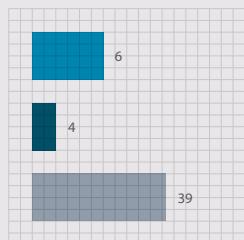
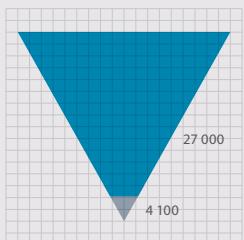


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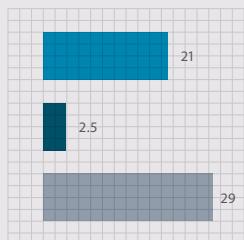
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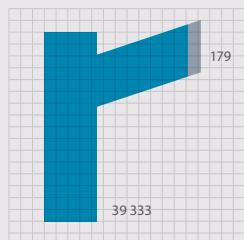
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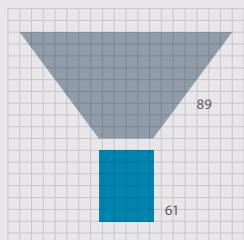
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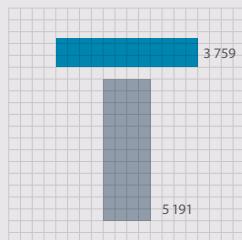
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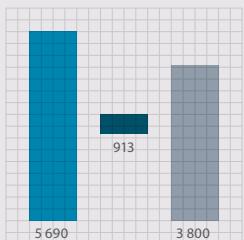
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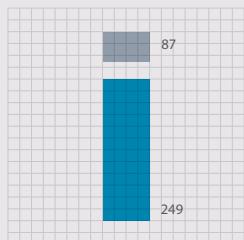
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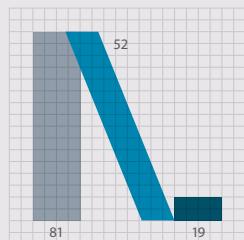
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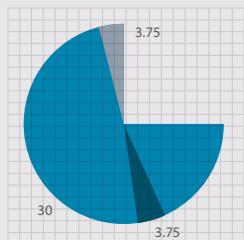
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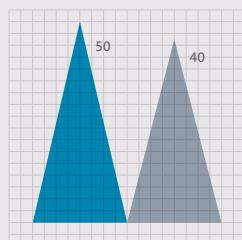
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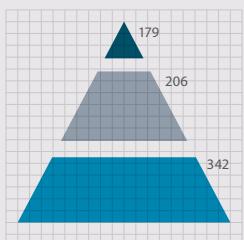
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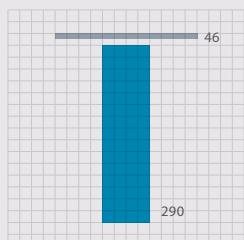
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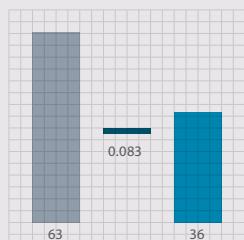
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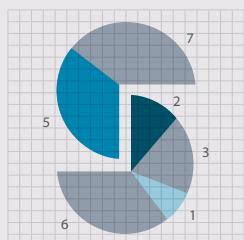
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# EVERYTHING MATHS

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Mathematics is commonly thought of as being about numbers but mathematics is actually a language! Mathematics is the language that nature speaks to us in. As we learn to understand and speak this language, we can discover many of nature's secrets. Just as understanding someone's language is necessary to learn more about them, mathematics is required to learn about all aspects of the world – whether it is physical sciences, life sciences or even finance and economics.

The great writers and poets of the world have the ability to draw on words and put them together in ways that can tell beautiful or inspiring stories. In a similar way, one can draw on mathematics to explain and create new things. Many of the modern technologies that have enriched our lives are greatly dependent on mathematics. DVDs, Google searches, bank cards with PIN numbers are just some examples. And just as words were not created specifically to tell a story but their existence enabled stories to be told, so the mathematics used to create these technologies was not developed for its own sake, but was available to be drawn on when the time for its application was right.

There is in fact not an area of life that is not affected by mathematics. Many of the most sought after careers depend on the use of mathematics. Civil engineers use mathematics to determine how to best design new structures; economists use mathematics to describe and predict how the economy will react to certain changes; investors use mathematics to price certain types of shares or calculate how risky particular investments are; software developers use mathematics for many of the algorithms (such as Google searches and data security) that make programmes useful.

But, even in our daily lives mathematics is everywhere – in our use of distance, time and money. Mathematics is even present in art, design and music as it informs proportions and musical tones. The greater our ability to understand mathematics, the greater our ability to appreciate beauty and everything in nature. Far from being just a cold and abstract discipline, mathematics embodies logic, symmetry, harmony and technological progress. More than any other language, mathematics is everywhere and universal in its application.

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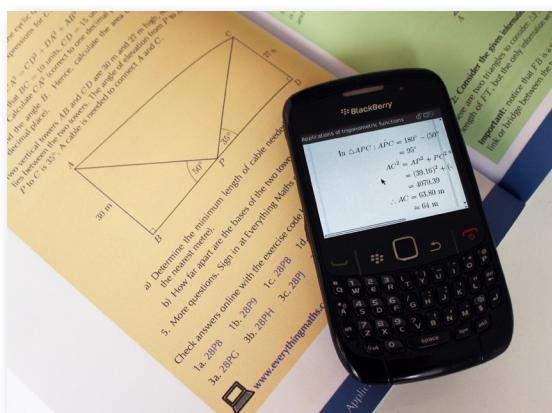
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### Example 2: Estimating surds

#### Question

Find the two consecutive integers such that  $\sqrt{49}$  lies between them.

Show me this worked solution

### Exercise 1:

#### Problem 1:

Determine between which two consecutive integers the following numbers lie, without using a calculator:

1.  $\sqrt{18}$
2.  $\sqrt{29}$
3.  $\sqrt[3]{5}$
4.  $\sqrt[3]{79}$

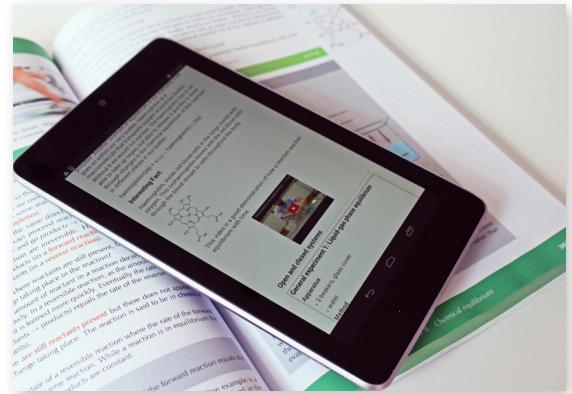
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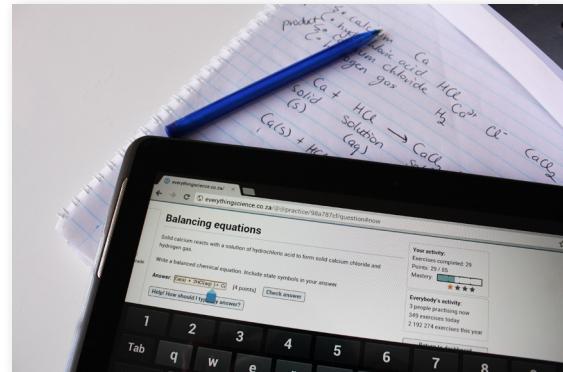
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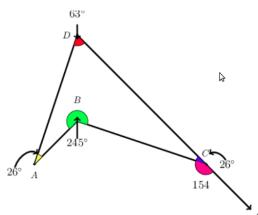
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### Angles in quadrilaterals

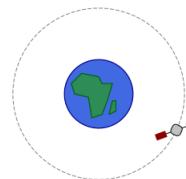
The diagram below represents quadrilateral ABCD with extended line  $\overline{CE}$ . Quadrilateral ABCD is a polygon with four sides and four angles. The sum of the interior angles in a quadrilateral =  $360^\circ$ . Angles on a straight line like  $\overline{CE} = 180^\circ$ .



### Effect of mass on gravitational force

The International Space Station (ISS) has a mass  $M$ , as it orbits the Earth, it experiences a gravitational force of  $F$ . A space shuttle docks onto the ISS. The gravitational force the ISS experiences once the mass of the shuttle is added increases by a factor of 3.

By what factor does the mass of the ISS increase for it to experience this increase of gravitational force? Write your answer as a fraction of the original mass  $M_{ISS}$  of the ISS.



Answer:  MISS [2 points] [Check answer](#)

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# CHAPTER



## *Algebraic expressions*

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# 1 Algebraic expressions

## 1.1 Introduction

EMA2

Over human history, all peoples and cultures have contributed to the field of Mathematics. Topics like algebra may seem obvious now, but for many centuries mathematicians had to make do without it. Over the next three grades, you will explore more advanced and abstract mathematics. It may not be obvious how this applies to everyday life, but the truth is, mathematics is required for nearly everything you will do one day. Enjoy your mathematical journey. Remember, there is no such thing as a “maths person” or “not a maths person”. We can all do mathematics, it just takes practice.

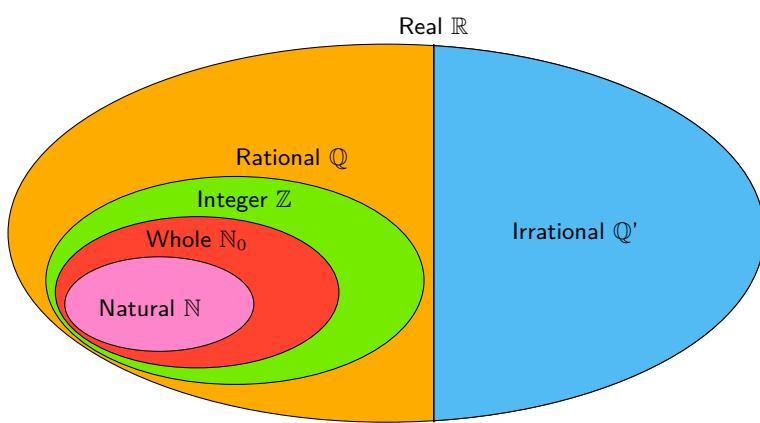


Figure 1.1: Some examples of early tally sticks. These were used to help people count things such as the number of days between events or the number of livestock they had.

In this chapter, we will begin by revising the real number system and then learn about estimating surds and rounding real numbers. We will also be expanding on prior knowledge of factorisation and delve into more complex calculations involving binomial and trinomial expressions.

## 1.2 The real number system

EMA3



We use the following definitions:

- $\mathbb{N}$ : natural numbers are  $\{1; 2; 3; \dots\}$
- $\mathbb{N}_0$ : whole numbers are  $\{0; 1; 2; 3; \dots\}$
- $\mathbb{Z}$ : integers are  $\{\dots; -3; -2; -1; 0; 1; 2; 3; \dots\}$

**VISIT:**

The following video shows an example of determining which of the above sets of numbers a particular number is in.

► See video: [2DBH](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

**NOTE:**

Not all numbers are real numbers. The square root of a negative number is called a non-real or imaginary number. For example  $\sqrt{-1}$ ,  $\sqrt{-28}$  and  $\sqrt{-5}$  are all non-real numbers.

## 1.3 Rational and irrational numbers

EMA4

**DEFINITION:** *Rational number*

A rational number ( $\mathbb{Q}$ ) is any number which can be written as:

$$\frac{a}{b}$$

where  $a$  and  $b$  are integers and  $b \neq 0$ .

The following numbers are all rational numbers:

$$\frac{10}{1}; \frac{21}{7}; \frac{-1}{-3}; \frac{10}{20}; \frac{-3}{6}$$

We see that all numerators and all denominators are integers.

This means that all integers are rational numbers, because they can be written with a denominator of 1.

**DEFINITION:** *Irrational numbers*

Irrational numbers ( $\mathbb{Q}'$ ) are numbers that cannot be written as a fraction with the numerator and denominator as integers.

Examples of irrational numbers:

$$\sqrt{2}; \sqrt{3}; \sqrt[3]{4}; \pi; \frac{1+\sqrt{5}}{2}$$

These are not rational numbers, because either the numerator or the denominator is not an integer.

## Decimal numbers

EMA5

All integers and fractions with integer numerators and non-zero integer denominators are rational numbers. Remember that when the denominator of a fraction is zero then the fraction is undefined.

You can write any rational number as a decimal number but not all decimal numbers are rational numbers. These types of decimal numbers are rational numbers:

- Decimal numbers that end (or terminate). For example, the fraction  $\frac{4}{10}$  can be written as 0,4.
- Decimal numbers that have a repeating single digit. For example, the fraction  $\frac{1}{3}$  can be written as 0,̄3 or 0,̄3. The dot and bar notations are equivalent and both represent recurring 3's, i.e.  $0,̄3 = 0,̄3 = 0,333\dots$
- Decimal numbers that have a recurring pattern of multiple digits. For example, the fraction  $\frac{2}{11}$  can also be written as 0,̄1̄8. The bar represents a recurring pattern of 1's and 8's, i.e.  $0,̄1̄8 = 0,181818\dots$

**NOTE:**

You may see a full stop instead of a comma used to indicate a decimal number. So the number 0,4 can also be written as 0.4

**Notation:** You can use a dot or a bar over the repeated digits to indicate that the decimal is a recurring decimal. If the bar covers more than one digit, then all numbers beneath the bar are recurring.

If you are asked to identify whether a number is rational or irrational, first write the number in decimal form. If the number terminates then it is rational. If it goes on forever, then look for a repeated pattern of digits. If there is no repeated pattern, then the number is irrational.

When you write irrational numbers in decimal form, you may continue writing them for many, many decimal places. However, this is not convenient and it is often necessary to round off.

**NOTE:**

Rounding off an irrational number makes the number a rational number that approximates the irrational number.

### Worked example 1: Rational and irrational numbers

#### QUESTION

Which of the following are not rational numbers?

1.  $\pi = 3,14159265358979323846264338327950288419716939937510\dots$
2. 1,4
3. 1,618033989...
4. 100
5. 1,7373737373...
6.  $0,\overline{02}$

#### SOLUTION

1. Irrational, decimal does not terminate and has no repeated pattern.
2. Rational, decimal terminates.
3. Irrational, decimal does not terminate and has no repeated pattern.
4. Rational, all integers are rational.
5. Rational, decimal has repeated pattern.
6. Rational, decimal has repeated pattern.

### Converting terminating decimals into rational numbers

EMA6

A decimal number has an integer part and a fractional part. For example, 10,589 has an integer part of 10 and a fractional part of 0,589 because  $10 + 0,589 = 10,589$ .

Each digit after the decimal point is a fraction with a denominator in increasing powers of 10.

For example:

- 0,1 is  $\frac{1}{10}$
- 0,01 is  $\frac{1}{100}$
- 0,001 is  $\frac{1}{1000}$

This means that

$$\begin{aligned}10,589 &= 10 + \frac{5}{10} + \frac{8}{100} + \frac{9}{1000} \\&= \frac{10\,000}{1000} + \frac{500}{1000} + \frac{80}{1000} + \frac{9}{1000} \\&= \frac{10\,589}{1000}\end{aligned}$$

**VISIT:**

The following two videos explain how to convert decimals into rational numbers.

Part 1

▶ See video: [2DBJ](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

Part 2

▶ See video: [2DBK](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## Converting recurring decimals into rational numbers

EMA7

When the decimal is a recurring decimal, a bit more work is needed to write the fractional part of the decimal number as a fraction.

### Worked example 2: Converting decimal numbers to fractions

#### **QUESTION**

Write  $0.\dot{3}$  in the form  $\frac{a}{b}$  (where  $a$  and  $b$  are integers).

#### **SOLUTION**

##### **Step 1: Define an equation**

Let  $x = 0,33333\dots$

##### **Step 2: Multiply by 10 on both sides**

$10x = 3,33333\dots$

##### **Step 3: Subtract the first equation from the second equation**

$$9x = 3$$

##### **Step 4: Simplify**

$$x = \frac{3}{9} = \frac{1}{3}$$

## Worked example 3: Converting decimal numbers to fractions

### QUESTION

Write  $5.\dot{4}\dot{3}\dot{2}$  as a rational fraction.

### SOLUTION

#### Step 1: Define an equation

$$x = 5,432432432\dots$$

#### Step 2: Multiply by 1000 on both sides

$$1000x = 5432,432432432\dots$$

#### Step 3: Subtract the first equation from the second equation

$$999x = 5427$$

#### Step 4: Simplify

$$x = \frac{5427}{999} = \frac{201}{37} = 5\frac{16}{37}$$

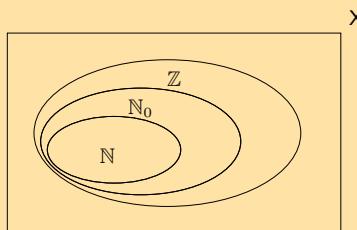
In the first example, the decimal was multiplied by 10 and in the second example, the decimal was multiplied by 1000. This is because there was only one digit recurring (i.e. 3) in the first example, while there were three digits recurring (i.e. 432) in the second example.

In general, if you have one digit recurring, then multiply by 10. If you have two digits recurring, then multiply by 100. If you have three digits recurring, then multiply by 1000 and so on.

Not all decimal numbers can be written as rational numbers. Why? Irrational decimal numbers like  $\sqrt{2} = 1,4142135\dots$  cannot be written with an integer numerator and denominator, because they do not have a pattern of recurring digits and they do not terminate.

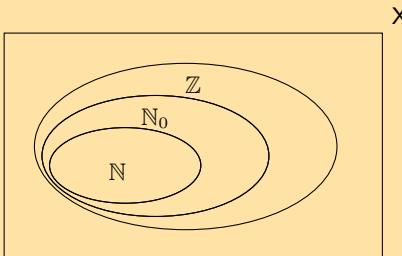
### Exercise 1 – 1:

1. The figure here shows the Venn diagram for the special sets  $\mathbb{N}$ ,  $\mathbb{N}_0$  and  $\mathbb{Z}$ .



- a) Where does the number  $-\frac{12}{3}$  belong in the diagram?

- b) In the following list, there are two false statements and one true statement. Which of the statements is **true**?
- Every integer is a natural number.
  - Every natural number is a whole number.
  - There are no decimals in the whole numbers.
2. The figure here shows the Venn diagram for the special sets  $\mathbb{N}$ ,  $\mathbb{N}_0$  and  $\mathbb{Z}$ .



- a) Where does the number  $-\frac{1}{2}$  belong in the diagram?
- b) In the following list, there are two false statements and one true statement. Which of the statements is **true**?
- Every integer is a natural number.
  - Every whole number is an integer.
  - There are no decimals in the whole numbers.
3. State whether the following numbers are real, non-real or undefined.
- a)  $-\sqrt{3}$       b)  $\frac{0}{\sqrt{2}}$       c)  $\sqrt{-9}$       d)  $\frac{-\sqrt{7}}{0}$       e)  $-\sqrt{-16}$       f)  $\sqrt{2}$
4. State whether the following numbers are rational or irrational. If the number is rational, state whether it is a natural number, whole number or an integer.
- a)  $-\frac{1}{3}$       b)  $0,651268962154862\dots$       c)  $\frac{\sqrt{9}}{3}$   
 d)  $\pi^2$       e)  $\pi^4$       f)  $\sqrt[3]{19}$   
 g)  $(\sqrt[3]{1})^7$       h)  $\pi + 3$       i)  $\pi + 0,858408346$
5. If  $a$  is an integer,  $b$  is an integer and  $c$  is irrational, which of the following are rational numbers?
- a)  $\frac{5}{6}$       b)  $\frac{a}{3}$       c)  $\frac{-2}{b}$       d)  $\frac{1}{c}$
6. For each of the following values of  $a$  state whether  $\frac{a}{14}$  is rational or irrational.
- a) 1      b) -10      c)  $\sqrt{2}$       d) 2,1
7. Consider the following list of numbers:

$$-3 ; 0 ; \sqrt{-1} ; -8\frac{4}{5} ; -\sqrt{8} ; \frac{22}{7} ; \frac{14}{0} ; 7 ; 1,\overline{34} ; 3,3231089\dots ; 3 + \sqrt{2} ; 9\frac{7}{10} ; \pi ; 11$$

Which of the numbers are:

- a) natural numbers      b) irrational numbers  
 c) non-real numbers      d) rational numbers  
 e) integers      f) undefined

8. For each of the following numbers:

- write the next three digits and
- state whether the number is rational or irrational.

a)  $1.\overline{15}$

d)  $3,324354\dots$

b)  $2,121314\dots$

e)  $3,3243\dot{5}\dot{4}$

c)  $1,242244246\dots$

9. Write the following as fractions:

a)  $0,1$

b)  $0,12$

c)  $0,58$

d)  $0,2589$

10. Write the following using the recurring decimal notation:

a)  $0,111111\dots$  b)  $0,1212121212\dots$  c)  $0,123123123123\dots$  d)  $0,11414541454145\dots$

11. Write the following in decimal form, using the recurring decimal notation:

a)  $\frac{25}{45}$

b)  $\frac{10}{18}$

c)  $\frac{7}{33}$

d)  $\frac{2}{3}$

e)  $1\frac{3}{11}$

f)  $4\frac{5}{6}$

g)  $2\frac{1}{9}$

12. Write the following decimals in fractional form:

a)  $0,\dot{5}$

b)  $0,6\dot{3}$

c)  $0,\dot{4}$

d)  $5,\overline{31}$

e)  $4,\overline{93}$

f)  $3,\overline{93}$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. <a href="#">2DBM</a>	2. <a href="#">2DBN</a>	3a. <a href="#">2DBP</a>	3b. <a href="#">2DBQ</a>	3c. <a href="#">2DBR</a>	3d. <a href="#">2DBS</a>	3e. <a href="#">2DBT</a>
3f. <a href="#">2DBV</a>	4a. <a href="#">2DBX</a>	4b. <a href="#">2DBY</a>	4c. <a href="#">2DC2</a>	4d. <a href="#">2DC3</a>	4e. <a href="#">2DC4</a>	4f. <a href="#">2DC5</a>
4g. <a href="#">2DC6</a>	4h. <a href="#">2DBZ</a>	4i. <a href="#">2DBW</a>	5. <a href="#">2DC7</a>	6. <a href="#">2DC8</a>	7. <a href="#">2DC9</a>	8a. <a href="#">2DCB</a>
8b. <a href="#">2DCC</a>	8c. <a href="#">2DCD</a>	8d. <a href="#">2DCF</a>	8e. <a href="#">2DCG</a>	9a. <a href="#">2DCH</a>	9b. <a href="#">2DCJ</a>	9c. <a href="#">2DCK</a>
9d. <a href="#">2DCM</a>	10a. <a href="#">2DCN</a>	10b. <a href="#">2DCP</a>	10c. <a href="#">2DCQ</a>	10d. <a href="#">2DCR</a>	11a. <a href="#">2DCS</a>	11b. <a href="#">2DCT</a>
11c. <a href="#">2DCV</a>	11d. <a href="#">2DCW</a>	11e. <a href="#">2DCX</a>	11f. <a href="#">2DCY</a>	11g. <a href="#">2DCZ</a>	12a. <a href="#">2DD2</a>	12b. <a href="#">2DD3</a>
12c. <a href="#">2DD4</a>	12d. <a href="#">2DD5</a>	12e. <a href="#">2DD6</a>	12f. <a href="#">2DD7</a>			



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## 1.4 Rounding off

EMA8

Rounding off a decimal number to a given number of decimal places is the quickest way to approximate a number. For example, if you wanted to round off 2,6525272 to three decimal places, you would:

- count three places after the decimal and place a | between the third and fourth numbers;
- round up the third digit if the fourth digit is greater than or equal to 5;
- leave the third digit unchanged if the fourth digit is less than 5;
- if the third digit is 9 and needs to be rounded up, then the 9 becomes a 0 and the second digit is rounded up.

So, since the first digit after the | is a 5, we must round up the digit in the third decimal place to a 3 and the final answer of 2,6525272 rounded to three decimal places is 2,653.

### VISIT:

The following video explains how to round off.

► See video: [2DD8](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## Worked example 4: Rounding off

### QUESTION

Round off the following numbers to the indicated number of decimal places:

1.  $\frac{120}{99} = 1,1\dot{2}$  to 3 decimal places.
2.  $\pi = 3,141592653\dots$  to 4 decimal places.
3.  $\sqrt{3} = 1,7320508\dots$  to 4 decimal places.
4. 2,78974526 to 3 decimal places.

### SOLUTION

#### Step 1: Mark off the required number of decimal places

If the number is not a decimal you first need to write the number as a decimal.

1.  $\frac{120}{99} = 1,212|121212\dots$
2.  $\pi = 3,1415|92653\dots$
3.  $\sqrt{3} = 1,7320|508\dots$
4. 2,789|74526

#### Step 2: Check the next digit to see if you must round up or round down

1. The last digit of  $\frac{120}{99} = 1,212|121212\dot{1}$  must be rounded down.
2. The last digit of  $\pi = 3,1415|92653\dots$  must be rounded up.
3. The last digit of  $\sqrt{3} = 1,7320|508\dots$  must be rounded up.
4. The last digit of 2,789|74526 must be rounded up.

Since this is a 9 we replace it with a 0 and round up the second last digit.

#### Step 3: Write the final answer

1.  $\frac{120}{99} = 1,212$  rounded to 3 decimal places.
2.  $\pi = 3,1416$  rounded to 4 decimal places.
3.  $\sqrt{3} = 1,7321$  rounded to 4 decimal places.
4. 2,790

### Exercise 1 – 2:

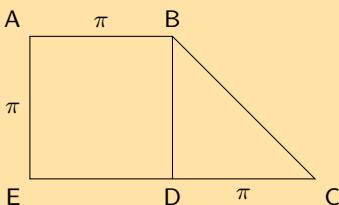
1. Round off the following to 3 decimal places:  
a) 12,56637061...      b) 3,31662479...      c) 0,26666666...  
d) 1,912931183...      e) 6,32455532...      f) 0,055555555...
2. Round off each of the following to the indicated number of decimal places:  
a) 345,04399906 to 4 decimal places.  
b) 1361,72980445 to 2 decimal places.  
c) 728,00905239 to 6 decimal places.

d)  $\frac{1}{27}$  to 4 decimal places.

e)  $\frac{45}{99}$  to 5 decimal places.

f)  $\frac{1}{12}$  to 2 decimal places.

3. Study the diagram below



a) Calculate the area of  $ABDE$  to 2 decimal places.

b) Calculate the area of  $BCD$  to 2 decimal places.

c) Using your answers in (a) and (b) calculate the area of  $ABCDE$ .

d) Without rounding off, what is the area of  $ABCDE$ ?

4. Given  $i = \frac{r}{600}$ ;  $r = 7,4$ ;  $n = 96$ ;  $P = 200\ 000$ .

a) Calculate  $i$  correct to 2 decimal places.

b) Using your answer from (a), calculate  $A$  in  $A = P(1 + i)^n$ .

c) Calculate  $A$  without rounding off your answer in (a), compare this answer with your answer in (b).

5. If it takes 1 person to carry 3 boxes, how many people are needed to carry 31 boxes?

6. If 7 tickets cost R 35,20, how much does one ticket cost?

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1a. [2DD9](#)   1b. [2DDB](#)   1c. [2DDC](#)   1d. [2DDD](#)   1e. [2DDF](#)   1f. [2DDG](#)   2a. [2DDH](#)   2b. [2DDJ](#)  
2c. [2DDK](#)   2d. [2DDM](#)   2e. [2DDN](#)   2f. [2DDP](#)   3. [2DDQ](#)   4. [2DDR](#)   5. [2DDS](#)   6. [2DDT](#)



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## 1.5 Estimating surds

EMA9

If the  $n^{\text{th}}$  root of a number cannot be simplified to a rational number, we call it a surd. For example,  $\sqrt{2}$  and  $\sqrt[3]{6}$  are surds, but  $\sqrt{4}$  is not a surd because it can be simplified to the rational number 2.

In this chapter we will look at surds of the form  $\sqrt[n]{a}$  where  $a$  is any positive number, for example,  $\sqrt{7}$  or  $\sqrt[3]{5}$ . It is very common for  $n$  to be 2, so we usually do not write  $\sqrt[2]{a}$ . Instead we write the surd as just  $\sqrt{a}$ .

It is sometimes useful to know the approximate value of a surd without having to use a calculator. For example, we want to be able to estimate where a surd like  $\sqrt{3}$  is on the number line. From a calculator we know that  $\sqrt{3}$  is equal to 1,73205.... It is easy to see that  $\sqrt{3}$  is above 1 and below 2. But to see this for other surds like  $\sqrt{18}$ , without using a calculator you must first understand the following:

If  $a$  and  $b$  are positive whole numbers, and  $a < b$ , then  $\sqrt[n]{a} < \sqrt[n]{b}$

A perfect square is the number obtained when an integer is squared. For example, 9 is a perfect square since  $3^2 = 9$ .

Similarly, a perfect cube is a number which is the cube of an integer. For example, 27 is a perfect cube, because  $3^3 = 27$ .

Consider the surd  $\sqrt[3]{52}$ . It lies somewhere between 3 and 4, because  $\sqrt[3]{27} = 3$  and  $\sqrt[3]{64} = 4$  and 52 is between 27 and 64.

### VISIT:

The following video explains how to estimate a surd.

► See video: [2DDV](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 5: Estimating surds

#### QUESTION

Find two consecutive integers such that  $\sqrt{26}$  lies between them. (Remember that consecutive integers are two integers that follow one another on the number line, for example, 5 and 6 or 8 and 9.)

#### SOLUTION

##### Step 1: Use perfect squares to estimate the lower integer

$5^2 = 25$ . Therefore  $5 < \sqrt{26}$ .

##### Step 2: Use perfect squares to estimate the upper integer

$6^2 = 36$ . Therefore  $\sqrt{26} < 6$ .

##### Step 3: Write the final answer

$5 < \sqrt{26} < 6$

### Worked example 6: Estimating surds

#### QUESTION

Find two consecutive integers such that  $\sqrt[3]{49}$  lies between them.

#### SOLUTION

##### Step 1: Use perfect cubes to estimate the lower integer

$3^3 = 27$ , therefore  $3 < \sqrt[3]{49}$ .

##### Step 2: Use perfect cubes to estimate the upper integer

$4^3 = 64$ , therefore  $\sqrt[3]{49} < 4$ .

##### Step 3: Write the answer

$3 < \sqrt[3]{49} < 4$

##### Step 4: Check the answer by cubing all terms in the inequality and then simplify

$27 < 49 < 64$ . This is true, so  $\sqrt[3]{49}$  lies between 3 and 4.

### Exercise 1 – 3:

1. Determine between which two consecutive integers the following numbers lie, without using a calculator:

a)  $\sqrt{18}$       b)  $\sqrt{29}$       c)  $\sqrt[3]{5}$       d)  $\sqrt[3]{79}$       e)  $\sqrt{155}$   
f)  $\sqrt{57}$       g)  $\sqrt{71}$       h)  $\sqrt[3]{123}$       i)  $\sqrt[3]{90}$       j)  $\sqrt[3]{81}$

2. Estimate the following surds to the nearest 1 decimal place, without using a calculator.

a)  $\sqrt{10}$       b)  $\sqrt{82}$       c)  $\sqrt{15}$       d)  $\sqrt{90}$

3. Consider the following list of numbers:

$\frac{27}{7}$ ;  $\sqrt{19}$ ;  $2\pi$ ; 0,45; 0,45;  $-\sqrt{\frac{9}{4}}$ ; 6;  $-\sqrt{8}$ ;  $\sqrt{51}$

Without using a calculator, rank all the numbers in ascending order.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

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1i. [2DF6](#)    1j. [2DF7](#)    2a. [2DF8](#)    2b. [2DF9](#)    2c. [2DFB](#)    2d. [2DFC](#)    3. [2DFD](#)



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## 1.6 Products

EMAB

Mathematical expressions are just like sentences and their parts have special names. You should be familiar with the following words used to describe the parts of mathematical expressions.

$$3x^2 + 7xy - 5^3$$

Name	Examples
term	$3x^2$ ; $7xy$ ; $-5^3$
expression	$3x^2 + 7xy - 5^3$
coefficient	3; 7
exponent	2; 1; 3
base	$x$ ; $y$ ; 5
constant	3; 7; 5
variable	$x$ ; $y$
equation	$3x^2 + 7xy - 5^3 = 0$

## Multiplying a monomial and a binomial

EMAC

A monomial is an expression with one term, for example,  $3x$  or  $y^2$ . A binomial is an expression with two terms, for example,  $ax + b$  or  $cx + d$ .

### Worked example 7: Simplifying brackets

#### QUESTION

Simplify:

$$2a(a - 1) - 3(a^2 - 1)$$

**SOLUTION**

$$\begin{aligned}2a(a - 1) - 3(a^2 - 1) &= 2a(a) + 2a(-1) + (-3)(a^2) + (-3)(-1) \\&= 2a^2 - 2a - 3a^2 + 3 \\&= -a^2 - 2a + 3\end{aligned}$$

**Multiplying two binomials**

EMAD

Here we multiply (or expand) two linear binomials:

$$(ax + b)(cx + d)$$

$$\begin{aligned}(ax + b)(cx + d) &= (ax)(cx) + (ax)d + b(cx) + bd \\&= acx^2 + adx + bcx + bd \\&= acx^2 + x(ad + bc) + bd\end{aligned}$$

**Worked example 8: Multiplying two binomials****QUESTION**

Find the product:  $(3x - 2)(5x + 8)$

**SOLUTION**

$$\begin{aligned}(3x - 2)(5x + 8) &= (3x)(5x) + (3x)(8) + (-2)(5x) + (-2)(8) \\&= 15x^2 + 24x - 10x - 16 \\&= 15x^2 + 14x - 16\end{aligned}$$

The product of two identical binomials is known as the square of the binomial and is written as:

$$(ax + b)^2 = a^2x^2 + 2abx + b^2$$

If the two terms are of the form  $ax + b$  and  $ax - b$  then their product is:

$$(ax + b)(ax - b) = a^2x^2 - b^2$$

This product yields the difference of two squares.

**Multiplying a binomial and a trinomial**

EMAF

A trinomial is an expression with three terms, for example,  $ax^2 + bx + c$ . Now we can learn how to multiply a binomial and a trinomial.

To find the product of a binomial and a trinomial, multiply out the brackets:

$$(A + B)(C + D + E) = A(C + D + E) + B(C + D + E)$$

### VISIT:

This video shows some examples of multiplying a binomial and a trinomial.

► See video: [2DFF](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 9: Multiplying a binomial and a trinomial

#### QUESTION

Find the product:  $(x - 1)(x^2 - 2x + 1)$

#### SOLUTION

##### Step 1: Expand the bracket

$$(x - 1)(x^2 - 2x + 1) = x(x^2 - 2x + 1) - 1(x^2 - 2x + 1) = x^3 - 2x^2 + x - x^2 + 2x - 1$$

##### Step 2: Simplify

$$(x - 1)(x^2 - 2x + 1) = x^3 - 3x^2 + 3x - 1$$

### Exercise 1 – 4:

1. Expand the following products:

- |                       |                           |                       |
|-----------------------|---------------------------|-----------------------|
| a) $2y(y + 4)$        | b) $(y + 5)(y + 2)$       | c) $(2 - t)(1 - 2t)$  |
| d) $(x - 4)(x + 4)$   | e) $-(4 - x)(x + 4)$      | f) $-(a + b)(b - a)$  |
| g) $(2p + 9)(3p + 1)$ | h) $(3k - 2)(k + 6)$      | i) $(s + 6)^2$        |
| j) $-(7 - x)(7 + x)$  | k) $(3x - 1)(3x + 1)$     | l) $(7k + 2)(3 - 2k)$ |
| m) $(1 - 4x)^2$       | n) $(-3 - y)(5 - y)$      | o) $(8 - x)(8 + x)$   |
| p) $(9 + x)^2$        | q) $(-7y + 11)(-12y + 3)$ | r) $(g - 5)^2$        |
| s) $(d + 9)^2$        | t) $(6d + 7)(6d - 7)$     | u) $(5z + 1)(5z - 1)$ |
| v) $(1 - 3h)(1 + 3h)$ | w) $(2p + 3)(2p + 2)$     | x) $(8a + 4)(a + 7)$  |
| y) $(5r + 4)(2r + 4)$ | z) $(w + 1)(w - 1)$       |                       |

2. Expand the following products:

- |                       |                       |                       |
|-----------------------|-----------------------|-----------------------|
| a) $(g + 11)(g - 11)$ | b) $(4b - 2)(2b - 4)$ | c) $(4b - 3)(2b - 1)$ |
| d) $(6x - 4)(3x + 6)$ | e) $(3w - 2)(2w + 7)$ | f) $(2t - 3)^2$       |

- g)  $(5p - 8)^2$       h)  $(4y + 5)^2$       i)  $(2y^6 + 3y^5)(-5y - 12)$   
 j)  $9(8y^2 - 2y + 3)$       k)  $(-2y^2 - 4y + 11)(5y - 12)$       l)  $(7y^2 - 6y - 8)(-2y + 2)$   
 m)  $(10y + 3)(-2y^2 - 11y + 2)$       n)  $(-12y - 3)(2y^2 - 11y + 3)$       o)  $(-10)(2y^2 + 8y + 3)$   
 p)  $(7y + 3)(7y^2 + 3y + 10)$       q)  $(a + 2b)(a^2 + b^2 + 2ab)$       r)  $(x + y)(x^2 - xy + y^2)$   
 s)  $3m(9m^2 + 2) + 5m^2(5m + 6)$       t)  $4x^2(10x^3 + 4) + 4x^3(2x^2 + 6)$       u)  $3k^3(k^2 + 3) + 2k^2(6k^3 + 7)$   
 v)  $(3x + 2)(3x - 2)(9x^2 - 4)$       w)  $(-6y^4 + 11y^2 + 3y)(y + 4)(y - 4)$   
 x)  $(x + 2)(x - 3)(x^2 + 2x - 3)$       y)  $(a + 2)^2 - (2a - 4)^2$

3. Expand the following products:

- a)  $(2x + 3)^2 - (x - 2)^2$       b)  $(2a^2 - a - 1)(a^2 + 3a + 2)$   
 c)  $(y^2 + 4y - 1)(1 - 4y - y^2)$       d)  $2(x - 2y)(x^2 + xy + y^2)$   
 e)  $3(a - 3b)(a^2 + 3ab - b^2)$       f)  $(2a - b)(2a + b)(2a^2 - 3ab + b^2)$   
 g)  $2(3x + y)(3x - y) - (3x - y)^2$       h)  $(x + y)(x - 3y) + (2x - y)^2$   
 i)  $\left(\frac{x}{3} - \frac{3}{x}\right)\left(\frac{x}{4} + \frac{4}{x}\right)$       j)  $\left(x - \frac{2}{x}\right)\left(\frac{x}{3} + \frac{4}{x}\right)$   
 k)  $\frac{1}{2}(10x - 12y) + \frac{1}{3}(15x - 18y)$       l)  $\frac{1}{2}a(4a + 6b) + \frac{1}{4}(8a + 12b)$

4. What is the value of  $b$ , in  $(x + b)(x - 1) = x^2 + 3x - 4$

5. What is the value of  $g$ , in  $(x - 2)(x + g) = x^2 - 6x + 8$

6. In  $(x - 4)(x + k) = x^2 + bx + c$ :

- a) For which of these values of  $k$  will  $b$  be positive?  
 $-3; -1; 0; 3; 5$
- b) For which of these values of  $k$  will  $c$  be positive?  
 $-3; -1; 0; 3; 5$
- c) For what real values of  $k$  will  $c$  be positive?
- d) For what real values of  $k$  will  $b$  be positive?

7. Answer the following:

- a) Expand  $\left(x + \frac{4}{x}\right)^2$ .
- b) Given that  $\left(x + \frac{4}{x}\right)^2 = 14$ , determine the value of  $x^2 + \frac{16}{x^2}$  without solving for  $x$ .

8. Answer the following:

- a) Expand:  $\left(a + \frac{1}{a}\right)^2$
- b) Given that  $\left(a + \frac{1}{a}\right) = 3$ , determine the value of  $\left(a + \frac{1}{a}\right)^2$  without solving for  $a$ .
- c) Given that  $\left(a - \frac{1}{a}\right) = 3$ , determine the value of  $\left(a + \frac{1}{a}\right)^2$  without solving for  $a$ .

9. Answer the following:

a) Expand:  $\left(3y + \frac{1}{2y}\right)^2$

b) Given that  $3y + \frac{1}{2y} = 4$ , determine the value of  $\left(3y + \frac{1}{2y}\right)^2$  without solving for  $y$ .

10. Answer the following:

a) Expand:  $\left(a + \frac{1}{3a}\right)^2$

b) Expand:  $\left(a + \frac{1}{3a}\right)\left(a^2 - \frac{1}{3} + \frac{1}{9a^2}\right)$

c) Given that  $a + \frac{1}{3a} = 2$ , determine the value of  $a^3 + \frac{1}{27a^3}$  without solving for  $a$ .

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1g. <a href="#">2DFP</a>	1h. <a href="#">2DFQ</a>	1i. <a href="#">2DFR</a>	1j. <a href="#">2DFS</a>	1k. <a href="#">2DFT</a>	1l. <a href="#">2DFV</a>
1m. <a href="#">2DFW</a>	1n. <a href="#">2DFX</a>	1o. <a href="#">2DFY</a>	1p. <a href="#">2DFZ</a>	1q. <a href="#">2DG3</a>	1r. <a href="#">2DG4</a>
1s. <a href="#">2DG5</a>	1t. <a href="#">2DG6</a>	1u. <a href="#">2DG7</a>	1v. <a href="#">2DG8</a>	1w. <a href="#">2DG9</a>	1x. <a href="#">2DGB</a>
1y. <a href="#">2DGC</a>	1z. <a href="#">2DGD</a>	2a. <a href="#">2DGF</a>	2b. <a href="#">2DGH</a>	2c. <a href="#">2DGJ</a>	2d. <a href="#">2DGK</a>
2e. <a href="#">2DGM</a>	2f. <a href="#">2DGN</a>	2g. <a href="#">2DGP</a>	2h. <a href="#">2DGQ</a>	2i. <a href="#">2DG2</a>	2j. <a href="#">2DGR</a>
2k. <a href="#">2DGS</a>	2l. <a href="#">2DGT</a>	2m. <a href="#">2DGV</a>	2n. <a href="#">2DGW</a>	2o. <a href="#">2DGX</a>	2p. <a href="#">2DGY</a>
2q. <a href="#">2DGZ</a>	2r. <a href="#">2DH2</a>	2s. <a href="#">2DH3</a>	2t. <a href="#">2DH4</a>	2u. <a href="#">2DH5</a>	2v. <a href="#">2DH6</a>
2w. <a href="#">2DH7</a>	2x. <a href="#">2DH8</a>	2y. <a href="#">2DH9</a>	3a. <a href="#">2DHB</a>	3b. <a href="#">2DHC</a>	3c. <a href="#">2DHD</a>
3d. <a href="#">2DHF</a>	3e. <a href="#">2DHG</a>	3f. <a href="#">2DHH</a>	3g. <a href="#">2DHJ</a>	3h. <a href="#">2DHK</a>	3i. <a href="#">2DHM</a>
3j. <a href="#">2DHN</a>	3k. <a href="#">2DHP</a>	3l. <a href="#">2DHQ</a>	4. <a href="#">2DHR</a>	5. <a href="#">2DHS</a>	6. <a href="#">2DHT</a>
7. <a href="#">2DHW</a>	8. <a href="#">2DHW</a>	9. <a href="#">2DHX</a>	10. <a href="#">2DHY</a>		



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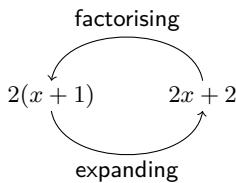


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## 1.7 Factorisation

EMAG

Factorisation is the opposite process of expanding brackets. For example, expanding brackets would require  $2(x + 1)$  to be written as  $2x + 2$ . Factorisation would be to start with  $2x + 2$  and end up with  $2(x + 1)$ .



The two expressions  $2(x + 1)$  and  $2x + 2$  are equivalent; they have the same value for all values of  $x$ .

In previous grades, we factorised by taking out a common factor and using difference of squares.

Factorising based on common factors relies on there being factors common to all the terms.

For example,  $2x - 6x^2$  can be factorised as follows:

$$2x - 6x^2 = 2x(1 - 3x)$$

And  $2(x - 1) - a(x - 1)$  can be factorised as follows:

$$(x - 1)(2 - a)$$

### VISIT:

The following video shows an example of factorising by taking out a common factor.

► See video: [2DHZ](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 10: Factorising using a switch around in brackets

#### QUESTION

Factorise:

$$5(a - 2) - b(2 - a)$$

#### SOLUTION

Use a “switch around” strategy to find the common factor.

Notice that  $2 - a = -(a - 2)$

$$\begin{aligned} 5(a - 2) - b(2 - a) &= 5(a - 2) - [-b(a - 2)] \\ &= 5(a - 2) + b(a - 2) \\ &= (a - 2)(5 + b) \end{aligned}$$

#### Exercise 1 – 5:

Factorise:

- |                         |                         |                           |
|-------------------------|-------------------------|---------------------------|
| 1. $12x + 32y$          | 2. $-2ab^2 - 4a^2b$     | 3. $18ab - 3bc$           |
| 4. $12kj + 18kq$        | 5. $-12a + 24a^3$       | 6. $-2ab - 8a$            |
| 7. $24kj - 16k^2j$      | 8. $-a^2b - b^2a$       | 9. $72b^2q - 18b^3q^2$    |
| 10. $125x^6 - 5y^2$     | 11. $6x^2 + 2x + 10x^3$ | 12. $2xy^2 + xy^2z + 3xy$ |
| 13. $12k^2j + 24k^2j^2$ | 14. $3a^2 + 6a - 18$    | 15. $7a + 4$              |

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on ‘Practise Maths’.

- |                         |                          |                          |                          |                          |                          |                          |                         |
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| 9. <a href="#">2DJB</a> | 10. <a href="#">2DJC</a> | 11. <a href="#">2DJD</a> | 12. <a href="#">2DJF</a> | 13. <a href="#">2DJG</a> | 14. <a href="#">2DJH</a> | 15. <a href="#">2DJJ</a> |                         |



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We have seen that  $(ax + b)(ax - b)$  can be expanded to  $a^2x^2 - b^2$ .

Therefore  $a^2x^2 - b^2$  can be factorised as  $(ax + b)(ax - b)$ .

For example,  $x^2 - 16$  can be written as  $x^2 - 4^2$  which is a difference of two squares. Therefore, the factors of  $x^2 - 16$  are  $(x - 4)$  and  $(x + 4)$ .

To spot a difference of two squares, look for expressions:

- consisting of two terms;
- with terms that have different signs (one positive, one negative);
- with each term a perfect square.

For example:  $a^2 - 1$ ;  $4x^2 - y^2$ ;  $-49 + p^4$ .

#### VISIT:

The following video explains factorising the difference of two squares.

► See video: [2DJK](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 11: The difference of two squares

#### QUESTION

---

Factorise:  $3a(a^2 - 4) - 7(a^2 - 4)$ .

#### SOLUTION

---

**Step 1: Take out the common factor ( $a^2 - 4$ )**

$$3a(a^2 - 4) - 7(a^2 - 4) = (a^2 - 4)(3a - 7)$$

**Step 2: Factorise the difference of two squares ( $a^2 - 4$ )**

$$(a^2 - 4)(3a - 7) = (a - 2)(a + 2)(3a - 7)$$

### Exercise 1 – 6:

Factorise:

- |                            |                             |                            |
|----------------------------|-----------------------------|----------------------------|
| 1. $4(y - 3) + k(3 - y)$   | 2. $a^2(a - 1) - 25(a - 1)$ | 3. $bm(b + 4) - 6m(b + 4)$ |
| 4. $a^2(a + 7) + 9(a + 7)$ | 5. $3b(b - 4) - 7(4 - b)$   | 6. $3g(z + 6) + 2(6 + z)$  |
| 7. $4b(y + 2) + 5(2 + y)$  | 8. $3d(r + 5) + 14(5 + r)$  | 9. $(6x + y)^2 - 9$        |

10.  $4x^2 - (4x - 3y)^2$       11.  $16a^2 - (3b + 4c)^2$       12.  $(b - 4)^2 - 9(b - 5)^2$   
 13.  $4(a - 3)^2 - 49(4a - 5)$       14.  $16k^2 - 4$       15.  $a^2b^2c^2 - 1$   
 16.  $\frac{1}{9}a^2 - 4b^2$       17.  $\frac{1}{2}x^2 - 2$       18.  $y^2 - 8$   
 19.  $y^2 - 13$       20.  $a^2(a - 2ab - 15b^2) - 9b^2(a^2 - 2ab - 15b^2)$

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## Factorising by grouping in pairs

EMAK

The taking out of common factors is the starting point in all factorisation problems. We know that the factors of  $3x + 3$  are 3 and  $(x + 1)$ . Similarly, the factors of  $2x^2 + 2x$  are  $2x$  and  $(x + 1)$ . Therefore, if we have an expression:

$$2x^2 + 2x + 3x + 3$$

there is no common factor to all four terms, but we can factorise as follows:

$$(2x^2 + 2x) + (3x + 3) = 2x(x + 1) + 3(x + 1)$$

We can see that there is another common factor  $(x + 1)$ . Therefore, we can write:

$$(x + 1)(2x + 3)$$

We get this by taking out the  $(x + 1)$  and seeing what is left over. We have  $2x$  from the first group and  $+3$  from the second group. This is called factorising by grouping.

### Worked example 12: Factorising by grouping in pairs

#### QUESTION

Find the factors of  $7x + 14y + bx + 2by$ .

#### SOLUTION

**Step 1: There are no factors common to all terms**

**Step 2: Group terms with common factors together**

7 is a common factor of the first two terms and  $b$  is a common factor of the second two terms. We see that the ratio of the coefficients 7 : 14 is the same as  $b : 2b$ .

$$\begin{aligned} 7x + 14y + bx + 2by &= (7x + 14y) + (bx + 2by) \\ &= 7(x + 2y) + b(x + 2y) \end{aligned}$$

### Step 3: Take out the common factor $(x + 2y)$

$$7(x + 2y) + b(x + 2y) = (x + 2y)(7 + b)$$

OR

### Step 4: Group terms with common factors together

$x$  is a common factor of the first and third terms and  $2y$  is a common factor of the second and fourth terms ( $7 : b = 14 : 2b$ ).

### Step 5: Rearrange the equation with grouped terms together

$$\begin{aligned} 7x + 14y + bx + 2by &= (7x + bx) + (14y + 2by) \\ &= x(7 + b) + 2y(7 + b) \end{aligned}$$

### Step 6: Take out the common factor $(7 + b)$

$$x(7 + b) + 2y(7 + b) = (7 + b)(x + 2y)$$

### Step 7: Write the final answer

The factors of  $7x + 14y + bx + 2by$  are  $(7 + b)$  and  $(x + 2y)$ .

### Exercise 1 – 7:

Factorise the following:

- |                                   |                                     |                                |
|-----------------------------------|-------------------------------------|--------------------------------|
| 1. $6d - 9r + 2t^5d - 3t^5r$      | 2. $9z - 18m + b^3z - 2b^3m$        | 3. $35z - 10y + 7c^5z - 2c^5y$ |
| 4. $6x + a + 2ax + 3$             | 5. $x^2 - 6x + 5x - 30$             | 6. $5x + 10y - ax - 2ay$       |
| 7. $a^2 - 2a - ax + 2x$           | 8. $5xy - 3y + 10x - 6$             | 9. $ab - a^2 - a + b$          |
| 10. $14m - 4n + 7jm - 2jn$        | 11. $28r - 20x + 7gr - 5gx$         | 12. $25d - 15m + 5yd - 3ym$    |
| 13. $45q - 18z + 5cq - 2cz$       | 14. $6j - 15v + 2yj - 5yv$          | 15. $16a - 40k + 2za - 5zk$    |
| 16. $ax - bx + ay - by + 2a - 2b$ | 17. $3ax + bx - 3ay - by - 9a - 3b$ |                                |

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

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Factorising is the reverse of calculating the product of factors. In order to factorise a quadratic, we need to find the factors which, when multiplied together, equal the original quadratic.

Consider a quadratic expression of the form  $ax^2 + bx$ . We see here that  $x$  is a common factor in both terms. Therefore  $ax^2 + bx$  factorises as  $x(ax + b)$ . For example,  $8y^2 + 4y$  factorises as  $4y(2y + 1)$ .

Another type of quadratic is made up of the difference of squares. We know that:

$$(a + b)(a - b) = a^2 - b^2$$

So  $a^2 - b^2$  can be written in factorised form as  $(a + b)(a - b)$ .

This means that if we ever come across a quadratic that is made up of a difference of squares, we can immediately write down the factors. These types of quadratics are very simple to factorise. However, many quadratics do not fall into these categories and we need a more general method to factorise quadratics.

We can learn about factorising quadratics by looking at the opposite process, where two binomials are multiplied to get a quadratic. For example:

$$\begin{aligned}(x + 2)(x + 3) &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6\end{aligned}$$

We see that the  $x^2$  term in the quadratic is the product of the  $x$ -terms in each bracket. Similarly, the 6 in the quadratic is the product of the 2 and 3 in the brackets. Finally, the middle term is the sum of two terms.

So, how do we use this information to factorise the quadratic?

Let us start with factorising  $x^2 + 5x + 6$  and see if we can decide upon some general rules. Firstly, write down the two brackets with an  $x$  in each bracket and space for the remaining terms.

$$(x \quad ) (x \quad )$$

Next, decide upon the factors of 6. Since the 6 is positive, possible combinations are: 1 and 6, 2 and 3,  $-1$  and  $-6$  or  $-2$  and  $-3$

Therefore, we have four possibilities:

<b>Option 1</b>	<b>Option 2</b>	<b>Option 3</b>	<b>Option 4</b>
$(x + 1)(x + 6)$	$(x - 1)(x - 6)$	$(x + 2)(x + 3)$	$(x - 2)(x - 3)$

Next, we expand each set of brackets to see which option gives us the correct middle term.

<b>Option 1</b>	<b>Option 2</b>	<b>Option 3</b>	<b>Option 4</b>
$(x + 1)(x + 6)$	$(x - 1)(x - 6)$	$(x + 2)(x + 3)$	$(x - 2)(x - 3)$
$x^2 + 7x + 6$	$x^2 - 7x + 6$	$x^2 + 5x + 6$	$x^2 - 5x + 6$

We see that Option 3,  $(x + 2)(x + 3)$ , is the correct solution.

The process of factorising a quadratic is mostly trial and error but there are some strategies that can be used to ease the process.

1. Take out any common factor in the coefficients of the terms of the expression to obtain an expression of the form  $ax^2 + bx + c$  where  $a$ ,  $b$  and  $c$  have no common factors and  $a$  is positive.
2. Write down two brackets with an  $x$  in each bracket and space for the remaining terms:  $(x \quad ) (x \quad )$
3. Write down a set of factors for  $a$  and  $c$ .
4. Write down a set of options for the possible factors for the quadratic using the factors of  $a$  and  $c$ .
5. Expand all options to see which one gives you the correct middle term  $bx$ .

**IMPORTANT!**

If  $c$  is positive, then the factors of  $c$  must be either both positive or both negative. If  $c$  is negative, it means only one of the factors of  $c$  is negative, the other one being positive. Once you get an answer, always multiply out your brackets again just to make sure it really works.

**VISIT:**

The following video summarises how to factorise expressions and shows some examples.

► See video: **2DKX** at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

**Worked example 13: Factorising a quadratic trinomial****QUESTION**

Factorise:  $3x^2 + 2x - 1$ .

**SOLUTION**

**Step 1: Check that the quadratic is in required form  $ax^2 + bx + c$**

**Step 2: Write down a set of factors for  $a$  and  $c$**

$$(x \quad ) (x \quad )$$

The possible factors for  $a$  are: 1 and 3

The possible factors for  $c$  are:  $-1$  and  $1$

Write down a set of options for the possible factors of the quadratic using the factors of  $a$  and  $c$ . Therefore, there are two possible options.

Option 1	Option 2
$(x - 1)(3x + 1)$	$(x + 1)(3x - 1)$
$3x^2 - 2x - 1$	$3x^2 + 2x - 1$

**Step 3: Check that the solution is correct by multiplying the factors**

$$\begin{aligned} (x + 1)(3x - 1) &= 3x^2 - x + 3x - 1 \\ &= 3x^2 + 2x - 1 \end{aligned}$$

**Step 4: Write the final answer**

$$3x^2 + 2x - 1 = (x + 1)(3x - 1)$$

### Exercise 1 – 8:

Factorise the following:

1.  $x^2 + 8x + 15$

4.  $2h^2 + 5h - 3$

7.  $x^2 - 2x - 15$

10.  $x^2 - x - 20$

13.  $6v^2 - 27v + 27$

16.  $3x^2 + 17x - 6$

19.  $a^2 - 7ab + 12b$

22.  $(x - 2)^2 - 7(x - 2) + 12$

25.  $3(b^2 + 5b) + 12$

2.  $x^2 + 9x + 8$

5.  $3x^2 + 4x + 1$

8.  $x^2 + 2x - 3$

11.  $2x^2 - 22x + 20$

14.  $6g^2 - 15g - 9$

17.  $7x^2 - 6x - 1$

20.  $3a^2 + 5ab - 12b^2$

23.  $(a - 2)^2 - 4(a - 2) - 5$

26.  $6(a^2 + 3a) - 168$

3.  $x^2 + 12x + 36$

6.  $3s^2 + s - 10$

9.  $x^2 + x - 20$

12.  $6a^2 + 14a + 8$

15.  $3x^2 + 19x + 6$

18.  $6x^2 - 15x - 9$

21.  $98x^4 + 14x^2 - 4$

24.  $(y + 3)^2 - 3(y + 3) - 18$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

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## Sum and difference of two cubes

EMAP

We now look at two special results obtained from multiplying a binomial and a trinomial:

Sum of two cubes:

$$\begin{aligned}(x+y)(x^2 - xy + y^2) &= x(x^2 - xy + y^2) + y(x^2 - xy + y^2) \\&= [x(x^2) + x(-xy) + x(y^2)] + [y(x^2) + y(-xy) + y(y^2)] \\&= x^3 - x^2y + xy^2 + x^2y - xy^2 + y^3 \\&= x^3 + y^3\end{aligned}$$

Difference of two cubes:

$$\begin{aligned}(x-y)(x^2 + xy + y^2) &= x(x^2 + xy + y^2) - y(x^2 + xy + y^2) \\&= [x(x^2) + x(xy) + x(y^2)] - [y(x^2) + y(xy) + y(y^2)] \\&= x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3 \\&= x^3 - y^3\end{aligned}$$

So we have seen that:

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

We use these two basic identities to factorise more complex examples.

## Worked example 14: Factorising a difference of two cubes

### QUESTION

Factorise:  $a^3 - 1$ .

### SOLUTION

#### Step 1: Take the cube root of terms that are perfect cubes

We are working with the difference of two cubes. We know that  $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ , so we need to identify  $x$  and  $y$ .

We start by noting that  $\sqrt[3]{a^3} = a$  and  $\sqrt[3]{1} = 1$ . These give the terms in the first bracket. This also tells us that  $x = a$  and  $y = 1$ .

#### Step 2: Find the three terms in the second bracket

We can replace  $x$  and  $y$  in the factorised form of the expression for the difference of two cubes with  $a$  and 1. Doing so we get the second bracket:

$$(a^3 - 1) = (a - 1)(a^2 + a + 1)$$

#### Step 3: Expand the brackets to check that the expression has been correctly factorised

$$\begin{aligned}(a - 1)(a^2 + a + 1) &= a(a^2 + a + 1) - 1(a^2 + a + 1) \\&= a^3 + a^2 + a - a^2 - a - 1 \\&= a^3 - 1\end{aligned}$$

## Worked example 15: Factorising a sum of two cubes

### QUESTION

Factorise:  $x^3 + 8$ .

### SOLUTION

#### Step 1: Take the cube root of terms that are perfect cubes

We are working with the sum of two cubes. We know that  $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ , so we need to identify  $x$  and  $y$ .

We start by noting that  $\sqrt[3]{x^3} = x$  and  $\sqrt[3]{8} = 2$ . These give the terms in the first bracket. This also tells us that  $x = x$  and  $y = 2$ .

#### Step 2: Find the three terms in the second bracket

We can replace  $x$  and  $y$  in the factorised form of the expression for the sum of two cubes with  $x$  and 2. Doing so we get the second bracket:

$$(x^3 + 8) = (x + 2)(x^2 - 2x + 4)$$

#### Step 3: Expand the brackets to check that the expression has been correctly factorised

$$\begin{aligned}(x + 2)(x^2 - 2x + 4) &= x(x^2 - 2x + 4) + 2(x^2 - 2x + 4) \\&= x^3 - 2x^2 + 4x + 2x^2 - 4x + 8 \\&= x^3 + 8\end{aligned}$$

## Worked example 16: Factorising a difference of two cubes

### QUESTION

Factorise:  $16y^3 - 432$ .

### SOLUTION

#### Step 1: Take out the common factor 16

$$16y^3 - 432 = 16(y^3 - 27)$$

#### Step 2: Take the cube root of terms that are perfect cubes

We are working with the difference of two cubes. We know that  $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ , so we need to identify  $x$  and  $y$ .

We start by noting that  $\sqrt[3]{y^3} = y$  and  $\sqrt[3]{27} = 3$ . These give the terms in the first bracket. This also tells us that  $x = y$  and  $y = 3$ .

#### Step 3: Find the three terms in the second bracket

We can replace  $x$  and  $y$  in the factorised form of the expression for the difference of two cubes with  $y$  and 3. Doing so we get the second bracket:

$$16(y^3 - 27) = 16(y - 3)(y^2 + 3y + 9)$$

#### Step 4: Expand the brackets to check that the expression has been correctly factorised

$$\begin{aligned} 16(y - 3)(y^2 + 3y + 9) &= 16[(y(y^2 + 3y + 9) - 3(y^2 + 3y + 9)] \\ &= 16[y^3 + 3y^2 + 9y - 3y^2 - 9y - 27] \\ &= 16y^3 - 432 \end{aligned}$$

## Worked example 17: Factorising a sum of two cubes

### QUESTION

Factorise:  $8t^3 + 125p^3$ .

### SOLUTION

#### Step 1: Take the cube root of terms that are perfect cubes

We are working with the sum of two cubes. We know that  $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ , so we need to identify  $x$  and  $y$ .

We start by noting that  $\sqrt[3]{8t^3} = 2t$  and  $\sqrt[3]{125p^3} = 5p$ . These give the terms in the first bracket. This also tells us that  $x = 2t$  and  $y = 5p$ .

## Step 2: Find the three terms in the second bracket

We can replace  $x$  and  $y$  in the factorised form of the expression for the difference of two cubes with  $2t$  and  $5p$ . Doing so we get the second bracket:

$$\begin{aligned}(8t^3 + 125p^3) &= (2t + 5p) \left[ (2t)^2 - (2t)(5p) + (5p)^2 \right] \\ &= (2t + 5p) (4t^2 - 10tp + 25p^2)\end{aligned}$$

## Step 3: Expand the brackets to check that the expression has been correctly factorised

$$\begin{aligned}(2t + 5p) (4t^2 - 10tp + 25p^2) &= 2t (4t^2 - 10tp + 25p^2) + 5p (4t^2 - 10tp + 25p^2) \\ &= 8t^3 - 20pt^2 + 50p^2t + 20pt^2 - 50p^2t + 125p^3 \\ &= 8t^3 + 125p^3\end{aligned}$$

### Exercise 1 – 9:

Factorise:

- |   |   |                            |
|---|---|----------------------------|
| 1. $w^3 - 8$  | 2. $g^3 + 64$                           | 3. $h^3 + 1$               |
| 4. $x^3 + 8$  | 5. $27 - m^3$                           | 6. $2x^3 - 2y^3$           |
| 7. $3k^3 + 81q^3$   | 8. $64t^3 - 1$                          | 9. $64x^2 - 1$             |
| 10. $125x^3 + 1$  | 11. $25x^3 + 1$                         | 12. $z - 125z^4$           |
| 13. $8m^6 + n^9$  | 14. $216n^3 - k^3$                      | 15. $125s^3 + d^3$         |
| 16. $8k^3 + r^3$  | 17. $8j^3k^3l^3 - b^3$                  | 18. $27x^3y^3 + w^3$       |
| 19. $128m^3 + 2f^3$                                       | 20. $p^{15} - \frac{1}{8}y^{12}$        | 21. $\frac{27}{t^3} - s^3$ |
| 22. $\frac{1}{64q^3} - h^3$                               | 23. $72g^3 + \frac{1}{3}v^3$            | 24. $1 - (x - y)^3$        |
| 25. $h^4(8g^6 + h^3) - (8g^6 + h^3)$                      | 26. $x(125w^3 - h^3) + y(125w^3 - h^3)$ |                            |
| 27. $x^2(27p^3 + w^3) - 5x(27p^3 + w^3) - 6(27p^3 + w^3)$ |   |                            |

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## 1.8 Simplification of fractions

EMAQ

We have studied procedures for working with fractions in earlier grades.

1.  $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$     ( $b \neq 0; d \neq 0$ )
2.  $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$     ( $b \neq 0$ )
3.  $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$     ( $b \neq 0; c \neq 0; d \neq 0$ )

**Note:** dividing by a fraction is the same as multiplying by the reciprocal of the fraction.

In some cases of simplifying an algebraic expression, the expression will be a fraction. For example,

$$\frac{x^2 + 3x}{x + 3}$$

has a quadratic binomial in the numerator and a linear binomial in the denominator. We have to apply the different factorisation methods in order to factorise the numerator and the denominator before we can simplify the expression.

$$\begin{aligned}\frac{x^2 + 3x}{x + 3} &= \frac{x(x + 3)}{x + 3} \\ &= x \quad (x \neq -3)\end{aligned}$$

If  $x = -3$  then the denominator,  $x + 3 = 0$  and the fraction is undefined.

### VISIT:

This video shows some examples of simplifying fractions.

