Brownian Motion

Modeling stock prices in a portfolio using multidimensional geometric Brownian Motion

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1 Abstract

Modelling and forecasting the price of stocks is an important topic in the field of financial analysis/studies and is one of the most challenging part in financial analysis due to the extremely complex characteristics of the stock market. In this project we try to model and forecast the net value of a portfolio using **Geometric Brownian Motion**.

2 Introduction to Brownian Motion

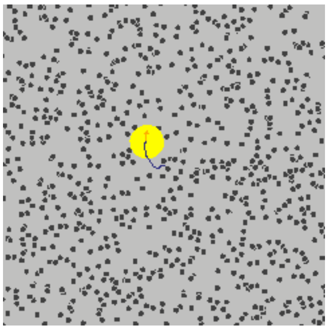

Figure 1: Screenshot of a dust particle simulated using Brownian motion.

[5]
Brownian Motion[1] is a stochastic process which is continuous in time and continuous in space, meaning the range of all the random variables in the stochastic process and the index set belongs to set of real numbers $\mathbb{R}$. Therefore this process is denoted by $\{X(t) \ : \ t \ \geq \ 0\}$.Now we shall derive the main properties required for a stochastic process to be a Brownian motion process by carefully applying limits to a symmetric discreet space and discreet time random walk.

2.1 Symmetric Random Walk

A symmetric random walk in one-dimension is where in each time unit an object is equally likely to move one unit distance to the left or to the right.
If the random variable X_n denotes the position of the object at time n then

$$X_0 = 0$$

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w.p. } \frac{1}{2} \\ X_n - 1 & \text{w.p. } \frac{1}{2} \end{cases} \tag{1}$$

Now instead of unit times and unit distances the object moves Δx distance equally likely to the left or to the right in each Δt time intervals.
Let $X(t)$ be the random variable which denotes the distance of the object at time t.

$$X(0) = 0$$

$$X(t) = \Delta x(X_1 + X_2 + \ldots + X_{\lfloor \frac{t}{\Delta t} \rfloor})$$

$$X_i = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases} \tag{2}$$

$\lfloor\rfloor$ is the floor function

$$E[X_i] = 1 \cdot Pr(X_i = 1) + (-1) \cdot Pr(X_i = -1) = 0$$
$$E[X_i^2] = 1^2 \cdot Pr(X_i = 1) + (-1)^2 \cdot Pr(X_i = -1) = 1$$
$$Var(X_i) = E[X_i^2] - E[X_i]^2 = 1$$

Now, using the above two values of $E[X_i]$,$Var(X_i)$ and using X_i,X_j are independent of each other($i \neq j$), we can calculate expectation and variance of $X(t)$.

$$E[X(t)] = E\left[\Delta x(X_1 + X_2 + \ldots + X_{\lfloor \frac{t}{\Delta t} \rfloor})\right] = \Delta x \sum_{i=1}^{\lfloor \frac{t}{\Delta t} \rfloor} E[X_i]$$

$$\boxed{E[X(t)] = 0}$$

$$Var(X(t)) = Var\left(\Delta x(X_1 + X_2 + \ldots + X_{\lfloor \frac{t}{\Delta t} \rfloor})\right) = (\Delta x)^2 Var\left(\sum_{i=1}^{\lfloor \frac{t}{\Delta t} \rfloor} X_i\right)$$
$$= (\Delta x)^2 \left(\sum_{i=1}^{\lfloor \frac{t}{\Delta t} \rfloor} Var(X_i) + \cancelto{0}{2\sum_{i<j} Cov(X_i, X_j)}\right)$$

$$\boxed{Var(X(t)) = (\Delta x)^2 \left(\lfloor \frac{t}{\Delta t} \rfloor\right)}$$

Now we shall apply the limit Δx and Δt tends to zero, however if we blindly apply this limit using any relationship between Δx and Δt, in many of the cases we get trivial limits. For example if $\Delta x = \Delta t$ then by applying the limit $\Delta t \to 0$, $E[X(t)]$ remains zero but $Var\,(X(t))$ would also be zero, this implies with $Pr = 1$ the value of $X(t) = 0$ (by central limit theorem). If we take $\boxed{\Delta x = \sigma\sqrt{\Delta t}}$ then we get a non-trivial value for $Var\,(X(t))$ which is $\boxed{Var\,(X(t)) = \sigma^2 t}$ and $\boxed{E[X(t)] = 0}$ remains. Using the central limit theorem we get the following distribution of this new random walk.

Let $\frac{t}{\Delta t} = n$

$$Pr\left(\frac{X(t) - E[X(t)]}{Var(X(t))} \leq a\right) = \lim_{n\to\infty} Pr\left(\frac{\Delta x\left(\sum_{i=1}^{n} X_i\right) - 0}{\sigma^2 t} \leq a\right)$$
$$= \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{a} e^{-\frac{x^2}{2\sigma^2 t}}\, dx$$

From the above proof we can we get the following conclusions of this new random walk.

