e^\theta)]}{(z - e^\theta)^4}\right\} = -ze^\theta \frac{z - e^\theta - 2z}{(z - e^\theta)^3} = \frac{z(z + e^\theta)e^\theta}{(z - e^\theta)^3}.$$

23.9 TWO BASIC THEOREMS

In applications, we often need the values of u_n for $n = 0$ or as $n \to \infty$ without requiring complete knowledge of u_n. We can find this as the behaviour of u_n for small values of n is related to the behaviour of $U(z)$ as $z \to \infty$ and *vice-versa*. The precise relationship is given by the following *initial and final value theorems :*

(1) Initial value theorem. *If $Z(u_n) = U(z)$, then $u_0 = \underset{z \to \infty}{Lt}\ U(z)$*

Proof. We know that $U(z) = Z(u_n) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \ldots$

Taking limits as $z \to \infty$, we get $\underset{z \to \infty}{\text{Lt}} [U(z)] = u_0$, as required.

Similarly additional initial values can be found successively, giving :

$$u_1 = \underset{z \to \infty}{\text{Lt}} \{z[U(z) - u_0]\}\ ;\ u_2 = \underset{z \to \infty}{\text{Lt}} \{z^2 [U(z) - u_0 - u_1 z^{-1}]\} \text{ and so on.}$$

(2) Final value theorem. *If $Z(u_n) = U(z)$, then*

$$\underset{n \to \infty}{Lt} (u_n) = \underset{z \to 1}{Lt} (z - 1) U(z)$$

Proof. By definition, $Z(u_{n+1} - u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$

or $$Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$$

or $$z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$$

or $$U(z)(z - 1) - u_0 z = \sum_{n=0}^{\infty} (u_{n+1} - u_n) z^{-n}$$

Taking limits of both sides as $z \to 1$, we get

$$\underset{z \to 1}{\text{Lt}} [(z - 1) U(z)] - u_0 = \sum_{n=0}^{\infty} (u_{n+1} - u_n) = \underset{n \to \infty}{\text{Lt}} [(u_1 - u_0) + (u_2 - u_1) + \ldots + (u_{n+1} - u_n)]$$

$$= \underset{n\to\infty}{\text{Lt}} [u_{n+1}] - u_0 = u_\infty - u_0$$

Hence $$u_\infty = \underset{z\to 1}{\text{Lt}} [(z-1)\,U(z)].$$ (*Anna, 2005 S*)

Example 23.10. *If* $U(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$, *evaluate* u_2 *and* u_3.

Solution. Writing $$U(z) = \frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4}$$

By initial value theorem, $$u_0 = \underset{z\to\infty}{\text{Lt}}\, U(z) = 0$$

Similarly, $$u_1 = \underset{z\to\infty}{\text{Lt}} \{z\,[U(z) - u_0]\} = 0$$

Now $$u_2 = \underset{z\to\infty}{\text{Lt}} \{z^2\,[U(z) - u_0 - u_1 z^{-1}]\} = 2 - 0 - 0 = 2$$

and $$u_3 = \underset{z\to\infty}{\text{Lt}}\, z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}] = \underset{z\to\infty}{\text{Lt}}\, z^3[U(z) - 0 - 0 - 2z^{-2}]$$

$$= \underset{z\to\infty}{\text{Lt}}\, z^3\left[\frac{2z^2 + 5z + 14}{(z-1)^4} - \frac{2}{z^2}\right] = \underset{z\to\infty}{\text{Lt}}\, z^3\left\{\frac{13z^3 + 2z^2 + 8z - 2}{z^2(z-1)^4}\right\} = 13.$$

PROBLEMS 23.1

1. Find the Z-transforms of the following sequences :

   (*i*) $\dfrac{a^n}{n!}$ $(n \geq 0)$ (*S.V.T.U., 2009*) (*ii*) $\dfrac{1}{(n+1)!}$ (*iii*) $(\cos\theta + i\sin\theta)^n$.

2. Using the linearity property, find the Z- transforms of the following functions :

   (*i*) $2n + 5\sin n\pi/4 - 3a^4$ (*ii*) $\dfrac{1}{2}(n-1)(n+2)$ (*S.V.T.U., 2007*)

   (*iii*) $(n+1)(n+2)$ (*Anna, 2008*) (*iv*) $(2n-1)^2$ (*V.T.U., 2011 S*)

3. Show that (*i*) $Z(\sinh n\theta) = \dfrac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$ (*V.T.U., 2011*) (*ii*) $Z(a^n \sinh n\theta) = \dfrac{az\sinh\theta}{z^2 - 2az\cosh\theta + a^2}$.

4. Show that (*i*) $Z(e^{-an}\cos n\theta) = \dfrac{ze^a(ze^a - \cos\theta)}{z^2e^{2a} - 2ze^a\cos\theta + 1}$; (*ii*) $Z(e^{-an}\sin n\theta) = \dfrac{ze^a\sin\theta}{z^2e^{2a} - 2ze^a\cos\theta + 1}$

   Also evaluate $Z(e^{3n}\sin 2n)$. (*S.V.T.U., 2007*)

5. Using $Z(n^2) = \dfrac{z^2 + z}{(z-1)^3}$, show that $Z(n+1)^2 = \dfrac{z^3 + z^2}{(z-1)^3}$.

6. Find the Z-transforms of (*i*) $\sin(n+1)\theta$, (*ii*) $\cos\left(\dfrac{k\pi}{8} + \alpha\right)$. (*Marathwada, 2008*)

7. Find the Z-transform of $\cos n\theta$ and hence find $Z(n\cos n\theta)$. (*Anna, 2009*)

8. Find the Z-transform of $\cos(n\pi/2)$ and $a^n\cos(n\pi/2)$. (*Anna, 2008 S*)

9. Find the Z-transforms of the following

   (*i*) e^{-an} (*ii*) e^{-2n} (*V.T.U., 2010 S*) (*iii*) $e^{-an}n^2$.

10. Show that (*i*) $Z[\delta(n+1)] = 1/z$ (*ii*) $(1/2)^n u(n) = \dfrac{2z}{2z-1}$.

11. Show that $Z({}^{n+p}C_p) = (1 - 1/z)^{-(p+1)}$. Using the damping rule, deduce that

    $$Z({}^{n+p}C_p\, a^n) = (1 + a/z)^{-(p+1)}.$$

12. If $Z(u_n) = \dfrac{z}{z-1} + \dfrac{z}{z^2+1}$, find the Z-transform of u_{n+2}. (*S.V.T.U., 2009*)

13. If $U(z) = \dfrac{2z^2 + 3z + 12}{(z-1)^4}$, find the value of u_2 and u_3.

14. Given that $Z(u_n) = \dfrac{2z^2 + 3z + 4}{(z-3)^3}$, $|z| > 3$, show that $u_1 = 2$, $u_2 = 21$, $u_3 = 139$.

15. Show that (i) $Z\left(\dfrac{1}{n}\right) = z \log \dfrac{z}{z-1}$. *(Madras, 2003 S)* (ii) $Z\left\{\dfrac{1}{n(n+1)}\right\}$. *(Anna, 2005 S)*

16. Using $Z(n) = \dfrac{z}{(z-1)^2}$, show that $Z(n \cos n\theta) = \dfrac{(z^3 + z)\cos\theta - 2z^2}{(z^2 - 2z\cos\theta + 1)^2}$.

23.10 SOME USEFUL Z-TRANSFORMS

Sr. No.	*Sequence u_n $(n \geq 0)$*	*Z-transform $U(z) = Z(u_n)$*
1.	k	$kz/(z-1)$
2.	$-k$	$kz/(z+1)$
3.	n	$z/(z-1)^2$
4.	n^2	$(z^2+z)/(z-1)^3$
5.	n^p	$-z\, d/dz\, [Z(n^{p-1})]$, p + ve integer.
6.	$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$	1
7.	$u(n) = \begin{Bmatrix} 0 & n < 0 \\ 1 & n \geq 0 \end{Bmatrix}$	$z/(z-1)$
8.	a^n	$z/(z-a)$
9.	na^n	$az/(z-a)^2$
10.	n^2a^n	$(az^2 + a^2z)/(z-a)^3$
11.	$\sin n\theta$	$\dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$
12.	$\cos n\theta$	$\dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$
13.	$a^n \sin n\theta$	$\dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$
14.	$a^n \cos n\theta$	$\dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$
15.	$\sinh n\theta$	$\dfrac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$
16.	$\cosh n\theta$	$\dfrac{z(z-\cosh\theta)}{z^2 - 2z\cosh\theta + 1}$
17.	$a^n \sinh n\theta$	$\dfrac{az\sinh\theta}{z^2 - 2az\cosh\theta + a^2}$
18.	$a^n \cosh n\theta$	$\dfrac{z(z - a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}$
19.	$a^n u_n$	$U(z/a)$
20.	u_{n+1}	$z[U(z) - u_0]$
	u_{n+2}	$z^2[U(z) - u_0 - u_1 z^{-1}]$
	u_{n+3}	$z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$
21.	u_{n-k}	$z^{-k}U(z)$
22.	nu_n	$-z\,d/dz[U(z)]$
23.	u_0	$\underset{z\to\infty}{\text{Lt}}\ U(z)$
24.	$\underset{n\to\infty}{\text{Lt}}\ (u_n)$	$\underset{z\to 1}{\text{Lt}}\ [(z-1)\,U(z)]$

23.11 SOME USEFUL INVERSE Z-TRANSFORMS

Sr. No.	*U(z)*	*Inverse Z-transform* $u_n = z^{-1}[U(z)]$
1.	$\frac{1}{z-a}$	a^{n-1}
2.	$\frac{1}{z+a}$	$(-a)^{n-1}$
3.	$\frac{1}{(z-a)^2}$	$(n-1)a^{n-2}$
4.	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(n-1)(n-2)a^{n-3}$
5.	$\frac{z}{z-a}$	a^n
6.	$\frac{z}{z+a}$	$(-a)^n$
7.	$\frac{z^2}{(z-a)^2}$	$(n+1)a^n$
8.	$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(n+1)(n+2)a^n u(n)$

23.12 CONVOLUTION THEOREM

If $Z^{-1}[U(z)] = u_n$ *and* $Z^{-1}[V(z)] = v_n$, *then*

$$Z^{-1}[U(z) \,.\, V(z)] = \sum_{m=0}^{n} u_m \,.\, v_{n-m} = u_n * v_n$$

where the symbol $*$ *denotes the convolution operation.*

Proof. We have $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, $V(z) = \sum_{n=0}^{\infty} v_n z^{-n}$

$\therefore$ $$U(z)\,V(z) = (u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots + u_n z^{-n} + \dots \infty) \times (v_0 + v_1 z^{-1} + v_2 z^{-2} + \dots + v_n z^{-n} + \dots \infty)$$

$$= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \dots + u_n v_0) z^{-n} = Z(u_0 v_n + u_1 v_{n-1} + \dots + u_n v_0)$$

whence follows the desired result.

Obs. The convolution theorem plays an important role in the solution of difference equations and in probability problems involving sums of two independent random variables.

Example 23.11. *Use convolution theorem to evaluate* $Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$.

Solution. We know that $Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$ and $Z^{-1}\left\{\frac{z}{z-b}\right\} = b^n$

$\therefore$ $$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = Z^{-1}\left\{\frac{z}{z-a} \cdot \frac{z}{z-b}\right\} = a^n * b^n$$

$$= \sum_{m=0}^{n} a^m \,.\, b^{n-m} = b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m \text{ which is a G.P.}$$

$$= b^n \,.\, \frac{(a/b)^{n+1} - 1}{a/b - 1} = \frac{a^{n+1} - b^{n+1}}{a - b}.$$

23.13 CONVERGENCE OF Z-TRANSFORMS

Z-transform operation is performed on a sequence u_n which may exist in the range of integers $-\infty < n < \infty$, and we write

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \qquad ...(1)$$

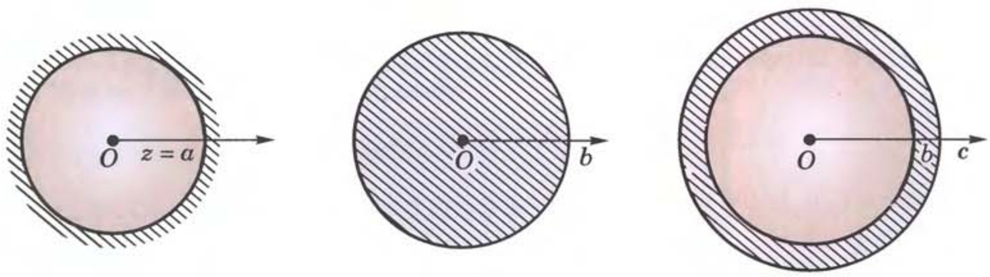
where u_n represents a number in the sequence for n = an integer. *The region of the z-plane in which (1) converges absolutely is known as the region of convergence (ROC) of U(z).*

We have so far discussed *one-sided Z-transform only for which* $n \geq 0$. Here the sequence is always *right-sided* and the *ROC* is always outside a prescribed circle say $|z| > |a|$ Fig. 23.2 (*i*)]. For a left-handed sequence, the ROC is always inside any prescribed contour $|z| < |b|$. [Fig. 23.2 (*ii*)].

23.14 TWO-SIDED Z-TRANSFORM OF u_n IS DEFINED BY

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \qquad ...(2)$$

In this case, the sequence is two-sided and the region of convergence for (2) is the *annular region* $|b| < |z| < |c|$ [Fig. 23.2 (*iii*)]. The inner circle bounds the terms in negative powers of z and the outer circle bounds the terms in positive powers of z. *The shaded annulus of convergence is necessary for the two sided sequence and its Z-transform to exist.*



$|z| > |a|$ (*i*) $\qquad |z| < |b|$ (*ii*) $\qquad |b| < |z| < |c|$ (*iii*)

Fig. 23.1

Example 23.12. *Find the Z-transfrom and region of convergence of*

(*a*) $u(n) = \begin{cases} 4^n & \text{for } n < 0 \\ 2^n & \text{for } n \geq 0 \end{cases}$ $\qquad$ (*b*) $u(n) = {}^nc_k,\ n \geq k.$

Solution. By definition $\quad Z[u(n)] = \sum_{-\infty}^{\infty} u(n) Z^{-n} = \sum_{-\infty}^{-1} 4^n z^{-n} + \sum_{n=0}^{\infty} 2^n z^{-n}$

Putting $-n = m$ in the first series, we get

$$Z[u(n)] = \sum_{1}^{\infty} 4^{-m} z^m + \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$\left\{\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + ...\right\} + \left\{1 + \frac{2}{z} + \frac{2^2}{z^2} + ...\right\}$$

$$= \frac{z}{4}\left\{1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + ...\right\} + \left\{1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + ...\right\} \qquad ...(i)$$

$$= \frac{z}{4} \cdot \frac{1}{1-(z/4)} + \frac{1}{1-(2/z)} = \frac{z}{4-z} + \frac{z}{z-2} = \frac{2z}{(4-z)(z-2)}$$

Now the two series in (*i*) being G.P. will be convergent if $| z/4 | < 1$ and $| 2/z | < 1$ *i.e.*, if $| z | < 4$ and $2 < | z |$. *i.e.* $2 < z < 4$.

Hence $Z[u(n)]$ is convergent if z lies between the annulus as shown shaded in Fig. 23.3. Hence ROC is $2 < z < 4$.

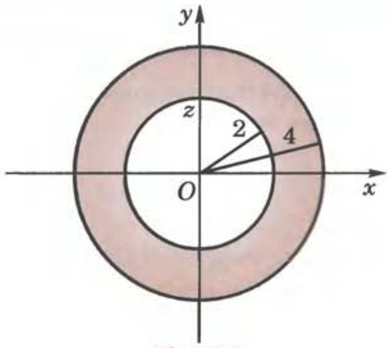


Fig. 23.2

(*b*) By definition, $Z[u(n)] = \sum_{-\infty}^{\infty} {}^nC_k z^{-n} = \sum_{n=k}^{\infty} {}^nC_k 2^n z^{-n}$

To find the sum of this series, we replace n by $k + r$

$$\therefore \quad Z[u(n)] = \sum_{r=0}^{\infty} {}^{k+r}C_k z^{-(k+r)} = z^{-k} \sum_{r=0}^{\infty} {}^{k+r}C_r z^{-r} \qquad [\because \ {}^kC_r = {}^kC_{k-r}]$$

$$= z^{-k}[1 + {}^{k+1}C_1 z^{-1} + {}^{k+1}C_2 z^{-2} + ...]$$

$$= z^{-k}(1 - 1/z)^{-(k+1)}$$

This series is convergence for $| 1/z | < 1$ *i.e.*, for $| z | > 1$.

Hence ROC is $| z | > 1$.

Example 23.13. *Find the Z-transform and the radius of convergence of*

(*a*) $f(n) = 2^n, n < 0$ (*b*) $f(n) = 5^n/n!, n \geq 0$. *(Mumbai, 2009)*

Solution. (*a*) Assuming that $f(n) = 0$ for $n \geq 0$ we have

$$Z[f(n)] = \sum_{-\infty}^{\infty} f(n) z^{-n} = \sum_{n=-\infty}^{-1} 2^n z^{-n} = \sum_{m=1}^{\infty} 2^{-m} z^m \quad \text{where} \quad m = -n$$

$$= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + ...\infty = \frac{z}{2} \{1 + (z/2) + (z/2)^2 + ... \infty\}$$

$$= \frac{z}{2} \cdot \frac{1}{1 - (z/2)} = \frac{z}{2 - z}$$

This series being a G.P. is convergent if $| z/2 | < 1$ *i.e.*, $| z | < 2$.

Hence ROC is $| z | < 2$.

(*b*) By definition, $Z[u(n)] = \sum_{n=0}^{\infty} \frac{5^n}{n!} \cdot z^{-n} = \sum_{0}^{\infty} \frac{(5/z)^n}{n!} = 1 + \left(\frac{5}{z}\right) + \frac{1}{2!}\left(\frac{5}{z}\right)^2 + \frac{1}{3!}\left(\frac{5}{z}\right)^3 + ... \infty$

$= e^{5/z}$

The above series is convergent for all values of z.

Hence ROC is the entire z-plane.

PROBLEMS 23.2

Find the Z-transform and its ROC in each of the following sequences :

1. $u(n) = 4^n, n \geq 0$.
2. $u(n) = 2^n, n < 0$.
3. $u(n) = 4^n$, for $n < 0$ and $= 3^n$ for $n \geq 0$.
4. $u(n) = n5^n, n \geq 0$.
5. $u(n) = 2^n/n, n > 1$.
6. $u(n) = 3^n/n!, n \geq 0$.
7. $u(n) = e^{an}, n \geq 0$.

23.15 EVALUATION OF INVERSE Z-TRANSFORMS

We can obtain the inverse Z-transforms using any of the following three methods :

I. **Power series method.** This is the simplest of all the methods of finding the inverse Z-transform. If $U(z)$ is expressed as the ratio of two polynomials which cannot be factorized, we simply divide the numerator by the denominator and take the inverse Z-transform of each term in the quotient.

Example 23.14. *Find the inverse Z-transform of log (z / z + 1) by power series method.*

Solution. Putting $z = 1/t$, $U(z) = \log\left(\dfrac{1/y}{1/y+1}\right) = -\log(1+y) = -y + \dfrac{1}{2}y^2 - \dfrac{1}{3}y^3 + \ldots$

$$= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + \ldots$$

Thus
$$u_n = \begin{cases} 0 & \text{for } n = 0 \\ (-1)^n/n & \text{otherwise} \end{cases}.$$

Example 23.15. *Find the inverse Z-transform of $z/(z+1)^2$ by division method.*

Solution.
$$U(z) = \frac{z}{z^2+2z+1} = z^{-1} - \frac{2+z^{-1}}{z^2+2z+1}, \text{ by actual division}$$

$$= z^{-1} - 2z^{-2} + \frac{3z^{-1}+2z^{-2}}{z^2+2z+1} = z^{-1} - 2z^{-2} + 3z^{-3} - \frac{4z^{-2}+3z^{-3}}{z^2+2z+1}$$

Continuing this process of division, we get an infinite series *i.e.*,

$$U(z) = \sum_{n=0}^{\infty} (-1)^{n-1}\, nz^{-n}$$

Thus $\quad u_n = (-1)^{n-1}\, n.$

II. **Partial fractions method.** This method is similar to that of finding the inverse Laplace transforms using partial fractions. The method consists of decomposing $U(z)/z$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

Example 23.16. *Find the inverse Z-transforms of*

(*i*) $\dfrac{2z^2+3z}{(z+2)(z-4)}$ (*V.T.U., 2008 S ; S.V.T.U., 2007*) (*ii*) $\dfrac{z^3-20z}{(z-2)^3(z-4)}$. (*V.T.U., 2011*)

Solution. (*i*) We write $U(z) = \dfrac{2z^2+3z}{(z+2)(z-4)}$ as $\dfrac{U(z)}{z} = \dfrac{2z+3}{(z+2)(z-4)} = \dfrac{A}{z+2} + \dfrac{B}{z-4}$

where $A = 1/6$ and $B = 11/6$

$\therefore$
$$U(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

On inversion, we have

$$u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n \qquad \text{[Using § 23.10 (9)]}$$

(*ii*) We write
$$U(z) = \frac{z^3-20z}{(z-2)^3(z-4)}$$

as
$$\frac{U(z)}{z} = \frac{z^2-20}{(z-2)^3(z-4)} = \frac{A+Bz+Cz^2}{(z-2)^3} + \frac{D}{z-4}$$

Readily we get $D = 1/2$. Multiplying throughout by $(z-2)^3(z-4)$, we get

$$z^2 - 20 = (A + Bz + Cz^2)(z-4) + D(z-2)^3.$$

Putting $z = 0, 1, -1$ successively and solving the resulting simultaneous equations, we get $A = 6$, $B = 0$, $C = 1/2$.

Thus
$$U(z) = \frac{1}{2}\cdot\frac{12z+z^3}{(z-2)^3} - \frac{z}{z-4} = \frac{1}{2}\frac{z(z-2)^2+4z^2+8z}{(z-2)^3} - \frac{z}{z-4}$$

$$= \frac{1}{2}\left\{\frac{z}{z-2} + 2\frac{2z^2+4z}{(z-2)^3}\right\} - \frac{z}{z-4}$$

On inversion, we get $u_n = \frac{1}{2}(2^n + 2 \,.\, n^2 2^n) - 4^n$ [Using § 23.10 (9) & (11)]

$$= 2^{n-1} + n^2 2^n - 4^n.$$

Example 23.17. *Find the inverse Z-transform of*

$2(z^2 - 5z + 6.5)/[(z-2)(z-3)^2]$, for $2 < |z| < 3$.

Solution. Splitting into partial fractions, we obtain

$$U(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2}$$ where $A = B = C = 1$

$$\therefore \quad U(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$= \frac{1}{2}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2}$$ so that $2/z < 1$ and $z/3 < 1$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \ldots\right) - \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \ldots\right) + \frac{1}{9}\left(1 + \frac{2z}{3} + \frac{3z^2}{9} + \frac{4z^3}{27} + \ldots\right)$$

where $2 < |z| < 3$.

$$= \left(\frac{1}{2} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \ldots\right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \ldots\right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + \ldots\right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1)\left(\frac{1}{3}\right)^{n+2} z^n$$

On inversion, we get $u_n = 2^{n-1}, n \geq 1$ and $u_n = -(n+2)3^{n-2}, n \leq 0$.

III. **Inversion integral method.** *The inverse Z-transform of $U(z)$ is given by the formula*

$$u_n = \frac{1}{2\pi i}\int_c U(z) z^{n-1}\, dz$$

= sum of residues of $U(z)\, z^{n-1}$ at the poles of $U(z)$ which are inside the contour C drawn according to the ROC given.

The following examples will illustrate the application of this formula :

Example 23.18. *Using the inversion integral method, find the inverse Z-transform of*

$$\frac{z}{(z-1)(z-2)}$$ *(V.T.U., 2010 S)*

Solution. Let $U(z) = \frac{z}{(z-1)(z-2)}$. Its poles are at $z = 1$ and $z = 2$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z) z^{n-1}\, dz,$$

where C is a circle large enough to enclose both the poles of $U(z)$.

= sum of residues of $U(z)\, z^{n-1}$ at $z = 1$ and $z = 2$.

Now $$Res\,[U(z)\, z^{n-1}]_{z=1} = \underset{z \to 1}{\text{Lt}} \left\{(z-1) \,.\, \frac{z^n}{(z-1)(z-2)}\right\} = -1$$

and $$Res\,[U(z)\, z^{n-1}]_{z=2} = \underset{z \to 2}{\text{Lt}} \left\{(z-2) \,.\, \frac{z^n}{(z-1)(z-2)}\right\} = 2^n$$

Thus the required inverse Z-transform $u_n = 2^n - 1, n = 0, 1, 2, \ldots$

Example 23.19. *Find the inverse Z-transform of $2z/[(z-1)(z^2+1)]$.* *(Madras, 2000 S)*

Solution. Let $U(z) = \dfrac{2z}{(z-1)(z+i)(z-i)}$. It has three poles at $z = 1, z = \pm i$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z)\,.\,z^{n-1}\,dz,$$ where C is a circle large enough to enclose the poles of $U(z)$.

$=$ sum of residues of $U(z)\,.\,z^{n-1}$ at $z = 1, z = \pm i$.

Now $$Res\,[U(z)\,z^{n-1}]_{z=1} = \underset{z\to 1}{\text{Lt}}\left\{(z-1)\frac{2z^n}{(z-1)(z^2+1)}\right\} = 1$$

$$Res\,[U(z)\,z^{n-1}]_{z=i} = \underset{z\to i}{\text{Lt}}\left\{(z-i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right\} = \frac{-(i)^n}{1+i}$$

$$Res\,[U(z)\,z^{n-1}]_{z=-i} = \underset{z\to -i}{\text{Lt}}\left\{(z+i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right\} = \frac{(-i)^n}{i-1}$$

Hence $$u_n = 1 - \frac{(i)^n}{1+i} - \frac{(-i)^n}{1-i}.$$

PROBLEMS 23.3

Using convolution theorem, evaluate the inverse Z-transforms of the following :

1. $\dfrac{z^2}{(z-1)(z-3)}$.
2. $\left(\dfrac{z}{z-a}\right)^2$ *(Madras, 2003)*
3. $\left(\dfrac{z}{z-1}\right)^3$.
4. Show that (*a*) $\dfrac{1}{n!} * \dfrac{1}{n!} = \dfrac{2^n}{n!}$ (*b*) $Z^{-1}\left(\dfrac{z^2}{(z+a)(z+b)}\right) = \dfrac{(-1)}{b-a}\,(b^{n+1} - a^{n+1})$. *(Anna, 2009)*

Find the inverse Z-transforms of the following :

5. $\dfrac{4z}{z-a}$, $|\,z\,| > |\,a\,|$. *(Kottayam, 2005)*
6. $\dfrac{5z}{(2-z)(3z-1)}$. *(Madras, 1999)*
7. $\dfrac{z}{(z-1)^2}$.
8. $\dfrac{18z^2}{(2z-1)(4z+1)}$. *(S.V.T.U., 2009)*
9. $\dfrac{8z-z^3}{(4-z)^3}$.
10. $\dfrac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$. *(Anna, 2005 S)*
11. $\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$. *(V.T.U., 2011 S)*
12. $\dfrac{z^3+3z}{(z-1)^2(z^2+1)}$. *(Anna, 2009)*
13. $\dfrac{(1-e^{at})z}{(z-1)(z-e^{-at})}$.
14. Obtain $Z^{-1}\{1/[(z-2)(z-3)]\}$ for (*i*) $|\,z\,| < 2$; (*ii*) $2 < |\,z\,| < 3$; (*iii*) $|\,z\,| > 3$. *(Marathwada, 2008)*
15. Evaluate $Z^{-1}\{(z-5)^{-3}\}$ for $|\,z\,| > 5$. *(Mumbai, 2009)*

Using inversion integral, find the inverse Z-transform of the following functions :

16. $\dfrac{z+3}{(z+1)(z-2)}$.
17. $\dfrac{(2z-1)z}{2(z-1)(z+0.5)}$.
18. $\dfrac{1}{z(z-1)(z+0.5)}$. *(S.V.T.U., 2008)*
19. $\dfrac{z^2+z}{(z-1)(z^2+1)}$. *(Madras, 2003)*
20. $\dfrac{2z(z^2-1)}{(z^2+1)^2}$.

23.16 (1) APPLICATION TO DIFFERENCE EQUATIONS

Just as the Laplace transforms method is quite effective for solving linear differential equations (§ 21.15), the Z-transforms are quite useful for solving linear difference equations.

The performance of discrete systems is expressed by suitable difference equations. Also Z-transform plays an important role in the analysis and representation of discrete-time systems. To determine the frequency response of such systems, the solution of difference equations is required for which Z-transform method proves useful.

(2) Working procedure *to solve a linear difference equation with constant coefficients by Z-transforms :*

1. *Take the Z-transform of both sides of the difference equations using the formulae of § 26.16 and the given conditions.*

2. *Transpose all terms without U(z) to the right.*

3. *Divide by the coefficient of U(z), getting U(z) as a function of z.*

4. *Express this function in terms of the Z-transforms of known functions and take the inverse Z-transform of both sides. This gives u_n as a function of n which is the desired solution.*

Example 23.20. *Using the Z-transform, solve*

$$u_{n+2} + 4u_{n+1} + 3u_n = 3^n \text{ with } u_0 = 0,\ u_1 = 1.$$ *(U.P.T.U., 2003)*

Solution. If $Z(u_n) = U(z)$, then $Z(u_{n+1}) = z[U(z) - u_0]$,

$$Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$$

Also $$Z(2^n) = z/(z-2)$$

$\therefore$ taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] + 4z\,[U(z) - u_0] + 3U(z) = z/(z-3)$$

Using the given conditions, it reduces to

$$U(z)\,(z^2 + 4z + 3) = z + z/(z-3)$$

$$\therefore \quad \frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)} = \frac{3}{8}\frac{1}{z+1} + \frac{1}{24}\frac{1}{z-3} - \frac{5}{12}\frac{1}{z+3},$$

on breaking into partial fractions.

$$U(z) = \frac{3}{8}\frac{z}{z+1} + \frac{1}{24}\frac{z}{z-3} - \frac{5}{12}\frac{z}{z+3}$$

On inversion, we obtain

$$u_n = \frac{3}{8}Z^{-1}\left(\frac{z}{z+1}\right) + \frac{1}{24}Z^{-1}\left(\frac{z}{z-3}\right) - \frac{5}{12}Z^{-1}\left(\frac{z}{z+3}\right) = \frac{3}{8}(-1)^n + \frac{1}{24}3^n - \frac{5}{12}(-3)^n.$$

Example 23.21. *Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$, using Z-transforms.*
(V.T.U., 2011 ; Anna, 2009 ; S.V.T.U., 2009)

Solution. If $Z(y_n) = Y(z)$, then $Z(y_{n+1}) = z\,[(Y(z) - y_0]$, $Z(y_{n+2}) = z^2[Y(z) - y_0 - y_1 z^{-1}]$

Also $Z(2^n) = z/(z-2)$.

Taking Z-transforms of both sides, we get

$$z^2[Y(z) - y_0 - y_1 z^{-1}] + 6z[Y(z) - y_0] + 9Y(z) = z/(z-2)$$

Since $y_0 = 0$, and $y_1 = 0$, we have $Y(z)\,(z^2 + 6z + 9) = z/(z-2)$

or $$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{1}{25}\left[\frac{1}{z-2} - \frac{1}{z+3} - \frac{5}{(z+3)^2}\right],$$ on splitting into partial fractions.

or $$Y(z) = \frac{1}{25}\left[\frac{z}{z-2} - \frac{z}{z+3} - 5\frac{z}{(z+3)^2}\right]$$

On taking inverse Z-transform of both sides, we obtain

$$y_n = \frac{1}{25}\left[Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z+3}\right) + \frac{5}{3}Z^{-1}\left(-\frac{3z}{(z+3)^2}\right)\right]$$

$$= \frac{1}{25}\{2^n - (-3)^n + \frac{5}{3}n(-3)^n\} \qquad \left[\because\ Z^{-1}\left\{\frac{az}{(z-a)^2}\right\} = na^n\right]$$

Example 23.22. *Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$, with $y_0 = 0, y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3, \ldots$ by Z-transform method.* (V.T.U., 2010)

Solution. Taking Z-transform of both sides of the given equation, we get

$$z^2(Y(z) - y_0 - y_1 z^{-1}) - 5z(Y(z) - y_0) + 6Y(z) = \frac{z}{z-1}$$

Substituting the values $y_0 = 0, y_1 = 1$, it reduces to

$$(z^2 - 5z + 6)\, Y(z) = \frac{z}{z-1} + z = \frac{z^2}{z-1}$$

or
$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)}$$

$$= \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3} \qquad \text{where } A = \frac{1}{2}, B = -2, C = \frac{3}{2}$$

so that
$$Y(z) = \frac{1}{2}\frac{z}{z-1} - 2\frac{z}{z-2} + \frac{3}{2}\frac{z}{z-3}$$

On inversion, we obtain $\quad y_n = \frac{1}{2} - 2(2)^n + \frac{3}{2}(3)^n$

Obs. The initial values given in the problem automatically appear in the generated sequence.

Example 23.23. *Solve the difference equation $y_n + \frac{1}{4}y_{n-1} = u_n + \frac{1}{3}u_{n-1}$ where u_n is a unit step sequence.*

Solution. Taking Z-transform of both sides of the given equation, we get

$$Y(z) + \frac{1}{4}z^{-1}\, Y(z) = 1 + \frac{1}{3}z^{-1}$$

or
$$Y(z) = \left(1 + \frac{1}{3}z^{-1}\right)\Big/\left(1 + \frac{1}{4}z^{-1}\right) = \left(z + \frac{1}{3}\right)\Big/\left(z + \frac{1}{4}\right)$$

There being only one simple pole at $z = -1/4$, consider the contour $|z| > 1/4$.

$$\therefore \quad Res\,[Y(z)\, z^{n-1}]_{z=-1/4} = \underset{z \to -1/4}{\text{Lt}} \left\{\left(z + \frac{1}{4}\right) \cdot \left(z + \frac{1}{3}\right) z^{n-1} \Big/ \left(z + \frac{1}{4}\right)\right\}$$

$$= \underset{z \to -1/4}{\text{Lt}} \left(z + \frac{1}{3}\right) z^{n-1} = \left(-\frac{1}{4} + \frac{1}{3}\right)\left(-\frac{1}{4}\right)^{n-1} = \frac{1}{12} \cdot \left(-\frac{1}{4}\right)^{n-1}$$

Hence by inversion integral method, we have

$$y_n = \frac{1}{12}\left(-\frac{1}{4}\right)^{n-1}.$$

Example 23.24. *Using the Z-transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.* (S.V.T.U., 2007)

Solution. Given equation is $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.

Taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] - 2z[U(z) - u_0] + U(z) = 3 \cdot \frac{z}{(z-1)^2} + 5 \cdot \frac{z}{z-1}$$

or
$$U(z)\,(z^2 - 2z + 1) = \frac{5z^2 - 2z}{(z-1)^2} + u_0(z^2 - 2z) + u_1 z$$

or
$$U(z) = \frac{5z^2 - 2z}{(z-1)^4} + u_0\frac{z^2 - 2z}{(z-1)^2} + u_1\frac{z}{(z-1)^2}$$

On inversion, we obtain

$$u_n = Z^{-1}\left\{\frac{5z^2 - 2z}{(z-1)^4}\right\} + u_0 Z^{-1}\left\{\frac{z^2 - 2z}{(z-1)^2}\right\} + u_1 Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} \qquad ...(i)$$

Noting that $Z(1) = \dfrac{z}{z-1}$, $Z(n) = \dfrac{z}{(z-1)^2}$

$$Z(n^2) = \frac{z^2 + z}{(z-1)^3}, \; Z(n^3) = \frac{z^3 + 4z^2 + z}{(z-1)^4}$$

We write $\dfrac{5z^2 - 2z}{(z-1)^4} \equiv A\dfrac{z^3 + 4z^2 + z}{(z-1)^4} + B\dfrac{z^2 + z}{(z-1)^3} + C\dfrac{z}{(z-1)^2} + D\dfrac{z}{z-1}$

Equating coefficients of like powers of z, we find

$$A = \frac{1}{2}, B = 1, C = -\frac{3}{2}, D = 0$$

$$\therefore \qquad Z^{-1}\left\{\frac{5z^2 - 2z}{(z-1)^4}\right\} = \frac{1}{2}n^3 + n^2 - \frac{3}{2}n = \frac{1}{2}n(n-1)(n+3)$$

Also $$Z^{-1}\left\{\frac{z^2 - 2z}{(z-1)^2}\right\} = Z^{-1}\left\{\frac{z}{z-1}\right\} - Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = 1 - n$$

and $$Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = n.$$

Substituting these values in (*i*) above, we get

$$u_n = \frac{1}{2}n(n-1)(n+3) + u_0(1-n) + u_1 n$$

$$= \frac{1}{2}n(n-1)(n+3) + c_0 + c_1 n. \qquad \text{where } c_0 = u_0, c_1 = u_1 - u_0$$

Example 23.25. *Using residue method, solve* $y_k + \dfrac{1}{9}y_{k-2} = \dfrac{1}{3^k}\cos\dfrac{k\pi}{2}, k \geq 0.$

Solution. Taking Z-transform of both sides of the given equation, we get

$$Z\left\{y_k + \frac{1}{9}y_{k-2}\right\} = Z\left\{\frac{1}{3^k}\cos\frac{k\pi}{2}\right\}$$

or $$Y(z) + \frac{1}{9}z^{-2}Y(z) = \frac{z^2}{z^2 + 1/9} \quad \text{or} \quad \left(1 + \frac{1}{9}z^{-2}\right)Y(z) = \frac{z^2}{z^2 + \frac{1}{9}}$$

or $$Y(z) = \frac{z^2}{\left(1 + \frac{1}{9}z^{-2}\right)\left(z^2 + \frac{1}{9}\right)} = \frac{z^4}{\left(z^2 + \frac{1}{9}\right)^2}$$

There are two poles of second order at $z = i/3$ and $z = -i/3$.

$$\therefore \qquad \text{Residue at } (z = i/3) = \left[\frac{d}{dz}\left\{\left(\frac{z-i}{3}\right)^2 \frac{z^{k-1}z^4}{(z^2 + 1/9)^2}\right\}\right]$$

$$= \left[\frac{d}{dz}\left\{\frac{z^{k+3}}{(z + i/3)^2}\right\}\right]_{z=i/3} = \left[\frac{(z + i/3)^2 (k+3)z^{k+2} - z^{(k+3)} \cdot 2(z + i/3)}{(z + i/3)^4}\right]_{z=i/3}$$

$$= \left[\frac{(z + i/3)(k+3)z^{k+2} - 2z^{k+3}}{(z + i/3)^3}\right]_{z=i/3} = \left(\frac{3}{2i}\right)^3\left[(2k+6)\left(\frac{i}{3}\right)^{k+3} - 2\left(\frac{i}{3}\right)^{k+3}\right]$$

$$= \frac{1}{8}(2k+4)\left(\frac{i}{3}\right)^k = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^k = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}\right) \quad ...(i)$$

Changing i to $-i$ in (i), we have

Residue at $(z = -i/3)$ $= \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{k\pi}{2} - i\sin\frac{k\pi}{2}\right) \quad ...(ii)$

Adding (i) and (ii), we obtain $y_k = \frac{1}{2}(k+2)\left(\frac{1}{3}\right)^k \cos\frac{k\pi}{2}$.

PROBLEMS 23.4

Solve the following difference equations using Z-transforms (1 – 8) :

1. $6y_{k+2} - y_{k+1} - y_k = 0$, given that $y(0) = y(1) = 1$. *(Kottayam, 2005)*
2. $y(n+2) + 2y(n+1) + y(n) = 0$, given that $y(0) = y(1) = 0$. *(V.T.U., 2008 S)*
3. $y_{n+2} - 4y_n = 0$ given that $y_0 = 0, y_1 = 2$. *(U.P.T.U., 2008)*
4. $f(n) + 3f(n-1) - 4f(n-2) = 0, n \geq 2$, given that $f(0) = 3, f(1) = -2$. *(Madras, 2003 S)*
5. $y_{(n+3)} - 3y(n+1) + 2y(n) = 0$, given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$. *(Anna, 2005 S)*
6. $y_{n+2} - 5y_{n+1} + 6y_n = 36$, given that $y(0) = y(1) = 0$. *(Anna, 2009)*
7. $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$.
8. $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.
9. $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n \quad (n \geq 0), y_0 = 0$.
10. $u_{x+2} + u_x = 5(2^x)$ given that $u_0 = 1, u_1 = 0$. *(Marathwada, 2008)*
11. $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0, y_1 = 1$. *(Madras, 2006)*
12. $u_{k+2} - 2u_{k+1} + u_k = 2^k$ with $y_0 = 2, y_1 = 1$.
13. $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$.
14. $y_k + \frac{1}{25}y_{k-2} = \left(\frac{1}{5}\right)^k \cos\frac{k\pi}{2}, \quad (k \geq 0)$.
15. Find the response of the system given by $y_n + 3y_{(n-1)} = u_n$ where u_n is a unit step sequence and $y_{(-1)} = 1$.
16. Find the impulse response of a system described by $y_{(n+1)} + 2y_{(n)} = \delta_n$; $y_0 = 0$.

23.1 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 23.5

Choose the correct answer or fill up the blanks in each of the following problems :

1. $Z(1) =$ ......
2. If u_n is defined for $n = 0, 1, 2,$ ...... only, then $Z(u_n) =$ ....
3. Z-transform of $n =$ .... *(Anna, 2009)*
4. $Z(na^n) =$ ......
5. $Z(\sin n\theta) =$ ......
6. Z-transform of $(1/n!)$ is
7. $Z(n^2) =$ ......
8. Linear property of Z-transform states that...
9. $Z^{-1}\left(\frac{1}{z-2}\right) =$ ....
10. $Z^{-1}\left\{\frac{z}{(z+1)^2}\right\} =$ ......
11. Initial value theorem on Z-transform states that ....
12. Z-transform is linear. (True or False)
13. If $Z(u_n) = u(z)$, then $\underset{n\to\infty}{\text{Lt}}(u_n) = \underset{z\to\infty}{\text{Lt}}(z-1)\,u(z)$. (True or False)
14. Z-transform of the sequence $\{2^k\}, k \geq 0$ is $z/(z-2)$. (True or False)
15. Z-transform of $\{a^k/k!\}, k \geq 0 = e^{a/z}$. (True or False)
16. Z-transform of $\{{}^nC_r\}, (0 \leq r \leq n)$ is $(1+z)^n$. (True or False)
17. Z-transform of unit impulse sequence $\delta(n) = \begin{cases} 1, & n < 0 \\ 0, & n \geq 0 \end{cases}$, is $z/z - 1$. (True or False)
18. Z-transform of unit step sequence $u(n) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0, \end{cases}$ is 1. (True or False)

APPENDIX 1

Useful Information

I. BASIC DATA

1. Basic Constants

$e = 2.7183$	$1/e = 0.3679$	$\log_e 2 = 0.6931$	$\log_e 3 = 1.0986$
$\pi = 3.1416$	$1/\pi = 0.3183$	$\log_e 10 = 2.3026$	$\log_{10} e = 0.4343$
$\sqrt{2} = 1.4142$	$\sqrt{3} = 1.732$	1 rad. = 57° 17′ 45″	1° = 0.0174 rad.

2. Conversion Factors

1 ft. = 30.48 cm = 0.3048 m	1 m = 100 cm = 3.2804 ft.
1 ft^2 = 0.0929 m^2	1 acre = 4840 yd^2 = 4046.77 m^2
1 ft^3 = 0.0283 m^3	1 m^3 = 35.32 ft^3
1 m/sec = 3.2804 ft/sec.	1 mile/h = 1.609 km/h.

3. Systems of Units

Quantity	*F.P.S. system*	*C.G.S. system*	*M.K.S. system*
Length	foot (ft)	centrimetre (cm)	metre (m)
Mass	pound (lb)	gram (gm)	kilogram (kg)
Time	second (sec)	second (sec)	second (sec)
Force	lb. wt.	dyne	newton (nt)

Note. The M.K.S. system is also known as the *International system of units (SI system).*

4. Greek Letters Used

α	alpha	θ	theta	κ	kappa	τ	tau
β	beta	ϕ	phi	μ	mu	χ	chi
γ	gamma	ψ	psi	ν	nu	ω	omega
δ	delta	ξ	xi	π	pi	Γ	cap. gamma
ε	epsilon	η	eta	ρ	rho	Δ	cap. delta
ι	iota	ζ	zeta	σ	sigma	Σ	cap. sigma
		λ	lambda				

5. Some Notations

$\in$	belongs to	$\cup$	union
$\notin$	does not belong to	$\cap$	intersection
$\Rightarrow$	implies	$\ni$	such that
$\Leftrightarrow$	implies & implied by		

Factorial n i.e., $n! = n(n-1)(n-2) \ldots\ldots 3 . 2 . 1.$

Double factorials : $(2n)!! = 2n(2n-2)(2n-4) \ldots\ldots 6 . 4 . 2.$

$(2n-1)!! = (2n-1)(2n-3)(2n-5) \ldots\ldots 5 . 3 . 1.$

Stirling's approximation. When n is large $n! \sim \sqrt{2\pi n} \cdot n^n e^{-n}$.

II. ALGEBRA

1. Quadratic equation : $ax^2 + bx + c = 0$ has roots

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac)}}{2a}, \beta = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}.$$

Roots are equal if $b^2 - 4ac = 0$

Roots are real and distinct if $b^2 - 4ac > 0$

Roots are imaginary if $b^2 - 4ac < 0$

2. Cubic equation : $x^3 + lx^2 + mx + n = 0$

Cardan's method :

(*i*) Remove x^2 term by putting $y = x - (-l/3)$

(*ii*) Equate coeffs. in the new equation and $y^3 - 3uvy - (u^3 + v^3) = 0$ [$\because \; y = u + v$

(*iii*) Find u^3 and v^3. Then find u and v.

(*iv*) Get $y = u + v$ and $x = y - l/3$.

3. Biquadratic equation : $x^4 + kx^3 + lx^2 + mx + n = 0$

I. *Ferrari's method :*

(*i*) Combine x^4 and x^3 terms into a perfect square by adding a term in λ.

(*ii*) Make R.H.S. a perfect square to find λ.

(*iii*) Solve resulting quadratic equations.

II. *Descarte's method :*

(*i*) Remove x^3 term by putting $y = x - (-k/4)$

(*ii*) Equate transformed expression to $(y^2 + py + q)(y^2 - py + q')$

(*iii*) Equate coeffs. of like powers from both sides.

(*iv*) Find p, q and q' and solve resulting quadratics.

4. Cross-multiplication :

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$

Then
$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

5. Method of least squares :

(*i*) *Straight line of best fit* $y = a + bx$.

Normal equations : $\Sigma y = na + b\Sigma x$, $\Sigma xy = a\Sigma x + b\Sigma x^2$.

To find a, b, solve these equations.

(*ii*) *Parabola of best fit* $y = a + bx + cx^2$

Normal equations : $\Sigma y = na + b\Sigma x + c\Sigma x^2$,

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3, \Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

To find a, b, c, solve these equations.

6. Progressions :

(*i*) Numbers $a, a + d, a + 2d, \ldots$ are said to be in *Arithmetic progression (A.P.)*

Its nth term $T_n = a + \overline{n-1}\, d$ and *sum* $S_n = \frac{n}{2}(2a + \overline{n-1}\, d)$

(*ii*) Numbers $a, ar, ar^2, \ldots\ldots$, are said to be in *Geometric progression (G.P.)*

Its nth term $T_n = ar^{n-1}$ and *sum* $S_n = \frac{a(1-r^n)}{1-r}$, $S_\infty = \frac{a}{1-r}$ $(r < 1)$

(*iii*) Numbers $1/a$, $1/(a + d)$, $1/(a + 2d)$, ... are said to be in *Harmonic progression (H.P.)* (*i.e.*, a sequence is said to be in H.P. if its reciprocals are in A.P.)

Its *n*th term $T_n = 1/(a + \overline{n-1}\, d)$

(*iv*) If a and b be two numbers then their

Arithmetic mean $= \frac{1}{2}(a + b)$, *Geometric mean* $= \sqrt{ab}$, *Harmonic mean* $= 2ab/(a + b)$

(*v*) *Natural numbers* are 1, 2, 3, ..., n.

$$\Sigma n = \frac{n(n+1)}{2},\ \Sigma n^2 = \frac{n(n+1)(2n+1)}{6},\ \Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

7. Permutations and Combinations

$${}^nP_r = \frac{n!}{(n-r)!};\ {}^nC_r = \frac{n!}{r!(n-r)!} = \frac{{}^nP_r}{r!};\ {}^nC_{n-r} = {}^nC_r;\quad {}^nC_0 = 1 = {}^nC_n.$$

8. Binomial theorem

(*i*) *When n is a positive integer*

$$(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots\ldots + {}^nC_n x^n.$$

(*ii*) *When n is a negative integer or a fraction*

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots\ldots \infty.$$

(*iii*) *Binomial coefficients* : ${}^nC_r = \frac{n!}{r!(n-r)!}$

9. Logarithms

(*i*) *Naturals logarithm* $\log x$ has base e and is inverse of e^x.