⌚ See video: [2DNV](https://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 18: Simplifying fractions

#### QUESTION

Simplify:

$$\frac{ax - b + x - ab}{ax^2 - abx}, \quad (x \neq 0; x \neq b)$$

#### SOLUTION

**Step 1: Use grouping to factorise the numerator and take out the common factor  $ax$  in the denominator**

$$\frac{(ax - ab) + (x - b)}{ax^2 - abx} = \frac{a(x - b) + (x - b)}{ax(x - b)}$$

**Step 2: Take out common factor  $(x - b)$  in the numerator**

$$= \frac{(x - b)(a + 1)}{ax(x - b)}$$

**Step 3: Cancel the common factor in the numerator and the denominator to give the final answer**

$$= \frac{a+1}{ax}$$

### Worked example 19: Simplifying fractions

#### QUESTION

Simplify:

$$\frac{x^2 - x - 2}{x^2 - 4} \div \frac{x^2 + x}{x^2 + 2x}, \quad (x \neq 0; x \neq \pm 2)$$

#### SOLUTION

**Step 1: Factorise the numerator and denominator**

$$= \frac{(x+1)(x-2)}{(x+2)(x-2)} \div \frac{x(x+1)}{x(x+2)}$$

**Step 2: Change the division sign and multiply by the reciprocal**

$$= \frac{(x+1)(x-2)}{(x+2)(x-2)} \times \frac{x(x+2)}{x(x+1)}$$

**Step 3: Write the final answer**

$$= 1$$

### Worked example 20: Simplifying fractions

#### QUESTION

Simplify:

$$\frac{x-2}{x^2-4} + \frac{x^2}{x-2} - \frac{x^3+x-4}{x^2-4}, \quad (x \neq \pm 2)$$

#### SOLUTION

**Step 1: Factorise the denominators**

$$\frac{x-2}{(x+2)(x-2)} + \frac{x^2}{x-2} - \frac{x^3+x-4}{(x+2)(x-2)}$$

**Step 2: Make all denominators the same so that we can add or subtract the fractions**

The lowest common denominator is  $(x - 2)(x + 2)$ .

$$\frac{x - 2}{(x + 2)(x - 2)} + \frac{(x^2)(x + 2)}{(x + 2)(x - 2)} - \frac{x^3 + x - 4}{(x + 2)(x - 2)}$$

**Step 3: Write as one fraction**

$$\frac{x - 2 + (x^2)(x + 2) - (x^3 + x - 4)}{(x + 2)(x - 2)}$$

**Step 4: Simplify**

$$\frac{x - 2 + x^3 + 2x^2 - x^3 - x + 4}{(x + 2)(x - 2)} = \frac{2x^2 + 2}{(x + 2)(x - 2)}$$

**Step 5: Take out the common factor and write the final answer**

$$\frac{2(x^2 + 1)}{(x + 2)(x - 2)}$$

**Worked example 21: Simplifying fractions****QUESTION**

Simplify:

$$\frac{2}{x^2 - x} + \frac{x^2 + x + 1}{x^3 - 1} - \frac{x}{x^2 - 1}, \quad (x \neq 0; x \neq \pm 1)$$

**SOLUTION****Step 1: Factorise the numerator and denominator**

$$\frac{2}{x(x - 1)} + \frac{(x^2 + x + 1)}{(x - 1)(x^2 + x + 1)} - \frac{x}{(x - 1)(x + 1)}$$

**Step 2: Simplify and find the common denominator**

$$\frac{2(x + 1) + x(x + 1) - x^2}{x(x - 1)(x + 1)}$$

**Step 3: Write the final answer**

$$\frac{2x + 2 + x^2 + x - x^2}{x(x - 1)(x + 1)} = \frac{3x + 2}{x(x - 1)(x + 1)}$$

### Exercise 1 – 10:

1. Simplify (assume all denominators are non-zero):

a)  $\frac{3a}{15}$   
 d)  $\frac{a^2 - 4a}{a - 4}$   
 g)  $\frac{6ab + 2a}{2b}$   
 j)  $\frac{9x^2 - 16}{6x - 8}$   
 m)  $\frac{x^2 - 2x - 15}{5x - 25}$   
 p)  $\frac{a^2 + 6a - 16}{a^3 - 8}$   
 s)  $\frac{2x^2 - x - 1}{x^3 - x}$   
 v)  $\frac{hx - hg + 13x - 13g}{x - g}$

b)  $\frac{2a + 10}{4}$   
 e)  $\frac{3a^2 - 9a}{2a - 6}$   
 h)  $\frac{16x^2y - 8xy}{12x - 6}$   
 k)  $\frac{b^2 - 81a^2}{18a - 2b}$   
 n)  $\frac{x^2 + 2x - 15}{x^2 + 8x + 15}$   
 q)  $\frac{a^2 - 4ab - 12b^2}{a^2 + 4ab + 4b^2}$   
 t)  $\frac{qz + qr + 16z + 16r}{z + r}$   
 w)  $\frac{f^2a - fa^2}{f - a}$

c)  $\frac{5a + 20}{a + 4}$   
 f)  $\frac{9a + 27}{9a + 18}$   
 i)  $\frac{4xyp - 8xp}{12xy}$   
 l)  $\frac{t^2 - s^2}{s^2 - 2st + t^2}$   
 o)  $\frac{x^2 - x - 6}{x^3 - 27}$   
 r)  $\frac{6a^2 - 7a - 3}{3ab + b}$   
 u)  $\frac{pz - pq + 5z - 5q}{z - q}$

2. Simplify (assume all denominators are non-zero):

a)  $\frac{b^2 + 10b + 21}{3(b^2 - 9)} \div \frac{2b^2 + 14b}{30b^2 - 90b}$   
 c)  $\frac{z^2 + 17z + 66}{3(z^2 - 121)} \div \frac{2z^2 + 12z}{24z^2 - 264z}$   
 e)  $\frac{a^2 - 5a}{2a + 10} \times \frac{4a}{3a + 15}$   
 g)  $\frac{24a - 8}{12} \div \frac{9a - 3}{6}$   
 i)  $\frac{p^2 + pq}{7p} \times \frac{21q}{8p + 8q}$   
 k)  $\frac{16 - x^2}{x^2 - x - 12} \times \frac{x + 3}{x + 4}$   
 m)  $\frac{a - 4}{a + 5a + 4} \times \frac{a^2 + 2a + 1}{a^2 - 3a - 4}$   
 o)  $\frac{a^2 - 2a + 8}{a^2 + 6a + 8} \times \frac{a^2 + a - 12}{3} - \frac{3}{2}$   
 q)  $\frac{x + 4}{3} - \frac{x - 2}{2}$

b)  $\frac{x^2 + 17x + 70}{5(x^2 - 100)} \div \frac{3x^2 + 21x}{45x^2 - 450x}$   
 d)  $\frac{3a + 9}{14} \div \frac{7a + 21}{a + 3}$   
 f)  $\frac{3xp + 4p}{8p} \div \frac{12p^2}{3x + 4}$   
 h)  $\frac{a^2 + 2a}{5} \div \frac{2a + 4}{20}$   
 j)  $\frac{5ab - 15b}{4a - 12} \div \frac{6b^2}{a + b}$   
 l)  $\frac{a^3 + b^3}{a^3} \times \frac{5a + 5b}{a^2 + 2ab + b^2}$   
 n)  $\frac{3x + 2}{x^2 - 6x + 8} \times \frac{x - 2}{3x^2 + 8x + 4}$   
 p)  $\frac{4x^2 - 1}{3x^2 + 10x + 3} \div \frac{6x^2 + 5x + 1}{4x^2 + 7x - 3} \times \frac{9x^2 + 6x + 1}{8x^2 - 6x + 1}$   
 r)  $\frac{p^3 + q^3}{p^2} \times \frac{3p - 3q}{p^2 - q^2}$

3. Simplify (assume all denominators are non-zero):

a)  $\frac{x - 3}{3} - \frac{x + 5}{4}$   
 c)  $1 + \frac{3x - 4}{4} - \frac{x + 2}{3}$   
 e)  $\frac{12}{x - 12} - \frac{6}{x - 6}$

b)  $\frac{2x - 4}{9} - \frac{x - 3}{4} + 1$   
 d)  $\frac{11}{a + 11} + \frac{8}{a - 8}$   
 f)  $\frac{12}{r + 12} + \frac{8}{r - 8}$

$$g) \frac{2}{xy} + \frac{4}{xz} + \frac{3}{yz}$$

$$i) \frac{k+2}{k^2+2} - \frac{1}{k+2}$$

$$k) \frac{3}{p^2-4} + \frac{2}{(p-2)^2}$$

$$m) \frac{1}{m+n} + \frac{3mn}{m^3+n^3}$$

$$o) \frac{x^2-1}{3} \times \frac{1}{x-1} - \frac{1}{2}$$

$$q) \frac{1}{(x-1)^2} - \frac{2x}{x^3-1}$$

$$s) \frac{x^2-3x+9}{x^3+27} + \frac{x-2}{x^2+4x+3} - \frac{1}{x-2}$$

$$h) \frac{5}{t-2} - \frac{1}{t-3}$$

$$j) \frac{t+2}{3q} + \frac{t+1}{2q}$$

$$l) \frac{x}{x+y} + \frac{x^2}{y^2-x^2}$$

$$n) \frac{h}{h^3-f^3} - \frac{1}{h^2+hf+f^2}$$

$$p) \frac{x^2-2x+1}{(x-1)^3} - \frac{x^2+x+1}{x^3-1}$$

$$r) \frac{t^2+2t-8}{t^2+t-6} + \frac{1}{t^2-9} + \frac{t+1}{t-3}$$

$$t) \frac{1}{a^2-4ab+4b^2} + \frac{a^2+2ab+b^2}{a^3-8b^3} - \frac{1}{a^2-4b^2}$$

4. What are the restrictions in the following:

$$a) \frac{1}{x-2}$$

$$b) \frac{3x-9}{4x+4}$$

$$c) \frac{3}{x} - \frac{1}{x^2-1}$$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1a. 2DNW

1b. 2DNX

1c. 2DNY

1d. 2DNZ

1e. 2DP2

1f. 2DP3

1g. 2DP4

1h. 2DP5

1i. 2DP6

1j. 2DP7

1k. 2DP8

1l. 2DP9

1m. 2DPB

1n. 2DPC

1o. 2DPD

1p. 2DPF

1q. 2DPG

1r. 2DPH

1s. 2DPJ

1t. 2DPK

1u. 2DPM

1v. 2DPN

1w. 2DPP

2a. 2DPQ

2b. 2DPR

2c. 2DPS

2d. 2DPT

2e. 2DPV

2f. 2DPW

2g. 2DPX

2h. 2DPY

2i. 2DPZ

2j. 2DQ2

2k. 2DQ3

2l. 2DQ4

2m. 2DQ5

2n. 2DQ6

2o. 2DQ7

2p. 2DQ8

2q. 2DQ9

2r. 2DQB

3a. 2DQC

3b. 2DQD

3c. 2DQF

3d. 2DQG

3e. 2DQH

3f. 2DQJ

3g. 2DQK

3h. 2DQM

3i. 2DQN

3j. 2DQP

3k. 2DQQ

3l. 2DQR

3m. 2DQS

3n. 2DQT

3o. 2DQV

3p. 2DQW

3q. 2DQX

3r. 2DQY

3s. 2DQZ

3t. 2DR2

4a. 2DR3

4b. 2DR4

4c. 2DR5



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## 1.9 Chapter summary

EMAR

► See presentation: [2DR6](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

- - $\mathbb{N}$ : natural numbers are  $\{1; 2; 3; \dots\}$
  - $\mathbb{N}_0$ : whole numbers are  $\{0; 1; 2; 3; \dots\}$
  - $\mathbb{Z}$ : integers are  $\{\dots; -3; -2; -1; 0; 1; 2; 3; \dots\}$
- A rational number is any number that can be written as  $\frac{a}{b}$  where  $a$  and  $b$  are integers and  $b \neq 0$ .
- The following are rational numbers:
  - Fractions with both numerator and denominator as integers
  - Integers
  - Decimal numbers that terminate
  - Decimal numbers that recur (repeat)

- Irrational numbers are numbers that cannot be written as a fraction with the numerator and denominator as integers.
- If the  $n^{\text{th}}$  root of a number cannot be simplified to a rational number, it is called a surd.
- If  $a$  and  $b$  are positive whole numbers, and  $a < b$ , then  $\sqrt[n]{a} < \sqrt[n]{b}$ .
- A binomial is an expression with two terms.
- The product of two identical binomials is known as the square of the binomial.
- We get the difference of two squares when we multiply  $(ax + b)(ax - b)$
- Factorising is the opposite process of expanding the brackets.
- The product of a binomial and a trinomial is:

$$(A + B)(C + D + E) = A(C + D + E) + B(C + D + E)$$

- Taking out a common factor is the basic factorisation method.
- We often need to use grouping to factorise polynomials.
- To factorise a quadratic we find the two binomials that were multiplied together to give the quadratic.
- The sum of two cubes can be factorised as:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

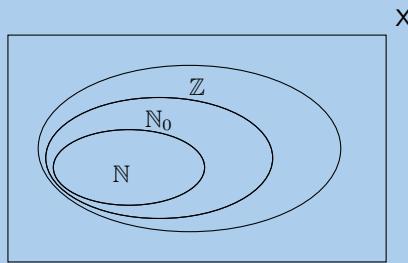
- The difference of two cubes can be factorised as:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

- We can simplify fractions by incorporating the methods we have learnt to factorise expressions.
- Only factors can be cancelled out in fractions, never terms.
- To add or subtract fractions, the denominators of all the fractions must be the same.

### End of chapter Exercise 1 – 11:

1. The figure here shows the Venn diagram for the special sets  $\mathbb{N}$ ,  $\mathbb{N}_0$  and  $\mathbb{Z}$ .



- a) Where does the number 2,13 belong in the diagram?
- b) In the following list, there are two false statements and one true statement. Which of the statements is **true**?
- Every natural number is an integer.
  - Every whole number is a natural number.
  - There are fractions in the integers.
2. State whether the following numbers are real, non-real or undefined.

a)  $-\sqrt{-5}$       b)  $\frac{\sqrt{8}}{0}$       c)  $-\sqrt{15}$       d)  $-\sqrt{7}$       e)  $\sqrt{-1}$       f)  $\sqrt{2}$

3. State whether each of the following numbers are rational or irrational.

a)  $\sqrt[3]{4}$

b)  $45\pi$

c)  $\sqrt{9}$

d)  $\sqrt[3]{8}$

4. If  $a$  is an integer,  $b$  is an integer and  $c$  is irrational, which of the following are rational numbers?

a)  $\frac{-b}{a}$

b)  $c \div c$

c)  $\frac{a}{c}$

d)  $\frac{1}{c}$

5. Consider the following list of numbers:

$$\sqrt[3]{26}; \frac{3}{2}; \sqrt{-24}; \sqrt{39}; 7,1\bar{1}; \pi^2; \frac{\pi}{2}; 7,12; -\sqrt{24}; \frac{\sqrt{2}}{0}; 3\pi; \sqrt{78}; 9; \pi$$

a) Which of the numbers are non-real numbers?

b) Without using a calculator, rank all the real numbers in ascending order.

c) Which of the numbers are irrational numbers?

d) Which of the numbers are rational numbers?

e) Which of the numbers are integers?

f) Which of the numbers are undefined?

6. Write each decimal as a simple fraction.

a) 0,12

b) 0,006

c)  $4,\overline{14}$

d) 1,59

e)  $12,27\dot{7}$

f)  $0,8\dot{2}$

g)  $7,\overline{36}$

7. Show that the decimal  $3,21\dot{1}\dot{8}$  is a rational number.

8. Write the following fractions as decimal numbers:

a)  $\frac{1}{18}$

b)  $1\frac{1}{2}$

9. Express  $0,\overline{78}$  as a fraction  $\frac{a}{b}$  where  $a, b \in \mathbb{Z}$  (show all working).

10. For each of the following numbers:

- write the next three digits;
- state whether the number is rational or irrational.

a) 1,11235...

b) 1,1

11. Write the following rational numbers to 2 decimal places.

a)  $\frac{1}{2}$

b) 1

c) 0,111111

d) 0,999999

12. Round off the following irrational numbers to 3 decimal places.

a) 3,141592654...

b) 1,618033989...

c) 1,41421356...

d) 2,71828182845904523536...

13. Round off the number 1523,00195593 to 4 decimal places.

14. Round off the number 1982,94028996 to 6 decimal places.

15. Round off the number 101,52378984 to 4 decimal places.

16. Use your calculator and write the following irrational numbers to 3 decimal places.

a)  $\sqrt{2}$

b)  $\sqrt{3}$

c)  $\sqrt{5}$

d)  $\sqrt{6}$

17. Use your calculator (where necessary) and write the following numbers to 5 decimal places. State whether the numbers are irrational or rational.

a)  $\sqrt{8}$

b)  $\sqrt{768}$

c)  $\sqrt{0,49}$

d)  $\sqrt{0,0016}$

e)  $\sqrt{0,25}$

f)  $\sqrt{36}$

g)  $\sqrt{1960}$

h)  $\sqrt{0,0036}$

i)  $-8\sqrt{0,04}$

j)  $5\sqrt{80}$

18. Round off:

a)  $\frac{\sqrt{2}}{2}$  to the nearest 2 decimal places.

b)  $\sqrt{14}$  to the nearest 3 decimal places.

19. Write the following irrational numbers to 3 decimal places and then write each one as a rational number to get an approximation of the irrational number.
- a) 3,141592654...      b) 1,618033989...  
 c) 1,41421356...      d) 2,71828182845904523536...
20. Determine between which two consecutive integers the following irrational numbers lie, without using a calculator.
- a)  $\sqrt{5}$       b)  $\sqrt{10}$       c)  $\sqrt{20}$       d)  $\sqrt{30}$       e)  $\sqrt[3]{5}$       f)  $\sqrt[3]{10}$   
 g)  $\sqrt[3]{20}$       h)  $\sqrt[3]{30}$       i)  $\sqrt{90}$       j)  $\sqrt{72}$       k)  $\sqrt[3]{58}$       l)  $\sqrt[3]{118}$
21. Estimate the following surds to the nearest 1 decimal place, without using a calculator.
- a)  $\sqrt{14}$       b)  $\sqrt{110}$       c)  $\sqrt{48}$       d)  $\sqrt{57}$
22. Expand the following products:
- a)  $(a + 5)^2$       b)  $(n + 12)^2$       c)  $(d - 4)^2$   
 d)  $(7w + 2)(7w - 2)$       e)  $(12q + 1)(12q - 1)$       f)  $-(-x - 2)(x + 2)$   
 g)  $(5k - 4)(5k + 4)$       h)  $(5f + 4)(2f + 2)$       i)  $(3n + 6)(6n + 5)$   
 j)  $(2g + 6)(g + 6)$       k)  $(4y + 1)(4y + 8)$       l)  $(d - 3)(7d + 2)$   
 m)  $(6z - 4)(z - 2)$       n)  $(5w - 11)^2$       o)  $(5s - 1)^2$   
 p)  $(3d - 8)^2$       q)  $5f^2(3f + 5) + 7f(3f^2 + 7)$       r)  $8d(4d^3 + 2) + 6d^2(7d^2 + 4)$   
 s)  $5x^2(2x + 2) + 7x(7x^2 + 7)$
23. Expand the following:
- a)  $(y^4 + 3y^2 + y)(y + 1)(y - 2)$       b)  $(x + 1)^2 - (x - 1)^2$   
 c)  $(x^2 + 2x + 1)(x^2 - 2x + 1)$       d)  $(4a - 3b)(16a^2 + 12ab + 9b^2)$   
 e)  $2(x + 3y)(x^2 - xy - y^2)$       f)  $(3a - 5b)(3a + 5b)(a^2 + ab - b^2)$   
 g)  $\left(y - \frac{1}{y}\right)\left(y + \frac{1}{y}\right)$       h)  $\left(\frac{a}{3} - \frac{3}{a}\right)\left(\frac{a}{3} + \frac{3}{a}\right)$   
 i)  $\frac{1}{3}(12x - 9y) + \frac{1}{6}(12x + 18y)$       j)  $(x + 2)(x - 2) - (x + 2)^2$
24. What is the value of  $e$  in  $(x - 4)(x + e) = x^2 - 16$ ?
25. In  $(x + 2)(x + k) = x^2 + bx + c$ :
- a) For which of these values of  $k$  will  $b$  be positive?  
 -6 ; -1 ; 0 ; 1 ; 6  
 b) For which of these values of  $k$  will  $c$  be positive?  
 -6 ; -1 ; 0 ; 1 ; 6  
 c) For what values of  $k$  will  $c$  be positive?  
 d) For what values of  $k$  will  $b$  be positive?
26. Answer the following:
- a) Expand:  $\left(3a - \frac{1}{2a}\right)^2$   
 b) Expand:  $\left(3a - \frac{1}{2a}\right)\left(9a^2 + \frac{3}{2} + \frac{1}{4a^2}\right)$   
 c) Given that  $3a - \frac{1}{2a} = 7$ , determine the value of  $27a^3 - \frac{1}{8a^3}$  without solving for  $a$ .

27. Solve by factorising:

a)  $17^2 - 15^2$       b)  $13^2 - 12^2$       c)  $120045^2 - 120035^2$       d)  $26^2 - 24^2$

28. Represent the following as a product of its prime factors:

a) 143      b) 168      c) 899      d) 99      e) 1599

29. Factorise:

a)  $a^2 - 9$       b)  $9b^2 - 81$   
c)  $m^2 - \frac{1}{9}$       d)  $5 - 5a^2b^6$   
e)  $16ba^4 - 81b$       f)  $a^2 - 10a + 25$   
g)  $16b^2 + 56b + 49$       h)  $-4b^2 - 144b^8 + 48b^5$   
i)  $16 - x^4$       j)  $7x^2 - 14x + 7xy - 14y$   
k)  $y^2 - 7y - 30$       l)  $1 - x - x^2 + x^3$   
m)  $-3(1 - p^2) + p + 1$       n)  $x^2 - 2x + 1 - y^4$   
o)  $4b(x^3 - 1) + x(1 - x^3)$       p)  $3m(v - 7) + 19(-7 + v)$   
q)  $3f(z + 3) + 19(3 + z)$       r)  $3p^3 - \frac{1}{9}$   
s)  $8x^6 - 125y^9$       t)  $(2 + p)^3 - 8(p + 1)^3$   
u)  $\frac{1}{3}a^3 - a^2b + 2a^2b - 6ab^2 + 3ab^2 - 9b^3$       v)  $6a^2 - 17a + 5$   
w)  $s^2 + 2s - 15$       x)  $16v + 24h + 2j^5v + 3j^5h$   
y)  $18h - 45g + 2m^3h - 5m^3g$       z)  $63d - 18s + 7u^2d - 2u^2s$

30. Factorise the following:

a)  $6a^2 + 14a + 8$       b)  $6g^2 - 15g - 9$   
c)  $125g^3 - r^3$       d)  $8r^3 + z^3$   
e)  $14m - 4n + 7jm - 2jn$       f)  $25d - 15m + 5yd - 3ym$   
g)  $g^3 - 27$       h)  $z^3 + 125$   
i)  $b^2 - (3a - 2b)^2$       j)  $9y^2 - (4x + 2y)^2$   
k)  $16x^6 - 3y^8$       l)  $\frac{1}{6}a^2 - 24b^4$   
m)  $4(a - 3) - 81x^2(a - 3)$       n)  $(2 + b)^2 - 11(2 + b) - 12$   
o)  $2x^2 + 7xy + 5y^2$       p)  $x^2 - 2xy - 15y^2$   
q)  $4x^4 + 11x^2 + 6$       r)  $6x^4 - 38x^2 + 40$   
s)  $9a^2x + 9a^2y + 27a^2 - b^2x - b^2y - 3b^2$       t)  $2(2y^2 - 5y) - 24$   
u)  $\frac{1}{2}x^3 - \frac{9}{2}x - 2x^2 + 18$       v)  $27r^3s^3 - 1$   
w)  $\frac{1}{125h^3} + r^3$       x)  $j(64n^3 - b^3) + k(64n^3 - b^3)$

31. Simplify the following:

a)  $(a - 2)^2 - a(a + 4)$       b)  $(5a - 4b)(25a^2 + 20ab + 16b^2)$   
c)  $(2m - 3)(4m^2 + 9)(2m + 3)$       d)  $(a + 2b - c)(a + 2b + c)$

e)  $\frac{m^2 + 11m + 18}{4(m^2 - 4)} \div \frac{3m^2 + 27m}{24m^2 - 48m}$

g)  $\frac{4 - b^2}{3b - 6}$

i)  $\frac{x^2 - 5x - 14}{3x + 6}$

k)  $\frac{a - 2}{a^2 + 4a + 3} \div \frac{(a - 1)(a + 1)}{a - 1} \times \frac{a^2 - 2a - 15}{a - 2}$

m)  $2 \div \frac{a + b}{a + 2b} \times \frac{b^2 - ba - 6a^2}{a^2 - 4b^2} \times \frac{a^2 - b - 2b^2}{3a - b}$

o)  $\frac{ny + nq + 8y + 8q}{y + q}$

q)  $\frac{2}{x} + \frac{x}{2} - \frac{2x}{3}$

s)  $\frac{x + 2}{2x^3} + 16$

u)  $\frac{1}{2}x + \frac{x - 2}{3} + 4$

w)  $\frac{b^2 + 6b + 9}{b^2 - 9} + \frac{b^2 - 6b + 8}{(b - 2)(b + 3)} + \frac{1}{b + 3}$

y)  $\frac{12}{z + 12} + \frac{5}{z - 5}$

f)  $\frac{t^2 + 9t + 18}{5(t^2 - 9)} \div \frac{4t^2 + 24t}{100t^2 - 300t}$

h)  $\frac{x^2 + 2x + 4}{x^3 - 8}$

j)  $\frac{d^2 + 23d + 132}{5(d^2 - 121)} \div \frac{4d^2 + 48d}{100d^2 - 1100d}$

l)  $\frac{a + 6}{a^2 + 12a + 11} \times \frac{a^2 + 14a + 33}{a + 3} \div \frac{a^3 + 216}{a + 1}$

n)  $\frac{st + sb + 31t + 31b}{t + b}$

p)  $\frac{p^2 - q^2}{p} \div \frac{p + q}{p^2 - pq}$

r)  $\frac{1}{a + 7} - \frac{a + 7}{a^2 - 49}$

t)  $\frac{1 - 2a}{4a^2 - 1} - \frac{a - 1}{2a^2 - 3a + 1} - \frac{1}{1 - a}$

v)  $\frac{1}{x^2 + 2x} + \frac{4x^2 - x - 3}{x^2 + 2x - 3}$

x)  $\frac{x^2 + 2x}{x^2 + x + 6} \times \frac{x^2 + 2x + 1}{x^2 + 3x + 2}$

z)  $\frac{11}{w - 11} - \frac{4}{w - 4}$

32. Show that  $(2x - 1)^2 - (x - 3)^2$  can be simplified to  $(x + 2)(3x - 4)$ .

33. What must be added to  $x^2 - x + 4$  to make it equal to  $(x + 2)^2$ ?

34. Evaluate  $\frac{x^3 + 1}{x^2 - x + 1}$  if  $x = 7,85$  without using a calculator. Show your work.

35. With what expression must  $(a - 2b)$  be multiplied to get a product of  $(a^3 - 8b^3)$ ?

36. With what expression must  $27x^3 + 1$  be divided to get a quotient of  $3x + 1$ ?

37. What are the restrictions on the following?

a)  $\frac{4}{3x^2 + 2x - 1}$

b)  $\frac{a}{3(b - a) + ab - a^2}$

For	more	exercises,	visit	<a href="http://www.everythingmaths.co.za">www.everythingmaths.co.za</a>	and	click	on	'Practise Maths'.
1.	<a href="#">2DR7</a>	<a href="#">2a. 2DR8</a>	<a href="#">2b. 2DR9</a>	<a href="#">2c. 2DRB</a>	<a href="#">2d. 2DRC</a>	<a href="#">2e. 2DRD</a>	<a href="#">2f. 2DRF</a>	
3a.	<a href="#">2DRG</a>	<a href="#">3b. 2DRH</a>	<a href="#">3c. 2DRJ</a>	<a href="#">3d. 2DRK</a>	<a href="#">4. 2DRM</a>	<a href="#">5. 2DRN</a>	<a href="#">6a. 2DRP</a>	
6b.	<a href="#">2DRQ</a>	<a href="#">6c. 2DRR</a>	<a href="#">6d. 2DRS</a>	<a href="#">6e. 2DRT</a>	<a href="#">6f. 2DRV</a>	<a href="#">6g. 2DRW</a>	<a href="#">7. 2DRX</a>	
8a.	<a href="#">2DRY</a>	<a href="#">8b. 2DRZ</a>	<a href="#">9. 2DS2</a>	<a href="#">10a. 2DS3</a>	<a href="#">10b. 2DS4</a>	<a href="#">11a. 2DS5</a>	<a href="#">11b. 2DS6</a>	
11c.	<a href="#">2DS7</a>	<a href="#">11d. 2DS8</a>	<a href="#">12a. 2DS9</a>	<a href="#">12b. 2DSB</a>	<a href="#">12c. 2DSC</a>	<a href="#">12d. 2DSD</a>	<a href="#">13. 2DSF</a>	
14.	<a href="#">2DSG</a>	<a href="#">15. 2DSH</a>	<a href="#">16a. 2DSJ</a>	<a href="#">16b. 2DSK</a>	<a href="#">16c. 2DSM</a>	<a href="#">16d. 2DSN</a>	<a href="#">17a. 2DSP</a>	
17b.	<a href="#">2DSQ</a>	<a href="#">17c. 2DSR</a>	<a href="#">17d. 2DSS</a>	<a href="#">17e. 2DST</a>	<a href="#">17f. 2DSV</a>	<a href="#">17g. 2DSW</a>	<a href="#">17h. 2DSX</a>	
17i.	<a href="#">2DSY</a>	<a href="#">17j. 2DSZ</a>	<a href="#">18a. 2DT2</a>	<a href="#">18b. 2DT3</a>	<a href="#">19a. 2DT4</a>	<a href="#">19b. 2DT5</a>	<a href="#">19c. 2DT6</a>	
19d.	<a href="#">2DT7</a>	<a href="#">20a. 2DT8</a>	<a href="#">20b. 2DT9</a>	<a href="#">20c. 2DTB</a>	<a href="#">20d. 2DTC</a>	<a href="#">20e. 2DTD</a>	<a href="#">20f. 2DTF</a>	
20g.	<a href="#">2DTG</a>	<a href="#">20h. 2DTH</a>	<a href="#">20i. 2DTJ</a>	<a href="#">20j. 2DTK</a>	<a href="#">20k. 2DTM</a>	<a href="#">20l. 2DTN</a>	<a href="#">21a. 2DTP</a>	
21b.	<a href="#">2DTQ</a>	<a href="#">21c. 2DTR</a>	<a href="#">21d. 2DTS</a>	<a href="#">22a. 2DTT</a>	<a href="#">22b. 2DTW</a>	<a href="#">22c. 2DTX</a>	<a href="#">22d. 2DTY</a>	
22e.	<a href="#">2DTZ</a>	<a href="#">22f. 2DV2</a>	<a href="#">22g. 2DV3</a>	<a href="#">22h. 2DV4</a>	<a href="#">22i. 2DV5</a>	<a href="#">22j. 2DV6</a>	<a href="#">22k. 2DV7</a>	
22l.	<a href="#">2DV8</a>	<a href="#">22m. 2DV9</a>	<a href="#">22n. 2DVB</a>	<a href="#">22o. 2DVC</a>	<a href="#">22p. 2DVD</a>	<a href="#">22q. 2DVF</a>	<a href="#">22r. 2DVG</a>	
22s.	<a href="#">2DVH</a>	<a href="#">23a. 2DVJ</a>	<a href="#">23b. 2DVK</a>	<a href="#">23c. 2DVM</a>	<a href="#">23d. 2DVN</a>	<a href="#">23e. 2DVP</a>	<a href="#">23f. 2DVQ</a>	
23g.	<a href="#">2DVR</a>	<a href="#">23h. 2DVS</a>	<a href="#">23i. 2DVT</a>	<a href="#">23j. 2DVV</a>	<a href="#">24. 2DVW</a>	<a href="#">25. 2DVX</a>	<a href="#">26. 2DVY</a>	

27a. <a href="#">2DVZ</a>	27b. <a href="#">2DW2</a>	27c. <a href="#">2DW3</a>	27d. <a href="#">2DW4</a>	28a. <a href="#">2DW5</a>	28b. <a href="#">2DW6</a>	28c. <a href="#">2DW7</a>
28d. <a href="#">2DW8</a>	28e. <a href="#">2DW9</a>	29a. <a href="#">2DWB</a>	29b. <a href="#">2DWC</a>	29c. <a href="#">2DWD</a>	29d. <a href="#">2DWF</a>	29e. <a href="#">2DWG</a>
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29m. <a href="#">2DWR</a>	29n. <a href="#">2DWS</a>	29o. <a href="#">2DWT</a>	29p. <a href="#">2DWV</a>	29q. <a href="#">2DWW</a>	29r. <a href="#">2DWX</a>	29s. <a href="#">2DWY</a>
29t. <a href="#">2DWZ</a>	29u. <a href="#">2DX2</a>	29v. <a href="#">2DX3</a>	29w. <a href="#">2DX4</a>	29x. <a href="#">2DX5</a>	29y. <a href="#">2DX6</a>	29z. <a href="#">2DX7</a>
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31l. <a href="#">2DYH</a>	31m. <a href="#">2DYJ</a>	31n. <a href="#">2DYK</a>	31o. <a href="#">2DYM</a>	31p. <a href="#">2DYN</a>	31q. <a href="#">2DYP</a>	31r. <a href="#">2DYQ</a>
31s. <a href="#">2DYS</a>	31t. <a href="#">2DYT</a>	31u. <a href="#">2DYV</a>	31v. <a href="#">2DYY</a>	31w. <a href="#">2DYW</a>	31x. <a href="#">2DZX</a>	31y. <a href="#">2DYY</a>
31z. <a href="#">2DYZ</a>	32. <a href="#">2DZ2</a>	33. <a href="#">2DZ3</a>	34. <a href="#">2DZ4</a>	35. <a href="#">2DZ5</a>	36. <a href="#">2DZ6</a>	37a. <a href="#">2DZ7</a>
37b. <a href="#">2DZ8</a>						



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# CHAPTER



## *Exponents*

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### 2.1 Introduction

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Exponential notation is a short way of writing the same number multiplied by itself many times. This is very useful in everyday life. You may have heard someone describe the size of an area in *square* metres or *square* kilometres. For example, the largest radio telescope in the world is being built in South Africa. The telescope is called the *square kilometre array*, or SKA. This is because the telescope will occupy an area of 1 kilometre by 1 kilometre or 1 kilometre squared.

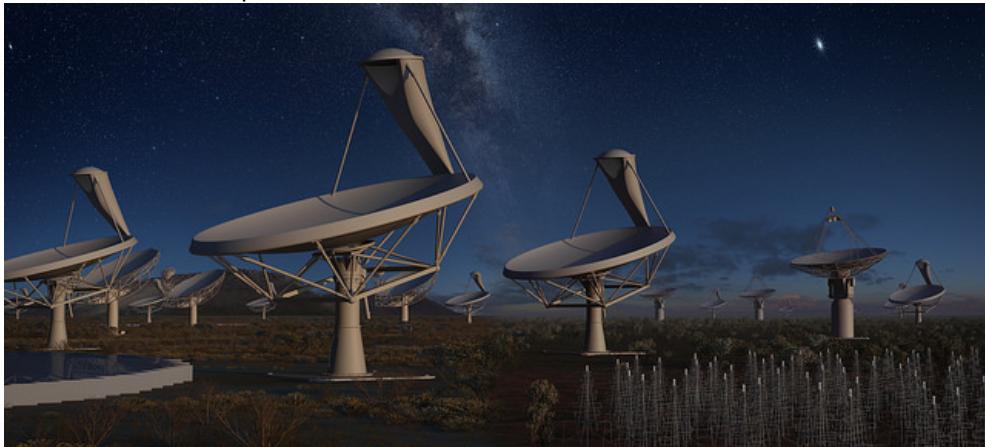


Figure 2.1: Antennas from the Square Kilometre Array (artist's concept).

Exponents are also very useful to describe very large and very small numbers. For example, the SKA will be detecting incredibly weak signals from objects which are so far away that to write out the strength of the signal or the number of kilometres away in full would be impractical. Outside of astronomy, exponents are used by many other professions such as computer programmers, engineers, economists, financial analysts, biologists and demographers.

**VISIT:**

If you want to know more about how exponents are used then take a look at the following presentation.

► See presentation: [2DZ9](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

You have already been introduced to exponents and exponent laws in previous grades. Remember that exponents can also be called indices or powers. Exponential notation is as follows:

base  $\leftarrow a$   $n \rightarrow$  exponent or index

For any real number  $a$  and natural number  $n$ , we can write  $a$  multiplied by itself  $n$  times as:  $a^n$ .

Remember the following identities:

1.  $a^n = a \times a \times a \times \cdots \times a$  ( $n$  times) ( $a \in \mathbb{R}, n \in \mathbb{N}$ )
2.  $a^0 = 1$  ( $a \neq 0$  because  $0^0$  is undefined)
3.  $a^{-n} = \frac{1}{a^n}$  ( $a \neq 0$  because  $\frac{1}{0}$  is undefined)
4. Similarly,  $\frac{1}{a^{-n}} = a^n$

**VISIT:**

Interested in finding out why raising a real number to the power of zero is one? Try work it out for yourself. If you get stuck, you can see an example of how to show this is true at this link.

► See video: [2DZB](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

Look at the following examples to see these identities in action:

1.  $3 \times 3 = 3^2 = 9$
2.  $5 \times 5 \times 5 \times 5 = 5^4$
3.  $p \times p \times p = p^3$
4.  $(3^x)^0 = 1$
5.  $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$
6.  $\frac{1}{5^{-x}} = 5^x$

**NOTE:**

If your final answer is easier to work out without a calculator, then write it out in full - not in exponential notation, as in examples 1 and 5.

**NOTE:**

It is convention to write your final answer with positive exponents.

In this chapter, we will revise the exponent laws and use these laws to simplify and solve more complex expressions and equations.

**VISIT:**

To revise what exponents are you can watch the following video.

► See video: [2DZC](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## 2.2 Revision of exponent laws

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There are several laws we can use to make working with exponential numbers easier. Some of these laws might have been done in earlier grades, but we list all the laws here for easy reference:

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $(a^m)^n = a^{mn}$

where  $a > 0, b > 0$  and  $m, n \in \mathbb{R}$

**VISIT:**

The following two videos explain the exponent laws.

Part 1:

► See video: [2DZD](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

Part 2:

► See video: [2DZF](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## Worked example 1: Applying the exponential laws

### QUESTION

Simplify:

$$1. 2^{3x} \times 2^{4x}$$

$$2. \frac{4x^3}{2x^5}$$

$$3. \frac{12p^2t^5}{3pt^3}$$

$$4. (3x)^2$$

$$5. (3^4 5^2)^3$$

$$6. 6p^0 \times (7p)^0$$

$$7. \left(\frac{2xp}{6x^2}\right)^3$$

$$8. (2^{-2})^{2x+1}$$

### SOLUTION

$$1. 2^{3x} \times 2^{4x} = 2^{3x+4x} = 2^{7x}$$

$$2. \frac{4x^3}{2x^5} = 2x^{3-5} = 2x^{-2} = \frac{2}{x^2}$$

$$3. \frac{12p^2t^5}{3pt^3} = 4p^{(2-1)}t^{(5-3)} = 4pt^2$$

$$4. (3x)^2 = 3^2x^2 = 9x^2$$

$$5. (3^4 \times 5^2)^3 = 3^{(4 \times 3)} \times 5^{(2 \times 3)} = 3^{12} \times 5^6$$

$$6. 6p^0 \times (7p)^0 = 6(1) \times 1 = 6$$

$$7. \left(\frac{2xp}{6x^2}\right)^3 = \left(\frac{p}{3x}\right)^3 = \frac{p^3}{27x^3}$$

$$8. (2^{-2})^{2x+1} = 2^{-2(2x+1)} = 2^{-4x-2}$$

### NOTE:

When you have a fraction that is one term over one term, use the method of Finding Prime Bases - in other words use prime factorisation on the bases.

## Worked example 2: Exponential expressions

### QUESTION

Simplify:  $\frac{2^{2n} \times 4^n \times 2}{16^n}$

### SOLUTION

#### Step 1: Change the bases to prime numbers

At first glance it appears that we cannot simplify this expression. However, if we reduce the bases to prime bases, then we can apply the exponent laws.

$$\frac{2^{2n} \times 4^n \times 2}{16^n} = \frac{2^{2n} \times (2^2)^n \times 2^1}{(2^4)^n}$$

## Step 2: Simplify the exponents

$$\begin{aligned}&= \frac{2^{2n} \times 2^{2n} \times 2^1}{2^{4n}} \\&= \frac{2^{2n+2n+1}}{2^{4n}} \\&= \frac{2^{4n+1}}{2^{4n}} \\&= 2^{4n+1-(4n)} \\&= 2\end{aligned}$$

## Worked example 3: Exponential expressions

### QUESTION

---

Simplify:

$$\frac{5^{2x-1} \cdot 9^{x-2}}{15^{2x-3}}$$

### SOLUTION

---

#### Step 1: Change the bases to prime numbers

$$\begin{aligned}\frac{5^{2x-1} \cdot 9^{x-2}}{15^{2x-3}} &= \frac{5^{2x-1} \cdot (3^2)^{x-2}}{(5 \times 3)^{2x-3}} \\&= \frac{5^{2x-1} \cdot 3^{2x-4}}{5^{2x-3} \cdot 3^{2x-3}}\end{aligned}$$

#### Step 2: Subtract the exponents (same base)

$$\begin{aligned}&= 5^{(2x-1)-(2x-3)} \times 3^{(2x-4)-(2x-3)} \\&= 5^{2x-1-2x+3} \times 3^{2x-4-2x+3} \\&= 5^2 \times 3^{-1}\end{aligned}$$

#### Step 3: Write the answer as a fraction

$$\begin{aligned}&= \frac{5^2}{3} \\&= \frac{25}{3}\end{aligned}$$

**NOTE:**

When working with exponents, all the laws of operation for algebra apply.

**Worked example 4: Simplifying by taking out a common factor****QUESTION**

Simplify:

$$\frac{2^t - 2^{t-2}}{3 \cdot 2^t - 2^t}$$

**SOLUTION****Step 1: Simplify to a form that can be factorised**

For each of the exponent laws we can “undo” the law - in other words we can work backwards. For this expression we can reverse the multiplication law to write  $2^{t-2}$  as  $2^t \cdot 2^{-2}$ .

$$\frac{2^t - 2^{t-2}}{3 \cdot 2^t - 2^t} = \frac{2^t - (2^t \cdot 2^{-2})}{3 \cdot 2^t - 2^t}$$

**Step 2: Take out a common factor**

$$= \frac{2^t (1 - 2^{-2})}{2^t (3 - 1)}$$

**Step 3: Cancel the common factor and simplify**

$$\begin{aligned} &= \frac{1 - 2^{-2}}{3 - 1} \\ &= \frac{1 - \frac{1}{4}}{2} \\ &= \frac{\frac{3}{4}}{\frac{1}{2}} \\ &= \frac{3}{8} \end{aligned}$$

**NOTE:**

When you have a fraction that has more than one term in the numerator or denominator, change to prime bases if necessary and then factorise.

**Worked example 5: Simplifying using difference of two squares****QUESTION**

Simplify:

$$\frac{9^x - 1}{3^x + 1}$$

## SOLUTION

### Step 1: Change the bases to prime numbers

$$\begin{aligned}\frac{9^x - 1}{3^x + 1} &= \frac{(3^2)^x - 1}{3^x + 1} \\ &= \frac{3^{2x} - 1}{3^x + 1} \quad \text{Recognise that } 3^{2x} = (3^x)^2\end{aligned}$$

### Step 2: Factorise using the difference of squares

$$= \frac{(3^x - 1)(3^x + 1)}{3^x + 1}$$

### Step 3: Cancel the common factor and simplify

$$= 3^x - 1$$

### Exercise 2 – 1:

Simplify without using a calculator:

1.  $16^0$

2.  $16a^0$

3.  $11^{9x} \times 11^{2x}$

4.  $10^{6x} \times 10^{2x}$

5.  $(6c)^3$

6.  $(5n)^3$

7.  $\frac{2^{-2}}{3^2}$

8.  $\frac{5}{2^{-3}}$

9.  $\left(\frac{2}{3}\right)^{-3}$

10.  $\frac{a^2}{a^{-1}}$

11.  $\frac{xy^{-3}}{x^4y}$

12.  $x^2x^{3t+1}$

13.  $3 \times 3^{2a} \times 3^2$

14.  $\frac{2^{m+20}}{2^{m+20}}$

15.  $\frac{2^{x+4}}{2^{x+3}}$

16.  $(2a^4)(3ab^2)$

17.  $(7m^4n)(8m^6n^8)$

18.  $2(-a^7b^8)(-4a^3b^6)(-9a^6b^2)$

19.  $(-9x^3y^6) \left(\frac{1}{9}x^8y^7\right) \left(\frac{1}{5}x^3y^6\right)$

20.  $\frac{a^{3x}}{a^x}$

21.  $\frac{20x^{10}a^4}{4x^9a^3}$

22.  $\frac{18c^{10}p^8}{9c^6p^5}$

23.  $\frac{6m^8a^{10}}{2m^3a^5}$

24.  $3^{12} \div 3^9$

25.  $\frac{7(a^3)^3}{a^7}$

26.  $\frac{9(ab^4)^8}{a^3b^5}$

27.  $\frac{2^2}{6^2}$

28.  $\left(\frac{a^6}{b^7}\right)^5$

29.  $(2t^4)^3$

30.  $(3^{n+3})^2$

31.  $\frac{3^n9^{n-3}}{27^{n-1}}$

32.  $\frac{13^c + 13^{c+2}}{3 \times 13^c - 13^c}$

33.  $\frac{3^{5x} \times 81^{5x} \times 3^3}{9^{8x}}$

$$34. \frac{16^x - 144^b}{4^x - 12^b}$$

$$37. \frac{9^3 \times 20^2}{4 \times 5^2 \times 3^5}$$

$$35. \frac{5^{2y-3} 2^{4y+4}}{10^{-5y+5}}$$

$$38. \frac{7^b + 7^{b-2}}{4 \times 7^b + 3 \times 7^b}$$

$$36. \frac{6^4 \times 12^3 \times 4^5}{30^3 \times 3^6}$$

$$39. \frac{12^y - 96^y}{3^y + 6^y}$$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |                          |                          |                          |                          |                          |                          |                          |                          |
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| 1. <a href="#">2DZG</a>  | 2. <a href="#">2DZH</a>  | 3. <a href="#">2DZJ</a>  | 4. <a href="#">2DZK</a>  | 5. <a href="#">2DZM</a>  | 6. <a href="#">2DZN</a>  | 7. <a href="#">2DZP</a>  | 8. <a href="#">2DZQ</a>  |
| 9. <a href="#">2DZR</a>  | 10. <a href="#">2DZS</a> | 11. <a href="#">2DZT</a> | 12. <a href="#">2DZV</a> | 13. <a href="#">2DZW</a> | 14. <a href="#">2DZX</a> | 15. <a href="#">2DZY</a> | 16. <a href="#">2DZZ</a> |
| 17. <a href="#">2F22</a> | 18. <a href="#">2F23</a> | 19. <a href="#">2F24</a> | 20. <a href="#">2F25</a> | 21. <a href="#">2F26</a> | 22. <a href="#">2F27</a> | 23. <a href="#">2F28</a> | 24. <a href="#">2F29</a> |
| 25. <a href="#">2F2B</a> | 26. <a href="#">2F2C</a> | 27. <a href="#">2F2D</a> | 28. <a href="#">2F2F</a> | 29. <a href="#">2F2G</a> | 30. <a href="#">2F2H</a> | 31. <a href="#">2F2J</a> | 32. <a href="#">2F2K</a> |
| 33. <a href="#">2F2M</a> | 34. <a href="#">2F2N</a> | 35. <a href="#">2F2P</a> | 36. <a href="#">2F2Q</a> | 37. <a href="#">2F2R</a> | 38. <a href="#">2F2S</a> | 39. <a href="#">2F2T</a> |                          |



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## 2.3 Rational exponents

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We can also apply the exponent laws to expressions with rational exponents.

### Worked example 6: Simplifying rational exponents

#### QUESTION

Simplify:

$$2x^{\frac{1}{2}} \times 4x^{-\frac{1}{2}}$$

#### SOLUTION

$$\begin{aligned} 2x^{\frac{1}{2}} \times 4x^{-\frac{1}{2}} &= 8x^{\frac{1}{2} - \frac{1}{2}} \\ &= 8x^0 \\ &= 8(1) \\ &= 8 \end{aligned}$$

### Worked example 7: Simplifying rational exponents

#### QUESTION

Simplify:

$$(0,008)^{\frac{1}{3}}$$

## SOLUTION

### Step 1: Write as a fraction and simplify

$$\begin{aligned}(0,008)^{\frac{1}{3}} &= \left(\frac{8}{1000}\right)^{\frac{1}{3}} \\&= \left(\frac{1}{125}\right)^{\frac{1}{3}} \\&= \left(\frac{1}{5^3}\right)^{\frac{1}{3}} \\&= \frac{1^{\frac{1}{3}}}{5^{(3 \cdot \frac{1}{3})}} \\&= \frac{1}{5}\end{aligned}$$

### VISIT:

Extension: the following video provides a summary of all the exponent rules and rational exponents.  
► See video: [2F2V](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Exercise 2 – 2:

Simplify without using a calculator:

1.  $t^{\frac{1}{4}} \times 3t^{\frac{7}{4}}$
2.  $\frac{16x^2}{(4x^2)^{\frac{1}{2}}}$
3.  $(0,25)^{\frac{1}{2}}$
4.  $(27)^{-\frac{1}{3}}$
5.  $(3p^2)^{\frac{1}{2}} \times (3p^4)^{\frac{1}{2}}$
6.  $12(a^4b^8)^{\frac{1}{2}} \times (512a^3b^3)^{\frac{1}{3}}$
7.  $((-2)^4a^6b^2)^{\frac{1}{2}}$
8.  $(a^{-2}b^6)^{\frac{1}{2}}$
9.  $(16x^{12}b^6)^{\frac{1}{3}}$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2F2W](#)
2. [2F2X](#)
3. [2F2Y](#)
4. [2F2Z](#)
5. [2F32](#)
6. [2F33](#)
7. [2F34](#)
8. [2F35](#)
9. [2F36](#)



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Exponential equations have the unknown variable in the exponent. Here are some examples:

$$3^{x+1} = 9$$

$$5^t + 3 \times 5^{t-1} = 400$$

If we can write a single term with the same base on each side of the equation, we can equate the exponents. This is one method to solve exponential equations.

**Important:** if  $a > 0$  and  $a \neq 1$  then:

$$a^x = a^y$$

then  $x = y$  (same base)

Also notice that if  $a = 1$ , then  $x$  and  $y$  can be different.

### Worked example 8: Equating exponents

#### QUESTION

---

Solve for  $x$ :  $3^{x+1} = 9$ .

#### SOLUTION

---

**Step 1: Change the bases to prime numbers**

$$3^{x+1} = 3^2$$

**Step 2: The bases are the same so we can equate exponents**

$$x + 1 = 2$$

$$\therefore x = 1$$

### Worked example 9: Equating exponents

#### QUESTION

---

Solve for  $t$ :  $3^t = 1$ .

#### SOLUTION

---

**Step 1: Solve for  $t$**

We know from the exponent identities that  $a^0 = 1$ , therefore:

$$3^t = 1$$

$$3^t = 3^0$$

$$\therefore t = 0$$

## Worked example 10: Solving equations by taking out a common factor

### QUESTION

Solve for  $t$ :  $5^t + 3 \cdot 5^{t+1} = 400$ .

### SOLUTION

#### Step 1: Rewrite the expression

$$5^t + 3(5^t \cdot 5) = 400$$

#### Step 2: Take out a common factor

$$5^t(1 + 3 \cdot 5) = 400$$

$$5^t(1 + 15) = 400$$

#### Step 3: Simplify

$$5^t(16) = 400$$

$$5^t = 25$$

#### Step 4: Change the bases to prime numbers

$$5^t = 5^2$$

#### Step 5: The bases are the same so we can equate exponents

$$\therefore t = 2$$

## Worked example 11: Solving equations by factorising a trinomial

### QUESTION

Solve for  $x$ :

$$3^{2x} - 80 \cdot 3^x - 81 = 0$$

### SOLUTION

#### Step 1: Factorise the trinomial

$$(3^x - 81)(3^x + 1) = 0$$

**Step 2: Solve for  $x$** 

$3^x = 81$  or  $3^x = -1$ . However  $3^x = -1$  is undefined, so:

$$\begin{aligned}3^x &= 81 \\3^x &= 3^4 \\x &= 4\end{aligned}$$

Therefore  $x = 4$

**Worked example 12: Solving equations by factorising a trinomial****QUESTION**

Solve for  $p$ :

$$p - 13p^{\frac{1}{2}} + 36 = 0$$

**SOLUTION****Step 1: Rewrite the equation**

We notice that  $(p^{\frac{1}{2}})^2 = p$  so we can rewrite the equation as:

$$(p^{\frac{1}{2}})^2 - 13p^{\frac{1}{2}} + 36 = 0$$

**Step 2: Factorise as a trinomial**

$$(p^{\frac{1}{2}} - 9)(p^{\frac{1}{2}} - 4) = 0$$

**Step 3: Solve to find both roots**

$$\begin{array}{lll}p^{\frac{1}{2}} - 9 = 0 & \text{or} & p^{\frac{1}{2}} - 4 = 0 \\p^{\frac{1}{2}} = 9 & & p^{\frac{1}{2}} = 4 \\(p^{\frac{1}{2}})^2 = (9)^2 & & (p^{\frac{1}{2}})^2 = (4)^2 \\p = 81 & & p = 16\end{array}$$

Therefore  $p = 81$  or  $p = 16$ .

**Worked example 13: Solving equations by factorisation****QUESTION**

Solve for  $x$ :

$$2^x - 2^{4-x} = 0$$

## SOLUTION

### Step 1: Rewrite the equation

In order to get the equation into a form which we can factorise, we need to rewrite the equation:

$$\begin{aligned}2^x - 2^{4-x} &= 0 \\2^x - 2^4 \cdot 2^{-x} &= 0 \\2^x - \frac{2^4}{2^x} &= 0\end{aligned}$$

Now eliminate the fraction by multiplying both sides of the equation by the denominator,  $2^x$ .

$$\begin{aligned}\left(2^x - \frac{2^4}{2^x}\right) \times 2^x &= 0 \times 2^x \\2^{2x} - 16 &= 0\end{aligned}$$

### Step 2: Factorise the equation

Now that we have rearranged the equation, we can see that we are left with a difference of two squares. Therefore:

$$\begin{aligned}2^{2x} - 16 &= 0 \\(2^x - 4)(2^x + 4) &= 0 \\2^x = 4 &\quad 2^x \neq -4 \quad (\text{a positive integer with an exponent is always positive}) \\2^x = 2^2 = 4 & \\x = 2 &\end{aligned}$$

Therefore  $x = 2$ .

### Exercise 2 – 3:

1. Solve for the variable:

- |                               |   |
|-------------------------------|---|
| a) $2^{x+5} = 32$             | b) $5^{2x+2} = \frac{1}{125}$                   |
| c) $64^{y+1} = 16^{2y+5}$     | d) $3^{9x-2} = 27$                              |
| e) $25 = 5^{z-4}$             | f) $-\frac{1}{2} \cdot 6^{\frac{m}{2}+3} = -18$ |
| g) $81^{k+2} = 27^{k+4}$      | h) $25^{1-2x} - 5^4 = 0$                        |
| i) $27^x \times 9^{x-2} = 1$  | j) $2^t + 2^{t+2} = 40$                         |
| k) $(7^x - 49)(3^x - 27) = 0$ | l) $(2 \cdot 2^x - 16)(3^{x+1} - 9) = 0$        |
| m) $(10^x - 1)(3^x - 81) = 0$ | n) $2 \times 5^{2-x} = 5 + 5^x$                 |
| o) $9^m + 3^{3-2m} = 28$      | p) $y - 2y^{\frac{1}{2}} + 1 = 0$               |
| q) $4^{x+3} = 0,5$            | r) $2^a = 0,125$                                |

$$s) 10^x = 0,001$$

$$u) \frac{8^x - 1}{2^x - 1} = 8.2^x + 9$$

$$t) 2^{x^2-2x-3} = 1$$

$$v) \frac{27^x - 1}{9^x + 3^x + 1} = -\frac{8}{9}$$

2. The growth of algae can be modelled by the function  $f(t) = 2^t$ . Find the value of  $t$  such that  $f(t) = 128$ .

3. Use trial and error to find the value of  $x$  correct to 2 decimal places

$$2^x = 7$$

4. Use trial and error to find the value of  $x$  correct to 2 decimal places

$$5^x = 11$$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1a. [2F37](#)    1b. [2F38](#)    1c. [2F39](#)    1d. [2F3B](#)    1e. [2F3C](#)    1f. [2F3D](#)

1g. [2F3F](#)    1h. [2F3G](#)    1i. [2F3H](#)    1j. [2F3J](#)    1k. [2F3K](#)    1l. [2F3M](#)

1m. [2F3N](#)    1n. [2F3P](#)    1o. [2F3Q](#)    1p. [2F3R](#)    1q. [2F3S](#)    1r. [2F3T](#)

1s. [2F3V](#)    1t. [2F3W](#)    1u. [2F3X](#)    1v. [2F3Y](#)    2. [2F3Z](#)    3. [2F42](#)

4. [2F43](#)



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## 2.5 Summary

EMAW

► See presentation: [2F44](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

• Exponential notation means writing a number as  $a^n$  where  $n$  is any natural number and  $a$  is any real number.

•  $a$  is the base and  $n$  is the exponent or index.

• Definition:

–  $a^n = a \times a \times \cdots \times a$  ( $n$  times)

–  $a^0 = 1$ , if  $a \neq 0$

–  $a^{-n} = \frac{1}{a^n}$ , if  $a \neq 0$

–  $\frac{1}{a^{-n}} = a^n$ , if  $a \neq 0$

• The laws of exponents:

–  $a^m \times a^n = a^{m+n}$

–  $\frac{a^m}{a^n} = a^{m-n}$

–  $(ab)^n = a^n b^n$

–  $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

–  $(a^m)^n = a^{mn}$

• When simplifying expressions with exponents, we can reduce the bases to prime bases or factorise.

• When solving equations with exponents, we can apply the rule that if  $a^x = a^y$  then  $x = y$ ; or we can factorise the expressions.

1. Simplify:

a)  $(8x)^3$

d)  $15^{3x} \times 15^{12x}$

g)  $\left(\frac{1}{7}a^2b^9\right)(6a^6b^2)(-3a^7b)$

j)  $\frac{2(x^4)^3}{x^{12}}$

m)  $\frac{6^{5p}}{9^p}$

p)  $\frac{5^{b-3}}{5^{b+1}}$

s)  $\frac{3^3}{9^3}$

v)  $\left(\frac{2x^{2a}}{y^{-b}}\right)^3$

y)  $\frac{(-3)^{-3}(-3)^2}{(-3)^{-4}}$

b)  $t^3 \times 2t^0$

e)  $\frac{7^{y+7}}{7^{y+6}}$

h)  $(b^{k+1})^k$

k)  $\frac{a^6b^5}{7(a^8b^3)^2}$

n)  $m^{-2t} \times (3m^t)^3$

q)  $\frac{2^{a-2}3^{a+3}}{6^a}$

t)  $\frac{x^{-1}}{x^4y^{-2}}$

w)  $\frac{2^{3x-1}8^{x+1}}{4^{2x-2}}$

z)  $(3^{-1} + 2^{-1})^{-1}$

c)  $5^{2x+y} \times 5^{3(x+z)}$

f)  $3(d^4)(7d^3)$

i)  $\frac{24c^8m^7}{6c^2m^5}$

l)  $\left(\frac{a^7}{b^4}\right)^2$

o)  $\frac{3x^{-3}}{(3x)^2}$

r)  $\frac{3^n9^{n-3}}{27^{n-1}}$

u)  $\frac{(-1)^4}{(-2)^{-3}}$

x)  $\frac{6^{2x}11^{2x}}{22^{2x-1}3^{2x}}$

2. Simplify:

a)  $\frac{9^{n-1} \cdot 27^{3-2n}}{81^{2-n}}$

d)  $\frac{2^{3p} + 1}{2^p + 1}$

g)  $\frac{13^a + 13^{a+2}}{6 \times 13^a - 13^a}$

j)  $\frac{11^{-4c-4}4^{4c-3}}{22^{-6c-2}}$

m)  $(0,81)^{\frac{1}{2}}$

p)  $\frac{a^{-1} - b^{-1}}{a - b}$

s)  $(a^{\frac{1}{2}} + a^{-\frac{1}{2}})^2 - (a^{\frac{1}{2}} - a^{-\frac{1}{2}})^2$

b)  $\frac{2^{3n+2} \cdot 8^{n-3}}{4^{3n-2}}$

e)  $(a^{10}b^6)^{\frac{1}{2}}$

h)  $\frac{3^{8z} \times 27^{8z} \times 3^2}{9^{6z}}$

k)  $\frac{12^4 \times 2^4}{16^6 \times 10}$

n)  $12(a^{10}b^{20})^{\frac{1}{5}} \times (729a^{12}b^{15})^{\frac{1}{3}}$

q)  $\left((x^{36})^{\frac{1}{2}}\right)^{\frac{1}{3}}$

c)  $\frac{3^{t+3} + 3^t}{2 \times 3^t}$

f)  $(9x^8y^4)^{\frac{1}{2}}$

i)  $\frac{121^b - 16^p}{11^b + 4^p}$

l)  $\frac{5^6 \times 3^{16} \times 2^7}{10^8 \times 9^6}$

o)  $2(p^{30}q^{20})^{\frac{1}{5}} \times (1331p^{12}q^6)^{\frac{1}{3}}$

r)  $\left(\frac{2}{3}\right)^{x+y} \cdot \left(\frac{3}{2}\right)^{x-y}$

3. Solve:

a)  $3^x = \frac{1}{27}$

c)  $5^{t-1} = 1$

e)  $-\frac{64}{3} = -\frac{4}{3}2^{-\frac{e}{3}+1}$

g)  $2^{m+1} = (0,5)^{m-2}$

i)  $z^{\frac{3}{2}} = 64$

b)  $121 = 11^{m-1}$

d)  $2 \times 7^{3x} = 98$

f)  $-\frac{1}{2}6^{-n-3} = -18$

h)  $3^{y+1} = 5^{y+1}$

j)  $16x^{\frac{1}{2}} - 4 = 0$

$$k) m^0 + m^{-1} = 0$$

$$l) t^{\frac{1}{2}} - 3t^{\frac{1}{4}} + 2 = 0$$

$$m) 3^p + 3^p + 3^p = 27$$

$$n) k^{-1} - 7k^{-\frac{1}{2}} - 18 = 0$$

$$o) x^{\frac{1}{2}} + 3x^{\frac{1}{4}} - 18 = 0$$

$$p) \frac{16^x - 1}{4^2 x + 1} = 3$$

$$q) (2^x - 8)(3^x - 9) = 0$$

$$r) (6^x - 36)(16 - 4^x) = 0$$

$$s) 5 \cdot 2^{x^2+1} = 20$$

$$t) 27^{x-2} = 9^{2x+1}$$

$$u) \frac{8^x - 1}{2^x - 1} = 7$$

$$v) \frac{35^x}{5^x} = \frac{1}{7}$$

$$w) \frac{a^{3x} \cdot a^{\frac{1}{x}}}{a^{-4}} = 1$$

$$x) 2x^{\frac{1}{2}} + 1 = -x$$

4. Use trial and error to find the value of  $x$  correct to 2 decimal places

$$4^x = 44$$

5. Use trial and error to find the value of  $x$  correct to 2 decimal places

$$3^x = 30$$

6. Explain why the following statements are false:

$$a) \frac{1}{a^{-1} + b^{-1}} = a + b$$

$$b) (a + b)^2 = a^2 + b^2$$

$$c) \left(\frac{1}{a^2}\right)^{\frac{1}{3}} = a^{\frac{2}{3}}$$

$$d) 2 \cdot 3^x = 6^x$$

$$e) x^{-\frac{1}{2}} = \frac{1}{-x^{\frac{1}{2}}}$$

$$f) (3x^4y^2)^3 = 3x^{12}y^6$$

7. If  $2^{2013} \cdot 5^{2015}$  is written out in full how many digits will there be?

$$8. \text{ Prove that } \frac{2^{n+1} + 2^n}{2^n - 2^{n-1}} = \frac{3^{n+1} + 3^n}{3^n - 3^{n-1}}$$

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1m. <a href="#">2F4K</a>	1n. <a href="#">2F4M</a>	1o. <a href="#">2F4N</a>	1p. <a href="#">2F4P</a>	1q. <a href="#">2F4Q</a>	1r. <a href="#">2F4R</a>
1s. <a href="#">2F4S</a>	1t. <a href="#">2F4T</a>	1u. <a href="#">2F4V</a>	1v. <a href="#">2F4W</a>	1w. <a href="#">2F4X</a>	1x. <a href="#">2F4Y</a>
1y. <a href="#">2F4Z</a>	1z. <a href="#">2F52</a>	2a. <a href="#">2F53</a>	2b. <a href="#">2F54</a>	2c. <a href="#">2F55</a>	2d. <a href="#">2F56</a>
2e. <a href="#">2F57</a>	2f. <a href="#">2F58</a>	2g. <a href="#">2F59</a>	2h. <a href="#">2F5B</a>	2i. <a href="#">2F5C</a>	2j. <a href="#">2F5D</a>
2k. <a href="#">2F5F</a>	2l. <a href="#">2F5G</a>	2m. <a href="#">2F5H</a>	2n. <a href="#">2F5J</a>	2o. <a href="#">2F5K</a>	2p. <a href="#">2F5M</a>
2q. <a href="#">2F5N</a>	2r. <a href="#">2F5P</a>	2s. <a href="#">2F5Q</a>	3a. <a href="#">2F5R</a>	3b. <a href="#">2F5S</a>	3c. <a href="#">2F5T</a>
3d. <a href="#">2F5V</a>	3e. <a href="#">2F5W</a>	3f. <a href="#">2F5X</a>	3g. <a href="#">2F5Y</a>	3h. <a href="#">2F5Z</a>	3i. <a href="#">2F62</a>
3j. <a href="#">2F63</a>	3k. <a href="#">2F64</a>	3l. <a href="#">2F65</a>	3m. <a href="#">2F66</a>	3n. <a href="#">2F67</a>	3o. <a href="#">2F68</a>
3p. <a href="#">2F69</a>	3q. <a href="#">2F6B</a>	3r. <a href="#">2F6C</a>	3s. <a href="#">2F6D</a>	3t. <a href="#">2F6F</a>	3u. <a href="#">2F6G</a>
3v. <a href="#">2F6H</a>	3w. <a href="#">2F6J</a>	3x. <a href="#">2F6K</a>	4. <a href="#">2F6M</a>	5. <a href="#">2F6N</a>	6a. <a href="#">2F6P</a>
6b. <a href="#">2F6Q</a>	6c. <a href="#">2F6R</a>	6d. <a href="#">2F6S</a>	6e. <a href="#">2F6T</a>	6f. <a href="#">2F6V</a>	7. <a href="#">2F6W</a>
8. <a href="#">2F6X</a>					



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# CHAPTER



## *Number patterns*

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# 3 Number patterns

## 3.1 Introduction

EMAX

In earlier grades you saw patterns in the form of pictures and numbers. In this chapter, we learn more about the mathematics of patterns. Patterns are repetitive sequences and can be found in nature, shapes, events, sets of numbers and almost everywhere you care to look. For example, seeds in a sunflower, snowflakes, geometric designs on quilts or tiles, or the number sequence 0; 4; 8; 12; 16; ...