1. $X(t)$ is normal with mean$= 0$ and variance$= \sigma^2 t$.
2. $X(t)$ possess independent increment property, take two intervals say (t_1, t_2) and t_3, t_4 such that $t_1 < t_2 < t_3 < t_4$, now $X(t_2) - X(t_1)$ would be completely independent $X(t_4) - X(t_3)$ as the summation of limit increments are independent and the intervals do not share a common increment X_i.

3. $X(t)$ possess stationary independent property as $X(t+s) - X(s)$ would only depend on t, this is because the number of independent limit increments would only depend on the length on the interval and not on it's starting and ending points.

2.2 Definition of Brownian Motion

A stochastic process $\{X(t) : t \geq 0\}$ is a Brownian motion process if and only if

1. $X(0) = 0$.
2. $\{X(t) : t \geq 0\}$ has both independent and stationary increment properties.
3. $X(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$ for all t.

When $\sigma = 1$ the Brownian motion process is called standard Brownian motion process, and of course any brownian motion may be converted to standard brownian motion by letting $B(t) = X(t)/\sigma$. The interpretation of brownian motion as the limit of a symmetric random walk suggests that $X(t)$ is continuous in t, the proof of this claim is done by Kolmogorov's continuity theorem, `https://math.stackexchange.com/questions/569801/proving-continuity-of-brownian-paths`.

2.2.1 Joint Probability density function

Let's calculate the joint probability density function of $X(t_1)$,$X(t_2)\ldots$,$X(t_n)$ for $t_1 < t_2 < \ldots < t_n$.

$$f(X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_n) = x_n)$$

$$=$$

$$f\left(X(t_1) = x_1, X(t_2) - X(t_1) = x_2 - x_1, \ldots, X(t_n) - X(t_{n-1}) = x_n - x_{n-1}\right)$$

By independent increment property.

$$f(X(t_1) = x_1) \times f(X(t_2) - X(t_1) = x_2 - x_1) \times \ldots \times f(X(t_n) - X(t_{n-1}) = x_n - x_{n-1})$$

By stationary increment property and normal distribution of $X(t)$ we get the joint probability distribution as-

$$= \frac{\exp\left(-\frac{1}{2}\left(\sum_{i=1}^{n}\left(\frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}\right)\right)\right)}{(2\pi)^{n/2}\left(\prod_{i=1}^{n}(t_i - t_{i-1})\right)^{1/2}}$$

2.3 Generalizing B.M to Geometric Brownian motion

[5]

2.3.1 Brownian motion with drift

A stochastic process $\{X(t) : t \geq 0\}$ is called a brownian motion with drift if it can shown as the following-

$$X(t) = \sigma B(t) + \mu t$$

Where $B(t)$ is the standard brownian motion, σ is the variance parameter and μ is drift parameter. Therefore we can say $X(t)$ is normally distributed with mean$=\mu t$ and variance$=\sigma^2 t$. It is trivial to notice that $X(t)$ also has independent increment property,stationary increment property and $X(0) = 0$.

2.3.2 Geometric Brownian Motion

Let $\{Y(t) : t \geq 0\}$ be a brownian motion with drift parameter μ and variance parameter σ i.e. $Y(t) = \sigma B(t) + \mu t$ where $B(t)$ is the standard brownian motion, then $\boxed{\{X(t) : t \geq 0\}}$ is a **geometric brownian motion** if and only if

$$\boxed{X(t) = e^{Y(t)}}$$

Few properties of G.B.M to consider when modelling stock prices
When modelling stock prices, we usually have data of the stock price for a certain time interval and wish to calculate the expected value of this stock price at a later time, that is what would be the expected value of the stock price given the stock prices for a certain duration. Say $X(t)$ is the random variable which denotes the stock price at time t and let $\{X(t) : t \geq 0\}$ follow geometric brownian motion, then we wish to calculate $E[X(t)|X(u), 0 \leq u \leq s]$-

$$\begin{aligned}
E[X(t)|X(u), 0 \leq u \leq s] &= E[e^{Y(t)}|e^{Y(u)}, 0 \leq u \leq s] \\
&= E[e^{Y(t)}|Y(u), 0 \leq u \leq s] \\
&= E[e^{Y(t)+Y(s)-Y(s)}|Y(u), 0 \leq u \leq s] \\
&= e^{Y(s)}E[e^{Y(t)-Y(s)}|Y(u), 0 \leq u \leq s] \\
&= X(s)E[e^{Y(t-s)}] \text{ Independent and stationary increment property}
\end{aligned}$$

Notice that $E[e^{Y(t-s)}]$ is the moment generating function of $Y(t-s)$ with the parameter $a = 1$, and $E[e^{aX}] = e^{aE[X]+a^2 Var(x)/2}$, this implies-

$$E[X(t)|X(u), 0 \leq u \leq s] = X(s)e^{\mu(t-s)+(t-s)\sigma^2/2}$$

$$\boxed{E[X(t)|X(u), 0 \leq u \leq s] = X(s)e^{(t-s)(\mu+\sigma^2/2)}}$$

Suppose X_n is the price of a stock at time n and $Y_n = \frac{X_n}{X_{n-1}}$ i.e the change in stock price relative to the previous price is an independent random variable, then geometric random variable is a good estimating method too forecast the expected price of the stock at later times.

$$X_n = Y_n X_{n-1}$$

$$X_n = Y_n Y_{n-1} Y_{n-2} \ldots Y_1 X_0$$

Apply $\log$ to the above equations we get-

$$\log(X_n) = \left(\sum_{i=1}^{n} \log(Y_i)\right) + \log(X_0)$$

The random variable $\log(X_n)$ is sum of n independent and identical random variable (i.e $\log(Y_i)$), for a large value of n the distribution of X_n would follow a normal distribution with some drift(from the central limit theorem). Removing the $\log X_n$ would follow geometric random variable.