Common logarithm $\log_{10} x = M \log x$ where $M = \log_{10} e = 0.4343$

(*ii*) $\log_a 1 = 0$; $\log_a 0 = -\infty\ (a > 1)$; $\log_a a = 1$.

(*iii*) $\log(mn) = \log m + \log n$; $\log(m/n) = \log m - \log n$; $\log(m^n) = n \log m$.

10. Partial Fractions

A fraction of the form $\frac{a_0x^m + a_1x^{m-1} + \ldots\ldots + a_m}{b_0x^n + b_1x^{n-1} + \ldots\ldots + b_n}$

in which m and n are positive integers, is called a *rational algebraic fraction*. When the numerator is of a lower degree than the denominator, it is called a *proper fraction*.

To resolve a given fraction into partial fractions, we first factorise the denominator into real factors. These will be either linear or quadratic, and some factors repeated. Then the proper fraction is resolved into a sum of partial fractions such that

(*i*) to a *non-repeated linear factor* $x - a$ in the denominator corresponds a partial fraction of the form $A/(x - a)$;

(*ii*) to a *repeated linear factor* $(x - a)^r$ in the denominator corresponds the sum of r partial fractions of the form $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \ldots + \frac{A_r}{(x-a)^r}$;

(*iii*) to a *non-repeated quadratic factor* $(x^2 + ax + b)$ in the denominator, corresponds a partial fraction of the form $\frac{Ax + B}{x^2 + ax + b}$;

(*iv*) to a *repeated quadratic factor* $(x^2 + ax + b)^r$ in the denominator, corresponds the sum of r partial fractions of the form $\frac{A_1x + B_1}{x^2 + ax + b} + \frac{A_2x + B_2}{(x^2 + ax + b)^2} + \ldots\ldots + \frac{A_rx + B_r}{(x^2 + ax + b)^r}$.

Then we have to determine the unknown constants A, A_1, B_1 etc.

To obtain the partial fraction corresponding to the non-repeated linear factor $x - a$ in the denominator, put $x = a$ everywhere in the given fraction except in the fac tor $x - a$ itself.

In all other cases, equate the given fraction to a sum of suitable partial fractions in accordance with (*i*) to (*iv*) above, having found the partial fractions corresponding to the non-repeated linear factors by the above rule. Then multiply both sides by the denominator of the given fraction and equate the coefficients of like powers of x or substitute convenient numerical values of x on both sides. Finally solve the simplest of the resulting equations to find the unknown constants.

11. Matrices

(*i*) *A system of mn numbers arranged in a rectangular array of m rows and n columns is called a* **matrix** *of order* $m \times n$.

In particular if $m = n$, it is called a *square matrix of order n*.

(*ii*) *Two matrices of the same order can be added or subtracted* by adding or subtracting the corresponding elements.

(*iii*) *Product of a matrix A by a scalar k* is a matrix whose each element is k times the corresponding elements of A.

(*iv*) *Two matrices can be multiplied* only when the number of columns in the first is equal to the number of rows in the second. If A is of order $m \times n$ and B is of order $n \times p$, then the product AB is a matrix of order $m \times p$, obtained by multiplying and adding the row elements of A with the corresponding column elements of B.

(*v*) *Transpose of a matrix A* is the matrix obtained by interchanging its rows and columns and is denoted by A'.

A square matrix A is said to be *symmetric* if $A = A'$ and *skew symmetric* if $A = -A'$.

(*vi*) If A and B are two square matrices such that $AB = I$ (*i.e.*, a unit matrix), then B is called the *inverse* of A and is denoted by A^{-1}. Then $AA^{-1} = A^{-1}A = I$.

(*vii*) *Rank of a matrix* is the largest order of any non-vanishing minor of the matrix.

(*viii*) *Consistency of a system of equations in n unknowns.*

If the rank of the coefficient matrix A be r and that of the augmented matrix K be r', then

(*a*) the equations are inconsistent (*i.e.* there is no solution) when $r \neq r'$,

(*b*) the equations are consistent when $r = r'$.

(*c*) the equations are consistent and there are infinite number of solutions when $r = r' < n$.

(*ix*) *Eigen values:* If A is any square matrix of order n, then the determinant of the matrix $A - \lambda I_n$ equated to zero is called the *Characterist equation* of A and its roots are called the *eigen values of A*.

(*x*) *Cayley Hamilton theorem*: Every square matrix satisfies its own characteristic equation.

12. Determinants

(*i*) A **determinant** is defined for a square matrix A and is denoted by $|A|$. Unlike a matrix it has a single value *e.g.*,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$$

In this way, determinant can be expanded in terms of any row or column.

(*ii*) **Properties :**

I. A determinant remains unaltered if its rows and columns are interchanged.

II. A determinant vanishes if two of its rows (or columns) are identical or proportional.

III. If each elements of a row (or column) consists of m terms, the determinant can be expressed as the sum of m determinants.

IV. If to each elements of a row (or column) be added equi-multiples of the corresponding elements of two or more rows (or columns), the determinant remains unaltered.

V. If the elements of a determinant Δ are functions of x and two parallel lines become identical when $x = a$, then $x - a$ is a factor of Δ.

III. GEOMETRY

1. Coordinates of a point : *Cartesian* (x, y) *and polar* (r, θ).

$$x = r\cos\theta, \qquad y = r\sin\theta$$

or $\quad r = \sqrt{(x^2 + y^2)}, \qquad \theta = \tan^{-1}(y/x)$. (Fig. 0.1).

Distance between two points (x_1, y_1) and $(x_2, y_2) = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$

Point of division of the line joining (x_1, y_1) and (x_2, y_2) in the ratio $m_1 : m_2$ is

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

In a triangle having vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3)

(*i*) *Area* $= \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

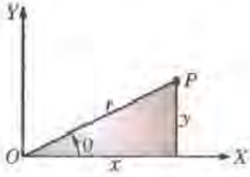


Fig. AP-1.1

(*ii*) *Centroid* (point of intersection of medians) is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

(*iii*) *Incentre* (point of intersection of the internal bisectors of the angles) is

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

where a, b, c are the lengths of the sides of the triangle.

(*iv*) *Circumcentre* is the point of intersection of the right bisectors of the sides of the triangle.

(*v*) *Orthocentre* is the point of intersection of the perpendiculars drawn from the vertices to the opposite sides of the triangle.

2. Straight Line

(*i*) *Slope of the line* joining the points (x_1, y_1) and $(x_2, y_2) = \dfrac{y_2 - y_1}{x_2 - x_1}$

Slope of the line $ax + by + c = 0$ is $-\dfrac{a}{b}$ *i.e.*, $-\dfrac{\text{coeff. of } x}{\text{coeff. of } y}$

(*ii*) *Equation of a line*

(*a*) having slope m and cutting an intercept c on y-axis is $y = mx + c$.

(*b*) cutting intercepts a and b from the axes is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(*c*) passing through (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$

(*d*) passing through (x_1, y_1) and making an $\angle\theta$ with the x-axis is $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$

(*e*) through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$

(*iii*) *Angle between two lines* having slopes m_1 and m_2 is $\tan^{-1}\dfrac{m_1 - m_2}{1 + m_1m_2}$

Two lines are parallel if $\quad m_1 = m_2$

Two lines are perpendicular if $\quad m_1m_2 = -1$

Any line parallel to the line $\quad ax + by + c = 0 \quad$ is $\quad ax + by + k = 0$

Any line perpendicular to $\quad ax + by + c = 0 \quad$ is $\quad bx - ay + k = 0$

(*iv*) *Length of the perpendicular* from (x_1, y_1) to the line $ax + by + c = 0$, is $\dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$.

3. Circle

(*i*) *Equation of the circle* having centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$

(*ii*) *Equation* $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle having centre $(-g, -f)$ and radius $= \sqrt{(g^2 + f^2 - c)}$.

(iii) *Equation of the tangent* at the point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(iv) *Condition for the line* $y = mx + c$ *to touch the circle* $x^2 + y^2 = a^2$ is $c = a\sqrt{(1 + m^2)}$.

(v) *Length of the tangent* from the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)}$.

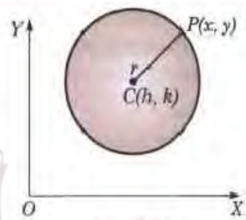


Fig. AP-1.2

4. Parabola

(i) *Standard equation of the parabola* $y^2 = 4ax$.

Its parametric equations are $x = at^2, y = 2at$.

Latus-rectum $LL' = 4a$, *Focus* is $S(a, 0)$

Directrix ZM is $x + a = 0$.

Focal distance of any point $P(x_1, y_1)$ on the parabola

$$y^2 = 4ax \text{ is } SP = x_1 + a$$

Equation of the tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1)$$

Condition for the line $y = mx + c$ *to touch the parabola*

$$y^2 = 4ax \text{ is } c = a/m.$$

Equation of the normal to the parabola $y^2 = 4ax$ in terms of its slope m is

$$y = mx - 2am - am^3.$$

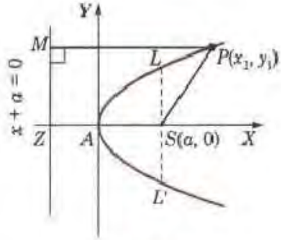
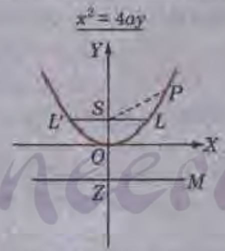
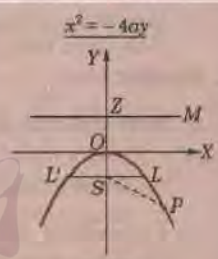


Fig. AP-1.3

(ii) *Other standard forms of parabola*

$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
		
Latus rectum : $LL' = 4a$	$LL' = 4a$	$LL' = 4a$
Focus : $S(-a, 0)$	$S(0, a)$	$S(0, -a)$
Directrix $ZM : x = a$	$ZM : y = -a$	$ZM : y = a$
Focal distance : $SP = a - x$	$SP = a + y$	$SP = a - y$

5. Ellipse

(i) *Standard equation of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \ (a > b > 0)$.

It parametric equations are $x = a\cos\theta, y = b\sin\theta$.

Eccentricity $e = \sqrt{(1 - b^2/a^2)}$,

Latus-rectum $LSL' = 2b^2/a$.

Foci $S(-ae, 0)$ and $S'(ae, 0)$

Directrices ZM $(x = -a/e)$ and $Z'M'$ $(x = a/e)$.

Sum of the focal distances of any point on the ellipse is equal to the major axis i.e.,

$$SP + S'P = 2a.$$

Equation of the tangent at the point (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

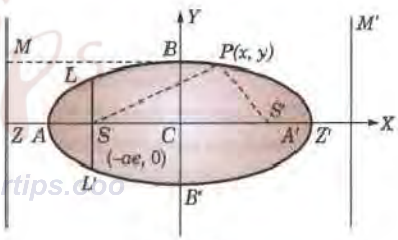


Fig. AP-1.4

Condition for the line $y = mx + c$ *to touch the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c = \sqrt{(a^2 m^2 + b^2)}$

(*ii*) *Another standard form of ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b > 0)$

Vertices: $A\ (0o, a)$; $A'\ (0, -a)$

Foci : $S\ (0, ae)$; $S'\ (0, -ae)$

Directrices : $ZM : y = a/e$, $Z'M' : y = -a/e$

Latus rectum : $LSL' = 2b^2/a$.

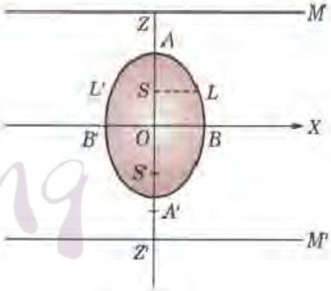


Fig. AP-1.4 (a)

6. Hyperbola

(*i*) *Standard equations of the hyperbola is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Its parametric equations are

$x = a \sec\theta, \quad y = b \tan\theta.$

Eccentricity $e = \sqrt{(1 + b^2/a^2)}$.

Latus-rectum $LSL' = 2b^2/a$.

Directrices

$ZM\ (x = a/e)$ and $Z'M'\ (x = -a/e)$.

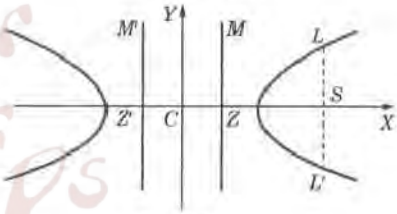


Fig. AP-1.5

(*ii*) *Equation of the tangent at the point* (x_1, y_1) to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(*iii*) *Condition for the line* $y = mx + c$ *to touch the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } c = \sqrt{(a^2 m^2 - b^2)}$$

(*iv*) *Asymptotes of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$.

(*v*) *Equation of the rectangular hyperbola with asymptotes as axes is* $xy = c^2$.

Its parametric equations are $x = ct$, $y = c/t$.

7. Nature of a conic

The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *represents*

(*i*) *a pair of lines, if* $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ $(= \Delta) = 0$

(*ii*) *a circle, if* $a = b$, $h = 0$, $\Delta \neq 0$

(*iii*) *a parabola, if* $ab - h^2 = 0$, $\Delta \neq 0$.

(*iv*) *an ellipse, if* $ab - h^2 > 0$, $\Delta \neq 0$.

(*v*) *a hyperbola, if* $ab - h^2 < 0$, $\Delta \neq 0$,

and a rectangular hyperbola if in addition, $a + b = 0$.

IV. SOLID GEOMETRY

1. (*i*) If l, m, n be the *direction cosines* of a line then $l^2 + m^2 + n^2 = 1$.

If a, b, c be the *direction ratios* of a line then $l = \frac{a}{\sqrt{\Sigma a^2}}$; $m = \frac{b}{\sqrt{\Sigma a^2}}$; $n = \frac{c}{\sqrt{\Sigma a^2}}$

(*ii*) If θ be the *angle between the lines having* d.c.'s l, m, n and l', m', n', then

$$\cos\theta = ll' + mm' + nn'$$

Lines are perpendicular if, $ll' + mm' + nn' = 0$

Lines are parallel if $l = l'$, $m = m'$, $n = n'$

(iii) *Projection of the line joining the points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *on a line having d.c.'s* l, m, n = $l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2)$.

2. Plane

(i) *Different forms of equation of a plane*

— *General form :* $ax + by + cz = d$

where a, b, c are the d.r.s of a normal to the plane.

— *Normal form :* $lx + my + nz = p$

— *Intercept form :* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

— *Any plane passing through the point* (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

(ii) *Angle* θ *between the planes* $ax + by + cz = d$ and $a'x + b'y + c'z = d'$ is given by

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(a'^2 + b'^2 + c'^2)}}$$

Planes are perpendicular if $aa' + bb' + cc' = 0$

Planes are parallel if $a/a' = b/b' = c/c'$

(iii) *Any plane parallel to the plane* $ax + by + cz = d$ is $ax + by + cz = k$.

3. Straight line

(i) *Equation of the line* through the point (x_1, y_1, z_1) having d.r.s a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \textit{(Symmetrical form)}$$

(ii) *Equation of the line* through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad \textit{(Two point form)}$$

(iii) *Angle* θ *between the plane* $ax + by + cz = d$ *and the line*

$$\frac{x - x_1}{a'} = \frac{y - y_1}{b'} = \frac{z - z_1}{c'}$$

is
$$\sin\theta = \frac{aa' + bb' + cc'}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(a'^2 + b'^2 + c'^2)}}$$

Line is parallel to the plane if $aa' + bb' + cc' = 0$

Line is perpendicular to the plane if $a/a' = b/b' = c/c'$

(iv) *Coplanar lines*

Two lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$

are coplanar if
$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and equation of the plane containing these lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

(v) *Shortest distance* between two skew lines

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

is $\qquad l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

where l, m, n are given by $ll_1 + mm_1 + nn_1 = 0$ and $ll_2 + mm_2 + nn_2 = 0$

Equation of the line of S.D. is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$$

4. Sphere

(*i*) *Equation of the sphere* having centre (a, b, c) and radius r is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

(*ii*) Equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere having *centre* $(-u, -v, -w)$ and *radius* $= \sqrt{(u^2 + v^2 + w^2 - d)}$

(*iii*) *Equation of the sphere* having the points (x_1, y_1, z_1) and (x_2, y_2, z_2) as the ends of a diameter is $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$

(*iv*) *Equation of a circle* (*i.e.*, section of a sphere $S = 0$ by the plane $U = 0$) is given by $S = 0$ and $U = 0$ taken together.

(*v*) *Equation of any sphere* through the circle of intersection of the sphere $S = 0$ and the plane $U = 0$ is $S + kU = 0$.

(*vi*) *Tangent plane* at any point (x_1, y_1, z_1) of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ is}$$

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

(*vii*) Two spheres $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and

$x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ *cut orthogonally if* $2uu' + 2vv' + 2ww' = d + d'$.

5. Cone

(*i*) *Equation of a cone* with vertex at the origin is a homogeneous equation of the second degree in x, y, z.

(*ii*) *Enveloping cone* of the sphere $S = 0$ with vertex (x_1, y_1, z_1) is $SS_1 = T^2$ where $S = x^2 + y^2 + z^2 - a^2$, $S_1 = x_1^2 + y_1^2 + z_1^2 - a^2$, $T = xx_1 + yy_1 + zz_1 - a^2$

6. Quadric surfaces

(*i*) *Ellipoid:* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

(*ii*) *Hyperboloid* of one sheet: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Hyperboloid of two sheets: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$

(*iii*) *Cone:* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$

(*iv*) *Elliptic paraboloid:* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2z}{c}$

Hyperbolic paraboloid: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{2z}{c}$

7. Volumes and surface areas

Solid	*Volume*	*Curved surface area*	*Total surface area*
Cube (side a)	a^3	$4a^2$	$6a^2$
Cuboid (length l, breadth b, height h)	lbh	$2(l + b)h$	$2(lb + bh + hl)$
Sphere (radius r)	$\frac{4}{3}\pi r^3$	—	$4\pi r^2$
Cylinder (base radius r, height h)	$\pi r^2 h$	$2\pi rh$	$2\pi r(r + h)$
Cone (base radius r, height h)	$\frac{1}{3}\pi r^2 h$	πrl	$\pi r(r + l)$
where slant height l is given by $l = \sqrt{(r^2 + h^2)}$.			

V. TRIGONOMETRY

1.

$\theta° =$	0	30	45	60	90	180	270	360
sin θ	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	– 1	0
cos θ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	– 1	0	1
tan θ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	– ∞	0

2. Signs and variations of t-ratios

Quadrant	*sin θ*	*cos θ*	*tan θ*
I	+ (0 to 1)	+ (1 to 0)	+ (0 to ∞)
II	+ (1 to 0)	– (0 to – 1)	– (– ∞ to 0)
III	– (0 to – 1)	– (– 1 to 0)	+ (0 to ∞)
IV	– (– 1 to 0)	+ (0 to 1)	– (– ∞ to 0)

3. *Any t-ratio of* $(n\,.\,90° \pm \theta) = \pm$ *same ratio of* θ, *when n is even.*

$= \pm$ *co-ratio of* θ, *when n is odd.*

The sign + or – is to be decided from the quadrant in which $n\,.\,90° \pm \theta$ *lies.*

e.g., $\sin 570° = \sin(6 \times 90° + 30°) = -\sin 30° = -\dfrac{1}{2}$,

$\tan 315° = \tan(3 \times 90° + 45°) = -\cot 45° = -1$

4. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\sin 2A = 2 \sin A \cos A = 2 \tan A/(1 + \tan^2 A)$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1 = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B};\ \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

5. $\sin A \cos B = \dfrac{1}{2}[\sin(A + B) + \sin(A - B)]$

$\cos A \sin B = \dfrac{1}{2}[\sin(A + B) - \sin(A - B)]$

$\cos A \cos B = \dfrac{1}{2}[\cos(A + B) + \cos(A - B)]$

$\sin A \sin B = \dfrac{1}{2}[\cos(A - B) - \cos(A + B)]$

6. $\sin C + \sin D = 2 \sin \dfrac{C + D}{2} \cos \dfrac{C - D}{2}$

$\sin C - \sin D = 2 \cos \dfrac{C + D}{2} \sin \dfrac{C - D}{2}$

$\cos C + \cos D = 2 \cos \dfrac{C + D}{2} \cos \dfrac{C - D}{2}$

$\cos C - \cos D = -2 \sin \dfrac{C + D}{2} \sin \dfrac{C - D}{2}$

7. $\sin 3A = 3\sin A - 4\sin^3 A$, $\cos 3A = 4\cos^3 A - 3\cos A$; $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

8. $$a\sin x + b\cos x = r\sin(x+\theta)$$
$$a\cos x + b\sin x = r\cos(x-\theta)$$
where $a = r\cos\theta$, $b = r\sin\theta$ so that $r = \sqrt{(a^2+b^2)}$, $\theta = \tan^{-1}(b/a)$

9. *In any* ΔABC :

(*i*) $a/\sin A = b/\sin B = c/\sin C$ (*Sine formula*)

(*ii*) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ (*Cosine formula*)

(*iii*) $a = b\cos C + c\cos B$ (*Projection formula*)

(*iv*) *Area of* $\Delta ABC = \dfrac{1}{2}bc\sin A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{1}{2}(a+b+c)$.

10. **Series**

(*i*) *Exponential series* : $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots \infty$

(*ii*) *Sin, cos, sinh, cosh series*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots \infty, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots \infty$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots \infty, \qquad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots \infty$$

(*iii*) *Log series*

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots \infty, \qquad \log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots \infty\right)$$

(*iv*) *Gregory series*

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots \infty, \qquad \tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots \infty$$

11. (*i*) **Complex number :** $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ [see Fig. AP-1.1]

(*ii*) **Euler's theorem :** $\cos\theta + i\sin\theta = e^{i\theta}$

(*iii*) **Demoivre's theorem :** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

12. (*i*) **Hyperbolic functions :** (*i*) $\sinh x = \dfrac{e^x - e^{-x}}{2}$; $\cosh x = \dfrac{e^x + e^{-x}}{2}$;

$$\tanh x = \frac{\sinh x}{\cosh x};\ \coth x = \frac{\cosh x}{\sinh x};\ \text{sech}\, x = \frac{1}{\cosh x};\ \text{cosech}\, x = \frac{1}{\sinh x}$$

(*ii*) *Relations between hyperbolic and circular functions :*

$$\sin ix = i\sinh x;\ \cos ix = \cosh x;\ \tan ix = i\tanh x.$$

(*iii*) *Inverse hyperbolic functions:*

$$\sinh^{-1} x = \log[x + \sqrt{x^2+1}];\ \cosh^{-1} x = \log[x + \sqrt{(x^2-1)}];\ \tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}.$$

VI. DIFFERENTIAL CALCULUS

1. **Standard limits :**

(*i*) $\underset{x\to a}{Lt}\dfrac{x^n - a^n}{x-a} = na^{n-1}$, n any rational number (*ii*) $\underset{x\to 0}{Lt}\dfrac{\sin x}{x} = 1$

(*iii*) $\underset{x\to 0}{Lt}(1+x)^{1/x} = e$ (*iv*) $\underset{x\to\infty}{Lt}x^{1/x} = 1$

(*v*) $\underset{x\to 0}{Lt}\dfrac{a^x - 1}{x} = \log_e a$

2. Differentiation

(i) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ $\quad$ $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du/dx - u\,dv/dx}{v^2}$

$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$ *(Chain Rule)* $\quad$ $\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1} \cdot a$

(ii) $\frac{d}{dx}(e^x) = e^x$ $\quad$ $\frac{d}{dx}(a^x) = a^x \log_e a$

$\frac{d}{dx}(\log_e x) = 1/x$ $\quad$ $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

(iii) $\frac{d}{dx}(\sin x) = \cos x$ $\quad$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad$ $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\quad$ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(iv) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}$ $\quad$ $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\quad$ $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{(x^2-1)}}$ $\quad$ $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{(x^2-1)}}$

(v) $\frac{d}{dx}(\sinh x) = \cosh x$ $\quad$ $\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ $\quad$ $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

(vi) $D^n (ax+b)^m = m(m-1)(m-2) \ldots\ldots (m-n+1)(ax+b)^{m-n} \cdot a^n$

$D^n \log(ax+b) = (-1)^{n-1}(n-1)!\, a^n/(ax+b)^n$

$D^n (e^{mx}) = m^n e^{mx}$ $\quad$ $D^n (a^{mx}) = m^n (\log a)^n \cdot a^{mx}$

$$D^n \begin{bmatrix} \sin(ax+b) \\ \cos(ax+b) \end{bmatrix} = a^n \begin{bmatrix} \sin(ax+b+n\pi/2) \\ \cos(ax+b+n\pi/2) \end{bmatrix}$$

$$D^n e^{ax} \begin{bmatrix} \sin(bx+c) \\ \cos(bx+c) \end{bmatrix} = (a^2+b^2)^{n/2} e^{ax} \begin{bmatrix} \sin(bx+c+n\tan^{-1} b/a) \\ \cos(bx+c+n\tan^{-1} b/a) \end{bmatrix}$$

(vii) *Leibnitz theorem:* $(uv)_n = u_n + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots\ldots + {}^nC_r u_{n-r} v_r + \ldots\ldots + {}^nC_n v_n$

3. (i) *Maclaurin's series* : $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$

(ii) *Taylor's series* : $f(x+a) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \ldots$

4. Curvature

(i) *Radius of curvature* $\rho = \frac{(1+{y_1}^2)^{3/2}}{y_2}$, $\rho = \frac{(r^2+{r_1}^2)^{3/2}}{r^2+2{r_1}^2 - rr_2}$; $\rho = r\frac{dr}{dp}$.

(ii) *Centre of curvature* : $\bar{x} = x - \frac{y_1(1+{y_1}^2)}{y_2}$, $\bar{y} = y + \frac{1}{y_2}(1+{y_1}^2)$.

(iii) **Evolute** *is the locus of the centre of curvature of a curve.* The curve is called the *involute* of the evolute.

(iv) **Envelope** *of a curve* $f(x, y, \alpha) = 0$ *is the 'α' eliminant from*

$$f(x, y, \alpha) = 0 \text{ and } \frac{\partial f}{\partial \alpha}(x, y, \alpha) = 0.$$

The envelope of the normals to a curve is its ***evolute.***

5. Asymptotes

(*i*) *Asymptotes parallel to x-axis are obtained by equating to zero the coefficient of the highest power of x in the equation,* provided this is not merely a constant.

Asymptotes parallel to y-axis are obtained by equating to zero the coefficient of highest power of y in the equation, provided this is not merely a constant.

(*ii*) *Oblique asymptotes are obtained as follows:*

Put $x = 1, y = m$ in the highest degrees terms getting $\phi_n(m)$

Put $\phi_n(m) = 0$ and find the values of m.

Find c from $c = -\phi_{n-1}(m)/\phi_n'(m)$

If two values of m are equal, then find c from

$$\frac{c^2}{2}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

The asymptotes is $y = mx + c$.

(*iii*) *Asymptotes of polar curve* $1/r = f(\theta)$ is $r \sin(\theta - \alpha) = 1/f'(\alpha)$ where α is a root of $f(\theta) = 0$.

6. Curve tracing

(*i*) *A curve is symmetrical about x-axis,* if only even powers of y occur in its equation.

(*ii*) *A curve is symmetrical about y-axis,* if only even powers of x occur in its equation.

(*iii*) *A curve is symmetrical about the line $y = x$,* if on interchanging x and y, its equation remains unchanged.

(*iv*) *A curve passes through the origin,* if there is no constant term in its equation.

(*v*) *Tangents to curve at the origin* are found by equating to zero the lowest degree terms.

7. Partial Differentiation

(*i*) *Euler's theorem.* If u is a homogeneous function in x and y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu\ ; \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u$$

(*ii*) *Chain rule :* $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt}$, if $u = f(x, y), x = \phi(t), y = \psi(t)$.

(*iii*) $\dfrac{dy}{dx} = -\dfrac{\partial \phi}{\partial x}\Big/\dfrac{\partial \phi}{\partial y}$, if $\phi(x, y) = c$

(*iv*) *Jacobian* $J\left(\dfrac{u, v}{x, y}\right) = \dfrac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$

If $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$, then $JJ' = 1$

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

(*v*) *Taylor's series :* $f(a + h, b + k) = f(a, b) + \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)f + \dfrac{1}{2!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f + \ldots\ldots$

(*vi*) *Maxima Minima* (*a*) $\dfrac{\partial f}{\partial x} = 0, \dfrac{\partial f}{\partial y} = 0$

(*b*) $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} > \dfrac{\partial^2 f}{\partial x \partial y}$; $\dfrac{\partial^2 f}{\partial x^2} < 0$ for maximum $\dfrac{\partial^2 f}{\partial x^2} > 0$ for minimum.

(*vii*) *Leibnitz's Rule* $\dfrac{d}{d\alpha}\left\{\displaystyle\int_a^b f(x, \alpha)\, d\alpha\right\} = \displaystyle\int_a^b \dfrac{\partial f(x, \alpha)}{\partial \alpha}\, d\alpha$

where $f(x, \alpha)$ and $\dfrac{\partial f(x, \alpha)}{\partial \alpha}$ are continuous functions of x and α and a, b are constants.

VII. INTEGRAL CALCULUS

1. Integration

(i) $\int x^n\,dx = \frac{x^{n+1}}{n+1}\ (n \neq -1)$ $\qquad$ $\int \frac{1}{x}\,dx = \log_e x$

$\int e^x\,dx = e^x$ $\qquad$ $\int a^x\,dx = a^x/\log_e a$

(ii) $\int \sin x\,dx = -\cos x$ $\qquad$ $\int \cos x\,dx = \sin x$

$\int \tan x\,dx = -\log\cos x$ $\qquad$ $\int \cot x\,dx = \log\sin x$

$\int \sec x\,dx = \log(\sec x + \tan x)$ $\qquad$ $\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x)$

$\int \sec^2 x\,dx = \tan x$ $\qquad$ $\int \operatorname{cosec}^2 x\,dx = -\cot x$

(iii) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ $\qquad$ $\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a}$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$ $\qquad$ $\int \frac{dx}{\sqrt{(a^2+x^2)}} = \sinh^{-1}\frac{x}{a}$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$ $\qquad$ $\int \frac{dx}{\sqrt{(x^2-a^2)}} = \cosh^{-1}\frac{x}{a}$

(iv) $\int \sqrt{(a^2-x^2)}\,dx = \frac{x\sqrt{(a^2-x^2)}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

$\int \sqrt{(a^2+x^2)}\,dx = \frac{x\sqrt{(a^2+x^2)}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

$\int \sqrt{(x^2-a^2)}\,dx = \frac{x\sqrt{(x^2-a^2)}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$

(v) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

(vi) $\int \sinh x\,dx = \cosh x$ $\qquad$ $\int \cosh x\,dx = \sinh x$

$\int \tanh x\,dx = \log\cosh x$ $\qquad$ $\int \coth x\,dx = \log\sinh x$

$\int \operatorname{sech}^2 x\,dx = \tanh x$ $\qquad$ $\int \operatorname{cosech}^2 x\,dx = -\coth x$

(vii) $\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \frac{(n-1)(n-3)(n-5)\ldots\ldots}{n(n-2)(n-4)\ldots\ldots} \times \left(\frac{\pi}{2}, \text{only if } n \text{ is even}\right)$

$\int_0^{\pi/2}\sin^m x\cos^n x\,dx = \frac{(m-1)(m-3)\ldots\ldots\times(n-1)(n-3)\ldots\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots} \times \left(\frac{\pi}{2}, \text{only if both } m \text{ and } n \text{ are even}\right)$

(viii) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx,$ $\qquad$ if $f(x)$ is an *even function*

$= 0,$ $\qquad$ if $f(x)$ is an *odd function.*

$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ $\qquad$ if $f(2a-x) = f(x)$

$= 0,$ $\qquad$ if $f(2a-x) = -f(x).$

2. Lengths of curves

(*i*) *Lenth of curve* $y = f(x)$ *between* $x = a$, $x = b$ *is* $\int_a^b \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx$

(*ii*) *Length of curve* $x = f(y)$ *between* $y = a$, $y = b$ *is* $\int_a^b \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy$

(*iii*) *Length of curve* $x = f(t)$, $y = \phi(t)$ *between* $t = t_1$, $t = t_2$ *is* $\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt$

(*iv*) *Length of curve* $r = f(\theta)$ *between* $\theta = \alpha$, $\theta = \beta$ *is* $\int_\alpha^\beta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta$

3. Areas of curves

(*i*) *Area bounded by* $y = f(x)$, *x-axis and* $x = a$, $x = b$ *is* $\int_a^b y\, dx$

(*ii*) *Area bounded by* $x = f(y)$, *y-axis* and $y = a$, $y = b$ *is* $\int_a^b x\, dy$

(*iii*) *Area bounded by* $r = f(\theta)$ and *lines* $\theta = \alpha$, $\theta = \beta$ *is* $\frac{1}{2}\int_\alpha^\beta r^2\, d\theta$

4. Volumes of revolution

(*i*) *Volume of revolution about x-axis of area bounded by* $y = f(x)$, *x-axis* and $x = a$, $x = b$ *is*

$$\int_a^b \pi y^2\, dx$$

(*ii*) *Volume of revolution about y-axis of area bounded by* $x = f(y)$, *y-axis and* $y = a$, $y = b$ *is*

$$\int_a^b \pi x^2\, dy$$

(*iii*) *Volume of revolution bounded by* $r = f(\theta)$ and $\theta = \alpha$, $\theta = \beta$

(*a*) *about OX* $= \int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta\, d\theta$ (*b*) *about OY* $= \int_\alpha^\beta \frac{2\pi}{3} r^3 \cos\theta\, d\theta$

5. Surface areas of revolution

(*i*) *Surface area of revolution about x-axis of curve* $y = f(x)$ *from* $x = a$ *to* $x = b$ *is*

$$S = \int_{x=a}^{x=b} 2\pi y\, ds$$

Cartesian form : $S = \int_a^b 2\pi y \frac{ds}{dx}\, dx$ where $\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$

Parametric form : $S = \int 2\pi y \frac{ds}{dt}\, dt$ where $\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$

Polar form : $S = \int 2\pi y \frac{ds}{d\theta}\, d\theta$ where $\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$

(*ii*) *Surface area of revolution about y-axis is* $\int 2\pi x\, ds$.

6. Multiple integrals

(*i*) *Area* $= \int_{x_1}^{x_2} \int_{y_1}^{y_2} dx\, dy$; *Volume* $= \int_{x_1}^{x_2} \int_{y_1}^{y_2} z\, dx\, dy$ *or* $\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} dx\, dy\, dz$

(*ii*) *C.G. of a plane lamina:* $\bar{x} = \frac{\iint x\rho\, dx\, dy}{\iint \rho\, dx\, dy}$, $\bar{y} = \frac{\iint y\rho\, dx\, dy}{\iint \rho\, dx\, dy}$

C.G. of a solid $\bar{x} = \dfrac{\iiint x\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}$, $\bar{y} = \dfrac{\iiint y\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}$, $\bar{z} = \dfrac{\iiint z\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}$

(*iii*) *Centre of pressure* $\bar{x} = \dfrac{\iint_A px\, dx\, dy}{\iint_A p\, dx\, dy}$, $\bar{y} = \dfrac{\iint_A py\, dx\, dy}{\iint_A p\, dx\, dy}$

(*iv*) *M.I. about x-axis i.e.,* $I_x = \iiint_V \rho\,(y^2 + z^2)\, dx\, dy\, dz$

M.I. about y-axis i.e., $I_y = \iiint_V \rho\,(z^2 + x^2)\, dx\, dy\, dz$

M.I. about z-axis i.e., $I_z = \iiint_V \rho\,(x^2 + y^2)\, dx\, dy\, dz$

7. *Gamma function* $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx = (n-1)\,!, \Gamma(n+1) = n\,\Gamma(n) = n\,!, \Gamma(1/2) = \sqrt{\pi}$

Beta function $\beta(m, n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\, dx = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ $(m > 0, n > 0)$

VIII. VECTORS

1. (*i*) If $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ then $r = |\,\mathbf{R}\,| = \sqrt{(x^2 + y^2 + z^2)}$

(*ii*) $\vec{PQ}$ = Position vector of Q – position vector of P.

2. If $\mathbf{A} = a_1\mathbf{I} + a_2\,\mathbf{J} + a_3\,\mathbf{K}$, $\mathbf{B} = b_1\,\mathbf{I} + b_2\,\mathbf{J} + b_3\,\mathbf{K}$, then

(*i*) *Scalar product:* $\mathbf{A}\,.\,\mathbf{B} = ab\cos\theta = a_1\,b_1 + a_2\,b_2 + a_3\,b_3$

(*ii*) *Vector product:* $\mathbf{A} \times \mathbf{B} = ab\sin\theta\,\hat{\mathbf{N}}$ = Area of the parallelogram having $\mathbf{A}$ and $\mathbf{B}$ as sides

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(*iii*) $\mathbf{B} \perp \mathbf{A}$ if $\mathbf{A}\,.\,\mathbf{B} = 0$ and $\mathbf{A}$ is parallel to $\mathbf{B}$ if $\mathbf{A} \times \mathbf{B} = \mathbf{0}$

3. (*i*) *Scalar triple product* $[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ = Volume of parallelopiped

(*ii*) If $[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = 0$, then $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are coplanar.

(*iii*) *Vector triple product* $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\,.\,\mathbf{C})\,\mathbf{B} - (\mathbf{A}\,.\,\mathbf{B})\,\mathbf{C}$

$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C}\,.\,\mathbf{A})\,\mathbf{B} - (\mathbf{C}\,.\,\mathbf{B})\,\mathbf{A}$

4. (*i*) *grad* $f = \nabla f = \dfrac{\partial f}{\partial x}\mathbf{I} + \dfrac{\partial f}{\partial y}\mathbf{J} + \dfrac{\partial f}{\partial z}\mathbf{K}$

div $\mathbf{F} = \nabla F = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$ where $\mathbf{F} = f_1\,\mathbf{I} + f_2\,\mathbf{J} + f_3\,\mathbf{K}$

(*ii*) If *div* $\mathbf{F} = 0$, then $\mathbf{F}$ is called a *solenoidal vector.*

(*iii*) If *curl* $\mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is called an *irrotational vector*

5. *Velocity* $= d\mathbf{R}/dt$; *Acceleration* $= d^2\mathbf{R}/dt^2$; *Tangent vector* $= d\mathbf{R}/dt$; *Normal vector* $= \nabla\,\phi$

6. **Green's theorem:** $\int_C (\phi\, dx + \psi\, dy) = \iint_C \left(\dfrac{\partial\psi}{\partial x} - \dfrac{\partial\phi}{\partial y}\right) dxdy$

Stoke's theorem: $\int_C \mathbf{F}\,.\,d\mathbf{R} = \int_S curl\,\mathbf{F}\,.\,\mathbf{N}\,ds$

Gauss divergence theorem: $\int_S \mathbf{F}\,.\,\mathbf{N}\,ds = \int_v div\,\mathbf{F}\,dv$

7. Coordinate systems

	Polar coordinates (r, θ)	*Cylindrical coordinates* (ρ, ϕ, z)	*Spherical polar coordinates* (r, θ, ϕ)
Coordinate transformations	$x = r\cos\theta$ $y = r\sin\theta$	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$
Jacobian	$\frac{\partial(x, y)}{\partial(r, \theta)} = r$	$\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$	$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$
(Arc-length)^2	$(ds)^2 = (dr)^2 + r^2(d\theta)^2$ $dxdy = r\,d\theta\,dr$	$(ds)^2 = (d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2$	$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2$
Volume-element		$dV = \rho\,d\rho\,d\phi\,dz$	$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

IX. DIFFERENTIAL EQUATIONS

1. Equations of first order

(*i*) *Variables separable* : $f(y)\,dy/dx = \phi(x)$, $\int f(y)\,dy = \int \phi(x)\,dx + c$.

(*ii*) *Homogeneous equation* $dy/dx = f(x, y)/\phi(x, y)$ where $f(x, y)$ and $\phi(x, y)$ are of the same degree. Put $y = vx$ so that $dy/dx = v + x\,dv/dx$.

(*iii*) *Equations reducible to homogenous form* : $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$

When $a/a' \neq b/b'$, put $x = X + h$, $y = Y + k$

When $a/a' = b/b'$, put $ax + by = t$.

(*iv*) *Leibnitz's linear equation* : $\dfrac{dy}{dx} + Py = Q$ where P, Q are functions of x.

$I.F. = e^{\int P\,dx}$, then *solution* is $y\,(I.F.) = \int Q\,(I.F.)\,dx + c$.

(*v*) *Bernoulli's equation* : $dy/dx + Py = Qy^n$, reducible to Leibnitz's equation by writing it as

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \text{ and putting } y^{1-n} = z.$$

(*vi*) *Exact equation* : $M(x, y)\,dx + N(x, y)\,dy = 0$

Solution is $\int_{(y\ cons.)} M\,dx + \int$ (terms of N not containing x) $dy = c$, provided $\partial M/\partial y = \partial N/\partial x$.

(*vii*) *Clairaut's equation* : $y = px + f(p)$ where $p = dy/dx$.

Solution is obtained on replacing p by c.

2. Linear equations with constant coefficients

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + k_{n-1}\frac{dy}{dx} + k_n y = X$$

Symbolic form : $(D^n + k_1 D^{n-1} + \ldots. + k_{n-1}D + k_n)\,y = X.$

I. To find C.F.

Roots of A.E.	C.F.
(i) $m_1, m_2, m_3, \ldots\ldots$	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$
(ii) $m_1, m_1, m_3, \ldots\ldots$	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \ldots$
(iii) $\alpha + i\beta, \alpha - i\beta, m_3, \ldots\ldots$	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots$
(iv) $\alpha \pm i\beta, \alpha \pm i\beta, m_5, \ldots\ldots$	$e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \ldots$

II. To find P.I.

(i) $X = e^{ax}$, $\quad$ P.I. $= \dfrac{1}{f(D)} e^{ax}$, put $D = a$, $\quad [f(a) \neq 0$

$= x \dfrac{1}{f'(D)} e^{ax}$, put $D = a$, $\quad [f(a) = 0, f'(a) \neq 0$

$= x^2 \dfrac{1}{f''(D)} e^{ax}$, put $D = a$, $\quad [f'(a) = 0, f''(a) \neq 0$

(ii) $X = \sin(ax + b)$ *or* $\cos(ax + b)$

P.I. $= \dfrac{1}{\phi(D^2)} \sin(ax + b)$ [or $\cos(ax + b)$], put $D^2 = -a^2$, $\quad [\phi(-a)^2 \neq 0]$

$= x \dfrac{1}{\phi'(D^2)} \sin(ax + b)$ [or $(\cos ax + b)$], put $D^2 = -a^2$, $\quad [\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$

$= x^2 \dfrac{1}{\phi''(D^2)} \sin(ax + b)$ [or $\cos(ax + b)$], put $D^2 = -a^2$, $\quad [\phi'(-a^2) = 0, \phi''(-a^2) \neq 0$

(iii) $X = x^m$, P.I. $= \dfrac{1}{f(D)} x^m = [f(D)]^{-1} x^m$. Expand $[f(D)]^{-1}$ in ascending powers of D as for as D^m and operate on x^m term by term.

(iv) $X = e^{ax} V$, P.I. $= \dfrac{1}{f(D)} e^{ax} V = e^{ax} \dfrac{1}{f(D + a)} V$.

III. Complete Solution : C.S. is y = C.F. + P.I.

3. Homogeneous linear equation : $x^3 \dfrac{d^3 y}{dx^3} + k_1 x^2 \dfrac{d^2 y}{dx^2} + k_2 x \dfrac{dy}{dx} + k_3 y = X$

reduces to linear equation with constant coefficients by putting

$$x = e^t, x\frac{dy}{dx} = Dy, x^2 \frac{d^2 y}{dx^2} = D(D-1)y, x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

4. Lagrange's linear partial differential equation

$Pp + Qq = R$, P, Q, R being functions of x, y, z.

To solve it (i) form the subsidiary equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

(ii) solve these equations giving $u = a, v = b$.

(iii) Complete solution is $\phi(u, v) = 0$ or $u = f(v)$.

5. Homogeneous linear partial differential equations with constant coefficients

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \ldots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

Symbolic form : $(D^n + k_1 D^{n-1} D' + \ldots + k_n D'^n) z = F(x, y)$

To find C.F.

Roots of A.E.	C.F.
(i) $m_1, m_2, m_3, \ldots\ldots$	$f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x) + \ldots\ldots$
(ii) $m_1, m_1, m_3, \ldots\ldots$	$f_1(y + m_1x) + xf_2(y + m_1x) + f_3(y + m_3x) + \ldots\ldots$
(iii) $m_1, m_1, m_1, \ldots\ldots$	$f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + \ldots\ldots$

To find P.I.

(i) $F(x, y) = e^{ax+by}$, $P.I. = \dfrac{1}{f(D, D')} e^{ax+by}$, put $D = a$, $D' = b$.

(ii) $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

$$P.I. = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos(mx + ny), \text{ put } D^2 = -m^2, DD' = -mn, D^2 = -n^2$$

(iii) $F(x, y) = x^m y^n$, $P.I. = [f(D, D')]^{-1} x^m y^n$. Expand $[f(D, D')]^{-1}$ and operate on $x^m y^n$.]

(iv) $F(x, y)$ is any function of x and y, $P.I. = \dfrac{1}{f(D, D')}$, $F(x, y)$.

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that $\dfrac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$.

Complete solution: C.S. is $y = C.F. + P.I.$

X. INFINITE SERIES

1. *Basic test:* If $\underset{n \to \infty}{Lt}\ u_n \neq 0$ then the series Σu_n diverges.
2. *G.P. Series:* $1 + r + r^2 + r^3 + \ldots\ldots \infty$ converge if $|r| < 1$; diverges if $r \geq 1$ and oscillates if $r \leq -1$.
3. *p-series:* $\dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \ldots\ldots \infty$ converge for $p > 1$; diverges for $p \leq 1$.
4. *Comparison test:* If two positive term series Σu_n and Σv_n be such that $\underset{n \to \infty}{Lt} \dfrac{u_n}{v_n}$ = *finite quantity* $(\neq 0)$, then Σu_n and Σv_n converge or diverge together.
5. *Ratio test:* In the positive term series Σu_n, if $\underset{n \to \infty}{Lt} \dfrac{u_n}{u_{n+1}} = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
6. *Raabes' test:* In the positive term series Σu_n, if $\underset{n \to \infty}{Lt}\ n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
7. *Logarithmic test:* In the positive term series Σu_n, if $\underset{n \to \infty}{Lt}\left(n \log \dfrac{u_n}{u_{n+1}}\right) = k$, then the series converges for $k > 1$, diverges for $k < 1$ and fails for $k = 1$.
8. If u_n/u_{n+1} does not involve n as an exponent or a logarithm, then the series Σu_n diverges.
9. *Cauchy's root test:* In a positive term series Σu_n if $\underset{n \to \infty}{Lt} (u_n)^{1/n} = \lambda$, then the series converges for $\lambda < 1$, diverges for $\lambda > 1$ and fails for $\lambda = 1$.

10. *Integral test:* A positive term series $\Sigma f(n)$ converges or diverges according as $\int_1^{\infty} f(x)\, dx$ is finite or infinite where $f(n)$ is continuous in $1 < x < \infty$ and decreases as n increases.
11. *Leibnitz's test for alternating series:* An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots \infty$ converges if each term is numerically less than the previous term and $\underset{n \to \infty}{Lt}\ u_n = 0$.

    if $\underset{n \to \infty}{Lt} \neq 0$, then the given series is oscillatory.

12. *General Ratio test:* In an arbitrary term series Σu_n if $\underset{n \to \infty}{Lt} \frac{|u_{n+1}|}{|u_n|} = |k|$, then Σu_n is absolutely convergent if $|k| < 1$ and divergent if $|k| > 1$ and the test fails if $|k| = 1$.