Figure 3.1: The pattern of seeds within a sunflower follows the Fibonacci sequence, or 1; 2; 3; 5; 8; 13; 21; 34; 55; 89; 144; ...

### VISIT:

Interested in learning more about the relationship between Fibonacci sequences and sunflowers?

► See video: [2F6Y](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

Try spot any patterns in the following sequences on your own:

1. 2; 4; 6; 8; 10; ...
2. 1; 2; 4; 7; 11; ...
3. 1; 4; 9; 16; 25; ...
4. 5; 10; 20; 40; 80; ...

### VISIT:

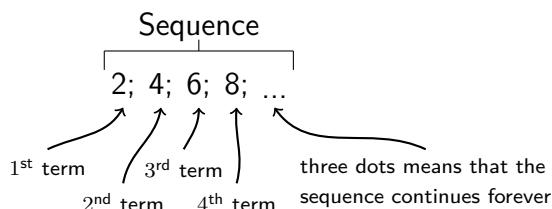
Identifying patterns in sequences.

► See video: [2F6Z](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## 3.2 Describing sequences

EMAY

A sequence is an ordered list of items, usually numbers. Each item which makes up a sequence is called a "term".



Sequences can have interesting patterns. Here we examine some types of patterns and how they are formed.

Examples:

1. 1; 4; 7; 10; 13; 16; 19; 22; 25; ...

There is difference of 3 between successive terms.

The pattern is continued by adding 3 to the previous term.

2. 13; 8; 3; -2; -7; -12; -17; -22; ...

There is a difference of -5 between successive terms.

The pattern is continued by adding -5 to (i.e. subtracting 5 from) the previous term.

3. 2; 4; 8; 16; 32; 64; 128; 256; ...

This sequence has a factor of 2 between successive terms.

The pattern is continued by multiplying the previous term by 2.

4. 3; -9; 27; -81; 243; -729; 2187; ...

This sequence has a factor of -3 between successive terms.

The pattern is continued by multiplying the previous term by -3.

5. 9; 3; 1;  $\frac{1}{3}$ ;  $\frac{1}{9}$ ;  $\frac{1}{27}$ ; ...

This sequence has a factor of  $\frac{1}{3}$  between successive terms.

The pattern is continued by multiplying the previous term by  $\frac{1}{3}$  which is equivalent to dividing the previous term by 3.

### Worked example 1: Study table

#### QUESTION

You and 3 friends decide to study for Maths and are sitting together at a square table. A few minutes later, 2 other friends arrive and would like to sit at your table. You move another table next to yours so that 6 people can sit at the table. Another 2 friends also want to join your group, so you take a third table and add it to the existing tables. Now 8 people can sit together.

Examine how the number of people sitting is related to the number of tables. Is there a pattern?

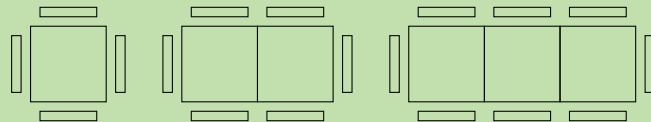


Figure 3.2: Two more people can be seated for each table added.

#### SOLUTION

##### Step 1: Make a table to see if a pattern forms

Number of tables, $n$	Number of people seated
1	$4 = 4$
2	$4 + 2 = 6$
3	$4 + 2 + 2 = 8$
4	$4 + 2 + 2 + 2 = 10$
:	:
$n$	$4 + 2 + 2 + 2 + \dots + 2$

##### Step 2: Describe the pattern

We can see that for 3 tables we can seat 8 people, for 4 tables we can seat 10 people and so on. We started out with 4 people and added two each time. So for each table added, the number of people increased by 2.

So the pattern formed is 4; 6; 8; 10; ....

To describe terms in a number pattern we use the following notation:

The first term of a sequence is  $T_1$ .

The fourth term of a sequence is  $T_4$ .

The tenth term of a sequence is  $T_{10}$ .

The general term is often expressed as the  $n^{\text{th}}$  term and is written as  $T_n$ .

A sequence does not have to follow a pattern but, when it does, we can write down the general formula to calculate any term. For example, consider the following linear sequence: 1; 3; 5; 7; 9; ...

The  $n^{\text{th}}$  term is given by the general formula:  $T_n = 2n - 1$

You can check this by substituting values into the formula:

$$T_1 = 2(1) - 1 = 1$$

$$T_2 = 2(2) - 1 = 3$$

$$T_3 = 2(3) - 1 = 5$$

$$T_4 = 2(4) - 1 = 7$$

$$T_5 = 2(5) - 1 = 9$$

If we find the relationship between the position of a term and its value, we can find a general formula which matches the pattern and find any term in the sequence.

## Common difference

EMAZ

Consider the following sequence:

$$6; 1; -4; -9; \dots$$

We can see that each term is decreasing by 5 but how would we determine the general formula for the  $n^{\text{th}}$  term? Let us try to do this with a table.

Term number	$T_1$	$T_2$	$T_3$	$T_4$	$T_n$
Term	6	1	-4	-9	$T_n$
Formula	$6 - 0 \times 5$	$6 - 1 \times 5$	$6 - 2 \times 5$	$6 - 3 \times 5$	$6 - (n - 1) \times 5$

You can see that the difference between the successive terms is always the coefficient of  $n$  in the formula. This is called a **common difference**.

Therefore, for sequences with a common difference, the general formula will always be of the form:  $T_n = dn + c$  where  $d$  is the difference between each term and  $c$  is some constant.

### NOTE:

Sequences with a common difference are called linear sequences.

### DEFINITION: Common difference

The common difference is the difference between any term and the term before it. The common difference is denoted by  $d$ .

For example, consider the sequence 10; 7; 4; 1; ...

To calculate the common difference, we find the difference between any term and the previous term.

Let us find the common difference between the first two terms.

$$\begin{aligned}d &= T_2 - T_1 \\&= 7 - 10 \\&= -3\end{aligned}$$

Let us check another two terms:

$$\begin{aligned}d &= T_4 - T_3 \\&= 1 - 4 \\&= -3\end{aligned}$$

We see that  $d$  is constant.

In general:  $d = T_n - T_{n-1}$

### IMPORTANT!

---

$d \neq T_{n-1} - T_n$  for example,  $d = T_2 - T_1$ , not  $T_1 - T_2$ .

---

### Worked example 2: Study table, continued

#### QUESTION

As before, you and 3 friends are studying for Maths and are sitting together at a square table. A few minutes later 2 other friends arrive so you move another table next to yours. Now 6 people can sit at the table. Another 2 friends also join your group, so you take a third table and add it to the existing tables. Now 8 people can sit together as shown below.

- Find an expression for the number of people seated at  $n$  tables.
- Use the general formula to determine how many people can sit around 12 tables.
- How many tables are needed to seat 20 people?

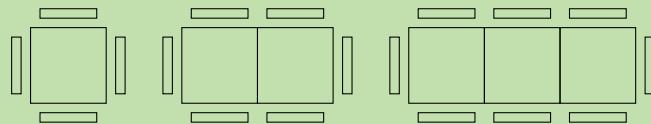


Figure 3.3: Two more people can be seated for each table added.

#### SOLUTION

##### Step 1: Make a table to see the pattern

Number of Tables, $n$	Number of people seated	Pattern
1	$4 = 4$	$= 4 + 2(0)$
2	$4 + 2 = 6$	$= 4 + 2(1)$
3	$4 + 2 + 2 = 8$	$= 4 + 2(2)$
4	$4 + 2 + 2 + 2 = 10$	$= 4 + 2(3)$
$\vdots$	$\vdots$	$\vdots$
$n$	$4 + 2 + 2 + 2 + \dots + 2$	$= 4 + 2(n - 1)$

**Note:** There may be variations in how you think of the pattern in this problem. For example, you may view this problem as the person on one end fixed, two people seated opposite each other per table and one person at the other end fixed. This results in  $1 + 2n + 1 = 2n + 2$ . Your formula for  $T_n$  will still be correct.

### Step 2: Describe the pattern

The number of people seated at  $n$  tables is  $T_n = 4 + 2(n - 1)$

### Step 3: Calculate the 12<sup>th</sup> term, in other words, find $T_n$ if $n = 12$

$$\begin{aligned}T_{12} &= 4 + 2(12 - 1) \\&= 4 + 2(11) \\&= 4 + 22 \\&= 26\end{aligned}$$

Therefore 26 people can be seated at 12 tables.

### Step 4: Calculate the number of tables needed to seat 20 people, in other words, find $n$ if $T_n = 20$

$$\begin{aligned}T_n &= 4 + 2(n - 1) \\20 &= 4 + 2(n - 1) \\20 &= 4 + 2n - 2 \\20 - 4 + 2 &= 2n \\18 &= 2n \\\frac{18}{2} &= n \\n &= 9\end{aligned}$$

Therefore 9 tables are needed to seat 20 people.

It is important to note the difference between  $n$  and  $T_n$ .  $n$  can be compared to a place holder indicating the position of the term in the sequence, while  $T_n$  is the value of the place held by  $n$ . From our example above, the first table holds 4 people. So for  $n = 1$ , the value of  $T_1 = 4$  and so on:

$n$	1	2	3	4	...
$T_n$	4	6	8	10	...

### Worked example 3: Data plans

#### QUESTION

Raymond subscribes to a limited data plan from Vodacell. The limited data plans cost R 120 for 1 gigabyte (GB) per month, R 135 for 2 GB per month and R 150 for 3 GB per month. Assume this pattern continues indefinitely.

1. Use a table to set up the pattern of the cost of the data plans.
2. Find the general formula for the sequence.
3. Use the general formula to determine the cost for a 30 GB data plan.
4. The cost of an unlimited data plan is R 520 per month. Determine the amount of data Raymond would have to use for it to be cheaper for him to sign up for the unlimited plan.

## SOLUTION

### Step 1: Make a table to see the pattern

Number of GB ( $n$ )	1	2	3	4
Cost (in Rands)	120	135	150	165
Pattern	120	$120 + (1)(15) = 135$	$120 + (2)(15) = 150$	$120 + (3)(15) = 165$

### Step 2: Use the observed pattern to determine the general formula.

The price of  $n$  GB of data is  $T_n = 120 + 15(n - 1)$

### Step 3: Determine the cost of 30 GB of data.

This question requires us to determine the value of the 30<sup>th</sup> term, in other words, find  $T_n$  if  $n = 30$ . Using the general formula, we get:

$$\begin{aligned}T_n &= 120 + 15(n - 1) \\ \therefore T_{30} &= 120 + 15(30 - 1) \\ &= 120 + 15(29) \\ &= 120 + 435 \\ &= 555\end{aligned}$$

Therefore the cost of a 30 GB data package is R 555.

### Step 4: Determine when it is cheaper to purchase the unlimited data plan

The final question of this worked example requires us to determine when it would be cheaper for Raymond to purchase an unlimited data plan instead of a limited plan. In other words, we need to find  $n$  where  $T_n$  is less than R 520.

We know that:

$$T_n = 120 + 15(n - 1)$$

Therefore, if  $T_n = 520 = 120 + 15(n - 1)$

Solving for  $n$ , we get:

$$\begin{aligned}520 &= 120 + 15(n - 1) \\ 520 &= 120 + 15n - 15 \\ 520 &= 105 + 15n \\ 405 &= 15n \\ \frac{405}{15} &= n \\ n &= 27\end{aligned}$$

Therefore it is cheaper for Raymond to purchase the unlimited data plan if he uses more than 27 GB per month.

### VISIT:

Learn more about number patterns.

See video: [2F72](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Exercise 3 – 1:

1. Use the given pattern to complete the table below.

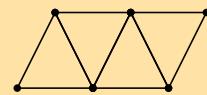
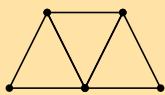
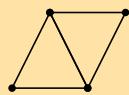
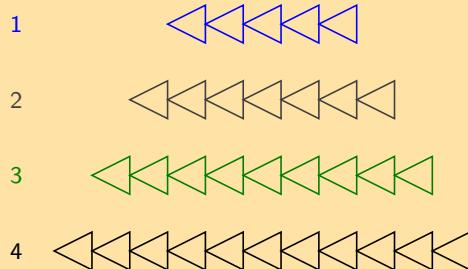


Figure number	1	2	3	4	$n$
Number of dots					
Number of lines					
Total					

2. Consider the sequence shown here:  $-4 ; -1 ; 2 ; 5 ; 8 ; 11 ; 14 ; 17 ; \dots$   
If  $T_n = 2$  what is the value of  $T_{n-1}$ ?
3. Consider the sequence shown here:  $C ; D ; E ; F ; G ; H ; I ; J ; \dots$   
If  $T_n = G$  what is the value of  $T_{n-4}$ ?
4. For each of the following sequences determine the common difference. If the sequence is not linear, write “no common difference”.
- a)  $9 ; -7 ; -8 ; -25 ; -34 ; \dots$       b)  $5 ; 12 ; 19 ; 26 ; 33 ; \dots$   
 c)  $2,93 ; 1,99 ; 1,14 ; 0,35 ; \dots$       d)  $2,53 ; 1,88 ; 1,23 ; 0,58 ; \dots$
5. Write down the next three terms in each of the following sequences:
- a)  $5 ; 15 ; 25 ; \dots$       b)  $-8 ; -3 ; 2 ; \dots$   
 c)  $30 ; 27 ; 24 ; \dots$       d)  $-13,1 ; -18,1 ; -23,1 ; \dots$   
 e)  $-9x ; -19x ; -29x ; \dots$       f)  $-15,8 ; 4,2 ; 24,2 ; \dots$   
 g)  $30b ; 34b ; 38b ; \dots$
6. Given a pattern which starts with the numbers:  $3 ; 8 ; 13 ; 18 ; \dots$  determine the values of  $T_6$  and  $T_9$ .
7. Given a sequence which starts with the letters:  $C ; D ; E ; F ; \dots$  determine the values of  $T_5$  and  $T_8$ .
8. Given a pattern which starts with the numbers:  $7 ; 11 ; 15 ; 19 ; \dots$  determine the values of  $T_5$  and  $T_8$ .
9. The general term is given for each sequence below. Calculate the missing terms (each missing term is represented by  $\dots$ ).
- a)  $0 ; 3 ; \dots ; 15 ; 24$        $T_n = n^2 - 1$   
 b)  $3 ; 2 ; 1 ; 0 ; \dots ; -2$        $T_n = -n + 4$   
 c)  $-11 ; \dots ; -7 ; \dots ; -3$        $T_n = -13 + 2n$   
 d)  $1 ; 10 ; 19 ; \dots ; 37$        $T_n = 9n - 8$   
 e)  $9 ; \dots ; 21 ; \dots ; 33$        $T_n = 6n + 3$
10. Find the general formula for the following sequences and then find  $T_{10}$ ,  $T_{50}$  and  $T_{100}$
- a)  $2; 5; 8; 11; 14; \dots$       b)  $0; 4; 8; 12; 16; \dots$       c)  $2; -1; -4; -7; -10; \dots$
11. The diagram below shows pictures which follow a pattern.



- a) How many triangles will there be in the 5<sup>th</sup> picture?

- b) Determine the formula for the  $n^{\text{th}}$  term.  
 c) Use the formula to find how many triangles are in the 25<sup>th</sup> picture of the diagram.
12. Study the following sequence: 15 ; 23 ; 31 ; 39 ; ...  
 a) Write down the next 3 terms.  
 b) Find the general formula for the sequence  
 c) Find the value of  $n$  if  $T_n$  is 191.
13. Study the following sequence: -44 ; -14 ; 16 ; 46 ; ...  
 a) Write down the next 3 terms.  
 b) Find the general formula for the sequence  
 c) Find the value of  $n$  if  $T_n$  is 406.
14. Consider the following list:  
 $-z - 5 ; -4z - 5 ; -6z - 2 ; -8z - 5 ; -10z - 5 ; \dots$   
 a) Find the common difference for the terms of the list. If the sequence is not linear (if it does not have a common difference), write "no common difference".  
 b) If you are now told that  $z = -2$ , determine the values of  $T_1$  and  $T_2$ .
15. Consider the following pattern:  
 $2n + 4 ; 1 ; -2n - 2 ; -4n - 5 ; -6n - 8 ; \dots$   
 a) Find the common difference for the terms of the pattern. If the sequence is not linear (if it does not have a common difference), write "no common difference".  
 b) If you are now told that  $n = -1$ , determine the values of  $T_1$  and  $T_3$ .
16. a) If the following terms make a linear sequence:  

$$\frac{k}{3} - 1 ; -\frac{5k}{3} + 2 ; -\frac{2k}{3} + 10 ; \dots$$
  
 Determine the value of  $k$ . If the answer is a non-integer, write the answer as a simplified fraction.  
 b) Now determine the numeric value of the first three terms. If the answers are not integers, write your answers as fractions.
17. a) If the following terms make a linear sequence:  

$$y - \frac{3}{2} ; -y - \frac{7}{2} ; -7y - \frac{15}{2} ; \dots$$
  
 find  $y$ . If the answer is a non-integer, write the answer as a simplified fraction.  
 b) Now determine the numeric value of the first three terms. If the answers are not integers, write your answers as fractions.
18. What is the 649<sup>th</sup> letter of the sequence:  
 PATTERNPATTERNPATTERNPATTERNPATTERNPATTERNPATTE.....?

For	more	exercises,	visit	www.everythingmaths.co.za	and	click	on	'Practise Maths'.
1. 2F73	2. 2F74	3. 2F75	4. 2F76	5a. 2F77	5b. 2F78			
5c. 2F79	5d. 2F7B	5e. 2F7C	5f. 2F7D	5g. 2F7F	6. 2F7G			
7. 2F7H	8. 2F7J	9a. 2F7K	9b. 2F7M	9c. 2F7N	9d. 2F7P			
9e. 2F7Q	10a. 2F7R	10b. 2F7S	10c. 2F7T	11. 2F7V	12. 2F7W			
13. 2F7X	14. 2F7Y	15. 2F7Z	16. 2F82	17. 2F83	18. 2F84			



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► See presentation: [2F85](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

- The general term is expressed as the  $n^{\text{th}}$  term and is written as  $T_n$ .
- We define the common difference  $d$  of a sequence as the difference between any two successive terms, where  $d = T_n - T(n - 1)$
- We can work out a general formula for each number pattern and use it to determine any term in the pattern.

#### End of chapter Exercise 3 – 2:

- Analyse the diagram and complete the table.

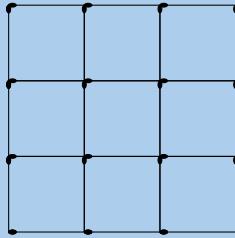


Figure number ( $n \times n$ )	$1 \times 1$	$2 \times 2$	$3 \times 3$	$4 \times 4$	$n \times n$
Number of horizontal matches					
Number of vertical matches					
Total number of matches					

- Given a list of numbers:  $7 ; 4 ; 1 ; -2 ; -5 ; \dots$  determine the common difference for the list (if there is one).
- For the pattern here:  $-0,55 ; 0,99 ; 2,49 ; 3,91 ; \dots$  calculate the common difference.  
If the pattern is not linear, write "no common difference". Otherwise, give your answer as a decimal.
- Consider the list shown here:  $2 ; 7 ; 12 ; 17 ; 22 ; 27 ; 32 ; 37 ; \dots$   
If  $T_5 = 22$  what is the value of  $T_{n-3}$ ?
- Write down the next three terms in each of the following linear sequences:
  - $-10,2 ; -29,2 ; -48,2 ; \dots$
  - $50r ; 46r ; 42r ; \dots$
- Given a sequence which starts with the numbers:  $6 ; 11 ; 16 ; 21 ; \dots$  determine the values of  $T_6$  and  $T_8$ .
- Given a list which starts with the letters:  $A ; B ; C ; D ; \dots$  determine the values of  $T_6$  and  $T_{10}$ .
- Find the sixth term in each of the following sequences:
  - $4 ; 13 ; 22 ; 31 ; \dots$
  - $5 ; 2 ; -1 ; -4 ; \dots$
  - $7,4 ; 9,7 ; 12 ; 14,3 ; \dots$
- Find the general formula for the following sequences and then find  $T_{10}$ ,  $T_{15}$  and  $T_{30}$ 
  - $-18 ; -22 ; -26 ; -30 ; -34 ; \dots$
  - $1 ; -6 ; -13 ; -20 ; -27 ; \dots$
- The general term is given for each sequence below. Calculate the missing terms (each missing term is represented by  $\dots$ ).
  - $10 ; \dots ; 14 ; \dots ; 18$        $T_n = 2n + 8$
  - $2 ; -2 ; -6 ; \dots ; -14$        $T_n = -4n + 6$
  - $8 ; \dots ; 38 ; \dots ; 68$        $T_n = 15n - 7$

11. Find the general term in each of the following sequences:

a)  $3 ; 7 ; 11 ; 15 ; \dots$

b)  $-2 ; 1 ; 4 ; 7 ; \dots$

c)  $11 ; 15 ; 19 ; 23 ; \dots$

d)  $\frac{1}{3} ; \frac{2}{3} ; 1 ; 1\frac{1}{3} ; \dots$

12. Study the following sequence  $-7 ; -21 ; -35 ; \dots$

a) Write down the next 3 terms;

b) Find the general formula for the sequence.

c) Find the value of  $n$  if  $T_n$  is  $-917$ .

13. What is the  $346^{\text{th}}$  letter of the sequence:

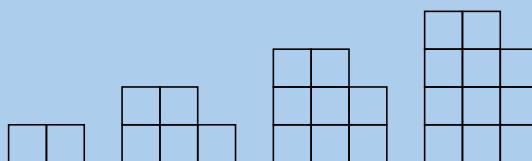
COMMONCOMMON.....?

14. What is the  $1000^{\text{th}}$  letter of the sequence:

MATHEMATICSMATHEMATICSMATHEMATICSMATHE .....

15. The seating of a sports stadium is arranged so that the first row has 15 seats, the second row has 19 seats, the third row has 23 seats and so on. Calculate how many seats are in the  $25^{\text{th}}$  row.

16. The diagram below shows pictures which follow a pattern.

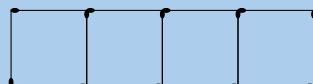


a) How many boxes will there be in the sixth picture?

b) Determine the formula for the  $n^{\text{th}}$  term.

c) Use the formula to find how many boxes are in the  $30^{\text{th}}$  picture of the diagram.

17. A single square is made from 4 matchsticks. Two squares in a row need 7 matchsticks and three squares need 10 matchsticks.



Answer the following questions for this sequence.

a) Determine the first term.

b) Determine the common difference.

c) Determine the general formula.

d) A row has twenty-five squares. How many matchsticks are there in this row?

18. You would like to start saving some money, but because you have never tried to save money before, you decide to start slowly. At the end of the first week you deposit R 5 into your bank account. Then at the end of the second week you deposit R 10 and at the end of the third week, R 15. After how many weeks will you deposit R 50 into your bank account?

19. Consider the following list:

$$-4y - 3 ; -y ; 2y + 3 ; 5y + 6 ; 8y + 9 ; \dots$$

a) Find the common difference for the terms of the list. If the sequence is not linear (if it does not have a common difference), write "no common difference".

b) If you are now told that  $y = 1$ , determine the values of  $T_1$  and  $T_2$ .

20. a) If the following terms make a linear sequence:

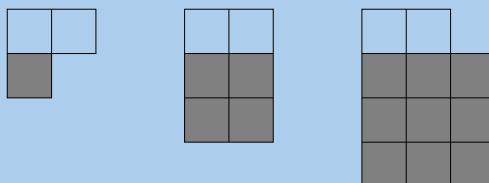
$$2n + \frac{1}{2}; 3n + \frac{5}{2}; 7n + \frac{11}{2}; \dots$$

Determine the value of  $n$ . If the answer is a non-integer, write the answer as a simplified fraction.

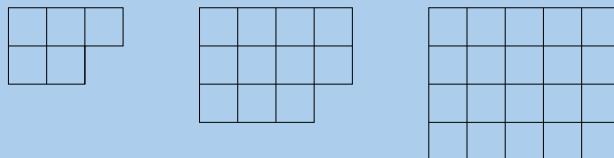
- b) Now determine the numeric value of the first three terms. If the answers are not integers, write your answers as fractions.

21. How many blocks will there be in the 85<sup>th</sup> picture?

Hint: Use the grey blocks to help.

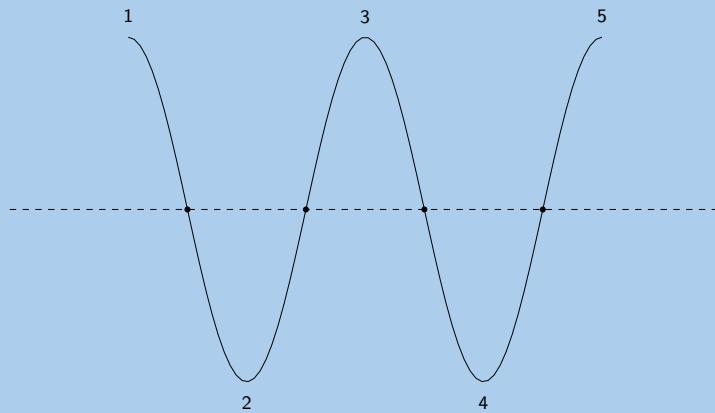


22. Analyse the picture below:



- a) How many blocks are there in the next picture?  
 b) Write down the general formula for this pattern.  
 c) How many blocks will there be in the 14th picture?

23. A horizontal line intersects a piece of string at 4 points and divides it into five parts, as shown below.



If the piece of string is intersected in this way by 19 parallel lines, each of which intersects it at 4 points, determine the number of parts into which the string will be divided.

24. Use a calculator to explore and then generalise your findings to determine the:

- a) units digit of  $3^{2007}$   
 b) tens digit of  $7^{2008}$   
 c) remainder when  $7^{250}$  is divided by 5

25. Analyse the diagram and complete the table.

The dots follow a triangular pattern and the formula is  $T_n = \frac{n(n + 1)}{2}$ .

The general formula for the lines is  $T_n = \frac{3n(n - 1)}{2}$ .

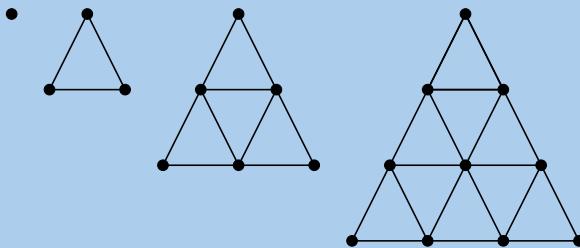


Figure number	1	2	3	4	5	20	$n$
Number of dots							
Number of lines							
Total							

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- |                           |                           |                           |                           |                           |                           |
|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 1. <a href="#">2F86</a>   | 2. <a href="#">2F88</a>   | 3. <a href="#">2F89</a>   | 4. <a href="#">2F8B</a>   | 5. <a href="#">2F8C</a>   | 6. <a href="#">2F8D</a>   |
| 7. <a href="#">2F8F</a>   | 8a. <a href="#">2F8G</a>  | 8b. <a href="#">2F8H</a>  | 8c. <a href="#">2F8J</a>  | 9. <a href="#">2F8K</a>   | 10. <a href="#">2F8M</a>  |
| 11a. <a href="#">2F8N</a> | 11b. <a href="#">2F8P</a> | 11c. <a href="#">2F8Q</a> | 11d. <a href="#">2F8R</a> | 12. <a href="#">2F8S</a>  | 13. <a href="#">2F8T</a>  |
| 14. <a href="#">2F8V</a>  | 15. <a href="#">2F8W</a>  | 16. <a href="#">2F8X</a>  | 17. <a href="#">2F8Y</a>  | 18. <a href="#">2F8Z</a>  | 19. <a href="#">2F92</a>  |
| 20. <a href="#">2F93</a>  | 21. <a href="#">2F94</a>  | 22. <a href="#">2F95</a>  | 23. <a href="#">2F96</a>  | 24a. <a href="#">2F97</a> | 24b. <a href="#">2F98</a> |
| 24c. <a href="#">2F99</a> | 25. <a href="#">2F87</a>  |                           |                           |                           |                           |



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# CHAPTER



## *Equations and inequalities*

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## 4 Equations and inequalities

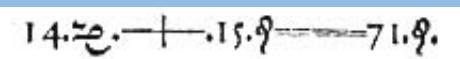
### 4.1 Introduction

EMA33

Equations are widely used to describe the world around us. In science equations are used to describe everything from how a ball rolls down a slope to how the planets move around the sun.

In this chapter we will explore different types of equations as well as looking at how these can be used to solve problems in the real world. We will also look at linear inequalities.

#### DID YOU KNOW?



The first use of an “equals” sign from *The Whetstone of Witte* by Robert Recorde 1557. This equation represents  $14x + 15 = 71$ . Recorde is also responsible for introducing the pre-existing “plus” sign (+) to the English-speaking world.

### 4.2 Solving linear equations

EMA34

The simplest equation to solve is a linear equation. A linear equation is an equation where the highest exponent of the variable is 1. The following are examples of linear equations:

$$\begin{aligned}2x + 2 &= 1 \\ \frac{2-x}{3x+1} &= 2 \\ 4(2x-9) - 4x &= 4 - 6x \\ \frac{2a-3}{3} - 3a &= \frac{a}{3}\end{aligned}$$

Solving an equation means finding the value of the variable that makes the equation true. For example, to solve the simple equation  $x + 1 = 1$ , we need to determine the value of  $x$  that will make the left hand side equal to the right hand side. The solution is  $x = 0$ .

The solution, also called the root of an equation, is the value of the variable that satisfies the equation. For linear equations, there is at most one solution for the equation.

To solve equations we use algebraic methods that include expanding expressions, grouping terms, and factorising.

For example:

$$\begin{aligned}2x + 2 &= 1 \\ 2x &= 1 - 2 \quad (\text{rearrange}) \\ 2x &= -1 \quad (\text{simplify}) \\ x &= -\frac{1}{2} \quad (\text{divide both sides by 2})\end{aligned}$$

Check the answer by substituting  $x = -\frac{1}{2}$ .

$$\begin{aligned}\text{LHS} &= 2x + 2 \\&= 2\left(-\frac{1}{2}\right) + 2 \\&= -1 + 2 \\&= 1 \\ \text{RHS} &= 1\end{aligned}$$

Therefore  $x = -\frac{1}{2}$

### VISIT:

The following video gives an introduction to solving linear equations.

► See video: 2F9B at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## Method for solving linear equations

EMA35

The general steps for solving linear equations are:

1. Expand all brackets.
2. Rearrange the terms so that all terms containing the variable are on one side of the equation and all constant terms are on the other side.
3. Group like terms together and simplify.
4. Factorise if necessary.
5. Find the solution and write down the answer.
6. Check the answer by substituting the solution back into the original equation.

### IMPORTANT!

---

An equation must always be balanced, whatever you do to the left-hand side, you must also do to the right-hand side.

---

### Worked example 1: Solving linear equations

#### QUESTION

Solve for  $x$ :

$$4(2x - 9) - 4x = 4 - 6x$$

#### SOLUTION

##### Step 1: Expand the brackets and simplify

$$\begin{aligned}4(2x - 9) - 4x &= 4 - 6x \\8x - 36 - 4x &= 4 - 6x \\8x - 4x + 6x &= 4 + 36 \\10x &= 40\end{aligned}$$

##### Step 2: Divide both sides by 10

$$x = 4$$

**Step 3: Check the answer by substituting the solution back into the original equation**

$$\begin{aligned}\text{LHS} &= 4[2(4) - 9] - 4(4) \\&= 4(8 - 9) - 16 \\&= 4(-1) - 16 \\&= -4 - 16 \\&= -20\end{aligned}$$

$$\begin{aligned}\text{RHS} &= 4 - 6(4) \\&= 4 - 24 \\&= -20\end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$ . Since both sides are equal, the answer is correct.

**Worked example 2: Solving linear equations****QUESTION**

Solve for  $x$ :

$$\frac{2-x}{3x+1} = 2$$

**SOLUTION****Step 1: Multiply both sides of the equation by  $(3x + 1)$** 

Division by 0 is undefined so there must be a restriction:  $(x \neq -\frac{1}{3})$ .

$$\begin{aligned}\frac{2-x}{3x+1} &= 2 \\(2-x) &= 2(3x+1)\end{aligned}$$

**Step 2: Expand the brackets and simplify**

$$\begin{aligned}2-x &= 6x+2 \\-x-6x &= 2-2 \\-7x &= 0\end{aligned}$$

**Step 3: Divide both sides by  $-7$** 

$$\begin{aligned}x &= \frac{0}{-7} \\x &= 0\end{aligned}$$

**Step 4: Check the answer by substituting the solution back into the original equation**

$$\begin{aligned}\text{LHS} &= \frac{2-(0)}{3(0)+1} \\&= 2 \\&= \text{RHS}\end{aligned}$$

Since both sides are equal, the answer is correct.

### Worked example 3: Solving linear equations

#### QUESTION

Solve for  $a$ :

$$\frac{2a - 3}{3} - 3a = \frac{a}{3}$$

#### SOLUTION

**Step 1:** Multiply the equation by the common denominator 3 and simplify

$$\begin{aligned} 2a - 3 - 9a &= a \\ -7a - 3 &= a \end{aligned}$$

**Step 2:** Rearrange the terms and simplify

$$\begin{aligned} -7a - a &= 3 \\ -8a &= 3 \end{aligned}$$

**Step 3:** Divide both sides by  $-8$

$$a = -\frac{3}{8}$$

**Step 4:** Check the answer by substituting the solution back into the original equation

$$\begin{aligned} \text{LHS} &= \frac{2(-\frac{3}{8}) - 3}{3} - 3\left(-\frac{3}{8}\right) & \text{RHS} &= \frac{-\frac{3}{8}}{3} \\ &= \frac{(-\frac{3}{4}) - \frac{12}{4}}{3} + \frac{9}{8} & &= \frac{-\frac{3}{8}}{3} \\ &= \left[-\frac{15}{4} \times \frac{1}{3}\right] + \frac{9}{8} & &= -\frac{3}{8} \times \frac{1}{3} \\ &= -\frac{5}{4} + \frac{9}{8} & &= -\frac{1}{8} \\ &= -\frac{10}{8} + \frac{9}{8} \\ &= -\frac{1}{8} \end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$ . Since both sides are equal, the answer is correct.

#### Exercise 4 – 1:

Solve the following equations (assume all denominators are non-zero):

1.  $2y - 3 = 7$
2.  $2c = c - 8$
3.  $3 = 1 - 2c$
4.  $4b + 5 = -7$
5.  $-3y = 0$
6.  $16y + 4 = -10$
7.  $12y + 0 = 144$
8.  $7 + 5y = 62$
9.  $55 = 5x + \frac{3}{4}$
10.  $5x = 2x + 45$
11.  $23x - 12 = 6 + 3x$
12.  $12 - 6x + 34x = 2x - 24 - 64$
13.  $6x + 3x = 4 - 5(2x - 3)$
14.  $18 - 2p = p + 9$
15.  $\frac{4}{p} = \frac{16}{24}$
16.  $-(-16 - p) = 13p - 1$
17.  $3f - 10 = 10$
18.  $3f + 16 = 4f - 10$
19.  $10f + 5 = -2f - 3f + 80$
20.  $8(f - 4) = 5(f - 4)$
21.  $6 = 6(f + 7) + 5f$
22.  $-7x = 8(1 - x)$
23.  $5 - \frac{7}{b} = \frac{2(b + 4)}{b}$
24.  $\frac{x+2}{4} - \frac{x-6}{3} = \frac{1}{2}$
25.  $1 = \frac{3a-4}{2a+6}$
26.  $\frac{2-5a}{3} - 6 = \frac{4a}{3} + 2 - a$
27.  $2 - \frac{4}{b+5} = \frac{3b}{b+5}$
28.  $3 - \frac{y-2}{4} = 4$
29.  $1,5x + 3,125 = 1,25x$
30.  $1,3(2,7x + 1) = 4,1 - x$
31.  $6,5x - 4,15 = 7 + 4,25x$
32.  $\frac{1}{3}P + \frac{1}{2}P - 10 = 0$
33.  $1\frac{1}{4}(x - 1) - 1\frac{1}{2}(3x + 2) = 0$
34.  $\frac{1}{5}(x - 1) = \frac{1}{3}(x - 2) + 3$
35.  $\frac{5}{2a} + \frac{1}{6a} - \frac{3}{a} = 2$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |                          |                          |                          |                          |                          |                          |                          |
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### 4.3 Solving quadratic equations

EMA36

A quadratic equation is an equation where the exponent of the variable is at most 2. The following are examples of quadratic equations:

$$\begin{aligned}2x^2 + 2x &= 1 \\3x^2 + 2x - 1 &= 0 \\0 &= -2x^2 + 4x - 2\end{aligned}$$

Quadratic equations differ from linear equations in that a linear equation has only one solution, while a quadratic equation has at most two solutions. There are some special situations, however, in which a quadratic equation has either one solution or no solutions.

We solve quadratic equations using factorisation. For example, in order to solve  $2x^2 - x - 3 = 0$ , we need to write it in its equivalent factorised form as  $(x + 1)(2x - 3) = 0$ . Note that if  $a \times b = 0$  then  $a = 0$  or  $b = 0$ .

**VISIT:**

The following video shows an example of solving a quadratic equation by factorisation.

► See video: [2FBM](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## Method for solving quadratic equations

EMA37

1. Rewrite the equation in the required form,  $ax^2 + bx + c = 0$ .
2. Divide the entire equation by any common factor of the coefficients to obtain an equation of the form  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$  and  $c$  have no common factors. For example  $2x^2 + 4x + 2 = 0$  can be written as  $x^2 + 2x + 1 = 0$ .
3. Factorise  $ax^2 + bx + c = 0$  to be of the form  $(rx + s)(ux + v) = 0$ .
4. The two solutions are  $(rx + s) = 0$  or  $(ux + v) = 0$ , so  $x = -\frac{s}{r}$  or  $x = -\frac{v}{u}$ , respectively.
5. Check the answer by substituting it back into the original equation.

### Worked example 4: Solving quadratic equations

#### QUESTION

Solve for  $x$ :

$$3x^2 + 2x - 1 = 0$$

#### SOLUTION

**Step 1: The equation is already in the required form,  $ax^2 + bx + c = 0$**

**Step 2: Factorise**

$$(x + 1)(3x - 1) = 0$$

**Step 3: Solve for both factors**

We have:

$$x + 1 = 0$$

OR

$$\therefore x = -1$$

$$3x - 1 = 0$$

$$\therefore x = \frac{1}{3}$$

**Step 4: Check both answers by substituting back into the original equation**

**Step 5: Write the final answer**

The solution to  $3x^2 + 2x - 1 = 0$  is  $x = -1$  or  $x = \frac{1}{3}$ .

## Worked example 5: Solving quadratic equations

### QUESTION

Find the roots:

$$0 = -2x^2 + 4x - 2$$

### SOLUTION

**Step 1:** The equation is already in the required form,  $ax^2 + bx + c = 0$

**Step 2:** Divide the equation by common factor  $-2$

$$\begin{aligned} -2x^2 + 4x - 2 &= 0 \\ x^2 - 2x + 1 &= 0 \end{aligned}$$

**Step 3:** Factorise

$$\begin{aligned} (x - 1)(x - 1) &= 0 \\ (x - 1)^2 &= 0 \end{aligned}$$

**Step 4:** The quadratic is a perfect square

This is an example of a special situation in which there is only one solution to the quadratic equation because both factors are the same.

$$\begin{aligned} x - 1 &= 0 \\ \therefore x &= 1 \end{aligned}$$

**Step 5:** Check the answer by substituting back into the original equation

**Step 6:** Write final answer

The solution to  $0 = -2x^2 + 4x - 2$  is  $x = 1$ .

### Exercise 4 – 2:

1. Write the following in standard form:

a)  $(r + 4)(5r - 4) = -16$       b)  $(3r - 8)(2r - 3) = -15$       c)  $(d + 5)(2d + 5) = 8$

2. Solve the following equations:

a) $x^2 + 2x - 15 = 0$	b) $p^2 - 7p - 18 = 0$
c) $9x^2 - 6x - 8 = 0$	d) $5x^2 + 21x - 54 = 0$
e) $4z^2 + 12z + 8 = 0$	f) $-b^2 + 7b - 12 = 0$
g) $-3a^2 + 27a - 54 = 0$	h) $4y^2 - 9 = 0$

- i)  $4x^2 + 16x - 9 = 0$   
 k)  $20m + 25m^2 = 0$   
 m)  $-75x^2 + 290x = 240$   
 o)  $x^2 - 4x = -4$   
 q)  $t^2 = 3t$   
 s)  $x^2 = 18$   
 u)  $4x^2 - 17x - 77 = 0$   
 w)  $2x^2 - 2x = 12$   
 y)  $(x - 6)^2 - 24 = 1$
- j)  $4x^2 - 12x = -9$   
 l)  $2x^2 - 5x - 12 = 0$   
 n)  $2x = \frac{1}{3}x^2 - 3x + 14\frac{2}{3}$   
 p)  $-x^2 + 4x - 6 = 4x^2 - 14x + 3$   
 r)  $x^2 - 10x = -25$   
 t)  $p^2 - 6p = 7$   
 v)  $14x^2 + 5x = 6$   
 x)  $(2a - 3)^2 - 16 = 0$

3. Solve the following equations (note the restrictions that apply):

a)  $3y = \frac{54}{2y}$   
 b)  $\frac{10z}{3} = 1 - \frac{1}{3z}$   
 c)  $x + 2 = \frac{18}{x} - 1$   
 d)  $y - 3 = \frac{5}{4} - \frac{1}{y}$   
 e)  $\frac{1}{2}(b - 1) = \frac{1}{3} \left( \frac{2}{b} + 4 \right)$   
 f)  $3(y + 1) = \frac{4}{y} + 2$   
 g)  $(x + 1)^2 - 2(x + 1) - 15 = 0$   
 h)  $z^4 - 1 = 0$   
 i)  $b^4 - 13b^2 + 36 = 0$   
 j)  $\frac{a+1}{3a-4} + \frac{9}{2a+5} + \frac{2a+3}{2a+5} = 0$   
 k)  $\frac{x^2 - 2x - 3}{x+1} = 0$   
 l)  $x + 2 = \frac{6x - 12}{x - 2}$   
 m)  $\frac{3(a^2 + 1) + 10a}{3a + 1} = 1$   
 n)  $\frac{3}{9a^2 - 3a + 1} - \frac{3a + 4}{27a^3 + 1} = \frac{1}{9a^2 - 1}$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

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## 4.4 Solving simultaneous equations

EMA38

Up to now we have solved equations with only one unknown variable. When solving for two unknown variables, two equations are required and these equations are known as simultaneous equations. The solutions are the values of the unknown variables which satisfy both equations simultaneously. In general, if there are  $n$  unknown variables, then  $n$  independent equations are required to obtain a value for each of the  $n$  variables.

An example of a system of simultaneous equations is:

$$\begin{aligned} x + y &= -1 \\ 3 &= y - 2x \end{aligned}$$

We have two independent equations to solve for two unknown variables. We can solve simultaneous equations algebraically using substitution and elimination methods. We will also show that a system of simultaneous equations can be solved graphically.

- Use the simplest of the two given equations to express one of the variables in terms of the other.
- Substitute into the second equation. By doing this we reduce the number of equations and the number of variables by one.
- We now have one equation with one unknown variable which can be solved.
- Use the solution to substitute back into the first equation to find the value of the other unknown variable.

**VISIT:**

The following video shows how to solve simultaneous equations using substitution.

► See video: [2FD5](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

**Worked example 6: Simultaneous equations****QUESTION**

Solve for  $x$  and  $y$ :

$$\begin{aligned}x - y &= 1 \quad \dots (1) \\3 &= y - 2x \quad \dots (2)\end{aligned}$$

**SOLUTION**

**Step 1: Use equation (1) to express  $x$  in terms of  $y$**

$$x = y + 1$$

**Step 2: Substitute  $x$  into equation (2) and solve for  $y$**

$$\begin{aligned}3 &= y - 2(y + 1) \\3 &= y - 2y - 2 \\5 &= -y \\\therefore y &= -5\end{aligned}$$

**Step 3: Substitute  $y$  back into equation (1) and solve for  $x$**

$$\begin{aligned}x &= (-5) + 1 \\\therefore x &= -4\end{aligned}$$

**Step 4: Check the solution by substituting the answers back into both original equations**

**Step 5: Write the final answer**

$$\begin{aligned}x &= -4 \\y &= -5\end{aligned}$$

## Worked example 7: Simultaneous equations

### QUESTION

Solve the following system of equations:

$$\begin{aligned} 4y + 3x &= 100 & \dots (1) \\ 4y - 19x &= 12 & \dots (2) \end{aligned}$$

### SOLUTION

**Step 1:** Use either equation to express  $x$  in terms of  $y$

$$\begin{aligned} 4y + 3x &= 100 \\ 3x &= 100 - 4y \\ x &= \frac{100 - 4y}{3} \end{aligned}$$

**Step 2:** Substitute  $x$  into equation (2) and solve for  $y$

$$\begin{aligned} 4y - 19\left(\frac{100 - 4y}{3}\right) &= 12 \\ 12y - 19(100 - 4y) &= 36 \\ 12y - 1900 + 76y &= 36 \\ 88y &= 1936 \\ \therefore y &= 22 \end{aligned}$$

**Step 3:** Substitute  $y$  back into equation (1) and solve for  $x$

$$\begin{aligned} x &= \frac{100 - 4(22)}{3} \\ &= \frac{100 - 88}{3} \\ &= \frac{12}{3} \\ \therefore x &= 4 \end{aligned}$$

**Step 4:** Check the solution by substituting the answers back into both original equations

**Step 5:** Write the final answer

$$\begin{aligned} x &= 4 \\ y &= 22 \end{aligned}$$

**Worked example 8: Simultaneous equations****QUESTION**

Solve the following system of equations:

$$3x + y = 2 \quad \dots (1)$$

$$6x - y = 25 \quad \dots (2)$$

**SOLUTION****Step 1: Make the coefficients of one of the variables the same in both equations**

The coefficients of  $y$  in the given equations are 1 and  $-1$ . Eliminate the variable  $y$  by adding equation (1) and equation (2) together:

$$\begin{array}{rcl} 3x + y & = & 2 \\ + 6x - y & = & 25 \\ \hline 9x + 0 & = & 27 \end{array}$$

**Step 2: Simplify and solve for  $x$** 

$$9x = 27$$

$$\therefore x = 3$$

**Step 3: Substitute  $x$  back into either original equation and solve for  $y$** 

$$\begin{aligned} 3(3) + y &= 2 \\ y &= 2 - 9 \\ \therefore y &= -7 \end{aligned}$$

**Step 4: Check that the solution  $x = 3$  and  $y = -7$  satisfies both original equations****Step 5: Write the final answer**

$$\begin{aligned} x &= 3 \\ y &= -7 \end{aligned}$$

**QUESTION**

Solve the following system of equations:

$$2a - 3b = 5 \quad \dots (1)$$

$$3a - 2b = 20 \quad \dots (2)$$

**SOLUTION****Step 1: Make the coefficients of one of the variables the same in both equations**

By multiplying equation (1) by 3 and equation (2) by 2, both coefficients of  $a$  will be 6.

$$\begin{array}{rcl} 6a - 9b & = & 15 \\ - (6a - 4b) & = & 40 \\ \hline 0 - 5b & = & -25 \end{array}$$

(When subtracting two equations, be careful of the signs.)

**Step 2: Simplify and solve for  $b$** 

$$\begin{aligned} b &= \frac{-25}{-5} \\ \therefore b &= 5 \end{aligned}$$

**Step 3: Substitute value of  $b$  back into either original equation and solve for  $a$** 

$$\begin{aligned} 2a - 3(5) &= 5 \\ 2a - 15 &= 5 \\ 2a &= 20 \\ \therefore a &= 10 \end{aligned}$$

**Step 4: Check that the solution  $a = 10$  and  $b = 5$  satisfies both original equations****Step 5: Write the final answer**

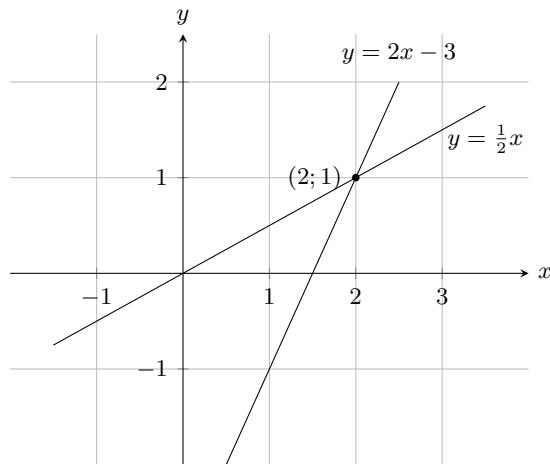
$$\begin{aligned} a &= 10 \\ b &= 5 \end{aligned}$$

Simultaneous equations can also be solved graphically. If the graphs of each linear equation are drawn, then the solution to the system of simultaneous equations is the coordinates of the point at which the two graphs intersect.

For example:

$$\begin{aligned}x &= 2y \quad \dots (1) \\y &= 2x - 3 \quad \dots (2)\end{aligned}$$

The graphs of the two equations are shown below.



The intersection of the two graphs is  $(2; 1)$ . So the solution to the system of simultaneous equations is  $x = 2$  and  $y = 1$ . We can also check the solution using algebraic methods.

Substitute equation (1) into (2):

$$\begin{aligned}x &= 2y \\ \therefore y &= 2(2y) - 3\end{aligned}$$

Then solve for  $y$ :

$$\begin{aligned}y - 4y &= -3 \\ -3y &= -3 \\ \therefore y &= 1\end{aligned}$$

Substitute the value of  $y$  back into equation (1):

$$\begin{aligned}x &= 2(1) \\ \therefore x &= 2\end{aligned}$$

Notice that both methods give the same solution.

#### VISIT:

You can use an online tool such as [graphsketch](#) to draw the graphs and check your solution.

## Worked example 10: Simultaneous equations

### QUESTION

Solve the following system of simultaneous equations graphically:

$$\begin{aligned}4y + 3x &= 100 & \dots (1) \\4y - 19x &= 12 & \dots (2)\end{aligned}$$

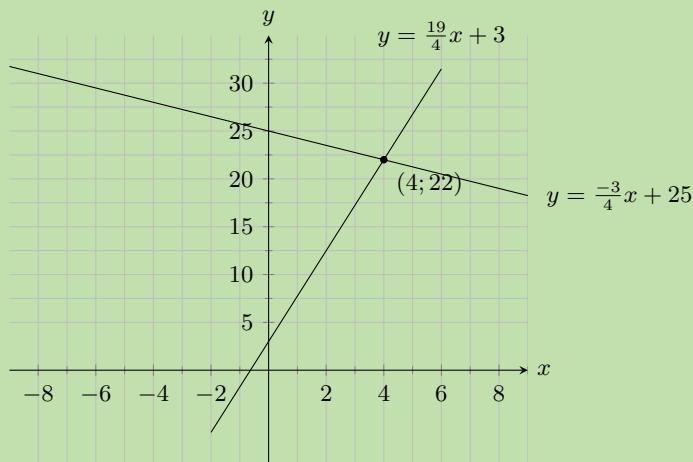
### SOLUTION

**Step 1: Write both equations in form  $y = mx + c$**

$$\begin{aligned}4y + 3x &= 100 \\4y &= 100 - 3x \\y &= -\frac{3}{4}x + 25\end{aligned}$$

$$\begin{aligned}4y - 19x &= 12 \\4y &= 19x + 12 \\y &= \frac{19}{4}x + 3\end{aligned}$$

**Step 2: Sketch the graphs on the same set of axes**



**Step 3: Find the coordinates of the point of intersection**

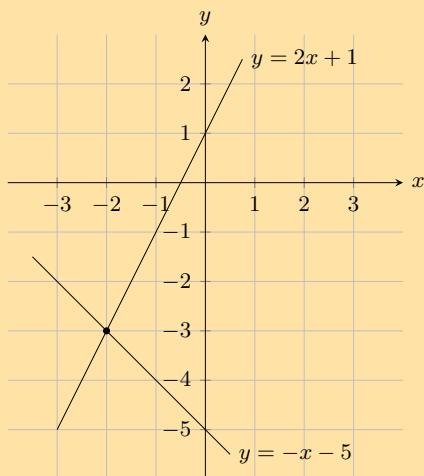
The two graphs intersect at  $(4; 22)$

**Step 4: Write the final answer**

$$\begin{aligned}x &= 4 \\y &= 22\end{aligned}$$

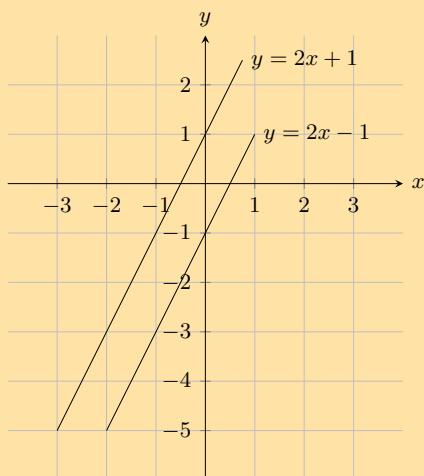
**Exercise 4 – 3:**

1. Look at the graph below:



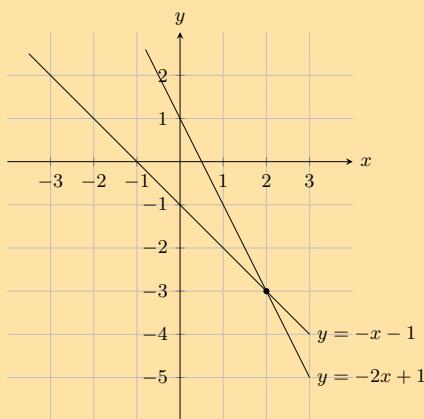
Solve the equations  $y = 2x + 1$  and  $y = -x - 5$  simultaneously.

2. Look at the graph below:



Solve the equations  $y = 2x - 1$  and  $y = 2x + 1$  simultaneously.

3. Look at the graph below:



Solve the equations  $y = -2x + 1$  and  $y = -x - 1$  simultaneously.

4. Solve for  $x$  and  $y$ :

- a)  $-10x = -1$  and  $-4x + 10y = -9$ .  
b)  $3x - 14y = 0$  and  $x - 4y + 1 = 0$   
c)  $x + y = 8$  and  $3x + 2y = 21$   
d)  $y = 2x + 1$  and  $x + 2y + 3 = 0$   
e)  $5x - 4y = 69$  and  $2x + 3y = 23$   
f)  $x + 3y = 26$  and  $5x + 4y = 75$   
g)  $3x - 4y = 19$  and  $2x - 8y = 2$   
h)  $\frac{a}{2} + b = 4$  and  $\frac{a}{4} - \frac{b}{4} = 1$   
i)  $-10x + y = -1$  and  $-10x - 2y = 5$   
j)  $-10x - 10y = -2$  and  $2x + 3y = 2$   
k)  $\frac{1}{x} + \frac{1}{y} = 3$  and  $\frac{1}{x} - \frac{1}{y} = 11$   
l)  $y = \frac{2(x^2 + 2) - 3}{x^2 + 2}$  and  $y = 2 - \frac{3}{x^2 + 2}$   
m)  $3a + b = \frac{6}{2a}$  and  $3a^2 = 3 - ab$

5. Solve graphically and check your answer algebraically:

- a)  $y + 2x = 0$  and  $y - 2x - 4 = 0$   
b)  $x + 2y = 1$  and  $\frac{x}{3} + \frac{y}{2} = 1$   
c)  $y - 2 = 6x$  and  $y - x = -3$   
d)  $2x + y = 5$  and  $3x - 2y = 4$   
e)  $5 = x + y$  and  $x = y - 2$

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## 4.5 Word problems

EMA3D

To solve word problems we need to write a set of equations that represent the problem mathematically. The solution of the equations is then the solution to the problem.

### Problem solving strategy

EMA3F

1. Read the whole question.
2. What are we asked to solve for?
3. Assign a variable to the unknown quantity, for example,  $x$ .
4. Translate the words into algebraic expressions by rewriting the given information in terms of the variable.
5. Set up an equation or system of equations to solve for the variable.
6. Solve the equation algebraically using substitution.
7. Check the solution.

#### VISIT:

The following video shows two examples of working with word problems.

► See video: [2FDX](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 11: Solving word problems

#### QUESTION

A shop sells bicycles and tricycles. In total there are 7 cycles (cycles include both bicycles and tricycles) and 19 wheels. Determine how many of each there are, if a bicycle has two wheels and a tricycle has three wheels.

## SOLUTION

---

### Step 1: Assign variables to the unknown quantities

Let  $b$  be the number of bicycles and let  $t$  be the number of tricycles.

### Step 2: Set up the equations

$$\begin{aligned} b + t &= 7 \quad \dots (1) \\ 2b + 3t &= 19 \quad \dots (2) \end{aligned}$$

### Step 3: Rearrange equation (1) and substitute into equation (2)

$$\begin{aligned} t &= 7 - b \\ \therefore 2b + 3(7 - b) &= 19 \\ -b &= -2 \\ \therefore b &= 2 \end{aligned}$$

### Step 4: Calculate the number of tricycles $t$

$$\begin{aligned} t &= 7 - b \\ &= 7 - 2 \\ &= 5 \end{aligned}$$

### Step 5: Write the final answer

There are 5 tricycles and 2 bicycles.

## Worked example 12: Solving word problems

## QUESTION

---

Bongani and Jane are friends. Bongani takes Jane's maths test paper and will not tell her what her mark is. He knows that Jane dislikes word problems so he decides to tease her. Bongani says: "I have 2 marks more than you do and the sum of both our marks is equal to 14. What are our marks?"

## SOLUTION

---

### Step 1: Assign variables to the unknown quantities

We have two unknown quantities, Bongani's mark and Jane's mark. Let Bongani's mark be  $b$  and Jane's mark be  $j$ .

### Step 2: Set up a system of equations

Bongani has 2 more marks than Jane.

$$b = j + 2 \quad \dots (1)$$

Both marks add up to 14.

$$b + j = 14 \quad \dots (2)$$

**Step 3: Use equation (1) to express  $b$  in terms of  $j$**

$$b = j + 2$$

**Step 4: Substitute into equation (2)**

$$\begin{aligned} b + j &= 14 \\ (j + 2) + j &= 14 \end{aligned}$$

**Step 5: Rearrange and solve for  $j$**

$$\begin{aligned} 2j &= 14 - 2 \\ &= 12 \\ \therefore j &= 6 \end{aligned}$$

**Step 6: Substitute the value for  $j$  back into equation (1) and solve for  $b$**

$$\begin{aligned} b &= j + 2 \\ &= 6 + 2 \\ &= 8 \end{aligned}$$

**Step 7: Check that the solution satisfies both original equations**

**Step 8: Write the final answer**

Bongani got 8 for his test and Jane got 6.

### Worked example 13: Solving word problems

#### QUESTION

A fruitshake costs R 2,00 more than a chocolate milkshake. If 3 fruitshakes and 5 chocolate milkshakes cost R 78,00, determine the individual prices.

#### SOLUTION

**Step 1: Assign variables to the unknown quantities**

Let the price of a chocolate milkshake be  $x$  and let the price of a fruitshake be  $y$ .

**Step 2: Set up a system of equations**

$$\begin{aligned} y &= x + 2 && \dots (1) \\ 3y + 5x &= 78 && \dots (2) \end{aligned}$$

**Step 3: Substitute equation (1) into (2)**

$$3(x + 2) + 5x = 78$$

**Step 4: Rearrange and solve for  $x$** 

$$\begin{aligned}3x + 6 + 5x &= 78 \\8x &= 72 \\\therefore x &= 9\end{aligned}$$

**Step 5: Substitute the value of  $x$  back into equation (1) and solve for  $y$** 

$$\begin{aligned}y &= x + 2 \\&= 9 + 2 \\&= 11\end{aligned}$$

**Step 6: Check that the solution satisfies both original equations****Step 7: Write final answer**

One chocolate milkshake costs R 9,00 and one fruitshake costs R 11,00.

**Worked example 14: Solving word problems****QUESTION**

The product of two consecutive negative integers is 1122. Find the two integers.

**SOLUTION****Step 1: Assign variables to the unknown quantities**

Let the first integer be  $n$  and let the second integer be  $n + 1$

**Step 2: Set up an equation**

$$n(n + 1) = 1122$$

**Step 3: Expand and solve for  $n$** 

$$\begin{aligned}n^2 + n &= 1122 \\n^2 + n - 1122 &= 0 \\(n + 34)(n - 33) &= 0 \\\therefore n &= -34 \\\text{or } n &= 33\end{aligned}$$

#### Step 4: Find the sign of the integers

It is given that both integers must be negative.

$$\begin{aligned}\therefore n &= -34 \\ n + 1 &= -34 + 1 \\ &= -33\end{aligned}$$

#### Step 5: Write the final answer

The two consecutive negative integers are  $-34$  and  $-33$ .

#### Exercise 4 – 4:

- Two jets are flying towards each other from airports that are  $1200$  km apart. One jet is flying at  $250 \text{ km} \cdot \text{h}^{-1}$  and the other jet at  $350 \text{ km} \cdot \text{h}^{-1}$ . If they took off at the same time, how long will it take for the jets to pass each other?
- Two boats are moving towards each other from harbours that are  $144$  km apart. One boat is moving at  $63 \text{ km} \cdot \text{h}^{-1}$  and the other boat at  $81 \text{ km} \cdot \text{h}^{-1}$ . If both boats started their journey at the same time, how long will they take to pass each other?
- Zwelibanzi and Jessica are friends. Zwelibanzi takes Jessica's civil technology test paper and will not tell her what her mark is. He knows that Jessica dislikes word problems so he decides to tease her. Zwelibanzi says: "I have  $12$  marks more than you do and the sum of both our marks is equal to  $148$ . What are our marks?"
- Kadesh bought  $20$  shirts at a total cost of R  $980$ . If the large shirts cost R  $50$  and the small shirts cost R  $40$ , how many of each size did he buy?
- The diagonal of a rectangle is  $25$  cm more than its width. The length of the rectangle is  $17$  cm more than its width. What are the dimensions of the rectangle?
- The sum of  $27$  and  $12$  is equal to  $73$  more than an unknown number. Find the unknown number.
- A group of friends is buying lunch. Here are some facts about their lunch:
  - a milkshake costs R  $7$  more than a wrap
  - the group buys  $8$  milkshakes and  $2$  wraps
  - the total cost for the lunch is R  $326$

Determine the individual prices for the lunch items.

- The two smaller angles in a right-angled triangle are in the ratio of  $1 : 2$ . What are the sizes of the two angles?
- The length of a rectangle is twice the breadth. If the area is  $128 \text{ cm}^2$ , determine the length and the breadth.
- If  $4$  times a number is increased by  $6$ , the result is  $15$  less than the square of the number. Find the number.
- The length of a rectangle is  $2$  cm more than the width of the rectangle. The perimeter of the rectangle is  $20$  cm. Find the length and the width of the rectangle.
- Stephen has  $1$  litre of a mixture containing  $69\%$  salt. How much water must Stephen add to make the mixture  $50\%$  salt? Write your answer as a fraction of a litre.
- The sum of two consecutive odd numbers is  $20$  and their difference is  $2$ . Find the two numbers.
- The denominator of a fraction is  $1$  more than the numerator. The sum of the fraction and its reciprocal is  $\frac{5}{2}$ . Find the fraction.