3 Simulation and Testing

[1]
For the simulation of geometric brownian motion we have taken help from the book P. Glasserman, Monte Carlo methods in financial engineering. Vol. 53 (2013). This books contains many algorithms used in monte-carlo simulations to replicate a stochastic process.
Chapter 3 of this book explains us how to simulate geometric brownian motion using some math transformation such as cholesky factorization, using the basic properties of a brownian motion mentioned in the previous section also the books also explains and prooves why we using $\log$ of prices i.e why they nicely approximate normal distributions.

Let $S(t)$ denote the price of the stock,μ be the drift parameter,σ be the variance parameter, and $W(t)$ be the standard brownian motion, then from the definition of geometric brownian motion.
Let-

$$X(t) = \mu t + \sigma W(t)$$

Then $X(t)$ satisfies the differential equation-

$$dX(t) = \mu dt + \sigma dW(t)$$

Now-

$$S(t) = S(0)e^{X(t)} \equiv f(X(t))$$

Using Ito's lemma and the differential equation above we obtain the differential equation for $S(t)$.

$$\frac{dS(t)}{S(t)} = \left(\mu - \sigma^2/2\right) dt + \sigma dW(t)$$

$$S(t) = S(0)e^{\left(\mu-\sigma^2/2\right)t+\sigma W(t)}$$

Dividing the above equation with $S(u)$ where $u < t$ we get a nice recursive formulation of $S(t)$.

$$S(t) = S(u)e^{\left(\mu-\sigma^2/2\right)(t-u)+\sigma(W(t)-W(u))}$$

$$\boxed{S(t) = S(u)e^{\left(\mu-\sigma^2/2\right)(t-u)+\sigma W(t-u)}}$$

Setting $t \equiv t+h$ and $t \equiv t$ we can get the the value of $S(t+h)$ i.e value at the next time step, if we know the value of $S(t)$-

$$S(t+h) = S(t)e^{\left(\mu-\sigma^2/2\right)(h)+\sigma W(h)}$$

$$\boxed{S(t+h) = S(t)e^{\left(\mu-\sigma^2/2\right)(h)+\sigma\sqrt{h}Z}}$$

Where $W(h)$ is replaced by $\sqrt{h}Z$ and $Z \sim Normal(0,1)$

The above definition is only for a single stock simulation, i.e one dimensional brownian motion, the below sub-section will explain the procedure to tackle multiple stocks where the correlation between two stocks is may or may not be non-zero, i.e multidimensional brownian motion.

3.1 Procedure followed to simulate multidimensional brownian motion

[1] [2] [3] [4]
Let there be n stocks in a portfolio, then for stock we can write the following equation-

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dX_i(t)$$

X_i is the standard one dimensional brownian motion defined for stock i. Let $\rho_{ij} \in [-1,1]$ denote the correlation between X_i and X_j then the covariance matrix $\sum$ would be defined as $\sim_{ij} = \sigma_i\sigma_j\rho_{ij}$. The actual covariance between $S_i(t)$ and $S_j(t)$ derived from the above equation of $S_i(t)$ is-

$$Cov[S_i(t), S_j(t)] = S_i(0)S_j(0)e^{(\mu_i+\mu_j)t}\,(e^{\rho_{ij}\sigma_i\sigma_j} - 1)$$

Now taking into the consideration of covariance due to presence of other stocks and using some linear algebra methods (Choslesky transformation)we can derive the differential equation of $S_i(t)$, the proof of this derivation is mentioned in the book mentioned above, necessary citations of the research papers have also been mentioned in the references section.

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{j=1}^{n} A_{ij} dW_j(t)$$

Where A is the Cholesky Matrix of the covariance matrix,i.e $[AA^T]_{ij} = Cov_{ij}$, and A is upper triangular
Using Ito's lemma and solving this differential equation,then dividing the equation with $S_i(u)$ where $u < t$ and then apply $t \equiv t+h$ and $u \equiv t$ we get a nice recursive formulation of $S_i(t)$.

$$\boxed{S_i(t+h) = S_i(t)\exp\left(\left(\mu_i - \sigma_i^2/2\right)h + \sqrt{h}\sum_{j=1}^{n} A_{ij}Z_{(k+1)j}\right)}$$

Where $Z_{ij} \sim Normal(0,1)\forall i,j$

Using this recursive relation we can simulate stocks in a portfolio.

4 Referencing Citations

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