XI. FOURIER SERIES

1. $f(x) = \frac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots$ in $(0\ 2\pi)$.

   where $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx\,, a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$

2. $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cos nx$ in any interval $(0, 2c)$,

   where $\quad a_0 = \frac{1}{c}\int_0^{2c} f(x)\, dx,\ a_n = \frac{1}{c}\int_0^{2c} f(x) \cos \frac{n\pi x}{c}\, dx\,, b_n = \frac{1}{c}\int_0^{c} f(x) \sin \frac{n\pi x}{c}\, dx$

3. *For even function f(x), Fourier expansion contains only cosine terms.*

   i.e., $\quad a_0 = \frac{2}{c}\int_0^{c} f(x)\, dx,\ a_n = \frac{2}{c}\int_0^{c} f(x) \cos \frac{n\pi x}{c}\, dx\,, b_n = 0.$

   For odd function f (x), Fourier expansion contains only sine terms.

   i.e., $\quad a_0 = 0, a_n = 0, b_n = \frac{2}{c}\int_0^{c} f(x) \sin \frac{n\pi x}{c}\, dx\,.$

XII. TRANSFORMS

1. **Laplace Transforms.** $L(f(t)) = \int_0^{\infty} e^{-st} f(t)\, dt$

   (*i*) $L(1) = \frac{1}{s}$

   (*ii*) $L(t^n) = \frac{n!}{s^{n+1}}$

   (*iii*) $L(e^{at}) = \frac{1}{s-a}$

   (*iv*) $L(\sin at) = \frac{a}{s^2+a^2}$

   (*v*) $L(\cos at) = \frac{s}{s^2+a^2}$

   (*vi*) $L(\sinh at) = \frac{a}{s^2-a^2}$

   (*vii*) $L(\cosh at) = \frac{s}{s^2-a^2}$

   (*viii*) $L\, e^{at} f(t) = F(s-a)$

   (*ix*) $L f'(t) = sL f(t) - f(0)$

   (*x*) $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$

   (*xi*) $L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} F(s)\, ds$

   (*xii*) $u(t-a) = \begin{cases} 0 \text{ when } t < a \\ 1 \text{ when } t > a \end{cases}$

   (*xiii*) $L[u(t-a)] = \frac{e^{-as}}{s}$

   (*xiv*) $L\delta(t-a) = e^{-as}$

   (*xv*) $L f(t) = \frac{\int_0^T e^{-st} f(t)\, dt}{1-e^{-sT}}$ where $f(t)$ is a periodic function of period T.

2. **Inverse Laplace Transforms**

   (*i*) $L^{-1}\left(\frac{1}{s}\right) = 1$

   (*ii*) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$

   (*iii*) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

   (*iv*) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a} \sin at$

   (*v*) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

   (*vi*) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \frac{1}{a} \sinh at$

(vii) $L^{-1}\left(\dfrac{s}{s^2-a^2}\right) = \cosh at.$

(viii) $L^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right) = \dfrac{1}{2a}\, t \sin at$

(ix) $L^{-1}\left[\dfrac{s^2-a^2}{(s^2+a^2)^2}\right] = t \cos at.$

3. Fourier Transforms : $F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, . dt$

Fourier sine transform : $F_s(s) = \int_0^{\infty} f(t) \sin st\, dt$

Fourier cosine transform : $F_c(s) = \int_0^{\infty} f(t) \cos st\, dt$

$$F\left(\frac{\partial^2 u}{\partial x^2}\right) = -s^2 F(u).$$

4. Z-Transforms : $Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

(i) $Z(1) = \dfrac{z}{z-1}$

(ii) $Z(n) = \dfrac{z}{(z-1)^2}$

(iii) $Z(n^2) = \dfrac{z^2+z}{(z-1)^3}$

(iv) $Z(a^n) = \dfrac{z}{z-a}$

(v) $Z(na^n) = \dfrac{az}{(z-a)^2}$

(vi) $Z(\sin n\theta) = \dfrac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$

(vii) $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

(viii) $Z(\sinh n\theta) = \dfrac{z \sinh\theta}{z^2 - 2z\cosh\theta + 1}$

(ix) $Z(\cosh n\theta) = \dfrac{z(z-\cosh\theta)}{z^2 - 2z\cosh\theta + 1}$

XIII. STATISTICS AND PROBABILITY

1. *A.M.* $\bar{x} = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$

2. *S.D.* $\sigma = \sqrt{\dfrac{\Sigma f_i (x_i - \bar{x})^2}{\Sigma f_i}}$

3. *Moments about the mean* : $\mu_0 = 1, \mu_1 = 0, \mu_2 = \sigma^2, \mu_r = \dfrac{\Sigma f_i (x_i - \bar{x})^r}{\Sigma f_i}$

4. *Coeff. of skewness* = (mean – mode)/σ which lies between – 1 and 1.

5. Kurtosis : $\beta_2 = \mu_4/\mu_2^2$.

6. *Coeff. of correlation* $r = \dfrac{n\,\Sigma d_x d_y - \Sigma d_x \Sigma d_y}{\sqrt{[\{n\,\Sigma d_x^2 - (\Sigma d_x)^2\}\{n\Sigma d_y^2 - (\Sigma d_y)^2\}]}}; -1 < r < 1$

7. *Line of regression of y on x* : $y - \bar{y} = r\,\dfrac{\sigma_y}{\sigma_x}(x - \bar{x})$

   Line of regression of x on y : $x - \bar{x} = \dfrac{r\,\sigma_x}{\sigma_y}(y - \bar{y})$

8. **Probability** $p(A) = \dfrac{\text{No. of ways favourable to } A}{\text{Total no. of equally likely ways}}, p + q = 1.$

   (i) $p(A \text{ or } B) = p(A) + p(B)$,

   (ii) $p(A \text{ and } B) = p(A) \,.\, p(B)$

9. **Binomial distribution :** $p(r) = {}^nC_r\, p^r q^{n-r}$

   Mean = np, *Variance* $(\sigma^2) = npq$

10. Poisson distribution : $p(r) = \dfrac{m^r}{r!} e^{-m}$

Mean $= m$, *Variance* $(\sigma^2) = m$.

11. Normal distribution : $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, *Standard variate* $= \dfrac{x-\mu}{\sigma}$

(*i*) *Probable error* $\lambda = 0.6745\ \sigma$.

(*ii*) 68% of values lie between $x = \mu - \sigma$ and $x = \mu + \sigma$.
95% of values lie between $x = \mu - 1.96\ \sigma$ and $x = \mu + 1.96\ \sigma$
99% of values lie between $x = \mu - 2.58\ \sigma$ and $x = \mu + 2.58\ \sigma$

XIV. NUMERICAL TECHNIQUES

1. Solution of equations

(*i*) *Bisection method* : $x_n = \dfrac{1}{2}(x_{n-1} + x_{n-2})$

(*ii*) *Method of False position* : $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$.

(*iii*) *Newton-Raphson method* : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(*iv*) *Iterative formula to find* $1/N$ is $x_{n+1} = x_n(2 - Nx_n)$

(*v*) *Iterative formula to find* $\sqrt{N}$ is $x_{n+1} = \dfrac{1}{2}(x_n + N/x_n)$

2. Solution of Linear Simultaneous equations

(*i*) *Matrix inversion method*. For the equations

$$a_1x + b_1y + c_1z = d_1,\ a_2x + b_2y + c_2z = d_2,\ a_3x + b_3y + c_3z = d_3$$

if $$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

where A_1, B_1, etc., are the co-factors of a_1, b_1, etc., in the determinant $|A|$.

(*ii*) *Gauss-elimination method*. In this method the coefficient matrix is transformed to *upper triangular matrix*.

(*iii*) *Gauss-Jordan method*. In this method the coefficient matrix is transformed to *diagonal matrix*.

(*iv*) *Gauss-Jordan method of finding the inverse of a matrix A*. The matrices A and I are written side by side and the same row transformations are performed on both till A is reduced to I. Then the other matrix represents A^{-1}.

3. Finite differences and Interpolation

(*i*) *Forward differences:* $\Delta y_r = y_{r+1} - y_r$
Backward differences: $\nabla y_r = y_r - y_{r-1}$
Central differences: $\delta y_{n-1/2} = y_n - y_{n-1}$

(*ii*) *Relations between operations :*

$$\Delta = E - 1;\ \nabla = 1 - E^{-1};\ \delta = E^{1/2} - E^{-1/2}$$

$$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2});\ \Delta = E\nabla = \nabla E = \delta E^{1/2};\ E = e^{hD}$$

(*iii*) *Factorial notation* $|\,x\,|^r = x(x-1)(x-2)\ldots(x-r+1)$.
Factorial polynomial $[x]^n = x(x-h)(x-2h)\ldots(x-h-1h)$

(*iv*) *Newton's forward interpolation formula*

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \text{ where } p = (x - x_0)/h.$$

(*v*) *Newton's backward interpolation formula:*

$$y_p = y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots\ldots \text{ where } p = (x - x_n)/h.$$

(*vi*) *Stirling's formula:*

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots$$

(*vii*) *Bessel's formula:*

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\,\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p - \frac{1}{2}\right)p\,(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\,\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots$$

(*viii*) *Lagrange's interpolation formula:*

$$y = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)}y_1 + \ldots\ldots +$$
$$\frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})}y_n$$

(*ix*) *Newton's divided difference formul*

$$y = f(x) = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2, x_3] + \ldots$$

where $[x_0, x_1]\ \dfrac{y_1 - y_0}{x_1 - x_0}$, $[x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_0, x_1]}{[x_2 - x_0]}$ and so on.

4. Numerical differentiation

(*i*) *Forward difference formulae:*

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\ldots\right] \text{ and so on.}$$

(*ii*) *Backward difference formulae:*

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \ldots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \ldots\ldots\right] \text{ and so on.}$$

(*iii*) *Central difference formulae:*

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\,\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \ldots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \ldots\ldots\right]$$

5. Numerical integration

(*i*) *Trapezoidal rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\,[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})]$$

(ii) *Simpon's 1/3 th rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0+y_n)+4(y_1+y_3+\ldots\ldots+y_{n-1})+2(y_2+y_4+\ldots\ldots+y_{n-2})]$$

(Number of sub intervals should be taken as even)

(iii) *Simpson's 3/8 th rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\ldots\ldots+y_{n-1})+2(y_3+y_5+\ldots\ldots+y_{n-2})]$$

(Number of sub-intervals should be taken as a multiple of 3)

(iv) *Weddle's rule:*

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}[y_0+5y_1+y_2+6y_3+y_4+5y_5+2y_6+5y_7+\ldots\ldots]$$

(Number of sub-intervals should be taken as multiple of (6)

6. Numerical solution of ordinary differential equations

(i) *Picard's method:* $y_1 = y_0 + \int_{x_0}^{x} (x, y_0)\,dx$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx \text{ etc.}$$

(ii) *Taylor's method:*

$$y = y_0 + (x-x_0)(y)_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \ldots$$

(iii) *Euler's method:* $y_2 = y_1 + h\,f(x_0+h, y_1)$
Repeat this process till y_2 is stationary. Then calculate y_3 and so on.

(iv) *Modified Euler's method:* $y_2 = y_1 + \frac{1}{2}[f(x_0+h, y_1) + f(x_0+2h, y_2)]$

Repeat this step till y_2 is stationary. Then calculate y_3 and so on.

(v) *Runge Kutta method:* $y_1 = y_0 + h$ where $h = \frac{1}{6}(k_1 + 2k_2, 2k_3 + k_4)$

such that $k_1 = h\,f(x_0, y_0)$; $k_2 = h\,f(x_0 + h/2, y_0 + k_1/2)$
$k_3 = h\,f(x_0 + h/2, y_0 + k_2/2)$; $k_4 = hf(x_0 + h, y_0 + k_3)$

(vi) *Milne's method*

Predictor formula: $y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

Corrector formula: $y_4 = y_2 + \frac{h}{3}(f_2 - 4f_3 + f_4)$

(Four prior values are required to find the next values)

(vii) *Adams-Bashforth method:*

Predictor formula: $y_1 = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$

Corrector formula: $y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} - f_{-2})$

(Four prior values are required to find the next values)

7. Numerical solution of partial differential equations

(i) *Classification of a second order equations:*

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + \left(F(x,y,u,)\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) = 0$$

is said to be *elliptic* if $B^2 - 4AC < 0$
parabolic if $B^2 - 4AC = 0$
hyperbolic if $B^2 - 4AC > 0$

(*ii*) *Laplace equation:* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Standard 5-point formula: $u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j}\, u_{i,j+1} + u_{i,j-1}]$

Diagonal 5-point formula: $u_{i,j} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j-1}\, u_{i+1,j+1} + u_{i-1,j-1}]$

(*iii*) *Poisson's equation:* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y).$

Standard 5-point formula:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

(*iv*) *One-dimensional heat equation:* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Schmidt formula: $u_{i,j+1} = \alpha\, u_{i-1,j} + (1 - 2d)\, u_{i,j} + \alpha\, u_{i+1,j}$ where $\alpha = kc^2/h^2$

when $\alpha = 1/2$, it reduces to

Bendre-Schmidt relation: $u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$

(*v*) *Wave equation:* $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Explict formula for solution is

$$u_{i,j+1} = 2(1 - \alpha^2 c^2)\, u_{i,j} + \alpha^2 c^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \text{ where } \alpha = k/h$$

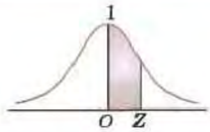
If α is so chosen that the coefficient of $u_{i,j}$ is zero *i.e.*, $k = h/c$ then the above explicit formula takes the *simplified form*

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

Table II : Bessel Functions

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000	3.0	−0.2601	0.3991
0.1	0.9975	0.0499	3.1	−0.2921	0.3009
0.2	0.9900	0.0995	3.2	−0.3202	0.2613
0.3	0.9776	0.1483	3.3	−0.3443	0.2207
0.4	0.9604	0.1960	3.4	−0.3643	0.1792
0.5	0.9385	0.2423	3.5	−0.3801	0.1374
0.6	0.9120	0.2867	3.6	−0.3918	0.0955
0.7	0.8812	0.3290	3.7	−0.3992	0.0538
0.8	0.8463	0.3688	3.8	−0.4026	0.0128
0.9	0.8075	0.4059	3.9	−0.4018	−0.0272
1.0	0.7652	0.4401	4.0	−0.3971	−0.0660
1.1	0.7196	0.4709	4.1	−0.3887	−0.1033
1.2	0.6711	0.4983	4.2	−0.3766	−0.1386
1.3	0.6201	0.5220	4.3	−0.3610	−0.1719
1.4	0.5669	0.5419	4.4	−0.3423	−0.2028
1.5	0.5118	0.5579	4.5	−0.3205	−0.2311
1.6	0.4554	0.5699	4.6	−0.2961	−0.2566
1.7	0.3980	0.5778	4.7	−0.2693	−0.2791
1.8	0.3400	0.5815	4.8	−0.2404	−0.2985
1.9	0.2818	0.5812	4.9	−0.2097	−0.3147
2.0	0.2239	0.5767	5.0	−0.1776	−0.3276
2.1	0.1666	0.5683	5.1	−0.1443	−0.3371
2.2	0.1104	0.5560	5.2	−0.1103	−0.3432
2.3	0.0555	0.5399	5.3	−0.0758	−0.3460
2.4	0.0025	0.5202	5.4	−0.0412	−0.3453
2.5	−0.0484	0.4971	5.5	−0.0068	−0.3414
2.6	−0.0968	0.4708	5.6	0.0270	−0.3343
2.7	−0.1424	0.4416	5.7	0.0599	−0.3241
2.8	−0.1850	0.4097	5.8	0.0917	−0.3110
2.9	−0.2243	0.3754	5.9	0.1220	−0.2951

Table III : Area under the Normal curve



x	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993

Table IV : Values of | t | with probability P and degrees of freedom ν

ν \ P	0.50	0.10	0.05	0.02	0.01
1	1.000	6.34	12.71	31.82	63.66
2	0.816	2.92	4.30	6.96	9.92
3	0.765	2.35	3.18	4.54	5.84
4	0.741	2.13	2.78	3.75	4.60
5	0.727	2.02	2.57	3.36	4.03
6	0.718	1.94	2.45	3.14	3.71
7	0.711	1.90	2.36	3.00	3.50
8	0.706	1.86	2.31	2.90	3.36
9	0.703	1.83	2.26	2.82	3.25
10	0.700	1.81	2.23	2.76	3.17
11	0.697	1.80	2.20	2.72	3.11
12	0.695	1.78	2.18	2.68	3.06
13	0.694	1.77	2.16	2.65	3.01
14	0.692	1.76	2.14	2.62	2.98
15	0.691	1.75	2.13	2.60	2.95
16	0.690	1.75	2.12	2.58	2.92
17	0.689	1.74	2.11	2.57	2.90
18	0.688	1.73	2.10	2.55	2.88
19	0.688	1.73	2.09	2.54	2.86
20	0.687	1.72	2.09	2.53	2.84
21	0.686	1.72	2.08	2.52	2.83
22	0.686	1.72	2.07	2.51	2.82
23	0.685	1.71	2.07	2.50	2.81
24	0.685	1.71	2.06	2.49	2.80
25	0.684	1.71	2.06	2.48	2.79
26	0.684	1.71	2.06	2.48	2.78
27	0.684	1.70	2.05	2.47	2.77
28	0.683	1.70	2.05	2.47	2.76
29	0.683	1.70	2.04	2.46	2.76
30	0.683	1.70	2.04	2.46	2.75

Table V : Values of χ^2 with probability P and df ν

ν \ P	0.99	0.95	0.50	0.30	0.20	0.10	0.05	0.01
1	0.0002	0.004	0.46	1.07	1.64	2.71	3.84	6.64
2	0.020	0.103	1.39	2.41	3.22	4.60	5.99	9.21
3	0.115	0.35	2.37	3.66	4.64	6.25	7.82	11.34
4	0.30	0.71	3.36	4.88	5.99	7.78	9.49	13.28
5	0.55	1.14	4.35	6.06	7.29	9.24	11.07	15.09
6	0.87	1.64	5.35	7.23	8.56	10.64	12.59	16.81
7	1.24	2.17	6.35	8.38	9.80	12.02	14.07	18.48
8	1.65	2.73	7.34	9.52	11.03	13.36	15.51	20.09
9	2.09	3.32	8.34	10.66	12.24	14.68	16.92	21.67
10	2.56	3.94	9.34	11.78	13.44	15.99	18.31	23.21
11	3.05	4.58	10.34	12.90	14.63	17.28	19.68	24.72
12	3.57	5.23	11.34	14.01	15.81	18.55	21.03	26.22
13	4.11	5.89	12.34	15.12	16.98	19.81	22.36	27.69
14	4.66	6.57	13.34	16.22	18.15	21.06	23.68	29.14
15	5.23	7.26	14.34	17.32	19.31	22.31	25.00	30.58
16	5.81	7.96	15.34	18.42	20.46	23.54	26.30	32.00
17	6.41	8.67	16.34	19.51	21.62	24.77	27.59	33.41
18	7.02	9.39	17.34	20.60	22.76	25.99	28.87	34.80
19	7.63	10.12	18.34	21.69	23.90	27.20	30.14	36.19
20	8.26	10.85	19.34	22.78	25.04	28.41	31.41	37.57
21	8.90	11.59	20.34	23.86	26.17	29.62	32.67	38.93
22	9.54	12.34	21.34	24.94	27.30	30.81	33.92	40.29
23	10.20	13.09	22.34	26.02	28.43	32.01	35.17	41.64
24	10.86	13.85	23.34	27.10	29.55	33.20	36.42	42.98
25	11.52	14.61	24.34	28.17	30.68	34.68	37.65	44.31
26	12.20	15.38	25.34	29.25	31.80	35.56	38.88	45.64
27	12.88	16.15	26.34	30.32	32.91	36.74	40.11	46.96
28	13.56	16.93	27.34	31.39	34.03	37.92	41.34	48.28
29	14.26	17.71	28.34	32.46	35.14	39.09	42.56	49.59
30	14.95	18.49	29.34	33.53	36.25	40.26	43.77	50.89

Table VI : 5% and 1% points of F

v_2 \ v_1	1	2	3	4	5	6	8	12	24	∞
2	18.51	19.00	19.16	19.25	19.30	19.32	19.37	19.41	19.45	19.50
	98.49	99.00	99.17	99.25	99.30	99.33	99.36	99.42	99.46	99.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.53
	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.12
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.63
	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.36
	16.26	13.27	12.06	11.39	10.97	10.67	10.27	9.89	9.47	9.02
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
	13.74	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
	12.25	9.55	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.07	2.90	2.71
	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54
	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
12	4.75	3.88	3.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
	8.86	6.51	5.56	5.03	4.69	4.46	4.14	3.80	3.43	3.00
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
	8.28	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.01	2.57
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.28	2.08	1.84
	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
	7.77	5.57	4.68	4.18	3.86	3.63	3.32	2.99	2.62	2.17
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62
	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.00	1.79	1.51
	7.31	5.18	4.31	3.83	3.51	3.29	2.99	2.66	2.29	1.81
60	4.00	3.15	2.76	2.52	2.37	2.25	2.10	1.92	1.70	1.39
	7.08	4.98	4.13	3.65	3.34	3.12	2.82	2.50	2.12	1.60

APPENDIX 3

Answers to Problems

Problems 1.1, page 5

1. $x^4 - 6x^3 + 3x^2 + 42x - 70 = 0$ **2.** (*i*) $-2, 1 + 3i, 1 - 3i$ (*ii*) $2 \pm \sqrt{3}, 3, -5$

5. Two roots between (1, 2) and (– 3, – 4)

6. $2, 2, -\frac{1}{3}$ **7.** 3 **8.** $a = 2, b = 1$

9. 6, 4, – 1 **10.** – 4, 2, 6 **11.** 1, 1, 2, 2

12. $\frac{1}{2}(3 \pm \sqrt{5}; \frac{1}{2}(5 \pm \sqrt{5})$ **13.** 1, 4, 7 **14.** $1, \frac{1}{2}, \frac{1}{4}$

16. (*i*) – 5, – 2, 1, 4 (*ii*) 1, – 2, 4, – 8 **17.** (*i*) $m^2 - 2ln$, (*ii*) $lm - n$

18. 36 **19.** (*i*) 4/3, (*ii*) 16/9.

Problems 1.2, page 8

1. $x^3 + 6x^2 - 36x + 27 = 0$ **2.** $6x^5 - 7x^4 - 13x^3 + 4x^2 - 2 = 0$ **3.** $10x^4 + 9x^3 + 8x^2 - 7x + 1 = 0$

4. $-\frac{1}{2}, \frac{1}{3}, 2$ **5.** (*i*) $x^3 - 9x^2 + 26x - 24 = 0$; (*ii*) $x^4 + 13x^3 + 60x^2 + 116x + 80 = 0$;

(*iii*) $x^5 + 7 = 0$

6. $x^3 + 15x^2 + 52x - 36 = 0$ **7.** $y^3 + (p^3 + 3q)y^2 + 3q^2y + q^3 = 0$ **8.** $3x^3 - 11x^2 + 9x - 2 = 0$

9. $y^3 - qy^2 + pry - r^2 = 0$

10. (*a*) $y^3 + 4my - 8n = 0$; (*b*) $nx^3 + m^2x^2 - 2mnx + n^2 = 0$; (*c*) $x(nx + m)^2 = n$

11. $y^3 - 30y^2 + 225y - 68 = 0$

12. (*i*) $\dfrac{-5 \pm \sqrt{21}}{2}, \dfrac{5 \pm \sqrt{91}i}{12}$ (*ii*) 2, 2, 1/2, 1/2 (*iii*) 1, – 2, 4, – 1/2, 1/4;

(*iv*) – 1, – 2, 3, – 1/2, 1/3; (*v*) $\pm 1, -3, -1/3, \dfrac{3 \pm \sqrt{5}}{2}$

13. $\frac{1}{2}(5 \pm \sqrt{21})$; $\frac{1}{2}(-3 \pm \sqrt{5})$ **14.** – 1, – 2, – 6, – 7.

Problems 1.3, page 11

1. – 6, 3, 3 **2.** $5, \frac{1}{2}(-5 \pm i\sqrt{3})$ **3.** $6, -3 \pm 2\sqrt{(-3)}$

4. $-1, -2, \frac{1}{2}$ **5.** $\dfrac{1}{2}, -\dfrac{1}{6}(3 \pm i\sqrt{3})$ **6.** $5, \dfrac{1}{2}(1 + i\sqrt{3})$

7. $2\cos\dfrac{2\pi}{9}, 2\cos\dfrac{8\pi}{9}, 2\cos\dfrac{14\pi}{9}$ **8.** $\dfrac{1}{2}, \dfrac{-7 \pm 9i\sqrt{3}}{6}$.

Problems 1.4, page 12

1. 1, 2, 3, 4 **2.** $-3, 1, \pm 2$ **3.** $4, -2, -1 \pm i$

4. $-1, 3, 3 \pm \sqrt{30}$ **5.** $1 \pm \sqrt{7}, 2 \pm \sqrt{3}$ **6.** $1 \pm 2i, -1 \pm \sqrt{2}$

7. $2 \pm \sqrt{3}, -2 \pm i\sqrt{3}$ **8.** $2, 4, 2 \pm 2i\sqrt{2}$.

Problems 1.5, page 15

1. 1.32 **2.** 2.29 **3.** 0.45

4. (*i*) 0.71 rad **5.** 1.81 rad **6.** 0.26.

Problems 1.6, page 15

1. (*d*) **2.** (*c*) **3.** (*c*)

4. (*c*) **5.** (*c*) **6.** (*a*)

7. (*d*) **8.** (*b*) **9.** (*c*)

10. (*a*) **11.** (*c*) **12.** minus

13. § 15.1 (v) **14.** p/q **15.** 21

16. -3 and -2 **17.** Conjugate pairs **18.** $f(x)$ is continuous in (a, b)

19. $x^3 - 9x^2 + 29x - 24 = 0$ **20.** $3, 6, -2$ **21.** $x^3 - 200x - 7000 = 0$

22. p/r **23.** $x^4 + 2x^3 - x^2 - 6x - 6 = 0$ **24.** 6

25. minus **26.** $pq = r$ **27.** $1, \frac{1}{2}(-1 \pm \sqrt{3}i)$

28. (*iii*) **29.** $1, 1, -2$ **30.** $x^3 - 7x^2 + 12x - 10 = 0$

31. Zero and 2 **32.** 21 **33.** True

34. True.

Problems 2.1 page 25

5. (*i*) 1 (*ii*) 0 **13.** $(a-b)(b-c)(c-a)$

14. $(a-b)(b-c)(c-a)(ab+bc+ca)$ **15.** $(b-c)(c-a)(a-b)(a-1)(b-1)(c-1)$

16. $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ **17.** $x = 0, \pm\sqrt{(a^2+b^2+c^2-ab-bc-ca)}$

18. $0, -\dfrac{1}{2}$.

Problems 2.2 page 31

1. $x = 0, 3$ **2.** $x = -3, y = -2, z = -4, a = 3$

3. $x = 3, y = 8$ **4.** $-2\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$

5. $AB = \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}$ **7.** (*i*) $[ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy]$

(*ii*) $\begin{bmatrix} 8 & 7 \\ 122 & 104 \\ -365 & -131 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 9 & 6 \\ -18 & -12 \\ 27 & 18 \end{bmatrix}$ **10.** $3I$

11. $\begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$ **15.** $\begin{bmatrix} 1 & 0 & 0 \\ 7/5 & 1 & 0 \\ 3/5 & 41/19 & 1 \end{bmatrix}\begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 0 & 327/19 \end{bmatrix}$

Problems 2.3, page 35

2. (i) $\begin{bmatrix} 3 & 0 & 5.5 \\ 0 & 7 & 1.5 \\ 5.5 & 1.5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 5 \\ 2 & 0 & -2.5 \\ -0.5 & 2.5 & 0 \end{bmatrix}$ (ii) $\begin{vmatrix} a & \frac{1}{2}(a+c) & \frac{1}{2}(b+c) \\ \frac{1}{2}(a+c) & b & \frac{1}{2}(a+b) \\ \frac{1}{2}(b+c) & \frac{1}{2}(a+b) & c \end{vmatrix} + \begin{vmatrix} 0 & \frac{1}{2}(a-c) & \frac{1}{2}(b-c) \\ \frac{1}{2}(c-a) & 0 & \frac{1}{2}(b-a) \\ \frac{1}{2}(c-b) & \frac{1}{2}(a-b) & 0 \end{vmatrix}$

4. (i) $\begin{bmatrix} 2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix}$ (ii) $\begin{bmatrix} 1/33 & -4/33 & 2/11 \\ -4/33 & 14/33 & 13/33 \\ 2/11 & 13/33 & -1/33 \end{bmatrix}$

5. $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **6.** (i) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$.

Problems 2.4, page 40

1. 3 **2.** 2 **3.** 3 **4.** 2 **5.** 3

6. No value of p is possible.

7. (i) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ (ii) $\frac{1}{21}\begin{bmatrix} 1 & 10 & -7 \\ 1 & -11 & 14 \\ -3 & 12 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} -1/9 & 2/9 & 2/9 \\ 2/9 & -1/9 & 2/9 \\ 2/9 & 2/9 & -1/9 \end{bmatrix}$ (iv) $\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$

8. $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 2/3 & -7/24 & 5/6 \\ 0 & -1/3 & 0 & 1/3 \\ 0 & 0 & -5/24 & 1/2 \\ 0 & 0 & -1/12 & 0 \end{bmatrix}$, Rank $(A) = 3$

9. $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$ **10.** (i) $P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(ii) $P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{6} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

11. (i) 3 (ii) 3 (iii) 2 (iv) 3.

Problems 2.5, Page 43

1. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ **2.** $\frac{1}{8}\begin{bmatrix} 2 & 2 & -2 \\ -9 & 11 & 5 \\ 5 & -7 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$ **4.** $\begin{bmatrix} 7 & -3 & 0 & -5 \\ 8 & 1 & -2 & 11 \\ -5 & 0 & 1 & 6 \\ 19 & 5 & -6 & -28 \end{bmatrix}$

Problems 2.6, page 45

1. $x = 1, y = 2, z = 1$ **2.** $x = 2, y = -1, z = 1/2$

3. $x = 1.2, y = 2.2, z = 3.2$ **4.** $x = y = z = e^2$

5. $u = 1, v = 1/2, w = 1/3$
6. $x_1 = 1, x_2 = -5, x_3 = 5$
7. $x = 2, y = 1, z = 0$
8. $x = 1/7, y = 10/7, z = 1/7$
9. $x = y = z = 2$
10. $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 3$
11. $x = 1, y = -1, z = 1$
12. $i_1 = 1.5, i_3 = 2.5$
13. $i_1 = 369/175, i_2 = 24/25, i_3 = 72/175$
14. $x_1 = 2, x_2 = 1/5, x_3 = 0, x_4 = 4/5$.

Problems 2.7, page 50

1. Consistent ; $x = 1, y = 3k - 2, z = k$ for all k
2. $k = 1, x = -3z, y = 2z + 1$; $k = 2, x = 1 - 3z, y = 2z$
3. (*i*) $\lambda = 3, \mu \neq 10$; (*ii*) $\lambda \neq 3$; (*iii*) $\lambda = 3, \mu = 10$
4. (*i*) Equations are inconsistent ; (*ii*) consistent ; $x = -1, y = 1, z = 2$;
(*iii*) Equations are inconsistent ; (*iv*) Consistent : $x = 2, y = 1, z = -4$
5. If $a = -1, b = 6$, equations will be consistent and have infinite number of solutions
If $a = -1, b \neq 6$, equations will be inconsistent ;
If $a \neq -1$, b has any value, equations will be consistent and have a unique solution
6. $\lambda \neq -5, x = 4/7, y = -9/7, z = 0$; $\lambda = -5, x = \frac{1}{7}(4 - 5k), y = \frac{1}{7}(13k - 9), z = k$ for all k
9. $\lambda = -1, 1, 12$; $x = -1/11, y = -15/11$; $x = -5, y = 1$; $x = \frac{1}{2}, y = 1$
11. $k = 3$ is the only real value for which $x = y = z$
12. $\lambda = 1, x_1 = 2t - s, x_2 = t, x_3 = s$; $\lambda = -3, x_1 = -t, x_2 = -2t, x_3 = t$
13. $\lambda = 1, -9$. For $\lambda = 1$, sol. is $x = k, y = -k, z = 2k$
For $\lambda = -9$, sol. is $x = 3k, y = 9k, z = -2k$
14. (*i*) Have infinite number of non-trivial solutions ; $x = \lambda - 5\mu/3, y = \lambda - 4\mu/3, z = \lambda, w = \mu$ for all values of λ and μ. (*ii*) $x = 11k_2 + 6k_1, y = -8k_2 - 3k_1, z = k_2, w = k_1$ where k_1, k_2 are arbitrary constants.

Problems 2.8, page 54

1. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$
2. $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, A^{-1} = A' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$
3. $Z = (BA)X$, where $BA = \begin{bmatrix} 1 & 4 & -1 \\ -1 & 9 & -1 \\ -3 & 14 & -1 \end{bmatrix}$
4. $x_1 = 19y_1 - 9y_2 + 2y_3$; $x_2 = -4y_1 + 2y_2 - y_3$; $x_3 = -2y_1 + y_2$
6. $a = \pm\frac{1}{\sqrt{2}}, b = \pm\frac{1}{\sqrt{6}}, c = \pm\frac{1}{\sqrt{3}}$
9. (*i*) No. (*ii*) No. (*iii*) Yes, $9x_1 - 12x_2 + 5x_3 - 5x_4 = 0$.

Problems 2.9, page 60

1. 10 ; 30
2. (*a*) 10, 3 ; (1, 2), (3, – 1) ; (*b*) – 1, 6 ; (1, 1), (2, – 5)
3. (*a*) 0, 3, 5 ; (1, 2, 2), (2, 1, – 2), (2, – 2, 1) (*b*) 1, 2, 3 ; (1, 0, – 1), (0, 1, 0), (1, 0, 1)
(*c*) 5, – 3, – 3 ; (1, 2, – 1), (2, – 1, 0), (3, 0, 1)
(*d*) 8, 2, 2 ; (2, – 1, 1), (1, 0, – 2), (1, 2, 0) (*e*) 2, 3, – 1 ; (3, 1, 1), (– 4, 1, – 3), (0, 5, 5)
5. (*i*) 8, 12, 6 (*ii*) 49, 121, 25
6. 1, 1, 1/5
9. (*i*) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$; (*ii*) $\frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$; (*iii*) $\frac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$; (*iv*) $\begin{bmatrix} 1 & 1/2 & -2/3 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$
10. $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$, $\frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$

11. (*i*) $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ (*ii*) $\frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$ (*iii*) $\frac{1}{27}\begin{bmatrix} 1 & 10 & -8 \\ -8 & 1 & 10 \\ 10 & -8 & 1 \end{bmatrix}$

12. $625I$

13. $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$, $\begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$

14. $\begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$

15. $A^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 6 & 6 \\ -1 & 6 & 2 \\ 1 & -10 & -6 \end{bmatrix}$, $A^{-2} = \frac{1}{16}\begin{bmatrix} 1 & -18 & -18 \\ -5 & 10 & -6 \\ 5 & 6 & 22 \end{bmatrix}$,

$A^{-3} = \frac{1}{64}\begin{bmatrix} 1 & 78 & 78 \\ -21 & 90 & 26 \\ 21 & -154 & -90 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$.

Problems 2.10, page 67

3. $\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$

4. $A^n = \begin{bmatrix} 2^n + 3.6^n & -3.2^n + 3.6^n \\ -2^n + 6^n & 3.2^n + 6^n \end{bmatrix}$; $A^4 = \begin{bmatrix} 976 & 960 \\ 320 & 336 \end{bmatrix}$

5. $\begin{bmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{bmatrix}$

7. $(1, 1, -1), (1, 1, -1), (2, -1, 1)$; $4x^2 + y^2 + z^2$

8. $x^2 + y^2 - 2z^2$

10. (*a*) 1, 2, 4 ; (1, 0, 0), (0, 1, 1), (0, 1, −1) ; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
(*b*) $x_1^2 + 2x_2^2 + 4x_3^2$

10. (*i*) $x_1^2 + 4x_2^2 + 4x_3^2$, $\begin{bmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$, positive definite ;

(*ii*) $3y^2 + 15z^2$, $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$, positive semidefinite

11. 2, 1

12. Indefinite.

Problems 2.11, Page 71

8. $\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + i\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix}$

Problems 2.12, page 72

1. (*b*) **2.** (*a*) **3.** (*c*) **4.** (*c*) **5.** (*c*) **6.** (*a*)
7. (*a*) **8.** (*b*) **9.** (*d*) **10.** (*a*) **11.** (*d*) **12.** (*c*)
13. (*b*) **14.** (*d*) **15.** (*c*) **16.** 2 **17.** sum **18.** 0, 8
19. 2 **20.** $\begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$ **21.** 0

22. All the eigen values are ≥ 0 and at least one eigen value is zero.

23. (*a*) $n = p$, (*b*) $m = p$, $n = q$ **24.** 8 **25.** (*b*)

26. $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ 6 & 9 & 3 \end{bmatrix}$ **27.** $x^2 + 4xy - 4y^2$ **28.** $x = y = z = 0$ **29.** 2, 2, 8
30. $A^2 = A$ **31.** 2 **32.** 1, 4, 9 **33.** (*iv*)
34. 4 **35.** zero **36.** Indefinite **37.** 1 – 1
38. The elements of its leading diagonal **39.** 2 **40.** $\lambda_i, i = 1, 2, ..., n$
41. (*c*) **42.** A or A^T **43.** 1, 1/2, 1/3 **44.** $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$
45. 1, 1/3 **46.** $x = 3 - 1$ **47.** Symmetric ; skew-symmetric **48.** 7 ; 5
49. $\begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}$ **50.** its determinant **51.** $\lambda^2 - 6\lambda + 3 = 0$ **52.** Augmented matrix
53. $\begin{bmatrix} 4 & -1 & 3 \\ -1 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ **54.** $\lambda_1^3, \lambda_2^3, \lambda_3^3$ **55.** $1/\lambda$ **56.** $\begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$
57. 38 **58.** 2 **59.** Index = 2, Signature = 1
60. False **61.** False **62.** False **63.** True
64. False **65.** True **66.** False **67.** True
68. False **69.** True **70.** True **71.** True.

Problems 3.1, page 80

1. (*i*) $\sqrt{159}$; $6/\sqrt{159}, 1/\sqrt{159}, 11/\sqrt{159}$; (*ii*) $\sqrt{131}$; $9/\sqrt{131}, -7/\sqrt{131}, 1/\sqrt{131}$
3. 90° **5.** $x = 1, y = -1$ **11.** 2 : 1.

Problems 3.2, page 88

1. 5 **2.** (*ii*) A, B, C form a Δ, rt. $\angle$*ed* at C **4.** $\cos^{-1}(.62)$
6. 2.11 **7.** 13 ; 12/13, 4/13, 3/13 **13.** 60°
14. $\cos^{-1}\left(-\frac{2}{3\sqrt{21}}\right)$.

Problems 3.3, page 89

1. $\mathbf{I} - 10\mathbf{J} - 18\mathbf{K}, \frac{1}{5\sqrt{17}}\mathbf{I} - \frac{2}{\sqrt{17}}\mathbf{J} - \frac{18}{5\sqrt{17}}\mathbf{K}, \sin^{-1}\left(\frac{5\sqrt{17}}{21}\right)$ **3.** $(2\mathbf{J} + \mathbf{K})/\sqrt{5}$
5. $\frac{1}{2}\sqrt{(94)}$ **6.** (*b*) $10\sqrt{3}$ **7.** $-2/\sqrt{26}$.

Problems 3.4, page 92

1. 40 **2.** 17 ; $-24\mathbf{I} + 13\mathbf{J} + 4\mathbf{K}$ **3.** 3.33
4. 70.5 **5.** $2\mathbf{I} - 7\mathbf{J} - 2\mathbf{K}$; $\sqrt{(57)}$ **6.** (1, 2, 2)
7. 6 **8.** 8.25 **9.** $\frac{5}{6}(-3\mathbf{I} + 2\mathbf{J} + 10\mathbf{K}), \frac{5}{6}\sqrt{113}$.

Problems 3.5, page 96

1. 7 **2.** –4 **3.** (*ii*) Yes **4.** Not linearly dependent
5. 5/6 **6.** (*i*) 15 ; (*ii*) $1\frac{1}{3}$ **11.** (*i*) $-7\mathbf{I} - 11\mathbf{J} + 5\mathbf{K}$; (*ii*) $-30\mathbf{I} - 15\mathbf{J} + 15\mathbf{K}$
15. (*b*) $\frac{1}{6}abc\begin{vmatrix} 1 & \cos\psi & \cos\phi \\ \cos\psi & 1 & \cos\theta \\ \cos\phi & \cos\theta & 1 \end{vmatrix}$.

Problems 3.6, page 101

1. $2x - y + 3z = 9$ **2.** $\mathbf{R} \cdot (2\mathbf{I} + 2\mathbf{J} + \mathbf{K}) = 5$ **3.** $\frac{1}{3}(2\mathbf{I} + 2 + \mathbf{K})$

4. 3 **5.** $4x - 3y + 2z = 3$ **7.** $x - 5y - 2z + 7 = 0$

8. 2 ; $2x + 2y - 3z = 6$ **9.** $\sum(x_1 - x_2)\left\{x - \frac{1}{2}(x_1 + x_2)\right\} = 0$ **10.** $y = 2$

11. $3x + 4y - 5z = 9$ **12.** $k = 10.2$; $5x - 15y - 21z = 34$ **13.** $1/6$. $2x - 3y + 6z + 5 = 0$

14. $\cos^{-1}(\sqrt{2}/3)$ **15.** (*i*) $\frac{5}{\sqrt{83}}, \frac{-7}{\sqrt{83}}, \frac{-3}{\sqrt{83}}$ (*ii*) 83.7° (*iii*) $5x - 7y - 3z + 7 = 0$

17. $6x + 3y - 2z = 18$; $2x - 3y - 6z = 6$ **20.** $x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$

21. $xyz = 6k^3$

22. (*i*) $25x + 17y + 62z - 78 = 0$; (*ii*) $x + 35y - 10z - 156 = 0$; (*iii*) bisects the acute angle.

Problems 3.7, page 105

1. $\frac{x-3}{1} = \frac{y-2}{3} = \frac{z-4}{-2}$ **2.** 43°3′ **3.** 90°

4. $x + 2 = \frac{y-3}{2} = \frac{z-4}{2}$ **5.** 3 **6.** $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z+1}{1}$

7. $\frac{x-1}{1} = \frac{y+2}{-19} = \frac{z-3}{3.5}$; $\frac{x-1}{11} = \frac{y+1}{13} = \frac{z-3}{-21.5}$ **8.** (3, 4, 5)

9. 4.1 **10.** 8.57

11. (3, 4, 5) ; (*ii*) (26/7, – 15/7, 17/7) **12.** 40°15′

13. $29x - 27y - 22z = 85$ **14.** $2 - x = y + 1 = (z + 1)/3$.

Problems 3.8, page 107

1. $7x - 2y - 3z = 0$ **2.** $2x + 3y + 6z = 38$ **3.** $11x + 12y - 8z = 5$

4. $3y - z = 2$ **5.** $\frac{x-4}{7} = \frac{y-6}{-13} = \frac{z+2/3}{9}$ **6.** $x + y + 2z = 1, x + y + (2/5)z = 1$

7. $\frac{x+4/15}{-11} = \frac{y-2/5}{9} = \frac{z}{15}$ **8.** $\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}$.

Problems 3.9, page 110

1. $x - 2y + z = 0$ **2.** (5, – 7, 6)

3. $\frac{a\alpha + b\beta + c\gamma + d}{al + bm + cn} = \frac{a'\alpha + b'\beta + c'\gamma + d'}{a'l + b'm + c'n}$ **4.** (0, 1, 2) ; $4x + y - 2z + 3 = 0$

7. $-\frac{1}{6}(x - 5) = y - 3 = \frac{1}{2}(z - 13)$ **8.** $\frac{x-2}{7} = \frac{y-3}{4} = \frac{z-1}{-5}$

9. (2, 8, – 3) ; (0, 1, 2) ; 8.83.

Problems 3.10, page 113

1. $1/\sqrt{6}$; $11x + 2y - 7z + 0 = 0$; $7x + y - 5z + 7 = 0$

2. 10.77 ; $\frac{1}{2}(x - 3) = \frac{1}{3}(y - 5) = \frac{1}{4}(z - 7)$; (3, 5, 7) ; (– 1, – 1, – 1)

3. $\frac{1}{\sqrt{5}}$; $3x - 10y + 6z - 1 = 0 = x + 2z$.

Problems 3.11, page 115

4. First and second planes cut along $x - 36 = -\frac{1}{2}(y + 22) = z$.

Problems 3.12, page 118

1. $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$; $(2, -3, -1)$; 3
2. $x^2 + y^2 + z^2 - 2x + 2y - 4z = 0$; $(1, -1, 2)$; $\sqrt{6}$
3. (a) $x^2 + y^2 + z^2 - 4x - 4y - 4z + 3 = 0$ (b) $3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$
7. (i) $x^2 + y^2 + z^2 - ax - by - cz = 0$,

   (ii) $x^2 + y^2 + z^2 - ax - by - cz = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$; $\left[\frac{a(b^{-2} + c^{-2})}{2\Sigma a^{-2}}, \frac{b(c^{-2} + a^{-2})}{2\Sigma a^{-2}}, \frac{c(a^{-2} + b^{-2})}{2\Sigma a^{-2}}\right]$
8. $(1, 3, 4)$; $\sqrt{7}$
10. $x^2 + y^2 + z^2 + 2(x + y + z + 1) = 0$
11. $13(x^2 + y^2 + z^2) - 35x - 21y + 43z + 176 = 0$
12. $3(x^2 + y^2 + z^2) - 7x - 8y + z + 10 = 0$
13. $x^2 + y^2 + z^2 + 7y - 8z + 24 = 0$.

Problems 3.13, page 120

1. (i) $x + 3 = 0, x - 7 = 0$ (ii) $x + 2y + 2z = 9,\ 2x + y - 2z = 9$
2. $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$ and $5(x^2 + y^2 + z^2) - 4x - 8y - 12z - 13 = 0$
3. (i) $x^2 + y^2 + z^2 - 10y - 10z - 31 = 0$
4. (ii) $x^2 + y^2 + z^2 - 14(x + y + z) + 98 = 0$
6. $\frac{x - 0.6}{-2} = \frac{y - 2.4}{7} = \frac{z}{5}$
8. $3\sqrt{6}, \sqrt{6}$
9. $3x + y + z + 6 = 0$
10. $(12/5, 4, 9/5)$.

Problems 3.14, page 124

1. $(\beta z - \gamma y)^2 = 4a(\alpha z - \gamma x)(z - \gamma)$
2. $528x^2 + 363y^2 + 76z^2 - 528xy - 264yz + 353zx + 704x + 1352z - 4436 = 0$
3. $5x^2 + 3y^2 + z^2 - 2xy - 6yz - 4zx + 6x + 8y + 10z - 26 = 0$
4. $x^2 + y^2 - 3z^2 - 2x - 2y + 6z - 1 = 0$
5. $x^2 + y^2 = z^2 \tan^2 \alpha$
6. $x^2 + 7y^2 + z^2 + 8xy + 8yz - 16zx = 0$
7. $4x^2 + 40y^2 + 19z^2 - 48xy - 72yz + 36zx = 0$
8. $x^2 - y^2 + z^2 + 4y - 4z = 0$
9. $yz \pm zx \pm xy = 0$, $\cos^{-1}(1/\sqrt{3})$; $x = y/\pm 1 = z/\pm 1$
10. $\cos^{-1} 4/\sqrt{(41)}$; $25x^2 - 16y^2 - 16z^2 = 0$
11. $4x^2 + 4y^2 - z^2 + 20z - 100 = 0$
12. $x = y/2 = z/-1$; $x/-2 = y = z$
14. $-2x^2 + y^2 - 2z^2 + 4xy - 8xz + 4yz + 8x - 10y + 8z - 3 = 0$.

Problems 3.15, page 126

1. $5x^2 + 8y^2 + 5z^2 + 4yz + 8xz - 4xy - 144 = 0$
2. $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$
3. $45x^2 + 40y^2 + 13z^2 + 12xy + 36yz - 24zx - 42x - 112y - 126z - 392 = 0$
4. $x^2 + y^2 + z^2 - yz - zx - xy = a^2$
5. $9x^2 + 5y^2 + 9z^2 + 12xy + 6yz - 36x - 30y - 18z + 36 = 0$; π units
6. $x^2 + y^2 - 2x - 4y - 11 = 0$
7. $a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(ny - mz)^2 + 2gn(nx - lz) + 2fn(ny - mz) + cn^2 = 0$.

Problems 3.16, page 131

1. Ellipsoid, 33.51
2. Hyperboloid of revolution of one sheet ; Hyperbola $5x^2 - y^2 = 6$. No area

3. Right circular cylinder with axis along z-axis
4. Hyperbolic paraboloid
5. Hyperboloid of two sheets
6. Parabolic cylinder
7. Right circular cylinder
8. Cone with vertex at the origin
9. Hyperbolic paraboloid
10. Hyperboloid of two sheets.