15. Masindi is 21 years older than her daughter, Mulivhu. The sum of their ages is 37. How old is Mulivhu?
16. Tshamano is now five times as old as his son Murunwa. Seven years from now, Tshamano will be three times as old as his son. Find their ages now.
17. If adding one to three times a number is the same as the number, what is the number equal to?
18. If a third of the sum of a number and one is equivalent to a fraction whose denominator is the number and numerator is two, what is the number?
19. A shop owner buys 40 sacks of rice and mealie meal worth R 5250 in total. If the rice costs R 150 per sack and mealie meal costs R 100 per sack, how many sacks of mealie meal did he buy?
20. There are 100 bars of blue and green soap in a box. The blue bars weigh 50 g per bar and the green bars 40 g per bar. The total mass of the soap in the box is 4,66 kg. How many bars of green soap are in the box?
21. Lisa has 170 beads. She has blue, red and purple beads each weighing 13 g, 4 g and 8 g respectively. If there are twice as many red beads as there are blue beads and all the beads weigh 1,216 kg, how many beads of each type does Lisa have?

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

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| 9. <a href="#">2FF8</a>  | 10. <a href="#">2FF9</a> | 11. <a href="#">2FFB</a> | 12. <a href="#">2FFC</a> | 13. <a href="#">2FFD</a> | 14. <a href="#">2FFF</a> | 15. <a href="#">2FFG</a> | 16. <a href="#">2FFH</a> |
| 17. <a href="#">2FFJ</a> | 18. <a href="#">2FFK</a> | 19. <a href="#">2FFM</a> | 20. <a href="#">2FFN</a> | 21. <a href="#">2FFP</a> |                          |                          |                          |



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## 4.6 Literal equations

EMA3G

A literal equation is one that has several letters or variables. Examples include the area of a circle ( $A = \pi r^2$ ) and the formula for speed ( $v = \frac{D}{t}$ ). In this section we solve literal equations in terms of one variable. To do this, we use the principles we have learnt about solving equations and apply them to rearranging literal equations. Solving literal equations is also known as changing the subject of the formula.

Keep the following in mind when solving literal equations:

- We isolate the unknown variable by asking "what is it joined to?" and "how is it joined?" We then perform the opposite operation to both sides as a whole.
- If the unknown variable is in two or more terms, then we take it out as a common factor.
- If we have to take the square root of both sides, remember that there will be a positive and a negative answer.
- If the unknown variable is in the denominator, we multiply both sides by the lowest common denominator (LCD) and then continue to solve.

### VISIT:

The following video shows an example of solving literal equations.

► See video: [2FFQ](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 15: Solving literal equations

#### QUESTION

The area of a triangle is  $A = \frac{1}{2}bh$ . What is the height of the triangle in terms of the base and area?

## SOLUTION

### Step 1: Isolate the required variable

We are asked to isolate the height, so we must rearrange the equation with  $h$  on one side of the equals sign and the rest of the variables on the other.

$$A = \frac{1}{2}bh$$

$$2A = bh$$

$$\frac{2A}{b} = h$$

### Step 2: Write the final answer

The height of a triangle is given by:  $h = \frac{2A}{b}$

## Worked example 16: Solving literal equations

### QUESTION

Given the formula:

$$h = R \times \frac{H}{R + r^2}$$

make  $R$  the subject of the formula.

### SOLUTION

### Step 1: Isolate the required variable

$$h(R + r^2) = R \times H$$

$$hR + hr^2 = HR$$

$$hr^2 = HR - hR$$

$$hr^2 = R(H - h)$$

$$\therefore R = \frac{hr^2}{H - h}$$

## Exercise 4 – 5:

1. Solve for  $x$  in the following formula:  $2x + 4y = 2$ .
2. Make  $a$  the subject of the formula:  $s = ut + \frac{1}{2}at^2$ .
3. Solve for  $n$ :  $pV = nRT$ .
4. Make  $x$  the subject of the formula:  $\frac{1}{b} + \frac{2b}{x} = 2$ .
5. Solve for  $r$ :  $V = \pi r^2 h$ .
6. Solve for  $h$ :  $E = \frac{hc}{\lambda}$ .

7. Solve for  $h$ :  $A = 2\pi rh + 2\pi r$ .
8. Make  $\lambda$  the subject of the formula:  $t = \frac{D}{f\lambda}$ .
9. Solve for  $m$ :  $E = mgh + \frac{1}{2}mv^2$ .
10. Solve for  $x$ :  $x^2 + x(a + b) + ab = 0$ .
11. Solve for  $b$ :  $c = \sqrt{a^2 + b^2}$ .
12. Make  $U$  the subject of the formula:  $\frac{1}{V} = \frac{1}{U} + \frac{1}{W}$ .
13. Solve for  $r$ :  $A = \pi R^2 - \pi r^2$ .
14.  $F = \frac{9}{5}C + 32$ ° is the formula for converting temperature in degrees Celsius to degrees Fahrenheit. Derive a formula for converting degrees Fahrenheit to degrees Celcius.
15.  $V = \frac{4}{3}\pi r^3$  is the formula for determining the volume of a soccer ball. Express the radius in terms of the volume.
16. Solve for  $x$  in:  $x^2 - ax - 3x = 4 + a$
17. Solve for  $x$  in:  $ax^2 - 4a + bx^2 - 4b = 0$
18. Solve for  $x$  in  $v^2 = u^2 + 2ax$  if  $v = 2$ ,  $u = 0,3$ ,  $a = 0,5$
19. Solve for  $u$  in  $f' = f \frac{v}{v-u}$  if  $v = 13$ ,  $f = 40$ ,  $f' = 50$
20. Solve for  $h$  in  $I = \frac{bh^2}{12}$  if  $b = 18$ ,  $I = 384$
21. Solve for  $r_2$  in  $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$  if  $R = \frac{3}{2}$ ,  $r_1 = 2$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

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| <a href="#">9. 2FG2</a>  | <a href="#">10. 2FG3</a> | <a href="#">11. 2FG4</a> | <a href="#">12. 2FG5</a> | <a href="#">13. 2FG6</a> | <a href="#">14. 2FG7</a> | <a href="#">15. 2FG8</a> | <a href="#">16. 2FG9</a> |
| <a href="#">17. 2FGB</a> | <a href="#">18. 2FGC</a> | <a href="#">19. 2FGD</a> | <a href="#">20. 2FGF</a> | <a href="#">21. 2FGG</a> |                          |                          |                          |



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## 4.7 Solving linear inequalities

EMA3H

A linear inequality is similar to a linear equation in that the largest exponent of a variable is 1. The following are examples of linear inequalities.

$$\begin{aligned} 2x + 2 &\leq 1 \\ \frac{2-x}{3x+1} &\geq 2 \\ \frac{4}{3}x - 6 &< 7x + 2 \end{aligned}$$

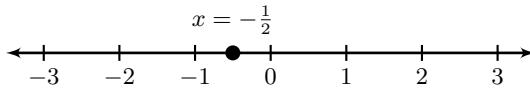
The methods used to solve linear inequalities are similar to those used to solve linear equations. The only difference occurs when there is a multiplication or a division that involves a minus sign. For example, we know that  $8 > 6$ . If both sides of the inequality are divided by  $-2$ , then we get  $-4 > -3$ , which is not true. Therefore, the inequality sign must be switched around, giving  $-4 < -3$ .

In order to compare an inequality to a normal equation, we shall solve an equation first.

Solve  $2x + 2 = 1$ :

$$\begin{aligned}2x + 2 &= 1 \\2x &= 1 - 2 \\2x &= -1 \\x &= -\frac{1}{2}\end{aligned}$$

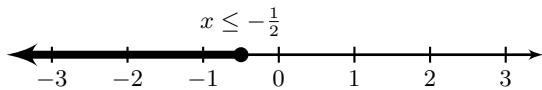
If we represent this answer on a number line, we get:



Now let us solve for  $x$  in the inequality  $2x + 2 \leq 1$ :

$$\begin{aligned}2x + 2 &\leq 1 \\2x &\leq 1 - 2 \\2x &\leq -1 \\x &\leq -\frac{1}{2}\end{aligned}$$

If we represent this answer on a number line, we get:



We see that for the equation there is only a single value of  $x$  for which the equation is true. However, for the inequality, there is a range of values for which the inequality is true. This is the main difference between an equation and an inequality.

**Remember:** when we divide or multiply both sides of an inequality by a negative number, the direction of the inequality changes. For example, if  $x < 1$ , then  $-x > -1$ . Also note that we cannot divide or multiply by a variable.

**NOTE:**

The following video provides an introduction to linear inequalities.

► See video: [2FGH](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## Interval notation

EMA3J

Examples:

(4; 12)	Round brackets indicate that the number is not included. This interval includes all real numbers greater than but not equal to 4 and less than but not equal to 12.
( $-\infty$ ; -1)	Round brackets are always used for positive and negative infinity. This interval includes all real numbers less than, but not equal to -1.
[1; 13)	A square bracket indicates that the number is included. This interval includes all real numbers greater than or equal to 1 and less than but not equal to 13.

It is important to note that this notation can only be used to represent an interval of real numbers.

We represent the above answer in interval notation as  $(-\infty; -\frac{1}{2}]$

## Worked example 17: Solving linear inequalities

### QUESTION

Solve for  $r$ :

$$6 - r > 2$$

Represent the answer on a number line and in interval notation.

### SOLUTION

#### Step 1: Rearrange and solve for $r$

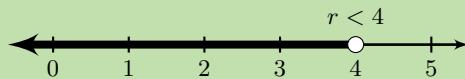
$$-r > 2 - 6$$

$$-r > -4$$

#### Step 2: Multiply by $-1$ and reverse inequality sign

$$r < 4$$

#### Step 3: Represent the answer on a number line



#### Step 4: Represent the answer in interval notation

$$(-\infty; 4)$$

## Worked example 18: Solving linear inequalities

### QUESTION

Solve for  $q$ :

$$4q + 3 < 2(q + 3)$$

Represent the answer on a number line and in interval notation.

### SOLUTION

#### Step 1: Expand the bracket

$$4q + 3 < 2(q + 3)$$

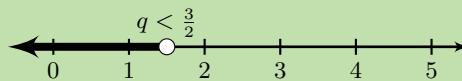
$$4q + 3 < 2q + 6$$

**Step 2: Rearrange and solve for  $q$** 

$$\begin{aligned}4q + 3 &< 2q + 6 \\4q - 2q &< 6 - 3 \\2q &< 3\end{aligned}$$

**Step 3: Divide both sides by 2**

$$\begin{aligned}2q &< 3 \\q &< \frac{3}{2}\end{aligned}$$

**Step 4: Represent the answer on a number line****Step 5: Represent the answer in interval notation**

$$(-\infty; \frac{3}{2})$$

**Worked example 19: Solving compound linear inequalities****QUESTION**

Solve for  $x$ :

$$5 \leq x + 3 < 8$$

Represent the answer on a number line and in interval notation.

**SOLUTION****Step 1: Subtract 3 from all the parts of the inequality**

$$\begin{array}{rclcl}5 - 3 & \leq & x + 3 - 3 & < & 8 - 3 \\2 & \leq & x & < & 5\end{array}$$

**Step 2: Represent the answer on a number line****Step 3: Represent the answer in interval notation**

$$[2; 5)$$

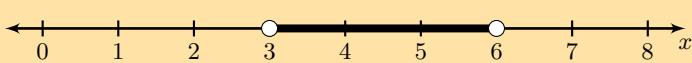
**Exercise 4 – 6:**

1. Look at the number line and write down the inequality it represents.

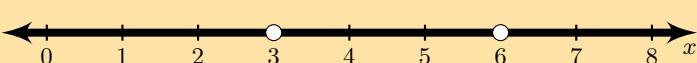
a)



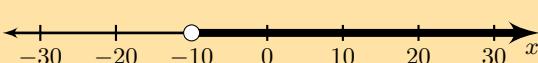
b)



c)



d)



2. Solve for  $x$  and represent the answer on a number line and in interval notation.

a)  $3x + 4 > 5x + 8$

b)  $3(x - 1) - 2 \leq 6x + 4$

c)  $\frac{x - 7}{3} > \frac{2x - 3}{2}$

d)  $-4(x - 1) < x + 2$

e)  $\frac{1}{2}x + \frac{1}{3}(x - 1) \geq \frac{5}{6}x - \frac{1}{3}$

f)  $-2 \leq x - 1 < 3$

g)  $-5 < 2x - 3 \leq 7$

h)  $7(3x + 2) - 5(2x - 3) > 7$

i)  $\frac{5x - 1}{-6} \geq \frac{1 - 2x}{3}$

j)  $3 \leq 4 - x \leq 16$

k)  $\frac{-7y}{3} - 5 > -7$

l)  $1 \leq 1 - 2y < 9$

m)  $-2 < \frac{x - 1}{-3} < 7$

3. Solve for  $x$  and show your answer in interval notation:

a)  $2x - 1 < 3(x + 11)$

b)  $x - 1 < -4(x - 6)$

c)  $\frac{x - 1}{8} \leq \frac{2(x - 2)}{3}$

d)  $\frac{x + 2}{4} \leq \frac{-2(x - 4)}{7}$

e)  $\frac{1}{5}x - \frac{5}{4}(x + 2) > \frac{1}{4}x + 3$

f)  $\frac{1}{5}x - \frac{2}{5}(x + 3) \geq \frac{4}{2}x + 3$

g)  $4x + 3 < -3 \quad \text{or} \quad 4x + 3 > 5$

h)  $4 \geq -6x - 6 \geq -3$

4. Solve for the unknown variable and show your answer on a number line.

a)  $6b - 3 > b + 2, \quad b \in \mathbb{Z}$

b)  $3a - 1 < 4a + 6, \quad a \in \mathbb{N}$

c)  $\frac{b - 3}{2} + 1 < \frac{b}{4} - 4, \quad b \in \mathbb{R}$

d)  $\frac{4a + 7}{3} - 5 > a - \frac{2}{3}, \quad a \in \mathbb{N}$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

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| 1a. <a href="#">2FGJ</a> | 1b. <a href="#">2FGK</a> | 1c. <a href="#">2FGM</a> | 1d. <a href="#">2FGN</a> | 2a. <a href="#">2FGP</a> | 2b. <a href="#">2FGQ</a> | 2c. <a href="#">2FGR</a> |
| 2d. <a href="#">2FGS</a> | 2e. <a href="#">2FGT</a> | 2f. <a href="#">2FGV</a> | 2g. <a href="#">2FGW</a> | 2h. <a href="#">2FGX</a> | 2i. <a href="#">2FGY</a> | 2j. <a href="#">2FGZ</a> |
| 2k. <a href="#">2FH2</a> | 2l. <a href="#">2FH3</a> | 2m. <a href="#">2FH4</a> | 3a. <a href="#">2FH5</a> | 3b. <a href="#">2FH6</a> | 3c. <a href="#">2FH7</a> | 3d. <a href="#">2FH8</a> |
| 3e. <a href="#">2FH9</a> | 3f. <a href="#">2FHB</a> | 3g. <a href="#">2FHC</a> | 3h. <a href="#">2FHD</a> | 4a. <a href="#">2FHF</a> | 4b. <a href="#">2FHG</a> | 4c. <a href="#">2FHH</a> |
| 4d. <a href="#">2FHJ</a> |                          |                          |                          |                          |                          |                          |



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► See presentation: [2FHK](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

- A linear equation is an equation where the exponent of the variable is 1. A linear equation has at most one solution.
- A quadratic equation is an equation where the exponent of the variable is at most 2. A quadratic equation has at most two solutions.
- To solve for two unknown variables, two equations are required. These equations are known as a system of simultaneous equations. There are two ways to solve linear simultaneous equations: algebraic solutions and graphical solutions. To solve algebraically we use substitution or elimination methods. To solve graphically we draw the graph of each equation and the solution will be the coordinates of the point of intersection.
- Literal equations are equations that have several letters and variables.
- Word problems require a set of equations that represent the problem mathematically.
- A linear inequality is similar to a linear equation and has the exponent of the variable equal to 1.
- If we divide or multiply both sides of an inequality by any number with a minus sign, the direction of the inequality changes.

**End of chapter Exercise 4 – 7:**

1. Solve:

a)  $5a - 7 = -2$

b)  $5m + 3 = -2$

c)  $1 = 4 - 3y$

d)  $2(p - 1) = 3(p + 2)$

e)  $3 - 6k = 2k - 1$

f)  $2,1x + 3 = 4,1 - 3,3x$

g)  $m + 6(-m + 1) + 8m = 0$

h)  $2k + 3 = 2 - 3(k + 2)$

i)  $3 + \frac{q}{5} = \frac{q}{2}$

j)  $\frac{1}{2} = \frac{4z + 1}{5z - 6}$

k)  $2 + \frac{a - 4}{2} - \frac{a}{3} = 7$

l)  $5 - \frac{2(m + 4)}{m} = \frac{7}{m}$

m)  $\frac{2}{t} - 2 - \frac{1}{2} = \frac{1}{2} \left(1 + \frac{2}{t}\right)$

2. Solve:

a)  $b^2 + 6b - 27 = 0$

b)  $-x^2 + 5x + 6 = 0$

c)  $-b^2 - 3b + 10 = 0$

d)  $2b - 15 = (b + 1)(b - 6) - b^2$

e)  $(5x + 1)(x - 3) = 0$

f)  $5t - 1 = t^2 - (t + 2)(t - 2)$

g)  $\frac{a + 2}{a - 3} = \frac{a + 8}{a + 4}$

h)  $\frac{n + 3}{n - 2} = \frac{n - 1}{n - 7}$

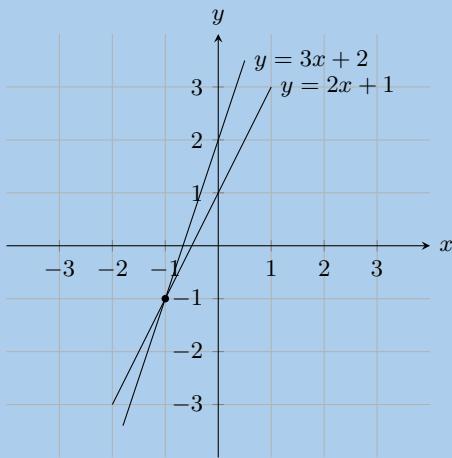
i)  $x^2 - 3x + 2 = 0$

j)  $y^2 + y = 6$

k)  $0 = 2x^2 - 5x - 18$

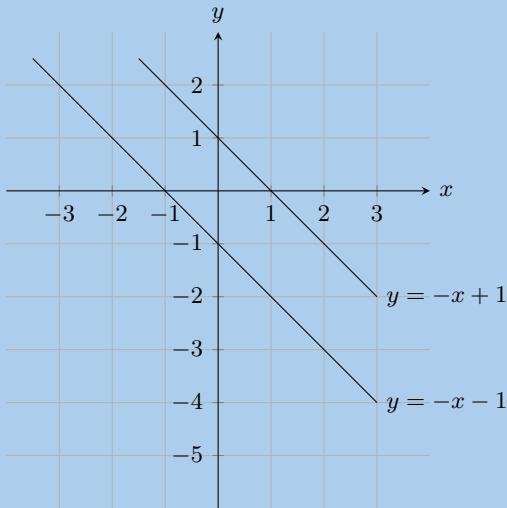
l)  $(d + 4)(d - 3) - d = (3d - 2)^2 - 8d(d - 1)$

3. Look at the graph below:



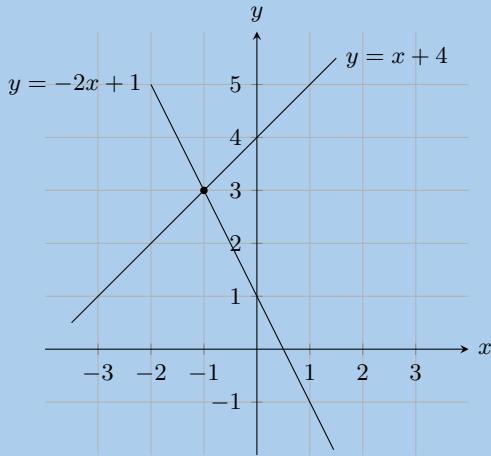
Solve the equations  $y = 3x + 2$  and  $y = 2x + 1$  simultaneously.

4. Look at the graph below:



Solve the equations  $y = -x + 1$  and  $y = -x - 1$  simultaneously.

5. Look at the graph below:



Solve the equations  $y = x + 4$  and  $y = -2x + 1$  simultaneously.

6. Solve the following simultaneous equations:

- a)  $7x + 3y = 13$  and  $2x - 3y = -4$   
b)  $10 = 2x + y$  and  $y = x - 2$   
c)  $7x - 41 = 3y$  and  $17 = 3x - y$   
d)  $2x - 4y = 32$  and  $7x + 2y = 32$   
e)  $7x + 6y = -18$  and  $4x + 12y = 24$   
f)  $3x - 4y = -15$  and  $12x + 5y = 66$   
g)  $x - 3y = -22$  and  $5x + 2y = -25$   
h)  $3x + 2y = 46$  and  $15x + 5y = 220$   
i)  $6x + 3y = -63$  and  $24x + 4y = -212$   
j)  $5x - 6y = 11$  and  $25x - 3y = 28$   
k)  $-9x + 3y = 4$  and  $2x + 2y = 6$   
l)  $3x - 7y = -10$  and  $10x + 2y = -6$   
m)  $2y = x + 8$  and  $4y = 2x - 44$   
n)  $2a(a - 1) - 4 + a - b = 0$  and  $2a^2 - a = b + 4$   
o)  $y = (x - 2)^2$  and  $x(x + 3) - y = 3x + 4(x - 1)$   
p)  $\frac{x+1}{y} = 7$  and  $\frac{x}{y+1} = 6$   
q)  $(x + 3)^2 + (y - 4)^2 = 0$

7. Find the solutions to the following word problems:

- a)  $\frac{7}{8}$  of a certain number is 5 more than  $\frac{1}{3}$  of the number. Find the number.  
b) Three rulers and two pens have a total cost of R 21,00. One ruler and one pen have a total cost of R 8,00. How much does a ruler cost and how much does a pen cost?  
c) A group of friends is buying lunch. Here are some facts about their lunch:
  - a hotdog costs R 6 more than a milkshake
  - the group buys 3 hotdogs and 2 milkshakes
  - the total cost for the lunch is R 143Determine the individual prices for the lunch items.

- d) Lefu and Monique are friends. Monique takes Lefu's business studies test paper and will not tell him what his mark is. She knows that Lefu dislikes word problems so she decides to tease him. Monique says: "I have 12 marks more than you do and the sum of both our marks is equal to 166. What are our marks?"  
e) A man runs to the bus stop and back in 15 minutes. His speed on the way to the bus stop is  $5 \text{ km}\cdot\text{h}^{-1}$  and his speed on the way back is  $4 \text{ km}\cdot\text{h}^{-1}$ . Find the distance to the bus stop.  
f) Two trucks are travelling towards each other from factories that are 175 km apart. One truck is travelling at  $82 \text{ km}\cdot\text{h}^{-1}$  and the other truck at  $93 \text{ km}\cdot\text{h}^{-1}$ . If both trucks started their journey at the same time, how long will they take to pass each other?  
g) Zanele and Piet skate towards each other on a straight path. They set off 20 km apart. Zanele skates at  $15 \text{ km}\cdot\text{h}^{-1}$  and Piet at  $10 \text{ km}\cdot\text{h}^{-1}$ . How far will Piet have skated when they reach each other?  
h) When the price of chocolates is increased by R 10, we can buy five fewer chocolates for R 300. What was the price of each chocolate before the price was increased?  
i) A teacher bought R 11 300 worth of textbooks. The text books were for Science and Mathematics with each of them being sold at R 100 per book and R 125 per book respectively. If the teacher bought 97 books in total, how many Science books did she buy?  
j) Thom's mom bought R 91,50 worth of easter eggs. The easter eggs came in 3 different colours blue, green and yellow. The blue ones cost R 2 each, green ones R 1,50 each and yellow ones R 1 each. She bought three times as many yellow eggs as the green ones and 72 eggs in total. How many blue eggs did she buy?  
k) Two equivalent fractions both have their numerator as one. The denominator of one fraction is the sum of two and a number, while the other fraction is twice the number less 3. What is the number?

8. Consider the following literal equations:

- a) Solve for  $x$ :  $a - bx = c$

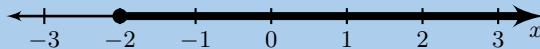
- b) Solve for  $I$ :  $P = VI$ .
- c) Make  $m$  the subject of the formula:  $E = mc^2$ .
- d) Solve for  $t$ :  $v = u + at$ .
- e) Make  $f$  the subject of the formula:  $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ .
- f) Solve for  $y$ :  $m = \frac{y - c}{x}$ .
- g) Solve for  $x$  in:  $ax - 4a + ab = 4b - bx - b^2 + 4c - cx - bc$
- h) Solve for  $r$  in  $S = \frac{a}{1-r}$  if  $a = 0,4$  and  $S = 3$
- i) Solve for  $b$  in  $I = \frac{1}{2}M(a^2 + b^2)$  if  $a = 4$ ,  $M = 8$ ,  $I = 320$

9. Write down the inequality represented by the following:

a)



b)



c)



10. Solve for  $x$  and show your answer in interval notation

a)  $-4x + 1 > -2(x - 15)$

b)  $\frac{x+2}{4} \leq \frac{-1(x+1)}{6}$

c)  $\frac{1}{4}x + \frac{2}{3}(x+1) \geq \frac{2}{5}x + 2$

d)  $3x - 3 > 14 \quad \text{or} \quad 3x - 3 < -2$

11. Solve and represent your answer on a number line

a)  $2x - 3 < \frac{3x - 2}{2}, x \in \mathbb{N}$

b)  $3(1 - b) - 4 + b > 7 + b, b \in \mathbb{Z}$

c)  $1 - 5x > 4(x + 1) - 3, x \in \mathbb{R}$

12. Solve for the unknown variable:

a)  $2 + 2\frac{1}{3}(x + 4) = \frac{1}{5}(3 - x) + \frac{1}{6}$

b)  $36 - (x - 4)^2 = 0$

c)  $64 - (a + 3)^2 = 0$

d)  $\frac{1}{2}x - \frac{2}{x} = 0$

e)  $a - 3 = 2 \left( \frac{6}{a} + 1 \right)$

f)  $a - \frac{6}{a} = -1$

g)  $(a + 6)^2 - 5(a + 6) - 24 = 0$

h)  $a^4 - 4a^2 + 3 = 0$

i)  $9y^4 - 13y^2 + 4 = 0$

j)  $\frac{(b+1)^2 - 16}{b+5} = 1$

k)  $\frac{a^2 + 8a + 7}{a+7} = 2$

l)  $5x + 2 \leq 4(2x - 1)$

m)  $\frac{4x - 2}{6} > 2x + 1$

n)  $\frac{x}{3} - 14 > 14 - \frac{x}{7}$

o)  $\frac{1-a}{2} - \frac{2-a}{3} \geq 1$   
 q)  $x-1 = \frac{42}{x}$   
 s)  $3ax + 2a - ax = 5ax - 6a$   
 u)  $3x^2 - xy - 2y^2 = 0$

w)  $\frac{2x-5}{(x+2)(x-4)} = \frac{1}{2(x-4)}$   
 y)  $\frac{x+4}{3} - 2 > \frac{x-3}{2} - \frac{x+1}{4}$

p)  $-5 \leq 2k+1 < 5$   
 r)  $(x+1)^2 = (x+1)(2x+3)$   
 t)  $\frac{ax}{b} - \frac{bx}{a} = \frac{a}{b} + 1$   
 v)  $x(2x+1) = 1$

x)  $x^2 + 1 = 0$

13. After solving an equation, Luke gave his answer as 4,5 rounded to one decimal digit. Show on a number line the interval in which his solution lay.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1a. <a href="#">2FHM</a>	1b. <a href="#">2FHN</a>	1c. <a href="#">2FHP</a>	1d. <a href="#">2FHQ</a>	1e. <a href="#">2FHR</a>	1f. <a href="#">2FHS</a>	1g. <a href="#">2FHT</a>
1h. <a href="#">2FHV</a>	1i. <a href="#">2FHW</a>	1j. <a href="#">2FHX</a>	1k. <a href="#">2FHY</a>	1l. <a href="#">2FHZ</a>	1m. <a href="#">2FJ2</a>	2a. <a href="#">2FJ3</a>
2b. <a href="#">2FJ4</a>	2c. <a href="#">2FJ5</a>	2d. <a href="#">2FJ6</a>	2e. <a href="#">2FJ7</a>	2f. <a href="#">2FJ8</a>	2g. <a href="#">2FJ9</a>	2h. <a href="#">2FJB</a>
2i. <a href="#">2FJC</a>	2j. <a href="#">2FJD</a>	2k. <a href="#">2FJF</a>	2l. <a href="#">2FJG</a>	3. <a href="#">2FJH</a>	4. <a href="#">2FJJ</a>	5. <a href="#">2FJK</a>
6a. <a href="#">2FJM</a>	6b. <a href="#">2FJN</a>	6c. <a href="#">2FJP</a>	6d. <a href="#">2FJQ</a>	6e. <a href="#">2FJR</a>	6f. <a href="#">2FJS</a>	6g. <a href="#">2FJT</a>
6h. <a href="#">2FJV</a>	6i. <a href="#">2FJW</a>	6j. <a href="#">2FJX</a>	6k. <a href="#">2FJY</a>	6l. <a href="#">2FJZ</a>	6m. <a href="#">2FK2</a>	6n. <a href="#">2FK3</a>
6o. <a href="#">2FK4</a>	6p. <a href="#">2FK5</a>	6q. <a href="#">2FK6</a>	7a. <a href="#">2FK7</a>	7b. <a href="#">2FK8</a>	7c. <a href="#">2FK9</a>	7d. <a href="#">2FKB</a>
7e. <a href="#">2FKC</a>	7f. <a href="#">2FKD</a>	7g. <a href="#">2FKF</a>	7h. <a href="#">2FGK</a>	7i. <a href="#">2FKH</a>	7j. <a href="#">2FKJ</a>	7k. <a href="#">2FKK</a>
8a. <a href="#">2FKM</a>	8b. <a href="#">2FKN</a>	8c. <a href="#">2FKP</a>	8d. <a href="#">2FKQ</a>	8e. <a href="#">2FKR</a>	8f. <a href="#">2FKS</a>	8g. <a href="#">2FKT</a>
8h. <a href="#">2FKV</a>	8i. <a href="#">2FKW</a>	9a. <a href="#">2FKX</a>	9b. <a href="#">2FKY</a>	9c. <a href="#">2FKZ</a>	10a. <a href="#">2FM2</a>	10b. <a href="#">2FM3</a>
10c. <a href="#">2FM4</a>	10d. <a href="#">2FM5</a>	11a. <a href="#">2FM6</a>	11b. <a href="#">2FM7</a>	11c. <a href="#">2FM8</a>	12a. <a href="#">2FM9</a>	12b. <a href="#">2FMB</a>
12c. <a href="#">2FMC</a>	12d. <a href="#">2FMD</a>	12e. <a href="#">2FMF</a>	12f. <a href="#">2FMG</a>	12g. <a href="#">2FMH</a>	12h. <a href="#">2FMJ</a>	12i. <a href="#">2FMK</a>
12j. <a href="#">2FMM</a>	12k. <a href="#">2FMN</a>	12l. <a href="#">2FMP</a>	12m. <a href="#">2FMQ</a>	12n. <a href="#">2FMR</a>	12o. <a href="#">2FMS</a>	12p. <a href="#">2FMT</a>
12q. <a href="#">2FMV</a>	12r. <a href="#">2FMW</a>	12s. <a href="#">2FMX</a>	12t. <a href="#">2FMY</a>	12u. <a href="#">2FMZ</a>	12v. <a href="#">2FN2</a>	12w. <a href="#">2FN3</a>
12x. <a href="#">2FN4</a>	12y. <a href="#">2FN5</a>	13. <a href="#">2FN6</a>				



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# CHAPTER



## *Trigonometry*

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## 5.1 Introduction

EMA3M

Trigonometry deals with the relationship between the angles and sides of a triangle. We will learn about trigonometric ratios in right-angled triangles, which form the basis of trigonometry.

There are many applications of trigonometry. Of particular value is the technique of triangulation, which is used in astronomy to measure the distances to nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. GPS (the global positioning system) would not be possible without trigonometry. Other fields which make use of trigonometry include acoustics, optics, analysis of financial markets, electronics, probability theory, statistics, biology, medical imaging (CAT scans and ultrasound), chemistry, cryptology, meteorology, oceanography, land surveying, architecture, phonetics, engineering, computer graphics and game development.



Figure 5.1: An artist's depiction of a GPS satellite orbiting the Earth. There are at least 24 GPS satellites operational at any one time. GPS uses an application of trigonometry, known as triangulation, to determine ones position. The accuracy of GPS is to within 15 metres.

### VISIT:

The following video covers a brief history of trigonometry and some of the uses of trigonometry.

► See video: [2FN7](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

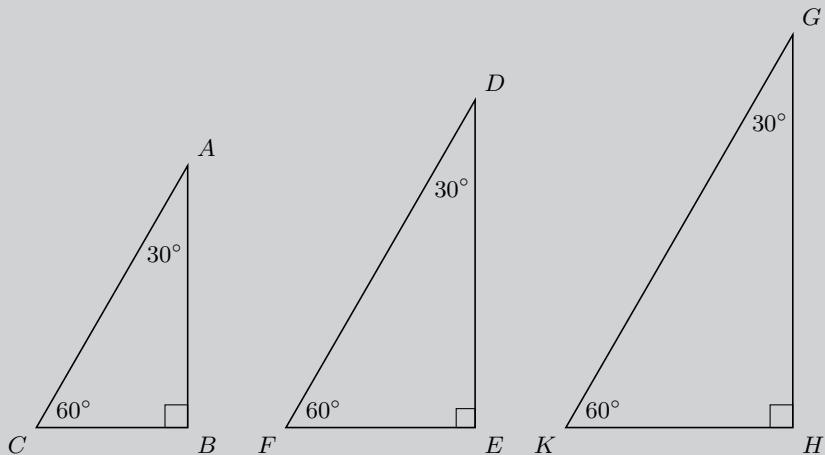
## 5.2 Similarity of triangles

EMA3N

Before we delve into the theory of trigonometry, complete the following investigation to get a better understanding of the foundation of trigonometry.

### Investigation: Ratios of similar triangles

Draw three similar triangles of different sizes using a protractor and a ruler, with each triangle having interior angles equal to  $30^\circ$ ,  $90^\circ$  and  $60^\circ$  as shown below. Measure the angles and lengths accurately in order to fill in the table (leave your answers as a simplified fraction):

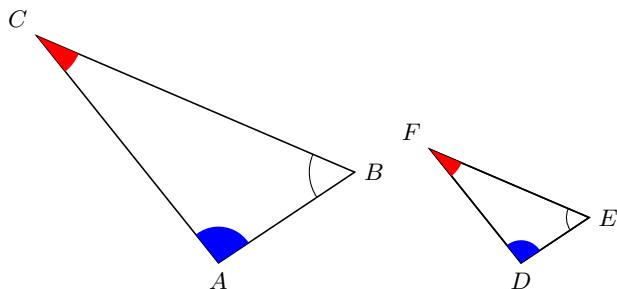


Dividing lengths of sides (ratios)		
$\frac{AB}{BC} =$	$\frac{AB}{AC} =$	$\frac{CB}{AC} =$
$\frac{DE}{EF} =$	$\frac{DE}{DF} =$	$\frac{FE}{DF} =$
$\frac{GH}{HK} =$	$\frac{GH}{GK} =$	$\frac{KH}{GK} =$

What observations can you make about the ratios of the sides?

Have you noticed that it does not matter what the lengths of the sides of the triangles are, if the angle remains constant, the ratio of the sides will always yield the same answer?

In the triangles below,  $\triangle ABC$  is similar to  $\triangle DEF$ . This is written as:  $\triangle ABC \parallel\!\!\!|| \triangle DEF$



In similar triangles, it is possible to deduce ratios between corresponding sides:

$$\begin{aligned}\frac{AB}{BC} &= \frac{DE}{EF} \\ \frac{AB}{AC} &= \frac{DE}{DF} \\ \frac{AC}{BC} &= \frac{DF}{EF} \\ \frac{AB}{DE} &= \frac{BC}{EF} = \frac{AC}{DF}\end{aligned}$$

Another important fact about similar triangles  $ABC$  and  $DEF$  is that the angle at vertex  $A$  is equal to the angle at vertex  $D$ , the angle at vertex  $B$  is equal to the angle at vertex  $E$ , and the angle at vertex  $C$  is equal to the angle at vertex  $F$ .

$$\hat{A} = \hat{D}$$

$$\hat{B} = \hat{E}$$

$$\hat{C} = \hat{F}$$

**NOTE:**

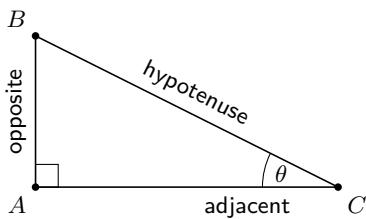
The order of letters for similar triangles is very important. Always label similar triangles in corresponding order. For example,

$\triangle ABC \parallel\!\!\!|| \triangle DEF$  is correct; but  
 $\triangle ABC \parallel\!\!\!|| \triangle DFE$  is incorrect.

## 5.3 Defining the trigonometric ratios

EMA3P

The ratios of similar triangles are used to define the trigonometric ratios. Consider a right-angled triangle  $ABC$  with an angle marked  $\theta$  (said 'theta').



In a right-angled triangle, we refer to the three sides according to how they are placed in relation to the angle  $\theta$ . The side opposite to the right-angle is labelled the hypotenuse, the side opposite  $\theta$  is labelled "opposite", the side next to  $\theta$  is labelled "adjacent".

You can choose either non- $90^\circ$  internal angle and then define the adjacent and opposite sides accordingly. However, the hypotenuse remains the same regardless of which internal angle you are referring to because it is *always* opposite the right-angle and *always* the longest side.

We define the trigonometric ratios: sine (sin), cosine (cos) and tangent (tan), of an angle, as follows:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

These ratios, also known as trigonometric identities, relate the lengths of the sides of a right-angled triangle to its interior angles. These three ratios form the basis of trigonometry.

**IMPORTANT!**

The definitions of opposite, adjacent and hypotenuse are only applicable when working with right-angled triangles! Always check to make sure your triangle has a right-angle before you use them, otherwise you will get the wrong answer.

You may also hear people saying “Soh Cah Toa”. This is a mnemonic technique for remembering the trigonometric ratios:

$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$
$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$
$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$

### VISIT:

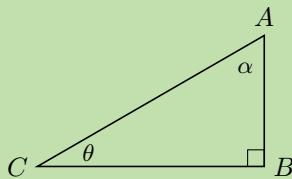
The following video introduces the three trigonometric ratios.

► See video: 2FN8 at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 1: Trigonometric ratios

#### QUESTION

Given the following triangle:



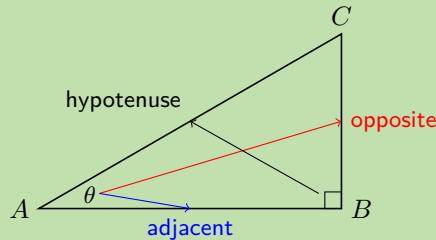
- Label the hypotenuse, opposite and adjacent sides of the triangle with respect to  $\theta$ .
- State which sides of the triangle you would use to find  $\sin \theta$ ,  $\cos \theta$  and  $\tan \theta$ .
- Label the hypotenuse, opposite and adjacent sides of the triangle with respect to  $\alpha$ .
- State which sides of the triangle you would use to find  $\sin \alpha$ ,  $\cos \alpha$  and  $\tan \alpha$ .

#### SOLUTION

##### Step 1: Label the triangle

First find the right angle, the hypotenuse is **always** directly opposite the right angle. The hypotenuse never changes position, it is always directly opposite the right angle and so we find this first.

The opposite and adjacent sides depend on the angle we are looking at. The opposite side relative to the angle  $\theta$  is directly opposite (as the word opposite suggests) the angle  $\theta$ . Finally the adjacent side is the remaining side of the triangle and must be one of the sides that forms the angle  $\theta$ .



## Step 2: Complete the trigonometric ratios

Now we can complete the trigonometric ratios for  $\theta$ :

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{CB}{AC}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{CB}{AB}$$

Therefore to find  $\sin \theta$  we would use sides  $CB$  (opposite side to  $\theta$ ) and  $AC$  (hypotenuse). To find  $\cos \theta$  we would use sides  $AB$  (adjacent side to  $\theta$ ) and  $AC$  (hypotenuse). To find  $\tan \theta$  we would use sides  $CB$  (opposite side to  $\theta$ ) and  $AB$  (adjacent side to  $\theta$ ).

And then we can complete the trigonometric ratios for  $\alpha$ . For angle  $\alpha$  the opposite and adjacent sides switch places (redraw the triangle above to help you see this). Notice how the hypotenuse is still  $AC$ .

$$\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AB}{AC}$$

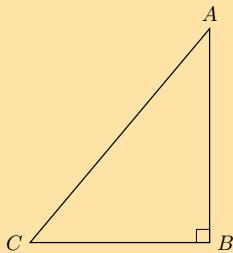
$$\cos \alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{CB}{AC}$$

$$\tan \alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{AB}{CB}$$

Therefore to find  $\sin \alpha$  we would use sides  $AB$  (opposite side to  $\alpha$ ) and  $AC$  (hypotenuse). To find  $\cos \alpha$  we would use sides  $CB$  (adjacent side to  $\alpha$ ) and  $AC$  (hypotenuse). To find  $\tan \alpha$  we would use sides  $AB$  (opposite side to  $\alpha$ ) and  $CB$  (adjacent side to  $\alpha$ ).

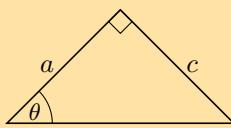
### Exercise 5 – 1:

1. Complete each of the following:

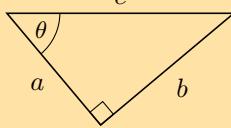


- a)  $\sin \hat{A} =$       b)  $\cos \hat{A} =$       c)  $\tan \hat{A} =$   
d)  $\sin \hat{C} =$       e)  $\cos \hat{C} =$       f)  $\tan \hat{C} =$
2. In each of the following triangles, state whether  $a$ ,  $b$  and  $c$  are the hypotenuse, opposite or adjacent sides of the triangle with respect to  $\theta$ .

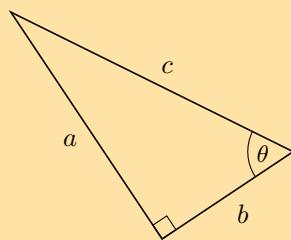
a)



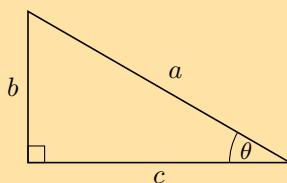
b)



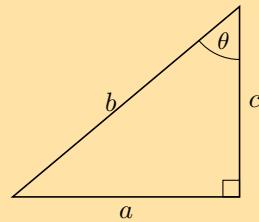
c)



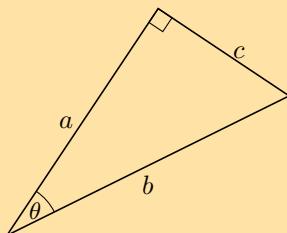
d)



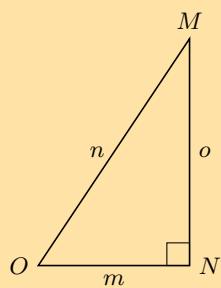
e)



f)



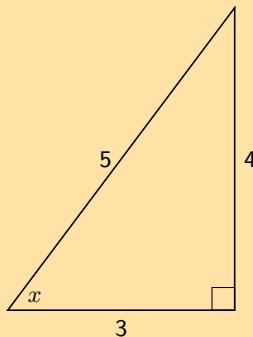
3. Consider the following diagram:



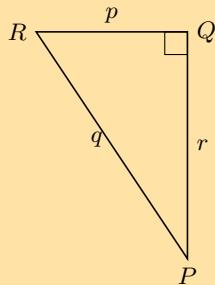
Without using a calculator, answer each of the following questions.

- Write down  $\cos \hat{O}$  in terms of  $m$ ,  $n$  and  $o$ .
- Write down  $\tan \hat{M}$  in terms of  $m$ ,  $n$  and  $o$ .
- Write down  $\sin \hat{O}$  in terms of  $m$ ,  $n$  and  $o$ .
- Write down  $\cos \hat{M}$  in terms of  $m$ ,  $n$  and  $o$ .

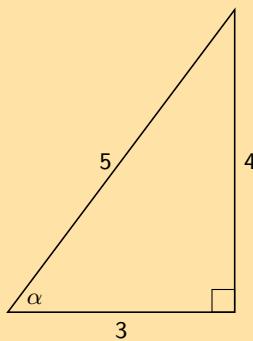
4. Find  $x$  in the diagram in three different ways. You do not need to calculate the value of  $x$ , just write down the appropriate ratio for  $x$ .



5. Which of these statements is true about  $\triangle PQR$ ?

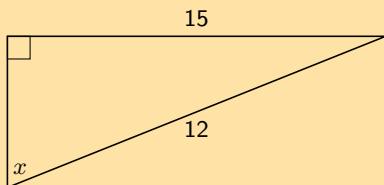


- a)  $\sin R = \frac{p}{q}$       b)  $\tan Q = \frac{r}{p}$       c)  $\cos P = \frac{r}{q}$       d)  $\sin P = \frac{p}{r}$
6. Sarah wants to find the value of  $\alpha$  in the triangle below. Which statement is a correct line of working?

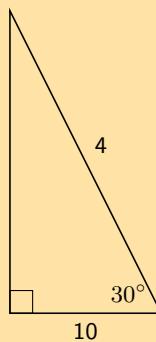


- a)  $\sin \alpha = \frac{4}{5}$       b)  $\cos \left( \frac{3}{5} \right) = \alpha$       c)  $\tan \alpha = \frac{5}{4}$       d)  $\cos 0,8 = \alpha$
7. Explain what is wrong with each of the following diagrams.

a)



b)



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |           |           |           |          |          |          |           |           |
|-----------|-----------|-----------|----------|----------|----------|-----------|-----------|
| 1. 2FN9   | 2a. 2FNB  | 2b. 2FNC  | 2c. 2FND | 2d. 2FNF | 2e. 2FNG | 2f. 2FNH  | 3a. 2F NJ |
| 3b. 2F NK | 3c. 2F NM | 3d. 2F NN | 4. 2F NP | 5. 2F NQ | 6. 2F NR | 7a. 2F NS | 7b. 2F NT |



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## 5.4 Reciprocal ratios

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Each of the three trigonometric ratios has a reciprocal. The reciprocals: cosecant (cosec), secant (sec) and cotangent (cot), are defined as follows:

$$\text{cosec } \theta = \frac{1}{\sin \theta}$$

$$\text{sec } \theta = \frac{1}{\cos \theta}$$

$$\text{cot } \theta = \frac{1}{\tan \theta}$$

We can also define these reciprocals for any right-angled triangle:

$$\text{cosec } \theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\text{sec } \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\text{cot } \theta = \frac{\text{adjacent}}{\text{opposite}}$$

Note that:

$$\sin \theta \times \text{cosec } \theta = 1$$

$$\cos \theta \times \text{sec } \theta = 1$$

$$\tan \theta \times \text{cot } \theta = 1$$

### VISIT:

This video covers the three reciprocal ratios for sin, cos and tan.

► See video: [2FNV](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### NOTE:

You may see cosecant abbreviated as csc.

In this section we will look at using a calculator to determine the values of the trigonometric ratios for any angle. For example we might want to know what the value of  $\sin 55^\circ$  is or what the value of  $\sec 34^\circ$  is.

When doing calculations involving the reciprocal ratios you need to convert the reciprocal ratio to one of the standard trigonometric ratios: sin, cos and tan as this is the only way to calculate these ratios on your calculator.

### **IMPORTANT!**

Most scientific calculators are quite similar but these steps might differ depending on the calculator you use. Make sure your calculator is in “degrees” mode.

#### **NOTE:**

Note that  $\sin^2 \theta = (\sin \theta)^2$ . This also applies for the other trigonometric ratios.

### **Worked example 2: Using your calculator**

#### **QUESTION**

Use your calculator to calculate the following (correct to 2 decimal places):

1.  $\cos 48^\circ$
2.  $2 \sin 35^\circ$
3.  $\tan^2 81^\circ$
4.  $3 \sin^2 72^\circ$
5.  $\frac{1}{4} \cos 27^\circ$
6.  $\frac{5}{6} \tan 34^\circ$
7.  $\sec 34^\circ$
8.  $\cot 49^\circ$

#### **SOLUTION**

##### **Step 1:**

The following shows the keys to press on a Casio calculator. Other calculators work in a similar way. On a Casio calculator ( is automatically added after pressing sin, cos and tan so you just need to press  $\boxed{)}$  after typing in the angle to close the brackets.

1. Press  $\boxed{\cos} \quad \boxed{48} \quad \boxed{)} \quad \boxed{=} \quad 0,66913... \approx 0,67$
2. Press  $\boxed{2} \quad \boxed{\sin} \quad \boxed{35} \quad \boxed{)} \quad \boxed{=} \quad 1,147152... \approx 1,15$
3. Press  $\boxed{(} \quad \boxed{\tan} \quad \boxed{81} \quad \boxed{)} \quad \boxed{)} \quad \boxed{x^2} \quad \boxed{=} \quad 39,86345... \approx 39,86$   
OR  
Press  $\boxed{\tan} \quad \boxed{81} \quad \boxed{)} \quad \boxed{=} \quad \boxed{ANS} \quad \boxed{x^2} \quad \boxed{ANS} \quad \boxed{=} \quad 39,86345... \approx 39,86$
4. Press  $\boxed{3} \quad \boxed{(} \quad \boxed{\sin} \quad \boxed{72} \quad \boxed{)} \quad \boxed{)} \quad \boxed{x^2} \quad \boxed{=} \quad 2,71352... \approx 2,71$   
OR  
Press  $\boxed{\sin} \quad \boxed{72} \quad \boxed{)} \quad \boxed{=} \quad \boxed{ANS} \quad \boxed{x^2} \quad \boxed{=} \quad \boxed{ANS} \quad \boxed{\times} \quad \boxed{3}$

5. Press  $( \quad 1 \quad \div \quad 4 \quad ) \quad \cos \quad 27 \quad ) \quad =$   $0,22275\dots \approx 0,22$   
 OR  
 Press  $\cos \quad 27 \quad ) \quad - \quad \text{ANS} \quad \div \quad 4 \quad =$   $0,22275\dots \approx 0,22$
6. Press  $( \quad 5 \quad \div \quad 6 \quad ) \quad \tan \quad 34 \quad ) \quad =$   $0,56209\dots \approx 0,56$   
 OR  
 Press  $\tan \quad 34 \quad = \quad \text{ANS} \quad \times \quad 5 \quad \div \quad 6 \quad =$   $0,56 \approx 0,56$
7. First write sec in terms of cos:  $\sec 34^\circ = \frac{1}{\cos 34^\circ}$  (since there is no “sec” button on your calculator).  
 Press  $1 \quad \div \quad \cos \quad 34 \quad ) \quad =$   $1,206217\dots \approx 1,21$
8. First write cot in terms of tan:  $\cot 49^\circ = \frac{1}{\tan 49^\circ}$  (since there is no “cot” button on your calculator).  
 Press  $1 \quad \div \quad \tan \quad 49 \quad ) \quad =$   $0,869286\dots \approx 0,87$

### Worked example 3: Calculator work using substitution

#### QUESTION

If  $x = 25^\circ$  and  $y = 65^\circ$ , use your calculator to determine whether the following statement is true or false:

$$\sin^2 x + \cos^2 (90^\circ - y) = 1$$

#### SOLUTION

##### Step 1: Calculate the left hand side of the equation

Press  $( \quad \sin \quad 25 \quad ) \quad ( \quad \sin \quad 90 \quad ) \quad - \quad 65 \quad ) \quad [x^2] \quad = \quad 1$

##### Step 2: Write the final answer

LHS = RHS therefore the statement is true.

### Exercise 5 – 2:

1. Use your calculator to determine the value of the following (correct to 2 decimal places):

- |                                |  |                                    |   |
|--------------------------------|--|------------------------------------|---|
| a) $\tan 65^\circ$             | b) $\sin 38^\circ$                       | c) $\cos 74^\circ$                 | d) $\sin 12^\circ$                                |
| e) $\cos 26^\circ$             | f) $\tan 49^\circ$                       | g) $\sin 305^\circ$                | h) $\tan 124^\circ$                               |
| i) $\sec 65^\circ$             | j) $\sec 10^\circ$                       | k) $\sec 48^\circ$                 | l) $\cot 32^\circ$                                |
| m) $\cosec 140^\circ$          | n) $\cosec 237^\circ$                    | o) $\sec 231^\circ$                | p) $\cosec 226^\circ$                             |
| q) $\frac{1}{4} \cos 20^\circ$ | r) $3 \tan 40^\circ$                     | s) $\frac{2}{3} \sin 90^\circ$     | t) $\frac{5}{\cos 4,3^\circ}$                     |
| u) $\sqrt{\sin 55^\circ}$      | v) $\frac{\sin 90^\circ}{\cos 90^\circ}$ | w) $\tan 35^\circ + \cot 35^\circ$ | x) $\frac{2 + \cos 310^\circ}{2 + \sin 87^\circ}$ |

$$y) \sqrt{4 \sec 99^\circ}$$

$$z) \sqrt{\frac{\cot 103^\circ + \sin 1090^\circ}{\sec 10^\circ + 5}}$$

2. If  $x = 39^\circ$  and  $y = 21^\circ$ , use a calculator to determine whether the following statements are true or false:

a)  $\cos x + 2 \cos x = 3 \cos x$

b)  $\cos 2y = \cos y + \cos y$

c)  $\tan x = \frac{\sin x}{\cos x}$

d)  $\cos(x + y) = \cos x + \cos y$

3. Solve for  $x$  in  $5^{\tan x} = 125$ .

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1a. 2FNW

1b. 2FNX

1c. 2FNY

1d. 2FNZ

1e. 2FP2

1f. 2FP3

1g. 2FP4

1h. 2FP5

1i. 2FP6

1j. 2FP7

1k. 2FP8

1l. 2FP9

1m. 2FPB

1n. 2FPC

1o. 2FPD

1p. 2FPF

1q. 2FPG

1r. 2FPH

1s. 2FPJ

1t. 2FPK

1u. 2FPM

1v. 2FPN

1w. 2FPP

1x. 2FPQ

1y. 2FPR

1z. 2FPS

2. 2FPT

3. 2FPV



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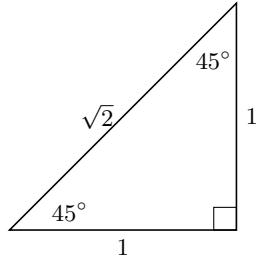
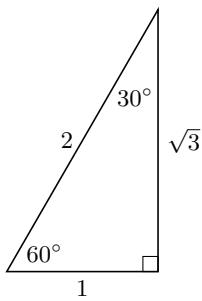
## 5.6 Special angles

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For most angles we need a calculator to calculate the values of sin, cos and tan. However, there are some angles we can easily work out the values for without a calculator as they produce simple ratios. The values of the trigonometric ratios for these special angles, as well as the triangles from which they are derived, are shown below.

### NOTE:

Remember that the lengths of the sides of a right-angled triangle must obey the Theorem of Pythagoras: the square of the hypotenuse equals the sum of the squares of the two other sides.



$\theta$	$30^\circ$	$45^\circ$	$60^\circ$
$\cos \theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\sin \theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\tan \theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

These values are useful when we need to solve a problem involving trigonometric ratios without using a calculator.

**Exercise 5 – 3:**

1. For each expression select the closest answer from the list provided:

a)  $\cos 45^\circ$

- $\frac{1}{2}$     $1$     $\sqrt{2}$     $\frac{1}{\sqrt{3}}$     $\frac{1}{\sqrt{2}}$

b)  $\sin 45^\circ$

- $\sqrt{2}$     $\frac{1}{2}$     $\frac{1}{\sqrt{2}}$     $\frac{\sqrt{3}}{1}$     $1$

c)  $\tan 30^\circ$

- $\frac{1}{2}$     $\frac{\sqrt{3}}{1}$     $\frac{1}{\sqrt{2}}$     $\frac{\sqrt{3}}{2}$     $\frac{1}{\sqrt{3}}$

d)  $\tan 60^\circ$

- $\frac{\sqrt{3}}{2}$     $\frac{1}{\sqrt{2}}$     $\frac{1}{\sqrt{3}}$     $\frac{\sqrt{3}}{1}$     $\frac{1}{1}$

e)  $\cos 45^\circ$

- $\frac{\sqrt{3}}{2}$     $\frac{1}{2}$     $\frac{1}{\sqrt{3}}$     $\frac{1}{\sqrt{2}}$     $\sqrt{2}$

f)  $\tan 30^\circ$

- $\frac{1}{\sqrt{2}}$     $\frac{1}{2}$     $\frac{\sqrt{3}}{2}$     $\frac{1}{1}$     $\frac{1}{\sqrt{3}}$

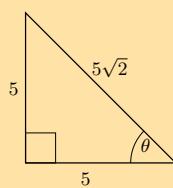
g)  $\tan 30^\circ$

- $\frac{1}{\sqrt{3}}$     $\frac{1}{2}$     $\frac{1}{\sqrt{2}}$     $\frac{\sqrt{3}}{2}$     $\frac{\sqrt{3}}{1}$

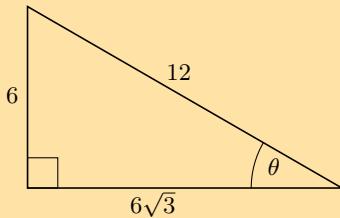
h)  $\cos 60^\circ$

- $\frac{1}{\sqrt{3}}$     $\frac{1}{2}$     $\frac{\sqrt{3}}{2}$     $\frac{\sqrt{3}}{1}$     $\frac{1}{\sqrt{2}}$

2. Solve for  $\cos \theta$  in the following triangle, in surd form:



3. Solve for  $\tan \theta$  in the following triangle, in surd form:



4. Calculate the value of the following without using a calculator:

a)  $\sin 45^\circ \times \cos 45^\circ$

b)  $\cos 60^\circ + \tan 45^\circ$

c)  $\sin 60^\circ - \cos 60^\circ$

5. Evaluate the following without using a calculator. Select the closest answer from the list provided.

a)  $\tan 45^\circ \div \sin 60^\circ$

$$\frac{2}{\sqrt{3}} \quad \frac{\sqrt{3}}{1} \quad \frac{\sqrt{2}}{\sqrt{3}} \quad \frac{1}{1} \quad \frac{1}{2}$$

b)  $\tan 30^\circ - \sin 60^\circ$

$$0 \quad \frac{1}{2} \quad -\frac{\sqrt{3}}{2} \quad -\frac{1}{2\sqrt{3}} \quad \frac{\sqrt{3}}{2}$$

c)  $\sin 30^\circ - \tan 45^\circ - \sin 30^\circ$

$$-\frac{\sqrt{3}}{2} \quad -1 \quad -\frac{1}{\sqrt{3}} \quad -\frac{\sqrt{3}}{1} \quad -\frac{7}{2\sqrt{3}}$$

d)  $\tan 30^\circ \div \tan 30^\circ \div \sin 45^\circ$

$$\frac{\sqrt{3}}{1} \quad \frac{2\sqrt{3}}{\sqrt{2}} \quad \frac{2}{\sqrt{3}} \quad \frac{\sqrt{2}}{1} \quad \frac{2\sqrt{2}}{\sqrt{3}}$$

e)  $\sin 45^\circ \div \sin 30^\circ \div \cos 45^\circ$

$$\frac{\sqrt{2}}{\sqrt{3}} \quad \frac{1}{\sqrt{2}} \quad \frac{4}{\sqrt{3}} \quad 2 \quad \frac{2\sqrt{2}}{\sqrt{3}}$$

f)  $\tan 60^\circ - \tan 60^\circ - \sin 60^\circ$

$$-\frac{1}{\sqrt{3}} \quad -\frac{1}{2} \quad -\frac{1}{\sqrt{2}} \quad -\frac{1}{1} \quad -\frac{\sqrt{3}}{2}$$

g)  $\cos 45^\circ - \sin 60^\circ - \sin 45^\circ$

$$-\frac{1}{2} \quad -\frac{1}{\sqrt{2}} \quad -\frac{7}{2\sqrt{3}} \quad -\frac{\sqrt{3}}{2} \quad -\frac{1}{\sqrt{3}}$$

6. Use special angles to show that:

a)  $\frac{\sin 60^\circ}{\cos 60^\circ} = \tan 60^\circ$       b)  $\sin^2 45^\circ + \cos^2 45^\circ = 1$       c)  $\cos 30^\circ = \sqrt{1 - \sin^2 30^\circ}$

7. Use the definitions of the trigonometric ratios to answer the following questions:

a) Explain why  $\sin \alpha \leq 1$  for all values of  $\alpha$ .

b) Explain why  $\cos \alpha$  has a maximum value of 1.

c) Is there a maximum value for  $\tan \alpha$ ?

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |                          |                          |                          |                          |                          |                          |                          |                          |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1a. <a href="#">2FPW</a> | 1b. <a href="#">2FPX</a> | 1c. <a href="#">2FPY</a> | 1d. <a href="#">2FPZ</a> | 1e. <a href="#">2FQ2</a> | 1f. <a href="#">2FQ3</a> | 1g. <a href="#">2FQ4</a> | 1h. <a href="#">2FQ5</a> |
| 2. <a href="#">2FQ6</a>  | 3. <a href="#">2FQ7</a>  | 4a. <a href="#">2FQ8</a> | 4b. <a href="#">2FQ9</a> | 4c. <a href="#">2FQB</a> | 5a. <a href="#">2FQC</a> | 5b. <a href="#">2FQD</a> | 5c. <a href="#">2FQF</a> |
| 5d. <a href="#">2FQG</a> | 5e. <a href="#">2FQH</a> | 5f. <a href="#">2FQJ</a> | 5g. <a href="#">2FQK</a> | 6a. <a href="#">2FQM</a> | 6b. <a href="#">2FQN</a> | 6c. <a href="#">2FQP</a> | 7a. <a href="#">2FQQ</a> |
| 7b. <a href="#">2FQR</a> | 7c. <a href="#">2FQS</a> |                          |                          |                          |                          |                          |                          |



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## 5.7 Solving trigonometric equations

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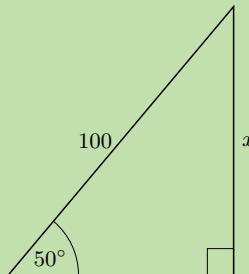
In this section we will first look at finding unknown lengths in right-angled triangles and then we will look at finding unknown angles in right-angled triangles. Finally we will look at how to solve more general trigonometric equations.

From the definitions of the trigonometric ratios and what we have learnt about determining the values of these ratios for any angle we can now use this to help us find unknown lengths in right-angled triangles. The following worked examples will show you how.

### Worked example 4: Finding lengths

#### QUESTION

Find the length of  $x$  in the following right-angled triangle using the appropriate trigonometric ratio (round your answer to two decimal places).



#### SOLUTION

##### Step 1: Identify the opposite and adjacent sides and the hypotenuse with reference to the given angle

Remember that the hypotenuse side is **always** opposite the right angle, it never changes position. The opposite side is opposite the angle we are interested in and the adjacent side is the remaining side.

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 50^\circ = \frac{x}{100}$$

##### Step 2: Rearrange the equation to solve for $x$

$$\begin{aligned}\sin 50^\circ \times 100 &= \frac{x}{100} \times 100 \\ \sin 50^\circ \times 100 &= x \\ x &= 100 \sin 50^\circ\end{aligned}$$

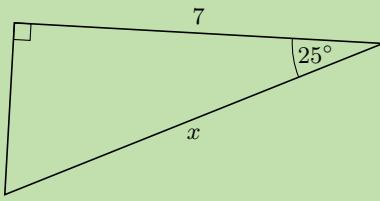
##### Step 3: Use your calculator to find the answer

$$\begin{aligned}x &= 76,60444... \\ x &\approx 76,60\end{aligned}$$

### Worked example 5: Finding lengths

#### QUESTION

Find the length of  $x$  in the following right-angled triangle using the appropriate trigonometric ratio (round your answer to two decimal places).



### SOLUTION

---

#### Step 1: Identify the opposite and adjacent sides and the hypotenuse with reference to the given angle

Remember that the hypotenuse side is **always** opposite the right angle, it never changes position. The opposite side is opposite the angle we are interested in and the adjacent side is the remaining side.

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 25^\circ = \frac{7}{x}$$

#### Step 2: Rearrange the equation to solve for $x$

$$\cos 25^\circ \times x = \frac{7}{x} \times x \quad \text{multiply both sides by } x$$

$$x \cos 25^\circ = 7$$

$$\frac{x \cos 25^\circ}{\cos 25^\circ} = \frac{7}{\cos 25^\circ} \quad \text{divide both sides by } \cos 25^\circ$$

$$x = \frac{7}{\cos 25^\circ}$$

#### Step 3: Use your calculator to find the answer

$$x = \frac{7}{0,90630\dots}$$

$$= 7,723645\dots$$

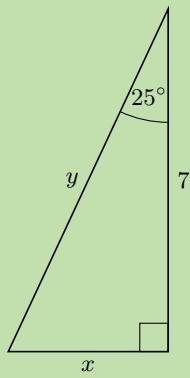
$$\approx 7,72$$

### Worked example 6: Finding lengths

#### QUESTION

---

Find the length of  $x$  and  $y$  in the following right-angled triangle using the appropriate trigonometric ratio (round your answers to two decimal places).



### SOLUTION

**Step 1: Identify the opposite and adjacent sides and the hypotenuse with reference to the given angle**

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 25^\circ = \frac{x}{7}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 25^\circ = \frac{7}{y}$$

**Step 2: Rearrange the equations to solve for  $x$  and  $y$**

$$x = 7 \times \tan 25^\circ$$

$$y = \frac{7}{\cos 25^\circ}$$

**Step 3: Use your calculator to find the answers**

$$x = 3,26415\dots$$

$$x \approx 3,26$$

$$y = 7,72364\dots$$

$$y \approx 7,72$$

### VISIT:

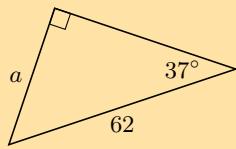
The following video shows an example of finding unknown lengths in a triangle using the trigonometric ratios.

▶ See video: [2FQT](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

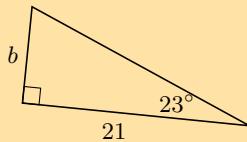
**Exercise 5 – 4:**

1. In each triangle find the length of the side marked with a letter. Give your answers correct to 2 decimal places.

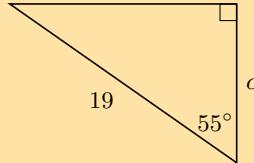
a)



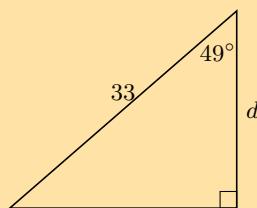
b)



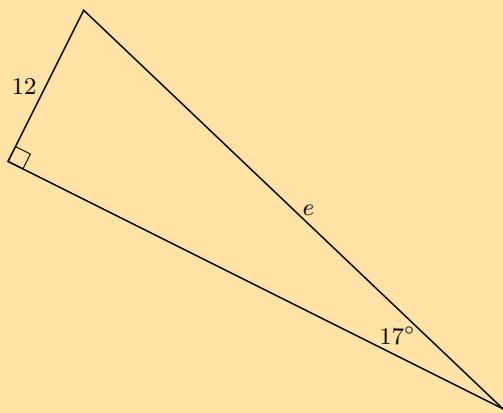
c)



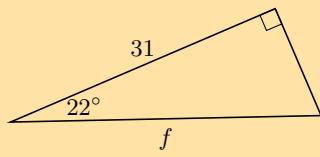
d)



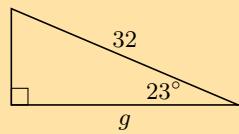
e)



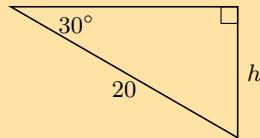
f)



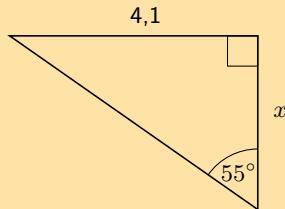
g)



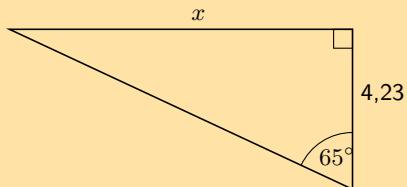
h)



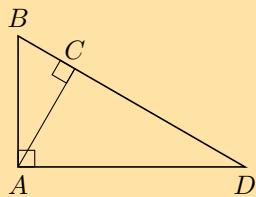
i)



j)



2. Write down two ratios for each of the following in terms of the sides:  $AB$ ;  $BC$ ;  $BD$ ;  $AD$ ;  $DC$  and  $AC$ .



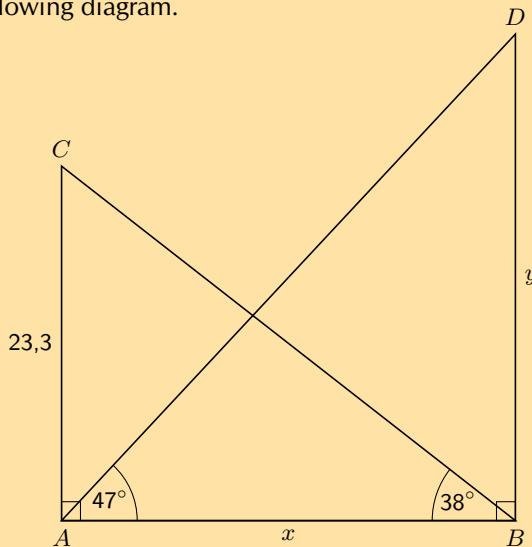
a)  $\sin \hat{B}$

b)  $\cos \hat{D}$

c)  $\tan \hat{B}$

3. In  $\triangle MNP$ ,  $\hat{N} = 90^\circ$ ,  $MP = 20$  and  $\hat{P} = 40^\circ$ . Calculate  $NP$  and  $MN$  (correct to 2 decimal places).

4. Calculate  $x$  and  $y$  in the following diagram.



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- 1a. [2FQV](#) 1b. [2FQW](#) 1c. [2FQX](#) 1d. [2FQY](#) 1e. [2FQZ](#) 1f. [2FR2](#) 1g. [2FR3](#)  
1h. [2FR4](#) 1i. [2FR5](#) 1j. [2FR6](#) 2. [2FR7](#) 3. [2FR8](#) 4. [2FR9](#)



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## Finding an angle

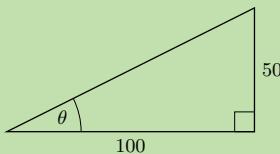
EMA3W

If the length of two sides of a triangle are known, the angles can be calculated using trigonometric ratios. In this section, we are finding angles inside right-angled triangles using the ratios of the sides.

### Worked example 7: Finding angles

#### QUESTION

Find the value of  $\theta$  in the following right-angled triangle using the appropriate trigonometric ratio.



#### SOLUTION

**Step 1: Identify the opposite and adjacent sides with reference to the given angle and the hypotenuse**  
In this case you have the opposite side and the adjacent side for angle  $\theta$ .

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan \theta = \frac{50}{100}$$

**Step 2: Use your calculator to solve for  $\theta$** 

To solve for  $\theta$ , you will need to use the inverse tangent function on your calculator. This works backwards by using the ratio of the sides to determine the angle which resulted in that ratio.

Press **SHIFT** **[tan]** **[50]** **[÷]** **[100]** **[=]** **[26,56505...]**  $\approx 26,6$

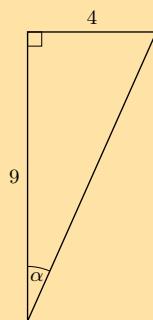
**Step 3: Write the final answer**

$$\theta \approx 26,6^\circ$$

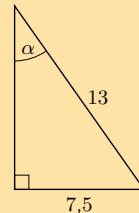
**Exercise 5 – 5:**

Determine  $\alpha$  in the following right-angled triangles:

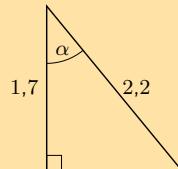
1.



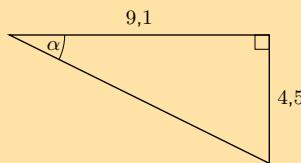
2.



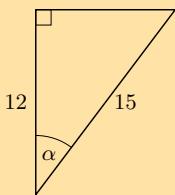
3.



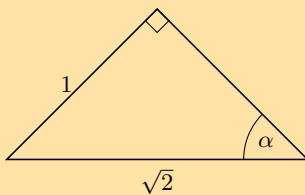
4.



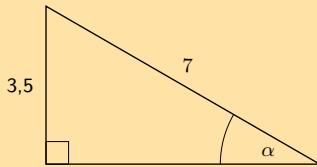
5.



6.



7.



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. 2FRB   2. 2FRC   3. 2FRD   4. 2FRF   5. 2FRG   6. 2FRH   7. 2FRJ



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We have now seen how to solve trigonometric equations in right-angled triangles. We can use the same techniques to help us solve trigonometric equations when the triangle is not shown.

### Worked example 8: Solving trigonometric equations

#### QUESTION

Find the value of  $\theta$  if  $\cos \theta = 0,2$ .

#### SOLUTION

##### Step 1: Use your calculator to solve for $\theta$

To solve for  $\theta$ , you will need to use the inverse cosine function on your calculator. This works backwards by using the ratio of the sides to determine the angle which resulted in that ratio.

Press [SHIFT] [cos] [0] [.] [2] [=] 78,46304  $\approx$  78,46

##### Step 2: Write the final answer

$\theta \approx 78,46^\circ$

## Worked example 9: Solving trigonometric equations

### QUESTION

Find the value of  $\theta$  if  $3 \sin \theta = 2,4$ .

### SOLUTION

#### Step 1: Rearrange the equation

We need to rearrange the equation so that  $\sin \theta$  is on one side of the equation.

$$\begin{aligned}3 \sin \theta &= 2,4 \\ \sin \theta &= \frac{2,4}{3}\end{aligned}$$

#### Step 2: Use your calculator to solve for $\theta$

To solve for  $\theta$ , you will need to use the inverse sine function on your calculator. This works backwards by using the ratio of the sides to determine the angle which resulted in that ratio.

Press **SHIFT** **sin** **(** **2**  **4** **)** **=** **53,1301...**  $\approx 53,13$ .

#### Step 3: Write the final answer

$\theta \approx 53,13^\circ$

### NOTE:

When you are solving trigonometric equations you might find that you get an error when you try to calculate sin or cos (remember that both the sine and cosine functions have a maximum value of 1). For these cases there is no solution to the equation.

## Worked example 10: Solving trigonometric equations

### QUESTION

Solve for  $\alpha$ :  $3 \sec \alpha = 1,4$ .

### SOLUTION

#### Step 1: Convert sec to cos

There is no “sec” button on the calculator and so we need to convert sec to cos so we can find  $\alpha$ .

$$\begin{aligned}3 \sec \alpha &= 1,4 \\ \frac{3}{\cos \alpha} &= 1,4\end{aligned}$$

#### Step 2: Rearrange the equation

We need to rearrange the equation so that we have  $\cos \alpha$  on one side of the equation.

$$\begin{aligned}\frac{3}{\cos \alpha} &= 1,4 \\ 3 &= 1,4 \cos \alpha \\ \frac{3}{1,4} &= \cos \alpha\end{aligned}$$

### Step 3: Use your calculator to solve for $\alpha$

To solve for  $\alpha$ , you will need to use the inverse cosine function on your calculator. This works backwards by using the ratio of the sides to determine the angle which resulted in that ratio.

Press **[SHIFT]** **[cos]** **[3]** **[÷]** **[1]** **[.]** **[4]** **[)** **[=]** *math error*

In this case we get an error when we try to do the calculation. This is because  $\frac{3}{1,4}$  is greater than 1 and the maximum value of the cosine function is 1. Therefore there is no solution. It is important in this case to write no solution and not math error.

### Step 4: Write the final answer

There is no solution.

### Exercise 5 – 6:

1. Determine the angle (correct to 1 decimal place):

- |                                 |  |                                    |                               |
|---------------------------------|--|------------------------------------|-------------------------------|
| a) $\tan \theta = 1,7$          | b) $\sin \theta = 0,8$                     | c) $\cos \alpha = 0,32$            | d) $\tan \beta = 4,2$         |
| e) $\tan \theta = 5\frac{3}{4}$ | f) $\sin \theta = \frac{2}{3}$             | g) $\cos \beta = 1,2$              | h) $4 \cos \theta = 3$        |
| i) $\cos 4\theta = 0,3$         | j) $\sin \beta + 2 = 2,65$                 | k) $2 \sin \theta + 5 = 0,8$       | l) $3 \tan \beta = 1$         |
| m) $\sin 3\alpha = 1,2$         | n) $\tan \frac{\theta}{3} = \sin 48^\circ$ | o) $\frac{1}{2} \cos 2\beta = 0,3$ | p) $2 \sin 3\theta + 1 = 2,6$ |

2. If  $x = 16^\circ$  and  $y = 36^\circ$ , use your calculator to evaluate each of the following, correct to 3 decimal places.

- |                         |                         |                                  |
|-------------------------|-------------------------|----------------------------------|
| a) $\sin(x - y)$        | b) $3 \sin x$           | c) $\tan x - \tan y$             |
| d) $\cos x + \cos y$    | e) $\frac{1}{3} \tan y$ | f) $\operatorname{cosec}(x - y)$ |
| g) $2 \cos x + \cos 3y$ | h) $\tan(2x - 5y)$      |                                  |

3. In each of the following find the value of  $x$  correct to two decimal places.

- |                               |                             |   |
|-------------------------------|-----------------------------|---|
| a) $\sin x = 0,814$           | b) $\sin x = \tan 45^\circ$ | c) $\tan 2x = 3,123$                    |
| d) $\tan x = 3 \sin 41^\circ$ | e) $\sin(2x + 45) = 0,123$  | f) $\sin(x - 10^\circ) = \cos 57^\circ$ |

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |                          |                          |                          |                          |                          |                          |                          |                          |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1a. <a href="#">2FRK</a> | 1b. <a href="#">2FRM</a> | 1c. <a href="#">2FRN</a> | 1d. <a href="#">2FRP</a> | 1e. <a href="#">2FRQ</a> | 1f. <a href="#">2FRR</a> | 1g. <a href="#">2FRS</a> | 1h. <a href="#">2FRT</a> |
| 1i. <a href="#">2FRV</a> | 1j. <a href="#">2FRW</a> | 1k. <a href="#">2FRX</a> | 1l. <a href="#">2FRY</a> | 1m. <a href="#">2FRZ</a> | 1n. <a href="#">2FS2</a> | 1o. <a href="#">2FS3</a> | 1p. <a href="#">2FS4</a> |
| 2a. <a href="#">2FS5</a> | 2b. <a href="#">2FS6</a> | 2c. <a href="#">2FS7</a> | 2d. <a href="#">2FS8</a> | 2e. <a href="#">2FS9</a> | 2f. <a href="#">2FSB</a> | 2g. <a href="#">2FSC</a> | 2h. <a href="#">2FSD</a> |
| 3a. <a href="#">2FSF</a> | 3b. <a href="#">2FSG</a> | 3c. <a href="#">2FSH</a> | 3d. <a href="#">2FSJ</a> | 3e. <a href="#">2FSK</a> | 3f. <a href="#">2FSM</a> |                          |                          |



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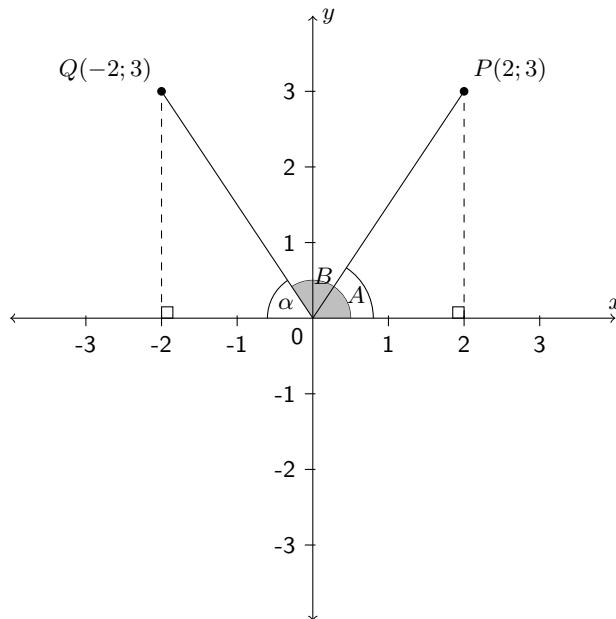


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We have defined the trigonometric ratios using right-angled triangles. We can extend these definitions to any angle, noting that the definitions do not rely on the lengths of the sides of the triangle, but on the size of the angle only. So if we plot any point on the Cartesian plane and then draw a line from the origin to that point, we can work out the angle between the positive  $x$ -axis and that line. We will first look at this for two specific points and then look at the more general case.

### Finding an angle for specific points

In the figure below points  $P$  and  $Q$  have been plotted. A line from the origin ( $O$ ) to each point is drawn. The dotted lines show how we can construct right-angled triangles for each point. The dotted line must always be drawn to the  $x$ -axis. Now we can find the angles  $A$  and  $B$ :



From the coordinates of  $P(2; 3)$ , we can see that  $x = 2$  and  $y = 3$ . Therefore, we know the length of the side opposite  $\hat{A}$  is 3 and the length of the adjacent side is 2. Using:

$$\tan \hat{A} = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{2}$$

we calculate that  $\hat{A} = 56,3^\circ$ .

We can also use the theorem of Pythagoras to calculate the hypotenuse of the triangle and then calculate  $\hat{A}$  using:

$$\sin \hat{A} = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{or} \quad \cos \hat{A} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Consider point  $Q(-2; 3)$ . We define  $\hat{B}$  as the angle formed between line  $OQ$  and the positive  $x$ -axis. This is called the standard position of an angle. Angles are always measured from the positive  $x$ -axis in an anti-clockwise direction. Let  $\alpha$  be the angle formed between the line  $OQ$  and the negative  $x$ -axis such that  $\hat{B} + \alpha = 180^\circ$ .

From the coordinates of  $Q(-2; 3)$ , we know the length of the side opposite  $\alpha$  is 3 and the length of the adjacent side is 2. Using:

$$\tan \alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{2}$$

we calculate that  $\alpha = 56,3^\circ$ . Therefore  $\hat{B} = 180^\circ - \alpha = 123,7^\circ$ .

Similarly, an alternative method is to calculate the hypotenuse using the theorem of Pythagoras and calculate  $\alpha$  using:

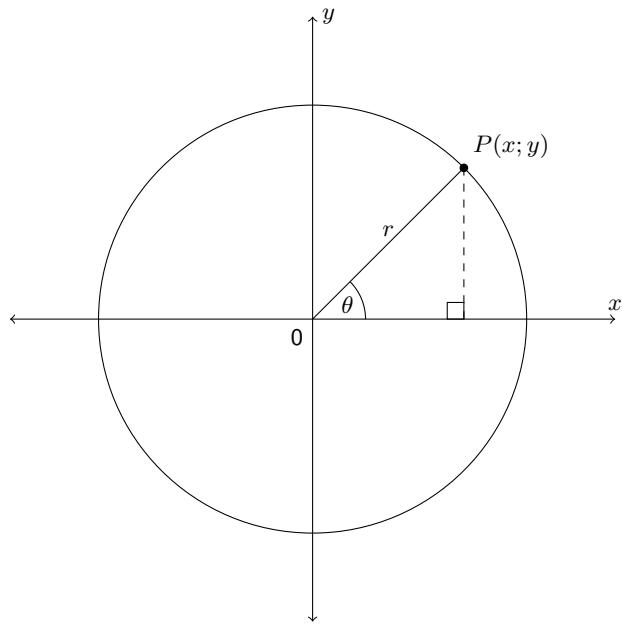
$$\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{or} \quad \cos \alpha = \frac{\text{adjacent}}{\text{hypotenuse}}$$

### NOTE:

We can also extend the definitions of the reciprocals in the same way.

### Finding any angle

If we were to draw a circle centred on the origin ( $O$ ) and passing through the point  $P(x; y)$ , then the length from the origin to point  $P$  is the radius of the circle, which we denote  $r$ . We denote the angle formed between the line  $OP$  and the  $x$ -axis as  $\theta$ .



We can rewrite all the trigonometric ratios in terms of  $x$ ,  $y$  and  $r$ . The general definitions for the trigonometric ratios are:

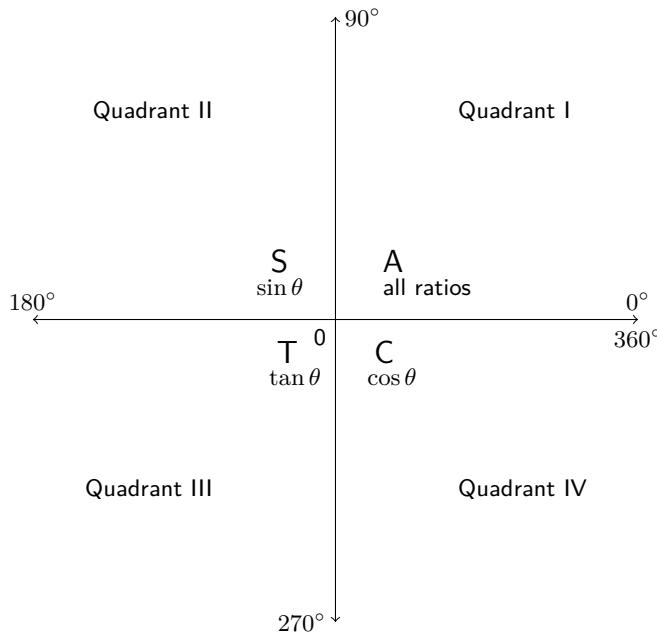
$$\sin \theta = \frac{y}{r} \quad \operatorname{cosec} \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r} \quad \sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$

### The CAST diagram

The Cartesian plane is divided into 4 quadrants in an anti-clockwise direction as shown in the diagram below. Notice that  $r$  is always positive but the values of  $x$  and  $y$  change depending on the position of the point in the Cartesian plane. As a result, the trigonometric ratios can be positive or negative. The letters C, A, S and T indicate which of the ratios are positive in each quadrant:



This diagram is known as the CAST diagram.

We note the following using the general definitions of the trigonometric ratios:

- **Quadrant I**  
Both the  $x$  and  $y$  values are positive so all ratios are positive in this quadrant.
- **Quadrant II**  
The  $y$  values are positive therefore sin and cosec are positive in this quadrant (recall that sin and cosec are defined in terms of  $y$  and  $r$ ).
- **Quadrant III**  
Both the  $x$  and the  $y$  values are negative therefore tan and cot are positive in this quadrant (recall that tan and cot are defined in terms of  $x$  and  $y$ ).
- **Quadrant IV**  
The  $x$  values are positive therefore cos and sec are positive in this quadrant (recall that cos and sec are defined in terms of  $x$  and  $r$ ).

### IMPORTANT!

The hypotenuse,  $r$ , is a length, and is therefore always positive.

#### VISIT:

The following video provides a summary of the trigonometric ratios in the Cartesian plane.

► See video: [2FSN](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Special angles in the Cartesian plane

When working in the Cartesian plane we include two other special angles in right-angled triangles:  $0^\circ$  and  $90^\circ$ .

Notice that when  $\theta = 0^\circ$  the length of the opposite side is equal to 0 and the length of the adjacent side is equal to the length of the hypotenuse.

Therefore:

$$\begin{aligned}\sin 0^\circ &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{0}{\text{hypotenuse}} = 0 \\ \cos 0^\circ &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{hypotenuse}}{\text{hypotenuse}} = 1 \\ \tan 0^\circ &= \frac{\text{opposite}}{\text{adjacent}} = \frac{0}{\text{adjacent}} = 0\end{aligned}$$

When  $\theta = 90^\circ$  the length of the adjacent side is equal to 0 and the length of the opposite side is equal to the length of the hypotenuse. Therefore:

$$\begin{aligned}\sin 90^\circ &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{hypotenuse}}{\text{hypotenuse}} = 1 \\ \cos 90^\circ &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{0}{\text{hypotenuse}} = 0 \\ \tan 90^\circ &= \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{opposite}}{0} = \text{undefined}\end{aligned}$$

Now we can extend our knowledge of special angles.

$\theta$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

### Worked example 11: Ratios in the Cartesian plane

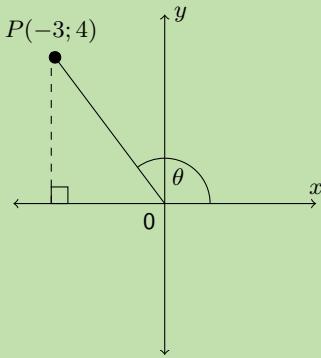
#### QUESTION

$P(-3; 4)$  is a point on the Cartesian plane with origin  $O$ .  $\theta$  is the angle between  $OP$  and the positive  $x$ -axis. Without using a calculator, determine the value of:

1.  $\cos \theta$
2.  $3 \tan \theta$
3.  $\frac{1}{2} \operatorname{cosec} \theta$

#### SOLUTION

##### Step 1: Sketch point $P$ in the Cartesian plane and label the angle $\theta$



**Step 2: Use the theorem of Pythagoras to calculate  $r$** 

$$\begin{aligned}r^2 &= x^2 + y^2 \\&= (-3)^2 + (4)^2 \\&= 25 \\\therefore r &= 5\end{aligned}$$

Note  $r$  is positive as it is the radius of the circle.

**Step 3: Substitute values for  $x$ ,  $y$  and  $r$  into the required ratios**

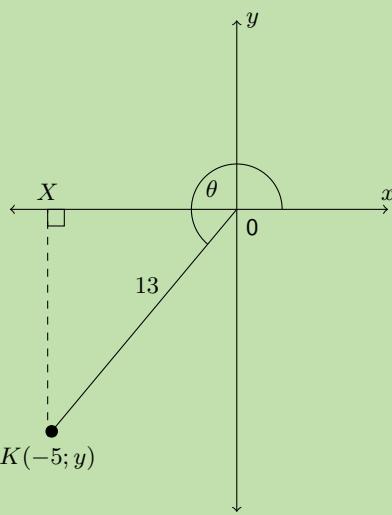
We note that  $x = -3$ ,  $y = 4$  and  $r = 5$ .

1.  $\cos \theta = \frac{x}{r} = \frac{-3}{5}$
2.  $3 \tan \theta = 3 \left( \frac{y}{x} \right) = 3 \left( \frac{4}{-3} \right) = -4$
3.  $\frac{1}{2} \operatorname{cosec} \theta = \frac{1}{2} \left( \frac{r}{y} \right) = \frac{1}{2} \left( \frac{5}{4} \right) = \frac{5}{8}$

**Worked example 12: Ratios in the Cartesian plane****QUESTION**

$X\hat{O}K = \theta$  is an angle in the third quadrant where  $X$  is a point on the positive  $x$ -axis and  $K$  is the point  $(-5; y)$ .  $OK$  is 13 units.

1. Determine, without using a calculator, the value of  $y$ .
2. Prove that  $\tan^2 \theta + 1 = \sec^2 \theta$  without using a calculator.

**SOLUTION****Step 1: Sketch point  $K$  in the Cartesian plane and label the angle  $\theta$** 

**Step 2: Use the theorem of Pythagoras to calculate  $y$** 

$$\begin{aligned}r^2 &= x^2 + y^2 \\y^2 &= r^2 - x^2 \\&= (13)^2 - (-5)^2 \\&= 169 - 25 \\&= 144 \\y &= \pm 12\end{aligned}$$

Given that  $\theta$  lies in the third quadrant,  $y$  must be negative. Therefore  $y = -12$

**Step 3: Substitute values for  $x$ ,  $y$  and  $r$  and simplify**

$x = -5$ ,  $y = -12$  and  $r = 13$ .

LHS

$$\begin{aligned}\tan^2 \theta + 1 &= \left(\frac{y}{x}\right)^2 + 1 \\&= \left(\frac{-12}{-5}\right)^2 + 1 \\&= \left(\frac{144}{25}\right) + 1 \\&= \frac{144 + 25}{25} \\&= \frac{169}{25}\end{aligned}$$

RHS

$$\begin{aligned}\sec^2 \theta &= \left(\frac{r}{x}\right)^2 \\&= \left(\frac{13}{-5}\right)^2 \\&= \frac{169}{25}\end{aligned}$$

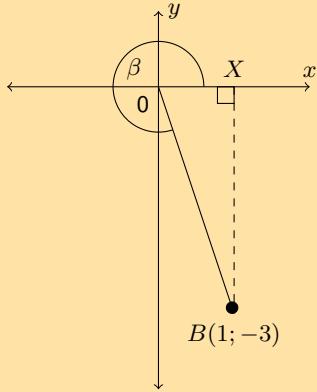
Therefore the LHS = RHS and we have proved that  $\tan^2 \theta + 1 = \sec^2 \theta$ .

**NOTE:**

Whenever you have to solve trigonometric problems without a calculator, it can be very helpful to make a sketch.

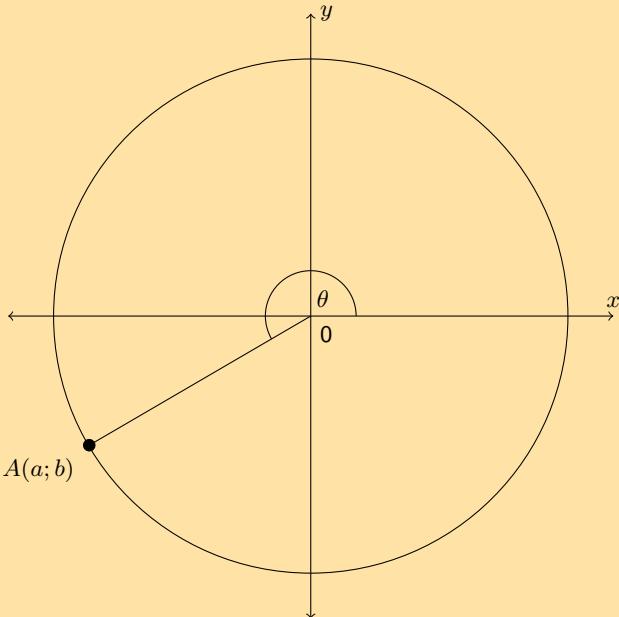
**Exercise 5 – 7:**

1.  $B$  is a point in the Cartesian plane. Determine without using a calculator:



- a)  $OB$       b)  $\cos \beta$       c)  $\operatorname{cosec} \beta$       d)  $\tan \beta$

2. If  $\sin \theta = 0,4$  and  $\theta$  is an obtuse angle, determine:
- $\cos \theta$
  - $\sqrt{21} \tan \theta$
3. Given  $\tan \theta = \frac{t}{2}$ , where  $0^\circ \leq \theta \leq 90^\circ$ . Determine the following in terms of  $t$ :
- $\sec \theta$
  - $\cot \theta$
  - $\cos^2 \theta$
  - $\tan^2 \theta - \sec^2 \theta$
4. Given:  $10 \cos \beta + 8 = 0$  and  $180^\circ < \beta < 360^\circ$ . Determine the value of:
- $\cos \beta$
  - $\frac{3}{\tan \beta} + 2 \sin^2 \beta$
5. If  $\sin \theta = -\frac{15}{17}$  and  $\cos \theta < 0$  find the following, without the use of a calculator:
- $\cos \theta$
  - $\tan \theta$
  - $\cos^2 \theta + \sin^2 \theta$
6. Find the value of  $\sin A + \cos A$  without using a calculator, given that  $13 \sin A - 12 = 0$ , where  $\cos A < 0$ .
7. If  $17 \cos \theta = -8$  and  $\tan \theta > 0$  determine the following with the aid of a diagram (not a calculator):
- $\frac{\cos \theta}{\sin \theta}$
  - $17 \sin \theta - 16 \tan \theta$
8.  $L$  is a point with co-ordinates  $(5; 8)$  on a Cartesian plane.  $LK$  forms an angle,  $\theta$ , with the positive  $x$ -axis. Set up a diagram and use it to answer the following questions.
- Find the distance  $LK$ .
  - Determine  $\sin \theta$ .
  - Determine  $\cos \theta$ .
  - Determine  $\tan \theta$ .
  - Determine  $\operatorname{cosec} \theta$ .
  - Determine  $\sec \theta$ .
  - Determine  $\cot \theta$ .
  - Determine  $\sin^2 \theta + \cos^2 \theta$ .
9. Given the following diagram and that  $\cos \theta = -\frac{24}{25}$ .



- State two sets of possible values of  $a$  and  $b$ .
- If  $OA = 100$ , state the values of  $a$  and  $b$ .
- Hence determine without the use of a calculator the value of  $\sin \theta$ .

10. If  $\tan \alpha = \frac{5}{-12}$  and  $0^\circ \leq \alpha \leq 180^\circ$ , determine **without the use of a calculator** the value of  $\frac{12}{\cos \alpha}$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. 2FSP 2. 2FSQ 3. 2FSR 4. 2FSS 5. 2FST 6. 2FSV  
7. 2FSW 8. 2FSX 9. 2FSY 10. 2FSZ



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## 5.9 Chapter summary

EMA3Y

► See presentation: [2FT2](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

- We can define three trigonometric ratios for right-angled triangles: sine (sin), cosine (cos) and tangent (tan)

These ratios can be defined as:

$$\begin{aligned} - \sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \\ - \cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \\ - \tan \theta &= \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \end{aligned}$$

- Each of these ratios have a reciprocal: cosecant (cosec), secant (sec) and cotangent (cot).

These ratios can be defined as:

$$\begin{aligned} - \operatorname{cosec} \theta &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{r}{y} \\ - \operatorname{sec} \theta &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x} \\ - \operatorname{cot} \theta &= \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y} \end{aligned}$$

- We can use the principles of solving equations and the trigonometric ratios to help us solve simple trigonometric equations.
- For some special angles ( $0^\circ$ ,  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$  and  $90^\circ$ ), we can easily find the values of sin, cos and tan without using a calculator.
- We can extend the definitions of the trigonometric ratios to any angle.

### End of chapter Exercise 5 – 8:

1. State whether each of the following trigonometric ratios has been written correctly.

a)  $\sin \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$

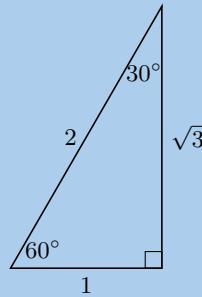
b)  $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$

c)  $\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$

2. Use your calculator to evaluate the following expressions to two decimal places:

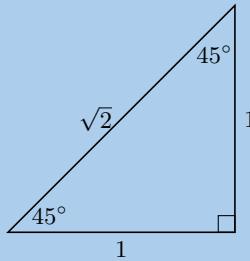
- a)  $\tan 80^\circ$       b)  $\cos 73^\circ$   
 c)  $\sin 17^\circ$       d)  $\tan 313^\circ$   
 e)  $\cos 138^\circ$       f)  $\sec 56^\circ$   
 g)  $\cot 18^\circ$       h)  $\operatorname{cosec} 37^\circ$   
 i)  $\sec 257^\circ$       j)  $\sec 304^\circ$   
 k)  $3 \sin 51^\circ$       l)  $4 \cot 54^\circ + 5 \tan 44^\circ$   
 m)  $\frac{\cos 205^\circ}{4}$       n)  $\sqrt{\sin 99^\circ}$   
 o)  $\sqrt{\cos 687^\circ + \sin 120^\circ}$       p)  $\frac{\tan 70^\circ}{\operatorname{cosec} 1^\circ}$   
 q)  $\sec 84^\circ + 4 \sin 0,4^\circ \times 50 \cos 50^\circ$       r)  $\frac{\cos 40^\circ}{\sin 35^\circ} + \tan 38^\circ$

3. Use the triangle below to complete the following:



- a)  $\sin 60^\circ =$       b)  $\cos 60^\circ =$       c)  $\tan 60^\circ =$   
 d)  $\sin 30^\circ =$       e)  $\cos 30^\circ =$       f)  $\tan 30^\circ =$

4. Use the triangle below to complete the following:



- a)  $\sin 45^\circ =$       b)  $\cos 45^\circ =$       c)  $\tan 45^\circ =$   
 5. Evaluate the following without using a calculator. Select the closest answer from the list provided.

a)  $\sin 60^\circ - \tan 60^\circ$

$$0 \quad -\frac{1}{2} \quad \frac{2}{\sqrt{3}} \quad -\frac{\sqrt{3}}{2} \quad -\frac{2}{\sqrt{3}}$$

b)  $\tan 30^\circ - \cos 30^\circ$

$$0 \quad -\frac{1}{2\sqrt{3}} \quad \frac{\sqrt{3}}{2} \quad -\frac{2}{\sqrt{3}} \quad -\frac{\sqrt{3}}{2}$$

c)  $\tan 60^\circ - \sin 60^\circ - \tan 60^\circ$

$$-\frac{\sqrt{3}}{2} \quad -\frac{\sqrt{3}}{1} \quad -\frac{1}{2} \quad -\frac{1}{1} \quad -\frac{1}{\sqrt{2}}$$

d)  $\sin 30^\circ \times \sin 30^\circ \times \sin 30^\circ$

$$\frac{1}{2} \quad \frac{1}{2\sqrt{3}} \quad \frac{1}{8} \quad \frac{1}{4} \quad \frac{\sqrt{3}}{4\sqrt{2}}$$

e)  $\sin 45^\circ \times \tan 45^\circ \times \tan 60^\circ$

$$\frac{3}{2\sqrt{2}} \quad \frac{\sqrt{3}}{8} \quad \frac{3}{4} \quad \frac{\sqrt{3}}{\sqrt{2}} \quad \frac{1}{4}$$

f)  $\cos 60^\circ \times \cos 45^\circ \times \tan 60^\circ$

$$\frac{\sqrt{3}}{2\sqrt{2}} \quad \frac{\sqrt{3}}{4} \quad \frac{3}{4\sqrt{2}} \quad \frac{1}{2} \quad \frac{1}{4\sqrt{3}}$$

g)  $\tan 45^\circ \times \sin 60^\circ \times \tan 45^\circ$

$$\frac{\sqrt{3}}{2} \quad \frac{3}{8} \quad \frac{1}{3} \quad \frac{\sqrt{3}}{2\sqrt{2}} \quad \frac{1}{4\sqrt{3}}$$

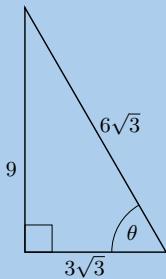
h)  $\cos 30^\circ \times \cos 60^\circ \times \sin 60^\circ$

$$\frac{3}{8} \quad \frac{3}{2\sqrt{2}} \quad \frac{\sqrt{3}}{4\sqrt{2}} \quad \frac{1}{2\sqrt{3}} \quad \frac{1}{4\sqrt{3}}$$

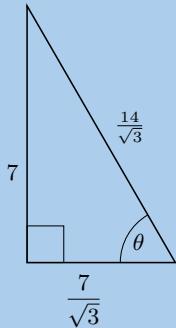
6. Without using a calculator, determine the value of:

$$\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ + \tan 45^\circ$$

7. Solve for  $\sin \theta$  in the following triangle, in surd form:

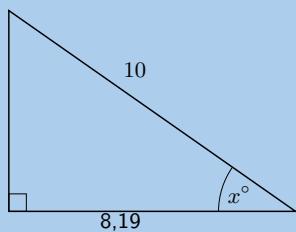


8. Solve for  $\tan \theta$  in the following triangle, in surd form:

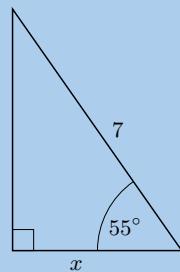


9. A right-angled triangle has hypotenuse 13 mm. Find the length of the other two sides if one of the angles of the triangle is  $50^\circ$ .
10. Solve for  $x$  to the nearest integer.

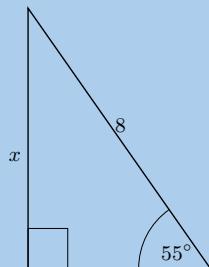
a)



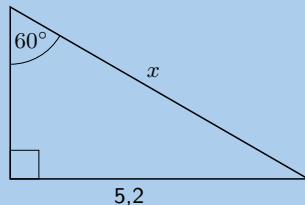
b)



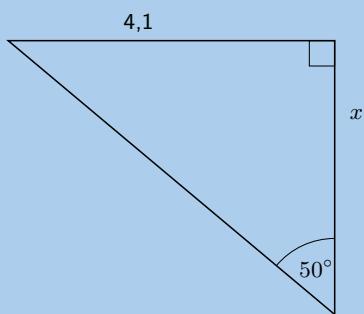
c)



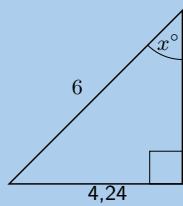
d)



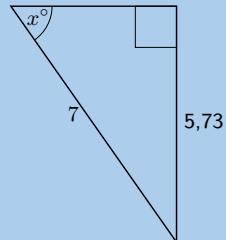
e)



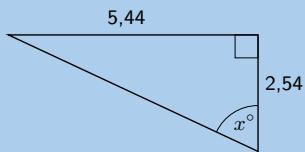
f)



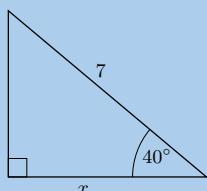
g)



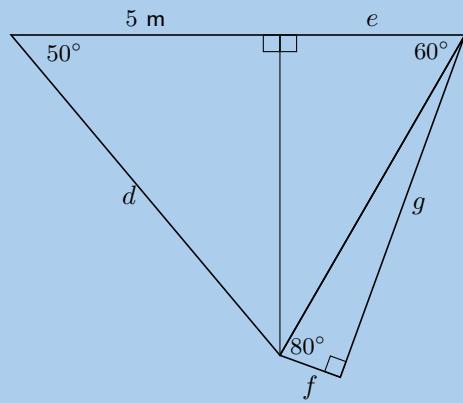
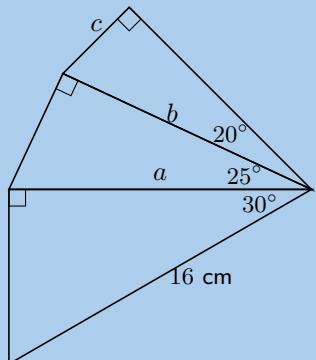
h)



i)



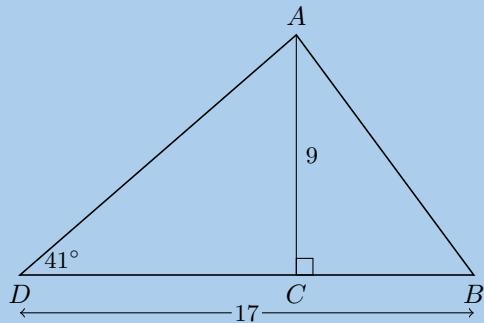
11. Calculate the unknown lengths in the diagrams below:



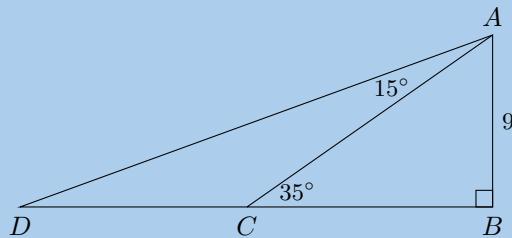
12. In  $\triangle PQR$ ,  $PR = 20$  cm,  $QR = 22$  cm and  $P\hat{R}Q = 30^\circ$ . The perpendicular line from  $P$  to  $QR$  intersects  $QR$  at  $X$ . Calculate:

- a) the length  $XR$       b) the length  $PX$       c) the angle  $Q\hat{P}X$

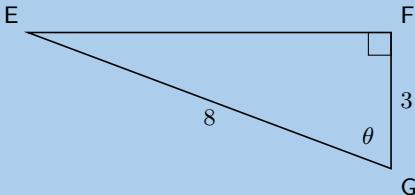
13. In the following triangle find the size of  $\hat{A}BC$ .



14. In the following triangle find the length of side  $CD$ :

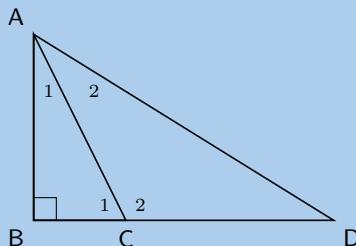


15. In the following triangle determine:



- a) the length of  $EF$       b)  $\tan(90^\circ - \theta)$       c) the value of  $\theta$

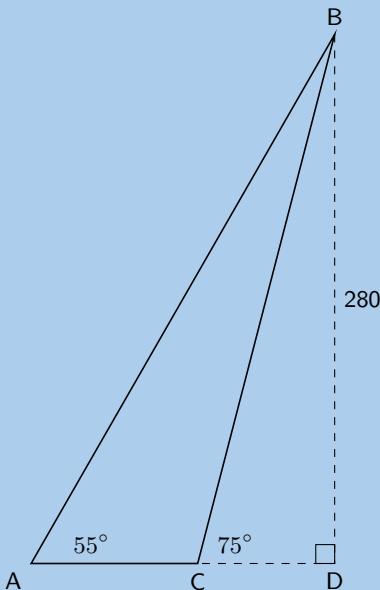
16. Given that  $\hat{D} = x$ ,  $\hat{C}_1 = 2x$ ,  $BC = 12,2$  cm,  $AB = 24,6$  cm. Calculate  $CD$ .



17. Solve for  $\theta$  if  $\theta$  is a positive, acute angle:

- |                              |  |                                   |
|------------------------------|--|-----------------------------------|
| a) $2 \sin \theta = 1,34$    | b) $1 - \tan \theta = -1$                | c) $\cos 2\theta = \sin 40^\circ$ |
| d) $\sec \theta = 1,8$       | e) $\cot 4\theta = \sin 40^\circ$        | f) $\sin 3\theta + 5 = 4$         |
| g) $\cos(4 + \theta) = 0,45$ | h) $\frac{\sin \theta}{\cos \theta} = 1$ |                                   |

18. If  $a = 29^\circ$ ,  $b = 38^\circ$  and  $c = 47^\circ$ , use your calculator to evaluate each of the following, correct to 2 decimal places.
- $\tan(a + c)$
  - $\operatorname{cosec}(c - b)$
  - $\sin(a \times b \times c)$
  - $\tan a + \sin b + \cos c$
19. If  $3 \tan \alpha = -5$  and  $0^\circ < \alpha < 270^\circ$ , use a sketch to determine:
- $\cos \alpha$
  - $\tan^2 \alpha - \sec^2 \alpha$
20. Given  $A(5; 0)$  and  $B(11; 4)$ , find the angle between the line through  $A$  and  $B$  and the  $x$ -axis.
21. Given  $C(0; -13)$  and  $D(-12; 14)$ , find the angle between the line through  $C$  and  $D$  and the  $y$ -axis.
22. Given the points  $E(5; 0)$ ,  $F(6; 2)$  and  $G(8; -2)$ . Find the angle  $F\hat{E}G$ .
23. A triangle with angles  $40^\circ$ ,  $40^\circ$  and  $100^\circ$  has a perimeter of 20 cm. Find the length of each side of the triangle.
24. Determine the area of  $\triangle ABC$ .



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1a. 2FT3	1b. 2FT4	1c. 2FT5	2a. 2FT6	2b. 2FT7	2c. 2FT8
2d. 2FT9	2e. 2FTB	2f. 2FTC	2g. 2FTD	2h. 2FTF	2i. 2FTG
2j. 2FTH	2k. 2FTJ	2l. 2FTK	2m. 2FTM	2n. 2FTN	2o. 2FTP
2p. 2FTQ	2q. 2FTR	2r. 2FTS	3. 2FTT	4. 2FTV	5a. 2FTW
5b. 2FTX	5c. 2FTY	5d. 2FTZ	5e. 2FV2	5f. 2FV3	5g. 2FV4
5h. 2FV5	6. 2FV6	7. 2FV7	8. 2FV8	9. 2FV9	10a. 2FVB
10b. 2FVC	10c. 2FVD	10d. 2FVF	10e. 2FVG	10f. 2FVH	10g. 2FVJ
10h. 2FVK	10i. 2FVM	11. 2FVN	12. 2FVP	13. 2FVQ	14. 2FVR
15. 2FVS	16. 2FVT	17a. 2FVV	17b. 2FVW	17c. 2FVX	17d. 2FVY
17e. 2FVZ	17f. 2FW2	17g. 2FW3	17h. 2FW4	18. 2FW5	19. 2FW6
20. 2FW7	21. 2FW8	22. 2FW9	23. 2FWB	24. 2FWC	



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# CHAPTER



## Functions

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# 6 Functions

## 6.1 Introduction

EMA3Z

Functions are mathematical building blocks for designing machines, predicting natural disasters, curing diseases, understanding world economies and for keeping aeroplanes in the air. Functions can take input from many variables, but always give the same output, unique to that function.

Functions also allow us to visualise relationships in terms of graphs, which are much easier to read and interpret than lists of numbers.



Figure 6.1: A cricket player facing a delivery. If a cricket player is hit on his batting pads and the umpire thinks that the ball would have hit the stumps behind him, he is given out LBW (leg before wicket). At professional levels of the game, sophisticated software is used to determine if the ball will hit the stumps. The software uses functions to predict the flight of the ball if the cricket players leg had not been in the way.

Some examples of functions include:

- Money as a function of time. You never have more than one amount of money at any time because you can always add everything to give one total amount. By understanding how your money changes over time, you can plan to spend your money sensibly. Businesses find it very useful to plot the graph of their money over time so that they can see when they are spending too much.
- Temperature as a function of various factors. Temperature is a very complicated function because it has so many inputs, including: the time of day, the season, the amount of clouds in the sky, the strength of the wind, where you are and many more. But the important thing is that there is only one temperature output when you measure it in a specific place.
- Location as a function of time. You can never be in two places at the same time. If you were to plot the graphs of where two people are as a function of time, the place where the lines cross means that the two people meet each other at that time. This idea is used in logistics, an area of mathematics that tries to plan where people and items are for businesses.

### **DEFINITION:** Function

A function is a mathematical relationship between two variables, where every input variable has one output variable.

## Dependent and independent variables

EMA42

In functions, the  $x$ -variable is known as the input or independent variable, because its value can be chosen freely. The calculated  $y$ -variable is known as the output or dependent variable, because its value depends on the chosen input value.

Examples:

$\{x : x \in \mathbb{R}, x > 0\}$	The set of all $x$ -values such that $x$ is an element of the set of real numbers and is greater than 0.
$\{y : y \in \mathbb{N}, 3 < y \leq 5\}$	The set of all $y$ -values such that $y$ is a natural number, is greater than 3 and is less than or equal to 5.
$\{z : z \in \mathbb{Z}, z \leq 100\}$	The set of all $z$ -values such that $z$ is an integer and is less than or equal to 100.

It is important to note that this notation can only be used to represent an interval of real numbers.

Examples:

(3; 11)	Round brackets indicate that the number is not included. This interval includes all real numbers greater than but not equal to 3 and less than but not equal to 11.
( $-\infty$ ; -2)	Round brackets are always used for positive and negative infinity. This interval includes all real numbers less than, but not equal to -2.
[1; 9)	A square bracket indicates that the number is included. This interval includes all real numbers greater than or equal to 1 and less than but not equal to 9.

This is a very useful way to express a function. Another way of writing  $y = 2x + 1$  is  $f(x) = 2x + 1$ . We say “ $f$  of  $x$  is equal to  $2x + 1$ ”. Any letter can be used, for example,  $g(x)$ ,  $h(x)$ ,  $p(x)$ , etc.

### 1. Determine the output value:

“Find the value of the function for  $x = -3$ ” can be written as: “find  $f(-3)$ ”.

Replace  $x$  with  $-3$ :

$$\begin{aligned} f(-3) &= 2(-3) + 1 = -5 \\ \therefore f(-3) &= -5 \end{aligned}$$

This means that when  $x = -3$ , the value of the function is  $-5$ .

### 2. Determine the input value:

“Find the value of  $x$  that will give a  $y$ -value of 27” can be written as: “find  $x$  if  $f(x) = 27$ ”.

We write the following equation and solve for  $x$ :

$$\begin{aligned} 2x + 1 &= 27 \\ \therefore x &= 13 \end{aligned}$$

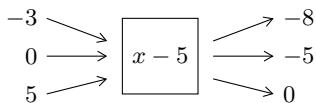
This means that when  $x = 13$  the value of the function is 27.

Functions can be expressed in many different ways for different purposes.

1. Words: "The relationship between two variables is such that one is always 5 less than the other."

2. Mapping diagram:

Input:      Function:      Output:



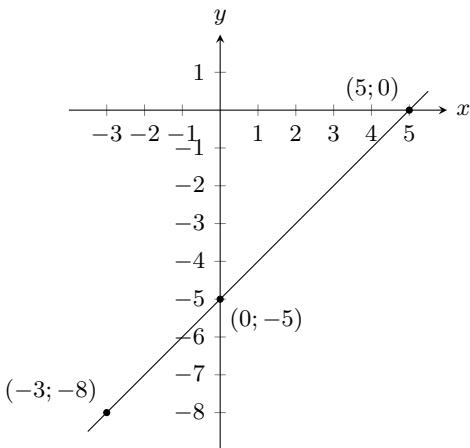
3. Table:

Input variable ( $x$ )	-3	0	5
Output variable ( $y$ )	-8	-5	0

4. Set of ordered number pairs:  $(-3; -8), (0; -5), (5; 0)$

5. Algebraic formula:  $f(x) = x - 5$

6. Graph:



## Domain and range

EMA47

The domain of a function is the set of all independent  $x$ -values from which the function produces a single  $y$ -value for each  $x$ -value.

The range is the set of all dependent  $y$ -values which can be obtained using an independent  $x$ -value.

### Exercise 6 – 1:

1. Write the following in set notation:

a)  $(-\infty; 7]$     b)  $[-13; 4)$     c)  $(35; \infty)$     d)  $[\frac{3}{4}; 21)$     e)  $[-\frac{1}{2}; \frac{1}{2}]$     f)  $(-\sqrt{3}; \infty)$

2. Write the following in interval notation:

a)  $\{p : p \in \mathbb{R}, p \leq 6\}$     b)  $\{k : k \in \mathbb{R}, -5 < k < 5\}$   
c)  $\{x : x \in \mathbb{R}, x > \frac{1}{5}\}$     d)  $\{z : z \in \mathbb{R}, 21 \leq z < 41\}$

3. Complete the following tables and identify the function.

a)

$x$	1	2	3	4	5	6
$y$	5	10		20		

b)

$x$	1		3	4		6
$y$	5	5			5	5

c)

x	2			8	10	12
y	1	2	3			6

4. Plot the following points on a graph.

a)

x	1	2	3	4	5	6
y	0,1	0,2	0,3	0,4	0,5	0,6

b)

x	1	2	3	4	5	6
y	5	9	13	17	21	25

5. Create a table of values from the function given and then plot the function. Your table must have at least 5 ordered pairs.

a)  $y = \frac{1}{2}x + 2$

b)  $y = x - 3$

6. If the functions  $f(x) = x^2 + 1$ ;  $g(x) = x - 4$ ;  $h(x) = 7 - x^2$ ;  $k(x) = 3$  are given, find the value of the following:

a)  $f(-1)$

b)  $g(-7)$

c)  $h(3)$

d)  $k(100)$

e)  $f(-2) + h(2)$

f)  $k(-5) + h(3)$

g)  $f(g(1))$

h)  $k(f(6))$

7. The cost of petrol and diesel per litre are given by the functions  $P$  and  $D$ , where:

$$P = 13,61V \quad D = 12,46V$$

Use this information to answer the following:

a) Evaluate  $P(8)$

b) Evaluate  $D(16)$

c) How many litres of petrol can you buy with R 300?

d) How many litres of petrol can you buy with R 275?

e) How much more expensive is petrol than diesel? Show your answer as a function.

8. A ball is rolling down a 10 m slope. The graph below shows the relationship between the distance and the time.



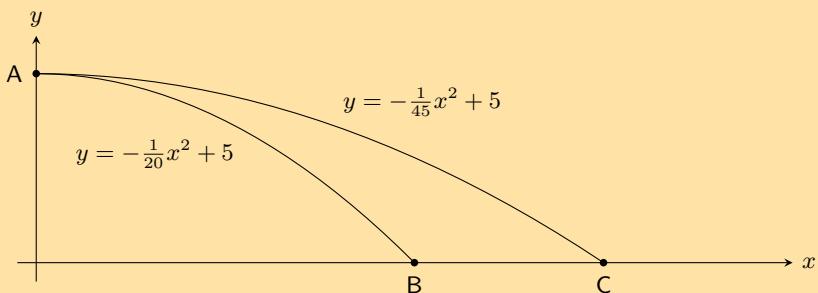
Use this information to answer the following:

a) After 6 s how much further does the ball have to roll?

b) What is the range of the function?

c) What is the domain of the function, and what does it represent?

9. James and Themba both throw a stone from the top of a building into a river. The path travelled by the stones can be described by quadratic equations.  $y = -\frac{1}{20}x^2 + 5$  describes the path of the stone thrown by James and  $y = -\frac{1}{45}x^2 + 5$  describes the path of Themba's stone.



- What is the height of the building that they stood on?
- How far did James throw his stone before it hit the river surface?
- How much farther did Themba throw his stone before it hit the river surface?

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |                          |                          |                          |                          |                          |                          |                          |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1a. <a href="#">2FWD</a> | 1b. <a href="#">2FWF</a> | 1c. <a href="#">2FWG</a> | 1d. <a href="#">2FWH</a> | 1e. <a href="#">2FWJ</a> | 1f. <a href="#">2FWK</a> | 2a. <a href="#">2FWM</a> |
| 2b. <a href="#">2FWN</a> | 2c. <a href="#">2FWP</a> | 2d. <a href="#">2FWQ</a> | 3a. <a href="#">2FWR</a> | 3b. <a href="#">2FWS</a> | 3c. <a href="#">2FWT</a> | 4a. <a href="#">2FWV</a> |
| 4b. <a href="#">2FWW</a> | 5a. <a href="#">2FWX</a> | 5b. <a href="#">2FWY</a> | 6. <a href="#">2FWZ</a>  | 7. <a href="#">2FX2</a>  | 8. <a href="#">2FX3</a>  | 9. <a href="#">2FX4</a>  |



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## 6.2 Linear functions

EMA48

### Functions of the form $y = x$

EMA49

Functions of the form  $y = mx + c$  are called straight line functions. In the equation,  $y = mx + c$ ,  $m$  and  $c$  are constants and have different effects on the graph of the function.

#### Worked example 1: Plotting a straight line graph

##### QUESTION

$$y = f(x) = x$$

Complete the following table for  $f(x) = x$  and plot the points on a set of axes.

$x$	-2	-1	0	1	2
$f(x)$	-2				

- Join the points with a straight line.
- Determine the domain and range.
- About which line is  $f$  symmetrical?
- Using the graph, determine the value of  $x$  for which  $f(x) = 4$ . Confirm your answer graphically.
- Where does the graph cut the axes?

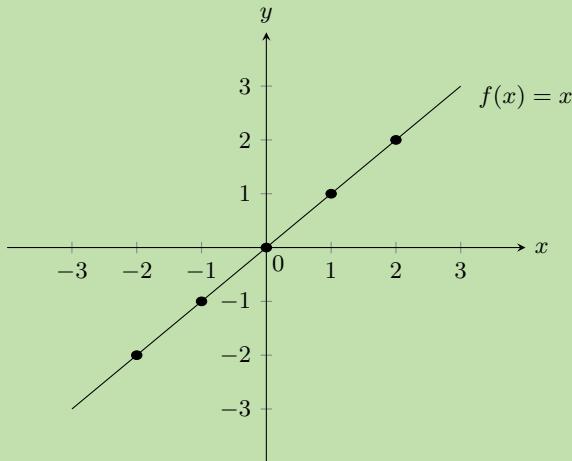
## SOLUTION

### Step 1: Substitute values into the equation

$x$	-2	-1	0	1	2
$f(x)$	-2	-1	0	1	2

### Step 2: Plot the points and join with a straight line curve

From the table, we get the following points and the graph:  $(-2; -2), (-1; -1), (0; 0), (1; 1), (2; 2)$



### Step 3: Determine the domain and range

Domain:  $x \in \mathbb{R}$

Range:  $f(x) \in \mathbb{R}$

### Step 4: Determine the value of $x$ for which $f(x) = 4$

From the graph we see that when  $f(x) = 4$ ,  $x = 4$ . This gives the point  $(4; 4)$ .

### Step 5: Determine the intercept

The function  $f$  intercepts the axes at the origin  $(0; 0)$ .

## Functions of the form $y = mx + c$

EMA4B

### Investigation: The effects of $m$ and $c$ on a straight line graph

On the same set of axes, plot the following graphs:

1.  $y = x - 2$
2.  $y = x - 1$
3.  $y = x$
4.  $y = x + 1$
5.  $y = x + 2$

Use your results to deduce the effect of different values of  $c$  on the graph.

On the same set of axes, plot the following graphs:

1.  $y = -2x$
2.  $y = -x$
3.  $y = x$
4.  $y = 2x$

Use your results to deduce the effect of different values of  $m$  on the graph.

### The effect of $m$

We notice that the value of  $m$  affects the slope of the graph. As  $m$  increases, the gradient of the graph increases.

If  $m > 0$  then the graph increases from left to right (slopes upwards).

If  $m < 0$  then the graph increases from right to left (slopes downwards). For this reason,  $m$  is referred to as the gradient of a straight-line graph.

### The effect of $c$

We also notice that the value of  $c$  affects where the graph cuts the  $y$ -axis. For this reason,  $c$  is known as the  $y$ -intercept.

If  $c > 0$  the graph shifts vertically upwards.

If  $c < 0$  the graph shifts vertically downwards.

	$m < 0$	$m = 0$	$m > 0$
$c > 0$			
$c = 0$			
$c < 0$			

Table 6.1: The effect of  $m$  and  $c$  on a straight line graph.

#### VISIT:

You can use this [Phet simulation](#) to help you see the effects of changing  $m$  and  $c$ .

The standard form of a straight line graph is the equation  $y = mx + c$ .

### Domain and range

The domain is  $\{x : x \in \mathbb{R}\}$  because there is no value of  $x$  for which  $f(x)$  is undefined.

The range of  $f(x) = mx + c$  is also  $\{f(x) : f(x) \in \mathbb{R}\}$  because  $f(x)$  can take on any real value.

### Intercepts

#### The $y$ -intercept:

Every point on the  $y$ -axis has an  $x$ -coordinate of 0. Therefore to calculate the  $y$ -intercept, let  $x = 0$ .

For example, the  $y$ -intercept of  $g(x) = x - 1$  is given by setting  $x = 0$ :

$$\begin{aligned} g(x) &= x - 1 \\ g(0) &= 0 - 1 \\ &= -1 \end{aligned}$$

This gives the point  $(0; -1)$ .

#### The $x$ -intercept:

Every point on the  $x$ -axis has a  $y$ -coordinate of 0. Therefore to calculate the  $x$ -intercept, let  $y = 0$ .

For example, the  $x$ -intercept of  $g(x) = x - 1$  is given by setting  $y = 0$ :

$$\begin{aligned} g(x) &= x - 1 \\ 0 &= x - 1 \\ \therefore x &= 1 \end{aligned}$$

This gives the point  $(1; 0)$ .

### Sketching graphs of the form $y = mx + c$

In order to sketch graphs of the form,  $f(x) = mx + c$ , we need to determine three characteristics:

1. sign of  $m$
2.  $y$ -intercept
3.  $x$ -intercept

### Dual intercept method

Only two points are needed to plot a straight line graph. The easiest points to use are the  $x$ -intercept and the  $y$ -intercept.

## Worked example 2: Sketching a straight line graph using the dual intercept method

### QUESTION

Sketch the graph of  $g(x) = x - 1$  using the dual intercept method.

### SOLUTION

#### Step 1: Examine the standard form of the equation

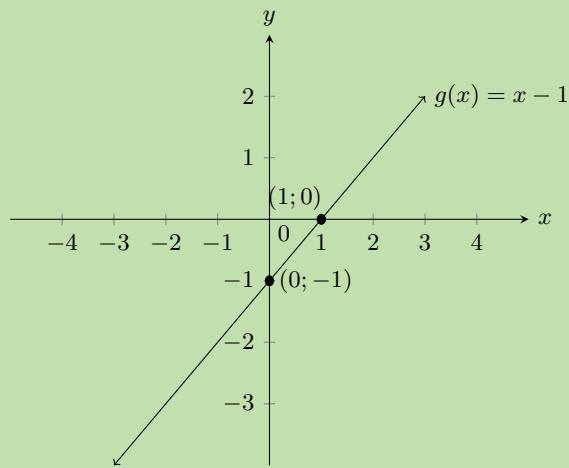
$m > 0$ . This means that the graph increases as  $x$  increases.

#### Step 2: Calculate the intercepts

For the  $y$ -intercept, let  $x = 0$ ; therefore  $g(0) = -1$ . This gives the point  $(0; -1)$ .

For the  $x$ -intercept, let  $y = 0$ ; therefore  $x = 1$ . This gives the point  $(1; 0)$ .

#### Step 3: Plot the points and draw the graph



## Gradient and $y$ -intercept method

EMA4G

We can draw a straight line graph of the form  $y = mx + c$  using the gradient ( $m$ ) and the  $y$ -intercept ( $c$ ).

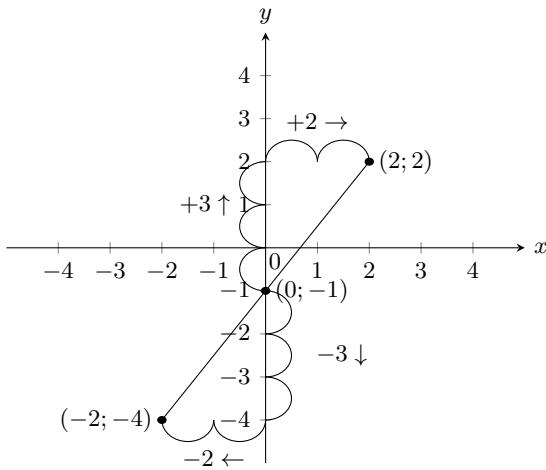
We calculate the  $y$ -intercept by letting  $x = 0$ . This gives us one point  $(0; c)$  for drawing the graph and we use the gradient to calculate the second point.

The gradient of a line is the measure of steepness. Steepness is determined by the ratio of vertical change to horizontal change:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{vertical change}}{\text{horizontal change}}$$

For example,  $y = \frac{3}{2}x - 1$ , therefore  $m > 0$  and the graph slopes upwards.

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{3 \uparrow}{2 \rightarrow} = \frac{-3 \downarrow}{-2 \leftarrow}$$



### Worked example 3: Sketching a straight line graph using the gradient–intercept method

#### QUESTION

Sketch the graph of  $p(x) = \frac{1}{2}x - 3$  using the gradient-intercept method.

#### SOLUTION

##### Step 1: Use the intercept

$c = -3$ , which gives the point  $(0; -3)$ .

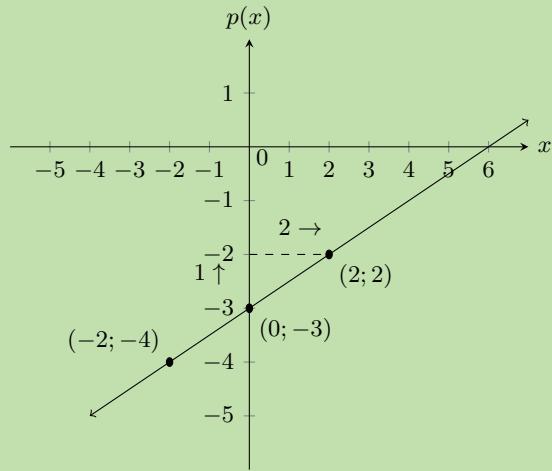
##### Step 2: Use the gradient

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{1 \uparrow}{2 \rightarrow} = \frac{-1 \downarrow}{-2 \leftarrow}$$

Start at  $(0; -3)$ . Move 1 unit up and 2 units to the right. This gives the second point  $(2; -2)$ .