Problems 3.17, page 131

1. (b)	2. (a)	3. (c)	4. (b)	5. (b)	6. (a)
7. (c)	8. (d)	9. (c)	10. (c)	11. (d)	12. (c)
13. (b)	14. (c)	15. (c)	16. (c)	17. (c)	18. (b)
19. (c)	20. (c)	21. (b)	22. (b)	23. (c)	24. (c)

25. (b)
26. (a)
27. $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$
28. 0, 0, 1
29. $x = 0, y = 0$
30. (– 3, 2, – 1)
31. 8 or – 10
32. $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$
33. $y^2 + z^2 = (bx/a)^2$
34. $12x + 31y - 20z = 66$
35. 523.6
36. $\cos^{-1}(6/\sqrt{42})$
37. $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = (x + y + z)^2$
38. (3, 5, 7), (5, 8, 11)
38. $-(x - 1) = y - 2 = \frac{1}{6}(z - 3)$
40. $x^2 + y^2 + z^2 + x - 6y - 7z + 9 = 0$

41. $al + bm + cn = 0, ax_1 + by_1 + cz_1 + d = 0$
42. $2/\sqrt{26}$
43. $(3/2, -2, 2), 3\sqrt{5}/2$
44. $\sqrt{44}/3$
45. $\frac{x + 1/3}{1} = \frac{y + 2/3}{-2} = \frac{z}{1}$
46. Parabolic cylinder
47. Hyperboloid of two sheets
48. $\cos^{-1}\left(\frac{7}{\sqrt{84}}\right)$
49. 6, – 4, 12
50. 6
51. True
52. True
53. True
54. Elliptic cylinder
55. $4(x^2 + y^2 + z^2) + 9(xy + yz + zx) = 0$
56. $(\mathbf{I} - 2\mathbf{J} - 8\mathbf{K})/\sqrt{69}$

Problems 4.1, page 135

6. $8t^3/(1 - t^2)^3$
7. $-\frac{3}{2}$
8. $\sin t/a \cos^4 t$.

Problems 4.2, page 138

1. $(-1)^{n-1}(n-1)!\,2^n[(2x+1)^{-n} + (2x-1)^{-n}]$
2. $(-1)^n\left\{\frac{n!}{(x+1)^{n+1}} - \frac{(n-1)!}{(x+2)^n} + \frac{(n-1)!}{(x+1)^n}\right\}$
3. $\frac{1}{16}[2\sin(x + n\pi/2) + 3^n\sin(3x + n\pi/2) - 5^n\sin(5x + n\pi/2)]$

4. $\frac{1}{256}[9^n\cos(9\theta + n\pi/2) + 9.7^n\cos(7\theta + n\pi/2) + 36.5^n\cos(5\theta + n\pi/2) + 84.3^n\cos(3\theta + n\pi/2) + 126\cos\theta]$
5. $\frac{(20)^{n/2}}{2}[e^{2x}\sin(2x + n\tan^{-1}2) - e^{-2x}\sin(4x - n\tan^{-1}2)]$
6. $\frac{1}{2}e^{5x}S\,(41)^{n/2}\cos[4x + n\tan^{-1}(0.8)] + (29)^{n/2}\cos[2x + n\tan^{-1}(0.4)]$

7. $\dfrac{(-1)^n n!}{3}\left\{\dfrac{4}{(x-1)^{n+1}}\dfrac{1}{(x+2)^{n+1}}\right\}$ **8.** $(-1)^n n!\left\{\dfrac{9(2)^{n-1}}{(2x+3)^{n+1}}-\dfrac{4}{(x+2)^{n+1}}\right\}$

9. $\dfrac{(-1)^n n!}{3}\left\{\dfrac{4}{(x+1)^{n+1}}+\dfrac{i-1}{4}\dfrac{1}{(x+i)^{n+1}}-\dfrac{i+1}{4}\dfrac{1}{(x-i)^{n+1}}\right\}$

10. $\dfrac{(-1)^n n!\cos(n+1)\theta}{(x^2+a^2)^{(n+1)/2}}$ where $\theta=\tan^{-1}(a/x)$

11. $2(-1)^{n-1}(n-1)!\sin n\alpha\sin^n\alpha$ where $\alpha=\cot^{-1}x$.

Problems 4.3, page 141

1. (*i*) $\dfrac{(-1)^{n-3}(n-3)!}{x^n}\{(n-1)(n-2)+n(3-n)x^2\}$;

(*ii*) $\dfrac{1}{256}[(\log 2)^n 2^x(\cos 9\theta+9\cos 7\theta+3\cos 5\theta+84\cos 3\theta+126\cos\theta)]+{}^nC_1(\log 2)^{n-1}2^x\{\cos 9\theta+\pi/2)$ $+9\cos(7\theta+n\pi/2)+36\cos(5\theta+\pi/2)+84\cos(3\theta+\pi/2)+126\cos\theta+\pi/2\}+\ldots\ldots+2^x\{\cos(9\theta+n\pi/2)$ $+9\cos(7\theta+n\pi/2)+36\cos(5\theta+n\pi/2)+84\cos(3\theta+n\pi/2)+126\cos(\theta+n\pi/2)\}]$

5. $y_{2m}(0)=0,\ y_{2m+1}(0)=(-1)^m.(2m)!$

7. $(y_n)_o=0$, if n is even

$=m(1^2-m^2)(3^2-m^2)\ldots\ldots[(2n-1)^2-m^2]$, if n is odd

8. $(y_{2n})_0=e^{m\pi/2}m^2(2^2+m^2)(4^2+m^2)\ldots\ldots[(2n-2)^2+m^2]$

$(y_{2n+1})_o=-e^{m\pi/2}m(1^2+m^2)(3^2+m^2)\ldots\ldots[(2n-1)^2+m^2]$

17. $\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots\ldots(m^2-2^2)m^2$, n even

$\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots\ldots(m^2-1^2)m$, n odd.

Problems 4.4, page 146

2. $x=(2m-1)a/(2m+2n-1)$

3. (*i*) $c=3.154, 0.846$; (*ii*) $c=\pi/2$; (*iii*) $c=e-1$. (*iv*) $c=0.5413$

6. 0.36 **12.** $\theta=0.25$.

Problems 4.5, page 150

1. $x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots\ldots\infty$ **2.** $x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\ldots\ldots\infty$

3. $1+x-\dfrac{x^2}{2}+\dfrac{x^3}{6}+\dfrac{x^4}{24}+\ldots\ldots$ **4.** $x+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5}+\dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^7}{7}+\ldots\ldots\infty$

5. $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots\ldots\infty$ **6.** $\dfrac{x^2}{2}+\dfrac{x^4}{12}+\dfrac{x^6}{45}+\ldots\ldots$

24. $\dfrac{m\sin\theta}{1!}-\dfrac{m(m^2-1^2)}{3!}\sin^3\theta+\dfrac{m(m^2-1^2)(m^2-3^2)}{5!}\sin^5\theta-\ldots\ldots$

25. $4+21(x-1)+13(x-1)^2+2(x-1)^3$ **26.** (*i*) $e\left\{1+\dfrac{(x-1)}{1!}+\dfrac{(x-1)^2}{2!}+\dfrac{(x-1)^3}{3!}+\ldots\ldots\right\}$

(*ii*) $\dfrac{\pi}{4}+\dfrac{1}{2}(x-1)-\dfrac{1}{4}(x-1)^2+\dfrac{1}{12}(x-1)^3$ **27.** $1-\dfrac{(x-\pi/2)^2}{2!}+\dfrac{(x-\pi/2)^4}{4!}-\ldots\ldots;\ .9998$

29. $\log(0.5) - \sqrt{3}(x - \pi/3) - 2(x - \pi/3)^2 - \frac{4\sqrt{3}}{3}(x - \pi/3)^3 + \ldots\ldots$

30. 0.8482 **31.** (*i*) 2.6121. (*ii*) 1.12.

Problems 4.6, page 154

1. $\log_e (a/b)$ **2.** $-1/3$ **3.** $1/3$ **4.** $a \log a$ **5.** 1 **6.** 1/18
7. 1/2 **8.** 1/12 **9.** 3/2 **10.** 0 **11.** 1/30 **12.** 1
13. 1/3 **14.** 2 **15.** 1 **16.** 1 **17.** 2 **18.** $11e/24$
19. $a = 2 ; 1$ **20.** $a = 5, b = -5$ **21.** $a = 1, b = 2, c = 1$.

Problems 4.7, page 156

1. $-1/3$ **2.** 1/2 **3.** -2 **4.** $-1/3$ **5.** 2/3 **6.** $1/e$
7. ae **8.** 1 **9.** e **10.** $1/\sqrt{e}$ **11.** $1/e$ **12.** 0
13. 0 **14.** 1 **15.** $e^{-1/6}$ **16.** e **17.** $e^{2/\pi}$ **18.** $e^{1/12}$
19. $-\frac{1}{2}$ **20.** $(6)^{1/3}$.

Problems 4.8, page 160

1. $x - 20y = 7 ; 20x + y = 140$ **2.** (a, b) **10.** $\pi/4$
14. $T = 2a \sin t/2 ; N = 2a \tan t/2. \sin t/2 ; S.T. = a \sin t ; S.N. = 2a \sin^2 t/2. \tan t/2$
15. $a \sin^3 \theta \tan \theta$.

Problems 4.9, page 162

7. (*i*) $\pi/2$; (*ii*) $\pi/2$.

Problems 4.10, page 166

4. $r^3 = 2ap^2$ **5.** $r^3 = a^2 p$ **6.** $pa^m = r^{m+1}$ **7.** $r^{m+1} = \sqrt{2}\, a^m p$
8. $(1 + m^2) p^2 = r^2$ **9.** (*i*) $\sqrt{(1 + 9x/4a)}$; (*ii*) $\cosh x/c$ **10.** $a\theta$
11. (*i*) $2a \sin \theta/2$; (*ii*) $a\sqrt{(\sec 2\theta)}$; (*iii*) $r\sqrt{(8r - 3)}$.

Problems 4.11, page 172

1. (*i*) $2a(1 + t^2)^{3/2}$; (*ii*) y^2/c (*iii*) $(1 + a^3)^{3b}/6a^2$
5. (*i*) $(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}/ab$; (*ii*) $4a \sin \theta/2$; (*iii*) at
11. (*i*) 3/2 ; (*ii*) 1 ; (*iii*) $\sqrt{2a}$
12. (*i*) $\frac{4a}{3} \sin \frac{\theta}{2}$; (*ii*) $a^n/(n + 1) r^{n-1}$ **14.** $2\sqrt{(r^3/a)}$

Problems 4.12, page 176

1. $a(2 + 3t)t^2, -4\sqrt{2}\, at^{3/2}$ **4.** (*i*) $x = a(t - \sin t), y - 2a = a(1 + \cos t)$, (*ii*) $x = a \cos \theta, y = a \sin \theta$
5. $(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$

7. (*i*) $(x - 3a/4)^2 + (y + 3a/4)^2 = a^2/2$ (*ii*) $x^2 + y^2 - \frac{21}{8}(x + y) + \frac{432}{128} = 0$
11. $y^2 = 4ax$ **12.** $(x/a)^2 + (y/b)^2 = 1$ **13.** $27ay^2 = 4(x - 2a)^3$.

14. $(x/a)^2 + (y/b)^2 = 1$ **15.** $y = \dfrac{u^2}{2g} - \dfrac{gx^2}{2u^2}$

16. (*i*) $\sqrt{x} + \sqrt{y} = \sqrt{c}$; (*ii*) $4xy = c^2$; (*iii*) $x^{2/3} + y^{2/3} = c^{2/3}$ **17.** $x^{2/3} + y^{2/3} = c^{2/3}$

Problems 4.13, page 181

2. $a = 1, b = 1/4$, Point of minima **4.** $x = 0.42l$ **5.** $v = (aw^2/3b)^{1/4}$

8. $\theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$; $(1 - \sin\alpha)/(1 + \sin\alpha)$ **10.** Sq. with side $\sqrt{2}a$

13. $8 + 2\sqrt{7}, 2 + 2\sqrt{7}, 5 - \sqrt{7}$ **14.** Depth is half the width **15.** $(a^{2/3} + b^{2/3})^{3/2}$

16. $3\sqrt{3}a/4$ **25.** 2.5 km/hr.

Problems 4.14, page 185

1. $x + y + a = 0$ **2.** $x = \pm a, y = \pm b$ **3.** $x = \pm a,\ y = \pm b$

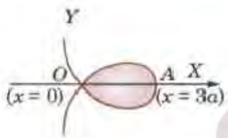
4. $y = 0$; $x + 1 = 0$; $x + y = 0$ **5.** $y = x, y + 2x = 0, y + 2x + 1 = 0$

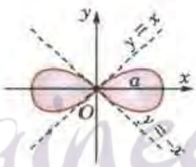
6. $x + a = 0$; $x - a = 0$; $x - y + \sqrt{2}a = 0$; $x - y - \sqrt{2}a = 0$

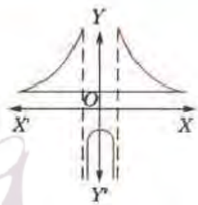
7. $x + 2y + 2 = 0, x + y = \pm\ 2\sqrt{2}$ **10.** $r\cos\theta = a$; $r\cos\theta = -a$ **11.** $r\cos\theta = 0$; $r\cos\theta = 2a$

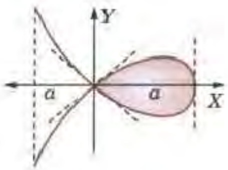
12. $r\sin\theta = 2$ **13.** $r\sin(\theta - m\pi/n) = a/n\cos m\pi$.

Problems 4.15, page 194

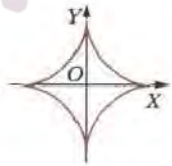
1.


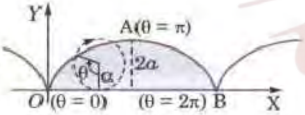
2.


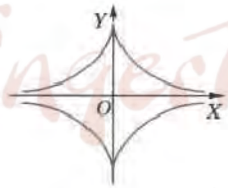
3.


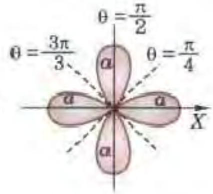
4.


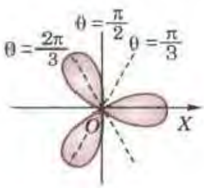
5.

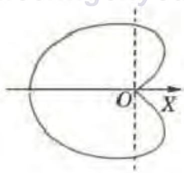

6.


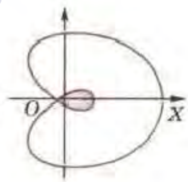
7.


8.


9.


10.


11.


12.


Problems 4.16, page 194

1. c **2.** $x^2 + 4ay = 0$ **3.** 1/5 **4.** (*c*) **5.** (*a*) **6.** (*b*)
7. (*c*) **8.** (*b*) **9.** (*b*) **10.** (*c*) **11.** (*b*) **12.** (*c*)
13. (*c*) **14.** (*b*) **15.** $x^2 = 4y$ **16.** of constant length

17. $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$ **18.** $\frac{1}{4}\left[2^n \cos\left(2x + n\frac{\pi}{2}\right) + 4^n \cos\left(4x + n\frac{\pi}{2}\right) + 6^n \cos\left(6x + n\frac{\pi}{2}\right)\right]$

19. $-32/3a$ **20.** True **21.** $-a$ **22.** $2a(1 + t^2)^{3/2}$
23. $(x - a)^2 + (y - b)^2 = k^{-2}$ **24.** envelope **25.** $xy = c^2$
26. α **27.** $2a$ **28.** $e^x(x^3 - 12x^2 - 36x - 24)$
29. (*iii*) **30.** $(x/a)^2 + (y/b)^2 = 1$ **31.** (B)
32. $c = 2.5$ **33.** $x = y$ **34.** node
35. Four loops of $r = a \sin 2\theta$ and three loops of $r = a \cos 3\theta$.
36. $y = \pm x$ **37.** $x = 4$ **38.** $4b$
39. (A) **40.** $r > a$ **41.** (D)
42. (D) **43.** (C).

Problems 5.1, page 198

1. 2/3 **2.** Does not exist **3.** Zero **4.** Does not exist
7. Discontinuous.

Problems 5.2, page 202

1. (*i*) $xy(2 - \cos xy) - \sin xy$; $x^2(1 - \cos xy)$;
(*ii*) $2x/(x^2 + y^2)$, $2y/(x^2 + y^2)$;
(*iii*) $(x^2 + 2xy - y^2)/\{(x^2 + y^2)^2 + (x + y)^2\}$; $(y^2 + 2xy - x^2)/\{(x^2 + y^2)^2 + (x + y)^2\}$;
(*iv*) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = \frac{z}{1 - z}$

11. $n = 2, -3$ **18.** $e^{xyz}(x^2y^2z^2 + 3xyz + 1)$.

Problems 5.4, page 208

13. $2u$.

Problems 5.5, page 211

2. $4a^2t(t^2 + 2)$ **2.** $-2/(e^{2t} + e^{-2t})$ **3.** zero
4. 6.5 sq. ft./sec **6.** $8e^{4t}$.

Problems 5.6, page 214

9. 0 ; 0.

Problems 5.7, page 218

6. zero **7.** $x(yv + 1 - w) + z - 2uv$ **10.** 0 ; $u = \tan v$
11. $u^2 - v^2 = 8w$

Problems 5.8, page 220

1. $4X + Y + Z = 6$; $\dfrac{X-2}{4} = Y - 1 = Z + 3$ **2.** $3Y + 2Z - X - 3 = 0$, $1 - X = \dfrac{Y-2}{3} = \dfrac{Z+1}{2}$

3. $\dfrac{X}{x_1} + \dfrac{Y}{y_1} + \dfrac{Z}{z_1} = 3$; $x_1(X - x_1) = y_1(Y - y_1) = z_1(Z - z_1)$

4. $7X - 3Y + 8Z = 26$; $\dfrac{X-1}{7} = \dfrac{Y+1}{-3} = \dfrac{Z-2}{8}$

5. $(-1, 2, 2/3)$ **7.** $\dfrac{X-x}{x} = \dfrac{Y-y}{y} = \dfrac{Z-z}{z}$.

Problems 5.9, page 226

1. (i) $x - \dfrac{1}{6}(x^3 + 3xy^2)$

(ii) $$\frac{1}{2\sqrt{2}}\Big[1 + \{(x+1) + (y - \pi/4)\} + \frac{1}{2}\{(x+1)^2 - 2(x+1)(y-\pi/4) + (y-\pi/4)^2\} + \frac{1}{6}\{(x+1)^3 + 3(x+1)^2(y-\pi/4) - 3(x+1)(y-\pi/4)^2 - (y-\pi/4)^3\} + \ldots\Big]$$

(iii) $1 + x + \dfrac{1}{2!}(x^2 - y^2) + \dfrac{1}{6}(y^3 - 3xy^2) + \ldots$

2. $1 + (x-1) + (x-1)(y-1) + \dfrac{1}{2}(x-1)^2(y-1) + \ldots$

3. -0.8232 **4.** -4500 units **5.** 2% **7.** 2%

11. $\dfrac{\alpha}{2}\cot\alpha + 2$ **12.** Rs. 43.20 **13.** $(p - 3q - 4r)$% **14.** $-1\frac{1}{3}$% **15.** $5r$.

Problems 5.10, page 233

1. (i) (a, a) gives maximum if $a < 0$ and minimum if $a > 0$
(ii) Min. at (a, a) (iii) Max. at $(4, 0)$, Min. at $(6, 0)$
(iv) Max. at $(\pm 1, 0)$; Min. at $(0, \pm 1)$ (v) Max. at $(\pi/3, \pi/3)$; Min. at $(2\pi/3, 2\pi/3)$

2. 4, 2, 1 **3.** (i) $3a^2$; (ii) $p^2/(a^2 + b^2 + c^2)$; (iii) $3a^2$ **4.** $12 \times 12 \times 6$ cm

6. $(0, 0, \pm 1)$ **8.** 4, 1 **9.** 50

10. 4, 8, 12 **11.** Two stationary values of u are given by $\dfrac{l^2}{au-1} + \dfrac{m^2}{bu-1} + \dfrac{n^2}{cu-1} = 0$.

Problems 5.11, page 236

1. $\dfrac{1}{2a^3}\tan^{-1}\dfrac{x}{a} + \dfrac{x}{2a^2(x^2 + a^2)}$ **2.** $\dfrac{(-1)^n\, n!}{(m+1)^{n+1}}$

3. $\pi \log\left[\dfrac{1}{2} + \dfrac{1}{2}\sqrt{(1 - a^2)}\right]$ **4.** $-\pi/(a^2 - 1)^{3/2}$.

Problems 5.12, page 236

1. zero	**2.** (a)	**3.** 1	**4.** (b)	**5.** (b)	**6.** (b)
7. (c)	**8.** (c)	**9.** (d)	**10.** (d)	**11.** (b)	**12.** (d)
13. (a)	**14.** (d)	**15.** (c)	**16.** (b)	**17.** zero	**18.** $2/(x + y)$

19. $rt - s^2 < 0$ **20.** (*d*) **21.** $4u$ **22.** $\partial/(u, v)/\partial(x, y)$

23. $f_x(a, b) = 0, f_y(a, b) = 0$ **24.** (*c*) **25.** $\frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

26. (*c*) **27.** -1 **28.** (*c*) **29.** equal

30. False.

Problems 6.1, page 244

1. (*i*) 128/315 ; (*ii*) 8/45 **2.** (*i*) 128/315 ; (*ii*) $11\pi/192$

3. (*i*) $\frac{(2n-3)(2n-5)\ldots\ldots 3.1}{(2n-2)(2n-4)\ldots\ldots 4.2}\frac{\pi}{2}$ (*ii*) $\frac{1}{8}\left(\frac{\pi}{8}+\frac{1}{6}\right)$ **4.** $35\pi/10240$

5. (*i*) $3\pi/512$ (*ii*) 1/144 **6.** (*i*) $5\pi/256$; (*ii*) 1/15 **7.** (*i*) $35\pi a^4/8$; (*ii*) $5\pi a^3/2$

8. (*i*) $5\pi/8$; (*ii*) 28π.

Problems 6.2, page 247

1. (*i*) $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$ (*ii*) $-\frac{1}{4}\cot^4 x + \frac{1}{2}\cot x + \log \sin x$

3. $\frac{1}{2}\log 2 - \frac{1}{4}$ **4.** $\frac{\pi}{4} - \frac{2}{3}$ **5.** $I_n = \frac{(2)^{(n-2)/2}}{n-1} + \frac{n-2}{n-1} + I_{n-2}$

6. (*i*) $\frac{1}{5}\sec^4 x \,.\, \tan x + \frac{4}{15}(\sec^2 x + 2)\tan x$ (*ii*) $\frac{11\sqrt{3}}{4} + \frac{3}{8}\log(2+\sqrt{3})$

(*iii*) $-\frac{1}{4}\cot x \operatorname{cosec} x - \frac{3}{8}\cot x \operatorname{cosec} x + \frac{3}{8}\log(\operatorname{cosec} x - \cot x)$

7. $\left\{\frac{67\sqrt{2}}{48} + \frac{5}{16}\log(1+\sqrt{2})\right\}a^6$ **8.** $\frac{t^5}{5} - \frac{t^3}{3} + t - \tan^{-1} t$.

Problems 6.3, page 250

1. $e^x(1 - x + x^2 - x^3 + x^4)$

3. $\int x^m (\log x)^n \, dx = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1}\, dx$; $\int_0^1 x^5 (\log x)^3 \, dx = -1/216$

5. 149/225 **6.** $3\pi^2/64 - 1/4$ **7.** $\frac{5}{16}\pi^4 - 15\pi^2 + 120$

11. $I_n = \frac{e^{ax}\cos^{n-1} x(a\cos x + n\sin x)}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2} I_{n-2}$; $\int_0^{\pi/2} e^{2x}\cos^3 x \, dx = \frac{2}{65}(3e^{\pi} - 11)$

12. 24/85.

Problems 6.4, page 254

7. (*i*) $3\pi/8$; (*ii*) $5\pi/8$; (*iii*) $3\pi/256$; (*iv*) $15\pi/640$ **8.** (*i*) $16\pi/35$; (*ii*) $8\pi/315$.

Problems 6.5, page 256

1. log 2 **2.** $\frac{1}{3}\log 2$ **3.** 1/3 **4.** $\pi/2$ **5.** $\frac{1}{4}\log 2$ **6.** $2e^{(\pi-4)/2}$.

Problems 6.6, page 260

1. (*i*) πab ; (*ii*) $8a^2/3$ **2.** $21\frac{1}{12}$ **3.** $2a^2/5$

4. (i) $8a^2/15\sqrt{3}$; (ii) $(2-\pi/2)a^2$ **5.** (i) πa^2 ; (ii) $\dfrac{\sqrt{2}}{3}a^2$ **6.** (i) πa^2 ; (ii) $4a^2$

8. $3\pi a^2$ **9.** $a^2/6$ **10.** $3\pi a^2$; πa^2

11. $\dfrac{11\pi}{3}-2\sqrt{3}$ **13.** $(3\pi-8)\,a/6$ **14.** $64a^2/3$

15. $1\frac{1}{8}$.

Problems 6.7, page 262

1. (i) $3\pi a^2/2$; (ii) a^2 **2.** (i) $\pi a^2/8$; (ii) $\pi a^2/12$ **5.** $(1-\pi/4)a^2$ **6.** $\pi a^2/2$.

Problems 6.8, page 265

1. $12\dfrac{11}{27}a$ **2.** (i) $\log(2+\sqrt{3})$, (ii) $\log_e(e+1/e)$

3. (i) $a\left[\sqrt{2}+\log(1+\sqrt{2})\right]$; (ii) $(15/16+\log 2)\,a$ **4.** (i) $4a/\sqrt{3}$; (ii) $4\sqrt{3}$

5. 37.85 **7.** (i) $8a$ **8.** $6a$

9. $4\sqrt{3}$ **11.** $2+\dfrac{1}{2}\log 3$ **12.** $8a$

13. $\sqrt{2}\,\pi a\left\{1+\left(\dfrac{1}{2}\right)^2+\left(\dfrac{1.3}{2.4}\right)^2.\left(\dfrac{1}{2}\right)^2+\left(\dfrac{1.3.5}{2.4.6}\right)^2.\left(\dfrac{1}{2}\right)^2+\ldots\ldots\right\}$ **14.** $2a\left[\sqrt{2}+\log(\sqrt{2}+1)\right]$.

Problems 6.9, page 269

1. $\pi c^3(1+\sinh 1\cosh 1)$ **2.** $\pi h^2(a-h/3)$ **3.** $2\pi a^3$

4. $\pi a^3/12$ **5.** (i) $\dfrac{4}{3}\pi ab^2$; (ii) $\dfrac{4}{3}\pi a^2b$ **6.** $\dfrac{\pi h}{3}(r^2+rR+R^2)$

7. $48\pi a^3$ **8.** (i) $2\pi a^3(\log 2-2/3)$; (ii) $\pi a^3/24$; (iii) $\pi/48$

9. (i) $5\pi^2a^3$; (ii) $5\pi^2a^3$ **10.** $32\pi a^3/105$ **11.** $4\pi^2a^3$

13. (i) $\dfrac{4}{3}\pi a^3$; (ii) $\dfrac{8}{3}\pi a^3$ **14.** $\dfrac{4}{3}\pi a(a^2+b^2)$ **15.** $\dfrac{\pi a^3}{4}\left\{\dfrac{1}{\sqrt{2}}\log(\sqrt{2}+1)-\dfrac{1}{3}\right\}$.

Problems 6.10, page 271

1. $\dfrac{\pi a^2}{2}(2+\sinh 2)$ **2.** $\dfrac{8\pi a^2}{3}(2\sqrt{2}-1)$ **3.** $2\pi ab\left\{\dfrac{b}{a}+\dfrac{a}{\sqrt{(a^2-b^2)}}\sin^{-1}\left[\sqrt{(a^2-b^2)}/a\right]\right\}$

4. $\dfrac{1}{2}\pi r^2h$; $\pi r\sqrt{(r^2+h^2)}$, where r is the base-radius and h the height of the cone

5. $4\pi a^2$ **8.** $\dfrac{64}{3}\pi a^2$ **9.** $\dfrac{64}{3}\pi a^2$ **12.** $4\pi a^2$

13. $\dfrac{32}{5}\pi a^2$ **14.** $4\pi a^2(1-1/\sqrt{2})$ **15.** $\pi a^2\,[3\sqrt{2}-\log(\sqrt{2}+1)]$.

Problems 6.11, page 271

1. (b) **2.** (c) **3.** (b) **4.** (c) **5.** (b) **6.** (c)
7. (c) **8.** (d) **9.** (b) **10.** (d) **11.** (c) **12.** (c)
13. (a) **14.** $\dfrac{3\pi a^2}{2}$ **15.** (iii) **16.** $\pi a^2/12$ **17.** 1 **18.** $7\pi/8$

19. (*iii*) **20.** (*iii*) **21.** $\pi h(r_1^2 + r_1 r_2 + r_2^2)$ **22.** (*c*) **23.** (*a*)
24. (*b*) or (*c*) **25.** (*a*).

Problems 7.1, page 280

1. 13 **2.** 3/35 **3.** $\frac{1}{2}(e-1)$ **4.** $\frac{1}{4}\pi \log(1+\sqrt{2})$

5. $a^4/8$ **6.** $\frac{\pi}{4}ab(a^2+b^2)$ **7.** 3/56 **8.** $\pi a/4$

9. 241/60 **10.** $1 - 1/\sqrt{2}$ **11.** $\frac{\pi a^2}{4}(\log e - \frac{1}{2})$ **12.** 1/24

13. $\frac{2}{3}a^4$ **14.** $\pi a^2/b$ **15.** 1

16. (*i*) $\int_0^{2a}\int_{y\sqrt{2a}}^{2a} f(x, y)\,dx\,dy - \int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} f(x, y)\,dx\,dy$, (*ii*) $\int_0^{\pi/2}\int_a^{ae^{\theta/2}} f(r, \theta)\,r dr\,d\theta$

18. $4a^2/3$ **19.** $45\pi/2$.

Problems 7.2, page 283

1. 4.5 **2.** 7/6 **3.** πa^2 **4.** $\frac{3}{2}\log_e 3 - \frac{2}{3}$
5. a^2 **6.** $a^2(1-\pi/4)$ **7.** 4/3 **8.** $a^2/4$.

Problems 7.3, page 284

1. $\frac{abc}{3}(a^2+b^2+c^2)$ **2.** $\frac{8}{3}abc(a^2+b^2+c^2)$ **3.** 4/35

4. $\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a - \frac{3}{8}$ **5.** $\frac{8}{3}\log 2 - \frac{19}{9}$ **6.** $\frac{1}{4}(13 - 8e + e^2)$
7. $5\pi a^3/64$.

Problems 7.4, page 291

1. $\pi/8$ **2.** $\pi/2$ **3.** $8\left(\frac{\pi}{2}-\frac{5}{3}\right)a^2$ **4.** $\frac{2^{n+3}}{n+4}$

5. $\frac{\pi}{4}-\frac{1}{2}$ **6.** 0 **7.** $\frac{15\pi a^4}{64}$ **9.** π

10. $\pi^2/8$ **11.** $4\pi a$ **12.** $\frac{1}{2}\left(\log 2 - \frac{5}{8}\right)$ **14.** $\pi a^8/12$

15. 3π **16.** 4π **17.** $\pi a^3(2-\sqrt{2})/3$ **18.** $16a^3/3$
19. $\pi a^3/8$ **20.** $128a^3/15$ **21.** $3\pi a^3$ **22.** $4\sqrt{3}\,\pi$
23. $8a^4/3$ **25.** $\frac{1}{4}$ **26.** $\frac{1}{6}abc$.

Problems 7.5, page 293

2. 64 **3.** $2(\pi-2)a^2$ **4.** $2\pi a^2$ **5.** $\frac{3\pi a^2}{4}$.

Problems 7.6, page 297

1. $182\frac{7}{24}\lambda$ 2. $21\,\pi\mu a^4/32$ 3. 30.375

4. $\left(\frac{3a}{20}, \frac{3a}{16}\right)$ 5. $\left[\frac{a(4a+3b)}{6(a+b)}, \frac{b(3a+b)}{6(a+b)}\right]$ 6. $\left(\frac{\pi a\sqrt{2}}{8}, 0\right)$

8. $\bar{x} = 3a/5$, $\bar{y} = 9a/40$ where $a = OA$ 9. (1/5, 1/5, 2/5).

10. $\bar{x} = 3/4$ 11. $\left(\frac{16a}{35}, \frac{16b}{35}, \frac{16c}{35}\right)$ 12. $\frac{27}{26}$ metres

13. $\left(\frac{3a}{8}, \frac{3\pi a}{16}\right)$ 14. Divides the diagonal in the ratio 7 : 5

15. $\left(\frac{a}{2}, \frac{2}{3}h\right)$ where a is the base, h the depth

16. C.P. lies on the radius $\perp$ to the bounding diameter at a depth $32a/(15\pi)$ from the centre.

Problems 7.7, page 301

1. $ab^3/12$ 2. $5Ma^2/4$ 3. $2M/9$ 4. $\frac{1}{3}M(a^2+b^2)$

5. $(21/32)\,\pi\rho a^4$ 6. $\frac{2}{5}MR^2$ 7. $\frac{1}{2}Mr^2$; $\frac{1}{12}\,M(3r^2+4h^2)$

8. (i) $\frac{3Mr^2}{10}$; (ii) $\frac{3M}{20}(r^2+4h^2)$; (iii) $\frac{M}{20}(3r^2+2h^2)$ 9. 104803770ρ

10. $\frac{1}{30}$ 11. $\frac{\pi\rho abc(a^2+b^2)}{30}$ 12. $\frac{\rho a^2 b^2}{8}$.

Problems 7.8, page 309

1. (i) 3.323, (ii) 11.629 ; (iii) $\pi\sqrt{2}$; (iv) 0.1964 ; (v) 0.1227

2. (i) $\sqrt{\pi}/2$; (ii) $\Gamma(5/4)$; (iii) $\sqrt{\pi}/3$; (iv) $2^{p+q-1}\beta(p, q)$

4. $\pi/4\sqrt{2}$ 7. $-3/8$ 9. $\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{n\Gamma\left(\frac{m}{n}+p+1+\frac{1}{n}\right)}$, (i) $\frac{1}{396}$ (ii) $\frac{\sqrt{\pi}}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

10. 16/35 15. $\frac{ka^2b^2c^2}{48}$.

Problems 7.9, page 312

2. $\frac{1}{2}K(\sqrt{3}/2)$ 3. $\frac{2}{\sqrt{3}}\left\{F\left(\sqrt{\frac{2}{3}}, \frac{1}{2}\pi\right) - F\left(\sqrt{\frac{2}{3}}, \frac{1}{4}\pi\right)\right\}$ 4. $2\sqrt{2}E(1/\sqrt{2}) - \sqrt{2}K(1/\sqrt{2})$

5. $erf(x) = \frac{2}{\pi}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\ldots\right)$; $erf(o) = 0$ 6. (i) 0.3248 ; (ii) 0.5204.

Problems 7.10, page 313

1. 4 2. Area of the triangle having vertices (0, 0), (0, 1), (1, 0)

3. $\sqrt{\pi}/2$ 4. 3.1416 5. $15\sqrt{\pi}/8$ 6. 92π

6. 26 8. $-1/3$ 9. $1/2\ \beta(4, 3/4)$ 10. 1

11. 3/4 **12.** $\frac{\pi a^3}{6}$ **13.** (*d*) **14.** 27/4

15. $\pi a^2/12$ **16.** – 1/3 **17.** ∞ **18.** 44/105

19. $r^2 \sin\theta\, dr\, d\theta\, d\phi$ **20.** $e^2 - 1$ **21.** $\frac{1}{4}\pi ab(a^2+b^2)$. **22.** $\frac{1}{4}\pi \log(1+\sqrt{2})$

23. 3/256 **24.** (*c*) **25.** $\frac{6}{25} + \frac{1}{2}\sin\frac{3}{5}$ **26.** 48/5

27. 1/6 **28.** 16/3 **29.** $\int_0^a \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\, dy\, dx$

30. $\sqrt{2}\,\pi$ **31.** $3\pi/256$ **32.** 1/2 **33.** $\int_0^{\pi/2} \int_0^{2a\cos\theta} r\, d\theta\, dr$

34. $\frac{7h}{12}$ **35.** $\left(\frac{3}{20}, \frac{3}{16}\right)$ **36.** 16 **37.** $3Mr^2/10$

38. 1 **39.** log 2 **40.** $\frac{1}{2}\sqrt{\pi}$ **41.** (*c*) **42.** (*b*).

Problems 8.1, page 318

3. (*i*) $t^3 \sin t + 7t^2 \cos t + 20t \sin t - 10t$; (*ii*) $(20t^3 + t\sin t - \cos t)\,\mathbf{I}$
$-(2t\cos t + 2\sin t + 75t^2)\,\mathbf{J} - t(t\sin t + 2t^2\cos t + 10\cos t)\mathbf{K}$

5. $-4(\mathbf{I} + 2\mathbf{J})$ **6.** (*i*) $(ua^2 \sec\alpha)$. (*ii*) $a^3\tan\alpha$; $(\cos t\,\mathbf{J} - \sin t\,\mathbf{I})\cos\alpha + \sin\alpha\,\mathbf{K}$

7. $[t\mathbf{I} + 2\mathbf{J} + (2t-3)\mathbf{K}]/\sqrt{(5t^2 - 12t + 13)}$; $\frac{1}{3}(2\mathbf{I} + 3\mathbf{J} + \mathbf{K})$

8. $(x - a/\sqrt{2}) = y - a/\sqrt{2} = \left(z - \frac{a\pi}{4}\tan\alpha\right) / \sqrt{2}\tan\alpha$

9. (*i*) $ab/(a^2\sin^2 t + b^2\cos^2 t)^{3/2}$; (*ii*) $1/4\sqrt{2}$

10. (*i*) $\mathbf{R} = (p+q)\,\mathbf{I} + q\mathbf{J} + 2q\mathbf{K}$; $\frac{2\mathbf{J} - \mathbf{K}}{\sqrt{5}}$

(*ii*) $\mathbf{R} = p\mathbf{I} + (p+2q)\,\mathbf{J} + (p+q)\mathbf{J} + (p+q)\mathbf{K}$; $(2\mathbf{K} - \mathbf{I} - \mathbf{J})/\sqrt{6}$.

Problems 8.2 page 321

1. $v(\text{at } t = 0) = \sqrt{37}$, $a(\text{at } t = 0) = \sqrt{325}$ **2.** $\alpha = \pm 1/\sqrt{6}$

3. $8\sqrt{14/7}$; $-\sqrt{14/7}$ **6.** (*a*) d^2s/dt^2 ; v^2/ρ ; (*b*) 0 ; 3

7. $\sqrt{(17)}$ m.p.h. in the direction $\tan^{-1}(0.25)$ North of East

8. 21.29 knots/hr. in the direction 74°47′ South of East.

Problems 8.3, page 325

1. (*a*) $2(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})/(x^2+y^2+z^2)$. (*b*) $\frac{6}{3} + c$ **2.** $(-\mathbf{I} + 3\mathbf{J} + 2\mathbf{K})/\sqrt{14}$

3. $12\frac{1}{3}$ **4.** $15/\sqrt{17}$ **5.** $a = 6, b = 24, c = -8$

6. $-260/(69)$; $\sqrt{(1056)}$ **7.** $a = \pm\frac{20}{9}, b = \pm\frac{55}{9}, c = \pm\frac{55}{9}$ **8.** $96(\mathbf{I} + 3\mathbf{J} - 3\mathbf{K})$; $96\sqrt{(19)}$

9. 9 **10.** $\frac{1}{3}(2\mathbf{I} + 2\mathbf{J} - \mathbf{K})$ **11.** $\cos^{-1}(1/\sqrt{22})$

12. $\cos^{-1}(-1/\sqrt{30})$ **13.** $a = -6, b = -10$.

Problems 8.4, page 333

1. (*i*) 12 ; $5\mathbf{I} - 16\mathbf{J} + 9\mathbf{K}$; (*ii*) 278 ; $5(27\,I - 54\,J + 8\,K)$; (*iii*) – 32 ; 0

4. $a = -2$; $4x(z - xy)\,\mathbf{I} + (y - 2yz + 4xy^2)\,\mathbf{J} + (2x^2 + y^2 - z^2 - z)\,\mathbf{K}$

13. (*i*) 0 ; (*ii*) $2(x + z)\,\mathbf{J} + 2y\,\mathbf{K}$

14. (*a*) $2n(2n - 1)/(x^2 + y^2 + z^2)^{n+1}$; $n = 1/2$

16. (*i*) $2(y^3 + 3x^2y - 6xy^2)z\mathbf{I} + 2(3xy^2 + x^3 - 6x^2y)\,z\mathbf{J} + 2(xy^2 + x^3 - 3x^2y)\,y\mathbf{K}$; (*ii*) Zero

17. $1724/\sqrt{21}$.

Problems 8.5, page 335

1. $75\frac{1}{3}\,\mathbf{I} + 360\,\mathbf{J} - 42\,\mathbf{K}$

2. $(t^3 - t + 2)\,\mathbf{I} + (1 - t^4)\mathbf{J} + (4 - 4\cos t - 3t)\,\mathbf{K}$

3. $\mathbf{V} = 6\sin 2t\mathbf{I} + 4(\cos 2t - 1)\mathbf{J} + 8t^2\mathbf{K}$; $\mathbf{R} = 3(1 - \cos 2t)\mathbf{I} + 2\sin 2t\mathbf{J} + \dfrac{8t^3}{3}\mathbf{K}$.

Problems 8.6, page 336

1. 0 **2.** 35 **3.** – 2/3 **4.** 5 **5.** $\dfrac{\pi^3\sqrt{2}}{3}$ **6.** zero

7. 303 **8.** $8\dfrac{8}{35}$ **9.** 9π **10.** $\left(2 - \dfrac{\pi}{4}\right)\mathbf{I} - (\pi - \frac{1}{2})\,\mathbf{J}$.

Problems 8.7, page 339

2. $3\dfrac{1}{3}$ **3.** 8.

Problems 8.8, page 341

3. πab **4.** πa^2 **5.** Zero **6.** 128/5 **7.** $35\pi a^4/16$.

Problems 8.9, page 345

3. $-2ab^2$ **5.** $\dfrac{19}{2}\pi$ **6.** Zero **10.** 2 **11.** 0 **12.** π.

Problems 8.10, page 350

4. 108π **7.** (*i*) $\frac{12}{5}\pi a^5$ (*ii*) $12(e - e^{-1})$ **8.** $4\pi a^3$

9. $\dfrac{\pi a^6}{12}$ **10.** $\dfrac{5}{4}\pi a^4 b$ **11.** -4π **12.** 8/3.

Problems 8.11, page 354

3. $14\frac{2}{3}$

4. $x^3y - y^2z^2 + z^3$

5. (*i*) $\frac{1}{3}(x^3 + y^3 + z^3 - 3xyz)$; (*ii*) $x^2y + y^2z + z$;

(*iii*) $xz^3 - yz + 3x^2y$. (*iv*) $x^2y^2 + y^2z^2 + xyz = 0$

6. (*i*) Yes, $\dfrac{a}{2}(x^2 + y^2 - 2z^2)$; (*ii*) Yes

7. $xy\sin z + \cos x + y^2z + c$

8. $x^2y + xz^3$; 202

9. $a = 4, b = 2, c = -1$

10. $a = 4$; $2x^2y - xz^3$; 47.

Problems 8.12, page 362

1. (i) $(\rho \sin 2\phi - z \sin \phi)\,\mathbf{T}_\rho - (2\rho \sin^2 \phi + z \cos \phi)\,\mathbf{T}_\phi + 3\rho \cos \phi\,\mathbf{T}_z$
   (ii) $(2p \cos^2 \phi - 3p^2 \sin^3 \phi)\,\mathbf{T}_p - (\rho \sin 2\phi + 3\rho^2 \sin^2 \phi \cos \phi)\,\mathbf{T}_\phi + \rho z \cos \phi\,\mathbf{T}_z$

2. (i) $r \sin \theta \Big[\{\sin \theta (1 + \sin^2 \phi) + r \cos^2 \theta \sin \phi\}\,\mathbf{T}_r + \{\cos \theta (1 + \sin^2 \phi) - r \sin \theta \cos \theta \sin \phi\}\,\mathbf{T}_\theta + \sin \phi \cos \phi\,\mathbf{T}_\phi \Big]$
   (ii) $r^2 \sin \theta \,\{(\sin^2 \theta \cos^2 \phi \sin \phi + \sin \theta \cos \theta \sin^2 \phi + \cos^2 \theta \cos \phi)\,\mathbf{T}_r + (\sin \theta \cos \theta \cos^2 \phi \sin \phi + \cos^2 \theta \sin^2 \phi - \sin \theta \cos \theta \cos \phi)\,\mathbf{T}_\theta + (\cos \theta \sin \phi \cos \phi - \sin \theta \sin^2 \phi \cos \phi)\,\mathbf{T}_\phi\}$

3. $\rho z \sin 2\phi\,\mathbf{T}_\rho + \rho z \cos 2\phi\,\mathbf{T}_\phi + \frac{1}{2}\rho^2 \sin 2\phi\,\mathbf{T}_z$.

Problems 8.13, page 363

1. $1/\sqrt{14}$, $2/\sqrt{14}$, 3/14
2. $\frac{1}{4}(x-2) = y - 1 = z + 3$
3. $du dv = \dfrac{1}{h_1 h_2}\, dx dy$
4. $4x - 3z + 2xz$
5. zero
6. $\frac{1}{2}\int_C (xdy - ydx)$
7. 3V
8. 3 ; 0
9. Irrotational
10. 4π
11. solenoidal
12. $-28/\sqrt{5}$
13. zero.
14. $-(y\mathbf{I} + z\mathbf{J} + x\mathbf{K})$.
15. $\dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$.
16. zero
17. $-(12\mathbf{I} + 5\mathbf{J} + 8\mathbf{K})$
18. zero
19. zero
20. zero
21. zero
22. $\mathbf{R}/r^2$; $nr^{n-2}\,\mathbf{R}$
23. §8.5(2)
24. $\dfrac{1}{\sqrt{21}}(2\mathbf{I} + 4\mathbf{J} - \mathbf{K})$.
25. 2, –2, 2
26. $6\dfrac{\sqrt{7}}{3}$
27. $2/r$
28. 7/3
29. zero
30. (c)
31. (c)
32. (b)
33. (c)
34. (a)
35. (a)
36. $5u$
37. zero
38. irrotational field
39. (a)
40. the rate at which fluid is originating at P per unit volume.
41. (a)
42. it gives the maximum rate of change of ϕ.
43. (iv)
44. (a)
45. (a)
46. (b)
47. (a)
48. (b)
49. zero
50. True
51. True.

Problems 9.1, page 366

1. Convergent	2. Convergent	3. Convergent	4. Divergent
5. Convergent	6. Convergent	7. Convergent	8. Divergent.

Problems 9.2, page 367

1. Convergent	2. Convergent	3. Oscillatory	4. Convergent
5. 15 m.			

Problems 9.3, page 372

1. Convergent	2. Convergent	3. Divergent	4. Divergent
5. Convergent	6. Conv. for $p > 2$; Div. for $p \le 2$.		7. Divergent
8. Convergent	9. Convergent	10. Convergent	11. Convergent

12. Divergent **13.** Divergent **14.** Convergent **15.** Convergent
16. Convergent **17.** Divergent **18.** Convergent.

Problems 9.4, page 376

1. Conv. for $x < 1$; Div. for $x \geq 1$ **2.** Conv. for $x < 1$; Div. for $x \geq 1$
3. Conv. for $x \leq 1$; Div. for $x > 1$ **4.** Conv. for ≥ 1 ; Div. for $x < 1$
5. Convergent for all values of p **6.** Convergent **7.** Convergent
8. Convergent **9.** Conv. for $x < 1$; Div. for $x \geq 1$ **10.** Convergent
11. Convergent **12.** Convergent **13.** Divergent
14. Convergent **15.** Conv. for $x < 1$, Div. for $x > 1$; Conv. for $p > 1$ and Div. for $p \leq 1$
16. Divergent **17.** Conv. if $\beta > \alpha > 0$; Div. if $\alpha \geq \beta > 0$.

Problems 9.5, page 379

1. Conv. for $x \leq 1$; Div. for $x > 1$ **2.** Conv. for $x \leq 1$; Div. for $x > 1$
3. Conv. for $x < 1$; Div. for $x \geq 1$ **4.** Conv. for $x < 2$; Div. for $x \geq 2$
5. Conv. for $x < e$; Div. for $x \geq e$ **6.** Conv. for $x \leq 1$; Div. for $x > 1$
7. Conv. for $x \leq 1$; Div. for $x > 1$ **8.** Conv. for $x^2 \leq 1$; Div. for $x^2 > 1$
9. Conv. for $x^2 < 4$; Div. for $x^2 \geq 4$ **10.** Convergent
11. Conv. for $x < 1/e$; Div. for $x \geq 1/e$ **12.** Conv. for $x < 1$; Div. for $x \geq 1$
13. Diverges

14. Conv. for $x < 1$; Div. for $x > 1$. When $x = 1$, Conv. for $b - a > 1$, Div. for $b - a \leq 1$.

Problems 9.6, page 381

1. Convergent **2.** Convergent **3.** Convergent
4. Convergent **5.** Conv. for $x < 1$; Div. for $x \geq 1$
6. Conv. for $x < \frac{1}{2}$; Div. for $x \geq \frac{1}{2}$ **7.** Convergent.

Problems 9.7, page 383

1. Oscillatory **2.** Convergent **3.** Convergent **4.** Convergent
5. Convergent **6.** Oscillatory **7.** Convergent **8.** Convergent
9. Convergent **10.** Oscillatory.