Or start at  $(0; -3)$ , move 1 unit down and 2 units to the left. This gives the second point  $(-2; -4)$ .

##### Step 3: Plot the points and draw the graph



Always write the function in the form  $y = mx + c$  and take note of  $m$ . After plotting the graph, make sure that the graph increases if  $m > 0$  and that the graph decreases if  $m < 0$ .

**Exercise 6 – 2:**

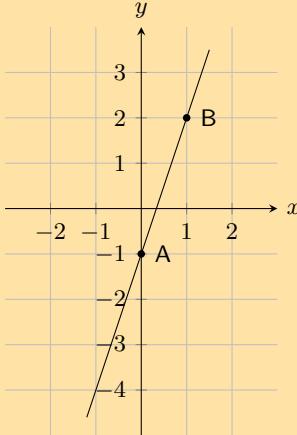
1. Determine the  $x$ -intercept and the  $y$ -intercept of the following equations.

a)  $y = x - 1$

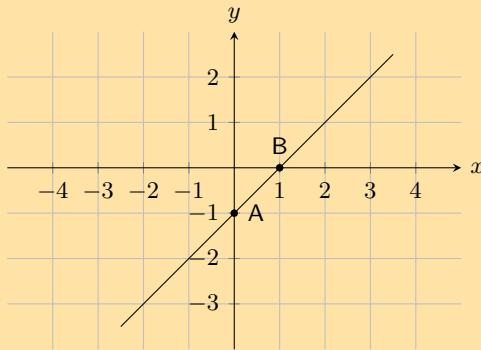
b)  $y = x + 2$

c)  $y = x - 3$

2. In the graph below there is a function with the equation  $y = mx + c$ . Determine the values of  $m$  (the gradient of the line) and  $c$  (the  $y$ -intercept of the line).



3. The graph below shows a function with the equation  $y = mx + c$ . Determine the values of  $m$  (the gradient of the line) and  $c$  (the  $y$ -intercept of the line).



4. List the  $x$  and  $y$ -intercepts for the following straight line graphs. Indicate whether the graph is increasing or decreasing:

a)  $y = x + 1$

b)  $y = x - 1$

c)  $h(x) = 2x + 1$

d)  $y + 3x = 1$

e)  $3y - 2x = 6$

f)  $k(x) = -3$

g)  $x = 3y$

h)  $\frac{x}{2} - \frac{y}{3} = 1$

5. State whether the following are true or not.

a) The gradient of  $2y = 3x - 1$  is 3.

b) The  $y$ -intercept of  $y = x + 4$  is 4.

c) The gradient of  $2 - y = 2x - 1$  is  $-2$ .

d) The gradient of  $y = \frac{1}{2}x - 1$  is  $-1$ .

e) The  $y$ -intercept of  $2y = 3x - 6$  is 6.

6. Write the following in standard form ( $y = mx + c$ ):

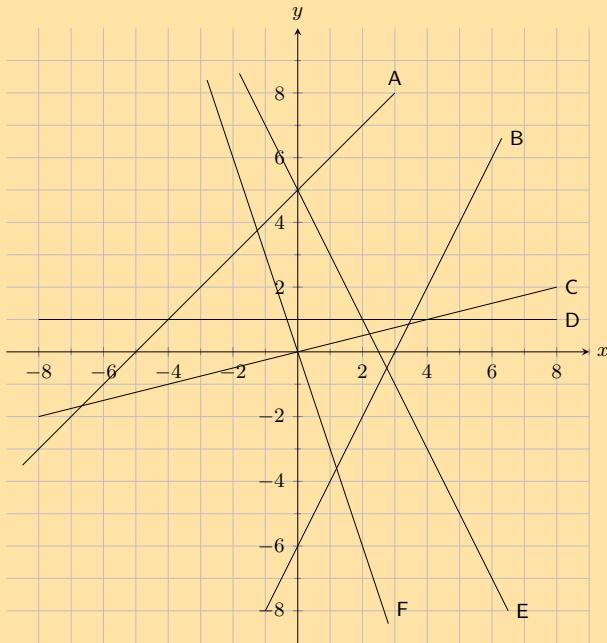
a)  $2y + 3x = 1$

b)  $3x - y = 5$

c)  $3y - 4 = x$

d)  $y + 2x - 3 = 1$

7. Look at the graphs below. Each graph is labelled with a letter. In the questions that follow, match any given equation with the label of a corresponding graph.



a)  $y = 5 - 2x$

d)  $y = -3x$

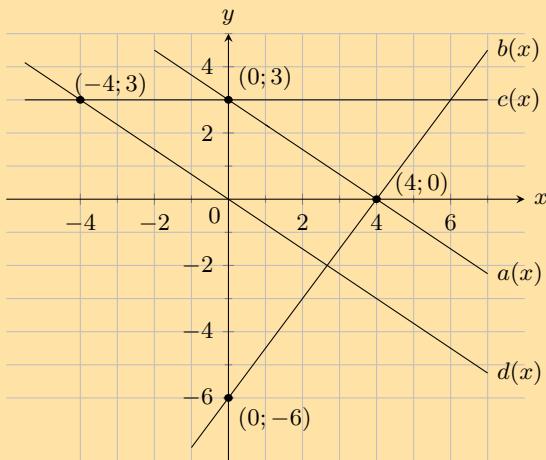
b)  $x + 5$

e)  $y = 1$

c)  $y = 2x - 6$

f)  $y = \frac{1}{2}x$

8. For the functions in the diagram below, give the equation of each line:



a)  $a(x)$

b)  $b(x)$

c)  $c(x)$

d)  $d(x)$

9. Sketch the following functions on the same set of axes, using the dual intercept method. Clearly indicate the coordinates of the intercepts with the axes and the point of intersection of the two graphs:  $x + 2y - 5 = 0$  and  $3x - y - 1 = 0$ .

10. On the same set of axes, draw the graphs of  $f(x) = 3 - 3x$  and  $g(x) = \frac{1}{3}x + 1$  using the gradient-intercept method.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |                          |                          |                          |                          |                          |                          |                          |                          |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1a. <a href="#">2FX5</a> | 1b. <a href="#">2FX6</a> | 1c. <a href="#">2FX7</a> | 2. <a href="#">2FX8</a>  | 3. <a href="#">2FX9</a>  | 4a. <a href="#">2FXB</a> | 4b. <a href="#">2FXC</a> | 4c. <a href="#">2FxD</a> |
| 4d. <a href="#">2FXF</a> | 4e. <a href="#">2FXG</a> | 4f. <a href="#">2FXH</a> | 4g. <a href="#">2FXJ</a> | 4h. <a href="#">2FK</a>  | 5a. <a href="#">2Fxm</a> | 5b. <a href="#">2FXN</a> | 5c. <a href="#">2Fxp</a> |
| 5d. <a href="#">2FXQ</a> | 5e. <a href="#">2FXR</a> | 6a. <a href="#">2FXS</a> | 6b. <a href="#">2FXT</a> | 6c. <a href="#">2FXV</a> | 6d. <a href="#">2FXW</a> | 7. <a href="#">2FXX</a>  | 8a. <a href="#">2FXY</a> |
| 8b. <a href="#">2FXZ</a> | 8c. <a href="#">2FY2</a> | 8d. <a href="#">2FY3</a> | 9. <a href="#">2FY4</a>  | 10. <a href="#">2FY5</a> |                          |                          |                          |



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Functions of the form  $y = x^2$ 

Functions of the general form  $y = ax^2 + q$  are called parabolic functions. In the equation  $y = ax^2 + q$ ,  $a$  and  $q$  are constants and have different effects on the parabola.

**Worked example 4: Plotting a quadratic function****QUESTION**

$$y = f(x) = x^2$$

Complete the following table for  $f(x) = x^2$  and plot the points on a system of axes.

$x$	-3	-2	-1	0	1	2	3
$f(x)$	9						

- Join the points with a smooth curve.
- The domain of  $f$  is  $x \in \mathbb{R}$ . Determine the range.
- About which line is  $f$  symmetrical?
- Determine the value of  $x$  for which  $f(x) = 6\frac{1}{4}$ . Confirm your answer graphically.
- Where does the graph cut the axes?

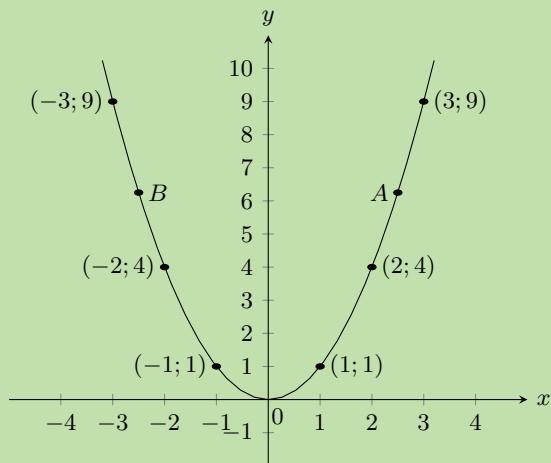
**SOLUTION****Step 1: Substitute values into the equation**

$$\begin{aligned}f(x) &= x^2 \\f(-3) &= (-3)^2 = 9 \\f(-2) &= (-2)^2 = 4 \\f(-1) &= (-1)^2 = 1 \\f(0) &= (0)^2 = 0 \\f(1) &= (1)^2 = 1 \\f(2) &= (2)^2 = 4 \\f(3) &= (3)^2 = 9\end{aligned}$$

$x$	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	0	4	9

**Step 2: Plot the points and join with a smooth curve**

From the table, we get the following points:  $(-3; 9), (-2; 4), (-1; 1), (0; 0), (1; 1), (2; 4), (3; 9)$



### Step 3: Determine the domain and range

Domain:  $x \in \mathbb{R}$

From the graph we see that for all values of  $x$ ,  $y \geq 0$ .

Range:  $\{y : y \in \mathbb{R}, y \geq 0\}$

### Step 4: Find the axis of symmetry

$f$  is symmetrical about the  $y$ -axis. Therefore the axis of symmetry of  $f$  is the line  $x = 0$ .

### Step 5: Determine the $x$ -value for which $f(x) = 6\frac{1}{4}$

$$\begin{aligned} f(x) &= \frac{25}{4} \\ \therefore \frac{25}{4} &= x^2 \\ x &= \pm \frac{5}{2} \\ &= \pm 2\frac{1}{2} \end{aligned}$$

See points  $A$  and  $B$  on the graph.

### Step 6: Determine the intercept

The function  $f$  intercepts the axes at the origin  $(0; 0)$ .

We notice that as the value of  $x$  increases from  $-\infty$  to  $0$ ,  $f(x)$  decreases.

At the turning point  $(0; 0)$ ,  $f(x) = 0$ .

As the value of  $x$  increases from  $0$  to  $\infty$ ,  $f(x)$  increases.

**Investigation: The effects of  $a$  and  $q$  on a parabola.**

Complete the table and plot the following graphs on the same system of axes:

1.  $y_1 = x^2 - 2$
2.  $y_2 = x^2 - 1$
3.  $y_3 = x^2$
4.  $y_4 = x^2 + 1$
5.  $y_5 = x^2 + 2$

$x$	-2	-1	0	1	2
$y_1$					
$y_2$					
$y_3$					
$y_4$					
$y_5$					

Use your results to deduce the effect of  $q$ .

Complete the table and plot the following graphs on the same system of axes:

1.  $y_6 = -2x^2$
2.  $y_7 = -x^2$
3.  $y_8 = x^2$
4.  $y_9 = 2x^2$

$x$	-2	-1	0	1	2
$y_6$					
$y_7$					
$y_8$					
$y_9$					

Use your results to deduce the effect of  $a$ .

**The effect of  $q$** 

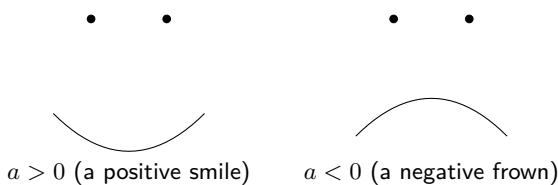
The effect of  $q$  is called a vertical shift because all points are moved the same distance in the same direction (it slides the entire graph up or down).

- For  $q > 0$ , the graph of  $f(x)$  is shifted vertically upwards by  $q$  units. The turning point of  $f(x)$  is above the  $y$ -axis.
- For  $q < 0$ , the graph of  $f(x)$  is shifted vertically downwards by  $|q|$  units. The turning point of  $f(x)$  is below the  $y$ -axis.

**The effect of  $a$** 

The sign of  $a$  determines the shape of the graph.

- For  $a > 0$ , the graph of  $f(x)$  is a “smile” and has a minimum turning point at  $(0; q)$ . The graph of  $f(x)$  is stretched vertically upwards; as  $a$  gets larger, the graph gets narrower.  
For  $0 < a < 1$ , as  $a$  gets closer to 0, the graph of  $f(x)$  gets wider.
- For  $a < 0$ , the graph of  $f(x)$  is a “frown” and has a maximum turning point at  $(0; q)$ . The graph of  $f(x)$  is stretched vertically downwards; as  $a$  gets smaller, the graph gets narrower.  
For  $-1 < a < 0$ , as  $a$  gets closer to 0, the graph of  $f(x)$  gets wider.



	$a < 0$	$a > 0$
$q > 0$		
$q = 0$		
$q < 0$		

Table 6.2: The effect of  $a$  and  $q$  on a parabola.

### VISIT:

You can use this [Phet simulation](#) to help you see the effects of changing  $a$  and  $q$  for a parabola.

## Discovering the characteristics

EMA4M

The standard form of the equation of a parabola is  $y = ax^2 + q$ .

### Domain and range

The domain is  $\{x : x \in \mathbb{R}\}$  because there is no value for which  $f(x)$  is undefined.

If  $a > 0$  then we have:

$$\begin{aligned} x^2 &\geq 0 && (\text{Perfect square is always positive}) \\ ax^2 &\geq 0 && (\text{since } a > 0) \\ ax^2 + q &\geq q && (\text{add } q \text{ to both sides}) \\ \therefore f(x) &\geq q \end{aligned}$$

Therefore if  $a > 0$ , the range is  $[q; \infty)$ . Similarly, if  $a < 0$  then the range is  $(-\infty; q]$ .

### Worked example 5: Domain and range of a parabola

#### QUESTION

If  $g(x) = x^2 + 2$ , determine the domain and range of the function.

## SOLUTION

### Step 1: Determine the domain

The domain is  $\{x : x \in \mathbb{R}\}$  because there is no value for which  $g(x)$  is undefined.

### Step 2: Determine the range

The range of  $g(x)$  can be calculated as follows:

$$\begin{aligned}x^2 &\geq 0 \\x^2 + 2 &\geq 2 \\g(x) &\geq 2\end{aligned}$$

Therefore the range is  $\{g(x) : g(x) \geq 2\}$ .

## Intercepts

### The $y$ -intercept:

Every point on the  $y$ -axis has an  $x$ -coordinate of 0, therefore to calculate the  $y$ -intercept let  $x = 0$ .

For example, the  $y$ -intercept of  $g(x) = x^2 + 2$  is given by setting  $x = 0$ :

$$\begin{aligned}g(x) &= x^2 + 2 \\g(0) &= 0^2 + 2 \\&= 2\end{aligned}$$

This gives the point  $(0; 2)$ .

### The $x$ -intercepts:

Every point on the  $x$ -axis has a  $y$ -coordinate of 0, therefore to calculate the  $x$ -intercept let  $y = 0$ .

For example, the  $x$ -intercepts of  $g(x) = x^2 + 2$  are given by setting  $y = 0$ :

$$\begin{aligned}g(x) &= x^2 + 2 \\0 &= x^2 + 2 \\-2 &= x^2\end{aligned}$$

There is no real solution, therefore the graph of  $g(x) = x^2 + 2$  does not have  $x$ -intercepts.

## Turning points

The turning point of the function of the form  $f(x) = ax^2 + q$  is determined by examining the range of the function.

- If  $a > 0$ , the graph of  $f(x)$  is a “smile” and has a minimum turning point at  $(0; q)$ .
- If  $a < 0$ , the graph of  $f(x)$  is a “frown” and has a maximum turning point at  $(0; q)$ .

## Axes of symmetry

The axis of symmetry for functions of the form  $f(x) = ax^2 + q$  is the  $y$ -axis, which is the line  $x = 0$ .

In order to sketch graphs of the form  $f(x) = ax^2 + q$ , we need to determine the following characteristics:

1. sign of  $a$
2.  $y$ -intercept
3.  $x$ -intercept
4. turning point

### Worked example 6: Sketching a parabola

#### QUESTION

Sketch the graph of  $y = 2x^2 - 4$ . Mark the intercepts and turning point.

#### SOLUTION

##### Step 1: Examine the standard form of the equation

We notice that  $a > 0$ . Therefore the graph is a “smile” and has a minimum turning point.

##### Step 2: Calculate the intercepts

For the  $y$ -intercept, let  $x = 0$ :

$$\begin{aligned}y &= 2x^2 - 4 \\&= 2(0)^2 - 4 \\&= -4\end{aligned}$$

This gives the point  $(0; -4)$ .

For the  $x$ -intercepts, let  $y = 0$ :

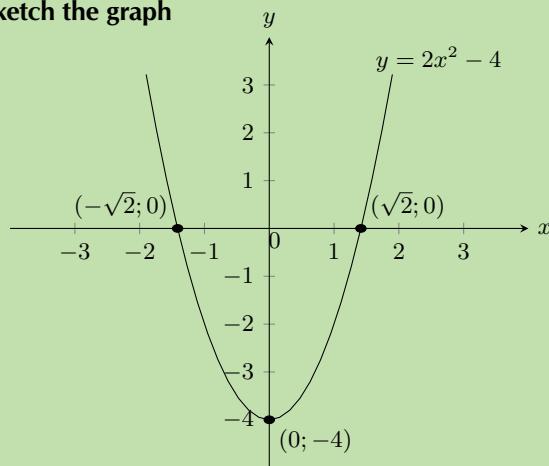
$$\begin{aligned}y &= 2x^2 - 4 \\0 &= 2x^2 - 4 \\x^2 &= 2 \\\therefore x &= \pm\sqrt{2}\end{aligned}$$

This gives the points  $(-\sqrt{2}; 0)$  and  $(\sqrt{2}; 0)$ .

##### Step 3: Determine the turning point

From the standard form of the equation we see that the turning point is  $(0; -4)$ .

##### Step 4: Plot the points and sketch the graph



Domain:  $\{x : x \in \mathbb{R}\}$

Range:  $\{y : y \geq -4, y \in \mathbb{R}\}$

The axis of symmetry is the line  $x = 0$ .

## Worked example 7: Sketching a parabola

### QUESTION

Sketch the graph of  $g(x) = -\frac{1}{2}x^2 - 3$ . Mark the intercepts and the turning point.

### SOLUTION

#### Step 1: Examine the standard form of the equation

We notice that  $a < 0$ . Therefore the graph is a “frown” and has a maximum turning point.

#### Step 2: Calculate the intercepts

For the  $y$ -intercept, let  $x = 0$ :

$$\begin{aligned} g(x) &= -\frac{1}{2}x^2 - 3 \\ g(0) &= -\frac{1}{2}(0)^2 - 3 \\ &= -3 \end{aligned}$$

This gives the point  $(0; -3)$ .

For the  $x$ -intercepts let  $y = 0$ :

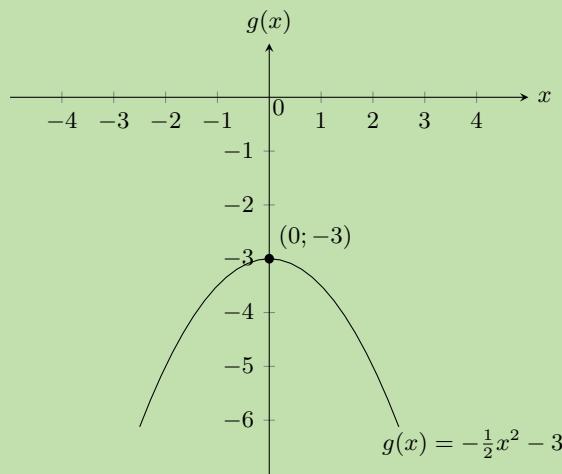
$$\begin{aligned} 0 &= -\frac{1}{2}x^2 - 3 \\ 3 &= -\frac{1}{2}x^2 \\ -2(3) &= x^2 \\ -6 &= x^2 \end{aligned}$$

There is no real solution, therefore there are no  $x$ -intercepts.

#### Step 3: Determine the turning point

From the standard form of the equation we see that the turning point is  $(0; -3)$ .

#### Step 4: Plot the points and sketch the graph

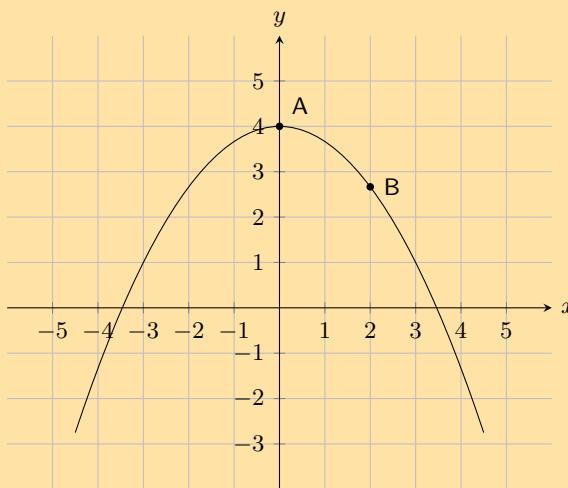


Domain:  $x \in \mathbb{R}$ . Range:  $y \in (-\infty; -3]$ . The axis of symmetry is the line  $x = 0$ .

**Exercise 6 – 3:**

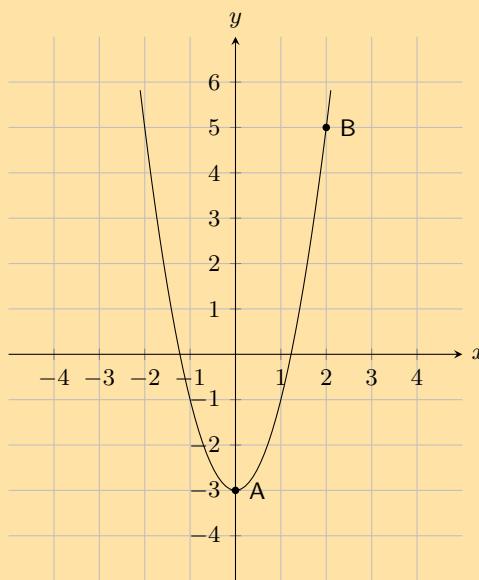
1. The graph below shows a quadratic function with the following form:  $y = ax^2 + q$ .

Two points on the parabola are shown: **Point A**, the turning point of the parabola, at  $(0; 4)$ , and **Point B** is at  $(2; \frac{8}{3})$ . Calculate the values of  $a$  and  $q$ .



2. The graph below shows a quadratic function with the following form:  $y = ax^2 + q$ .

Two points on the parabola are shown: **Point A**, the turning point of the parabola, at  $(0; -3)$ , and **Point B** is at  $(2; 5)$ . Calculate the values of  $a$  and  $q$ .



3. Given the following equation:

$$y = 5x^2 - 2$$

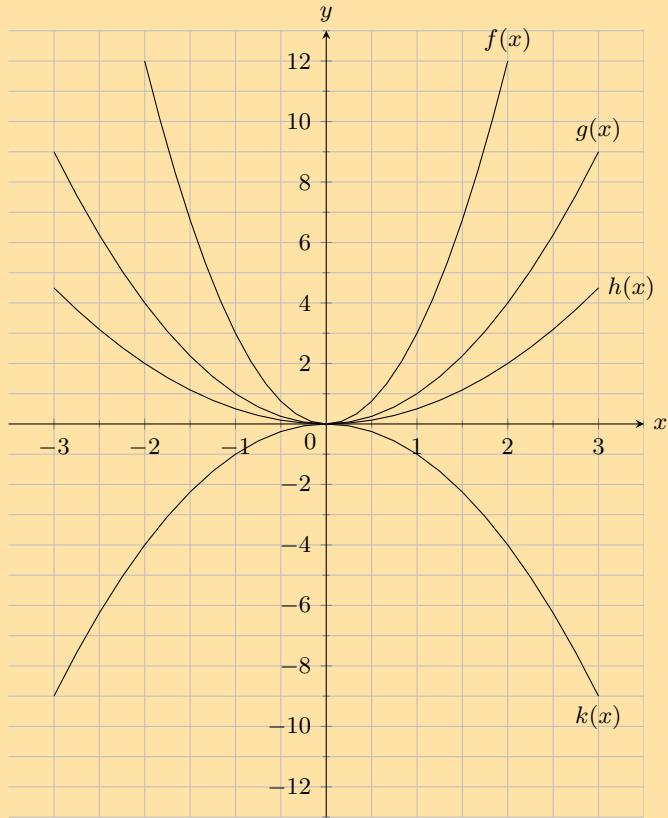
- Calculate the  $y$ -coordinate of the  $y$ -intercept.
- Now calculate the  $x$ -intercepts. Your answer must be correct to 2 decimal places.

4. Given the following equation:

$$y = -2x^2 + 1$$

- Calculate the  $y$ -coordinate of the  $y$ -intercept.
- Now calculate the  $x$ -intercepts. Your answer must be correct to 2 decimal places.

5. Given the following graph, identify a function that matches each of the following equations:



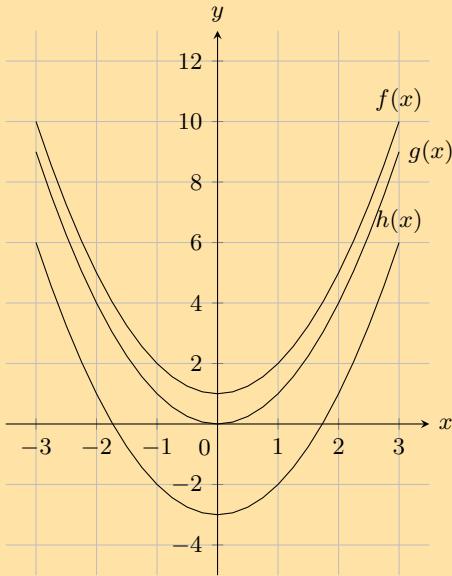
a)  $y = 0,5x^2$

b)  $y = x^2$

c)  $y = 3x^2$

d)  $y = -x^2$

6. Given the following graph, identify a function that matches each of the following equations:

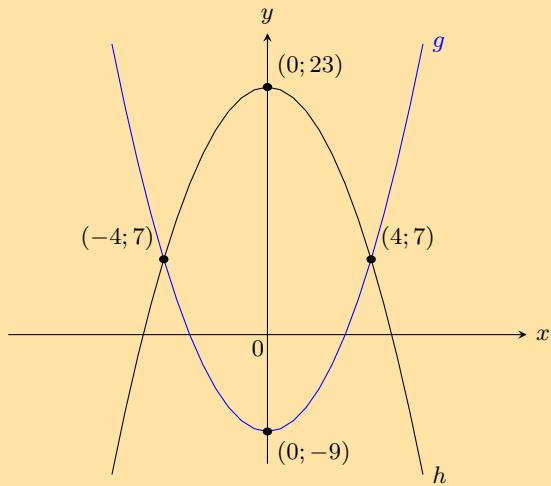


a)  $y = x^2 - 3$

b)  $y = x^2 + 1$

c)  $y = x^2$

7. Two parabolas are drawn:  $g : y = ax^2 + p$  and  $h : y = bx^2 + q$ .



- Find the values of  $a$  and  $p$ .
  - Find the values of  $b$  and  $q$ .
  - Find the values of  $x$  for which  $g(x) \geq h(x)$ .
  - For what values of  $x$  is  $g$  increasing?
8. Show that if  $a < 0$  the range of  $f(x) = ax^2 + q$  is  $\{f(x) : f(x) \leq q\}$ .
9. Draw the graph of the function  $y = -x^2 + 4$  showing all intercepts with the axes.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. 2FY6   2. 2FY7   3. 2FY8   4. 2FY9   5. 2FYB   6. 2FYC   7. 2FYD   8. 2FYF   9. 2FYG



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## 6.4 Hyperbolic functions

EMA4P

Functions of the form  $y = \frac{1}{x}$

EMA4Q

Functions of the general form  $y = \frac{a}{x} + q$  are called hyperbolic functions.

### Worked example 8: Plotting a hyperbolic function

#### QUESTION

$$y = h(x) = \frac{1}{x}$$

Complete the following table for  $h(x) = \frac{1}{x}$  and plot the points on a system of axes.

$x$	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
$h(x)$	$-\frac{1}{3}$										

1. Join the points with smooth curves.
2. What happens if  $x = 0$ ?
3. Explain why the graph consists of two separate curves.
4. What happens to  $h(x)$  as the value of  $x$  becomes very small or very large?
5. The domain of  $h(x)$  is  $\{x : x \in \mathbb{R}, x \neq 0\}$ . Determine the range.
6. About which two lines is the graph symmetrical?

### SOLUTION

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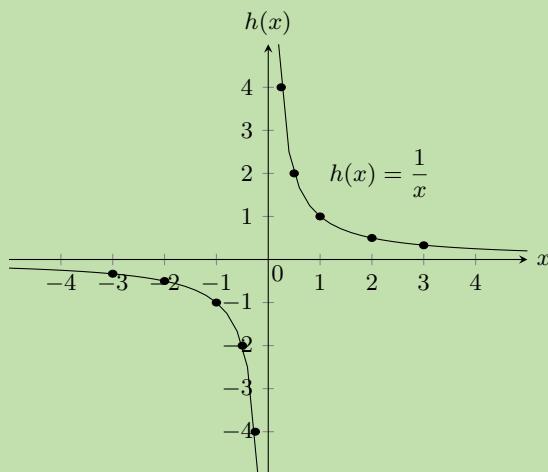
#### Step 1: Substitute values into the equation

$$\begin{aligned}
 h(x) &= \frac{1}{x} \\
 h(-3) &= \frac{1}{-3} = -\frac{1}{3} \\
 h(-2) &= \frac{1}{-2} = -\frac{1}{2} \\
 h(-1) &= \frac{1}{-1} = -1 \\
 h\left(-\frac{1}{2}\right) &= \frac{1}{-\frac{1}{2}} = -2 \\
 h\left(-\frac{1}{4}\right) &= \frac{1}{-\frac{1}{4}} = -4 \\
 h(0) &= \frac{1}{0} = \text{undefined} \\
 h\left(\frac{1}{4}\right) &= \frac{1}{\frac{1}{4}} = 4 \\
 h\left(\frac{1}{2}\right) &= \frac{1}{\frac{1}{2}} = 2 \\
 h(1) &= \frac{1}{1} = 1 \\
 h(2) &= \frac{1}{2} = \frac{1}{2} \\
 h(3) &= \frac{1}{3} = \frac{1}{3}
 \end{aligned}$$

$x$	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
$h(x)$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	undefined	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$

#### Step 2: Plot the points and join with two smooth curves

From the table we get the following points:  $(-3; -\frac{1}{3})$ ,  $(-2; -\frac{1}{2})$ ,  $(-1; -1)$ ,  $(-\frac{1}{2}; -2)$ ,  $(-\frac{1}{4}; -4)$ ,  $(\frac{1}{4}; 4)$ ,  $(\frac{1}{2}; 2)$ ,  $(1; 1)$ ,  $(2; \frac{1}{2})$ ,  $(3; \frac{1}{3})$



For  $x = 0$  the function  $h$  is undefined. This is called a discontinuity at  $x = 0$ .

$y = h(x) = \frac{1}{x}$  therefore we can write that  $x \times y = 1$ . Since the product of two positive numbers **and** the product of two negative numbers can be equal to 1, the graph lies in the first and third quadrants.

### Step 3: Determine the asymptotes

As the value of  $x$  gets larger, the value of  $h(x)$  gets closer to, but does not equal 0. This is a horizontal asymptote, the line  $y = 0$ . The same happens in the third quadrant; as  $x$  gets smaller  $h(x)$  also approaches the negative  $x$ -axis asymptotically.

We also notice that there is a vertical asymptote, the line  $x = 0$ ; as  $x$  gets closer to 0,  $h(x)$  approaches the  $y$ -axis asymptotically.

### Step 4: Determine the range

Domain:  $\{x : x \in \mathbb{R}, x \neq 0\}$

From the graph, we see that  $y$  is defined for all values except 0.

Range:  $\{y : y \in \mathbb{R}, y \neq 0\}$

### Step 5: Determine the lines of symmetry

The graph of  $h(x)$  has two axes of symmetry: the lines  $y = x$  and  $y = -x$ . About these two lines, one half of the hyperbola is a mirror image of the other half.

Functions of the form  $y = \frac{a}{x} + q$

EMA4R

### Investigation: The effects of $a$ and $q$ on a hyperbola.

On the same set of axes, plot the following graphs:

$$1. \quad y_1 = \frac{1}{x} - 2$$

$$2. \quad y_2 = \frac{1}{x} - 1$$

3.  $y_3 = \frac{1}{x}$
4.  $y_4 = \frac{1}{x} + 1$
5.  $y_5 = \frac{1}{x} + 2$

Use your results to deduce the effect of  $q$ .

On the same set of axes, plot the following graphs:

1.  $y_6 = \frac{-2}{x}$
2.  $y_7 = \frac{-1}{x}$
3.  $y_8 = \frac{1}{x}$
4.  $y_9 = \frac{2}{x}$

Use your results to deduce the effect of  $a$ .

### The effect of $q$

The effect of  $q$  is called a vertical shift because all points are moved the same distance in the same direction (it slides the entire graph up or down).

- For  $q > 0$ , the graph of  $f(x)$  is shifted vertically upwards by  $q$  units.
- For  $q < 0$ , the graph of  $f(x)$  is shifted vertically downwards by  $|q|$  units.

The horizontal asymptote is the line  $y = q$  and the vertical asymptote is always the  $y$ -axis, the line  $x = 0$ .

### The effect of $a$

The sign of  $a$  determines the shape of the graph.

- If  $a > 0$ , the graph of  $f(x)$  lies in the first and third quadrants.

For  $a > 1$ , the graph of  $f(x)$  will be further away from the axes than  $y = \frac{1}{x}$ .

For  $0 < a < 1$ , as  $a$  tends to 0, the graph moves closer to the axes than  $y = \frac{1}{x}$ .

- If  $a < 0$ , the graph of  $f(x)$  lies in the second and fourth quadrants.

For  $a < -1$ , the graph of  $f(x)$  will be further away from the axes than  $y = -\frac{1}{x}$ .

For  $-1 < a < 0$ , as  $a$  tends to 0, the graph moves closer to the axes than  $y = -\frac{1}{x}$ .

	$a < 0$	$a > 0$
$q > 0$		
$q = 0$		
$q < 0$		

Table 6.3: The effects of  $a$  and  $q$  on a hyperbola.

The standard form of a hyperbola is the equation  $y = \frac{a}{x} + q$ .

### Domain and range

For  $y = \frac{a}{x} + q$ , the function is undefined for  $x = 0$ . The domain is therefore  $\{x : x \in \mathbb{R}, x \neq 0\}$ .

We see that  $y = \frac{a}{x} + q$  can be rewritten as:

$$\begin{aligned}y &= \frac{a}{x} + q \\y - q &= \frac{a}{x} \\ \text{If } x \neq 0 \text{ then: } (y - q)x &= a \\x &= \frac{a}{y - q}\end{aligned}$$

This shows that the function is undefined only at  $y = q$ .

Therefore the range is  $\{f(x) : f(x) \in \mathbb{R}, f(x) \neq q\}$

### Worked example 9: Domain and range of a hyperbola

#### QUESTION

If  $g(x) = \frac{2}{x} + 2$ , determine the domain and range of the function.

#### SOLUTION

##### **Step 1: Determine the domain**

The domain is  $\{x : x \in \mathbb{R}, x \neq 0\}$  because  $g(x)$  is undefined only at  $x = 0$ .

##### **Step 2: Determine the range**

We see that  $g(x)$  is undefined only at  $y = 2$ . Therefore the range is  $\{g(x) : g(x) \in \mathbb{R}, g(x) \neq 2\}$

### Intercepts

#### **The $y$ -intercept:**

Every point on the  $y$ -axis has an  $x$ -coordinate of 0, therefore to calculate the  $y$ -intercept let  $x = 0$ .

For example, the  $y$ -intercept of  $g(x) = \frac{2}{x} + 2$  is given by setting  $x = 0$ :

$$\begin{aligned}y &= \frac{2}{x} + 2 \\y &= \frac{2}{0} + 2\end{aligned}$$

which is undefined, therefore there is no  $y$ -intercept.

### The $x$ -intercept:

Every point on the  $x$ -axis has a  $y$ -coordinate of 0, therefore to calculate the  $x$ -intercept, let  $y = 0$ .

For example, the  $x$ -intercept of  $g(x) = \frac{2}{x} + 2$  is given by setting  $y = 0$ :

$$\begin{aligned}y &= \frac{2}{x} + 2 \\0 &= \frac{2}{x} + 2 \\\frac{2}{x} &= -2 \\x &= \frac{2}{-2} \\&= -1\end{aligned}$$

This gives the point  $(-1; 0)$ .

### Asymptotes

There are two asymptotes for functions of the form  $y = \frac{a}{x} + q$ .

The horizontal asymptote is the line  $y = q$  and the vertical asymptote is always the  $y$ -axis, the line  $x = 0$ .

### Axes of symmetry

There are two lines about which a hyperbola is symmetrical:  $y = x + q$  and  $y = -x + q$ .

Sketching graphs of the form  $y = \frac{a}{x} + q$

EMA4T

In order to sketch graphs of functions of the form,  $y = f(x) = \frac{a}{x} + q$ , we need to determine four characteristics:

1. sign of  $a$
2.  $y$ -intercept
3.  $x$ -intercept
4. asymptotes

### Worked example 10: Sketching a hyperbola

#### QUESTION

Sketch the graph of  $g(x) = \frac{2}{x} + 2$ . Mark the intercepts and the asymptotes.

#### SOLUTION

##### Step 1: Examine the standard form of the equation

We notice that  $a > 0$  therefore the graph of  $g(x)$  lies in the first and third quadrant.

## Step 2: Calculate the intercepts

For the  $y$ -intercept, let  $x = 0$ :

$$g(x) = \frac{2}{x} + 2$$
$$g(0) = \frac{2}{0} + 2$$

This is undefined, therefore there is no  $y$ -intercept.

For the  $x$ -intercept, let  $y = 0$ :

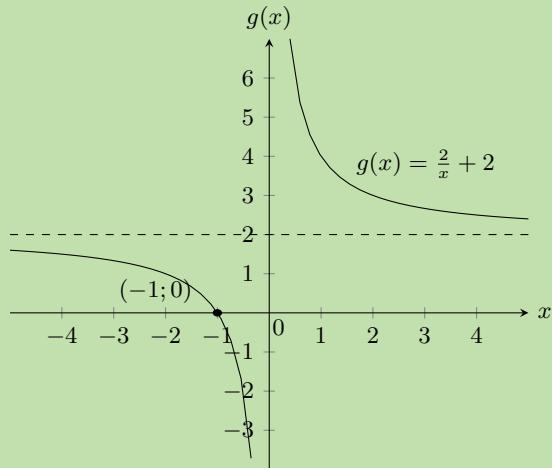
$$g(x) = \frac{2}{x} + 2$$
$$0 = \frac{2}{x} + 2$$
$$\frac{2}{x} = -2$$
$$\therefore x = -1$$

This gives the point  $(-1; 0)$ .

## Step 3: Determine the asymptotes

The horizontal asymptote is the line  $y = 2$ . The vertical asymptote is the line  $x = 0$ .

## Step 4: Sketch the graph



Domain:  $\{x : x \in \mathbb{R}, x \neq 0\}$

Range:  $\{y : y \in \mathbb{R}, y \neq 2\}$

## Worked example 11: Sketching a hyperbola

### QUESTION

Sketch the graph of  $y = \frac{-4}{x} + 7$ .

## SOLUTION

### Step 1: Examine the standard form of the equation

We see that  $a < 0$  therefore the graph lies in the second and fourth quadrants.

### Step 2: Calculate the intercepts

For the  $y$ -intercept, let  $x = 0$ :

$$\begin{aligned}y &= \frac{-4}{x} + 7 \\&= \frac{-4}{0} + 7\end{aligned}$$

This is undefined, therefore there is no  $y$ -intercept.

For the  $x$ -intercept, let  $y = 0$ :

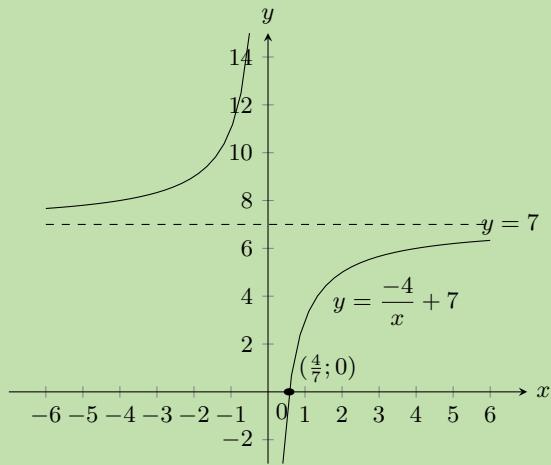
$$\begin{aligned}y &= \frac{-4}{x} + 7 \\0 &= \frac{-4}{x} + 7 \\\frac{-4}{x} &= -7 \\\therefore x &= \frac{4}{7}\end{aligned}$$

This gives the point  $\left(\frac{4}{7}; 0\right)$ .

### Step 3: Determine the asymptotes

The horizontal asymptote is the line  $y = 7$ . The vertical asymptote is the line  $x = 0$ .

### Step 4: Sketch the graph



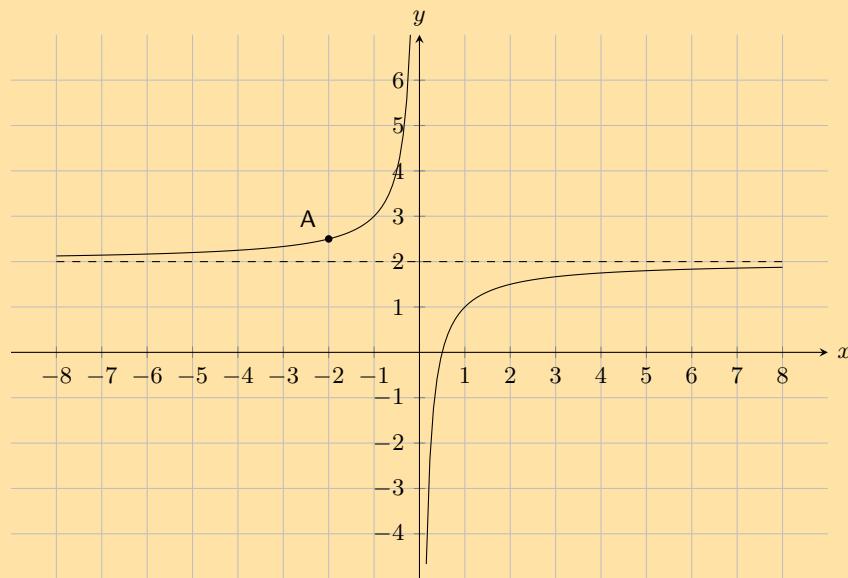
Domain:  $\{x : x \in \mathbb{R}, x \neq 0\}$

Range:  $\{y : y \in \mathbb{R}, y \neq 7\}$

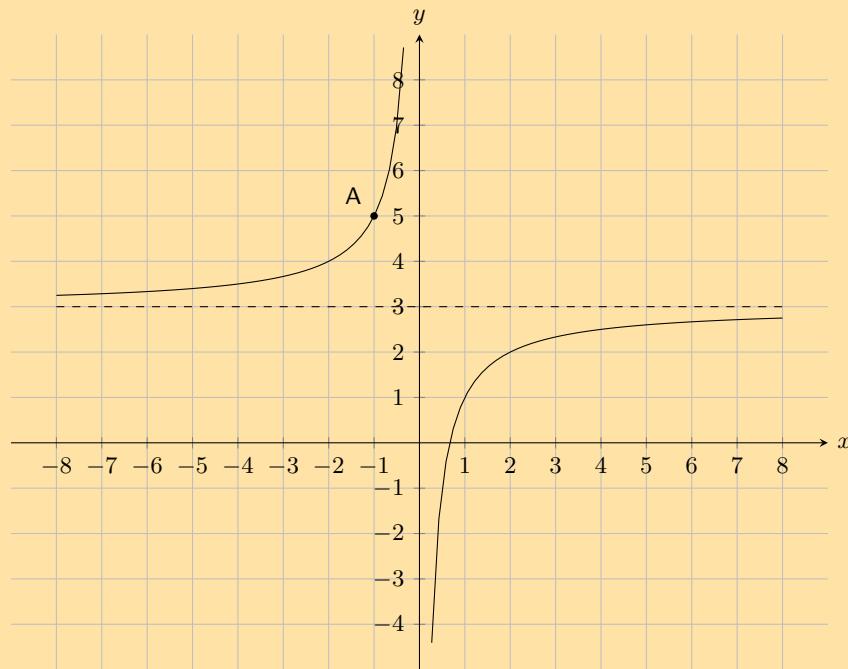
Axis of symmetry:  $y = x + 7$  and  $y = -x + 7$

**Exercise 6 – 4:**

1. The following graph shows a hyperbolic equation of the form  $y = \frac{a}{x} + q$ . **Point A** is shown at  $(-2; \frac{5}{2})$ . Calculate the values of  $a$  and  $q$ .



2. The following graph shows a hyperbolic equation of the form  $y = \frac{a}{x} + q$ . **Point A** is shown at  $(-1; 5)$ . Calculate the values of  $a$  and  $q$ .



3. Given the following equation:

$$y = \frac{3}{x} + 2$$

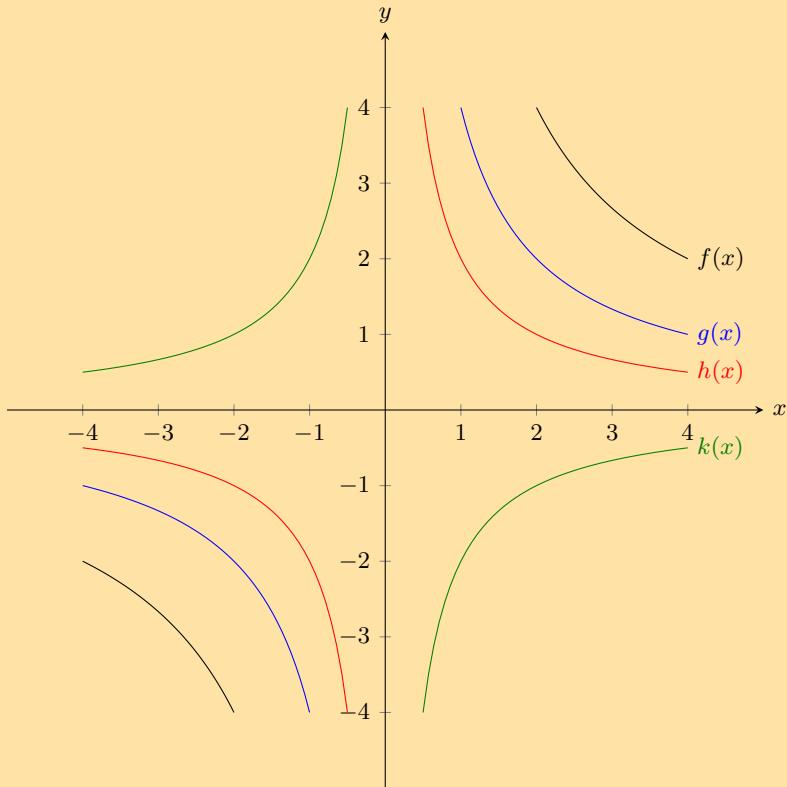
- a) Determine the location of the  $y$ -intercept.  
 b) Determine the location of the  $x$ -intercept. Give your answer as a fraction.

4. Given the following equation:

$$y = -\frac{2}{x} - 2$$

- a) Determine the location of the  $y$ -intercept.
- b) Determine the location of the  $x$ -intercept.

5. Given the following graph, identify a function that matches each of the following equations:



a)  $y = \frac{2}{x}$

b)  $y = \frac{4}{x}$

c)  $y = -\frac{2}{x}$

d)  $y = \frac{8}{x}$

6. Given the function:  $xy = -6$ .

- a) Draw the graph.
- b) Does the point  $(-2; 3)$  lie on the graph? Give a reason for your answer.
- c) If the  $x$ -value of a point on the graph is 0,25 what is the corresponding  $y$ -value?
- d) What happens to the  $y$ -values as the  $x$ -values become very large?
- e) Give the equation of the asymptotes.
- f) With the line  $y = -x$  as a line of symmetry, what is the point symmetrical to  $(-2; 3)$ ?

7. Given the function:  $h(x) = \frac{8}{x}$ .

- a) Draw the graph.

b) How would the graph of  $g(x) = \frac{8}{x} + 3$  compare with that of  $h(x) = \frac{8}{x}$ ? Explain your answer fully.

c) Draw the graph of  $y = \frac{8}{x} + 3$  on the same set of axes, showing asymptotes, axes of symmetry and the coordinates of one point on the graph.

8. Sketch the functions given and describe the transformation performed on the first function to obtain the second function. Show all asymptotes.

- a)  $y = \frac{1}{x}$  and  $\frac{3}{x}$
- b)  $y = \frac{6}{x}$  and  $\frac{6}{x} - 1$
- c)  $y = \frac{5}{x}$  and  $-\frac{5}{x}$
- d)  $y = \frac{1}{x}$  and  $\frac{1}{2x}$

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- 1. [2FYH](#)
- 2. [2FYJ](#)
- 3. [2FYK](#)
- 4. [2FYM](#)
- 5. [2FYN](#)
- 6. [2FYP](#)
- 7. [2FYQ](#)
- 8a. [2FYR](#)
- 8b. [2FYS](#)
- 8c. [2FYT](#)
- 8d. [2FYV](#)



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## 6.5 Exponential functions

EMA4V

### Functions of the form $y = b^x$

EMA4W

Functions of the general form  $y = ab^x + q$  are called exponential functions. In the equation  $a$  and  $q$  are constants and have different effects on the function.

#### Worked example 12: Plotting an exponential function

##### QUESTION

$$y = f(x) = b^x \text{ for } b > 0 \text{ and } b \neq 1$$

Complete the following table for each of the functions and draw the graphs on the same system of axes:  $f(x) = 2^x$ ,  $g(x) = 3^x$ ,  $h(x) = 5^x$ .

	-2	-1	0	1	2
$f(x) = 2^x$					
$g(x) = 3^x$					
$h(x) = 5^x$					

1. At what point do these graphs intersect?
2. Explain why they do not cut the  $x$ -axis.
3. Give the domain and range of  $h(x)$ .
4. As  $x$  increases, does  $h(x)$  increase or decrease?
5. Which of these graphs increases at the slowest rate?
6. For  $y = k^x$  and  $k > 1$ , the greater the value of  $k$ , the steeper the curve of the graph. True or false?

Complete the following table for each of the functions and draw the graphs on the same system of axes:

$$F(x) = \left(\frac{1}{2}\right)^x, G(x) = \left(\frac{1}{3}\right)^x, H(x) = \left(\frac{1}{5}\right)^x$$

	-2	-1	0	1	2
$F(x) = \left(\frac{1}{2}\right)^x$					
$G(x) = \left(\frac{1}{3}\right)^x$					
$H(x) = \left(\frac{1}{5}\right)^x$					

- Give the  $y$ -intercept for each function.
- Describe the relationship between the graphs  $f(x)$  and  $F(x)$ .
- Describe the relationship between the graphs  $g(x)$  and  $G(x)$ .
- Give the domain and range of  $H(x)$ .
- For  $y = \left(\frac{1}{k}\right)^x$  and  $k > 1$ , the greater the value of  $k$ , the steeper the curve of the graph. True or false?
- Give the equation of the asymptote for the functions.

### SOLUTION

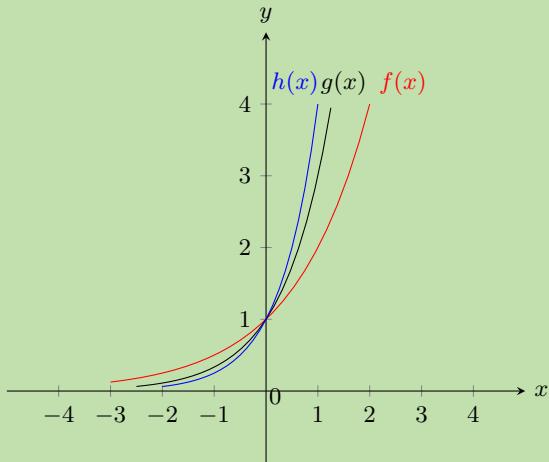
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#### Step 1: Substitute values into the equations

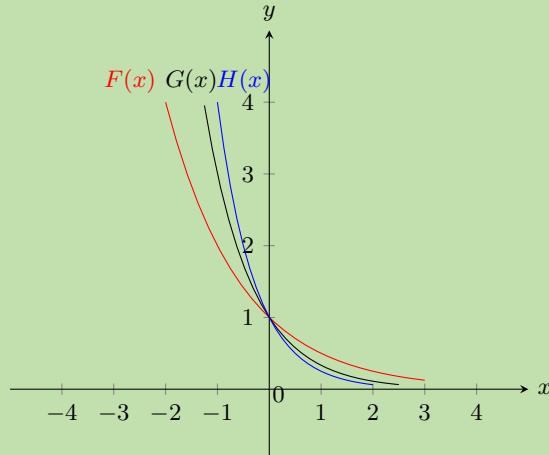
	-2	-1	0	1	2
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$g(x) = 3^x$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$h(x) = 5^x$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25

	-2	-1	0	1	2
$F(x) = \left(\frac{1}{2}\right)^x$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
$G(x) = \left(\frac{1}{3}\right)^x$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$
$H(x) = \left(\frac{1}{5}\right)^x$	25	5	1	$\frac{1}{5}$	$\frac{1}{25}$

#### Step 2: Plot the points and join with a smooth curve



- We notice that all graphs pass through the point  $(0; 1)$ . Any number with exponent 0 is equal to 1.
- The graphs do not cut the  $x$ -axis because you can never get 0 by raising any non-zero number to the power of any other number.
- Domain:  $\{x : x \in \mathbb{R}\}$   
Range:  $\{y : y \in \mathbb{R}, y > 0\}$
- As  $x$  increases,  $h(x)$  increases.
- $f(x) = 2^x$  increases at the slowest rate because it has the smallest base.
- True: the greater the value of  $k$  ( $k > 1$ ), the steeper the graph of  $y = k^x$ .



- The  $y$ -intercept is the point  $(0; 1)$  for all graphs. For any real number  $z$ :  $z^0 = 1 \quad z \neq 0$ .
- $F(x)$  is the reflection of  $f(x)$  about the  $y$ -axis.
- $G(x)$  is the reflection of  $g(x)$  about the  $y$ -axis.
- Domain:  $\{x : x \in \mathbb{R}\}$   
Range:  $\{y : y \in \mathbb{R}, y > 0\}$
- True: the greater the value of  $k$  ( $k > 1$ ), the steeper the graph of  $y = (\frac{1}{k})^x$ .
- The equation of the horizontal asymptote is  $y = 0$ , the  $x$ -axis.

## Functions of the form $y = ab^x + q$

EMA4X

### Investigation: The effects of $a$ , $q$ and $b$ on an exponential graph.

On the same set of axes, plot the following graphs ( $a = 1$ ,  $q = 0$  and  $b$  changes):

- $y_1 = 2^x$
- $y_2 = (\frac{1}{2})^x$
- $y_3 = 6^x$
- $y_4 = (\frac{1}{6})^x$

	-2	-1	0	1	2
$y_1 = 2^x$					
$y_2 = (\frac{1}{2})^x$					
$y_3 = 6^x$					
$y_4 = (\frac{1}{6})^x$					

Use your results to deduce the effect of  $b$ .

On the same set of axes, plot the following graphs ( $b = 2$ ,  $a = 1$  and  $q$  changes):

1.  $y_5 = 2^x - 2$
2.  $y_6 = 2^x - 1$
3.  $y_7 = 2^x$
4.  $y_8 = 2^x + 1$
5.  $y_9 = 2^x + 2$

	-2	-1	0	1	2
$y_5 = 2^x - 2$					
$y_6 = 2^x - 1$					
$y_7 = 2^x$					
$y_8 = 2^x + 1$					
$y_9 = 2^x + 2$					

Use your results to deduce the effect of  $q$ .

On the same set of axes, plot the following graphs ( $b = 2$ ,  $q = 0$  and  $a$  changes).

1.  $y_{10} = 1 \times 2^x$
2.  $y_{11} = 2 \times 2^x$
3.  $y_{12} = -1 \times 2^x$
4.  $y_{13} = -2 \times 2^x$

	-2	-1	0	1	2
$y_{10} = 1 \times 2^x$					
$y_{11} = 2 \times 2^x$					
$y_{12} = -1 \times 2^x$					
$y_{13} = -2 \times 2^x$					

Use your results to deduce the effect of  $a$ .

### The effect of $q$

The effect of  $q$  is called a vertical shift because all points are moved the same distance in the same direction (it slides the entire graph up or down).

- For  $q > 0$ , the graph is shifted vertically upwards by  $q$  units.
- For  $q < 0$ , the graph is shifted vertically downwards by  $q$  units.

The horizontal asymptote is shifted by  $q$  units and is the line  $y = q$ .

### The effect of $a$

The sign of  $a$  determines whether the graph curves upwards or downwards.

For  $0 < b < 1$ :

- For  $a > 0$ , the graph curves downwards. It reflects the graph about the horizontal asymptote.
- For  $a < 0$ , the graph curves upwards.

For  $b > 1$ :

- For  $a > 0$ , the graph curves upwards.
- For  $a < 0$ , the graph curves downwards. It reflects the graph about the horizontal asymptote.

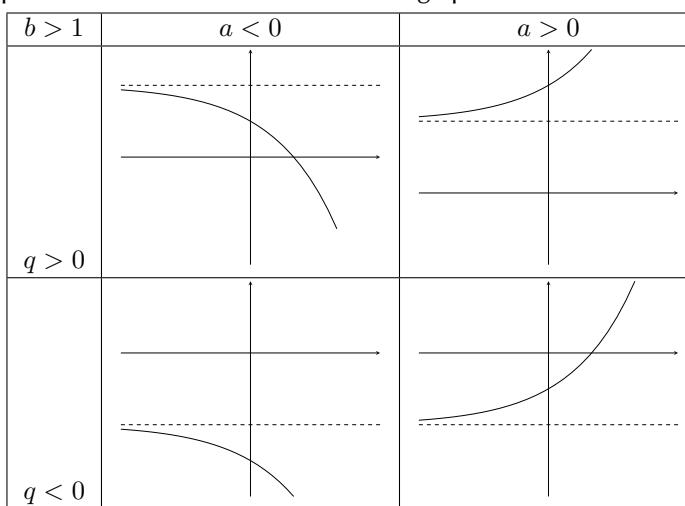


Table 6.4: The effect of  $a$  and  $q$  on an exponential graph when  $b > 1$ .

$0 < b < 1$	$a < 0$	$a > 0$
$q > 0$		
$q < 0$		

Table 6.5: The effect of  $a$  and  $q$  on an exponential graph when  $0 < b < 1$ .

## Discovering the characteristics

EMA4Y

The standard form of an exponential function is  $y = ab^x + q$ .

### Domain and range

For  $y = ab^x + q$ , the function is defined for all real values of  $x$ . Therefore the domain is  $\{x : x \in \mathbb{R}\}$ .

The range of  $y = ab^x + q$  is dependent on the sign of  $a$ .

For  $a > 0$ :

$$\begin{aligned} b^x &> 0 \\ ab^x &> 0 \\ ab^x + q &> q \\ f(x) &> q \end{aligned}$$

Therefore, for  $a > 0$  the range is  $\{f(x) : f(x) > q\}$ .

For  $a < 0$ :

$$\begin{aligned} b^x &> 0 \\ ab^x &< 0 \\ ab^x + q &< q \\ f(x) &< q \end{aligned}$$

Therefore, for  $a < 0$  the range is  $\{f(x) : f(x) < q\}$ .

## Worked example 13: Domain and range of an exponential function

### QUESTION

Find the domain and range of  $g(x) = 5 \cdot 2^x + 1$

### SOLUTION

#### Step 1: Find the domain

The domain of  $g(x) = 5 \times 2^x + 1$  is  $\{x : x \in \mathbb{R}\}$ .

#### Step 2: Find the range

$$\begin{aligned}2^x &> 0 \\5 \times 2^x &> 0 \\5 \times 2^x + 1 &> 1\end{aligned}$$

Therefore the range is  $\{g(x) : g(x) > 1\}$ .

### Intercepts

#### The $y$ -intercept.

For the  $y$ -intercept, let  $x = 0$ :

$$\begin{aligned}y &= ab^x + q \\&= ab^0 + q \\&= a(1) + q \\&= a + q\end{aligned}$$

For example, the  $y$ -intercept of  $g(x) = 5 \times 2^x + 1$  is given by setting  $x = 0$ :

$$\begin{aligned}y &= 5 \times 2^x + 1 \\&= 5 \times 2^0 + 1 \\&= 5 + 1 \\&= 6\end{aligned}$$

This gives the point  $(0; 6)$ .

#### The $x$ -intercept:

For the  $x$ -intercept, let  $y = 0$ .

For example, the  $x$ -intercept of  $g(x) = 5 \times 2^x + 1$  is given by setting  $y = 0$ :

$$\begin{aligned}y &= 5 \times 2^x + 1 \\0 &= 5 \times 2^x + 1 \\-1 &= 5 \times 2^x \\2^x &= -\frac{1}{5}\end{aligned}$$

There is no real solution. Therefore, the graph of  $g(x)$  does not have any  $x$ -intercepts.

## Asymptotes

Exponential functions of the form  $y = ab^x + q$  have a single horizontal asymptote, the line  $x = q$ .

### Sketching graphs of the form $y = ab^x + q$

EMA4Z

In order to sketch graphs of functions of the form,  $y = ab^x + q$ , we need to determine four characteristics:

1. sign of  $a$
2.  $y$ -intercept
3.  $x$ -intercept
4. asymptote

#### VISIT:

The following video shows some examples of sketching exponential functions.

▶ See video: [2FYWW](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 14: Sketching an exponential function

#### QUESTION

Sketch the graph of  $g(x) = 3 \times 2^x + 2$ . Mark the intercept and the asymptote.

#### SOLUTION

##### Step 1: Examine the standard form of the equation

From the equation we see that  $a > 1$ , therefore the graph curves upwards.  $q > 0$  therefore the graph is shifted vertically upwards by 2 units.

##### Step 2: Calculate the intercepts

For the  $y$ -intercept, let  $x = 0$ :

$$\begin{aligned}y &= 3 \times 2^x + 2 \\&= 3 \times 2^0 + 2 \\&= 3 + 2 \\&= 5\end{aligned}$$

This gives the point  $(0; 5)$ .

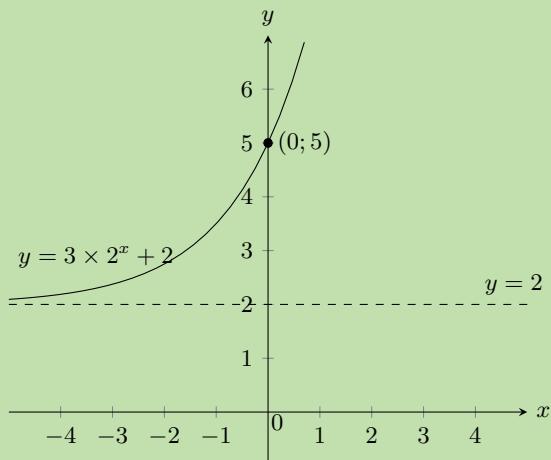
For the  $x$ -intercept, let  $y = 0$ :

$$\begin{aligned}y &= 3 \times 2^x + 2 \\0 &= 3 \times 2^x + 2 \\-2 &= 3 \times 2^x \\2^x &= -\frac{2}{3}\end{aligned}$$

There is no real solution, therefore there is no  $x$ -intercept.

**Step 3: Determine the asymptote**

The horizontal asymptote is the line  $y = 2$ .

**Step 4: Plot the points and sketch the graph**

Domain:  $\{x : x \in \mathbb{R}\}$  Range:  $\{g(x) : g(x) > 2\}$

Note that there is no axis of symmetry for exponential functions.

**Worked example 15: Sketching an exponential graph****QUESTION**

Sketch the graph of  $y = -2 \times 3^x + 6$

**SOLUTION****Step 1: Examine the standard form of the equation**

From the equation we see that  $a < 0$  therefore the graph curves downwards.  $q > 0$  therefore the graph is shifted vertically upwards by 6 units.

**Step 2: Calculate the intercepts**

For the  $y$ -intercept, let  $x = 0$ :

$$\begin{aligned} y &= -2 \times 3^x + 6 \\ &= -2 \times 3^0 + 6 \\ &= 4 \end{aligned}$$

This gives the point  $(0; 4)$ .

For the  $x$ -intercept, let  $y = 0$ :

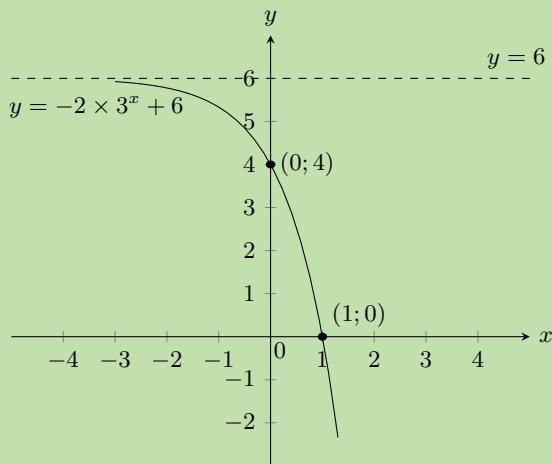
$$\begin{aligned} y &= -2 \times 3^x + 6 \\ 0 &= -2 \times 3^x + 6 \\ -6 &= -2 \times 3^x \\ 3^1 &= 3^x \\ \therefore x &= 1 \end{aligned}$$

This gives the point  $(1; 0)$ .

### Step 3: Determine the asymptote

The horizontal asymptote is the line  $y = 6$ .

### Step 4: Plot the points and sketch the graph



Domain:  $\{x : x \in \mathbb{R}\}$  Range:  $\{g(x) : g(x) < 6\}$

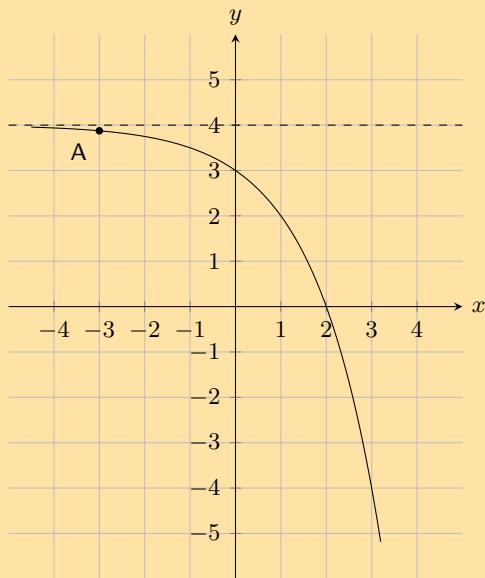
### Exercise 6 – 5:

- Given the following equation:

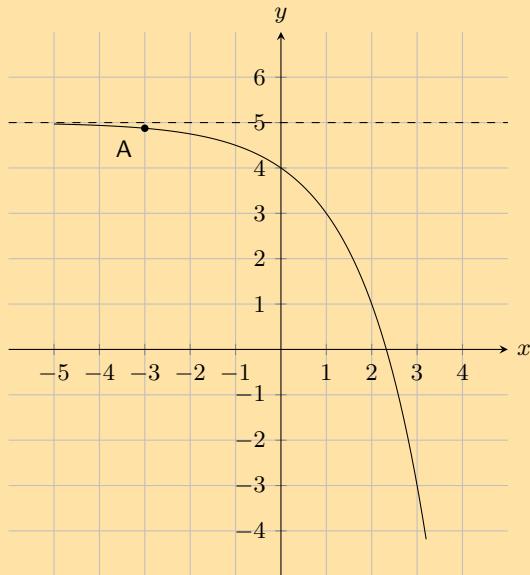
$$y = -\frac{2}{3} \cdot (3)^x + 1$$

- Calculate the  $y$ -intercept. Your answer must be correct to 2 decimal places.
- Now calculate the  $x$ -intercept. Estimate your answer to one decimal place if necessary.

- The graph here shows an exponential function with the equation  $y = a \cdot 2^x + q$ . One point is given on the curve: **Point A** is at  $(-3; 3,875)$ . Determine the values of  $a$  and  $q$ , correct to the nearest integer.



3. Below you see a graph of an exponential function with the equation  $y = a \cdot 2^x + q$ . One point is given on the curve: **Point A** is at  $(-3; 4,875)$ . Calculate the values of  $a$  and  $q$ , correct to the nearest integer.

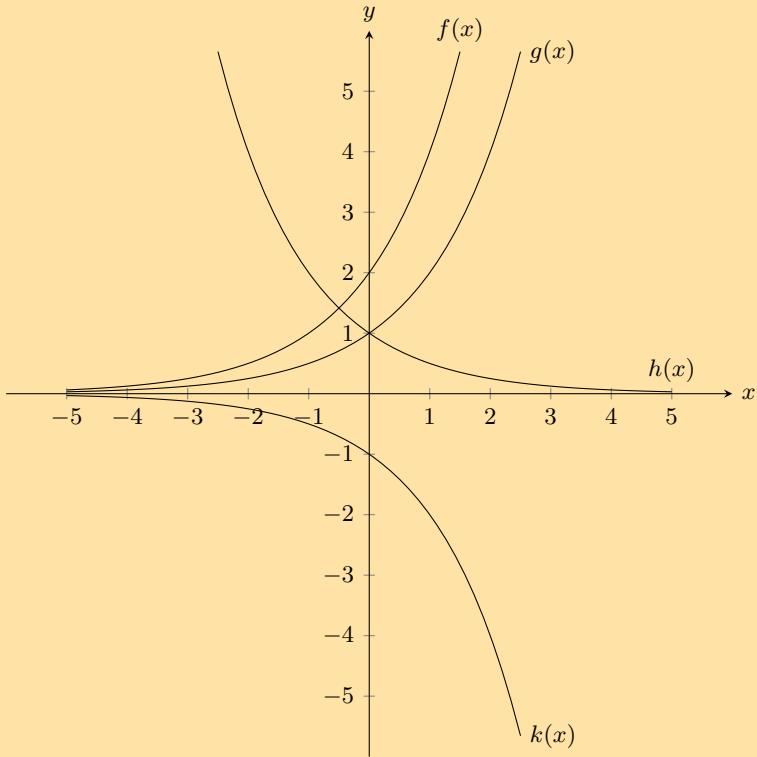


4. Given the following equation:

$$y = \frac{1}{4} \cdot (4)^x - 1$$

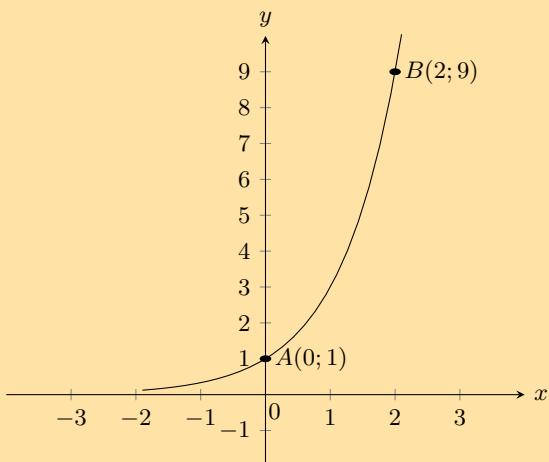
- a) Calculate the  $y$ -intercept. Your answer must be correct to 2 decimal places.  
 b) Now calculate the  $x$ -intercept.

5. Given the following graph, identify a function that matches each of the following equations:



- a)  $y = 2^x$       b)  $y = -2^x$       c)  $y = 2 \cdot 2^x$       d)  $y = \left(\frac{1}{2}\right)^x$

6. Given the functions  $y = 2^x$  and  $y = \left(\frac{1}{2}\right)^x$ .
- Draw the graphs on the same set of axes.
  - Is the  $x$ -axis an asymptote or an axis of symmetry to both graphs? Explain your answer.
  - Which graph can be described by the equation  $y = 2^{-x}$ ? Explain your answer.
  - Solve the equation  $2^x = \left(\frac{1}{2}\right)^x$  graphically and check your answer is correct by using substitution.
7. The curve of the exponential function  $f$  in the accompanying diagram cuts the  $y$ -axis at the point  $A(0; 1)$  and passes through the point  $B(2; 9)$ .



- Determine the equation of the function  $f$ .
- Determine the equation of the function  $h(x)$ , the reflection of  $f(x)$  in the  $x$ -axis.
- Determine the range of  $h(x)$ .
- Determine the equation of the function  $g(x)$ , the reflection of  $f(x)$  in the  $y$ -axis.
- Determine the equation of the function  $j(x)$  if  $j(x)$  is a vertical stretch of  $f(x)$  by +2 units.
- Determine the equation of the function  $k(x)$  if  $k(x)$  is a vertical shift of  $f(x)$  by -3 units.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. 2FYX   2. 2FYY   3. 2FYZ   4. 2FZ2   5. 2FZ3   6. 2FZ4   7. 2FZ5



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## 6.6 Trigonometric functions

EMA52

This section describes the graphs of trigonometric functions.

### Sine function

EMA53

### Functions of the form $y = \sin \theta$

EMA54

#### Worked example 16: Plotting a sine graph

#### QUESTION

$$y = f(\theta) = \sin \theta \quad [0^\circ \leq \theta \leq 360^\circ]$$

Use your calculator to complete the following table.

Choose an appropriate scale and plot the values of  $\theta$  on the  $x$ -axis and of  $\sin \theta$  on the  $y$ -axis. Round answers to 2 decimal places.

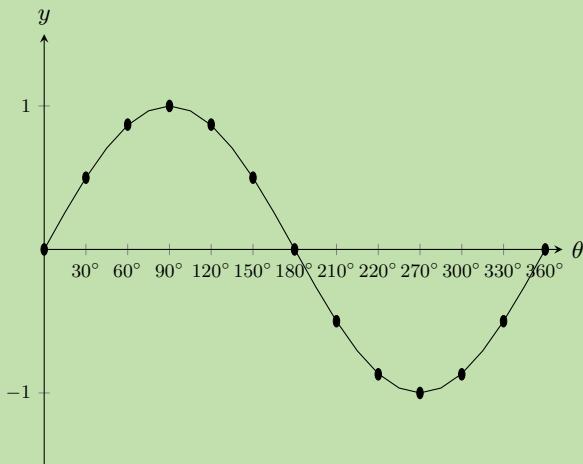
$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$	$360^\circ$
$\sin \theta$													

### SOLUTION

#### Step 1: Substitute values for $\theta$

$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$	$360^\circ$
$\sin \theta$	0	0,5	0,87	1	0,87	0,5	0	-0,5	-0,87	-1	-0,87	-0,5	0

#### Step 2: Plot the points and join with a smooth curve



Notice the wave shape of the graph. Each complete wave takes  $360^\circ$  to complete. This is called the period. The height of the wave above and below the  $x$ -axis is called the graph's amplitude. The maximum value of  $y = \sin \theta$  is 1 and the minimum value is -1.

Domain:  $[0^\circ; 360^\circ]$

Range:  $[-1; 1]$

$x$ -intercepts:  $(0^\circ; 0)$ ,  $(180^\circ; 0)$ ,  $(360^\circ; 0)$

$y$ -intercept:  $(0^\circ; 0)$

Maximum turning point:  $(90^\circ; 1)$

Minimum turning point:  $(270^\circ; -1)$

**Investigation: The effects of  $a$  and  $q$  on a sine graph**

In the equation,  $y = a \sin \theta + q$ ,  $a$  and  $q$  are constants and have different effects on the graph. On the same set of axes, plot the following graphs for  $0^\circ \leq \theta \leq 360^\circ$ :

1.  $y_1 = \sin \theta - 2$
2.  $y_2 = \sin \theta - 1$
3.  $y_3 = \sin \theta$
4.  $y_4 = \sin \theta + 1$
5.  $y_5 = \sin \theta + 2$

Use your results to deduce the effect of  $q$ .

On the same set of axes, plot the following graphs for  $0^\circ \leq \theta \leq 360^\circ$ :

1.  $y_6 = -2 \sin \theta$
2.  $y_7 = -\sin \theta$
3.  $y_8 = \sin \theta$
4.  $y_9 = 2 \sin \theta$

Use your results to deduce the effect of  $a$ .

**The effect of  $q$** 

The effect of  $q$  is called a vertical shift because the whole sine graph shifts up or down by  $q$  units.

- For  $q > 0$ , the graph is shifted vertically upwards by  $q$  units.
- For  $q < 0$ , the graph is shifted vertically downwards by  $q$  units.

**The effect of  $a$** 

The value of  $a$  affects the amplitude of the graph; the height of the peaks and the depth of the troughs.

- For  $a > 1$ , there is a vertical stretch and the amplitude increases.  
For  $0 < a < 1$ , the amplitude decreases
- For  $a < 0$ , there is a reflection about the  $x$ -axis.  
For  $-1 < a < 0$ , there is a reflection about the  $x$ -axis and the amplitude decreases.  
For  $a < -1$ , there is a reflection about the  $x$ -axis and the amplitude increases.

Note that amplitude is always positive.

Effect of $a$	
$a > 1$ : vertical stretch, amplitude increases	
$a = 1$ : basic sine graph	
$0 < a < 1$ : vertical contraction, amplitude decreases	
$-1 < a < 0$ : reflection about $x$ -axis of $0 < a < 1$	
$a < -1$ : reflection about $x$ -axis of $a > 1$	

Table 6.6: The effect of  $a$  on a sine graph.

Effect of $q$	
$q > 0$ : vertical shift upwards by $q$ units	
$q = 0$ : basic sine graph	
$q < 0$ : vertical shift downwards by $q$ units	

Table 6.7: The effect of  $q$  on a sine graph.

## Discovering the characteristics

EMA56

### Domain and range

For  $f(\theta) = a \sin \theta + q$ , the domain is  $[0^\circ; 360^\circ]$

The range of  $f(\theta) = a \sin \theta + q$  depends on the values of  $a$  and  $q$ .

For  $a > 0$ :

$$\begin{aligned} -1 &\leq \sin \theta \leq 1 \\ -a &\leq a \sin \theta \leq a \\ -a + q &\leq a \sin \theta + q \leq a + q \\ -a + q &\leq f(\theta) \leq a + q \end{aligned}$$

For all values of  $\theta$ ,  $f(\theta)$  is always between  $-a + q$  and  $a + q$ .

Therefore for  $a > 0$ , the range of  $f(\theta) = a \sin \theta + q$  is  $\{f(\theta) : f(\theta) \in [-a + q, a + q]\}$

Similarly, for  $a < 0$ , the range of  $f(\theta) = a \sin \theta + q$  is  $\{f(\theta) : f(\theta) \in [a + q, -a + q]\}$

## Period

The period of  $y = a \sin \theta + q$  is  $360^\circ$ . This means that one sine wave is completed in  $360^\circ$ .

## Intercepts

The  $y$ -intercept of  $f(\theta) = a \sin \theta + q$  is simply the value of  $f(\theta)$  at  $\theta = 0^\circ$

$$\begin{aligned}y &= f(0^\circ) \\&= a \sin 0^\circ + q \\&= a(0) + q \\&= q\end{aligned}$$

This gives the point  $(0; q)$

**Important:** when sketching trigonometric graphs, always start with the basic graph and then consider the effects of  $a$  and  $q$ .

### Worked example 17: Sketching a sine graph

#### QUESTION

Sketch the graph of  $f(\theta) = 2 \sin \theta + 3$  for  $\theta \in [0^\circ; 360^\circ]$ .

#### SOLUTION

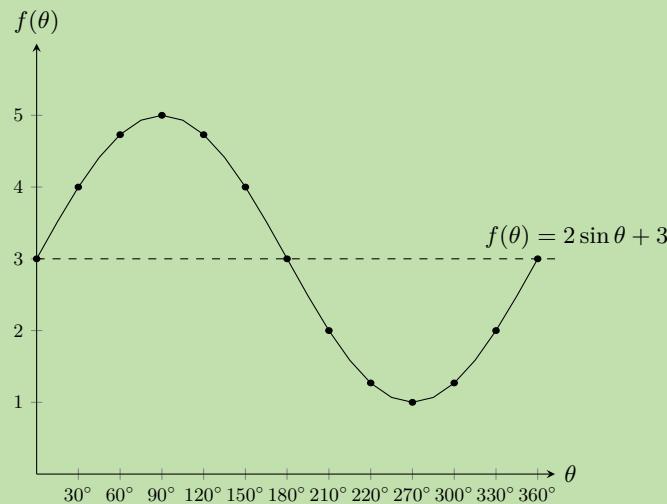
##### Step 1: Examine the standard form of the equation

From the equation we see that  $a > 1$  so the graph is stretched vertically. We also see that  $q > 0$  so the graph is shifted vertically upwards by 3 units.

##### Step 2: Substitute values for $\theta$

$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$	$360^\circ$
$f(\theta)$	3	4	4,73	5	4,73	4	3	2	1,27	1	1,27	2	3

##### Step 3: Plot the points and join with a smooth curve



Domain:  $[0^\circ; 360^\circ]$  Range:  $[1; 5]$

$x$ -intercepts: none  $y$ -intercepts:  $(0^\circ; 3)$

Maximum turning point:  $(90^\circ; 5)$  Minimum turning point:  $(270^\circ; 1)$

**Worked example 18: Plotting a cosine graph****QUESTION**

$$y = f(\theta) = \cos \theta \quad [0^\circ \leq \theta \leq 360^\circ]$$

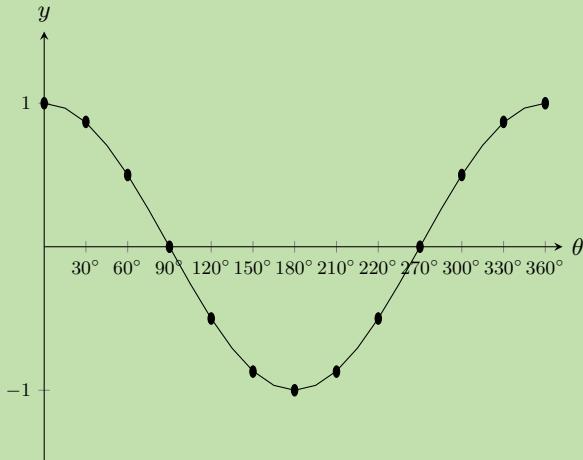
Use your calculator to complete the following table.

Choose an appropriate scale and plot the values of  $\theta$  on the  $x$ -axis and  $\cos \theta$  on the  $y$ -axis. Round your answers to 2 decimal places.

$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$	$360^\circ$
$\cos \theta$													

**SOLUTION****Step 1: Substitute values for  $\theta$** 

$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$	$360^\circ$
$\cos \theta$	1	0,87	0,5	0	-0,5	-0,87	-1	-0,87	-0,5	0	0,5	0,87	1

**Step 2: Plot the points and join with a smooth curve**

Notice the similar wave shape of the graph. The period is also  $360^\circ$  and the amplitude is 1. The maximum value of  $y = \cos \theta$  is 1 and the minimum value is -1.

Domain:  $[0^\circ; 360^\circ]$  Range:  $[-1; 1]$

$x$ -intercepts:  $(90^\circ; 0)$ ,  $(270^\circ; 0)$   $y$ -intercept:  $(0^\circ; 1)$

Maximum turning points:  $(0^\circ; 1)$ ,  $(360^\circ; 1)$  Minimum turning point:  $(180^\circ; -1)$

**Investigation: The effects of  $a$  and  $q$  on a cosine graph**

In the equation,  $y = a \cos \theta + q$ ,  $a$  and  $q$  are constants and have different effects on the graph.

On the same set of axes, plot the following graphs for  $0^\circ \leq \theta \leq 360^\circ$ :

1.  $y_1 = \cos \theta - 2$
2.  $y_2 = \cos \theta - 1$
3.  $y_3 = \cos \theta$
4.  $y_4 = \cos \theta + 1$
5.  $y_5 = \cos \theta + 2$

Use your results to deduce the effect of  $q$ .

On the same set of axes, plot the following graphs for  $0^\circ \leq \theta \leq 360^\circ$ :

1.  $y_6 = -2 \cos \theta$
2.  $y_7 = -\cos \theta$
3.  $y_8 = \cos \theta$
4.  $y_9 = 2 \cos \theta$

Use your results to deduce the effect of  $a$ .

**The effect of  $q$** 

The effect of  $q$  is called a vertical shift because the whole cosine graph shifts up or down by  $q$  units.

- For  $q > 0$ , the graph is shifted vertically upwards by  $q$  units.
- For  $q < 0$ , the graph is shifted vertically downwards by  $q$  units.

**The effect of  $a$** 

The value of  $a$  affects the amplitude of the graph; the height of the peaks and the depth of the troughs.

- For  $a > 0$ , there is a vertical stretch and the amplitude increases.  
For  $0 < a < 1$ , the amplitude decreases.
- For  $a < 0$ , there is a reflection about the  $x$ -axis.  
For  $-1 < a < 0$ , there is a reflection about the  $x$ -axis and the amplitude decreases.  
For  $a < -1$ , there is a reflection about the  $x$ -axis and the amplitude increases.

Note that amplitude is always positive.

Effect of $a$	
$a > 1$ : vertical stretch, amplitude increases	
$a = 1$ : basic cosine graph	
$0 < a < 1$ : amplitude decreases	
$-1 < a < 0$ : reflection about $x$ -axis, amplitude decreases	
$a < -1$ : reflection about $x$ -axis, amplitude increases	

Table 6.8: The effect of  $a$  on a cosine graph.

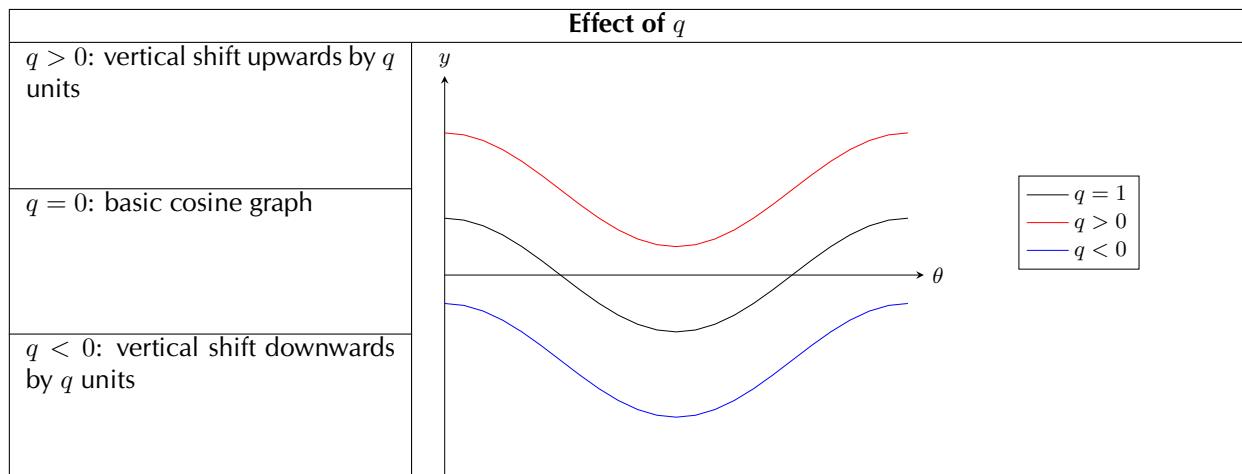


Table 6.9: The effect of  $q$  on a cosine graph.

## Discovering the characteristics

EMA5B

### Domain and range

For  $f(\theta) = a \cos \theta + q$ , the domain is  $[0^\circ; 360^\circ]$

It is easy to see that the range of  $f(\theta)$  will be the same as the range of  $a \sin \theta + q$ . This is because the maximum and minimum values of  $a \cos \theta + q$  will be the same as the maximum and minimum values of  $a \sin \theta + q$ .

For  $a > 0$  the range of  $f(\theta) = a \cos \theta + q$  is  $\{f(\theta) : f(\theta) \in [-a + q; a + q]\}$

For  $a < 0$  the range of  $f(\theta) = a \cos \theta + q$  is  $\{f(\theta) : f(\theta) \in [a + q; -a + q]\}$

### Period

The period of  $y = a \cos \theta + q$  is  $360^\circ$ . This means that one cosine wave is completed in  $360^\circ$ .

### Intercepts

The  $y$ -intercept of  $f(\theta) = a \cos \theta + q$  is calculated in the same way as for sine.

$$\begin{aligned}
 y &= f(0^\circ) \\
 &= a \cos 0^\circ + q \\
 &= a(1) + q \\
 &= a + q
 \end{aligned}$$

This gives the point  $(0^\circ; a + q)$ .

### Worked example 19: Sketching a cosine graph

#### QUESTION

Sketch the graph of  $f(\theta) = 2 \cos \theta + 3$  for  $\theta \in [0^\circ; 360^\circ]$ .

#### SOLUTION

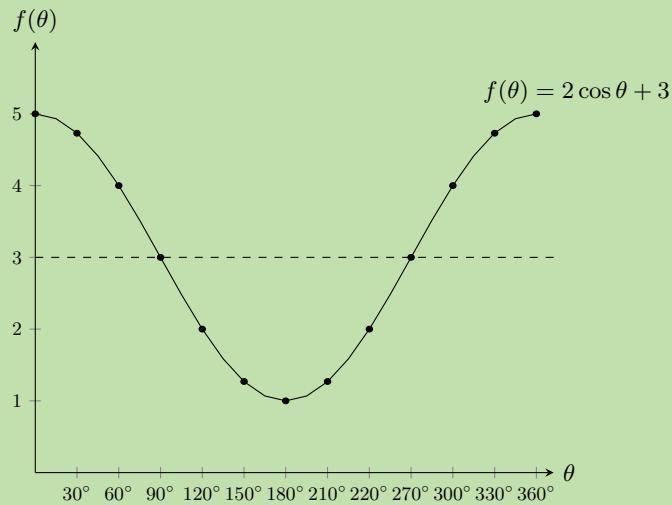
##### Step 1: Examine the standard form of the equation

From the equation we see that  $a > 1$  so the graph is stretched vertically. We also see that  $q > 0$  so the graph is shifted vertically upwards by 3 units.

##### Step 2: Substitute values for $\theta$

$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$	$360^\circ$
$f(\theta)$	5	4,73	4	3	2	1,27	1	1,27	2	3	4	4,73	5

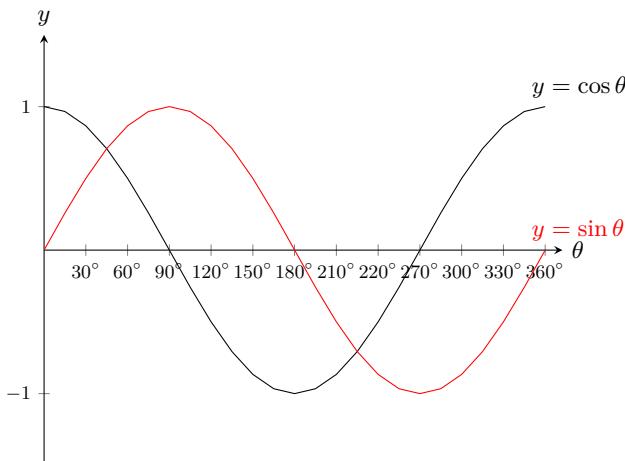
##### Step 3: Plot the points and join with a smooth curve



Domain:  $[0^\circ; 360^\circ]$  Range:  $[1; 5]$

$x$ -intercepts: none  $y$ -intercept:  $(0^\circ; 5)$

Maximum turning points:  $(0^\circ; 5), (360^\circ; 5)$  Minimum turning point:  $(180^\circ; 1)$



Notice that the two graphs look very similar. Both waves move up and down along the  $x$ -axis. The distances between the peaks for each graph is the same. The height of the peaks and the depths of the troughs are also the same.

If you shift the whole cosine graph to the right by  $90^\circ$  it will overlap perfectly with the sine graph. If you shift the sine graph  $90^\circ$  to the left it would overlap perfectly with the cosine graph. This means that:

$$\begin{aligned} \sin \theta &= \cos(\theta - 90^\circ) && \text{(shift the cosine graph to the right)} \\ \cos \theta &= \sin(\theta + 90^\circ) && \text{(shift the sine graph to the left)} \end{aligned}$$

### Worked example 20: Plotting a tangent graph

#### QUESTION

$$y = f(\theta) = \tan \theta \quad [0^\circ \leq \theta \leq 360^\circ]$$

Use your calculator to complete the following table.

Choose an appropriate scale and plot the values with  $\theta$  on the  $x$ -axis and  $\tan \theta$  on the  $y$ -axis. Round your answers to 2 decimal places.

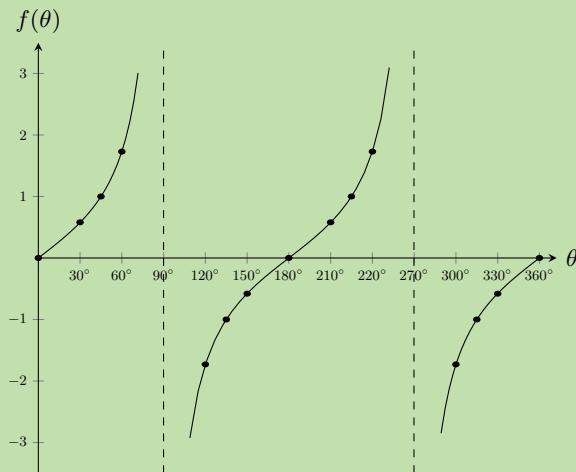
$\theta$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$
$\tan \theta$									
$\theta$	$210^\circ$	$235^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$315^\circ$	$330^\circ$	$360^\circ$	
$\tan \theta$									

#### SOLUTION

##### Step 1: Substitute values for $\theta$

$\theta$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$
$\tan \theta$	0	0,58	1	1,73	undefined	-1,73	-1	-0,58	0
$\theta$	$210^\circ$	$235^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$315^\circ$	$330^\circ$	$360^\circ$	
$\tan \theta$	0,58	1	1,73	undefined	-1,73	-1	-0,58	0	

## Step 2: Plot the points and join with a smooth curve



There is an easy way to visualise the tangent graph. Consider our definitions of  $\sin \theta$  and  $\cos \theta$  for right-angled triangles:

$$\begin{aligned}\frac{\sin \theta}{\cos \theta} &= \frac{\left(\frac{\text{opposite}}{\text{hypotenuse}}\right)}{\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)} \\ &= \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}} \\ &= \frac{\text{opposite}}{\text{adjacent}} \\ &= \tan \theta\end{aligned}$$

So for any value of  $\theta$ :  $\tan \theta = \frac{\sin \theta}{\cos \theta}$

So we know that for values of  $\theta$  for which  $\sin \theta = 0$ , we must also have  $\tan \theta = 0$ . Also if  $\cos \theta = 0$  the value of  $\tan \theta$  is undefined as we cannot divide by 0. The dashed vertical lines are at the values of  $\theta$  where  $\tan \theta$  is not defined and are called the asymptotes.

Asymptotes: the lines  $\theta = 90^\circ$  and  $\theta = 270^\circ$

Period:  $180^\circ$

Domain:  $\{\theta : 0^\circ \leq \theta \leq 360^\circ, \theta \neq 90^\circ, 270^\circ\}$  Range:  $\{f(\theta) : f(\theta) \in \mathbb{R}\}$

$x$ -intercepts:  $(0^\circ; 0)$ ,  $(180^\circ; 0)$ ,  $(360^\circ; 0)$   $y$ -intercept:  $(0^\circ; 0)$

## Functions of the form $y = a \tan \theta + q$

EMA5G

### Investigation: The effects of $a$ and $q$ on a tangent graph

On the same set of axes, plot the following graphs for  $0^\circ \leq \theta \leq 360^\circ$ :

1.  $y_1 = \tan \theta - 2$
2.  $y_2 = \tan \theta - 1$
3.  $y_3 = \tan \theta$
4.  $y_4 = \tan \theta + 1$
5.  $y_5 = \tan \theta + 2$

Use your results to deduce the effect of  $q$ .

On the same set of axes, plot the following graphs for  $0^\circ \leq \theta \leq 360^\circ$ :

1.  $y_6 = -2 \tan \theta$
2.  $y_7 = -\tan \theta$
3.  $y_8 = \tan \theta$
4.  $y_9 = 2 \tan \theta$

Use your results to deduce the effect of  $a$ .

### The effect of $q$

The effect of  $q$  is called a vertical shift because the whole tangent graph shifts up or down by  $q$  units.

- For  $q > 0$ , the graph is shifted vertically upwards by  $q$  units.
- For  $q < 0$ , the graph is shifted vertically downwards by  $q$  units.

### The effect of $a$

The value of  $a$  affects the steepness of each of the branches of the graph. The greater the value of  $a$ , the quicker the branches of the graph approach the asymptotes.

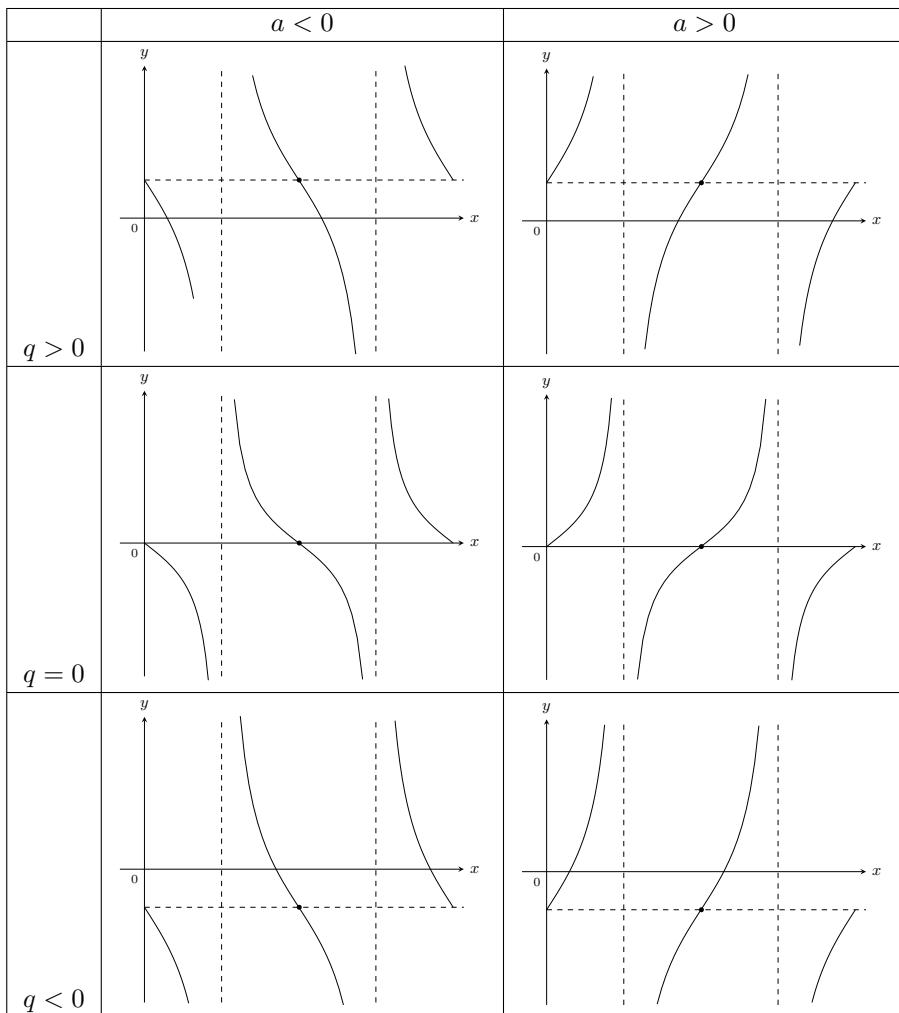


Table 6.10: The effect of  $a$  and  $q$  on a tangent graph.

## Domain and range

From the graph we see that  $\tan \theta$  is undefined at  $\theta = 90^\circ$  and  $\theta = 270^\circ$ .

Therefore the domain is  $\{\theta : 0^\circ \leq \theta \leq 360^\circ, \theta \neq 90^\circ; 270^\circ\}$ .

The range is  $\{f(\theta) : f(\theta) \in \mathbb{R}\}$ .

## Period

The period of  $y = a \tan \theta + q$  is  $180^\circ$ . This means that one tangent cycle is completed in  $180^\circ$ .

## Intercepts

The  $y$ -intercept of  $f(\theta) = a \tan \theta + q$  is simply the value of  $f(\theta)$  at  $\theta = 0^\circ$ .

$$\begin{aligned} y &= f(0^\circ) \\ &= a \tan 0^\circ + q \\ &= a(0) + q \\ &= q \end{aligned}$$

This gives the point  $(0^\circ; q)$ .

## Asymptotes

The graph has asymptotes at  $\theta = 90^\circ$  and  $\theta = 270^\circ$ .

### Worked example 21: Sketching a tangent graph

#### QUESTION

Sketch the graph of  $y = 2 \tan \theta + 1$  for  $\theta \in [0^\circ; 360^\circ]$ .

#### SOLUTION

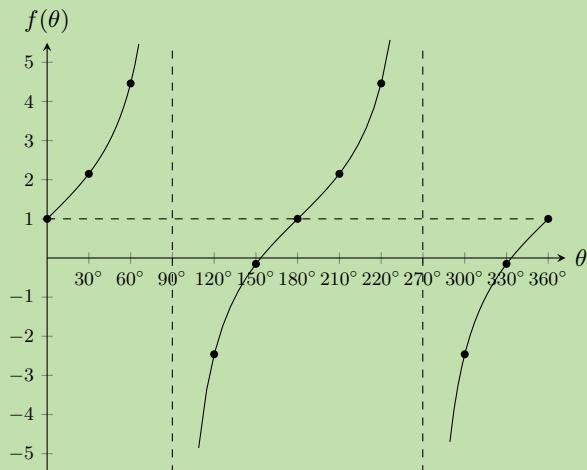
##### Step 1: Examine the standard form of the equation

We see that  $a > 1$  so the branches of the curve will be steeper. We also see that  $q > 0$  so the graph is shifted vertically upwards by 1 unit.

##### Step 2: Substitute values for $\theta$

$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$	$360^\circ$
$y$	1	2,15	4,46	—	-2,46	-0,15	1	2,15	4,46	—	-2,46	-0,15	1

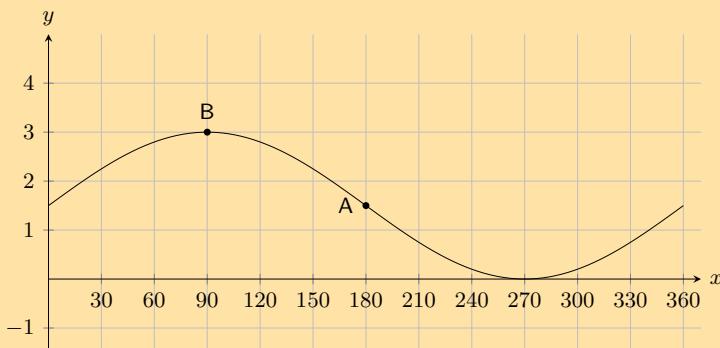
**Step 3: Plot the points and join with a smooth curve**



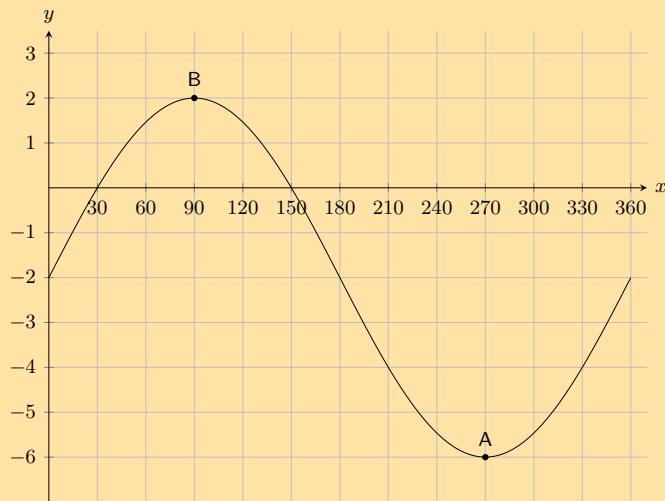
Domain:  $\{\theta : 0^\circ \leq \theta \leq 360^\circ, \theta \neq 90^\circ; 270^\circ\}$  Range:  $\{f(\theta) : f(\theta) \in \mathbb{R}\}$

**Exercise 6 – 6:**

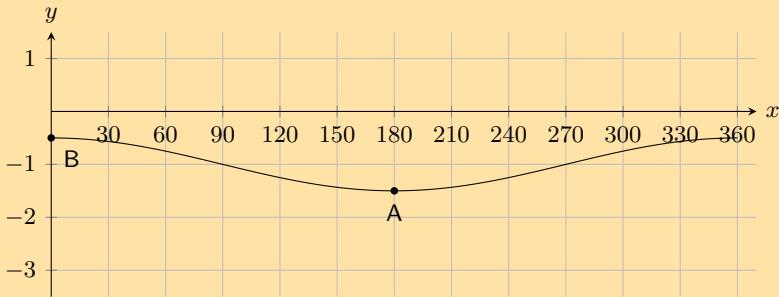
1. Show the following graph of the following form:  $y = a \sin \theta + q$  where **Point A** is at  $(180^\circ; 1.5)$ , and **Point B** is at  $(90^\circ; 3)$ , find the values of  $a$  and  $q$ .



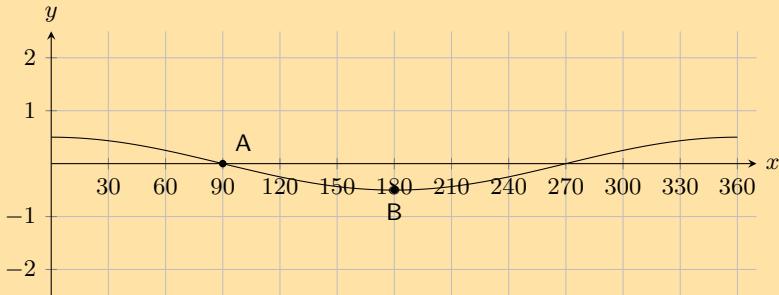
2. Show the following graph of the following form:  $y = a \sin \theta + q$  where **Point A** is at  $(270^\circ; -6)$ , and **Point B** is at  $(90^\circ; 2)$ , determine the values of  $a$  and  $q$ .



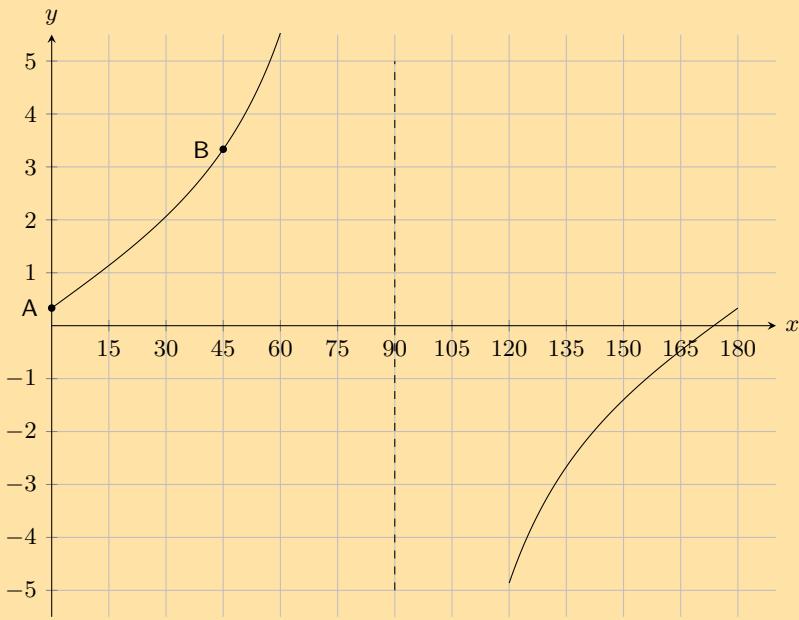
3. The graph below shows a trigonometric equation of the following form:  $y = a \cos \theta + q$ . Two points are shown on the graph: **Point A** at  $(180^\circ; -1,5)$ , and **Point B**:  $(0^\circ; -0,5)$ . Calculate the values of  $a$  (the amplitude of the graph) and  $q$  (the vertical shift of the graph).



4. The graph below shows a trigonometric equation of the following form:  $y = a \cos \theta + q$ . Two points are shown on the graph: **Point A** at  $(90^\circ; 0,0)$ , and **Point B**:  $(180^\circ; -0,5)$ . Calculate the values of  $a$  (the amplitude of the graph) and  $q$  (the vertical shift of the graph).

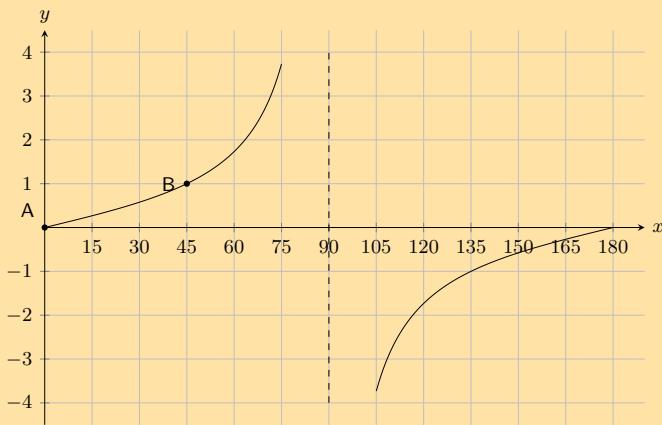


5. On the graph below you see a tangent curve of the following form:  $y = a \tan \theta + q$ . Two points are labelled on the curve: **Point A** is at  $(0^\circ; \frac{1}{3})$ , and **Point B** is at  $(45^\circ; \frac{10}{3})$ . Calculate, or otherwise determine, the values of  $a$  and  $q$ .

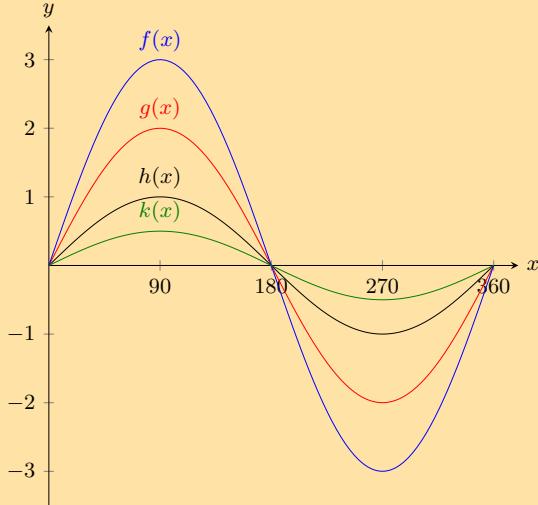


6. The graph below shows a tangent curve with an equation of the form  $y = a \tan \theta + q$ . Two points are labelled on the curve: **Point A** is at  $(0^\circ; 0)$ , and **Point B** is at  $(45^\circ; 1)$ .

Find  $a$  and  $q$ .

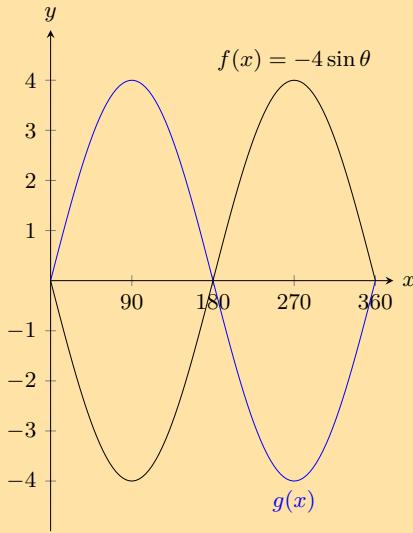


7. Given the following graph, identify a function that matches each of the following equations:



- a)  $y = \sin \theta$       b)  $y = \frac{1}{2} \sin \theta$       c)  $y = 3 \sin \theta$       d)  $y = 2 \sin \theta$

8. The graph below shows functions  $f(x)$  and  $g(x)$



What is the equation for  $g(x)$ ?

9. With the assistance of the table below sketch the three functions on the same set of axes.

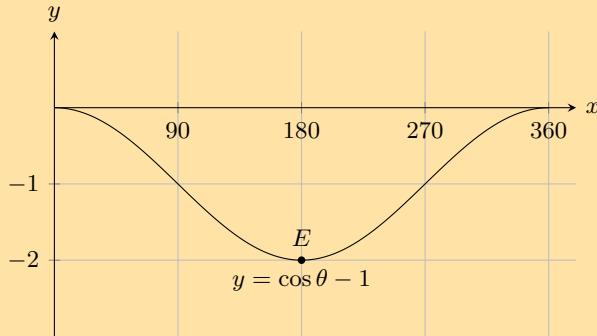
$\theta$	$0^\circ$	$45^\circ$	$90^\circ$	$135^\circ$	$180^\circ$	$225^\circ$	$270^\circ$	$315^\circ$	$360^\circ$
$\tan \theta$	0	1	undefined	-1	0	1	undefined	-1	0
$3 \tan \theta$	0	3	undefined	-3	0	3	undefined	-3	0
$\frac{1}{2} \tan \theta$	0	$\frac{1}{2}$	undefined	$-\frac{1}{2}$	0	$\frac{1}{2}$	undefined	$-\frac{1}{2}$	0

10. With the assistance of the table below sketch the three functions on the same set of axes.

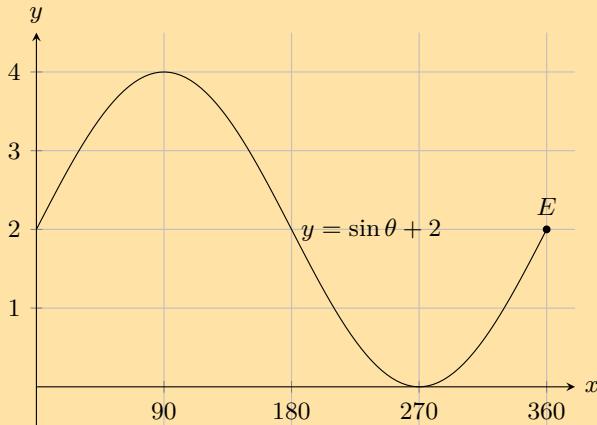
$\theta$	$0^\circ$	$90^\circ$	$180^\circ$	$270^\circ$	$360^\circ$
$\cos \theta - 2$	-1	-2	-3	-2	-1
$\cos \theta + 4$	5	4	2	4	5
$\cos \theta + 2$	3	2	1	2	3

11. State the coordinates at  $E$  and the range of the function.

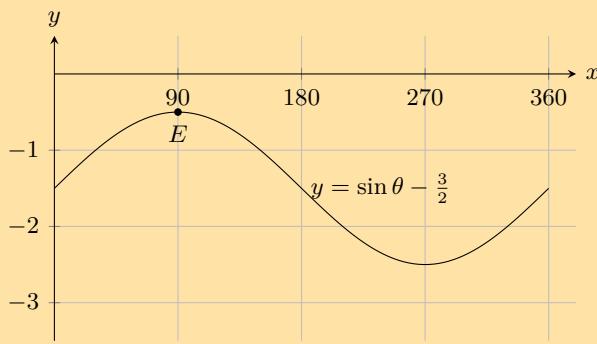
a)



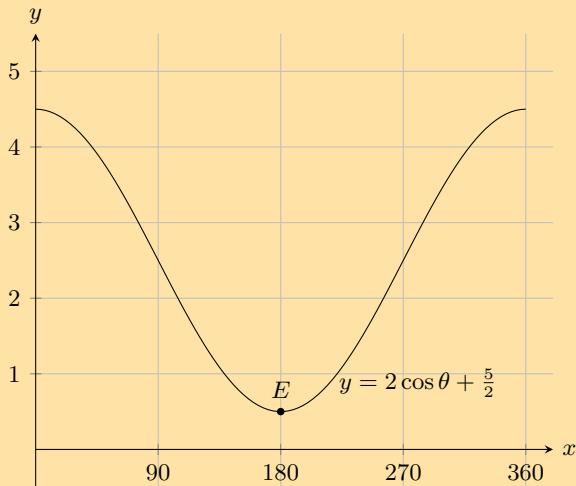
b)



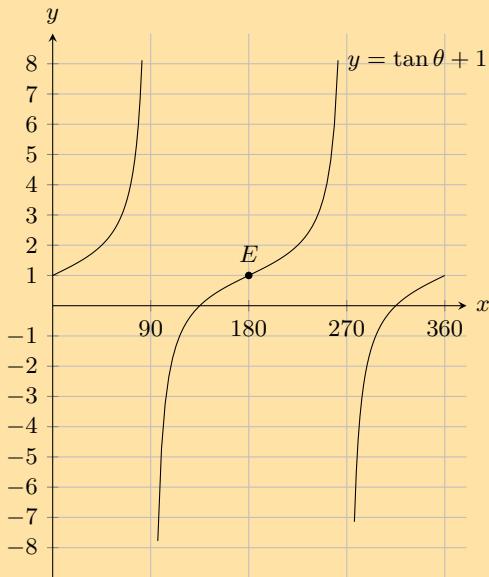
c)



d)



12. State the coordinates at E and the domain and range of the function in the interval shown.

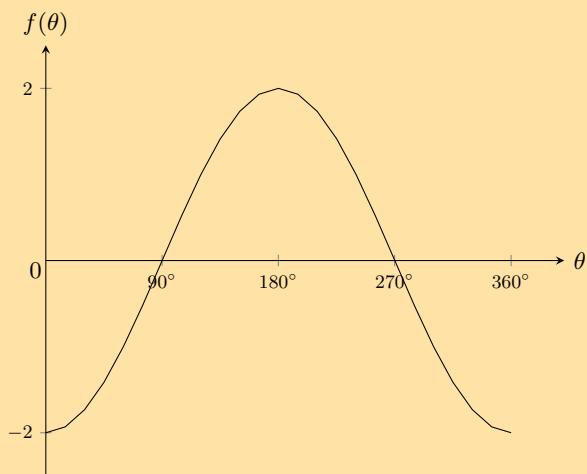


13. Using your knowledge of the effects of  $a$  and  $q$ , sketch each of the following graphs, without using a table of values, for  $\theta \in [0^\circ; 360^\circ]$

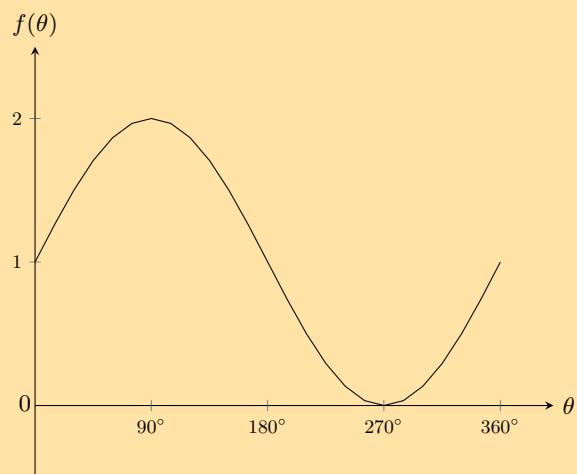
- a)  $y = 2 \sin \theta$
- b)  $y = -4 \cos \theta$
- c)  $y = -2 \cos \theta + 1$
- d)  $y = \sin \theta - 3$
- e)  $y = \tan \theta - 2$
- f)  $y = 2 \cos \theta - 1$

14. Give the equations for each of the following graphs:

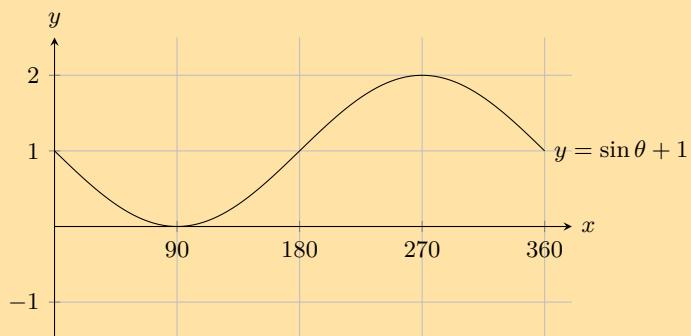
a)



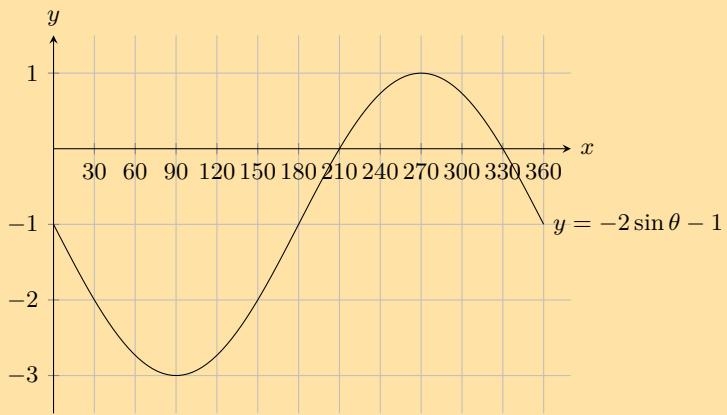
b)



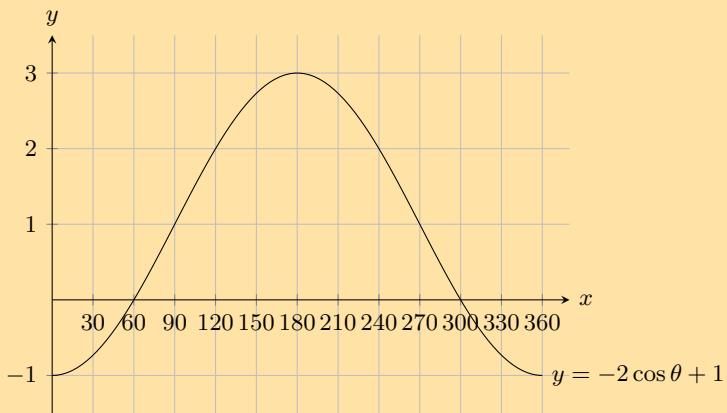
15. For which values of  $\theta$  is the function increasing, in the interval shown?



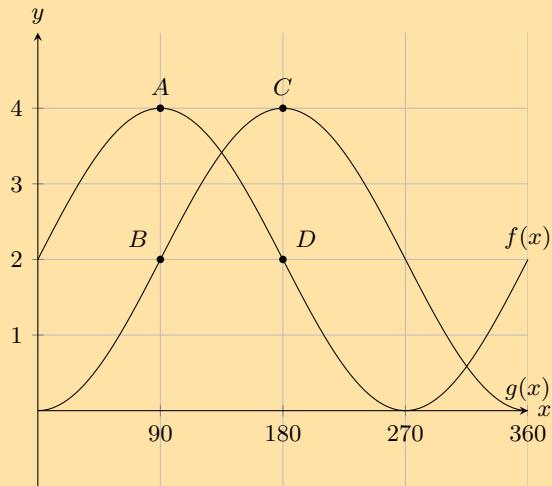
16. For which values of  $\theta$  is the function negative, in the interval shown?



17. For which values of  $\theta$  is the function positive, in the interval shown?

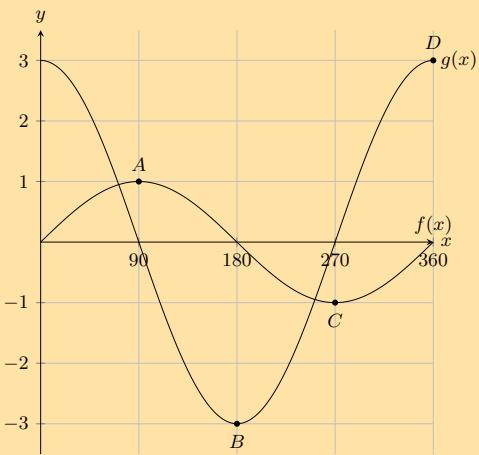


18. Given the following graph.

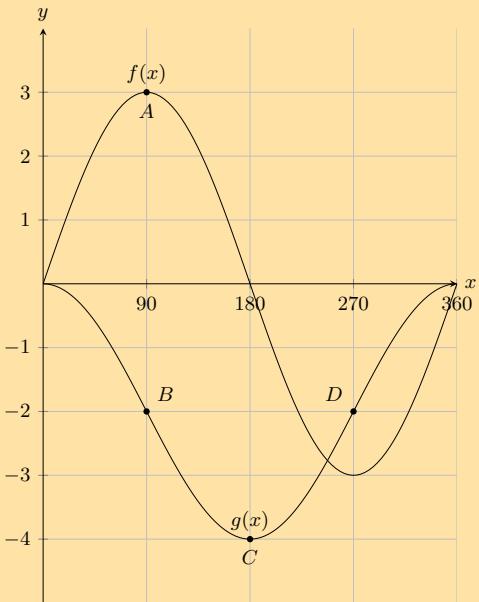


- State the coordinates at  $A$ ,  $B$ ,  $C$  and  $D$ .
- How many times in this interval does  $f(x)$  intersect  $g(x)$ .
- What is the amplitude of  $f(x)$ .
- Evaluate:  $f(360^\circ) - g(360^\circ)$ .

19. Given the following graph.



- a) State the coordinates at  $A$ ,  $B$ ,  $C$  and  $D$ .  
 b) How many times in this interval does  $f(x)$  intersect  $g(x)$ .  
 c) What is the amplitude of  $g(x)$ .  
 d) Evaluate:  $f(90^\circ) - g(90^\circ)$ .  
 20. Given the following graph:



- a) State the coordinates at  $A$ ,  $B$ ,  $C$  and  $D$ .  
 b) How many times in this interval does  $f(x)$  intersect  $g(x)$ .  
 c) What is the amplitude of  $g(x)$ .  
 d) Evaluate:  $f(270^\circ) - g(270^\circ)$ .

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. <a href="#">2FZ6</a>	2. <a href="#">2FZ7</a>	3. <a href="#">2FZ8</a>	4. <a href="#">2FZ9</a>	5. <a href="#">2FZB</a>	6. <a href="#">2FZC</a>	7. <a href="#">2FZD</a>	8. <a href="#">2FZF</a>
9. <a href="#">2FZG</a>	10. <a href="#">2FZH</a>	11a. <a href="#">2FZJ</a>	11b. <a href="#">2FZK</a>	11c. <a href="#">2FZM</a>	11d. <a href="#">2FZN</a>	12. <a href="#">2FZP</a>	13a. <a href="#">2FZQ</a>
13b. <a href="#">2FZR</a>	13c. <a href="#">2FZS</a>	13d. <a href="#">2FZT</a>	13e. <a href="#">2FZV</a>	13f. <a href="#">2FZW</a>	14a. <a href="#">2FZX</a>	14b. <a href="#">2FZY</a>	15. <a href="#">2FZZ</a>
16. <a href="#">2G22</a>	17. <a href="#">2G23</a>	18. <a href="#">2G24</a>	19. <a href="#">2G25</a>	20. <a href="#">2G26</a>			



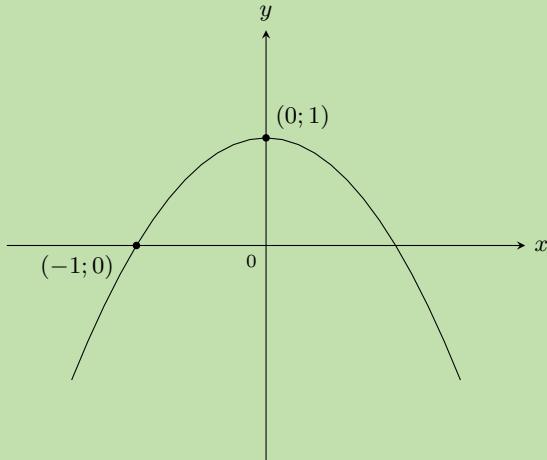
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**Worked example 22: Determining the equation of a parabola****QUESTION**

Use the sketch below to determine the values of  $a$  and  $q$  for the parabola of the form  $y = ax^2 + q$ .

**SOLUTION****Step 1: Examine the sketch**

From the sketch we see that the shape of the graph is a “frown”, therefore  $a < 0$ . We also see that the graph has been shifted vertically upwards, therefore  $q > 0$ .

**Step 2: Determine  $q$  using the  $y$ -intercept**

The  $y$ -intercept is the point  $(0; 1)$ .

$$\begin{aligned}y &= ax^2 + q \\1 &= a(0)^2 + q \\\therefore q &= 1\end{aligned}$$

**Step 3: Use the other given point to determine  $a$** 

Substitute point  $(-1; 0)$  into the equation:

$$\begin{aligned}y &= ax^2 + q \\0 &= a(-1)^2 + 1 \\\therefore a &= -1\end{aligned}$$

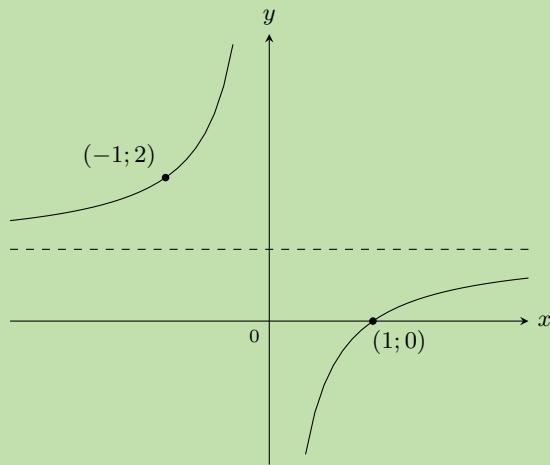
**Step 4: Write the final answer**

$a = -1$  and  $q = 1$ , so the equation of the parabola is  $y = -x^2 + 1$ .

## Worked example 23: Determining the equation of a hyperbola

### QUESTION

Use the sketch below to determine the values of  $a$  and  $q$  for the hyperbola of the form  $y = \frac{a}{x} + q$ .



### SOLUTION

#### Step 1: Examine the sketch

The two curves of the hyperbola lie in the second and fourth quadrant, therefore  $a < 0$ . We also see that the graph has been shifted vertically upwards, therefore  $q > 0$ .

#### Step 2: Substitute the given points into the equation and solve

Substitute the point  $(-1; 2)$ :

$$\begin{aligned}y &= \frac{a}{x} + q \\2 &= \frac{a}{-1} + q \\2 &= -a + q\end{aligned}$$

Substitute the point  $(1; 0)$ :

$$\begin{aligned}y &= \frac{a}{x} + q \\0 &= \frac{a}{1} + q \\0 &= a + q \\\therefore a &= -q\end{aligned}$$

#### Step 3: Solve the equations simultaneously using substitution

$$\begin{aligned}2 &= -a + q \\&= q + q \\&= 2q \\\therefore q &= 1 \\\therefore a &= -q \\&= -1\end{aligned}$$

#### Step 4: Write the final answer

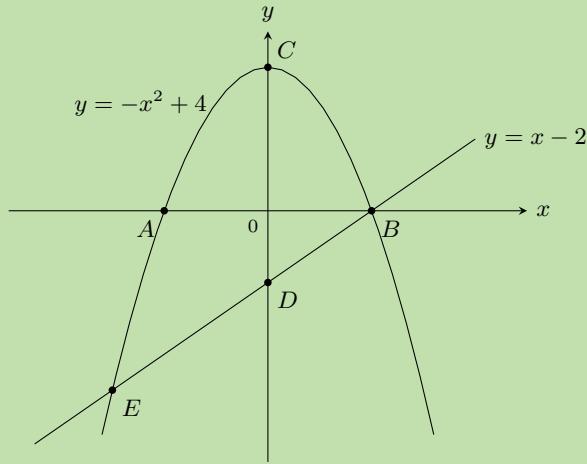
$a = -1$  and  $q = 1$ , therefore the equation of the hyperbola is  $y = \frac{-1}{x} + 1$ .