Problems 9.8, page 387

1. (*i*) and (*ii*) conditionally convergent
3. (*i*) Conditionally convgt. for $0 < p \leq 1$; (*ii*) Conditionally convgt
4. Absolutely convergent for (*i*) $0 < x < 1$; (*ii*) $-1 < x \leq 1$; (*iii*) $|x| \leq 1$.
5. Convergent for $x \leq 1$ and not convergent for $x > 1$
6. (*i*) $-1 < x \leq 1$; (*ii*) $-1 < x \leq 1$;
7. $-e < x \leq e$ **8.** (*i*) Absolutely convergent (*ii*) convergent
9. Absolutely convergent.

Problems 9.9, page 388

1. Conv. for $x < 1$; Div. for $x \geq 1$ **2.** Convergent **3.** Divergent
4. Convergent **5.** Divergent **6.** Conv. for $x < 1$; Div. for $x \geq 1$
7. Conv. for $x < 1$; Div. for $x \geq 1$ **8.** Conv. for $x < 1$; Div. for $x \geq 1$
9. Conv. for $x < 1/4$; Div. for $x \geq 1/4$ **10.** Conv. for $x < 2$; Div. for $x \geq 2$

11. Convergent for all x **12.** Conv. for $x < 1$; Div. for $x \geq 1$ **13.** Convergent
14. Absolutely convergent **15.** Convergent
16. Convergent for $p > 1$; divergent for $p \leq 1$.

Problems 9.10, page 391

1. Uniformly convergent for $0 \leq x \leq 1$. **2.** to
5. Uniformly convergent for all real values of x **6.** Uniformly convergent for $0 \leq x \leq 1/a$
10. (*i*) and (*ii*) Both converge uniformly for all real values of x.

Problems 9.11, page 392

1. (*c*)	**2.** (*d*)	**3.** (*a*)	**4.** (*b*)	**5.** (*c*)	**6.** (*d*)
7. (*a*)	**8.** (*b*)	**9.** (*b*)	**10.** (*d*)	**11.** (*c*)	
12. (*a*) $(-1, 1)$	(*b*) $(-1/2, 1/2)$	**13.** $-1 < x \leq 1$	**14.** $k > 1$	**15.** $a_n < k$	**16.** Oscillatory
17. All values of x		**18.** $k < 1$	**19.** Convergent.	**20.** Divergent.	**21.** $q - p > 1$
22. Divergent	**23.** Convergent.	**24.** $0 < x < 4$	**25.** yes	**26.** True	**27.** Convergent
28. Divergent	**29.** $x > 1$	**30.** $0 \leq x \leq 1$	**31.** (*b*)	**32.** (*c*)	**33.** (*d*)
34. (*b*)	**35.** True.				

Problems 10.1, page 400

1. $\dfrac{2 \sinh a\pi}{\pi} \left\{ \left(\dfrac{1}{2a} - \dfrac{a \cos x}{1^2 + a^2} + \dfrac{a \cos 2x}{2^2 + a^2} - \ldots \right) + \left(\dfrac{\sin x}{1^2 + a^2} - \dfrac{2 \sin 2x}{2^2 + a^2} + \dfrac{3 \sin 3x}{3^2 + a^2} - \ldots \right) \right\}$

$\dfrac{\pi}{\sinh \pi} = 2\left[\dfrac{1}{2^2 + 1} - \dfrac{1}{3^2 + 1} + \dfrac{1}{4^2 + 1} - \ldots \right]$.

Problems 10.2, page 401

1. No **2.** No **3.** Yes.

Problems 10.3, page 404

1. $\dfrac{1}{2}\pi - \dfrac{4}{\pi}\left(\cos x + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \ldots\ldots \right)$ **2.** $\dfrac{I_0}{\pi} + \dfrac{1}{2} I_0 \sin x - \dfrac{2I_0}{\pi} \sum_{n=1}^{\infty} \dfrac{\cos 2nx}{4n^2 - 1}, \dfrac{1}{2}$

3. $\dfrac{\pi^2}{6} - 2\left(\cos x - \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} - \ldots\ldots \right) - \dfrac{1}{\pi}\left\{ \left(\dfrac{2}{1^3} - \dfrac{\pi^2}{1} \right) \sin x - \left(\dfrac{2}{2^3} - \dfrac{\pi^2}{2} \right) \sin 2x + \ldots \right\}$

4. $2\left(\pi - \dfrac{4}{\pi} \right) \sin x - \pi \sin 2x + \dfrac{2}{3}\left(\pi - \dfrac{4}{9\pi} \right) \sin 3x - \dfrac{\pi}{2} \sin 4x + \ldots\ldots$

5. $\dfrac{4}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \ldots\ldots \infty \right)$.

Problems 10.4, page 408

1. $-\dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{\sin n\pi x}{n}$

2. (i) $\frac{a^2}{3}\ \frac{a^2}{11^2}\left\{\frac{1}{12}\cos\frac{2\pi x}{a}+\frac{1}{2^2}\cos 4\pi\frac{x}{a}+......\right\}\ \frac{a^2}{\pi}\left\{\frac{1}{1}\sin\frac{2\pi x}{a}\ \frac{1}{2}\sin\frac{4\pi x}{a}+......\right\}$

(ii) $f(t)=\frac{2}{3}+\frac{4}{\pi^2}\left(\cos\pi t-\frac{\cos 2\pi t}{2^2}+\frac{\cos 3\pi t}{3^2}-......\right)$

4. $\frac{3}{2}-\frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left\{\frac{\cos(2n-1)\pi x}{3}\right\}$

5. $\frac{E}{\pi}+\frac{E}{2}\sin\omega t-\frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2\omega t+\frac{1}{3.5}\cos 4\omega t+\frac{1}{5.7}\cos 6\omega t+...\right]$

6. $f(x)=\frac{\pi}{4}+2\sum_{n=1}^{\infty}(-1)^{n+1}\sin n\pi x$; put $x=1/2$.

Problems 10.5, page 412

1. $\frac{a^2}{3}+\sum_{n=1}^{\infty}\frac{4a^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{a}$

3. $1-\frac{1}{2}\cos x-\frac{2}{1.3}\cos 2x+\frac{2}{2.4}\cos 3x-\frac{2}{3.5}\cos 4x-...$

5. $\frac{1}{2}\pi-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+......\right)$

6. (i) $\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+......+\frac{\cos 2nx}{4n^2-1}+......\right)$ (ii) $\frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{4(-1)^{m+1}}{(4m^2-1)\pi}\cos\frac{2m\pi x}{l}$

7. $\frac{\pi}{2}+1-\frac{4}{\pi}\left(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+...\right);\ \frac{\pi^2}{8}$

8. $\frac{4k}{\pi}\left(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+......\infty\right)$.

Problems 10.6, page 416

2. $\frac{\pi}{2}-\frac{4}{\pi}\left[\cos x+\frac{1}{3^2}.\cos 3x+\frac{1}{5^2}.\cos 5x+......\right]$; $2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-......\right]$

3. $\frac{\pi^2}{3}-4\left[\cos x-\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x-\frac{1}{4^2}\cos 4x+...\infty\right]$

4. $\sum_{n=2}^{\infty}\frac{1}{n}\sin 2nx$

5. $\frac{1}{3}+\frac{4}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\cos n\pi x$

6. $\frac{8}{\pi^3}\left(\frac{\sin\pi t}{1^3}+\frac{\sin 3\pi t}{3^3}+\frac{\sin 5\pi t}{5^3}+......\right)$

7. $\frac{1}{n\pi}[1-(-1)^n]$

8. $-\frac{1}{2}\sin x+\sum_{n=2}^{\infty}\frac{(-1)^n.2n}{n^2-1}\sin nx$

9. $\sum_{n=1}^{\infty}\frac{2n\pi}{1+n^2\pi^2}(1-e\cos n\pi)\sin n\pi x$

10. $\sum_{n=1}^{\infty}\frac{4}{\pi n^3}[1-(-1)^n]\sin nx$

12. $\frac{l}{4}+\sum_{n=1}^{\infty}\frac{2l}{(n\pi)^2}\left\{2\cos\frac{n\pi}{2}-1-(-1)^n\right\}\cos\frac{n\pi x}{l}$

13. $\frac{8}{\pi}\cos\frac{\pi}{4}\left[\frac{\sin 2x}{1.3}-\frac{\sin 6x}{5.7}+\frac{\sin 10x}{9.11}+......\right]$

14. $\frac{2l^2h}{a(l-a)\pi^2}\left[\sin\frac{\pi a}{l}\sin\frac{\pi x}{l}+\frac{1}{2^2}\sin\frac{2\pi a}{l}\sin\frac{2\pi x}{l}+\frac{1}{3^2}\sin\frac{3\pi a}{l}\sin\frac{3\pi x}{l}+......\right]$.

Problems 10.7, page 419

3. $\pi^4/96$.

Problems 10.8, page 420

1. $\sum_{-\infty}^{\infty} \frac{(\sinh al \cos n\pi l - i \cosh al \sin n\pi l)(a + in\pi)}{(a^2 + n^2\pi^2)} e^{in\pi x/l}$

2. $\frac{2}{\pi}\left\{1 - \frac{e^{2it} + e^{-2it}}{1.3} - \frac{e^{4it} + e^{-4it}}{3.5} - \frac{e^{6it} + e^{-6it}}{5.7} - \ldots\ldots\right\}$

3. $\frac{a}{\pi} \sin a\pi \sum_{-\infty}^{\infty} \frac{(-1)^n e^{inx}}{a^2 - n^2}$

4. $\sin h\, 9 \sum_{-\infty}^{\infty} \frac{(-1)^n (9 + n\pi i)}{81 + (n\pi)^2} e^{n\pi i x/3}$

5. $\frac{a}{2} - \frac{a}{\pi}\left[(e^u - e^{-u}) + \frac{1}{3}(e^{3u} - e^{-3u}) + \frac{1}{5}(e^{5u} - e^{-5u}) + \ldots\ldots\right]$ where $u = i\pi x/l$.

Problems 10.9, page 423

1. $11.733 - 7.733 \cos 2x - 2.833 \cos 4x + \ldots\ldots - 1.566 \sin 2x - 0.116 \sin 4x + \ldots\ldots$
2. $1.45 + (-0.37 \cos x + 0.17 \sin x) - (0.1 \cos 2x + 0.06 \sin 2x)$
3. $a_0 = 41.66, a_1 = -8.33, b_1 = -1.15$
4. -0.0731
5. $y = 2.102 + 0.558 \cos x + 1.531 \sin x + 0.354 \cos 2x + 0.145 \sin 2x$
6. $7.8 \sin \theta + 1.5 \sin 2\theta - 9.2 \sin 3\theta + 11.6 \sin 4\theta - \ldots\ldots$.

Problems 10.10, page 424

1. $2\pi/3$
2. $\frac{1}{2}\{f(c-0) + f(c+0)\}$
3. $(-1, 1)$ such that $f(x) = -f(-x)$
4. $f(x) = A$ when $0 < x < \pi$ and $f(x) = -A$ when $\pi < x < 2\pi$
5. Sine
6. § 10.11 (3)
7. Zero
8. not defined
9. odd
10. Cosine
11. even
12. $x = k/n$
13. Zero
14. Cosine
15. Zero
16. $\int_0^2 x^2 \cos \frac{n\pi x}{2}\, dx$
17. $\frac{1}{T}\int_{\alpha}^{\alpha+2T} f(x) \sin \frac{n\pi x}{T}\, dx$
18. § 10.3
19. Zero
20. $a_0 = \frac{2}{l}\int_0^l f(x)\, dx,\ a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$
21. $\frac{4}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right\}$
22. π
23. Zero
24. $2l$
25. $\sum_{n=-\infty}^{\infty} (-1)^n \frac{(1 - in\pi)}{1 + n^2\pi^2} \sinh 1 e^{in\pi x}$
26. False
27. $-\pi/2$
28. odd
29. Zero
30. 3.5355
31. zero
32. $-1/2$
33. $\frac{1}{2}a_n$
34. $\frac{\pi^2}{8}$
35. $\frac{\pi}{2} - \frac{4}{\pi}\left\{\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right\}$
36. $f(x) = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos 4x$
37. $x^2 - x$
38. $x(l + x)$
39. True
40. False
41. False.

Problems 11.1, page 429

1. $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 0$ **2.** $\dfrac{d^2y}{dx^2} + 4y = 0$ **3.** $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy = 0$

4. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$ **5.** $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} + 6y = 0$ **6.** $2xy\dfrac{dy}{dx} + x^2 - y^2 = 0$

7. $(x^2 - 25)\left(\dfrac{dy}{dx}\right)^2 + x^2 = 0$ **8.** $y\dfrac{dy}{dx} = 2a.$ **10.** $y''' - 3y'' + 3y' - y = 0.$

Problems 11.2, page 431

1. $\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = c$ **2.** $\log\dfrac{x}{y} - \dfrac{1}{x} - \dfrac{1}{y} = c$ **3.** $\tan x \tan y = c$

4. $(1+x^2)^{3/2} - 3\sqrt{(1+y^2)} = c$ **5.** $\tan y = c(1-e^x)$ **6.** $2e^{-y} = e^{-x^2} + 1$

7. $x = 2\cos y$ **8.** $(x^2+1)(y^2+1) = c$ **9.** $3e^{2x} - 2e^{3y} + 8x^3 = c$

10. $(1-ay)(a+x) = cy$ **11.** $(x+1)(2-e^y) = c$ **12.** $a\log\left(\dfrac{x-y-a}{x-y+a}\right) = 2y + c.$

13. $y = \tan^{-1}(x+y+1) + c$ **14.** $\tan(x+y) = \sec(x+y) + x + c$

15. $x = \operatorname{cosec}(x+y+1) - \cot(x+y+1) + c$ **16.** $\log\sin(y-x) = \dfrac{1}{2}x^2 + cz$

17. $\cos xy + \dfrac{1}{2x^2} = c.$

Problems 11.3, page 432

1. $x(x^2 - 3y^2) = c$ **2.** $cy^3 = x^2e^{-x/y}$ **3.** $(x/y)^3 = 3\log cy$

4. $y + \sqrt{(x^2+y^2)} = c$ **5.** $y^2 = 2x[y + x\log(cx)]$ **6.** $x(c+y) = ay^2$

7. $y = 2x\tan^{-1}(cx)$ **8.** $e^{x/y} = y + c$ **9.** $\log y - \dfrac{x^2}{4y^2}\left(z\log\dfrac{y}{x} + 1\right) = c$

10. $\log x = \dfrac{1}{2}\left[\dfrac{y}{x} - \dfrac{1}{2}\sin\left(\dfrac{2y}{x}\right)\right] + c$ **11.** $xy\cos(y/x) = c.$

Problems 11.4, page 434

1. $(X^2 + 2Y^2)^2 = c\left(\dfrac{\sqrt{2}Y - X}{\sqrt{2}Y + X}\right)$ where $X = x+1, Y = y-1$

2. $(y-x)^3 = c(y+x-2)$ **3.** $(x+y)^7 = c(x-y-2/3)^3$

4. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ **5.** $3(2y-x) + \log(3x+3y+4) = c$

6. $x - y + \dfrac{3}{4}\log(8x - 12y - 5) = c$ **7.** $\log\left(x + y + \frac{1}{3}\right) + \frac{3}{2}(y-x) = c.$

Problems 11.5, page 437

1. $y = ce^{-\tan x} + \tan x - 1$ **2.** $y = \log x + c/\log x$ **3.** $y\sec^2 x = \sec x - 2$

4. $y\cosh x = c + \dfrac{2}{3}\cosh^3 x$ **5.** $y\sqrt{(1-x^2)} = \sin^{-1}x + c$ **6.** $y = c(1-x)^2 + (1-x^2)$

7. $y(1+\sin x) = c - x^2/2$ **8.** $2r\sin^2\theta + \sin^4\theta = c$ **9.** $ye^{x^2} = 2x + c$

10. $x = y^3 + cy$ **11.** $x = \sin^{-1} y - 1 + ce^{-\sin^{-1} y}$ **12.** $xy^{-2} = c - e^{-y}$

13. $xe^{\tan^{-1} y} = \tan^{-1} y + c$ **14.** $xe^y = c + \tan y$.

Problems 11.6, page 439

1. $y^{-1} \sec x = \tan x + c$ **2.** $1/r = \sin \theta + c \cos \theta$ **3.** $x^2 + (4x^5 + c)y^4 = 0$

4. $1/y = x^2 - 2 + ce^{-x^2/2}$ **5.** $y^2 = x^2 + cx - 1$ **6.** $y/x = \log y + c$

7. $\sin y = (1 + x)(e^x + c)$ **8.** $e^{x+y} = \frac{1}{2} e^{2x} + c$ **9.** $\tan y = x^3 - 3x^2 + 6x - 6 + ce^{-x}$

10. $\cos y = \cos x (\sin x + c)$ **11.** $\sqrt{x} = \sqrt{y}(\log \sqrt{y} + c)$ **12.** $y^{-1} = \frac{1}{2} \log x + \frac{1}{4} + cx^2$.

Problems 11.7, page 442

1. $x^3 + y^3 - 3axy = c$ **2.** $x^4 + 2x^2y^2 - y^4 - 2a^2x^2 - 2b^2y^2 = c$

3. $x^3 - 6x^2y - 6xy^2 + y^3 = c$ **4.** $\frac{x^5}{5} - x^2 y^2 + xy^4 + \cos y = c$ **5.** $e^{xy} + y^2 = c$

6. $x^5 + x^3y^2 - x^2y^3 - y^5 = c$ **7.** $x^3 + 3x^2y^2 + y^4 = c$ **8.** $x^2 - y^2 = cy^3$

9. $3y \cos 2x + 6y + 2y^3 = c$ **10.** $e^x = \sec x \tan y + c$ **11.** $x^2 y + xy - x \tan y + \tan y = c$.

Problems 11.8, page 445

1. $ax + \tan^{-1} y/x = c$ **2.** $x^2 + y^2 - 2a^2 \tan^{-1} (y/x) = c$ **3.** $y + cx + \log x + 1 = 0$

4. $3 \log x - (y/x)^3 = c$ **5.** $\log (y/x) + \frac{1}{2} x^2y^2 = c$ **6.** $xy + \log (x/y) - (1/xy) = c$

7. $(y + 2/y^2)x + y^2 = c$ **8.** $4x^4y + 4x^3 y^2 - x^4 = c$ **9.** $2 \cos (xy) + x^{-2} = c$

10. $\log (x/y) = c + xy$ **11.** $(x/y) + e^{x^3} = c$ **12.** $4(xy)^{1/3} - \frac{2}{3} (x/y)^{3/2} = c$

13. $4y \log x = y^2 + c$.

Problems 11.9, page 446

1. $(x - y + c)(x^2 + y^2 + c) = 0$ **2.** $(2y - x^2 + c)(y + x + ce^{-x} - 1) = 0$

3. $x^2 + y^2 = cx$ **4.** $(y - cx)(y^2 - x^2 - c) = 0$

5. $(y - c)(y + x^2 - c)(xy + cy + 1) = 0$.

Problems 11.10, page 448

1. $x + c = \frac{a}{2}\left[\log \frac{p-1}{\sqrt{(1+p^2)}} - \tan^{-1} p\right]$, with the given relation

2. $xy = c^2x + c$ **3.** $y = 2\sqrt{(xc)} + c^2$ **4.** $2cy = c^2x^2 + 1$

5. $x = (\log p - p + c)(p - 1)^2$, with the given relation

6. $x = \sin p + c$, with the given relation.

Problems 11.11, page 449

1. $y = c(x - c)^2$ **2.** $y^2 = 2cx + c^3$

3. $(y + ap)\sqrt{(p^2 - 1)} + a \cosh^{-1} p = c$, with the given relation

4. $y + (1 + p^2)^{-1} = c$, with the given relation.

Problems 11.12, page 450

1. (*i*) Gen. sol. : $y = cx + a/c^2$; Singular sol. : $2ax^2 = (2ac + x)^3$
(*ii*) Gen. sol. : $c = \log(cx - y)$; Singular sol. : $y = x(\log x - 1)$
(*iii*) Gen. sol. $y = cx + \sqrt{(a^2c^2 + b^2)}$; Singular sol. $y + \sqrt{1 - x^2} = 0$
(*iv*) Gen. sol. $y = cx - \sin^{-1} c$; Singular sol. $y = \sqrt{x^2 - 1} - \sin^{-1} \frac{\sqrt{x^2 - 1}}{x}$

2. $y = cx + (c - 2c^2)$
3. $(y - cx)(c - 1) = c$
4. $(y - cx)(c + 1) + ac^2 = 0$
5. $y^2 = cx^2 + c^2$ [**Hint :** Put $x^2 = u, y^2 = v$]
6. $xy = cy - c^2$ [**Hint :** Put $u = y, v = xy$]
7. $y^2 = cx^2 - \frac{2c}{1+c}$.

Problems 11.13, page 450

1. (*i*)
2. (*ii*)
3. (*iii*)
4. (*i*)
5. $\log y + c = x^2/2y^2$
6. $yx^2 = x^3 + c$
7. $e^x + x^2y + cy = 0$.
8. (*iii*)
9. $x^2 + y^2 + 2 \tan^{-1} y/x = c$
10. $\log x + c = y^3/3x^3$
11. (*i*)
12. $y^2 + 1/x + ce^{-y^2/2} = 2$
13. $y = cx + a/c^2$
14. $c = \log(cx - y)$
15. $xy = c$ or $x^2 - y^2 = c$
16. 2
17. $xy = c$
18. (*b*)
19. (*b*)
20. $(1 + x^2)^{3/2} + (1 + y^2)^{3/2} = c$
21. $y = 5e^{-x}$
22. $x + y = u$
23. x^{-5}
24. § 11.11 (3)
25. $5x^4y^2 + 2(x^5 + y^5) = c$
26. $\sin(y/x) = cx$
27. (*a*)
28. (*c*)
29. $x + y\, dy/dx = 0$
30. $e^{-x^2} + 2 \cos y = c$
31. (*c*)
32. False
33. False.

Problems 12.1, page 454

1. (*i*) $9y + 4x^2 = 0$; (*ii*) $3(x + 3y) = 2(1 - e^{3x})$
2. $y + 1 = 2e^{x^2/2}$
3. $x^2 + y^2 = cx$
4. $y = \sqrt{(a^2 - x^2)} + a \log\left(\frac{\alpha - \sqrt{a^2 - x^2}}{x}\right) + c$
5. $y^2 = 4x$
6. $y = ae^{cx}$
7. $y = ax + b$
8. $x = 3y^2$
9. (*i*) $r(\theta - \alpha) = c$; (*ii*) $r = a + b \cos\theta$
10. $r^2 = a^2 \sin 2\theta$
11. $c^2x^2 = 2cy + 1$
12. $r = ae^{\theta \cot \alpha}$.

Problems 12.2, page 457

1. $2x^2 + y^2 = c$
2. $x^2 + 2y^2 = c^2$
3. $3y^2 + 2x^2 = c^2$
4. $x^2 + y^2 + 2\mu y - c = 0$
5. This system is self-orthogonal
6. $r = c(1 - \cos\theta)$
7. $r = b(\cos\theta - \sin\theta)$
8. $r = 2b/(1 - \cos\theta)$
9. $r^2 = c^2 \sin 2\theta$
10. $r^n\, n \sin\theta = b$
13. $x^2 + y^2 + cx + 1 = 0$
14. $y = cx$.

Problems 12.3, page 462

1. $V = \sqrt{\left(\frac{mg}{k}\right)} \tanh\left(\frac{9k}{m}t + c\right)$
3. $\frac{1}{k} \log_e 2$
5. $2\sqrt{v_o}/k$
6. $v^2 = 2gx - \frac{\lambda}{m} x^2$
10. $y = (\sqrt{150} - 0.001328t)^2$; t_1 = 45 min. 1 sec., t_2 = 1 hr. 16 min. 51 sec., t_3 = 1 hr. 38 min. 13 sec
11. 17 min. 4 sec.

Problems 12.4, page 465

1. 0.0006931 sec
2. $\frac{10}{L^2 + R^2}(R \sin t - L \cos t + Le^{-Rt/L})$
3. $i = \frac{1}{5}(1 - e^{-100})$
4. $i = ke^{-t/RC} + \frac{\omega CE_m}{\sqrt{(1 + R^2C^2\omega^2)}} \sin(\omega t + \theta)$ where $\theta = \cot^{-1}(RC\omega)$.

Problems 12.5, page 467

1. 52.5 mts
2. 48°C
3. B drinks hotter coffee
4. 490,000 cal
5. 2.16 cm.

Problems 12.6, page 469

1. 604.9
2. 2 log 3/log 2
3. $(1 - 1/p)^{21}$ times the original amount
4. 64.5 days
5. 21.5 gm
6. $t = 300 - 5 \log 2 + 5 \log \frac{0.7 - x}{0.5 - x}$
7. 3 hr. 50 min. 16 sec
8. $100(2 - e^{-t/20})$; 13.9 min.

Problems 12.7, page 469

1. $6(1 - e^{-3})$
2. 54 m
3. 90.25%
4. $r(\theta - \alpha) = c$
5. $y = ae^{cx}$
6. rectangular hyperbola
7. $x^2 - y^2 = c$
8. The system is self-orthogonal
9. $2\sqrt{v_0/k}$
10. 2 log 3/log 2
11. Sunil
12. (*d*)
13. (*c*)
14. (*d*)
15. 2.21
16. (*c*)
17. (*c*)
18. (*a*)
19. False
20. True.

Problems 13.1, page 474

1. $\frac{2}{3}e^{2t} \sin 3t$
2. $y = e^x(4 \cos 3x - \sin 3x)$
3. $y = c_1 + (c_2 + c_3x)e^{-x/2}$
4. $y = c_1e^{-x} + e^{x/2}\left(c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2}\right)$
5. $y = (c_1 + c_2x + c_3x^2)e^x$
6. $y = (c_1 + c_2x) \cos 2x + (c_3 + c_4x) \sin 2x$
7. $y = c_1e^{-x} + c_2e^{2x} + e^{x/2}(c_3 \cos x/\sqrt{2} + c_4 \sin x/\sqrt{2})$
8. $y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x + c_5e^x$.

Problems 13.2, page 486

1. $y = (c_1 + c_2x)e^{3x} + 3x^2e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}\log 2$
2. $y = \frac{3}{5}e^{-2x}(\cos x + 3 \sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2}$
3. $x = c_1 \cos nt + c_2 \sin nt + \frac{kx}{2n} \sin(nt + \alpha)$
4. $x = e^{-t}(c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t) + \frac{1}{4}(\sin t - \cos t)$
5. $y = c_1e^{-x} + c_2e^{-2x} + 1 + \frac{1}{10}(3 \sin 2x - \cos 2x)$

6. $y = c_1e^x + c_2e^{3x} + \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(\sin x + 2\cos x)$

7. $y = c_1 + (c_2 + c_3x)e^{-x} - \frac{x^2}{2}e^{-x} + \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$

8. $y = (c_1 + c_2x)e^{-x} + \frac{1}{2} + \frac{1}{5}(2\sin 2x + \cos 2x)$

9. $y = (c_1 + c_2x)e^x + c_3e^{3x} + \frac{1}{8}(xe^{3x} - x^2e^x)$ 10. $y = c_1e^x + c_2e^{-x} + \frac{e^x}{12}(2x^3 - 3x^2 + 9x)$

11. $y = c_1 + c_2e^x + c_3e^{-x} + xe^x - (x^2 + x) - 2\sin x$

12. $y = e^{3x}(c_1\cos 4x + c_2\sin 4x) + \frac{1}{17}e^{2x} + \frac{1}{565}(23\sin x + 6\cos x) + \frac{x}{25} + \frac{6}{625}$

13. $y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x + x^4 - 24x^2 + 72 + \frac{1}{225}\sin 4x - \frac{1}{9}\sin 2x$

14. $y = c_1e^{-2x} + c_2e^{-3x} - \frac{e^{-2x}}{10}(\cos 2x + 2\sin 2x)$

15. $y = e^{-x/2}\left\{(c_1 + x/4)\cos(x\sqrt{3/2}) + (c_2 + x/4\sqrt{3})\sin(x\sqrt{3/2}\right\} + e^{x/2}\left\{c_3\cos\sqrt{3x/2} + c_4\sin\sqrt{3x/2}\right\}$

16. $y = e^{-x}(c_1\cos\sqrt{2x} + c_2\sin\sqrt{2x}) + \frac{e^x}{41}(4\sin x + 5\cos x)$.

17. $y = c_1e^{-x} + c_2e^{-3x} - \frac{e^{-x}}{5}(\sin x + 2\cos x) + \frac{e^{3x}}{22}\left(x - \frac{5}{11}\right)$

18. $y = c_1\cos\sqrt{2x} + c_2\sin\sqrt{2x} + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11}x + \frac{50}{121}\right) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$

19. $y = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x - (1/5)\cos x\cosh x$

20. $y = c_1 + (c_2 + c_3x)e^{-x} + \frac{e^{2x}}{18}\left(x^2 - \frac{7x}{8} + \frac{11}{6}\right) + \frac{1}{100}(3\sin 2x + 4\cos 2x)$

21. $y = c_1\cos 4x + c_2\sin 4x + \frac{1}{7}\left(x\sin 3x - \frac{6}{7}\cos 3x\right)$

22. $y = (c_1 + c_2x)e^{-x} + \frac{1}{2}\cos x + \frac{1}{2}(x - 1)\sin x$

23. $y = c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + (xe^x/12)(2x^2 - 3x + 9)$

24. $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x} + e^{-2x}.\ e^{e^x}$

25. $y = c_1\cos ax + c_2\sin ax - \frac{1}{a^2}\cos ax\log(\sec ax + \tan ax)$.

Problems 13.3, page 490

1. $y = (c_1 - x/a)\cos ax + [c_2 + (1/a^2)\log\sin ax]\sin ax$
2. $y = c_1\cos x + c_2\sin x + \cos x\log(\cos x) + x\sin x$
3. $y = c_1\cos x + c_2\sin x - \cos x\log(\sec x + \tan x)$
4. $y = c_1\cos x + c_2\sin x + \frac{x}{2}\sin x - \frac{x^2}{4}\cos x$ 5. $y = (c_1 + c_2x)e^x + xe^x\log x$
6. $y = (e^x + e^{2x})\log(1 + e^x) + (c_1 - 1 - x)e^x + (c_2 - x)e^{2x}$
7. $y = e^x(c_1\cos x + c_2\sin x) - e^x\cos x\log(\sec x + \tan x)$
8. $y = c_1 + c_2e^{2x} - \frac{1}{2}e^x\sin x$ 9. $y = c_1\cos x + c_2\sin x + \sin x\log(1 + \sin x) - x\cos x - 1$
10. $y = c_1e^x + c_2e^{2x} + \frac{1}{2}(x^2 + 3x + 3.5 - 2xe^x)$ 11. $y = c_1\cos x + c_2\sin x - x\sin x$

12. $y = c_1 e^{2x} + c_2 e^{3x} + xe^{3x} + \frac{1}{10}(\sin x + \cos x)$ **13.** $y = c_1 e^{x} + c_2 e^{-2x} - \frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x)$

14. $y = e^{x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) + \frac{1}{27}(9x^3 + 18x^2 + 6x - 8) + \frac{1}{4}(\cos x - \sin x)$

15. $y = c_1 + c_2 e^{2x} - \frac{1}{2} e^{x} \sin x.$

Problems 13.4, page 495

1. $y = c_1 x^2 + c_2 x^3 - x^2 \log x$ **2.** $y = c_1 x^4 + c_2 x^{-1} + \frac{x^4}{5} \log x$

3. $y = (c_1 + c_2 \log x)x^2 + \frac{1}{4} + 2x + \frac{1}{2}x^2 (\log x)^2$

4. $y = c_1 x^2 + c_2 x^{-1} + \frac{1}{3}(x^2 - 1/x)\log x$ **5.** $u = \frac{kr}{8}(a^2 - r^2)$

6. $c_1 x^{-1} + c_2 x^{-2} + \frac{1}{2}\log x - \frac{3}{4}$

7. $y = c_1 x^{-1} + \sqrt{x}\{c_2 \cos[(\sqrt{3}/2)\log x] + c_3 \sin[(\sqrt{3}/2)\log x]\} + \frac{1}{2}x + \log x$

8. $y = c_1 x^{-2} + x[c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x)] + 8\cos(\log x) - \sin(\log x)$

9. $y = c_1 x^{-1} + \{c_2 \cos(\log x) + c_3 \sin(\log x)\}x + 5x + 10\log x/x$

10. $y = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{x}\log\frac{x}{1-x}$ **11.** $y = x^{-2}(c_1 + c_2 \log x) + \frac{x}{9}\left(\log x - \frac{2}{3}\right)$

12. $y = c_1 x^3 + c_2 x^{-4} + \frac{x^3}{98}\log x(7\log x - 2)$

13. $y = c_1(2x+3)^a + c_2(2x+3)^b - \frac{3}{14}(2x+3) + \frac{3}{4}$ where $a, b = \frac{3 \pm \sqrt{57}}{4}$

14. $y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x+1) + 1$

15. $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \frac{1}{3}\sin[2\log(1+x)]$

16. $y = c_1(3x+2)^{1/3} + c_2(3x+2)^{-1} + \frac{1}{27}\left[\frac{1}{15}(3x+2)^2 + \frac{1}{4}(3x+2) - 7\right]$

Problems 13.6, page 499

1. $x = (c_1 + c_2 x)e^{3x}$; $y = [(1-2x)(c_2 - 2c_1)]e^{3x}$

2. $x = e^{t} + e^{-t}, y = e^{-t} - e^{t} + \sin t$ **3.** $x = c_1 e^{t} + c_2 e^{-5t} + \frac{6}{7}e^{2t}$; $y = c_2 e^{-5t} - c_1 e^{t} + \frac{8}{7}e^{2t}$

4. $x = e^{6t}(c_1 \cos t + c_2 \sin t), y = e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t]$

5. $x = \frac{1}{5}e^{t} + \frac{2}{5}e^{-t} - c_1 \sin 2t + c_2 \cos 2t, y = \frac{2}{5}e^{t} + \frac{1}{5}e^{-t} + c_1 \cos 2t + c_2 \sin 2t$

6. $x = c_1 e^{t} + c_2 e^{-5t} + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{13}{25}, y = c_1 e^{t} - c_2 e^{-5t} - \frac{4}{7}e^{2t} - \frac{3t}{5} - \frac{12}{25}$

7. $x = -t - \frac{2}{3}, y = \frac{1}{2}t^2 + \frac{4}{3}t + c$ **8.** $y = c_1 e^{x} + c_2 e^{-2x} + 2e^{-x}, z = 3c_1 e^{x} + 2c_2 e^{-2x} + 3e^{-x}$

9. $x = c_1 e^{-t} + c_2 e^{3t} - \frac{1}{5}(\cos t - 2\sin t)$; $y = 2c_1 e^{-t} - 2c_2 e^{3t} + \frac{1}{5}(\sin t + 2\cos t)$

10. $x = \frac{1}{2}\left(t + \frac{1}{t}\right), y = \frac{1}{2}\left(-t + \frac{1}{t}\right)$

11. $x = c_1 e^{-t} + c_2 e^{3t} - \frac{1}{5}(\cos t - 2\sin t), y = 2c_1 e^{-t} - 2c_2 e^{3t} + \frac{2}{5}\cos t + \frac{1}{5}\sin t$

12. $x = (c_1 + c_2 t)e^{-t} + (c_3 + c_4 t)e^{t}, y = -\frac{1}{2}[c_1 + c_2(1+t)]e^{-t} + \frac{1}{2}[c_4(1-t) - c_3]e^{t}$

13. $x = c_1e^t + c_2e^{-t} + c_3 \cos t + c_4 \sin t - \frac{t}{4} \cos t + \frac{t}{4} \sin t$

$y = -c_1e^t - c_2e^{-t} + c_3 \cos t + c_4 \sin t + \frac{1}{4}(2+t)(\sin t - \cos t)$

14. $x = \frac{8}{9}\left(1 - \cos \frac{3}{2}t\right), y = \frac{4}{3}t - \frac{8}{9} \sin \frac{3}{2}t.$

Problems 13.7, page 500

1. $y = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax$

2. $-\frac{1}{25}(3 \sin 2x + 4 \cos 2x)$ **3.** 1/6 **4.** $e^x(x-1)$

5. (*b*) **6.** $y = c_1 + (c_2 + c_3 x + c_4 x^2) e^{2x}$ **7.** (*a*)

8. $y = e^x (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$ **9.** $y = \cos x + 2 \sin x$

10. (*ii*) **11.** $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$

12. $\frac{1}{10} \cosh 3x$ **13.** $y = a \log x + 6$

14. $y = (c_1 + c_2 x)\, e^{\sqrt{2}x} + (c_3 + c_4 x)\, e^{-\sqrt{2}x}$ **15.** $\frac{1}{6}x^3e^{-x}$

16. $\frac{1}{2}e^{2x}$ **17.** $\sin 2x$ **18.** $\frac{1}{2}x^2 e^{-x}$

19. $y = (c_1 + c_2 x) e^{-x/2} + c_3$ **20.** (*c*) **21.** (*a*)

22. $x e^{-t}$ **23.** $\frac{d^2y}{dt^2} + 7y = 2e^t.$ **24.** (*c*)

25. (*a*) **26.** (*b*)

27. $y = (c_1 + c_2 \log x)\, x$ **28.** $x^2y_2 + xy_1 - y = 0$ **29.** $\frac{1}{9} \log 2$

30. e^t **31.** (*d*)

32. $y = c_1e^{-x} + c_2 e^{2(1+\sqrt{2})x} + c_3 e^{2(1-\sqrt{2})x}$ **33.** $\frac{d^3y}{dx^3} + 2$

34. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ **35.** False **36.** False.

Problems 14.1, page 506

1. 38 sec **4.** $x = \frac{ue^{-\lambda nt}}{n\sqrt{1-\lambda^2}} \sin [nt\sqrt{(1-\lambda^2)}]$

6. It must be shortened by 1/8640 of its length

7. It must be increased by 0.0074 ft./sec^2 **8.** 4321/4319

9. $k^2 > 4\mu, \theta = c_1 e^{\frac{-k+\lambda}{2}t} + c_2 e^{\frac{-k-\lambda}{2}t}$ where $\lambda = \sqrt{(k^2 - 4\mu)}$

$k^2 = 4\mu, \theta = (c_1 + c_2t)\, e^{-kt/2}$

$k^2 < 4\mu, \theta = c_1\, e^{-kt/2} \cos \left(\frac{\sqrt{4\mu - k^2}}{2}t + c_2\right)$

10. $x = \frac{F_0}{2n^2}(\sin nt - n \cos nt).$

Problems 14.2, page 513

2. 1 ft. ; $\pi/2\sqrt{2}$ sec ; $4\sqrt{2}$ ft./sec

4. $\pi/\sqrt{7}$

5. $x = e^{-5t}\left\{\cos\sqrt{220t} + (5/\sqrt{220})\sin\sqrt{220}\,t\right\}$.

6. 0.45 sec. ; 1.15 sec

8. $x = \frac{10}{21}e^{-t} - \frac{17}{27}e^{-8t} - \frac{\sqrt{2}}{3}\sin\left(9t + \frac{\pi}{4}\right)\frac{\sqrt{2}}{3}, \frac{2\pi}{9}$ sec., $9/2\pi$ cycles/sec

9. $0.8\,(2\sin 4t - \cos 4t)$

10. (i) $x = Ae^{-kt}\cos\left\{t\sqrt{(b^2 - k^2)} + B\right\} + \{e^{-kt}/(b^2 + k^2 - n^2)\}\sin nt$

(ii) $x = Ae^{-kt}\cos\left\{t\sqrt{(b^2 - k^2)} + B\right\} - (te^{-kt}/2n)\cos nt$

Problems 14.3, page 517

2. $i = I\sin\,(T/\sqrt{LC})$

3. $i = 2Eke^{-RT/2L}\sin\,(kt/2L)$, where $k = \sqrt{\left(\frac{4L - CR^2}{C}\right)}$

4. $R^2 > 4L/C$ for over damping ; $R^2 = 4L/C$ for critical damping ; $R^2 < 4L/C$ for under damping ; critical resistance $= 2\sqrt{L/C}$

5. $q = e^{-500t}(0.002\cos 1323t + 0.0008\sin 1323t)$

8. (i) $i = Ae^{-\alpha t}\cosh(\beta t + \gamma)$; (ii) $i = Ae^{-\alpha t}\cos(\beta t + \gamma) + \frac{E}{R}\cos\phi\sin(pt + \phi)$

where $\alpha = -\frac{R}{2L}, \beta = \pm\left\{\left(\frac{R}{2L}\right)^2 - \frac{1}{CL}\right\}$ and $\phi = \tan^{-1}\{(1 - CLp^2)/CRp\}$.

Problems 14.4, page 525

4. $y = \frac{wl}{2Pn}\operatorname{cosec}\frac{nl}{2}\cos\left(nx - \frac{nl}{2}\right) - \frac{wl}{2nP}\cot\frac{nl}{2} + \frac{w}{2P}(x^2 - lx)$

6. $y = \frac{F}{P}[n\sin nx - l\cos nx + l - x]$

7. $\pi^2 EI/4l^2$

8. $\frac{W}{2a^2}\left(\operatorname{sech}\frac{al}{2} - \sec al\right)$

Problems 14.5, page 528

1. $\frac{2u\sin\alpha}{g}; \frac{u^2\sin 2\alpha}{g}$

2. (i) $\frac{2u^2\sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta}$; (iii) $\frac{u^2}{g(1 + \sin\beta)}$

4. $4x^2 + k^2y^2 = 4$

5. $i_1 = \frac{a}{p + \omega}\sin pt, i_2 = \frac{a}{p + \omega}\cos pt.$

6. $i_1 = \frac{E}{R}\left(\frac{2}{3} - \frac{1}{2}e^{-Rt/L} - \frac{1}{6}e^{-3Rt/L}\right); i_2 = \frac{E}{R}\left(\frac{1}{3} - \frac{1}{2}e^{-Rt/L} + \frac{1}{6}e^{-3Rt/L}\right)$

7. $x = a(nt - \sin nt), y = a(1 - \cos nt)$

8. $x = \frac{E}{H\omega}(1 - \cos\omega t), y = \frac{E}{H\omega}(\omega t - \sin\omega t)$, where $\omega = eH/m$.

Problems 14.6, page 529

1. (b) 2. (b) 3. (c) 4. (b)
5. (b) 6. (b) 7. (b) 8. 60 sec
9. $30/\pi\sqrt{LC}$ 10. 0.0074 sec 11. resonance 12. $EI\dfrac{d^2y}{dx^2} - Py = \dfrac{w}{2}(x^2 - lx)$
13. $y = 0$ and $\dfrac{dy}{dx} = 0$

Problems 15.1, page 531

1. $y = -x^2 \sin x - 4x \cos x + c_1x + c_2$
2. $y = \dfrac{x^4}{24} + \dfrac{x^3}{6}\log x - \dfrac{11}{35}x^3 + c_1x^2 + c_2x + c_3$

Problems 15.2, page 533

1. $2(y^{1/4} - 1) = x$
2. $\sqrt{(y^2 - 8y)} + 4\cosh^{-1}(\frac{1}{4}y - 1) = 3x$
3. $r = \dfrac{\sqrt{(v^2 - a^2\omega^2)}}{\omega}\sinh\left[\omega t + \sinh^{-1}\dfrac{a\omega}{\sqrt{(v^2 - a^2\omega^2)}}\right]$
4. $t = \dfrac{h^{3/2}}{\sqrt{(2g)}a}\left[\cos^{-1}\sqrt{\dfrac{x}{h}} - \sqrt{\left(\dfrac{hx - x^2}{h}\right)}\right]$

Problems 15.3 page 534

1. $y = c_1 - x^2 - c_2/x$
2. $y = c_1x + (c_1^2 + 1)\log(x - c_1) + c_2$
3. $15c_1^2y = 4(c_1x + a^2)^{5/2} + c_2x + c_3$
4. $x^2 + y^2 = a^2$
5. $\theta = \dfrac{m}{\mu a}\log\left(1 + \dfrac{\mu a \omega t}{m}\right)$
6. $v = \dfrac{1}{r_1 - r_2}\left[v_1r_1 - v_2r_2 - \dfrac{(v_2 - v_1)r_1r_2}{r}\right]$.

Problems 15.4, page 536

1. $y = 2x - 2\log(1 - c_1e^{2x}) + c_2$
2. $y^2 = x^2 + c_1x + c_2$
3. $\log y = c_1e^x + c_2e^{-x}$
4. $(\log y - 1)(c_1x + c_2) = 1$
5. $(x - a)^2 + y^2 = c^2$, circles whose centres are on the x-axis.

Problems 15.5, page 537

1. $y = e^{x^2}(c_1x + c_2)$
2. $y = (x^2 - x + c_1x)e^x + c_2x$
3. $y = e^x(c_1 \log x + x + c_2)$
4. $cy = 1 + (k - x)\cot x$
5. $y = \left[c_1 - \dfrac{1}{2}\cos x - \dfrac{1}{5}c_2e^{-2x}(\cos x + 2\sin x)\right]e^x$.

Problems 15.6, Page 539

1. $y = c_1\cos(\sin x) + c_2\sin(\sin x)$
2. $y = c_1\cos(1/x) + c_2\sin(1/x)$
3. $y = c_1e^t + c_2e^{-t} - t$ where $t = \cos x$
4. $y = c_1\cos(2\tan^{-1}x) + c_2\sin(2\tan^{-1}x)$
5. $y = c_1e^{\sqrt{2}\sin x} + c_2e^{-\sqrt{2}\sin x} + \sin^2 x$.

Problems 15.7, page 540

1. $nx - lz = c(mz - ny)$
2. $x^2 + y^2 + z^2 = cx$
3. $xy^2 = cz^3$
4. $x^2 + y^2 - xz = cz$
5. $y(x + z) = c(x + y + z)$
6. $x + y + z + \log (xyz) = c.$

Problems 15.8, page 541

1. $x^3 - y^3 = c_1, x^2 - z^2 = c_2$
2. $lx + my + nz = c_1, x^2 + y^2 + z^2 = c_2$
3. $\frac{x-y}{y-z} = c_1, \frac{z-x}{y-z} = c_2$
4. $x^2 - y - 2xy = c_1,\ x^2 - y^2 - z^2 = c_2$
5. $xyz = c_1, x^2 + y^2 + z^2 = c_2$
6. $y = c_1 z, x^2 + y^2 + z^2 = c_2 z.$

Problems 16.1, page 544

1. $y = a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$

2. $y = a_0\left(1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \frac{x^{12}}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} + \ldots\ldots\right)$
$+ a_1\left(x - \frac{x^5}{5 \cdot 4} + \frac{x^9}{9 \cdot 8 \cdot 5 \cdot 4} - \frac{x^{13}}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} + \ldots\ldots\right)$

3. $y = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \ldots\ldots\right) + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \frac{x^7}{3 \cdot 5 \cdot 7} + \ldots\ldots\right)$

4. $y = 4 + 5x - 4x^2 - \frac{5}{3}x^3 - \frac{x^5}{3} - \frac{x^7}{7} - \ldots\ldots$

5. $y = a_0\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \ldots\right) + a_1 x$

6. $y = a_0(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{5}x^6 + \ldots\ldots) + a_1 x.$

Problems 16.2, page 550

1. $y = c_1 \cos\sqrt{x} + c_2 \sin\sqrt{x}$

2. $y = a_0\left(1 - x^2 + \frac{x^4}{4} \ldots\ldots\right) + a_1\left(x - \frac{x^3}{2} + \frac{3x^5}{10} \ldots\ldots\right)$

3. $y = (c_1 + c_2 \log x)\left[1 + x + \frac{1}{(2\,!)^2}x^2 + \frac{1}{(3\,!)^3}x^3 + \ldots\ldots\right] - 2c_2\left[x + \frac{1}{(2\,!)^2}(1+\frac{1}{2})x^2 + \frac{1}{(3\,!)^2}(1+\frac{1}{2}+\frac{1}{3})x^3 + \ldots\ldots\right]$

4. $y = c_1(1 + x + x^2/4 + x^3/4\,.\,7 + \ldots) + c_2 x^{2/3}(1 + \frac{1}{3}x + x^2/3\,.\,6 + x^3/3\,.\,6\,.\,9 + \ldots)$

5. $y = a_0\left(1 - 2x + \frac{3}{2!}x^2 - \frac{4}{3!}x^3 + \ldots\right) + a_1\left[y_1 \log x + a_0\left(3x - \frac{13}{4}x^2 + \ldots\right)\right]$

6. $y = a_0 x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \ldots\right) + a_1 x^{-1/2}\left(1 - x - \frac{x^2}{2} + \ldots\right)$

7. $y = c_1 x^{-\frac{1}{2}}\left(1 + \frac{x}{2} + \frac{x^2}{40} + \ldots\ldots\right) + c_2 x^{1/4}\left(1 + \frac{x}{14} + \frac{x^2}{616} + \ldots\ldots\right)$

$+ c_2\sqrt{x}(x + x^2/2\,.\,3 + x^4/2\,.\,4\,.\,3\,.\,7 + x^6/2\,.\,4\,.\,6\,.\,3\,.\,7\,.\,11 + \ldots)$

8. $y = a_0\sqrt{x}(1 - x) + a_1\left(1 - 3x + \frac{3x^2}{1 \cdot 3} + \frac{3x^3}{3 \cdot 5} + \frac{3x^4}{5 \cdot 7} + \ldots\right)$

9. $y = a_0(1 - \frac{2}{3}x + \frac{1}{3}x^2 + \ldots) + a_1 x^4(1 - 2x + 3x^2 - 4x^3 + \ldots)$

10. $y = c_1\left(1 + 3x^2 + \frac{3}{5}x^4 \ldots\ldots\right) + c_2 x^{3/2}\left(1 + \frac{3}{8}x^2 - \frac{1.3}{8.16}x^4 + \frac{1.3.5}{8.16.24}x^6 + \ldots\ldots\right).$

Problems 16.3, page 557

1. 0.224, 0.44.

Problems 16.4, page 562

1. $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$

2. $y = c_1 J_{2/5}(x) + c_2 J_{-2/5}(x)$

3. $y = x^n[c_1 J_n(kx) + c_2 y_n(kx)]$ where $n = \frac{1}{2}(1 - \alpha)$

4. $y = x[c_1 J_1(2x) + c_2 Y_1(2x)]$

5. $y = c_1\sqrt{x}J_1(2\sqrt{x}) + c_2\sqrt{x}\,Y_1(2\sqrt{x})$

7. $y = c_1\sqrt{x}J_n(x) + c_2\sqrt{x}\,J_{-n}(x)$

11. $x^2 = \sum_{n=1}^{\infty} \frac{2}{\alpha_n^2} \cdot \frac{1}{J_2^2(3\alpha_n)}(3\alpha_n J_1(3\alpha_n) - 2J_2(2\alpha_n))$

Problems 16.5, page 570

3. (*i*) $2P_3 + 4P_1$;

(*ii*) $\frac{2}{5}P_3 + \frac{4}{3}P_2 - \frac{2}{5}P_1 - \frac{7}{3}P_0$;

(*iii*) $\frac{8}{5}P_3 - 4P_2 + \frac{47}{5}P_1 + 4$

(*iv*) $\frac{8}{35}P_4 + \frac{6}{5}P_3 - \frac{2}{21}P_2 + \frac{34}{5}P_1 - \frac{224}{105}P_0$

9. (*i*) $f(x) = -\frac{7}{3}P_0(x) - \frac{2}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{5}P_3(x)$;

(*ii*) $f(x) = -\frac{32}{15}P_0(x) - \frac{4}{5}P_1(x) - \frac{40}{21}P_2(x) + \frac{2}{5}P_3(x) + \frac{8}{35}P_4(x)$.

Problems 16.6, page 572

1. $x^3 = \frac{1}{4}(3T_1 + T_3)$.

Problems 16.7, page 575

1. $y_n(x) = \sin nx,\ n = 1, 2, \ldots$

2. $y_n(x) = \sin[(2n+1)\pi x/2l],\ n = 0, 1, 2, \ldots$

3. $y_n(x) = \cos nx,\ n = 0, 1, 2, \ldots$

4. 1, $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, ...

5. $y_n(x) = \sin\left[(2n+1)\frac{\pi}{2}\log|x|\right],\ n = 0, 1, 2, \ldots$

6. $[xe^{-x}y']' + ne^{-x^2}y = 0,\ p(x) = e^{-x}$

7. $[e^{-x^2}y']' + 2ne^{-x^2}y = 0,\ p(x) = e^{-x^2}$.

Problems 16.8, page 575

1. $\frac{1}{3}(10 - 9P_1 + 8P_2)$

2. $\sqrt{(2/\pi x)}\cos x$

3. $\frac{2}{(2n+1)}$

4. zero

5. zero

6. $J_n(x) = \frac{x}{2n}(J_{n+1}(x) + J_{n-1}(x))$

7. $\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = 0$

8. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$

9. $\sqrt{(2/\pi x)}\sin x$

10. $x^n J_{n-1}(x)$
11. $\frac{1}{2}(3x^2 - 1)$
12. zero
13. True
14. $P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n}(x^2 - 1)^n$
15. $\alpha \neq \beta$
16. $2P_3 + 4P_1$
17. $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$
18. $-J_1(x)$
19. False
20. True
21. True
22. True
23. True
24. False
25. (*b*)
26. (*c*)
27. (*iv*)
28. (*iii*)
29. (*iii*)
30. (*iv*)
31. (*iii*)
32. (*iii*)
33. (*iii*)
34. (*iii*)
35. (*iv*)
36. 0, 1.

Problems 17.1, page 579

1. $z = px + qy + p^2 + q^2$
2. $z^2(p^2 + q^2 + 1) = c^2$
3. $p^2 + q^2 = \tan^2 \alpha$
4. $p + q = px + qy$
5. $z^2(p^2 + q^2 + 1) = 9$
6. $py - qx = 0$
7. $py + qx = 0$
8. $qx - py = x + y$
9. $xys = px + py - z$
10. $xyr = 2 (px + qy - 2z)$ y
11. $\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}$
12. $x(y - z)p + y(z - x)q = z(x - y)$
13. $z\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$
14. $p + q = mz$
15. $px^2 + qy = 2y^2$
16. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$
17. $\frac{\partial^2 v}{\partial t^2} = \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right)$
18. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0$
19. $p(x - 2z) + q(2z - y) = y - x$
20. $(y - z)p + (z - x)q = x - y.$

Problems 17.2, page 581

1. $z = \frac{x^2}{2} \log x + axy + \phi(x) + \psi(y)$
2. $z = \frac{1}{6}x^3 y + xf(y) + \phi(y)$
3. $u = -e^{-t} \sin x + \phi(x) + \psi(t)$
4. $z = f(x) + x\phi(y) + \psi(y) - \frac{1}{12} \sin (2x + 3y)$
5. $z = e^x \cosh y + e^{-x} \sinh x$
6. $z = \sin x + e^y \cos x.$

Problems 17.3, page 584

1. $x = z^3 f(x/y)$
2. $\sqrt{x} - \sqrt{y} = f(\sqrt{x} - \sqrt{z})$
3. $x^2 + y^2 + z^2 = f(x + y + z)$
4. $[\cos (x + y) + \sin (x + y)]e^{y-x} = \phi\left[z^{\sqrt{2}} \tan\left(\frac{x + y}{2} + \frac{\pi}{8}\right)\right]$
5. $x^2 - y^2 = f(y^2 - z^2)$
6. $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$
7. $x \log (x + y) - z = f(x + y)$
8. $x^2 + y^2 + 2z = [\log (xy)]$
9. $x^2 + y^2 - z^2 = f(x + y + z)$
10. $x + y + z = f(xyz)$
11. $\phi(x^2 + y^2 + z^2, xyz) = 0$
12. $x^2 + y^2 = f(y^2 - yz)$
13. $\phi(y/z, x^2 + y^2 + z^2) = 0$
14. $x^2 + y^2 + z^2 = f(y^2 - 2yz - z^2)$
15. $f\left(\frac{y}{z}, \frac{z}{x} - \frac{y}{x} + x^2\right) = 0.$

Problems 17.4, page 587

1. $z = ax - ay/(1 + a) + b$
2. $z = ax + \sqrt{(1 - a^2)}y + c$
3. $4z(1 + a^2) = (x + ay + b)^2$
4. $(1 - a + az) = (x + ay + b)^2$
5. $2z = ay^2 - [a/(a + 1)]x^2 + b$
6. $z = a(x - y) - (\cos x + \cos y) + b$
7. $\frac{8}{9}z = (x + a)^{3/2} + (y + a)^{3/2} + b$
8. $3z = (x + a)^3 + (y - a)^3 + b$
9. $z = \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x_1\sqrt{(x^2+a^2)}}{2} + \frac{y\sqrt{(y^2-a^2)}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + b$
10. $z = ax + by + \sin(a + b)$
11. $z = \frac{1}{6}(zx + a)^3 + a^2y + b$
12. $z = ax + by - 2\sqrt{(ab)}$
13. $z = axy + a^2(x + y) + b.$

Problems 17.5, page 590

1. $z = \{\sqrt{(ax)} + \sqrt{(b + y)}\}^2/(1 + a)$
2. $z = ax^b y^{1/b}$
3. $\frac{z^2}{2} \pm \left\{\frac{z}{2}\sqrt{z^2 - 4a^2} - 2a^2\log\left(z + \sqrt{z^2 - 4a^2}\right)\right\} = 2ax + 2y + b$
4. $\log(z - ax) = y - a\log(a + y) + b$
5. $2\sqrt{(z - a - b)} = \sqrt{ax} + \frac{1}{\sqrt{a}}y + c$
6. $z = axe^{-y} - \frac{1}{2}a^2e^{-2y} + b.$

Problems 17.6, page 595

1. $z = f_1(y) + f_2(y + 2x) + xf_3(y + 2x)$
2. $z = f_1(y - x) + f_2(y + 2x) + xf_3(y + 2x) + \frac{e^{x+2y}}{27}$
3. $z = f_1(x + y) + xf_2(x + y) + \frac{x^2}{2} \times e^{x+y}$
4. $z = f_1(y + x) + zf_2(y + x) + f_3(y + 2x) - e^{2x+y}$
5. $z = f_1(y + x) + xf_2(y + x) - \sin x$
6. $y = f_1(x - at) + f_2(x + at) - \frac{E}{p^2}\sin pt$
7. $z = f_1(y) + f_2(y + 2x) + xf_3(y + 2x) + 3x\cos(3x + 2y)$
8. $f_1(y x) + f_2(y - 2x) + f_3(y + 3x) + \frac{1}{75}\sin(x + 2y) + \frac{2}{3}x^3.$
9. $z = f_1(y + x) + f_2(y + 2x) + \frac{1}{12}e^{2x-y} - xe^{x+y} - \frac{1}{3}\cos(x + 2y)$
10. $z = f_1(y) + f_2(y + x) + \frac{1}{3}(\sin x\cos 2y + 2\cos x\sin 2y)$
11. $z = f_1(y) + f_2(y + x) + \frac{1}{2}[\sin(x + 2y) + \cos(x + 2y)] - \frac{1}{6}[\sin(x - 2y) + \cos(x - 2y)]$
12. $z = f_1(y + x) + f_2(y - x) + \frac{3}{28}e^{x-y}[\sin(x + 2y) - 2\cos(x + 2y)]$
13. $z = f_1(y - x) + f_2(y - 2x) + 4x^3y - 3x^4$
14. $z = f_1(y - x) + xf_2(y - x) + \frac{1}{4}(x^4 - 2x^3y + 2x^2y^2)$
15. $z = f_1(y - x) + f_2(y + 2x) + ye^x$
16. $z = f_1(y - x) + xf_2(y - x) + f_3(y + x) + \frac{e^x}{25}(\cos 2y + 2\sin 2y)$
17. $z = f_1(y - x) + xf_2(y - x) + x\sin y.$

Problems 17.7, page 597

1. $z = e^{-x}\phi_1(y) + e^x\phi_2(y-x) - \dfrac{xe^{-x}}{2}$
2. $z = e^x\phi_1(y+x) + e^{2x}\phi_2(y+x) + \frac{1}{2}e^{2x-y}$
2. $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-2x) + x + 2y + 6$
4. $z = f_1(y) + e^{-x}f_2(y+x) + \frac{1}{3}x^3 - x^2 + xy^2 + 6x$
5. $z = e^x\phi_1(y) + e^{-x}\phi_2(x+y) + \frac{1}{2}\cos(x+2y)$
6. $z = f_1(x) + e^{3y}f_2(2y-x) + \dfrac{3}{50}[4\cos(3x-2y) + 3\sin(3x-2y)]$

Problems 17.8, page 598

1. $z = \phi_1(x) + \phi_2(x+y+z)$
2. $z = \phi_1(y+\sin x) + \phi_2(y-\sin x)$
3. $z = \phi_1(xy^2) + \phi_2(x^2y)$
4. $z = \phi_1(y/x) + \phi_2(x^2+y^2) + xy$
5. $y = \phi_1(z) + e^x\phi_2(z)$
6. $y = \phi_1(x+y+z) + x\phi_2(x+y+z)$.

Problems 17.9, page 598

1. order two & degree two
2. $z = f_1(y+2x) + xf_2(y+2x)$
3. $z = -x^2\sin xy + yf(x) + \phi(x)$
4. $x^2 + y^2 + z^2 = f(x+y+z)$
5. $-\dfrac{1}{2}\sin(x+y)$
6. $xp + yq = z$
7. $z = ax + (1-\sqrt{a})^2y + c$
8. $\sqrt{x} - \sqrt{y} = f(6/x - \sqrt{z})$
9. $x\log(x+y) = z + f(x+y)$
10. First
11. $z = 2x + y\log x + f(xy)$
12. $\partial z/\partial x = \partial z/\partial y$

13. $z = f_1(y) + f_2(y+x) + f_3(y+2x)$
14. $4y^2p = q^2$
15. $u = \int f(y)\,dy + \phi(x)$
16. $c = 1$
17. $u = \dfrac{1}{6}x^3y + xf(y) + \phi(y)$
18. $f_1(y+x) + f_2(y+6x)$
19. (*iv*)
20. (*iii*)
21. (*iii*)
22. (*iii*)
23. (*ii*)
24. (*iv*)
25. (*iv*)
26. (*i*)
27. False
28. False
29. True
30. True
31. False.

Problems 18.1, page 601

1. $z = ce^{4ax^3}.e^{-3ay^4}$
2. $u = ce^{k(1/y - 1/x)}$
3. $u = 8e^{-12x-3y}$
4. $u = 3e^{x-y} - e^{2x-5y}$
5. $u = 3e^{-5x-3y} + 2e^{-3x-2y}$
6. $u = \dfrac{1}{\sqrt{2}}\sinh\sqrt{2x} + e^{-3y}\sin x$.

Problems 18.2, page 610

1. $y(x,t) = \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}\cos\dfrac{n\pi ct}{l}$, when $b_n = \dfrac{2}{l}\int_0^b f(x).\sin\dfrac{n\pi x}{l}dx$
2. $y(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\dfrac{n\pi ct}{l} + b_n\sin\dfrac{n\pi ct}{l}\right)\sin\dfrac{n\pi x}{l}$ where
$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx,\ b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}dx$$
3. $y = \dfrac{8h}{\pi^2}\left(\sin\dfrac{\pi x}{l}\cos\dfrac{\pi ct}{l} - \dfrac{1}{3^2}\sin\dfrac{3\pi x}{l}\cos\dfrac{3\pi ct}{l} + \ldots\right)$

4. $y = \dfrac{8h}{\pi^2}\left(\sin\dfrac{\pi x}{l}\cos\dfrac{\pi ct}{l} - \dfrac{1}{3^2}\sin\dfrac{3\pi x}{l}\cos\dfrac{3\pi ct}{l} + \ldots\right)$

6. $y(x, t) = \dfrac{4l^2c}{a\pi^3}\left\{\sin\dfrac{\pi x}{l}\sin\dfrac{\pi at}{l} - \dfrac{1}{33}\sin\dfrac{3\pi x}{l}\sin\dfrac{3\pi at}{l}\ldots\right\}$

7. (*i*) $y(x, t) = a(x - x^2 - c^2t^2)$; (*ii*) $y(x, t) = \dfrac{a}{2}(1 - \cos 2\pi x \cos 2\pi ct)$.

Problems 18.3, page 617

1. $u(x, t) = \dfrac{400}{\pi^2}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)^2}e^{-[(2n+1)c\pi/100]^2t}\sin\dfrac{(2n+1)\pi x}{100}$

2. $\sum_{n=odd}\dfrac{8a}{n^3\pi^3}\sin\dfrac{n\pi x}{l}e^{(n\pi c/l)^2t}$

3. $u(x, t) = \dfrac{40}{\pi}\sum_{n=1}^{\infty}\dfrac{[1-4(-1)^n]}{n}\sin\left(\dfrac{n\pi x}{30}\right)\overline{e}\,\dfrac{(\alpha n\pi)^2t}{900}$

4. $u(x, t) = -3x + 90 - \dfrac{80}{\pi}\sum_{1}^{\infty}\dfrac{1}{n}\sin\dfrac{n\pi x}{5}e^{-c^2n^2\pi^2t/25}$

5. $u(x, t) = \dfrac{5}{2} - \dfrac{4l}{\pi^2}\sum_{1}^{\infty}\dfrac{1}{(2n-1)^2}\cos\dfrac{(2n-1)\pi x}{l}\,.\,e^{-(2n-1)^2c^2\pi^2t/25}$

6. $u(x, t) = 50 - \dfrac{400}{\pi^2}\sum_{1}^{\infty}\dfrac{2}{(2n-1)^2}\cos\dfrac{(2n-1)\pi x}{50}\,.\,e^{-c^2\pi^2(2n-1)^2t/2500}$

7. $\theta = \dfrac{4\theta_0}{\pi}\left[e^{-(\pi/2l)^2kt}\cos\dfrac{\pi x}{2l} - \dfrac{1}{3}e^{-(3\pi/2l)^2kt}\cos\dfrac{3\pi}{2l}x + \dfrac{1}{5}e^{-(5\pi/2l)^2kt}\cos\dfrac{5\pi}{2l}x - \ldots\ldots\right]$

8. $V = V_0e^{-\sqrt{(n/2k)}x}\sin[nt - \sqrt{(n/2k)}\,x]$.

Problems 18.4, page 623

1. $u = -\dfrac{8}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\dfrac{\sin nx\sinh n(\pi - y)}{n(n^2-4)\sinh nx}$

5. $u(x, y) = \dfrac{3200}{\pi^3}\sum_{1}^{\infty}\dfrac{\sin\dfrac{(2n-1)\pi x}{20}\sinh\dfrac{(n-1)\pi y}{20}}{(2n-1)^2\sinh(2n-1)\pi}$

8. $u(x, y) = u_0\cosh\dfrac{\pi x}{a}\cosh\dfrac{\pi}{a}(b - y)\,\mathrm{sech}\,\dfrac{\pi b}{a}$.

Problems 18.5, page 626

1. $u(r, \theta) = \dfrac{8k}{\pi}\sum_{1}^{\infty}\left(\dfrac{r}{a}\right)^{2n-1}\dfrac{\sin(2n-1)\theta}{(2n-1)^3}$

2. $u(r, \theta) = \dfrac{3200}{\pi^2}\sum_{1}^{\infty}\left(\dfrac{r}{10}\right)^{2n-1}\dfrac{\sin(2n-1)\theta}{(2n-1)^3}$

3. $u(r, \pi) = \dfrac{2}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\left(\dfrac{a}{r}\right)^{2n}\dfrac{r^{4n}-b^{4n}}{a^{4n}-b^{4n}}\cdot\dfrac{\sin 2n\theta}{n^3}$

4. $u(r, \theta) = \sum\dfrac{2k}{n\pi}\left(\dfrac{r}{a}\right)^{4n}(1 - \cos n\pi)\sin 4n\theta$

5. $u(r, \theta) = 50 - \dfrac{200}{\pi}\sum_{1}^{\infty}\dfrac{1}{2n-1}\left(\dfrac{r}{a}\right)^{2n-1}\sin(2n-1)\theta$

6. $u(r, \theta) = \cos\theta\left(\dfrac{200}{r} - \dfrac{r}{2}\right) + \sin\theta\left(2r - \dfrac{200}{r}\right)$.

7. $u(r, \theta) = 4\cos\theta\,(r - 1/r) + 4\sin\theta\,(r + 1/r)$.

Problems 18.6, page 630

1. $u = \sum_{m=1}^{\infty}\dfrac{J_2(\alpha_m)}{\alpha_m^2J_1^2(\alpha_m)}\cos\alpha_m t\,J(\alpha_m r)$.

Problems 18.7, page 634

1. $e = e_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{(LC)}}$; $i = i_0 - e_0\sqrt{\left(\frac{C}{L}\right)} \cos \frac{\pi x}{l} \sin \frac{\pi t}{l\sqrt{CL}}$

3. $v = \frac{20(l-x)}{l} + \frac{24}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l}\, exp\,(-n^2\pi^2 t / RCl^2)$

$i = \frac{20}{lR} + \frac{24}{lR}\sum_{n=1}^{\infty} (-1)^n \frac{n\pi x}{l}\, exp\,(-n^2\pi^2 t / RCl^2)$

4. $v = V_0 \cos \{pt - px\sqrt{(LC)}\}$.

Problems 18.9, page 638

1. $\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}, \frac{\partial^2 i}{\partial t^2} = LC\frac{\partial^2 i}{\partial t^2}$

2. $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

3. If $u(x, t)$ is the temperature, then temperature gradient at a point is $\partial u/\partial x$ for all t.

4. elliptic

5. $\partial^2 y/\partial t^2 = c^2\partial^2 y/\partial x^2$

6. $u = \frac{10}{l}x + 30$

7. parabolic partial differential equation

8. $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$

9. $y = \frac{2h}{l}x, 0\ x < \frac{l}{2}; y = \frac{l}{2h}(2h - y), \frac{l}{2} < x < l$

10. $y(0, t) = 0, y(l, t) = 0, \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

11. zero

12. $u(0, y) = 0, v(a, y) = 0, 0 < y < a; u_x(x, 0) = 0$ for all t and $u\ (x, a) = u$ for $0 < x < a$

13. $\frac{\partial u\,(0,t)}{\partial x} = 0, \frac{\partial u(l,t)}{\partial x} = 0$ for all t

14. $y(x, t) = f(x + ct) + f(x - ct)$

15. $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ where c^2 is the diffusivity

16. $u(x) = x^2 + 20$

17. § 18.8-(6), (7), (8)

18. $u = 8e^{-12x-3y}$

19. $z = 4e^{3x+t}$

20. $\alpha^2\ (= k/s\rho)$ is called the diffusivity of the substance (cm²/sec)

21. $\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}, \frac{\partial^2 i}{\partial x^2} = RC\frac{\partial c}{\partial t}$

22. $u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$

23. § 18.7 – (3), (4), (5)

25. False.

Problems 19.1, page 646

1. (*i*) $\sqrt{2}\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$. (*ii*) $-8i/25$

2. $\frac{-y}{x^2 + y^2 - 2x + 1}$

3. $x = \pm 1.5, y = \pm 2$

5. A circle: centre (–1, 1) radius $\sqrt{2}$

8. $-1 + i\sqrt{3}, -1 - i\sqrt{3}, 1 - i\sqrt{3}; 4\sqrt{3}$

9. $-2 + 0.\,i, 1 - i\sqrt{3}$

10. $-1 - i, \sqrt{2}(\mp \sin 15° \pm i \cos 15°), \sqrt{2}(\mp \cos 15° \pm i \sin 15°)$

11. (*i*) Annular region between the circles of radii 2 and 4 with centre $(-3, 0)$ including boundary of inner circle; (*ii*) Region of complex plane above the line $y = 2$; (*iii*) Infinite region bounded by the lines $\theta = \pi/3$ and $\theta = \pi/2$; (*iv*) Real axis and region above it between $x = \pm 2$

13. (*i*) Ellipse with foci at $z = \pm 1$ and major for axis $= 3$

(*ii*) (*a*) Right bisector of the line joining $z = 3$ and $z = -1$;

(*b*) Circle through the points $z = 3$ and $z = -1$;

14. (*i*) Right bisector of the points 0 and 2; (*ii*) Circle through the points $\frac{1}{3}$ and 3

15. (*i*) A straight line; (*ii*) Circle with centre (1, 1/2) and radius $\sqrt{5/2}$.

Problems 19.2, page 650

5. $4m\pi/n(n+1)$

7. $2^{n+1}\sin^n \dfrac{\alpha-\beta}{2}\cos n\left(\dfrac{\pi+\alpha+\beta}{2}\right)$.

Problems 19.3, page 653

1. (*i*) $(2)^{1/8}[0.98 \pm i(0.195)]6$; $(2)^{1/2}[-0.195 \pm i(0.98)]$

(*ii*) $(2)^{1/5}\cos\dfrac{4n+3}{10}\pi$, where $n = 0, 1, 2, 3, 4$; (*iii*) $\pm 2\sqrt{2}$

(*iv*) $2^3\sqrt{2}\cos r\pi/9$, where $r = 1, 7$ or 13

3. $\pm i$, $\frac{1}{2}(\sqrt{3} \pm i)$, $\frac{1}{2}(-\sqrt{3} \pm i)$

5. (*i*) $1, -1, \cos(\pm\pi/5), \cos(\pm 3\pi/5)$

(*ii*) $-1, \frac{1}{2}(1 \pm i\sqrt{3}), \pm(1+i)/\sqrt{2}, \pm(-1+i)/\sqrt{2}$

(*iii*) $\dfrac{1\pm i}{\sqrt{2}}, \dfrac{-1\pm i}{\sqrt{2}}$; $1, \cos\left(\pm\dfrac{2\pi}{5}\right), \cos\left(\pm\dfrac{4\pi}{5}\right)$.

(*iv*) $\cos(2n+1)\pi/6$, where $n = 0, 1, 2, 3, 4, 5$; $2(2m+1)\pi/3$, where $m = 0, 1, 2$

6. $(-1+i)/\sqrt{2}, (1-i)/\sqrt{2}$

7. $\pm 1, \pm i, \pm\left(\cos\dfrac{\pi}{6} \pm i\sin\dfrac{\pi}{6}\right), \pm\left(\cos\dfrac{\pi}{3} \pm i\sin\dfrac{\pi}{3}\right)$; Last four values

9. $x^3 + x^2 - 2x - 1 = 0$.

Problems 19.4, page 655

1. $32\cos^5\theta - 24\cos^3\theta + 6\cos\theta$

13. $-(2)^{-11}(\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta)$

14. $\sin^5\theta = A\sin\theta - B\sin 3\theta + C\sin 5\theta$.

Problems 19.5, page 661

1. (*i*) $e^{x^2-y^2}\cos 2xy$, $e^{x^2-y^2}\sin 2xy$ (*ii*) ie^5. (*iii*) $e^{16}\cos 30$, $e^{16}\sin 30$

5. $[(pq'-p'q)(qr'-q'r)]^2 = [(pq'-p'q)^2 + (qr'-q'r)^2](pr'-p'r)^2$

10. $\dfrac{1}{64}(\cosh 7\theta + 7\cosh 5\theta + 21\cosh 3\theta + 35\cosh\theta)$

17. $-\log^3$

18. $-13/12$.

Problems 19.6, page 664

10. $\pm \frac{\pi}{4} + \frac{i}{4} \log \frac{1+\sin\theta}{1-\sin\theta}$ according as $\cos\theta$ is + ve or – ve

11. $\sin^{-1}\left(\sqrt{\sin\theta}\right) + i\log\left[\sqrt{(1+\sin\theta)} - \sqrt{\sin\theta}\right]$.

Problems 19.7, page 667

1. (i) $\log_e 10 + i\,[\tan^{-1}(4/3) \pm 2n\pi]$; (ii) $\log_e 1 + i\,(\pi + 2n\pi)$

4. (i) $\sqrt{2}e^{-(2n-\frac{1}{4})\pi}, (2n-\frac{1}{4})\pi + \log\sqrt{2}$; (ii) $e^{-\pi^2/8}, (\pi/4)\log_e 2$

9. $\sqrt{[\frac{1}{2}(\cos 2x + \cosh 2y)]} - i\tan^{-1}(\tan x \tanh y)$

10. (i) $2n\pi \pm i\log(2+\sqrt{3})$; (ii) $-\frac{1}{2}\log 3 + (n+\frac{1}{2})i\pi$.

Problems 19.8, page 669

1. $e^{\sin\theta\cos\theta}\cos(\theta + \sin^2\theta)$

2. $\sin\alpha\cos(\cos\beta)\cosh(\sin\beta) - \cos\alpha\sin(\cos\beta)\sinh(\sin\beta)$

3. $\tan^{-1}\frac{x\sin\alpha}{1+x\cos\alpha}$, except when $x = 1$ and $\alpha = (2n+1)\pi$

4. $\log(2\cos\theta/2)$

5. $-\frac{1}{2}\tan^{-1}(\cos\beta\ \text{cosech}\ \alpha)$

6. $\frac{1}{2}\tan^{-1}\frac{2c\sin\alpha}{1-c^2}$

7. $(2\cos\theta)^{-1/2}\cos\theta/2$

8. $\sin\frac{n(\pi-\alpha)}{2}/(2\sin\alpha/2)^n$

9. $\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2}$

10. $\dfrac{\cos(\alpha + \frac{1}{2}(n-1)\beta)\sin n\frac{\beta}{2}}{\sin\frac{1}{2}\beta}$

11. $\dfrac{\sin\alpha(\cos\alpha - \sin\alpha)}{1-\sin 2\alpha + \sin^2\alpha}$

12. $\dfrac{1 - x\cos\theta - x^n\cos n\theta + x^{n+1}\cos\overline{n-1}\,\theta}{1-2x\cos\theta + x^2}$.

Problems 19.9, page 670

2. 0.053 radians
3. 1°59′
4. 39.7.

Problems 19.10, page 670

1. (b)
2. (c)
3. (b)
4. (c)
5. (b)
6. (d)
7. odd
8. $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$
9. $6(1-2i)$
10. $2i\sin n\theta$
11. $\frac{1}{25}(-6+17i)$
12. $\cosh x\cos y$
13. $\frac{1}{19}$ radians
14. 1
15. $-\cos x\sinh y$
16. $e^{-\pi/4\sqrt{2}}$
17. $\alpha+\beta+\gamma$
18. $2i\,n\pi$
19. real
20. $\sinh\phi$
21. $\sinh 2\phi/(\cosh 2\theta + \cosh 2\phi)$
22. $16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$
23. an equilateral
24. $\pi/2$
25. $x = \pm 1, y = -4$.
26. a circle
27. True
28. True
29. True
30. True
31. False
32. True
33. False.

Problems 20.1, page 682

4. $a = 1, b = -6, c = 1, d = 2, e = 4$

6. (*i*) and (*ii*) Not analytic. (*iii*) Analytic

7. $p = -1$

11 & 12. $f(z)$ is not analytic at origin although C-R equations are satisfied there

14. (*i*) $z^3 + 3z^2 + 1 + ic$; (*ii*) $\cos z$; (*iii*) $\log z$; (*iv*) i/z ; (*v*) $e^z + i(c - z)$; (*vi*) $ze^{2z} + ic$; (*vii*) $z \sin z$; (*viii*) $x^2e^z + ic$

15. (*i*) $(1 + i)/z + c$; (*ii*) $\cos z + c$; (*iii*) e^z ; (*iv*) $\bar{z}e^{-\bar{z}} + c$; (*v*) $1 + ize^{-z}$; (*vi*) $(2 \cos x \cosh y)/(\cos 2x + \cosh 2y)$

16. (*i*) $f(z) = c - iz^3$; (*ii*) $f(z) = \left(\frac{1}{2} - \cot\frac{z}{2}\right)$ (*iii*) $\frac{\cot z}{1 + c} + c\left(\frac{1 + 3i}{5}\right)e^z + c$

17. $\psi = 3xy^2 - x^3 + c$

18. $2 \tan^{-1}(y/x)$; $2 \log z + c$

20. $v = C - e^{-2xy} \cos(x^2 - y^2)$; $f(z) = C - i\, e^{iz^2}$

22. (*i*) $x^4 - 6x^2y^2 + y^4 = c$ (*ii*) $x^2 - y^2 + 2e^x \sin y = c$ (*iii*) $r^2 \sin 2\theta = c'$

23. (*i*) $x/(x^2 + y^2)$; (*ii*) $\frac{1}{2} \log(x^2 + y^2)$

24. (*i*) $a(1 + \cos\theta + i \sin\theta \log r)$; (*ii*) $(r + 1/r)\cos\theta + (r - 1/r)\sin\theta + c$

27. $-2 \tan^{-1}[(y - 2)/(x - 1)]$, $2i \log(z - 1 - 2i)$.

Problems 20.2, page 687

1. $z = \pm i$

2. $w = -1/z$

3. $w = (2i - 6z)/(iz - 3)$; fixed points $z = i, 2i$; no critical points

4. $w = \dfrac{(20 + 18i) - (32 + 12i)z}{(29 + 17i) - (11 - 3i)z}$

5. (*i*) $w = i(1 - z)/(1 + z)$; (*ii*) $w = \dfrac{(4i - 2) + (5 - 3i)z}{2i + (1 + i)z}$ (*iii*) $w = (1 - z)/(1 + z)$.

Problems 20.3, page 692

1. (*i*) $I(w) > 0$; (*ii*) Region bounded by the parabolas $v^2 = 4(1 \pm u)$; and $u^2 = 1 - 2v$; (*iii*) Region bounded by parabolas $v^2 = \frac{1}{4} \pm u$, $v^2 = 4(1 \pm u)$; (*iv*) Region boundary $\rho = 2\sqrt{\rho}\cos\frac{\phi}{2} + 3$.

2. $w = z^6$

3. Lines parallel to x and y axes map into two families of rectangular hyperbolas in the w-plane which cut each other at right angles. Lines parallel to u and v axes map into two families of parabolas in the z plane which cut each other orthogonally. It is conformal at $z = 0$

4. (*a*) Line $4v + 1 = 0$

5. (*b*) Every circle through the origin ($z = 0$) transforms into a st. line not passing through the origin ($w = 0$). If a line passes through $z = 0$, its image is a line through $w = 0$. (*b*) Circle with centre (1/2, 1/2) and radius $(1/\sqrt{2})$ (*c*) Lemniscate $p^2 = \cos 2\phi$

10. $z = \pm a$

11. See § 20.10 (3)

14. See § 20.10(4).

Problems 20.4, page 694

1. $z = \cosh w$

3. $w = \sin z$

4. $w = \log z$.

Problems 20.5, page 696

1. (*a*) $(5 - i)/6$ (*b*) $(5 + i)/6$

2. (*i*) $4 + (25/3)i$; (*ii*) $4 + 8i$

3. $\frac{1}{6}(64i - 103)$

6. (*i*) i ; (*ii*) $2i$; (*iii*) 0

7. (*a*) $\frac{1}{3}(i - 1)$, (*b*) $\frac{1}{6}(5i - 3)$

9. (*i*) $\frac{2}{3}$; (*ii*) $-\frac{2}{3}$.

Problems 20.6, page 702

1. (*i*) 0 ; (*ii*) $2\pi i$
2. 0
3. (*a*) zero ; (*b*) zero
4. (*i*) $5\pi i$; (*ii*) $\pi i/2$ (*iii*) $4\pi i$
5. (*a*) $-10\pi i$ (*b*) $2\pi i e$
6. (*i*) $4\pi i$; (*ii*) $2\pi i e^{-1}$; (*iii*) $-\pi i$
7. (*i*) $8\pi i$; (*ii*) 0
8. (*i*) 0 ; (*ii*) $2\pi(6+13i)$; (*iii*) $12\pi i$
9. zero.

Problems 20.7, page 709

1. $\sum_{n=1}^{\infty} (-1)^{n+1} n(z-1)^n$; Convgt. in $|z-1|<1$

2. $f(z) = \frac{1}{3} + \frac{u}{9}(z+i) - \frac{7}{27}(z+1)^2 + \ldots\ldots$ Region of convergence is $|z+i|<1$

3. (*i*) $\frac{1}{2}(z-1) - \frac{1}{2^2}(z-1)^2 + \frac{1}{2^3}(z-1)^3 - \ldots\ldots$ (*ii*) $-(z-\pi/2) + (z-\pi/2)^3/3! - (z-\pi/2)^5/5! + \ldots$

   (*iii*) (*a*) $f(z) = -\frac{1}{5}\sum_{n=0}^{\infty}(-1)^n (z+1)^n - \frac{1}{20}\sum_{n=0}^{\infty}\left(\frac{z+1}{4}\right)^n$ *in the region* $|z+1|<1$. Also, $(z+1)<4$

   (*b*) $f(z) = -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{z-1}{5}\right)^n \frac{1}{10}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$ in the region $|z-1|<2$. Also $|z-1|<3$

4. (*i*) $\frac{1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \ldots + \frac{1}{2}\left[1 + \frac{z+1}{2} + \frac{(z+1)^2}{2^2} + \frac{(z+1)^3}{2^3}\right]$

   (*ii*) $\frac{1}{4z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right) - \frac{1}{12}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \ldots\right)$. (*iii*) $-\frac{1}{2(z-1)} - 3\sum_{n=1}^{\infty}\frac{(z-1)^{n-1}}{2^{n+1}}$

5. (*i*) $e\left[(z-1)^{-2} + (z-1)^{-1} + \frac{1}{2!} + \frac{1}{3!}(z-1) + \frac{1}{4!}(z-1)^2 + \ldots\right.$

   (*ii*) $e^2(z-1)^{-3} + 2e^2(z-1)^{-2} + 2e^2(z-1)^{-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \ldots$

6. (*i*) $\sum_{n=1}^{\infty}(-1)^{n-1} n\,.\,(z-1)^{-n}$ for $|z-1|>1$. (*ii*) $-\sum_{n=2}^{\infty}\frac{z^{2n-5}}{2(n-1)!}$

7. (*i*) $1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \ldots\ldots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \ldots\ldots\right)$

   (*ii*) $\frac{2}{z+2} + \frac{3}{(z+2)^2} + \frac{3^2}{(z+2)^3} + \ldots + \frac{1}{5}\left(1 + \frac{z+2}{5} + \frac{(z+2)^2}{5^2} + \frac{(z+2)^3}{5^3} + \ldots\right)$

   (*iii*) (*a*) $\frac{7}{z} - \frac{9}{z^2} - \frac{45}{z^3} - \frac{81}{z^4} + \ldots\ldots$ (*b*) $\frac{5}{2(z-3)} + \frac{7}{12} - \frac{z-3}{24} - \frac{5(z-3)^2}{432} + \frac{7(z-3)^2}{864} + \ldots\ldots$

8. (*a*) $\frac{z}{4} - \frac{5z^2}{16} + \frac{21}{64}z^5 - \ldots\ldots$; (*b*) $\frac{1}{3}\left(\frac{1}{z^5} - \frac{1}{z^3} - \frac{z}{4} + \frac{z^3}{16} - \frac{z^5}{64} + \ldots\ldots\right)$; (*c*) $\frac{1}{z^3} - \frac{3}{z^5} + \ldots\ldots$

9. $z=0, z=2$ are the isolated singularities
10. $z=0$ is an isolated essential singularity
11. $z=0$, is a non-isolated essential singularity
12. $z=1$ is a pole of order 2
13. $z=1$ is a pole of order 4
14. $z=a$ is a double pole and $z=0, \pm 1, \pm 2, \ldots\ldots$ are simple poles.

Problems 20.8, page 715

1. $-\frac{1}{t} - 2i + 3t + 4i\,t^2 + \ldots$ where $t = z - i$; -1

59. essential singularity **60.** zero

61. $\sin z = \dfrac{1}{\sqrt{2}}\left\{1 + (z - \pi/4) - \dfrac{(z-\pi/4)^2}{2!} - \dfrac{(z-\pi/4)^3}{3!} + \dfrac{(z-\pi/4)^4}{4!} + \ldots\right\}$

62. a circle with centre (3, 2) and radius 2 in w-plane. **63.** $n\pi$, n an integer

64. $\phi(a)/\psi'(a)$ **65.** circles **66.** True **67.** False **68.** True **69.** True
70. False **71.** True **72.** True **73.** True **74.** True **75.** True
76. True **77.** False **78.** False **79.** False **80.** True **81.** True
82. True.

Problems 21.1, page 732

1. $\dfrac{1}{s-2} + \dfrac{24}{s^4} + \dfrac{3(s-2)}{s^4+9}$. **2.** $\dfrac{1}{s} + \dfrac{\sqrt{\pi}}{s^{3/2}} + \sqrt[3]{\left(\dfrac{\pi}{s}\right)}$ **3.** $\dfrac{3s-20}{s^2-25}$

4. $\dfrac{s\cos b - a\sin b}{s^2+a^2}$ **5.** $\dfrac{s^2-2s+4}{s(s^2+4)}$ **6.** $\dfrac{2(s^2-5)}{(s^2+1)(s^2+25)}$

7. $\dfrac{\sqrt{\pi} - e^{1(1/4s)}}{2s^3/2}$ **8.** $\dfrac{5}{4}\left\{\dfrac{1}{s^2+1} - \dfrac{3/2}{s^2+9} + \dfrac{1/2}{s^2+25}\right\}$ **9.** $\dfrac{s(s^2+28)}{(s^2+4)(s^2+36)}$

10. $\dfrac{b}{(s+a)^2-b^2}$ **11.** $\dfrac{60}{s-2} - \dfrac{s-2}{s^2-4s+20}$ **12.** $\dfrac{30(s+3)}{(s^2+6s+13)(s^2+6s+73)}$

13. $\dfrac{2}{(s+1)(s^2+2s+5)}$ **14.** $\dfrac{1}{8}\left\{\dfrac{3}{s-2} - \dfrac{4(s-2)}{s^2-4s+8} + \dfrac{s-4}{s^2-8s+32}\right\}$

15. $\dfrac{a(s^2+2a^2)}{s^4+4a^4}$ **16.** $\dfrac{3}{2}\left[\dfrac{1}{s^2-9} + \dfrac{s^2-13}{s^4-10s^2+169}\right]$ **17.** $\dfrac{2}{(s+2)^3}$

18. $\dfrac{1}{n} + \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^4}$ **19.** $4\dfrac{(4s^2+4s-1)}{(4s^2+1)^2}$

20. $\dfrac{4}{s} - \dfrac{e^{-s}}{s}$ **21.** $\dfrac{1+e^{-\pi s}}{s^2+1}$ **22.** $\dfrac{e^{-\pi s/3}}{s^2+1}$

23. $e^{-2\pi s/3}\dfrac{s}{s^2+1}$ **24.** $\dfrac{2}{s^3} - \dfrac{e^{-2s}}{s^3}(2+3s+3s^2) + \dfrac{e^{-3s}}{s^2}(5s-1)$

25. $\dfrac{4}{(s-1)(s^2-2s+5)}$.

Problems 21.2, page 734

1. $(1/s^2T) - e^{-sT}/s(1-e^{sT})$ **2.** $[Ew/(s^2+w^2)]\coth(\pi s/2w)$ **3.** $(a/s)\tanh(as/2)$

4. $(1/s^2)\tanh\frac{1}{2}as$ **5.** $\dfrac{1}{\sqrt{(s^2+a^2)}}$ **6.** $(s^2-2as+a^2+b^2)^{-1/2}$

7. $\dfrac{2}{(s-2)\sqrt{(s+1)}}$.

Problems 21.3, page 740

1. $\dfrac{s+1}{s(s^2+2s+2)}$ **4.** $\dfrac{2(3s^2+4)}{s^2(s^2+4)^2}$ **5.** $\dfrac{16}{(s^2+4)^2}$

6. $\dfrac{2s^3-6a^2s}{(s^2+a^2)^3}$ 7. $\dfrac{2as}{(s^2-a^2)^2}$ 8. $\dfrac{6(s-2)}{(s^2-4s+13)^2}$

9. $\dfrac{8(s+2)}{s^2+4s+20}$ 10. $\dfrac{2(s^3+6s^2+9s+2)}{(s^2+4s+5)^3}$ 11. $\log\{(s+b)/(s+a)\}$

12. $\cot^{-1}(s)$ 13. $\frac{1}{2}\log\{(s^2+36)/(s^2+16)\}$ 14. $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{(s-a)^2}\right)$

15. $\cot^{-1}(s+1)$ 16. $\dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2}\right)$ 17. $\cot^{-1}s-\frac{1}{2}s\log(1+s^{-2})$

18. $\dfrac{1}{s-\log 2}+\dfrac{2s}{(s^2+1)^2}+\dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+4}\right)$ 19. (*i*) $\log 2/3$; (*ii*) $\pi/8$; (*iii*) $12/169$; (*iv*) $\dfrac{8(s+1)}{s(s^2+2s+17)}$

21. (*i*) $\dfrac{1}{s}\cot^{-1}(s)$; (*ii*) $\dfrac{1}{s}\cdot\dfrac{s+1}{s^2+2s+2}$; (*iii*) $\dfrac{\cot^{-1}(s-1)}{s}$.

Problems 21.4, page 743

1. $\dfrac{1}{2}\left(\cos\dfrac{5t}{2}-\sin\dfrac{5t}{2}\right)-4\cosh 3t+6\sinh 3t$ 2. $e^{3t}-e^{2t}$

3. $3e^{t/2}+2e^{t/3}$ 4. $e^{2t}+2e^{-4t}$ 5. $\dfrac{1}{3}(8e^{2t}-e^{-t})$

6. $\cosh t$ 7. $e^t+e^{-2t}-2e^{3t}$ 8. $2e^{3t}-\frac{3}{5}e^{2t}-\frac{2}{5}e^{7t}$

9. $\dfrac{1}{2}e^t-e^{2t}+\dfrac{5}{2}e^{3t}$ 10. $\frac{1}{2}t\sinh t$ 11. $\frac{1}{3}t(e^t-e^{-2t})$

12. $\frac{1}{13}(3e^{3t}-3\cos 2t+2\sin 2t)$ 13. $\frac{1}{2}(\sin t-te^{-t})$ 14. $\frac{1}{2}[\cos at+\cosh at]$

15. $(\frac{1}{3}a^2)[e^{at}-e^{-at/2}\{\cos(\sqrt{3}at/2)+\sqrt{3}\sin(\sqrt{3}at/2)\}]$ 16. $\frac{1}{3}(5\sin t-\sin 2t)$

17. $\frac{1}{3}e^{-2t}(6\cos 3t-7\sin 3t)$ 18. $\dfrac{1}{5}(1+e^{-t})\sin t+\dfrac{3}{5}(1-e^{-t})\cos t$

19. $\frac{1}{3}e^{-t}(\sin t+\sin 2t)$ 20. $(2/\sqrt{3})\sinh(\frac{1}{2}t)\sin(\frac{1}{2}\sqrt{3}t)$ 21. $\cos at\sinh at$.

Problems 21.5, page 750

1. $\dfrac{1}{25}(e^{-5t}+5t-1)$ 2. $\dfrac{1}{8}-\dfrac{1}{4}\left(t^2+t+\dfrac{1}{2}\right)e^{-2t}$ 3. $\dfrac{1}{a^2}\cos\left(\dfrac{bt}{a}\right)$

4. $\dfrac{1}{a^2}\left(t-\dfrac{1}{a}\sin at\right)$ 5. $\dfrac{1}{2}t^2+\cos t+1$ 6. $\dfrac{1}{2}te^{-2t}\sin 2t$

7. $t\sin at$ 8. $\dfrac{1}{2}(a^2t^2-4at+2)e^{-at}$ 9. $\dfrac{1-e^t}{t}$

10. $\dfrac{1}{t}(e^{-bt}-e^{-at})$ 11. $e^{-t}-e^{-2t}-e^{-3t}$ 12. $\dfrac{1}{t}(\cos at-\cos bt)$

13. $\dfrac{2}{t}(1-\cosh at)$ 14. $\dfrac{2}{t}(e^t-\cos t)$ 15. $\dfrac{\sin 2t}{t}$

16. $\dfrac{\sin t}{t}$ 17. $\dfrac{2(\sinh t-t\cosh t)}{t^2}$ 18. $\dfrac{e^{-bt}-e^{-at}}{a-b}$

19. $\frac{1}{2a^3}(\sin at - at\cos at)$ 20. $\frac{1}{a^3}(at - \sin at)$ 21. $t(e^{-t}+1)+2(e^{-t}-1)$

22. $\frac{1}{16}(e^{2t}-e^{-2t}-4te^{-2t})$ 23. $\frac{1}{13}(3\sin 3t + 2\cos 2t - 2e^{-2t})$ 24. $\frac{t^2}{2}+\cos t - 1$

25. $\frac{e^{-2t}}{54}(\sin 3t - 3t\cos 3t)$.

Problems 21.6, page 754

1. $y = \frac{7}{4}e^{-t} - \frac{3}{4}e^{-3t} - \frac{1}{2}te^{-t}$ 2. $x = \frac{at}{2}\sinh t$ 3. $y = t - 3\sin t + \cos t$

4. $y = 2t + 3 + \frac{1}{2}(e^{3t}-e^{t}) - 2e^{2t}$ 5. $y = 4e^{2t}(1+t) - 7e^{t}$ 6. $y = 2\cos 5t + t\sin 5t$

7. $y = \frac{1}{2\omega}\sin \omega t$ 8. $y = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(2\sin t + \cos t)$

9. $y = \frac{1}{2}(\cos kt + \cosh kt)$ 10. $y = \frac{1}{8}[(3-t^2)\sin t - 3t\cos t]$ 11. $y = \frac{11}{3}e^{-t}(\sin t + \sin 2t)$

12. $y = \frac{-12}{5} + \frac{12}{5}e^{-t}\cos 2t + \frac{7}{10}e^{-t}\sin 2t$ 13. $y = e^{2t}(x^2-6x+12) - e^{t}(15x^2+7x+11)$

14. $x = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{1}{3}t\cos 2t$ 15. $y = \frac{1}{2}\left(\frac{3\sin t}{t} - \cos t\right)$

16. $y = e^{2t}$ 17. $y = t$ 18. $y = 3J_0(2t)$

21. $(n\sin at - a\sin nt)F_0/mn(n^2-a^2)$, where $n^2 = k/m$.

Problems 21.7, page 756

1. $x = \frac{1}{2}(e^{t} + \cos t + 2\sin t - t\cos t)$, $y = \frac{1}{2}(t\sin t - e^{t} + \cos t - \sin t)$

2. $x = e^{t} + e^{-t}$, $y = e^{-t} - e^{t} + \sin t$ 3. $x = 2 + t^2/2$, $y = -1 - t^2/2$

4. $x = \frac{1}{10}(5 - 2e^{-t} - 3e^{-6t/11})$, $y = \frac{1}{5}(e^{-t} - e^{-6t/11})$ 5. $x = e^{6}(1+2t) + 2e^{3t}$, $y = \sinh t + \cosh t - e^{-3t} - te^{t}$

6. $i_1 = \frac{a}{p+\omega}(\sin \omega t + \sin pt)$; $i_2 = \frac{a}{p-\omega}(\cos \omega t - \cos pt)$.

Problems 21.8, page 762

1. $\frac{2}{s^2+4}(e^{-2\pi s} - e^{-4\pi s})$

2. (i) $(1-2t)u(t-\pi) + 2tu(t)$, $\frac{2}{s^2} + \left(\frac{1-2\pi}{s} - \frac{2}{s^2}\right)e^{-as}$

(ii) $t^2[u(t)-u(t-2)]$, $\frac{2(1-e^{-2s})}{s^3} - \frac{4e^{-2s}(1+s)}{s^2}$

(iii) $\{u(t)-u(t-T)\}\cos(\omega t + \phi)$; $[(s\cos\phi - \omega\sin\phi) - e^{-sT}\times\{s\cos(\phi+\omega T) - \omega\sin(\phi+\omega T)\}]/(s^2+\omega^2)$

3. (i) $\frac{s}{s^2+1} + \left(\frac{1}{s} + \frac{s}{s^2+1}\right)e^{-\pi s} - \left(\frac{1}{s} - \frac{1}{s^2+1}\right)e^{-2\pi s}$

(ii) $\frac{1}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} - \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$

(iii) $\frac{2}{s^3}\{1 + e^{-2s}(2s^2-1) - 2e^{-4s}(1+4s)\}$

4. (*i*) $e^{-s}/(s-1)$; (*ii*) $2e^{-s}/s^3$; (*iii*) $e^{-2s}\left\{\frac{24}{s^4}+\frac{42}{s^3}+\frac{28}{s^2}+\frac{25}{s}\right\}$; (*iv*) $e^{-s}(2+2s+s^2+s^3)/s^3$

5. $20e^{-2}$

6. (*i*) $-\sin t \,.\, u(t-\pi)$; (*ii*) $\frac{1}{3}e^{-4(t-2)}\sin 3(t-2)\,.\,u(t-2)$;

(*iii*) $\frac{1}{2}e^{-(t-1)}(t-1)^2 u(t-1)$ (*iv*) $3-4(t-1)u(t-1)+4(t-3)u(t-3)$

7. $y=\frac{1}{2}\sin 2t+\frac{1}{4}(1-\cos 2t)-\left\{\frac{1}{4}1-\cos(t-1)u(t-1)\right\}$

8. $x=3-2\cos t+2\{t-4-\sin(t-4)\}u(t-4)$.

9. $y(x)=\begin{cases}\dfrac{2Wx^2(3l-5x)}{81EI}, & 0<x<l/3\\[2mm] \dfrac{2Wx^2(3l-5x)}{81EI}+\dfrac{W}{6EI}\left(x-\dfrac{l}{3}\right)^3, & \dfrac{l}{3}<x<l\end{cases}$

10. $y(x)=\frac{wl^2}{16EI}x^2-\frac{wl}{12EI}x^3+\frac{w}{24EI}x^4-\frac{w}{24EI}(x-l/2)^4u(x-l/2)$

11. $x=\frac{I}{mn}e^{-\mu t/2m}\sin nt$; $\frac{dx}{dt}=\frac{I}{m}e^{-\mu t/2m}\left(\cos nt-\frac{\mu}{2mn}\sin nt\right)$, where $n^2=\frac{k}{m}-\frac{\mu^2}{4m^2}$.

Problems 21.9, page 764

1. $\frac{2s}{(s^2+1)^2}$ **2.** (*d*) **3.** (*b*)

4. $\frac{1}{s^2-4s+5}$ **5.** te^{-2t} **6.** $\frac{1}{3}e^{-2t}\sin 3t$

7. $s\bar{f}(s)-f(0)$ **8.** $\frac{1}{8}(2-3t)e^{-3t/2}$ **9.** $\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-16}\right]$

10. $\frac{1}{s-\log 2}$ **11.** $\frac{k!}{(s+1)^{k+1}}$ **12.** $\frac{2}{13}$

13. $e^{-at/s}$ **14.** $\Gamma(3/2)/s^{3/2}$ **15.** $s^2f(s)-sf(0)-f'(0)$

16. $\frac{1}{4}\left[\frac{3s}{s^2+16}+\frac{s}{s^2+144}\right]$ **17.** $\cot^{-1}(s/a)$ **18.** $\frac{e^{-3t}t^4}{24}$

19. $\frac{s\cos 3-2\sin 3}{s^2+4}$ **20.** (*c*) **21.** $f(t-a)u(t-a)$

22. e^{-as} **23.** $1-3t+2t^2$ **24.** $\int_0^t f(t)\,dt$

25. $\int_0^T\frac{e^{-st}f(t)\,dt}{(1-e^{-st})}$ **26.** $\frac{1}{(s+1)^2(s+2)}$ **27.** $\frac{2(s-3)}{s^2-6s+34}+\frac{12}{s^2-6s+25}$

28. $1/(s-\log 4)$ **29.** $\frac{e^{-3t}}{\sqrt{(\pi t)}}$. **30.** (*d*)

31. $\frac{t}{2}\sin\frac{t}{2}$ **32.** (*c*) **33.** (*c*) **34.** (*ii*)

35. (*iii*) **36.** (*ii*) **37.** (*iv*) **38.** (*i*)

39. $\int_a^b f(x)dx$ **40.** (*ii*) **41.** (*iii*) **42.** (*d*)

43. True **44.** False **45.** False.

Problems 22.1, page 776

1. $\frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda\,;\ \frac{\pi}{2}$ for $|x|<1$, $\frac{\pi}{4}$ for $|x|=1$, 0 for $|x|>1$

2. (*i*) $\frac{4}{\pi}\int_0^\infty \frac{\sin\omega-\omega\cos\omega}{\omega^3}\cos\omega x\,d\omega$ (*ii*) $\frac{2}{\pi}\int_0^\infty \cos\omega x\,\frac{a}{a+\omega^2}\,d\omega$

4. (*i*) $\frac{2\sin as}{s}$, π ; (*ii*) $2\{(a^2s^2-2)\sin as+2as\cos as\}/s^3$

5. $\frac{4}{s^3}(\sin sa-sa\cos sa)$ **6.** (*i*) $\sqrt{(3\pi)}\,e^{-3s^2/4}$ (*ii*) $\frac{\sqrt{\pi}}{2}\,e^{(3is-s^2/4)}$

7. $\frac{1-\cos 2s}{s}, \frac{\sin 2s}{s}$ **8.** $\sqrt{(\pi/2)}\,e^{-as}$ **9.** $\frac{a}{a^2+s^2}\,;\ \frac{\pi}{2a}e^{-a\lambda}$

10. 1 **13.** (*i*) $\frac{\pi}{2a^2}(1-e^{-as})$ (*ii*) $\tan^{-1}(s/a)$

11. $\frac{1}{a\sqrt{2}}e^{-s^2/4a^2}\,;\ \frac{1}{2a^3\sqrt{2}}e^{-s^2/4a^2}$

14. (*i*) $\frac{1}{2}\left\{\frac{\sin[a(1-s)]}{1-s}-\frac{\sin[a(1+s)]}{1+s}\right\}$ (*ii*) $(2\cos s-\cos 4s-1)/s^2-(2\sin s)/s$

15. $2/(\pi s^2)$ **16.** $F_s(p)=-32(-1)^p/p\pi\,;\ F_c(p)=32\frac{(-1)^p-1}{p^2\pi^2}$

17. $(\pi/s)\cos s/c$ **19.** $f(x)=(2+2\cos x-4\cos 2x)/\pi x$ **20.** $2/\pi(1+x^2)$.

Problems 22.2, page 780

2. $\frac{1}{4}\int_{-\infty}^{\infty} e^{-\{|x-t|+|t|\}}\,dt$ **5.** $2\left(\frac{1-\cos s}{s^2}\right)$.

Problems 22.3, page 783

1. $\frac{1}{9}[e^{-t}+e^{2t}(3t-1)]$ **2.** $\frac{1}{5}(e^{-2t}-2\sin t-\cos t)$ **3.** $(\sinh at-at)/a^2$

4. $\frac{1}{2}[e^x(x-1)+\cos x]$ **5.** $\frac{1}{2}(\sin t-t\cos t)$.

Problems 22.4, page 791

1. $y=30\,e^{-75t}\cos 5x$ **2.** $\sum_{n=1}^{\infty}\frac{V_0}{n\pi}(1-\cos n\pi)\,e^{-n^2t}\sin nx_0$

3. $u(x,t)=\frac{2}{\pi}\int_0^\infty e^{-s^2t}\left\{\frac{\sin(1+x)s+\sin(1-x)s}{s}\right\}ds$

6. $\theta(x,t)=\theta_0\,erf\left(\frac{x}{2\sqrt{(kt)}}\right)+\theta_0\sum_{n=1}^{\infty}(-1)^n\left\{erf\,\frac{nl-x}{2\sqrt{(kt)}}-erf\,\frac{nl+x}{2\sqrt{(kt)}}\right\}$.

Problems 22.5, page 792

1. $F_c(s) = \int_0^\infty f(t) \cos st \, dt$ 2. $s^2/2$

3. The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

4. $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$ 5. $\int_{-\infty}^{\infty} t^n f(t) e^{ist} \, dt$ 6. $e^{isa} F(s)$

7. $\frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t \, dt \, d\lambda$ 8. $\frac{1}{a}$ 9. $-s^2 [F(u)]$

10. $\frac{2}{\pi} \int_0^\infty \sin(\lambda x) \, d\lambda \int_0^\pi \sin(\lambda t) \, dt$ 11. $-\frac{n\pi}{l} \int_0^1 f(x) \cdot \cos \frac{n\pi x}{l} \, dx$ 12. $\frac{1}{a} F\left(\frac{\lambda}{a}\right)$

13. $f(x) = \frac{2}{\pi^3} \sum_{p=1}^{\infty} \left(\frac{1 - \cos p\pi}{p^2} \right) \sin px$ 14. $\frac{1}{2} F(s/2)$

15. $1/(s^2 + 1)$ 16. True 17. False 18. True
19. True 20. False 21. True.

Problems 23.1, page 800

1. (i) $e^{a/z}$; (ii) $z(e^{-z} - 1)$; (iii) $\frac{z}{z - e^{i\theta}}$

2. (i) $\frac{2z}{(z-1)^2} + \frac{z/\sqrt{2}}{z^2 - \sqrt{2}z + 1} + \frac{z}{z-1}$; (ii) $\frac{z^3 - 3z^2 + 4z}{(z-1)^3}$; (iii) $\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}$

6. (i) $\frac{z^2 \sin \theta}{z^2 - 2z \cos \theta + 1}$ (ii) $\cos \alpha \frac{z(z - \cos \pi/8)}{z^2 \; 2z \cos \pi/8 + 1} - \sin \alpha \frac{z \sin \pi/8}{z^2 - 22 \cos \pi/8 + 1}$

7. $\frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}, |z| > 1 ; \frac{z(z^2 \cos \theta - 2z + \cos \theta)}{(z^2 - 2z \cos \theta + 1)^2}$ 8. $\frac{z^2}{z^2 + 1}, |z| > 1 ; \frac{z^2}{z^2 + a^2}$

9. (i) $\frac{z}{z - e^{-a}}$; (ii) $\frac{z e^{-a}}{(z - e^{-a})^2}$; (iii) $\frac{(z + e^{-a}) e^{-a}}{(z - e^{-a})^3}$

12. $\frac{z^2(1 + 3z^2)}{(1 - z)(1 + z^2)}$ 13. $u_2 = 2, u_3 = 11$.

Problems 23.2 page 804

1. $\frac{z}{z-4}$; $|z| > 4$ 2. $\frac{z}{2-z}$; $|z| < 2$ 3. $\frac{3z}{(4-z)(z-3)}$; $3 < |z| < 4$.

4. $\frac{5z}{(4-5)^2}$; $|z| > 5$ 5. $-\log(1 - 3/z)$; $|z| > 3$ 6. $e^{3/z}$, ROC is z-plane

7. $(1 - e^a/z)^{-1}$; $|z| > |e^a|$.

Problems 23.3, page 807

1. $\frac{1}{2}(3^{n+1} - 1)$ 2. $(n + 1) a^n$ 3. $\frac{1}{2} n (n - 1)$

5. $4a^n$ 6. $(1/3)^n - 2^n$ 7. n

8. $\frac{3}{4} \left\{ \frac{1}{(2)^{n-1}} + \frac{1}{(-4)^n} \right\}$ 9. $(n^2 + 7n + 4)(4)^{n-1}$

13. $u_n = (2)^{n-1} + (3)^{n-1} + (4)^{n-1}\ (n > 0)$

11. $2 + (2)^n + 3(n-1)\,2^n, (n \geq 1)$

12. $(-1)^{n+1} - 2n + \cos n\pi/2$

13. $1 - e^{-at}$

14. (*i*) $\left(-\frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \frac{z^3}{3^4} \cdots\right) + \left(\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \frac{z^3}{2^4} + \cdots\right)$; (*ii*) $(-2^{n-1})\,z^{-n}, n > 0$

(*iii*) $(3^{n-1} - 2^{n-1})\,z^{-n}, n \geq 1, 0, n \leq 0$

15. $\frac{1}{2}\ (n-1)(n-2)5^{n-3}, n \geq 3$ and $= 0, n < 3$

16. $2(-i)^{n-1} - (-2)^{n-1}$

17. $\frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n$

18. $\frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^{n-1}$

19. $u_n = 1 + \frac{1}{2}\ [(i)^{n-2} + (-i)^{n-2}]$

20. $2n \sin (n\pi/2), n = 0, 1, 2, \ldots$

Problems 23.4, page 811

1. $y_k = \frac{8}{5}\left(\frac{1}{2}\right)^k - \frac{3}{5}\left(\frac{-1}{3}\right)^k$

2. $y(n) = (n-1)(-1)^{n-2}\,y(n-2) - 2^n$

3. $y_n = 2^{n-1} + (-2)^{n-1}$

4. $f(n) = 2 + (-4)^n$

5. $y_n = \frac{4}{3}\left[2(-1)^n + (2)^n\right]$

6. $36\left[\frac{1}{2} - (2)^n + \left(\frac{1}{2}\right)^n\right]$

7. $y_n = (c_1 + c_2 n)\,3^n + \frac{1}{2}\ n(n-1)\,3^{n-2}$

8. $y_n = c_1 + c_2 \,.\, 3^n + 5^n/8$

9. $y_n = 2\left[\left(\frac{1}{4}\right)^n - \left(-\frac{1}{4}\right)^n\right]$

10. 5.2^n

11. $y_n = \frac{1}{3}(-1)^n - \frac{2}{5}(-3)^n + \frac{1}{15}(2)^n$

12. $y_k = 1 - 2k + 2^k$

13. $y_n = c_1\,4^n + \left(c_2 - \frac{n}{4}\right)2^n + 2n - \frac{8}{3}$

14. $y_k = \frac{1}{2}(k+2)\frac{1}{5^k}\cos\frac{k\pi}{2}$

15. $y_n = \left[\frac{1}{4} - \frac{9}{4}(-3)^n\right]\mu(n)$

16. $y_n = (-2)^{n-1}, (n \geq 1)$.

Problems 23.5, page 811

1. $z/(z-1)$

2. $\sum_{n=0}^{\infty} u_n\, z^{-n}$

3. $z/(z-1)^2$

4. $az/(z-a)^2$

5. $\frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$

6. $e^{1/z}$

7. $(z^2 + z)/(z-1)^3$

8. $Z(au_n + bv_n + cw_n) = aZ(u_n) + bZ(v_n) + cZ(w_n)$

9. 2^{n-1}

10. $(-1)^{n-1}\,n$

11. $u_0 = \underset{z \to \infty}{Lt}\ \{Z(u_n)\}$

12. False

13. False

14. True

15. True

16. False

17. False

18. False.

Problems 24.1, page 815

1. $a = 2.28, b = 6.1879, p = 30.46$

2. $a = 1120, b = 55.1$

3. $a = 0.2, b = 0.0044$

4. $a = 0.5012, n = 0.5$

5. $a = 0.115, b = 11.8$

6. $a = 4.1, b = 0.43$

7. $a = 0.0498, b = -0.02$.

Problems 24.2, page 819

1. $y = 13.6x$
2. 15.2 thousand tons
3. $Y = 0.004P + 0.048$
4. $R = 70.052 + 0.292t$
5. $a = 0.545, b = 0.636$
6. (a) $y = 4.193 + 1.117x$ (b) $y = 8 - 0.5x$
7. $y = 1.243 - 0.004x + 0.22x^2$
8. $y = 0.34 - 0.78x + 0.99x^2$
9. $y = 18.866 + 66.158x - 4.333$
10. $R = 3.48 - 0.002V + 0.0029V^2$
11. $V = 2.593 - 0.326T + 0.023\ T^2$.

Problems 24.3, page 823

1. 6.32, $b = 0.0095$
2. $a = 1.52, b = 0.49$
3. $a = 3, b = 2$
4. $y = 7.187 - 5.16\dfrac{1}{x}$; 4.894
5. $a = 0.988, b = 3.275$
6. $y = 2.978\ x^{0.5143}$; 5.8769
7. $a = 0.1839, b = 0.0221$
8. $f(t) = 0.678\ e^{-3t} + 0.312\ e^{-2t}$
9. $a = 146.3, k = -0.412$
10. $a = 99.86, b = 1.2$.

Problems 24.4, page 826

1. $a = 11.1, b = 0.71$
2. $y = 46.05 + 6.1\ x$
3. $a = 0.0028, b = 0.01, c = 4.18$
4. $a = 15.8, b = 2.1, c = -0.5$
5. $a = 1.459, b = 0.062$.

Problems 24.5, page 828

1. $y = 0.12 + 0.47x$
2. $y = 1.184 + .523x$
3. $y = 1.53 + 0.063x + 0.074x^2$
4. $y = 0.485 + 0.397x + 0.124x^2$.

Problems 24.6, page 829

1. $Y = aX + b$ where $X = x, Y = y/x$
2. $Y = A + BX$, where $X = \log_{10} p, Y = \log_{10} v, A = \dfrac{1}{r}\log b, B = -1/r$
3. § 24.4
4. $y = aX + c$, where $X = x^b$
5. (ii)
6. $\Sigma y = nA + B\Sigma x, \Sigma xy = A\Sigma x + B\Sigma x^2$ where $y = \log_{10} y, A = \log_{10} a, B = \log_{10} b$
7. § 24.12
8. Zero
9. $y = aX + b$ where $X = x^2/\log_{10} z, Y = y/\log_{10} x$
10. $a = 0.0167, b = 1.05$.
11. The moments of the observed values of y are respectively equal to the moments of the calculated values of y
12. $a = 1.7, b = 1.26$
13. $y = a + bx$ where $x = 1/x, y = 1/y$
14. (r)
15. (b)

Problems 25.1, page 837

1. 336.79
2. 64% get more than 50 marks ; median 54.7, $Q_1 = 46, Q_3 = 61.5$
3. Mean = 27.9 ; Median = 25.66 ; Mode = 21.85
4. Mean = 32.58 ; Median = 32.6 ; Mode = 35.1
5. 3.1%
6. 1.3%
7. 192 km/hr
8. 60 km/hr
9. 38.6 ; 36.2
10. Median = 12.2 days ; Mode = 11.4 days

Problems 25.2, page 842

1. 4.45, 0.39
3. 4, 7
4. 10.04, 10.13, 11.69, 5.54, 2.35

5. 32, 32.6, 12.4
6. Q.D. = 10.9, S.D. = 15.26
7. 1.845 ; 1.8175
8. 0.55, 1.24 ; first, yes
9. Height
10. A
11. B is a better player and more consistent
12. A is more efficient, B is more consistent
13. 161.3, 5.68.

Problems 25.3, page 845

1. $\mu_1 = \mu_3 = 0, \mu_2 = 2, \mu_4 = 11$; $\beta_1 = 0, \beta_2 = 2.75$
2. 8.85 ; 5.25 ; 0.32 ; 1.09
3. – 0.2064
4. 0.22 ; 1.157
6. 0 ; 2.9
7. $\beta_1 = 0.493$; $\beta_2 = 0.655$; platykurtic.

Problems 25.4, page 854

1. $r = 0.81$; $x = 0.5y + 0.5, y = 1.3x + 1.1$
2. $r = 0.96$
3. $r = 0.92$
4. $r = -0.055$
5. 0.7291
6. $r = 0.4517$
7. $r = 0.632$; $y = 0.467 + 0.8x, x = 0.167 + 0.5y$
9. $m = (\beta - b)/(a - \alpha)$
10. $\bar{x} = 4, \bar{y} = 7, r = -0.5$
11. $\bar{x} = 9.06, \bar{y} = 5.52, r = 0.46$
12. $r = 0.7395$; $\bar{x} = -0.1034$; $\bar{y} = 0.5172$
13. 134.5
14. 1.28 inch
15. 0.8545
16. 0.932.

Problems 25.5, page 855

1. (*d*)
2. (*d*)
3. (*a*)
4. (*a*)
5. (*b*)
6. (*a*)
7. (*b*)
8. Coefficient of correlation
9. No
10. 13.83
11. $\bar{x} = 2, \bar{y} = 3, r = \sqrt{3}$
12. Zero
13. $\frac{1}{2}(Q_3 - Q_1)$
14. –1
15. § 25.9
16. $\frac{\Sigma XY}{n\,\sigma_x\,\sigma_y}$
17. $(\bar{x}, \bar{y})$
18. Reliability or consistency
19. $\sqrt{\beta_1}$
20. degree of peakedness
21. $y - \bar{y} = r\,\frac{\sigma_y}{\sigma_x}(x - \bar{x})$
22. $100\,\sigma/\bar{x}$
23. $\tan^{-1}\left\{\frac{1-r^2}{r}\cdot\frac{\sigma_x\,\sigma_y}{\sigma_x^2 + \sigma_y^2}\right\}$
24. 3
25. Two regression coefficients
26. perpendicular
27. greater
28. ± 1
29. $1 - \frac{6\,\Sigma d_i^2}{n^3 - n}$
30. Coefficient of standard deviation
31. zero
32. – 1 and 1
33. – 0.6
34. False
35. True
36. False

Problems 26.1, page 858

1. 4096
2. 360 ; 120
3. 120 ; 115
4. (*a*) 676000 (*b*) 468000

Problems 26.2, page 868

1. (*i*) 7/12 ; (*ii*) $P(A/B) = 3/4$, $P(A \cup B) = 7/12$, $P(A'\,B') = 3/8$
2. (*a*) 1/36 ; (*b*) 1/6, Yes
3. 36 : 30 : 25
4. 1/7
5. (*i*) 1/4, (*ii*) 7/8, (*iii*) 11/16
6. 15/1024

7. 3/28 **8.** 20/81 **9.** (*i*) 2816/4165 ; (*ii*) 2197/2025,
10. 0.11 **11.** (*a*) 6.739 ; (*b*) 0.024 **12.** (*a*) 1/114 ; (*b*) 685/1140
13. 2/801 **14.** 10/21 **15.** $1-(1-p_1)(1-p_2)\ldots(1-p_n)$; 0.518
16. 15/17 **17.** 1/2 **18.** 5/12 **19.** (*i*) 83/110 (*ii*) 25/83
20. $1-2/(n-1)$ **21.** 7/20 **22.** (*a*) 1/6 ; (*b*) 3/4 **23.** 61/90
24. 0.72 **25.** 0.2223 **26.** 0.88

Problems 26.3, page 871

1. 3/11 **2.** 25/69, 28/69, 16/69 **3.** 0.3175, 0.254 **4.** 15/59.

Problems 26.4, page 878

1. $k = 1$; $\mu = .8$, $\sigma^2 = 2.232$ **2.** $2\sqrt{5}$
3. $F(x) = 0, -\infty < x < 0$
$= 1/8, 0 \le x \le 1$
$= 1/2, 1 \le x \le 2$
$= 7/8, 2 \le x \le 3$
$= 1, 3 \le x \le \infty$
5. ₹ 32
6. 2 **7.** $f(x)$ is a p.d.f. $\bar{x} = \frac{1}{2}, \sigma^2 = \frac{1}{20}$ **8.** (*i*) 9/16, 7/16 ; (*ii*) $k = 0.45$

9. (*i*) 0.37, (*ii*) 0.63 **10.** 4/9
12. $y_0 = 3/4$; Mean = 1 ; Variance = 1/5. **13.** 0.2
14. 1/3, 2/9 **15.** $F(x) = 0, x < x_1$; $(x - x_1)/(x_2 - x_1), x_1 \le x < x_2$; $1, x \ge x_2$.

Problems 26.5, page 881

1. $n = 4, p = q = \frac{1}{2}; \frac{15}{16}$ **2.** ${}^4C_r(1/6)^{4-r}(5/6)^r$; $r = 0, 1, 2, 3, 4$
3. (*a*) 0.02579 ; (*b*) 0.04571 ; (*c*) 1.024×10^{-7} **4.** 0.3456
5. 45927/50000 **6.** (*i*) 0.246 ; (*ii*) 0.345
7. (*i*) ${}^{20}C_1(1/20)(19/20)^{19}$; (*ii*) $\sum_{r=0}^{5} {}^{20}C_r(1/20)^r(19/20)^{20-r}$ (*iii*) 19
8. (*a*) 0.08 ; (*b*) 0.26 ; (*c*) 0.92 **9.** (*a*) 250 ; (*b*) 25 ; (*c*) 500
10. (*i*) 0.59049 ; (*ii*) 0.32805 ; (*iii*) 0.08146 **11.** 11
12. 99.83 **13.** 0.3585, 0.3773, 0.1887 ; 0.0596
14. 600 **15.** $100\,(.432 + .568)^5$
16. $200\,(0.554 + 0.446)^6$.

Problems 26.6, page 884

1. (*i*) 2 ; (*ii*) $\frac{2}{3}e^{-2}$ **2.** $P(0) = 0.2636, P(3) = 0.1041, P(>3) = 0.1506$

4. $(10)^{15}e^{-10}/15! = 0.035$ **5.** 0.08
6. (*i*) 0.2231 (*ii*) 0.1913 **7.** 0.0008
8. $m = 0.51 = \sigma^2$; Poisson frequencies of 0, 1, 2, 3, 4 accidents are 180.1, 91.9, 23.4, 4, 0.6
9. 0.6 **11.** Theoretical frequencies are 44, 43, 21, 7, 1
12. Theoretical frequencies are 109, 142, 92, 40, 13, 3, 1, 0, 0, 0, 0.

Problems 26.7, page 890

2. (*i*) 0.1644 ; (*ii*) 0.7686
3. (*i*) 0.095 ; (*ii*) – 0.995
4. 36.4
5. (*i*) 16, (*ii*) 2
6. 294
7. 543
8. (*i*) 79 ; (*ii*) 35% ; (*iii*) 11
10. 52
11. 67
12. ₹ 866
13. $y = \frac{100}{\sqrt{(3.4\pi)}} e^{-\frac{(x-2)^2}{3.4}}$
14. $\mu = 13.64$. $\sigma = 3.98$.

Problems 26.8, page 892

1. (*a*) 0.0287, (*b*) 0.9672, (*c*) 0.5111
2. (*a*) 0.7854, (*b*) 0.1815, (*c*) 0.1815
3. (*a*) 0.97815, (*b*) 0.00595, (*c*) 0.01209.

Problems 26.9, page 893

3. $\frac{1}{2}(n+1), \frac{1}{12}(n^2-1)$
5. Mean $= a + b$, variance $= b^2$
6. $[(1-e^t)/t]^2$
8. $(1-t)^{3/4}$.

Problems 26.10, page 894

1. (*a*)
2. (*b*)
3. (*d*)
4. (*b*)
5. 1/7
6. 1/2
7. (*b*)
8. (*b*)
9. (*a*)
10. (*c*)
11. 0.1288
12. 2
13. 0.21
14. 0.24

15.

X:	0	1	2
$p(x)$:	1/4	2/4	1/4

16. § 25.5
17. 0.7837
18. zero
19. equal
20. $P(A) + P(B)$
21. $\beta_1 = 0, \beta_2 = 3$
22. 120
23. 0.2646
24. 1/9
25. 0.2222
26. 1/6
27. e^{-3}
28. 5/36
29. 2
30. symmetrical
31. $1 - e^{-m}$
32. six
33. 0.6915
34. $(q + pe^t)^n$
35. 4 : 5
36. 1/6
37. 3.5
38. $\sqrt{2}$
39. unity
40. $n \to \infty, p \to 0$ such that np is fixed
41. $P(A) + P(B)$
42. ${}^yC_x / {}^{y+z}C_x$
43. np
44. $P(A \cup B) = 0.72, P(A \cap B') = 0.1653$
45. $\left(\frac{1}{3} + \frac{2}{3}\right)^{18}$
46. 1
47. $\mu_r' = \left[\frac{d^r}{dt^r}(\Sigma\, p_i\, e^{tx_i})\right]_{t=0}$
48. 1/6
49. 3/4
50. 1/2
51. 0.2
52. 2/7
53. $(q + p)^n$
54. 2/3
55. Mean and S.D.
56. 1/13
57. 2
58. § 26.6
59. $e^{-4/3}$
60. 1/3
61. True
62. False
63. True
64. False
65. False
66. True
67. False
68. False
69. $k = 2$
70. 8
71. $f(x) = \lambda e^{-\lambda x}$ for $x > 0$, λ is a parameter
72. 25/12
73. 2
74. l
75. 4
76. 8
77. n, m degrees of freedom
78. $\int_{-\infty}^{\infty} e^{t(x-a)} f(x)\, dx$

79. – 5/7 **80.** $\dfrac{e^{-x}x^{l-1}}{\Gamma(l)}, 0 < x < \infty$ **81.** $\dfrac{1}{2}(b+a)$

82. 50% **83.** 3/8 **84.** 1 **85.** 3/4

86. (*iii*) **87.** $F(x) = \int_{-\infty}^{x} f(x)\,dx$ **88.** $\dfrac{\lambda^{2r}e^{-\lambda}}{(2r)!}$ **89.** $-\infty < t < \infty$

90. 6 **91.** 1/9 **92.** False **93.** 2

94. $4(1-x)^3$ **95.** 0.264

Problems 27.1, page 901

1. Die is biased **2.** No **3.** Yes

4. 8.91% and 15.07% **5.** Consistent **6.** $p = 65/500$, S.E. = 0.015

7. 37.5% ; 30.3 and 44.7 respectively **8.** No

9. Difference is not significant

10. $Z \sim 6.56$ so that the difference is significant **11.** No.

Problems 27.2, page 904

1. No **2.** Mean weight lies between 64.6 and 69.4 lbs.

3. 0.0774 **5.** 62 **6.** 2.696 **7.** No

8. No **9.** (*i*) Yes, (*ii*) No **10.** Yes.

Problems 27.3, page 910

1. 0.25 **2.** $t = 0.62$, Yes **3.** 11.887 and 12.113 cm

4. Refute the claim **5.** Process is not under control **6.** No

7. Sample mean = 575.2 kg., S.E. = 2.75 kg **8.** Accept null hypothesis

9. Yes with 75% confidence. **10.** No **11.** No

12. No

Problems 27.4, page 914

1. 0.41 **2.** Hypothesis is correct

3. Significant at 5% level **4.** Yes

5. f_e : 33.82 161.78 315.98 308.48 150.54 29.4

$\chi^2 = 7.97$. Binomial distribution gives a good fit at 5% level

6.

f_o :	305	365	210	80	18	12
f_e :	301.2	361.4	216.8	86.7	26	7.9 ; $\chi^2 = 3.097$

Poisson distribution gives a good fit at 5% level

7.

f_o :	314	355	204	85	29	12
f_e :	301	362.2	217.3	86.9	26.1	6.5

Poisson distribution can be fitted to the data

8. $\chi^2 = 1.2$. The fit is quite good at 5% level.

Problems 27.5, page 917

1. First variance cannot be regarded as significantly greater than the second

2. Not significant as $F = 2.1$ and $F_{0.05} = 4.15$ **4.** Not significant as $F = 2.4$ and $F_{0.01} = 3.2$

5. Product of firm B cannot be said to be of better quality than those of firm A.

6. Not significant at 1% level and just significant at 5% level as $F = 2$, $F_{0.01} = 2.62$ and $F_{0.05} = 1.98$
7. $F = 1.49$, Not-significant **8.** $F = 1.025$; Yes

Problems 27.6, page 917

1. § 27.3 (3) **2.** § 27.15 **3.** § 27.3 (2)
4. We are testing the hypothesis that one process is better than another
5. § 27.11 **6.** 1 **7.** 50 **8.** $(0, \infty)$
9. II **10.** 8 ; 16 **11.** $r = n - 1$ **12.** False
13. Less than 30 **14.** § 27.17 **15.** $-\infty < t < \infty$ **16.** (*ii*)
17. True

Problems 28.1, page 926

1. (*i*) 2.687, (*ii*) 1.46, (*iii*) 2.375, (*iv*) 2.875 **2.** (*i*) 0.519, (*ii*) 2.875, (*iii*) 1.146, (*iv*) 0.367
3. (*i*) – 0.686, (*ii*) 2.7065, (*iii*) 0.686, (*iv*) 1.4036
4. (*i*) 0.853, (*ii*) 0.607, (*iii*) 2.798, (*iv*) 3.789, (*v*) – 0.134
5. 1.861 **6.** (*i*) 1.532, (*ii*) 2.095, (*iii*) 1.834, (*iv*) 1.226
7. (*i*) 1.855 (*ii*) 2.198 (*iii*) 1.662 **8.** – 16.56
9. (*i*) 0.853, (*ii*) – 1.9338, (*iii*) 2.7985, (*iv*) 4.545
10. (*i*) 0.518, (*ii*) 0.052, (*iii*) 0.695, (*iv*) 2.911 **11.** $x_{n+1} = \frac{1}{2}(x_n + N/x_n)$; (*i*) 3.605 (*ii*) 3.162
12. 3.4482 **13.** 2.3784
14. (*i*) 0.055 (*ii*) 0.258 (*iii*) 0.4347

Problems 28.2, page 929

1. (*i*) 1.532, (*ii*) 0.684, (*iii*) 3.18, (*iv*) 1.168 **2.** 1.674
3. 2.231 **4.** – 1.328 **5.** 2.924

Problems 28.3, page 936

1. $x = 7, y = -9, z = 5$ **2.** $x = -51/4, y = 115/8, z = 35/4$ **3.** $x = 1, y = 2, z = 3$
4. $x_1 = 2, x_2 = -1, x_3 = 3$ **5.** $x_1 = 1, x_2 = 2, x_3 = -1, x_4 = -2$ **6.** $x = 1, y = 3, z = 5$
7. $x = 8.7, y = 5.7, z = -1.3$ **8.** $x = 1, y = 2, z = 3$ **9.** $x = 7, y = -9, z = 5$
10. $x_1 = 2, x_2 = 1/5, x_3 = 0, x_4 = 4/5$ **11.** $x = y = z = 1$ **12.** $x = 1, y = 2, z = 3$
13. $x = 35/18, y = 29/18, z = 5/18$ **14.** $x_1 = -1, x_2 = 0, x_3 = 1, x_4 = 2$ **15.** $\begin{bmatrix} 1.2 & -0.4 & 0.2 \\ -0.2 & -0.1 & 0.3 \\ -0.4 & 0.3 & 0.1 \end{bmatrix}$.

Problems 28.4, page 942

1. $x = 2.556, y = 1.722, z = -1.055$
2. (*a*) $x = 2.426, y = 3.573, z = 1.926$ (*b*) $x = 2.426, y = 3.573, z = 1.926$
3. $x = 1, y = 1, z = 1$ **4.** $x = 0.998, y = 1.723, z = 2.024$
6. $x = 1.052, y = 1.369, z = 1.962$ **8.** $x = -13.223, y = 16.766, z = -2.306$
9. $x = 1, y = 2, z = 3, u = 4$ **10.** $x = 1.36, y = 2.103, z = 2.845$
11. $x = y = z = 1$ **12.** $x = 52.5, y = 44.5, z = 59.7$
13. $x = 1.93, y = 3.57, z = 2.43$

Problems 28.5, page 943

1. $x = 2, y = 1$ **3.** $x = 0.7974, y = 0.4006$ **4.** $x = 3.162, y = 6.45$
2. $x = -1.853, y = -1.927$ **5.** $x = -3.131, y = 2.362$

Problems 28.6, page 945

1. (*a*) 5.38, $\begin{bmatrix} 0.46 \\ 1 \end{bmatrix}$; (*b*) 4.418, $\begin{bmatrix} 1 \\ 0.618 \end{bmatrix}$ **2.** 3.41 ; [0.74, – 1, 0.67]′

3. (*a*) 6, [1, 1, – 1]′ (*b*) 8, [1, – 0.5, 0.5]′

4. (*a*) 7 ; [2.099/7, 0.467/7, 1] (*b*) 25.182, [1, 0.045, 0.068]′ (*d*) 11.66 [0.025, 0.422, 1.000].

Problems 28.7, page 945

1. Newton-Raphson method **2.** $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$ **3.** Chord AB

4. $x_{n+1} = \dfrac{1}{2}(x_n + N/x_n)$ **5.** initial approximation x_0 is chosen sufficiently close to the root

6. diagonal **7.** (*c*) **8.** (*a*)

9. $x_{n+1} = \dfrac{1}{3}\,(2x_n + N/x^2{}_n)$ **10.** $x_{n+1} = \dfrac{1}{2}\,(x_n + x_{n-1})$ **11.** $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)}\, f(x_0)$

12. (*a*) **13.** $x_{n+1} = x_n(2 - Nx_n)$ **14.** (*b*)
15. Newton-Raphson method **16.** § 28.6 **17.** Upper triangular matrix
18. False **19.** True **20.** $x = 1, y = 1$.

Problems 29.1, page 952

1. 0.4 **2.** – 7459 **5.** 239 **6.** 4.68, 2.68, 55.8, 99.88

8. (*i*) $1 - 2 \sin (x + 1/2) \sin 1/2$; (*ii*) $\tan^{-1} (1/2n^2)$;
(*iii*) $192\,[x(x+4)(x+8)(x+12)(x+16)]$ (*iv*) $-2/[(x+2)(x+3)(x+4)]$

9. (*i*) $e^{3x}\,[e^3 \log (1 + 1/x) + (e^3 - 1) \log 2x]$ (*ii*) $2^x (1 - x)/(1 + x)$
(*iii*) $(a-1)^n a^x$; (*iv*) $(-1)^n\, n!/[x(x+1)(x+2) \ldots (x+n)]$.

12. (*i*) – 36 ; (*ii*) $24 \times 2^{10} \times 10\,!$ **14.** $u = [x]^4 - 6[x]^3 + 13[x]^2 + x + 9$

15. $4x^3 - 12x^2 + 8x + 1$; $12x(x - 1)$ **16.** $\dfrac{1}{2}[x]^4 + 3[x]^3 + 4[x] + c$

17. $y(4) = 74, y(6) = 261$ **19.** 15.

Problems 29.2, page 957

1. $\left(\dfrac{\Delta^2}{E}\right) u_x = u_{x+h} - 2u_x + u_{x-h}$; $\dfrac{\Delta^2 u_x}{E u_x} = \dfrac{u_{x+2h} - 2u_{x+h} + u_x}{u_{x+h}}$

2. (*i*) $2(\cos h - 1) \sin x$; (*ii*) $6x$; (*iii*) $2(\cos h - 1)\,[\sin (x + h) + 1]$; (*iv*) 8

8. *Error* = 10 **9.** 31 **10.** $f(1.5) = 0.222, f(5) = 22.022$

11. $y(4) = 74, y(6) = 261$ **12.** – 99 **13.** $y_4 = 1$ approx

15. (*i*) $n(3n^2 + 6n + 1)$; (*ii*) $\dfrac{n(n+1)(n+2)(n+3)}{4}$ **16.** $2/(1 - x)^3$.

Problems 29.3, page 961

1. 5.54 **2.** 6.36 **3.** 1.1312 **4.** 0.788
5. ₹ 110.52 **6.** 8666 **7.** 352 ; 219 **8.** 0.9623, 0.2903
9. 24 approx **10.** $f(x) = 9x - 4x^2$ **11.** 1.625 **12.** 0.1955
13. 4.219 **14.** 2530 **15.** $y_1 = 0.1, y_{10} = 100$ **16.** $u_2 = 42, u_4 = 49$
17. 10, 22 **18.** 755.

Problems 29.4, page 971

1. 19.4 **2.** 12.826 **3.** 54000 **4.** 3.2219
5. 3.0375 **6.** 395 **7.** 3.347 **8.** 9
9. 3250.875 **11.** 2.5283 by all formulae.

Problems 29.5, page 974

1. 14.63 **2.** 2.8168 **3.** 0.89 **4.** 100
5. $648 + 30x - x^2$ **6.** $x^3 - 3x^2 + 5x - 6$ **7.** $x^5 - 9x^4 + 18x^3 - x^2 + 19x - 18$
8. 3 **9.** $\frac{0.5}{x-1} - \frac{0.5}{x+1} + \frac{1}{x-2}$

Problems 29.6, page 977

1. 1 **2.** 3.09 **3.** 448, 3150 **4.** 133.19
5. $f(x) = \frac{1}{24}x^3 - 25x + 24 - \frac{7}{6}x^2 + \frac{557}{60}x - 25$ **6.** $f(x) = \frac{1}{20}x^3$ **7.** $f(x) = x^4 - 3x^3 + 5x^2 - 6$
8. 31.

Problems 29.7, page 978

1. 11.5 **2.** 6.5928 **3.** 37.23

Problems 29.8, page 978

1. \$ 7.3 **2.** (*b*)

3.

x	$f(x)$	I.D.D.	II.D.D.
5	7		
		2.9	
15	36		0.87
22	160	17.7	

4. Intermediate value of the variables. **5.** \$ 7.8
6. $\frac{[x_1, x_2, x_3, x_4] - [x_0, x_1, x_2, x_3]}{x_4 - x_0}$ **7.** $\frac{-1}{4}$ and $\frac{1}{4}$ **8.** \$ 7.14

9. $f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + \frac{(x - x_0)(x - x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$

10. $\frac{13}{5}$ **11.** (*c*) **12.** 1.857
13. *Extrapolation* is the process of estimating the value of a function outside the given range of values.
14. $1/(abc)$ **15.** (*a*) **16.** $x^3 - 7x^2 - 18x - 12$
17. (*b*) **18.** $6h^2(x + h)$.

Problems 30.1, page 987

1. – 27.9, 117.67
2. 4.75, 9
3. 0.63, 6.6
4. (a) 0.493, – 1.165 (b) 0.4473, – 0.1583 ; (c) 0.4662, – 0.2043
5. 2.8326
6. – 0.06 ; 0.5
7. 0.175
8. 13.13 m/sec
9. (i) – 52.4, (ii) – 0.0191
10. 44.92
11. 0.085
12. 3.82 rad/sec, 6.75 rad/sec^2
13. 0.2561
15. 0.0186
16. 135
17. $y_{max}(1) = 0.25$, $y_{min}(0) = 0$
18. Max $f(10.04) = 1340.03$

Problems 30.2, page 995

1. (i) 0.695 (ii) 0.693 (iii) 0.693
2. (i) 0.7854 (ii) 0.7854, (iii) 0.78535, (iv) 0.7854
3. 1.61
4. – 6.436
5. (i) 70.16 (ii) 0.635
6. 0.6305
7. (i) 2.0009 ; (ii) 1.1873
8. (i) 1.1249 (ii) 0.911
9. (a) 1.8276551, .0001924 ; (b) 1.8278472, .0000003 ; (c) 1.8278470, 0000005 ; (d) 1.8278474, .00000001
10. 1.3028
11. 403.67
12. 7.78
13. 710 sq.ft
14. 3.032
15. 408.8 cub. cm.
16. 1.063 sec ; 1.064 sec
17. 552 m ; 3 m/sec^2.
18. 30.87 m/sec.
19. 29 min nearly.

Problems 30.3, page 996

1. (c)
2. $\frac{1}{h}\left[\Delta f(a) - \frac{1}{2}\Delta^2 f(a) + \frac{1}{3}\Delta^3 f(a) - \cdots\right]$
3. h should be small
4. 0.775
5. $2\frac{2}{3}$
6. 30.8
7. (b)
8. larger number of sub-intervals
9. 0.7854
10. (d)
11. $y'_{x_n} \frac{1}{h} = \left[\nabla_{y_n} + \frac{1}{2}\nabla^2{}_{y_n} + \frac{1}{3}\nabla^3{}_{y_n} + \cdots\right]$
12. a multiple of 6
13. (c)
14. 0.783
15. 0.69
16. 1.36125
17. 1.36
18. if the entire curve is itself a parabola
19. even and multiples of 3
20. False

Problems 31.1, page 999

1. $y_{x+3} - 2y_{x+2} + 2y_{x+1} = 0$
2. $\Delta y_n = (-1)^{n+1}/(n+1)$
3. $u_{n+1} - 2u_n = 0$
4. (i) $(x+2)y_{x+2} - 2(2x+1)y_{x+1} + xy_x = 0$; (ii) $(x^2+x)y_{x+2} - (2x^2+4x)y_{x+1} + (x^2+3x+2)y_x = 0$
5. (ii) $y_{n+2} - 8y_{n+1} + 15y_n = 0$; (ii) $y_{n+2} - 6y_{n+1} + 4y_n = 0$
6. (i) $(x-1)y_{x+2} - (3x-2)y_{x+1} + 2xy_x = 0$; (ii) $y_{x+2} - 4y_x = 0$; (iii) $y_{x+3} - 6y_{x+2} + 11y_{x+1} - 6y_x = 0$.

Problems 31.2, page 1002

1. $u_p = (c_1 + c_2 p)\, 3p$
2. $y_n = c_1 \cos\frac{2n\pi}{3} + c_2 \sin\frac{2n\pi}{3}$
3. $u_n = c_1 \cos n\pi/2 + c_2 \sin n\pi/2$
4. $y_n = c_1 . 2^n + c_2 . 3^n$.
5. $y_n = (2)^{n-1} + (-2)^{n-1}$
6. $u_k = c_1(-1)^k + (c_2 + c_3 k)\, 2^k$.
7. $f(x) = (c_1 + c_2 x)(-1)^x + c_3 . 2^x$
8. $u_n = 2n + (-2)^n$.
9. $y_n = 6 + (n-3)\, 2^n$
10. $u_n = 2^{n/2}[c_1 \cos n\pi/4 + c_2 \sin n\pi/4]$
11. $y_m = 2^m\left\{c_1 \cos\frac{m\pi}{4} + c_2 + \sin\frac{m\pi}{4} + c_3 \cos\frac{3m\pi}{4} + c_4 \sin\frac{3m\pi}{4}\right\}$
14. $y_n = c_1(-1)^n + c_2(10)^n$

Problems 31.3, page 1005

1. $y_n = c_1(-1)^n + c_2\,(6)^n - 2^n/12$
2. $y_n = \left(\frac{n}{15} - \frac{1}{25}\right)(-3)^n + \frac{2^n}{25}$
3. $y_p = c_1 + c_2 p + c_3 p^2 + \frac{1}{6}\,p(p-1)(p-2)$
4. $y_n = 2^n\left(\frac{2}{\sqrt{3}}\sin\frac{n\pi}{3} - 2\cos\frac{n\pi}{3}\right) + 2$
5. $y = c_1 + c_2\,.\,3^x + \frac{1}{2}\,x\,.\,3^{x-1}$
6. $y_x = (c_1 + c_2 x)\,2^x + 3x(x-1)\,2^{x-3} + 5\,.\,4^{x-1}$
7. $u_n = c_1 + c_2(-1)^n + \frac{1}{2}\,\frac{\cos\left(\frac{n}{2}-1\right) - \cos\frac{n}{2}}{1-\cos 1}$
8. $y_p = c_1\cos\frac{p}{2} + c_2\sin\frac{p}{2} + \frac{p\cos\left(p-\frac{1}{2}\right)}{2\sin\frac{1}{2}}$
9. $y_x = c_1 + 2^x + c_2(-2)^x - \frac{1}{27}\,(9x^2 + 12x + 11)$
10. $y_n = c_1(-1)^n + c_2\cos\frac{n\pi}{3} + c_3 + \sin\frac{n\pi}{3} + \frac{1}{2}\,n(n-3)$
11. $y_n = (c_1 + c_2 n)(3)^n + c_3(-1)^n + \frac{1}{3}(2)^n - \frac{3n}{4}$
12. $y_n = (c_1 + c_2 n)\,2^{-n} + \frac{2^n}{9} + n(n-1)\left(\frac{1}{2}\right)^{n-1}$
13. $y_n = c_1(-2)^n + c_2(-3)^n + \frac{n}{12} - \frac{7}{144}$
14. $u_x + (c_1 + c_2 x)(-3)^x + \frac{2^x}{25}(5x-2) + \frac{2}{4}\,x^{x-2} + \frac{7}{16}$
15. $y_n = c_1(-2)^n + 2^n\,(c_2\cos n\pi/3 + c_3\sin n\pi/3) + \frac{3}{16}\,(2)^n + 2^{n-4}\,(2n+3)$
16. $u_n = \left\{c_1 + c_2 n + \frac{1}{48}\,n\,(n-1)^2\,(n-2)\right\}2^n.$
17. $y_k = c_1\,.\,2^k + c_2\,.\,3^k + \frac{4^k}{2}(k^2 - 13k + 61)$
18. $y_n = 2^n\left\{(c_1 + n)\cos\frac{n\pi}{3} + c_2 + \sin\frac{n\pi}{3}\right\}.$

Problems 31.4, page 1006

1. $y_x = a + b\,(-1)^x + x,\ z_x = a + b\,(-1)^{x+1} - (x+1)$
2. $y_x = (a + bx)(-1)^x - \frac{1}{9}\,.\,2^{x+2},\ z_x = \frac{2^x}{9} - (-1)^x\left[a + b(x - \frac{1}{2})\right]$
3. $u_n = 2.4^n - 2\frac{1}{2}\,n(n-1),\ v_n = 4^n + 2 + \frac{1}{n} + n(n+1)$
4. $u_x = -2a + b(-2)^x - c + \frac{1}{2x}\,(3-x),\ v_x = a + c + b\,(-2)^x + \frac{1}{2}x(x-1).$

Problems 31.5, page 1007

1. $y_{i+1} - 2y_i + y_{i-1} = -\frac{l_m}{P}\,y_i$. Solve it for y_i.

Problems 31.6, page 1007

1. $y_{n+2} - 5y_{n+1} + 6y_n = 0$
2. $u_n = c_1 + c_2 n + c_3 n^2$
3. $u_n = c_1 + c_2(-2)^n + c_3(3)^n$
4. $y_n = c + 2^n.$
5. $y_n = c(2)^n - (n+1)$
6. $(x^2 + x)y_{x+2} - (2x^2 + 4x)y_{x+1} - (x^2 + 3x + 2)y_x = 0$
7. $y_n = c(2)^k + 1$
8. $y_n = (2)^{n+2} + (-2)^{n-1}$
9. Third

10. $(x+2)y_{n+2} - 2(n+1)y_{n+1} + ny_n = 0$
11. Second
12. $y_n = (C_1 + C_2 n)2^n$
13. $\frac{1}{2}x(x-1)(3)^{x-2}$
14. $y_{n+2} - 6y_{n+1} + 9y_n = 0$
15. True.

Problems 32.1, page 1012

1. $y = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$
2. 0.0214
3. $y = \frac{1}{3}x^3 - \frac{1}{81}x^9 + \ldots$
4. (*a*) 0.9138, (*b*) 0.1938
5. $y(1.1) = 0.1103$, $y(1.2) = 0.2428$. Exact $y(1.1) = 0.1103$ and $y(1.2) = 0.2428$
6. 1.1053425
7. 1.1272
8. 1.005.

Problems 32.2, page 1017

1. 1.1831808
2. 1.1448
3. 4.5559
4. $y(0.1) = 0.095$, $y(0.2) = 0.181$, $y(0.3) = 0.259$
5. $y(0.2) = 1.2046$, $y(0.4) = 1.4644$
6. 2.2352
7. 1.0928
8. 5.051.

Problems 32.3, page 1021

1. 1.7278
2. 1.1749
3. 1.0207, 1.0438
4. 2.5005
5. $y(0.1) = 0.9052$, $y(0.2) = 0.8213$
6. $y(0.1) = 2.9919$, $y(0.2) = 2.9627$
7. 0.3487
8. 0.8489
9. 1.1678
10. 1.0911, 1.1677, 1.2352, 1.2902, 1.338.

Problems 32.4, page 1026

1. 3.795
2. 1.2797
3. $y(1.4) = 3.0794$
4. $y(4.5) = 1.023$
5. $y(0.4) = 2.162$
6. $y(0.4) = 0.441$.

Problems 32.5, page 1030

1. 0.2416
2. 1.0408
3. 0.6897
4. $y(4.4) = 1.019$
5. 2.5751
6. $y(1.4) = 0.949$.

Problems 32.6, page 1034

1. $y_3 = 1 + \frac{x}{2} + \frac{3}{40}x^5 + \frac{1}{40}x^6 + \frac{1}{192}x^9$. $z_3 = \frac{1}{2} + \frac{3}{8}x^4 + \frac{1}{10}x^5 + \frac{3}{34}x^8 + \frac{7}{340}x^9 + \frac{1}{256}x^{12}$
2. $y(0.1) = 0.105$, $z(0.1) = 0.999$; $y(0.2) = 0.22$, $Z(0.2) = 0.997$
3. $y(0.1) = 2.0845$, $z(0.1) = 0.5867$
4. $y_2 = 1 + \frac{1}{2}x + \frac{3}{40}x^5$
5. $y(0.1) = 0.5075$
6. $y(0.2) = 0.9802$, $y(0.2) = -0.196$
7. -0.5159
8. $\theta(0.2) = 0.8367$, $(d\theta/dt)_{0.2} = 3.6545$.

Problems 32.7, page 1038

1. 0.14031
2. $y(.25) = y(.75) = 2.4$, $y(.5) = 3.2$
3. $y(1.25) = 1.3513$, $y(1.5) = 1.635$, $y(1.75) = 1.8505$
4. $y(.25) = -0.3473$, $y(.5) = -0.9508$, $y(.75) = -1.7257$
5. $n = 2$, $y(0.5) = 0.1389$, true value $= 0.1505$; $n = 4$, $y(0.5) = 0.147$

6. $y(0.25) = 0.062, y(0.5) = 0.25, y(0.75) = 0.562$
7. $y(1) = 1.0171, y(2) = 1.094$
8. $y(1/3) = 1.1539, y(2/3) = 3.9231 ; y(1) = 7.4615.$

Problems 32.8, page 1038

1. (*b*)
2. $y_{i+1} - 2y_i + y_{i-1} + h^2 y_i = 5h^2$
3. $1 + x + x^2 + x^3/6$
4. § 31.4
5. § 31.3
6. (*b*)
7. $y_4 = y_2 + \dfrac{h}{3}(f_2 + 4f_3 + f_4)$
8. Modified Euler's method
9. $y_1 = y_0 + \dfrac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$
10. Milne's method and Adam-Bashforth method
11. four
12. $y = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{8}$
13. (*b*)
14. $y_4 = y_0 + \dfrac{4h}{3}(2f_1 - f_2 + 2f_3)$
15. $y_1 = y_0 + \dfrac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$
16. 1.1818
17. $\dfrac{dy}{dx} = z, \dfrac{dz}{dx} + y(1 + yz) = 0$
18. § 31.7
19. starting values
20. Picards and Runge-Kutta methods
21. It agrees with Taylor's series solution upto the term in h^4
22. (*d*)
23. $y_{i+1} + 2(h^2 - 1)\, y_i + y_{i-1} = 0$
24. (*a*)
25. True
26. False
27. False.

Problems 33.1, page 1041

1. Parabolic
2. Hyperbolic
3. (*i*) Parabolic (*ii*) Elliptic (*iii*) Elliptic
4. Outside the ellipse $(x/0.5)^2 + (y/0.25)^2 = 1$.

Problems 33.2, page 1050

1. $u_1 = 1.999, u_2 = 2.999, u_3 = 3.999$
2. 2.37, 5.6, 9.87, 2.89, 6.14, 9.89, 3.02, 6.17, 9.51
3. $u_1 = 10.188, u_2 = 0.5, u_3 = 1.188, u_4 = 0.25, u_5 = 0.625, u_6 = 1.25$
4. $u_1 = 26.66, u_2 = 33.33, u_3 = 43.33, u_4 = 46.66$
5. $u_1 = 0.99, u_2 = 1.49, u_3 = 0.49$
6. $u_1 = 1.57, u_2 = 3.71, u_3 = 6.57, u_4 = 2.06, u_5 = 4.69, u_6 = 8.06, u_7 = 2, u_8 = 4.92, u_9 = 9$
7. $u_1 = -3, u_2 = -2, u_3 = -2.$

Problems 33.3, page 1054

1.

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	2	3	2	0
2	0	1.5	2	1.5	0
3	0	1	1.5	1	0
4	0	0.75	1	0.75	0
5	0	0.5	0.75	0.5	0

2.

j \ i	0	1	2	3	4	5	6	7	8	9	10
0	0	0.09	0.16	0.21	0.24	0.25	0.24	0.21	0.16	0.09	0
1	0	0.08	0.15	0.20	0.23	0.24	0.23	0.20	0.15	0.08	0
2	0	0.075	0.14	0.19	0.22	0.23	0.22	0.19	0.14	0.075	0
3	0	0.07	0.133	0.18	0.21	0.22	0.21	0.18	0.133	0.07	0

3.

j \ i	0	1	2	3	4	5
0	0	24	84	144	144	0
1	0	42	84	114	72	0
2	0	42	78	78	57	0
3	0	39	60	67.5	39	0
4	0	30	53.25	49.5	33.75	0
5	0	26.6	39.75	43.5	24.75	0
6	0	19.88	35.06	32.25	21.75	0

4.

j \ i	0	1	2	3	4
0	0	0.5	1	0.5	0
1	0	0.5	0.5	0.5	0
2	0	0.25	0.5	0.25	0
3	0	0.25	0.25	0.25	0

Problems 33.4, page 1060

1.

$t = 0.3$; $x =$	0.1	0.2	0.3	0.4	0.5
Numerical sol. $u =$	0.02	0.04	0.06	0.075	0.08
Exact sol. $u =$	0.02	0.04	0.06	0.075	0.08

2.

j \ i	0	1	2	3	4	5
0	0	20	15	10	5	0
1	0	7.5	15	10	5	0
2	0	– 5	2.5	10	5	0
3	0	– 5	– 10	– 2.5	5	0
4	0	– 5	– 10	– 15	– 7.5	0
5	0	– 5	– 10	– 15	– 20	0

3.

j \ i	0	1	2	3	4	5	6	7	8	9	10
0	0	10.19	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	0
1	0	5.08	10.15	10.20	10.23	10.24	10.23	10.20	10.15	5.08	0
2	0	0.06	5.12	10.17	10.20	10.21	10.20	10.17	10.12	0.06	0
3	0	0.04	0.08	5.12	10.15	10.16	10.15	10.12	10.08	0.04	0
4	0	0.02	0.04	0.06	5.08	10.09	10.08	10.06	10.04	0.02	0
5	0	0	0	0	0	0	0	0	0	−0.02	0

4.

j \ i	0	0.1	0.2	0.3	0.4	0.5
0.1	0	0.037	0.07	0.096	0.113	0.119
0.2	0	0.031	0.059	0.082	0.096	0.101
0.3	0	0.023	0.043	0.059	0.07	0.074
0.4	0	0.012	0.023	0.031	0.037	0.039
0.5	0	0	0	0	0	0

Problems 33.5, page 1060

1. (*b*) **2.** False **3.** a hyperbolic equation

4. Poisson's equation **5.** $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ **6.** $(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})/h^2$

7. § 32.8 (2) **8.** $\frac{1}{4}(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1})$

9. hyperbolic **10.** Bendre-Schmidt

11. $u_{i,j+1} = 2(1 - 4\alpha^2)\, u_{i,j} + 4\alpha^2 (u_{i-1,j} + u_{i+1,j} - u_{i,j-1})$ where $\alpha = k/h$.

12. $u_{i,j+1} = \frac{1}{2}\,(u_{i-1,j} + u_{i+1,j})$ **13.** $33.5(2) **14.** $\lambda < \frac{1}{2}$

15. Elliptic

16. $u(0, t) = 0 = u\,(1, t), (t > 0)\,;\, u(x, 0) = f(x), 0 < x < l\,;\, \delta x/\delta t\,(x, 0) = 0, 0 < x < l$

17. $u_{i,j+1} = u_{i+1,j} - u_{i-1,j} - u_{i,j-1}$ **18.** $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = f\,(x, y)$

19. 100 **20.** $\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0$

21. $y < 0$ **22.** $k = 1/4$.

Problems 34.1, page 1063

1. Max. $Z = 1.2x_1 + 1.4x_2$; subject to $40x_1 + 25x_2 \le 1000$,
$35x_1 + 28x_2 \le 980$, $25x_1 + 35x_2 \le 875$ and $x_1, x_2 \ge 0$

2. Max. $Z = 3x_1 + 2x_2 + 4x_3$; subject to $4x_1 + 3x_2 + 5x_3 \le 2000$,
$2x_1 + 2x_2 + 4x_3 \le 2500$, $100 \le x_1 \le 150$, $200 \le x_2$ and $50 \le x_3$

3. Max. $Z = 3x_1 + 2x_2 + x_3$; subject to $3x_1 + 4x_2 + 3x_3 \le 42$,
$5x_1 + 3x_3 \le 45$, $3x_1 + 6x_2 + 2x_3 \le 41$ and $x_1, x_2, x_3 \ge 0$.

4. Max. $Z = 400\,x + 300\,y$; subject to $x + y \le 200$, $x \ge 20$, $y \ge 4x$, $x \ge 0$, $y \ge 0$

5. Min. $Z = x_1 + x_2$; subject to $2x_1 + x_2 \ge 12$, $5x_1 + 8x_2 \ge 74$,
$x_1 + 6x_2 \ge 24$ and $x_1, x_2, x_3 \ge 0$.

6. Max. $Z = 0.15x_1 + 0.25x_2$; subject to
$2x_1 + 5x_2 \le 480{,}000$, $5x_1 + 4x_2 \le 720{,}000$,
$8x_1 + 16x_2 \le 300{,}000$, $0 \le x_1 \le 25{,}000$ and $0 \le x_2 \le 7{,}000$
7. Min. $Z = 41x_1 + 35x_2 + 96x_3$; subject to
$2x_1 + 3x_2 + 7x_3 \ge 1250$, $x_1 + x_2 \ge 250$, $5x_1 + 3x_2 \ge 900$,
$6x_1 + 25x_2 + x_3 \ge 232.5$ and $x_1, x_2, x_3 \ge 0$
8. Min. $Z = 100x_1 + 250x_2 + 160x_3$; subject to
$0.94x_1 + x_2 + 1.04x_3 \le 0.98$, $10x_1 + 15x_2 + 17x_3 \ge 14$,
$470x_1 + 500x_2 + 520x_3 \ge 495$, $x_1 + x_2 + x_3 = 1$ and $x_1, x_2, x_3 \ge 0$.

Problems 34.2, page 1069

1. $x_1 = 100, x_2 = 50$; max. $Z = 550$
2. $x_1 = 8/15, x_2 = 12/5$, max. $Z = 24.8$
3. $x_1 = 15, x_2 = 0$; max. $Z = 300$
4. $x_1 = 1000, x_2 = 500$; max. $Z = 5500$
5. 450 units of product B only ; max. profit = ₹ 1800
6. $X = 2, Y = 4.5$; max. profit = ₹ 37
7. A = 1.18 units, B = 0.53 units ; max. profit = ₹ 14.50 approx.
8. 2000/11 units of product A and 1000/11 units of B ; max. profit = ₹ 10,000
9. $x_1 = 4, x_2 = 0$; max. $z = 8$
10. Unbounded solution
11. $x_1 = 2, x_2 = 4$; min. $Z = 64$
12. Production cost will be min. if G and J run for 12 and 4 days respectively.

Problems 34.3, page 1074

1. Max. $Z = 3x_1 + 5x_2 + 8x_3$; subject to $2x_1 - 5x_2 + s_1 = 6$,
$3x_1 + 2x_2 + x_3 - s_2 = 5$, $3x_1 + 4x_3 + s_3 = 3$;
$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$
2. Min. $Z = 3x_1 + 2x_2 + 5x_3$, subject to $-5x_1 + 2x_2 + s_1 = 5$,
$2x_1 + 3x_2 + 4x_3 - s_2 = 7$, $2x_1 + 5x_3 + s_3 \ge 3$
$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$.
3. Max. $Z = 3x_1 - 2x_2 + 4x_4 - 4x_5$; subject to
$x_1 + 2x_2 + x_4 - x_5 + s_1 = 8$, $2x_1 - x_2 + x_4 - x_5 - s_2 = 2$,
$-4x_1 + 2x_2 + 3x_4 - 3x_5 = 6$; $x_1, x_2, x_4, x_5, s_1, s_2 \ge 0$
4. (*i*) $x_1 = 2, x_3 = 1$ (Basic) ; $x_3 = 0$ (Non-basic). (*ii*) $x_1 = 5$,
$x_3 = -1$ (Basic) ; $x_2 = 0$ (Non-basic) ; (*iii*) $x_2 = 5/3, x_3 = 2/3$ (Basic) ; $x_1 = 0$ (Non-basic). All the three basic solutions are non-degenerate
6. Basic solutions are (*i*) $x_1 = 2, x_2 = 1$ (Basic) and $x_3 = 0$;
(*ii*) $x_1 = x_3 = 1$ (Basic) and $x_2 = 0$; (*iii*) $x_2 = -1, x_3 = 2$ (Basic) and $x_1 = 0$
(*a*) First two solutions are non-degenerate basic feasible solutions
(*b*) First solution is optimal and $Z_{max} = 5$.

Problems 34.4, page 1081

1. $x_1 = 2, x_2 = 4$, max. $Z = 14$
2. $x_1 = 0, x_2 = 20$; max. $Z = 200$
3. $x_1 = 7/3, x_2 = 4/3$; max. $Z = 16$
4. $x_1 = 5, x_2 = x_3 = 0$; max $Z = 50$
5. $x_1 = 0, x_2 = 100, x_3 = 230$; max. $Z = 1350$
6. $x_1 = 89/41, x_2 = 50/41, x_3 = 62/41$; max. Z = ₹ 765/41
7. $x_1 = 4, x_2 = 5, x_3 = 0$; min $Z = -11$
8. $x_1 = 280/13, x_2 = 0, x_3 = 20/13, x_4 = 180/13$; max. $Z = 2280/13$
9. $x_1 = 0, x_2 = 400$ units ; max. profit = ₹ 1200
10. $x_1 = 125, x_2 = 250$ units ; max profit = ₹ 2250
11. $x_1 = 400$ gms, $x_2 = 0$; min. cost = ₹ 2
12. $x_1 = 0, x_2 = x_3 = 50$; max. profit = ₹ 700

13. $x_1 = 0.5, x_2 = x_3 = 0.04$ units ; min. cost = ₹ 5.80
14. Averages for corn, wheat, soyabeans are 250, 625, zero respectively to achieve a max. profit of ₹ 32,000.

Problems 34.5, page 1088

1. $x_1 = 0, x_2 = 2, x_3 = 0$; max. $Z = 4$
2. $x_1 = 3, x_2 = 2, x_3 = 0$; max. $Z = 8$
3. $x_1 = x_2 = -6/15$; max. $Z = -48/5$
4. $x_1 = 23/3, x_2 = 5, x_3 = 0$; max $Z = 85/3$
5. $x_1 = x_2 = x_3 = 5/2, x_4 = 0$; max. $Z = 15$
6. $x_1 = 21/13, x_2 = 10/13$; max. $Z = 31/13$
7. Infeasible
8. $x_1 = 23/3, x_2 = 5, x_3 = 0$; max. $Z = 85/3$
9. $x_1 = 55/7, x_2 = 30/7, x_3 = 0$; max. $Z = 155/7$
10. $x_1 = 2, x_2 = 0$; max. $Z = 18$
11. Degenerate solution : $x_1 = 0$ (non-basic) ; $x_2 = 1, x_3 = 0$ (basic) ; max. $Z = 3$.

Problems 34.6, page 1091

1. Min. $W = 26y_1 + 7y_2$; subject to $6y_1 + 4y_2 \geq 10$
$5y_1 + 2y_2 \geq 13, 3y_1 + 5y_2 \geq 19$; $y_1, y_2, y_3 \geq 0$
2. Max. $W = 11y_1 + 7y_2 + y_3 + 5y_4$; subject to $3y_1 + 2y_2 - y_3 + 3y_4 \leq 2$,
$4y_1 + 3y_2 + 2y_3 + 2y_4 \leq 4, y_1 - 2y_2 + 3y_3 + 2y_4 \leq 3$;
$y_1, y_2, y_3, y_4 \geq 0$
3. Min. $W = -3y_1 + y_2 + 4y_3$; subject to $y_1 + 3y_2 - 2y_3 \leq -3$,
$y_1 + y_3 \geq 16, y_1 - 2y_2 + y_3 \leq -7$;
$y_1, y_2 \geq 0, y_3$, unrestricted in sign
4. Max. $W = -5y_1 + 9y_2 + 8y_3$; subject to $-2y_1 + 4y_2 - 8y_3 \leq 3$,
$3y_1 - 2y_2 + 4y_3 \leq -2, -y_1 + 3y_3 = 1$;
$y_1, y_2 \geq 0, y_3$ unrestricted
5. Min. $y = 3y_1 + 4y_2 + y_3 + 6y_4$; subject to $5y_1 - 2y_2 + y_3 - 3y_4 \geq 2$,
$6y_1 + y_2 - 5y_3 - 3y_4 \geq 5, -y_1 + 4y_2 + 3y_3 + 7y_4 \geq 6, y_1, y_2, y_3, y_4 \geq 0$.

Problems 34.7, page 1094

1. $x_1 = x_2 = 0, x_3 = 5/2$; min. $Z = 2.5$
2. $x_1 = 4, x_2 = 2$; max. $Z = 10$.
3. $x_1 = 7, x_2 = 0$, max. $Z = 21$
4. $x_1 = 0, x_2 = 100, x_3 = 230$; max. $Z = 1350$.

Problems 34.8, page 1097

1. $x_1 = 0, x_2 = 1$; max. $Z = -1$
2. $x_1 = 3/5, x_2 = 6/5$; min. $Z = 12/5$
3. $x_1 = 6, x_2 = 2, x_3 = 0$; min. $Z = 10$
4. $x_1 = 65/23, x_2 = 0, x_3 = 20/23, x_4 = 0$; min. $Z = 215/23$.

Problems 34.9, page 1104

1. $x_{11} = 200, x_{12} = 50, x_{22} = 175, x_{24} = 125, x_{33} = 275, x_{34} = 125$; min. cost = ₹ 12075
2. $x_{13} = 14, x_{21} = 6, x_{22} = 5, x_{23} = 1, x_{32} = 5$; min. cost = 143
3. $x_{11} = 50, x_{12} = 100, x_{21} = 150, x_{33} = 150, x_{42} = 100, x_{43} = 50$; min. tonnage = 3300
4. $x_{11} = 140, x_{13} = 60, x_{21} = 40, x_{22} = 120, x_{33} = 90$; min. cost = ₹ 5920
5. $x_{11} = 5, x_{14} = 2, x_{22} = 3, x_{23} = 7, x_{32} = 5, x_{34} = 13$;
min. cost = ₹ 799 and maximum saving = ₹ 201
6. $x_{11} = 150, x_{13} = 20, x_{22} = 160, x_{24} = 40, x_{33} = 90, x_{34} = 90$; max. profit = 4920
7. $x_{13} = 2, x_{22} = 1, x_{23} = 2, x_{31} = 4, x_{33} = 1$; min. cost = 33
8. $x_{13} = 0, x_{15} = 800, x_{21} = 400, x_{24} = 100, x_{32} = 400, x_{33} = 200$,
$x_{34} = 300, x_{43} = 300$; min. cost = 9200.

Problems 34.10, page 1109

1. $x_{11} = x_{22} = x_{33} = 1$; min. cost = ₹ 18
2. $A \to 2, B \to 3, C \to 4, D \to 1$; min. $Z = 38$
3. $I \to B, II \to A, III \to D, IV \to C$; min. cost = ₹ 49
4. $A \to$ Dyn. Prog., $B \to$ Queuing Th., $C \to$ Reg. Analysis, $D \to$ L.P. ; min. time = 28 hrs
5. (*i*) $A \to I, B \to II, C \to III, D \to IV$; (*ii*) $A \to I, B \to III, C \to II, D \to IV$
6. $1 \to IV, 2 \to II, 3 \to VI, 4 \to I, 5 \to III, 6 \to V$; max. profit = ₹ 270

Problems 34.11, page 1110

1. § 34.5 Def. 2
2. it provides an optimality test
3. § 34.11
4. § 34.16 (1)
5. § 34.13
6. § 34.6 (1)
7. Min. $W = 7y_1 + 5y_2$, subject to $2y_1 + 3y_2 \le 4$,
   $3y_1 - 2y_2 \le 9$, $2y_1 + 4y_2 \le 2$, $y_1 \ge 0$, y_2 is unrestricted in sign
8. § 34.12 (2)
9. § 34.14
10. Minimize $Z = (2x_{11} + 3x_{12} + 11x_{13} + 4x_{14}) + (5x_{21} + 6x_{22} + 8x_{23} + 7x_{24})$,
    subject to $x_{11} + x_{12} + x_{13} + x_{14} = a_1 (= 15)$, $x_{21} + x_{22} + x_{23} + x_{24} = a_2 (= 20)$,
    $x_{11} + x_{21} = b_1 (= 10)$, $x_{12} + x_{22} = b_2 (= 5)$; $x_{13} + x_{23} = b_3 (= 12)$; $x_{14} + x_{24} = b_4 (= 8)$ and $x_{ij} \ge 0$.
    [$\because \quad \Sigma a_i = \Sigma b_j = 35$]
11. (*i*) $x_1 = 3, x_2 = 5, x_3 = 0$; (*ii*) $x_1 = 0.5, x_2 = 0, x_3 = 2.5$
12. § 34.5 (Def. 4)
13. § 34.15
14. balanced
15. § 34.9
16. § 34.7 (3)
17. optimal
18. Minimize $y = 5y_4 - 3y_3$, subject to $y_4 + y_3 = 5$, $2y_4 - 5y_3 \ge 6$, $y_3 \ge 0$ and y_4 unrestricted
19. 5
20. Max. $Z = 5/19$
21. § 34.7
22. § 34.16
23. § 34.7 [2 (*ii*)]
24. *Min.* $W = 2y_1 + 4y_2 + 3y_3$, subject to $-y_1 + y_2 + y_3 \ge 2$, $2y_1 + y_2 \ge 1$, $y_1, y_2 \ge 0$
25. North west corner rule and Vogeli approximation method
26. Slash or surplus variables.

Problems 35.1, page 1118

1. (*i*) $y = \frac{1}{4}x^2 + c_1 x + c_2$; (*ii*) $y = c_1 x^{-1} + c_2$
3. $y = c_1 e^x + c_2 e^{-x} + \frac{xe^x}{2}$
4. $y = -x \cos x/2$
5. $y = (c + x) \sin x$
6. $y = \sinh (c_1 x + c_2)$
7. $y = x^2 - 1$
8. $(x + c)^2 + y^2 = k^2$
9. $y = 2 \sin x$
12. The spirals of the family $r = a \sec (\phi \sin \alpha + b)$.

Problems 35.2, page 1120

1. $y = \pm 2 \sin m\pi x$, where m is an integer
2. $y = \lambda x^2 + ax + b$, where λ, a, b are determined from the isoperimetric and boundary conditions.
4. $y(x) = \frac{1}{2}(1 - \cos x) + \frac{1}{4}(2 - \pi) \sin x$.

Problems 35.3, page 1124

2. $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$,
   $z = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x - 2c_2 \sin x + 2c_4 \cos x$
3. $y = a_n \sin nx, n = 1, 2, 3 \ldots$
4. $y = \cos x$
5. $y = -\frac{\lambda}{24\mu}(x^2 - a^2)^2$

6. (*i*) $y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$;
(*ii*) $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 x^5 + x^7/7\,!$.

Problems 35.4, page 1126

1. (*i*) and (*ii*) $\bar{y} = \frac{5}{18} x(1-x)$
2. $\bar{y} = \frac{x}{4}(5x-1)$
4. $\bar{y} = 0.58 + 0.27x$
5. (*i*) and (*ii*) $c_1 = 0.93, c_2 = -0.05$
6. 0.05
7. $c_1 = 3.27, c_2 = -2.69$
8. $y = \frac{1}{2}(5x^2 - 3x)$.

Problems 36.1, page 1134

1. $y(x) = 6x - 5 + \int_0^x (5 - 6x + 6t)\, y(t)\, dt$.
2. $y(x) = x - \sin x + e^x (x-1) + \int_0^x [\sin x - e^x (x-t)]\, y(t)\, dt$
3. $y(x) = \int_0^x t(t-x)\, y(t)\, dt + \frac{1}{2}x^2$
4. $y(x) + \int_0^x [1 + x - 2t + (x-t)e^{-t}]\, y(t)\, dt = \frac{x^5}{20} - \frac{5x^3}{6} + x - 3$
5. $y(x) = \cos x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{2}\int_0^x (x-t)^2\, y(t)\, dt$
6. $y(x) = \sin x - \frac{1}{2} + \int_0^x \left\{\frac{1}{2}x(x-t)^2 - 1\right\} y(t)\, dt$
7. $y(x) - \int_0^x \{4 - 6(x-t) + 2(x-t)^2 - \frac{1}{6}(x-t)^3\}\, y(t)\, dt = \frac{1}{4}\cos 2x - \frac{19}{12} + \frac{32}{3}x - \frac{85}{6}x^2 + \frac{20}{3}x^3$
8. $y''(x) - 2xy'(x) - 3y(x) = 0$; $y(0) = 1, y'(0) = 0$
9. $y''(x) - y(x) + 3\sin x = 0$; $y(0) = 3, y'(0) = 0$
10. $y'''(x) + 6y(x) = 0$; $y(0) = 4, y'(0) = -3, y''(0) = 2$
11. $y'''(x) - 3y''(x) + 4y'(x) + 2y(x) + e^{-x} = 0$; $y(0) = 1, y'(0) = 2, y''(0) = 3$.

Problems 36.2, page 1137

1. $y(x) = \int_0^1 G(x,t)\, t y(t)\, dt + \frac{1}{2}x(x-1)$, where $G(x,t) = x(1-t), (x < t)$ and $= t(1-x), x > t$
2. $y(x) = \int_0^1 G(x,t)\, y(t)\, dt + \frac{1}{6}(x^3 - 3x + 6)$, where $G(x,t) = x, x < t$ and $= t, x > t$
3. $u(x) = \int_0^1 G(x,t)\, e^t\, u(t)\, dt + \frac{1}{6}(x^3 + x)$, where $G(x,t) = x(1-2t), x < t$ and $= t(1-2x), x > t$
4. $G(x,t) = \frac{\sinh x \sinh (t-1)}{\sinh 1}, x < t$ and $= \frac{\sinh t \sinh (x-1)}{\sinh 1}, x > t$
5. $u(x) = \lambda \int_0^1 G(x,t)\, t\,.\, u(t)\, dt$, where $G(x,t) = \frac{1}{2}x\left(\frac{1}{t-t}\right), x < t$ and $= \frac{1}{2}\cdot t\left(\frac{1}{x-x}\right), x > t$

Problems 36.3, page 1141

5. $y(x) = \frac{1}{2}(\sin x + \sinh x)$
6. $y(x) = x^2 + \frac{1}{12}x^4$
7. $y(x) = 1$
8. $y(x) = \pm 6J_0(4x)$
9. $y(x) = J_1(2x) \quad (x > 0)$

10. $y(x) = \dfrac{1}{2} e^{-2x} (\cos x + 3 \sin x) - \dfrac{1}{2} e^{-x}$

11. $y(x) = (1 - x) e^{-x} + 1/2 \sin x$

12. $y(x) = 1 + x^2 + x^4/24.$

13. $y(x) = 1 + x^2/2$

14. $y(x) = 1/2$

15. $y(x) = \dfrac{3\sqrt{3}}{4\pi} x^{1/3} (3x + 2).$

Problems 36.4, page 1145

1. Has no eigen values and eigen functions
2. Eigen value $\lambda = 1/4$; eigen function is $y(x) = x^2 + 3x/2$
3. Eigen values $\lambda = 8/(\pi - 2)$; eigen function is $y(x) = \sin^2 (x)$
4. Eigen value $\lambda = 1/\pi$, $y = \sin x$
5. Has no eigen values or eigen functions
6. Eigen values are $\lambda = \pm 1/\pi$; eigen functions are $y(x) = \cos x + \sin x$, $y(x) = \cos x - \sin x$
7. $y(x) = x + \dfrac{\lambda [(6 - \lambda) x - 4]}{12 + \lambda^2}$
8. $y(x) = x + \dfrac{\lambda}{12 (1 - 2\lambda) - \lambda^2} [10 + (6 + \lambda) x]$
9. $y(x) = x + \dfrac{\lambda}{(1 - \lambda\pi)\left(1 - \frac{1}{2}\lambda\pi\right) + 4\lambda^2} \left\{2\lambda\pi + \frac{1}{2}\pi^2 \left(1 - \frac{1}{2}\lambda\pi\right) + \pi(1 - 2\lambda\pi) \sin x\right\}$
10. $y(x) = 2x - \pi + \dfrac{\pi^2 \sin^2 x}{\pi - 1}$
11. $y(x) = \dfrac{2}{2 - \lambda} \sin x$, $\lambda \neq 2$
12. $y(x) = x + \dfrac{2\lambda\pi}{1 + 2\lambda^2\pi^2} (\lambda\pi x - 4\lambda\pi \sin x + \cos x)$
13. There is no solution to the integral equation when $\lambda = 3$
14. $\lambda_1 = 2, \lambda_2 = -2$; $y_1(x) = 1 - x$, $y_2(x) = 1 - 3x$
15. (*i*) When $F(x) = x$, solution is $y(x) = x + \lambda \left\{\dfrac{2\lambda\pi^2}{\lambda^2\pi^2 - 1} \sin x + \dfrac{2\pi}{\lambda^2\pi^2 - 1} \cos x\right\}$

    (*ii*) When $F(x) = 1$, solution is $y(x) = 1$.

Problems 36.5, page 1148

1. $y(x) = 1 - \dfrac{3\lambda x}{2(3 + \lambda)}$ $(\lambda \neq -3)$
2. $y(x) = \dfrac{\sin x}{1 + \lambda\pi}$ only if $|\lambda| < \dfrac{1}{\pi}$
3. $y(x) = \dfrac{4 + 2\lambda (2 - 3x)}{4 - \lambda^2}$ $(\lambda \neq 2)$
4. $y = e^x$
5. $y(x) = 1$
6. $y = \sin x$
7. $y(x) = 2.$

Problems 37.1, page 1154

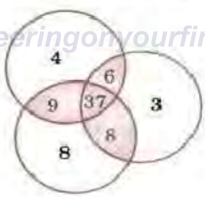
2. (*i*) *True*, since {*a*} is a subset of the set {*a*, *b*, *c*} ; (*ii*) and (*iii*) *False*, since the element *a* cannot be a subset of the set {*a*, *b*, *c*} ; (*iv*) *True*, since the set {*a*, *b*} is a subset of the set {*a*, *b*, *c*} ; (*v*) *False*, since the set {*a*, *b*} is not an element of the set {*a*, *b*, *c*} ; (*vi*) *True*, since the null set ϕ is a subset of every set.

17. 20

18. 105

19. 136

20. Number of students not taking any of these courses is 71.



Problems 37.2, page 1160

1. (a) It is not true that Sam is a teacher and John is an honest boy ; (b) Sam is a teacher and John is not an honest boy ; (c) Sam is not a teacher iff John is an honest boy ; (d) If Sam is a teacher then John is not an honest boy.
2. (a) $(p \vee q) \Rightarrow r$ where $p \equiv I$ have no car, $q = I$ do not wear good dress, $r = I$ am not, a millionaire.
3.

p	q	$\sim q$	$p \Rightarrow q$	$p \Rightarrow q \wedge \sim q$
1	1	0	1	0
1	0	1	0	0
0	1	0	1	0
0	0	1	1	1

(b)

p	q	r	$p \Leftrightarrow q$	$r \vee q$	$(p \Leftrightarrow q) \wedge (r \vee q)$
1	1	1	1	1	1
1	1	0	1	1	0
1	0	1	0	1	0
1	0	0	0	0	0
0	1	1	0	1	0
0	1	0	0	1	0
0	0	1	1	1	1
0	0	0	1	0	0

7. (i) $T_p = [2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31]$
$T_q = [1, 3, 9, 27, ...]$, $T_r = [1, 3, 9, 7]$
(ii) $T_r \le T_q$

10. (i)

p	q	$q \to p$	$p \to (q \to p)$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

15. (i) Dual of $(p \wedge q) \vee r$ is $(p \wedge q) \vee r$ (ii) Dual of $(p \wedge q) \vee t$ is $(p \vee q) \wedge t$

Problems 37.3, page 1166

1. (a) $(\forall\, x \in A)\,(x + 2 < 10)$ (b) $(1\, x \in A)\,(x + 2 = 10)$
2. (a) $\forall\, x, (x^3 \neq x)$ (b) $\to x, (x + 5 \le x)$
(c) None of the students are 26 or older (d) Some students do not live in the hostels.
3. $\forall\, x\, P(x)$ is false
4. $\forall\, (x_1, x_2)\, Q(x_1 + x_2)Q$
5. $\forall\, (a, b)\, R(a + x = b)$
6. (a) $\forall\, x[Q(x) \to R(x)]$ (b) $\sim \forall\, x[Q(x) - R(x)]$, (c) $\exists\, x[Q(x) \wedge R(x)]$, (d) $\exists x\, [Q(x) \wedge \sim R(x)]$
8. $(\sim A \vee \sim A) \wedge (B \vee \sim A) \wedge (\sim A \vee C) \wedge (B \vee C)$

10. 1. $p \vee q$ (Premise), 2. $\sim p \to q$ (conditional equivalence)
3. $q \to s$ (Premise) 4. $\sim p \to s$ (2, 3 chain rule)
5. $p \to r$ (Premise) 6. $\sim s \to p$ (4, conditional equivalence)
7. $\sim s \to r$(5, 6 chain rule) 8. $s \vee r$ (7, conditional equivalence)

12. $(b\, x)\, R\, (x = \sqrt{Z}\,)$
13. (a) Conclusion is not valid (b) Conclusion is not valid

Problems 37.4, page 1170

1.

x	y	z	$x \wedge y$	z'	$y \wedge z'$	$(x \wedge y) \vee (y \wedge z')$
0	0	0	0	1	0	0
0	0	1	0	0	0	0
0	1	0	0	1	1	1
1	0	0	0	1	0	0
1	1	0	1	1	1	1
1	0	1	0	0	0	0
0	1	1	0	0	0	0
1	1	1	1	0	0	1

2. $x \vee z' \wedge y = x \wedge y$ **3.** (*i*) $x' \vee y' \vee z'$ (*ii*) 0

14.

x_1	x_2	x_3	$x_1 \vee x_3$	x_3'	$x_2 \vee x_3'$	$x_1 \wedge (x_2 \vee x_3')$	P
0	0	1	1	0	0	0	0
0	1	0	0	1	1	0	0
0	1	0	0	1	1	0	0
1	0	0	1	1	1	1	1
1	1	0	1	1	1	1	1
0	1	1	1	0	1	0	0
1	0	1	1	0	0	0	0
1	1	1	1	0	1	1	1

Problems 37.5, page 1172

1. (*i*) 0 (*ii*) 0 **2.** (*i*) $(x \vee y \vee z) \wedge (x \vee y \vee z')$ (*ii*) $x \vee y \wedge (x \vee y') \wedge (x' \vee y)$

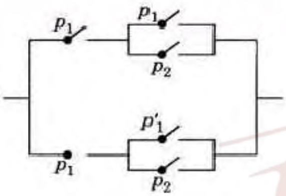
4. $(x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x \vee y' \vee z') \wedge (x' \vee y' \vee z) \wedge (x' \vee y' \vee z')$

5. $(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z')$

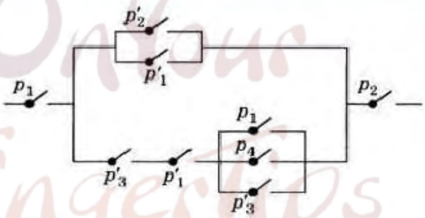
6. $F' = (x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x' \wedge y \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$

Problems 37.6, page 1174

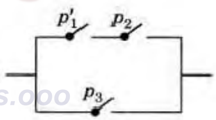
1. (*i*)



(*ii*)

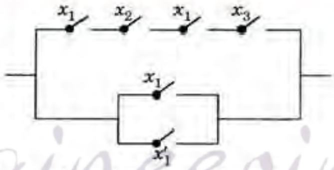


2. (*i*) $p_1 \vee [p_2' \wedge (p_1 \vee p_2')]$ (*ii*) $[(p_1 \vee p_2) \vee (p_1 \vee p_3)] \wedge (p_1 \wedge p_2')$



3. $x \wedge y$ **4.** $p_1' \wedge p_2 \vee p_3$;

6. (a) $x_1 \wedge x_2'$ (b)



(c) $(x_1 \vee x_2' \vee x_3) \wedge (x_1 \vee x_2' \vee x_3') \wedge (x_3 \vee x_2' \vee x_1) \wedge (x_3 \vee x_2' \vee x_1')$

Problems 37.7, page 1179

1. $F_2 = F_3$
2. F^c = [0.2 Ram, 0.7 Sham, 0.4 John, 0.3 Charu]
3. $F \cup G = [0.4\, x_1, 0.7\, x_2, 0.5\, x_3, 0.9\, x_4]$
   $F \cap G = [0.3\, x_1, 0.6\, x_2, 0.1\, x_3, 0.8\, x_4]$
4. (i) Truth value of 'F is not rich' is 0.2
   (ii) Truth value of 'G is not fat' is 0.4
   (iii) Truth value of 'Mary is not beautiful' is 0.3
5. (i) $F \neq G$ (ii) F is not a subset of G ; G is not a subset of F.
   (iii) F^c = [1, 1, 1, 1, 0.9, 0.7, 0.5, 0.1, 0, 0]
   $F \cap G$ = [0, 0.1, 0.3, 0.5]
   $F \cup G$ = [0.1, 0.5, 0.9, 1, 0.9, 0.9, 1, 1]
6. (i) Truth value of the cojunction of 'Latif and John are good players' is 0.6.
   (ii) Truth value of the disjunction of 'Latif and John are good players' is 0.7.

8. Members and its degree of membership.

Problems 38.1, page 1186

1. (i) $\dfrac{d\phi}{dt} = \dfrac{d\phi}{\partial x^i} \cdot \dfrac{dx^i}{dt}$; (ii) $x^i x^i$
2. (i) $a_{11}(x^1)^2 + a_{22}(x^2)^2 + a_{33}(x^3)^2 + (a_{12} + a_{21})\, x^1x^2 + (a_{13} + a_{31})x^1x^3 + (a_{23} + a_{32})x^2x^3$
   (ii) $g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2g_{12}dx^1dx^2 + 2g_{23}dx^2dx^3 + 2g_{31}dx^3\, dx^1$.
   (iii) $g_{11} = g_{12}\, g^{2p} + \dots + g_{ln}\, g^{np}$
3. (i) δ^i_h ; (ii) δ^p_s.
6. (i) $\bar{A}_p^{\;qr} = \dfrac{\partial \bar{x}^q}{\partial x^j} \dfrac{\partial \bar{x}^r}{\partial x^k} \dfrac{\partial x^i}{\partial \bar{x}^p} A_i^{jk}$; (ii) $\bar{C}_{pq} = \dfrac{\partial x^m}{\partial \bar{x}^p} \dfrac{\partial x^n}{\partial \bar{x}^q} C_{mn}$
7. Yes, $A^{ij}{}_{kl}{}^{m}$, contravariant order 3, covariant order 2, Rank 5
9. (a) $2\rho \cos^2 \phi - z \cos \phi + \rho^3 \sin^2 \phi \cos^2 \phi$, $-\rho^2 \sin 2\phi + \rho z \sin \phi + \rho^4 \sin \phi \cos^3 \phi$, $\rho z \sin \phi$;
   (b) $2r \sin^2 \theta \cos^2 \phi - r \sin \theta \cos \theta \cos \phi + r^3 \sin^4 \theta \sin^2 \phi \cos^2 \phi + r^2 \sin \theta \cos^2 \theta \sin \phi$;
   $2r^2 \sin \theta \cos \theta \cos^2 \phi - r^2 \cos^2 \theta \cos \phi + r^4 \sin^3 \theta \cos \theta \sin^2 \phi \cos^2 \phi - r^3 \sin^2 \theta \cos \theta \sin \phi$;
   $-2r^2 \sin^2 \theta \sin \phi \cos \phi + r^2 \sin \theta \cos \theta \sin \phi + r^4 \sin^4 \theta \sin \phi \cos^3 \phi$
11. $(a \cos \phi + b \sin \phi) \sin \theta + c \cos \theta$; $[(a \cos \phi + b \sin \phi) \cos \theta - c \sin \theta]/r$; $(b \cos \phi - a \sin \phi)/r \sin \theta$.

Problems 38.2, page 1189

6. Rank = 1

Problems 38.3, page 1193

1. $g = 4, g^{11} = 2, g^{22} = 5, g^{33} = 1.5, g^{12} = 3, g^{23} = -2.5, g^{13} = -1.5,$
2. $g_{11} = 1, g_{22} = \rho^2, g_{33} = 1, g_{ij} = 0\ (i \neq j)$; $g^{11} = 1, g^{22} = \rho^{-2}, g^{33} = 1, g^{ij} = 0\ (i \neq j)$
3. $g = r^4 \sin^2 \theta/(1 - r^2/R^2)$; $g^{11} = 1 - r^2/R^2, g^{22} = 1/r^2, g^{33} = (r \sin \theta)^{-2}, g^{ij} = 0\ (i \neq j)$.

Problems 36.4, page 1199

1. (a) $[ii, i] = \frac{1}{2}\partial g^{ij}/\partial x^i$, $[ii, k] = -\frac{1}{2}\partial g_{ii}/\partial x^k$,

$[ik, k] = [ki, k] = \frac{1}{2}\partial g_{kk}/\partial x^i$, $[ij, k] = 0$, when i, j, k are all different

(b) $\begin{Bmatrix} i \\ ii \end{Bmatrix} = \frac{1}{2}g^{ij}\frac{\partial g_{ii}}{\partial x^i}, \begin{Bmatrix} k \\ ii \end{Bmatrix} = -\frac{1}{2}g^{kk}\frac{\partial g_{ii}}{\partial x^k}$

$\begin{Bmatrix} k \\ ik \end{Bmatrix} = \begin{Bmatrix} k \\ ki \end{Bmatrix} = \frac{1}{2}g^{kk}\frac{\partial g_{kk}}{\partial x^i}$ (no summation over i or k)

$\begin{Bmatrix} k \\ ij \end{Bmatrix} = 0$, when i, j, k are all different

2. (a) All are zero

(b) $[21, 2] = \rho = [12, 2]$; $[22, 1] = \rho$, all others are zero

(c) $[21, 2] = r = [12, 2]$; $[31, 3] = r\sin^2\theta = [13, 3]$; $[32, 3]$
$= r^2\sin\theta\cos\phi = [23, 3]$; $[22, 1] = -r$; $[33, 1] = -r\sin^2\theta$;
$[33, 2] = -r^2\sin\theta\cos\phi$; all others are zero

3. (a) All are zero

(b) $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\rho, \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{\rho}$, all others are zero

(c) $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -r, \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = -r\sin^2\theta, \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{r}$,

$\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = -\sin\theta\cos\theta, \begin{Bmatrix} 3 \\ 31 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{1}{r}$,

$\begin{Bmatrix} 3 \\ 32 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \cot\theta$, all others are zero

5. (a) $r^2\sin\theta\cos\theta$; $r^2\sin\theta\cos\theta$ (b) $-\sin\theta\cos\theta$; $\cot\theta$

6. (a) $-r\sin^2\theta$; $r^2\sin\theta\cos\theta$ (b) $-r\sin^2\theta$; $\cot\theta$

8. (a) $u^{ij}{}_{,k} = \frac{\partial u^{ij}}{\partial x^k} + \begin{Bmatrix} i \\ ks \end{Bmatrix} u^{sj} + \begin{Bmatrix} i \\ ks \end{Bmatrix} u^{is}$; (b) $A^h{}_{ij,k} = \frac{\partial A^h{}_{ij}}{\partial x^k} - \begin{Bmatrix} s \\ ik \end{Bmatrix} A^h{}_{sj} + \begin{Bmatrix} h \\ ks \end{Bmatrix} A^s{}_{ij}$

10. $A^j{}_{k,q}B_n{}^{lm} + A^j{}_k B^{lm}{}_{n,q}$

11. (a) $\frac{1}{\rho}\left[\frac{\partial}{\partial p}(\rho A_\rho) + \frac{\partial}{\partial\phi}(A_\phi) + \frac{\partial}{\partial z}(\rho A_z)\right]$; (b) $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial r}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

12. (a) $\frac{\partial^2 v}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial\phi^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{\rho}\frac{\partial v}{\partial\rho} = 0$ (b) $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial v}{\partial\theta} = 0.$