## Worked example 24: Interpreting graphs

### QUESTION

The graphs of  $y = -x^2 + 4$  and  $y = x - 2$  are given. Calculate the following:

1. coordinates of  $A, B, C, D$
2. coordinates of  $E$
3. distance  $CD$



### SOLUTION

#### Step 1: Calculate the intercepts

For the parabola, to calculate the  $y$ -intercept, let  $x = 0$ :

$$\begin{aligned}y &= -x^2 + 4 \\&= -0^2 + 4 \\&= 4\end{aligned}$$

This gives the point  $C(0; 4)$ .

To calculate the  $x$ -intercept, let  $y = 0$ :

$$\begin{aligned}y &= -x^2 + 4 \\0 &= -x^2 + 4 \\x^2 - 4 &= 0 \\(x + 2)(x - 2) &= 0 \\\therefore x &= \pm 2\end{aligned}$$

This gives the points  $A(-2; 0)$  and  $B(2; 0)$ .

For the straight line, to calculate the  $y$ -intercept, let  $x = 0$ :

$$\begin{aligned}y &= x - 2 \\&= 0 - 2 \\&= -2\end{aligned}$$

This gives the point  $D(0; -2)$ .

**Step 2: Calculate the point of intersection E**

At E the two graphs intersect so we can equate the two expressions:

$$\begin{aligned}x - 2 &= -x^2 + 4 \\ \therefore x^2 + x - 6 &= 0 \\ \therefore (x - 2)(x + 3) &= 0 \\ \therefore x = 2 \text{ or } -3\end{aligned}$$

At E,  $x = -3$ , therefore  $y = x - 2 = -3 - 2 = -5$ . This gives the point  $E(-3; -5)$ .

**Step 3: Calculate distance CD**

$$\begin{aligned}CD &= CO + OD \\ &= 4 + 2 \\ &= 6\end{aligned}$$

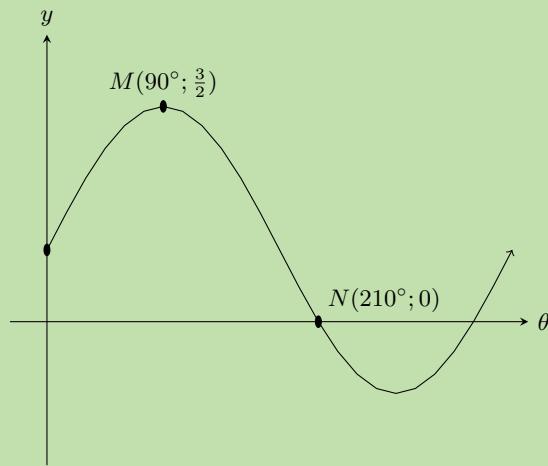
Distance CD is 6 units.

**Step 4: Write the final answer**

1. coordinates of  $A(-2; 0), B(2; 0), C(0; 4), D(0; -2)$
2. coordinates of  $E(-3; -5)$
3. distance  $CD = 6$  units

**Worked example 25: Interpreting trigonometric graphs****QUESTION**

Use the sketch to determine the equation of the trigonometric function  $f$  of the form  $y = af(\theta) + q$ .



## SOLUTION

### Step 1: Examine the sketch

From the sketch we see that the graph is a sine graph that has been shifted vertically upwards. The general form of the equation is  $y = a \sin \theta + q$ .

### Step 2: Substitute the given points into equation and solve

At  $N$ ,  $\theta = 210^\circ$  and  $y = 0$ :

$$\begin{aligned}y &= a \sin \theta + q \\0 &= a \sin 210^\circ + q \\&= a \left(-\frac{1}{2}\right) + q \\\therefore q &= \frac{a}{2}\end{aligned}$$

At  $M$ ,  $\theta = 90^\circ$  and  $y = \frac{3}{2}$ :

$$\begin{aligned}\frac{3}{2} &= a \sin 90^\circ + q \\&= a + q\end{aligned}$$

### Step 3: Solve the equations simultaneously using substitution

$$\begin{aligned}\frac{3}{2} &= a + q \\&= a + \frac{a}{2} \\3 &= 2a + a \\3a &= 3 \\\therefore a &= 1 \\\therefore q &= \frac{a}{2} \\&= \frac{1}{2}\end{aligned}$$

### Step 4: Write the final answer

$$y = \sin \theta + \frac{1}{2}$$

### Exercise 6 – 7:

1. Plot the following functions on the same set of axes and clearly label all the points at which the functions intersect.

a)  $y = x^2 + 1$  and  $y = 3^x$

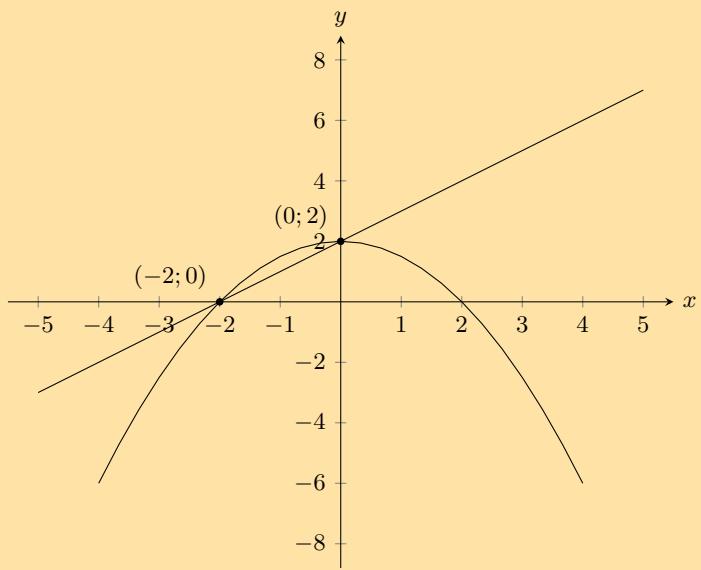
b)  $y = x$  and  $y = \frac{2}{x}$

c)  $y = x^2 + 3$  and  $y = 6$

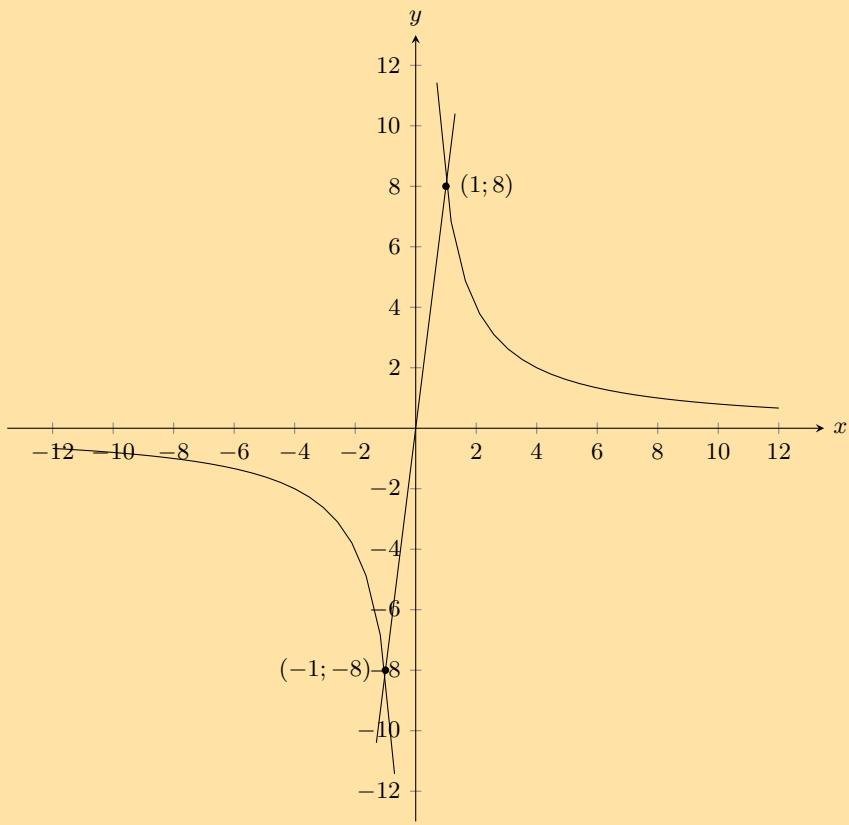
d)  $y = -x^2$  and  $y = \frac{8}{x}$

2. Determine the equations for the graphs given below.

a)



b)



3. Choose the correct answer:

- a) The range of  $y = 2 \sin \theta + 1$  is:  
i.  $1 \leq \theta \leq 2$       ii.  $-2 \leq \theta \leq 2$       iii.  $-1 \leq \theta \leq 3$       iv.  $-2 \leq \theta \leq 3$
- b) The range of  $y = 2 \cos \theta - 4$  is:  
i.  $-6 \leq \theta \leq 2$       ii.  $-4 \leq \theta \leq -2$       iii.  $-6 \leq \theta \leq 1$       iv.  $-6 \leq \theta \leq -2$

- c) The  $y$ -intercept of  $2^x + 1$  is:
- 3
  - 1
  - 2
  - 0
- d) Which of the following passes through  $(1; 7)$ ?
- $y = \frac{7}{x}$
  - $y = 2x + 3$
  - $y = \frac{4}{x}$
  - $y = x^2 + 1$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- 1a. [2G27](#) 1b. [2G28](#) 1c. [2G29](#) 1d. [2G2B](#) 2a. [2G2C](#) 2b. [2G2D](#)  
 3a. [2G2F](#) 3b. [2G2G](#) 3c. [2G2H](#) 3d. [2G2J](#)



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## 6.8 Chapter summary

EMA5K

► See presentation: [2G2K](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

- Characteristics of functions:
  - The given  $x$ -value is known as the independent variable, because its value can be chosen freely. The calculated  $y$ -value is known as the dependent variable, because its value depends on the  $x$ -value.
  - The domain of a function is the set of all  $x$ -values for which there exists at most one  $y$ -value according to that function. The range is the set of all  $y$ -values, which can be obtained using at least one  $x$ -value.
  - An asymptote is a straight line, which the graph of a function will approach, but never touch.
- Special functions and their properties:
  - Linear functions of the form  $y = ax + q$ .
  - Parabolic functions of the form  $y = ax^2 + q$ .
  - Hyperbolic functions of the form  $y = \frac{a}{x} + q$ .
  - Exponential functions of the form  $y = ab^x + q$ .
  - Trigonometric functions of the form  $y = a \sin \theta + q$        $y = a \cos \theta + q$        $y = a \tan \theta + q$

### End of chapter Exercise 6 – 8:

1. Complete the following tables and identify the function.

a)

$x$		2	3	4		6
$y$	3	6		12	15	

b)

$x$	1			4	5	6
$y$	-3	-2	-1		1	2

2. Plot the following points on a graph.

a)

$x$	1	2	3	4	5	6
$y$	1	2	3	4	5	6

b)

$x$	50	100	150	200	250	300
$y$	1	2	3	4	5	6

3. Create a table of values from the function given and then plot the function. Your table must have at least 5 ordered pairs.

a)  $x^2 - 4$

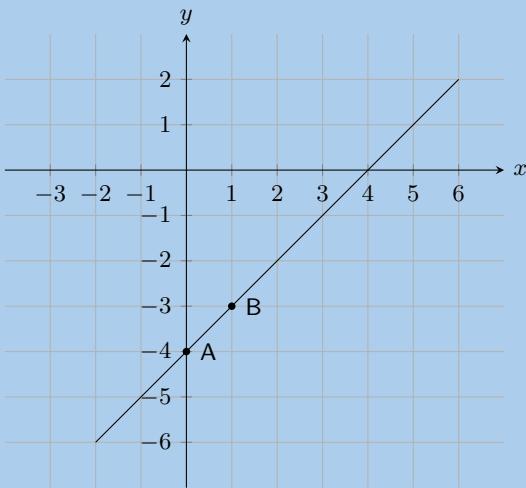
b)  $y = 4x - 1$

4. Determine the  $y$ -intercept and the  $x$ -intercepts of the function.

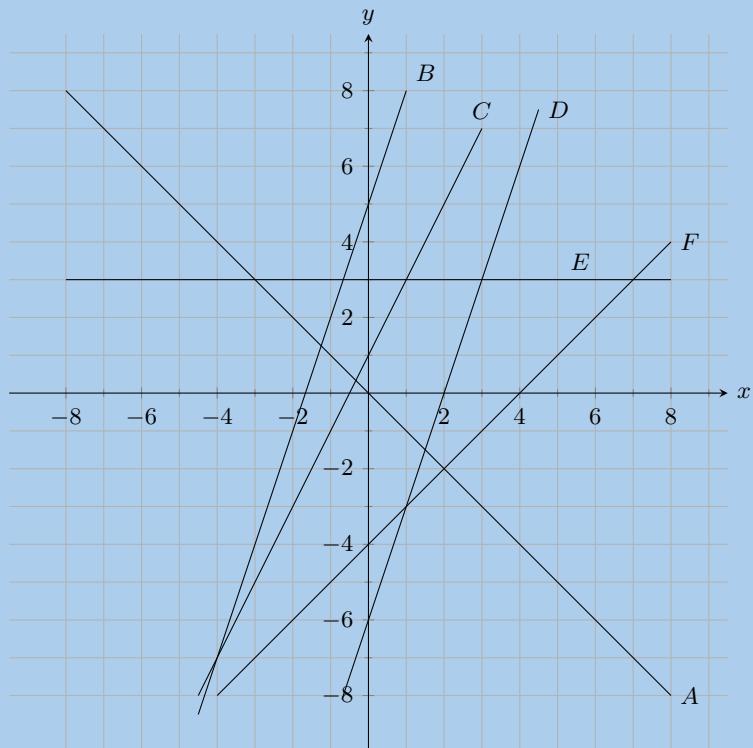
a)  $y = -3x - 5$

b)  $y = 2x + 4$

5. The graph below shows an equation, which has the form  $y = mx + c$ . Calculate or otherwise find the values of  $m$  (the gradient of the line) and  $c$  (the  $y$ -intercept of the line).



6. Look at the graphs below. Each graph is labelled with a letter. In the questions that follow, match any given equation with the label of a corresponding graph.



a)  $y = 3$

b)  $y = 3x + 5$

c)  $y = -x$

d)  $y = 2x + 1$

e)  $y = x - 4$

f)  $y = 3x - 6$

7. State whether the following is true or not

- The  $y$ -intercept of  $y + 5 = x$  is  $-5$ .
- The gradient of  $-y = x + 2$  is  $1$ .
- The gradient of  $-4y = 3$  is  $1$ .

8. Write the following in standard form:

- $2y - 5x = 6$
- $6y - 3x = 5x + 1$

9. Sketch the graphs of the following:

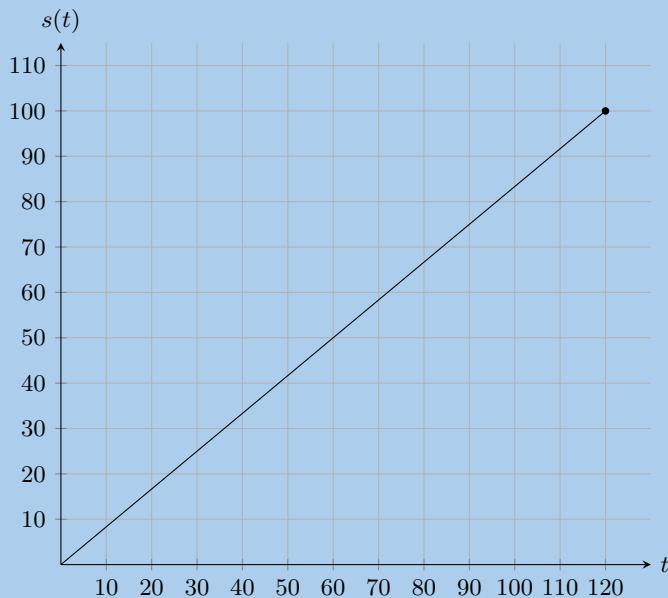
- $y = 2x + 4$
- $y - 3x = 0$
- $2y = 4 - x$

10. The function for how much water a tap dispenses is given by:  $V = 60t$ , where  $x$  and  $V$  are in seconds and mL respectively.

Use this information to answer the following:

- Evaluate  $V(2)$ .
- Evaluate  $V(10)$ .
- How long will it take to fill a 2 L bottle of water?
- How much water can the tap dispense in 4 s?

11. The graph below shows the distance travelled by a car over time, where  $s(t)$  is distance in km and  $t$  is time in minutes.

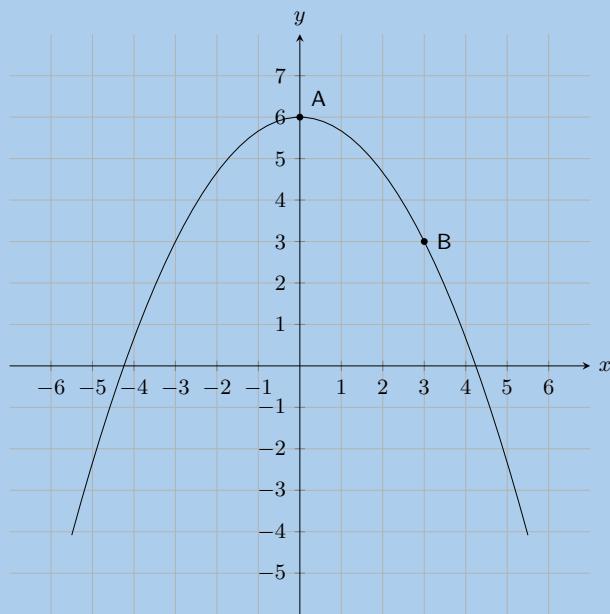


Use this information to answer the following:

- What distance did the car travel in an hour?
- What is the domain of the function?
- What is the range of the function? What does it represent?

12. On the graph here you see a function of the form:  $y = ax^2 + q$ .

Two points on the parabola are shown: **Point A**, the turning point of the parabola, at  $(0; 6)$ , and **Point B** is at  $(3; 3)$ . Calculate the values of  $a$  and  $q$ .

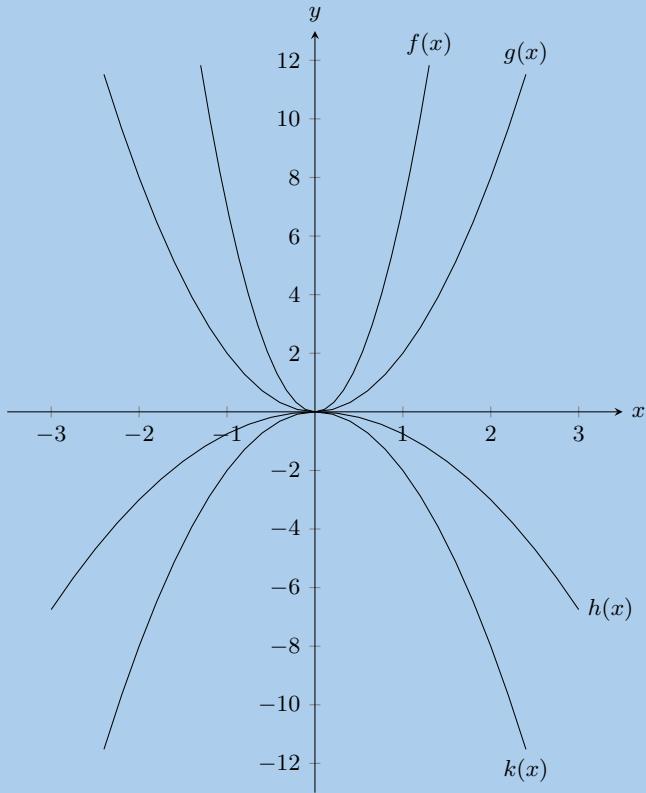


13. Given the following equation:

$$y = -5x^2 + 3$$

- a) Calculate the  $y$ -coordinate of the  $y$ -intercept.
- b) Now calculate the  $x$ -intercepts. Your answer must be correct to 2 decimal places.

14. Given the following graph, identify a function that matches each of the given equations:



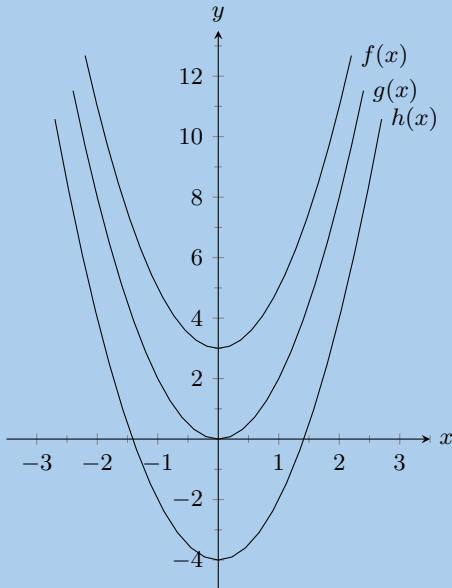
a)  $y = -2x^2$

b)  $y = 2x^2$

c)  $y = -0,75x^2$

d)  $y = 7x^2$

15. Given the following graph, identify a function that matches each of the given equations:



a)  $y = 2x^2$

b)  $y = 2x^2 + 3$

c)  $y = 2x^2 - 4$

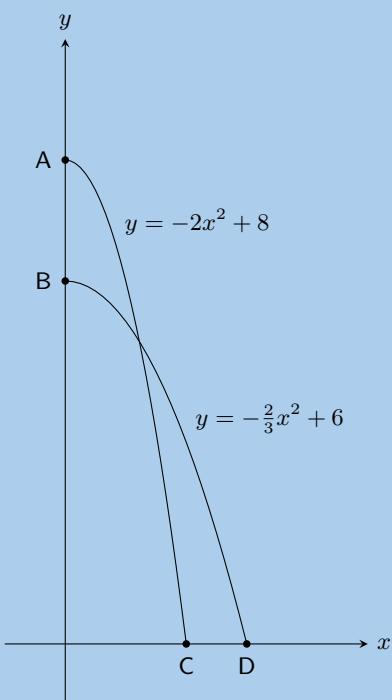
16. Sketch the following functions:

a)  $y = x^2 + 3$

b)  $y = \frac{1}{2}x^2 + 4$

c)  $y = 2x^2 - 4$

17. Sebastian and Lucas dive into a pool from different heights. Their midair paths can be described by the following quadratic equations:  $y = -2x^2 + 8$  for Sebastian and  $y = -\frac{2}{3}x^2 + 6$  for Lucas.



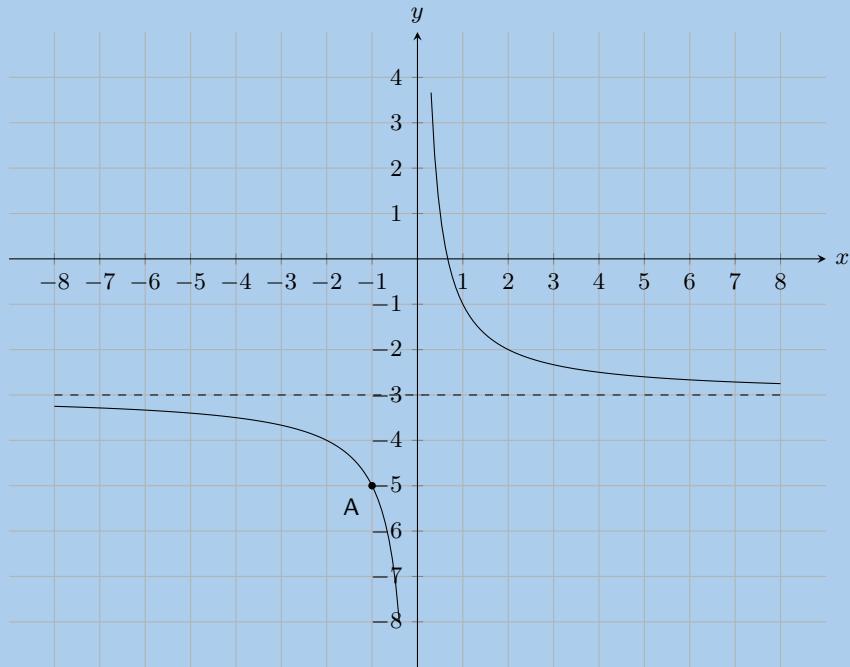
a) From what height did Sebastian dive?

b) From what height did Lucas dive?

c) How far from the pool wall did Lucas land?

d) How much closer to the pool wall did Sebastian land compared to Lucas?

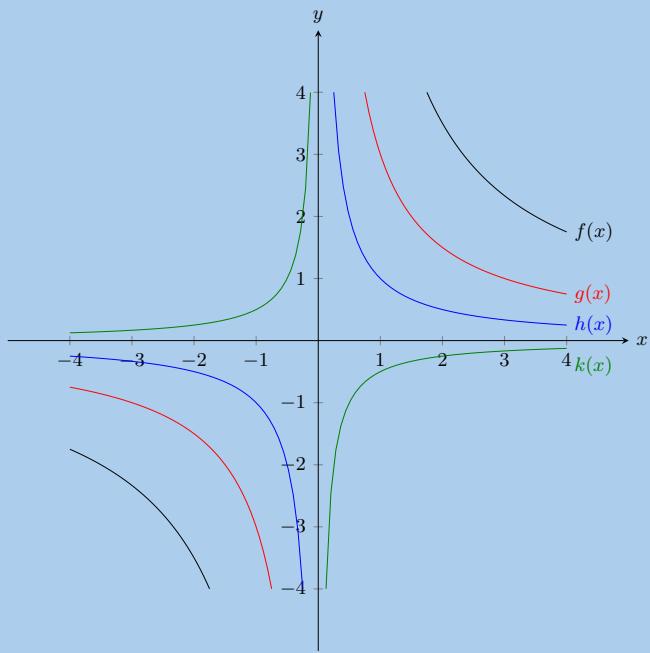
18. The following graph shows a hyperbolic equation of the form  $y = \frac{a}{x} + q$ . **Point A** is shown at  $(-1; -5)$ . Calculate the values of  $a$  and  $q$ .



19. Given the following equation:  $y = -\frac{3}{x} + 4$

- Determine the location of the  $y$ -intercept.
- Determine the location of the  $x$ -intercept.

20. Given the following graph, identify a function that matches each of the given equations:



a)  $y = -\frac{1}{2x}$

b)  $y = \frac{7}{x}$

c)  $y = \frac{3}{x}$

d)  $y = \frac{1}{x}$

21. Sketch the following functions and identify the asymptotes:

a)  $y = \frac{3}{x} + 4$

b)  $y = \frac{1}{x}$

c)  $y = \frac{2}{x} - 2$

22. Sketch the functions given and describe the transformation used to obtain the second function. Show all asymptotes.

a)  $y = \frac{2}{x}$  and  $\frac{2}{x} + 2$

b)  $y = \frac{2}{x}$  and  $\frac{1}{2x}$

c)  $y = \frac{3}{x}$  and  $y = \frac{3x+3}{x}$

d)  $y = \frac{3}{x}$  and  $y = -\frac{3}{x}$

23. Given the following equation:

$$y = -\frac{1}{2} \cdot (4)^x + 3$$

a) Calculate the  $y$ -intercept. Your answer must be correct to 2 decimal places.

b) Now calculate the  $x$ -intercept. Estimate your answer to one decimal place if necessary.

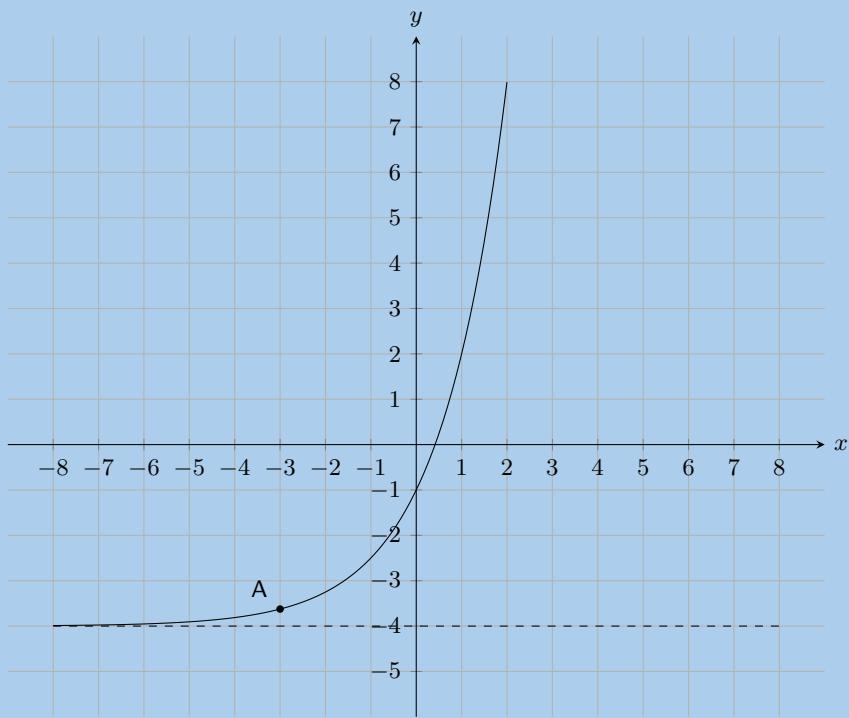
24. Sketch the following functions and identify the asymptotes:

a)  $y = 3^x + 2$

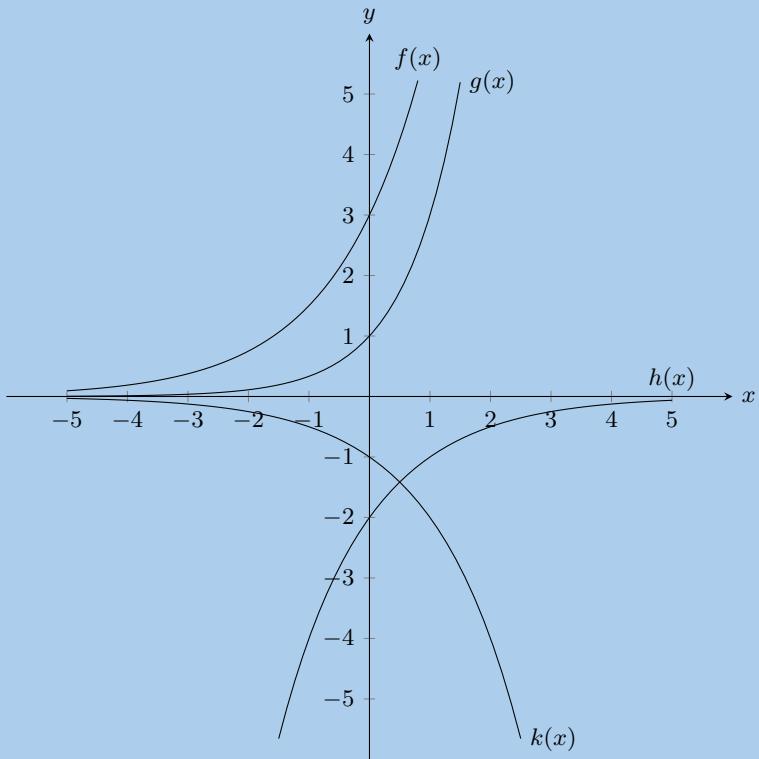
b)  $y = -4 \times 2^x$

c)  $y = \left(\frac{1}{3}\right)^x - 2$

25. The form of the curve graphed below is  $y = a \cdot 2^x + q$ . One point is given on the curve: **Point A** is at  $(-3; -3,625)$ . Find the values of  $a$  and  $q$ , correct to the nearest integer.



26. Given the following graph, identify a function that matches each of the given equations



a)  $y = -2 \left(\frac{1}{2}\right)^x$       b)  $y = 3 \cdot 2^x$       c)  $y = -2^x$       d)  $y = 3^x$

27. Use the functions  $f(x) = 3 - x$ ,  $g(x) = 2x^2 - 4$ ;  $h(x) = 3^x - 4$ ;  $k(x) = \frac{3}{2x} - 1$ , to find the value of the following:

a)  $f(7)$       b)  $g(1)$       c)  $h(-4)$       d)  $k(5)$   
 e)  $f(-1) + h(-3)$       f)  $h(g(-2))$       g)  $k(f(6))$

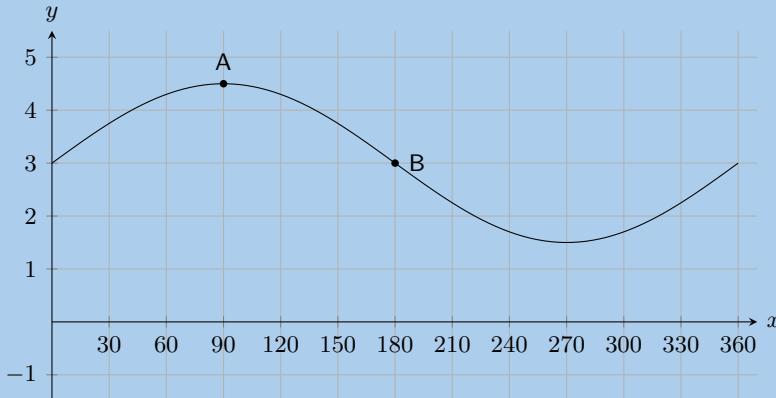
28. Determine whether the following statements are true or false. If the statement is false, give reasons why.

- a) The given or chosen  $y$ -value is known as the independent variable.
- b) A graph is said to be continuous if there are breaks in the graph.
- c) Functions of the form  $y = ax + q$  are straight lines.
- d) Functions of the form  $y = \frac{a}{x} + q$  are exponential functions.
- e) An asymptote is a straight line which a graph will intersect at least once.
- f) Given a function of the form  $y = ax + q$ , to find the  $y$ -intercept let  $x = 0$  and solve for  $y$ .

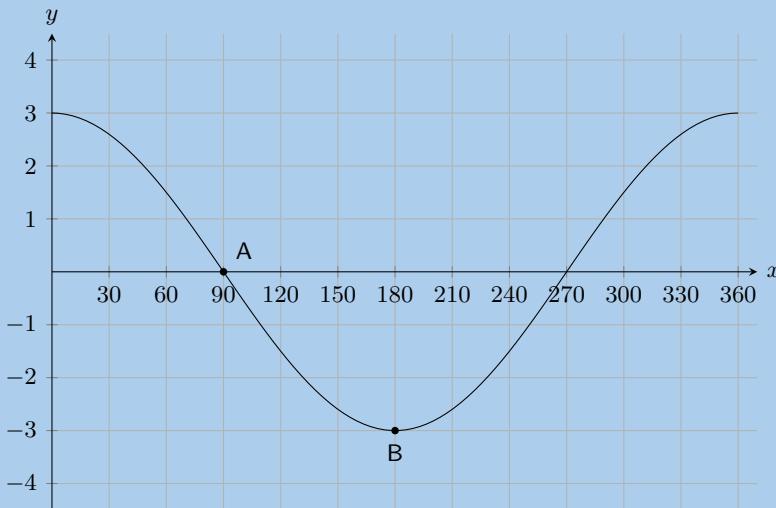
29. Given the functions  $f(x) = 2x^2 - 6$  and  $g(x) = -2x + 6$ .

- a) Draw  $f$  and  $g$  on the same set of axes.
- b) Calculate the points of intersection of  $f$  and  $g$ .
- c) Use your graphs and the points of intersection to solve for  $x$  when:
  - i.  $f(x) > 0$
  - ii.  $g(x) < 0$
  - iii.  $f(x) \leq g(x)$
- d) Give the equation of the reflection of  $f$  in the  $x$ -axis.

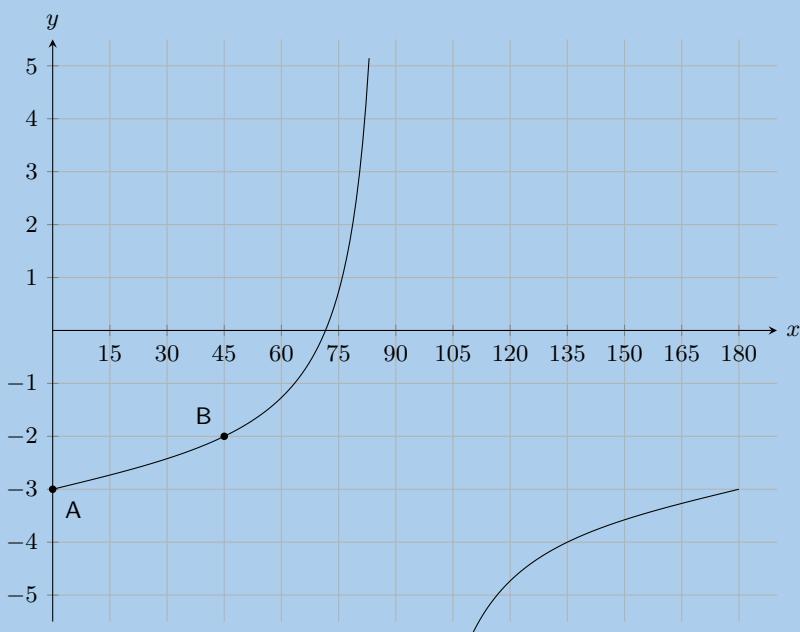
30. After a ball is dropped, the rebound height of each bounce decreases. The equation  $y = 5(0,8)^x$  shows the relationship between the number of bounces  $x$  and the height of the bounce  $y$  for a certain ball. What is the approximate height of the fifth bounce of this ball to the nearest tenth of a unit?
31. Mark had 15 coins in R 5 and R 2 pieces. He had 3 more R 2 coins than R 5 coins. He wrote a system of equations to represent this situation, letting  $x$  represent the number of R 5 coins and  $y$  represent the number of R 2 coins. Then he solved the system by graphing.
- Write down the system of equations.
  - Draw their graphs on the same set of axes.
  - Use your sketch to determine how many R 5 and R 2 pieces Mark had.
32. Shown the following graph of the following form:  $y = a \sin \theta + q$  where **Point A** is at  $(90^\circ; 4,5)$ , and **Point B** is at  $(180^\circ; 3)$ , determine the values of  $a$  and  $q$ .



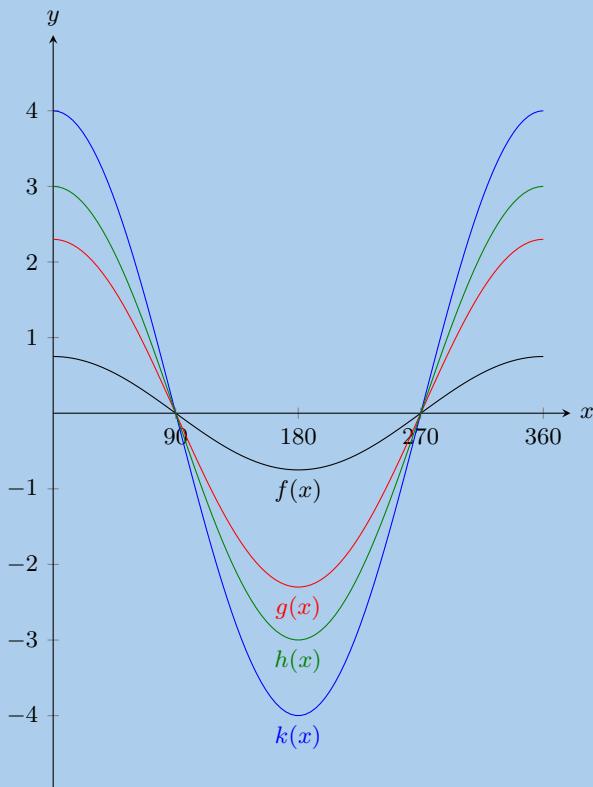
33. The graph below shows a trigonometric equation of the following form:  $y = a \cos \theta + q$ . Two points are shown on the graph: **Point A** at  $(90^\circ; 0)$ , and **Point B**:  $(180^\circ; -3)$ . Calculate the values of  $a$  (the amplitude of the graph) and  $q$  (the vertical shift of the graph).



34. On the graph below you see a tangent curve of the following form:  $y = a \tan \theta + q$ . Two points are labelled on the curve: **Point A** is at  $(0^\circ; -3)$ , and **Point B** is at  $(45^\circ; -2)$ . Calculate, or otherwise determine, the values of  $a$  and  $q$ .

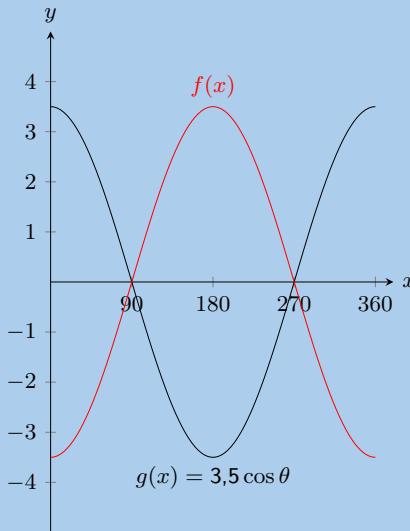


35. Given the following graph, identify a function that matches each of the given equations:



- a)  $y = 2,3 \cos \theta$
- b)  $y = 0,75 \cos \theta$
- c)  $y = 4 \cos \theta$
- d)  $y = 3 \cos \theta$

36. The graph below shows functions  $f(x)$  and  $g(x)$ .



What is the equation for  $f(x)$ ?

37. With the assistance of the table below sketch the three functions on the same set of axes.

$\theta$	$0^\circ$	$45^\circ$	$90^\circ$	$135^\circ$	$180^\circ$	$225^\circ$	$270^\circ$	$315^\circ$	$360^\circ$
$\tan \theta$	0	1	undefined	-1	0	1	undefined	-1	0
$2 \tan \theta$	0	2	undefined	-2	0	2	undefined	-2	0
$\frac{1}{3} \tan \theta$	0	$\frac{1}{3}$	undefined	$-\frac{1}{3}$	0	$\frac{1}{3}$	undefined	$-\frac{1}{3}$	0

38. With the assistance of the table below sketch the three functions on the same set of axes.

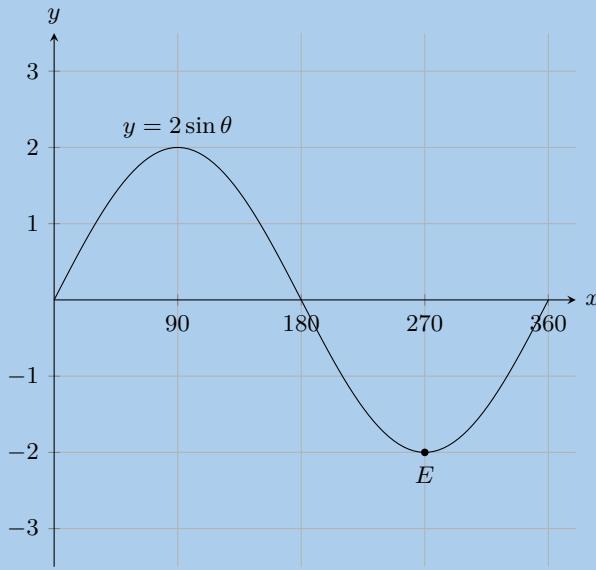
$\theta$	$0^\circ$	$90^\circ$	$180^\circ$	$270^\circ$	$360^\circ$
$\sin \theta + 1$	1	2	1	0	1
$\sin \theta + 2$	3	2	2	1	2
$\sin \theta - 2$	-2	-1	-2	-3	-2

39. Sketch graphs of the following trigonometric functions for  $\theta \in [0^\circ; 360^\circ]$ . Show intercepts and asymptotes.

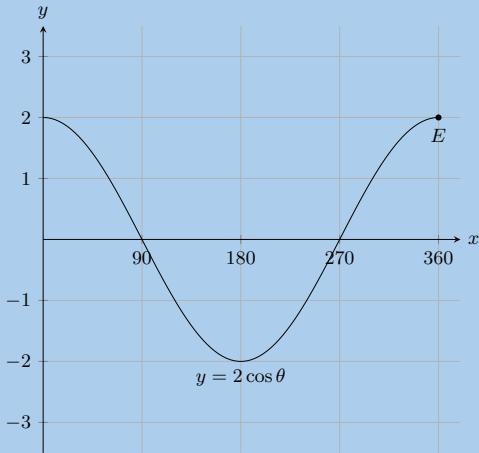
a)  $y = -4 \cos \theta$     b)  $y = \sin \theta - 2$     c)  $y = -2 \sin \theta + 1$     d)  $y = \tan \theta + 2$     e)  $y = \frac{\cos \theta}{2}$

40. State the coordinates at  $E$  and the range of the function.

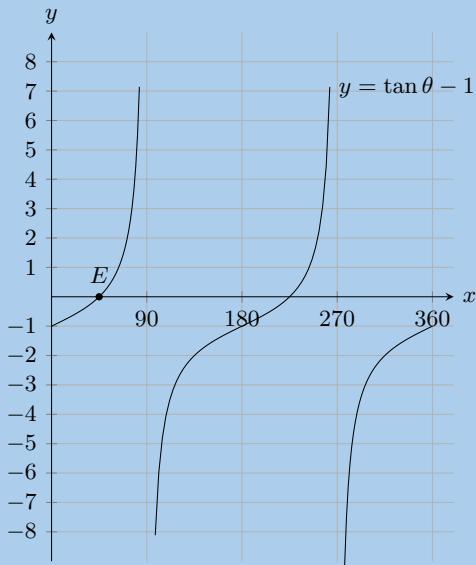
a)



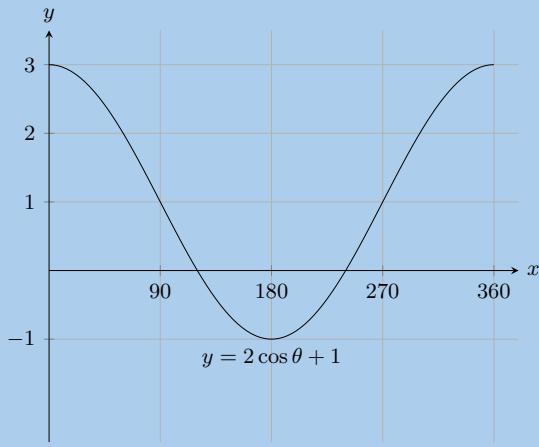
b)



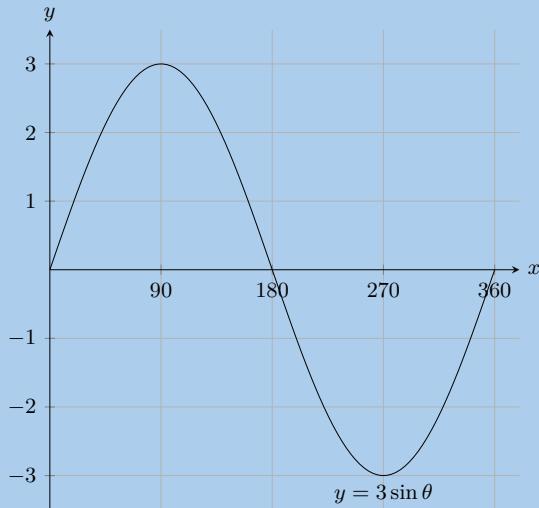
41. State the coordinates at  $E$  and the domain and range of the function in the interval shown.



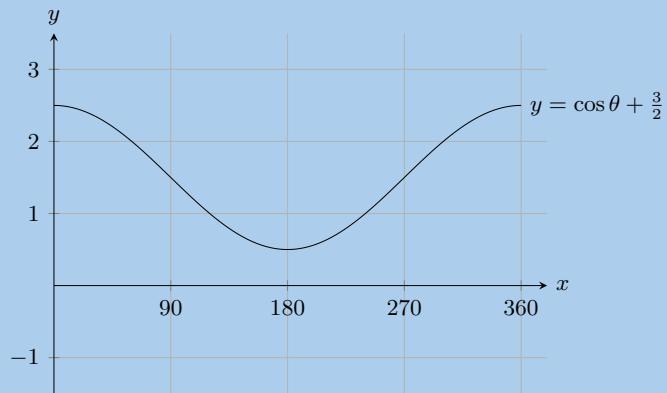
42. For which values of  $\theta$  is the function decreasing, in the interval shown?



43. For which values of  $\theta$  is the function increasing, in the interval shown?

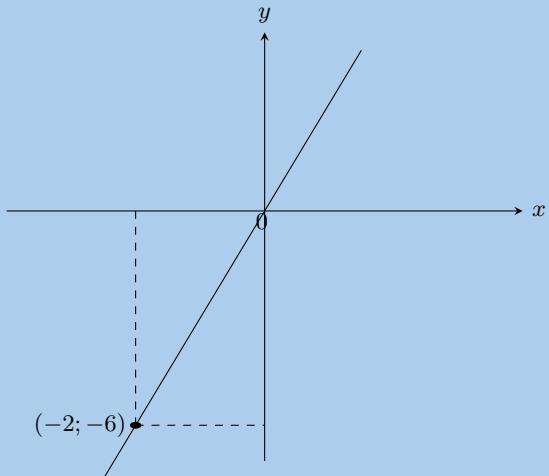


44. For which values of  $\theta$  is the function positive, in the interval shown?

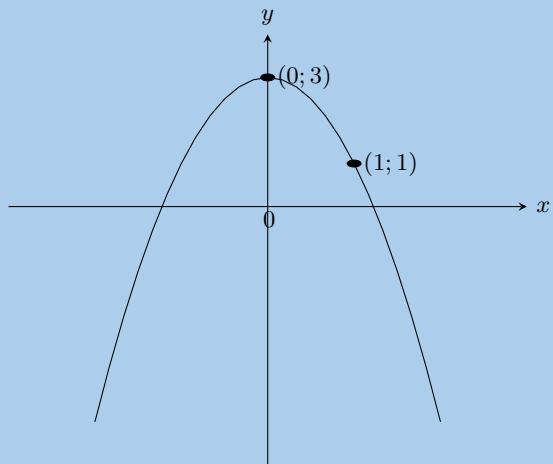


45. Given the general equations  $y = mx + c$ ,  $y = ax^2 + q$ ,  $y = \frac{a}{x} + q$ ,  $y = a \cdot b^x + q$ ,  $y = a \sin \theta + q$ ,  $y = a \cos \theta + q$  and  $y = a \tan \theta$ , determine the specific equations for each of the following graphs.

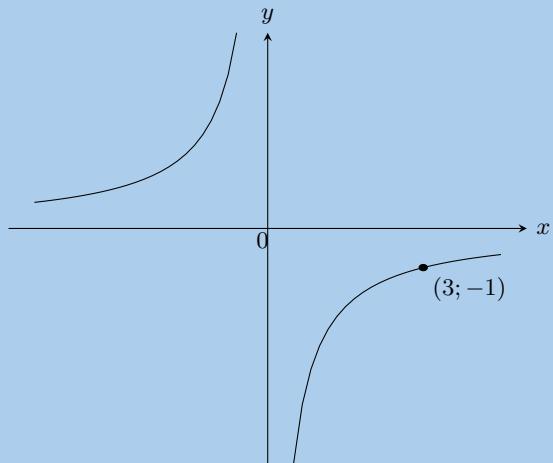
a)



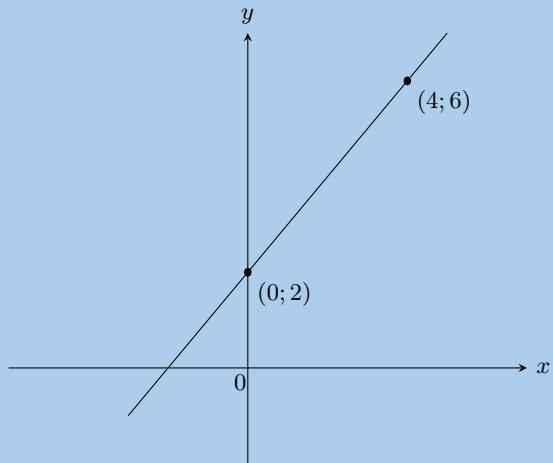
b)



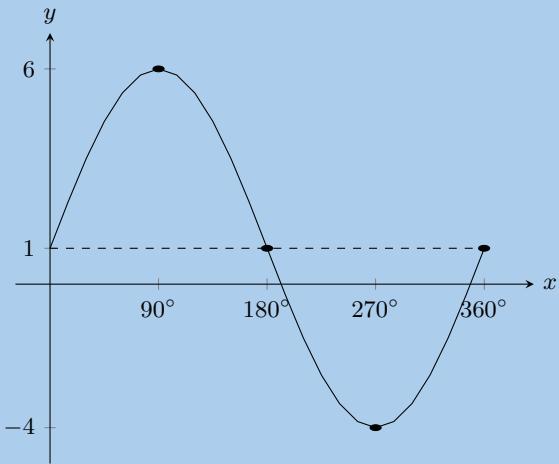
c)



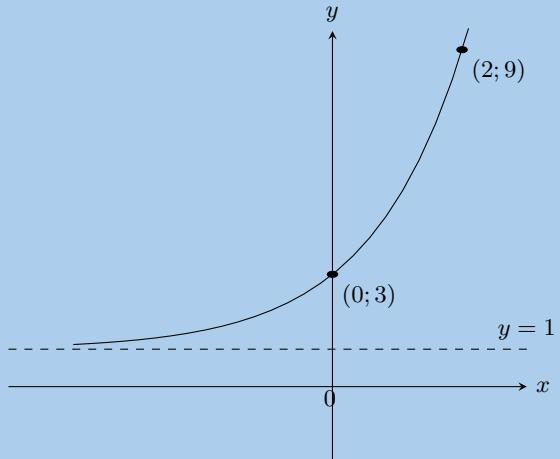
d)



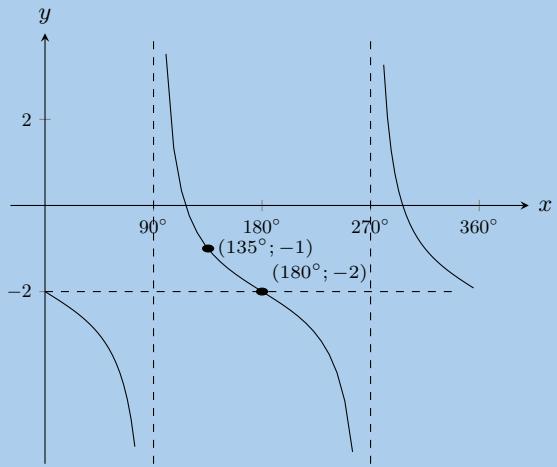
e)



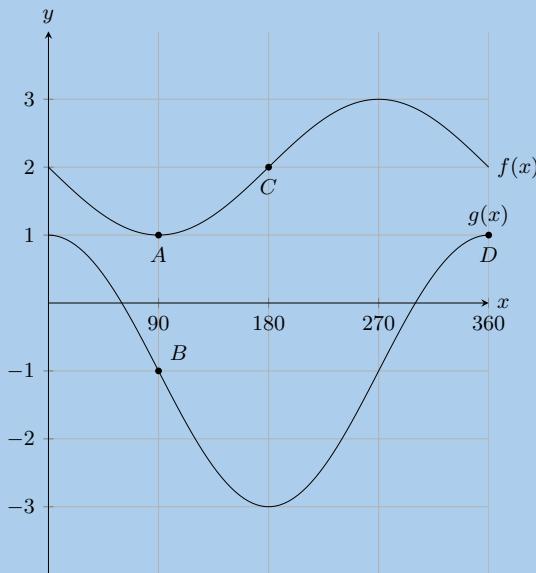
f)



g)

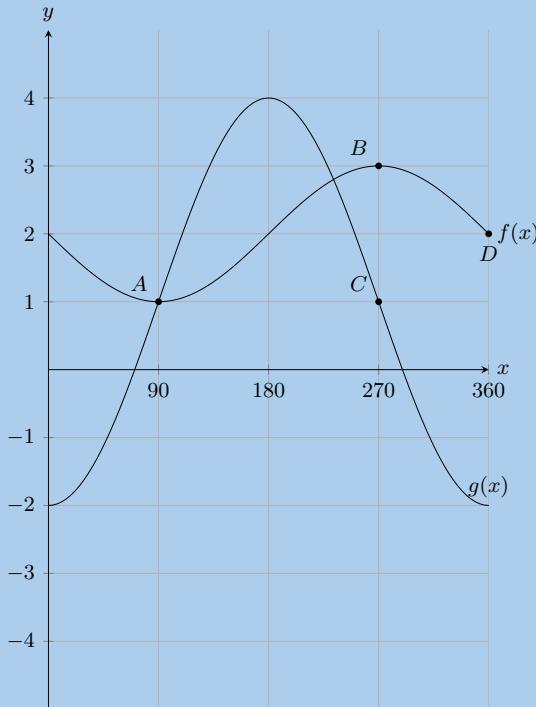


46.



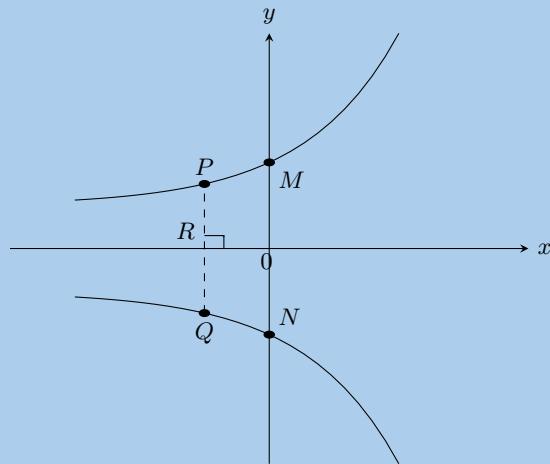
- State the coordinates at  $A$ ,  $B$ ,  $C$  and  $D$ .
- How many times in this interval does  $f(x)$  intersect  $g(x)$ .
- What is the amplitude of  $f(x)$ .
- Evaluate:  $f(180^\circ) - g(180^\circ)$ .

47.



- State the coordinates at  $A$ ,  $B$ ,  $C$  and  $D$ .
- How many times in this interval does  $f(x)$  intersect  $g(x)$ .
- What is the amplitude of  $g(x)$ .
- Evaluate:  $g(180^\circ) - f(180^\circ)$ .

48.  $y = 2^x$  and  $y = -2^x$  are sketched below. Answer the questions that follow.



- Calculate the coordinates of  $M$  and  $N$ .
  - Calculate the length of  $MN$ .
  - Calculate the length of  $PQ$  if  $OR = 1$  unit.
  - Give the equation of  $y = 2^x$  reflected about the  $y$ -axis.
  - Give the range of both graphs.
49. Plot the following functions on the same set of axes and clearly label all points of intersection.

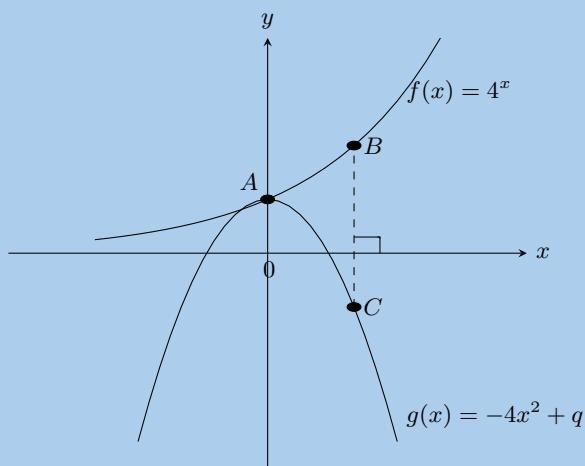
a)  $y = -2x^2 + 3$

$y = 2x + 4$

b)  $y = x^2 - 4$

$y = 3x$

50.  $f(x) = 4^x$  and  $g(x) = -4x^2 + q$  are sketched below. The points  $A(0; 1)$  and  $B(1; 4)$  are given. Answer the questions that follow.



- Determine the value of  $q$ .
- Calculate the length of  $BC$ .
- Give the equation of  $f(x)$  reflected about the  $x$ -axis.
- Give the equation of  $f(x)$  shifted vertically upwards by 1 unit.
- Give the equation of the asymptote of  $f(x)$ .
- Give the ranges of  $f(x)$  and  $g(x)$ .

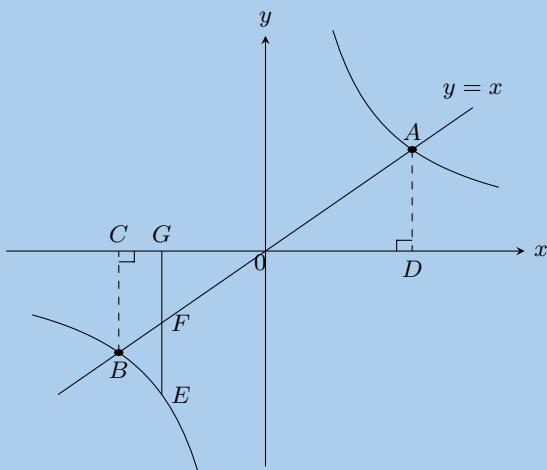
51. Given  $h(x) = x^2 - 4$  and  $k(x) = -x^2 + 4$ . Answer the questions that follow.

- Sketch both graphs on the same set of axes.
- Describe the relationship between  $h$  and  $k$ .
- Give the equation of  $k(x)$  reflected about the line  $y = 4$ .
- Give the domain and range of  $h$ .

52. Sketch the graphs of  $f(\theta) = 2 \sin \theta$  and  $g(\theta) = \cos \theta - 1$  on the same set of axes. Use your sketch to determine:

- $f(180^\circ)$
- $g(180^\circ)$
- $g(270^\circ) - f(270^\circ)$
- The domain and range of  $g$ .
- The amplitude and period of  $f$ .

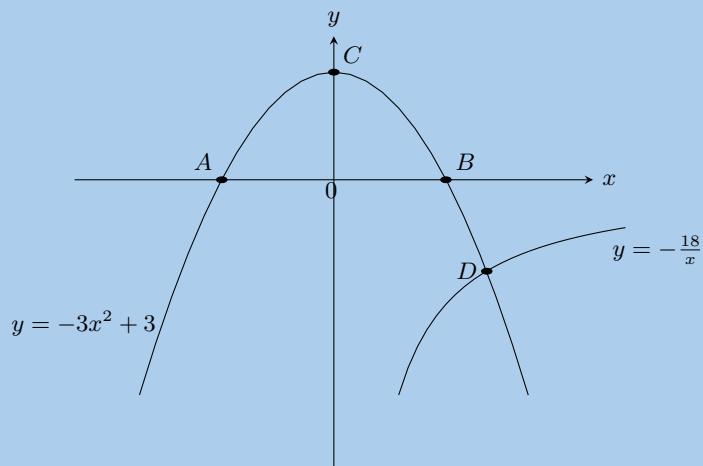
53. The graphs of  $y = x$  and  $y = \frac{8}{x}$  are shown in the following diagram.



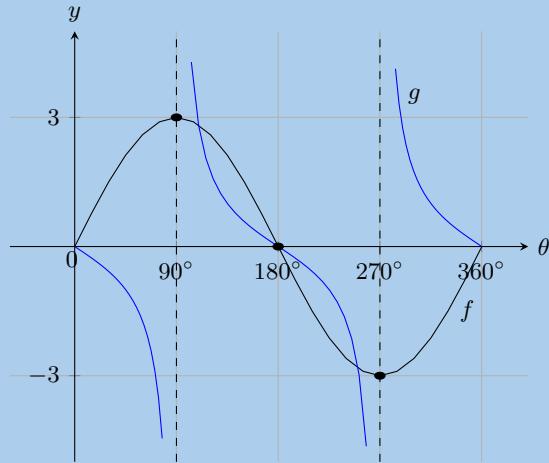
Calculate:

- The coordinates of points  $A$  and  $B$ .
- The length of  $CD$ .
- The length of  $AB$ .
- The length of  $EF$ , given  $G(-2; 0)$ .

54. Given the diagram with  $y = -3x^2 + 3$  and  $y = -\frac{18}{x}$ .

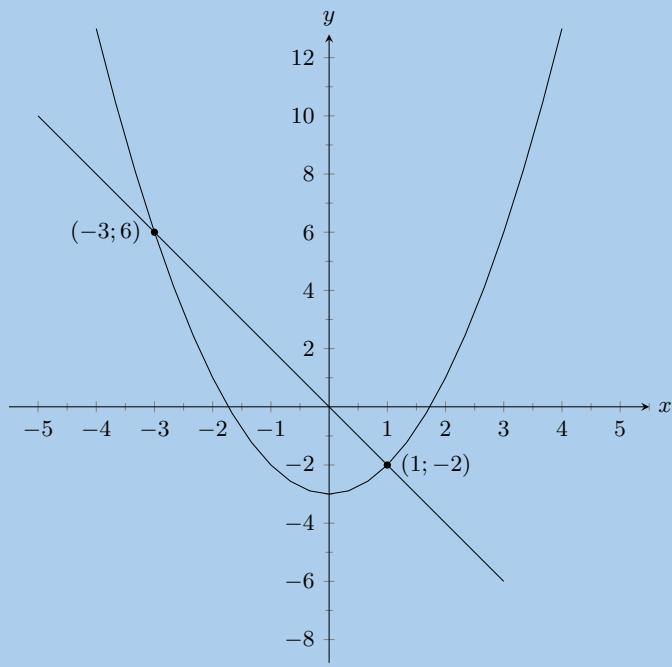


- a) Calculate the coordinates of  $A$ ,  $B$  and  $C$ .
- b) Describe in words what happens at point  $D$ .
- c) Calculate the coordinates of  $D$ .
- d) Determine the equation of the straight line that would pass through points  $C$  and  $D$ .
55. The diagram shows the graphs of  $f(\theta) = 3 \sin \theta$  and  $g(\theta) = -\tan \theta$

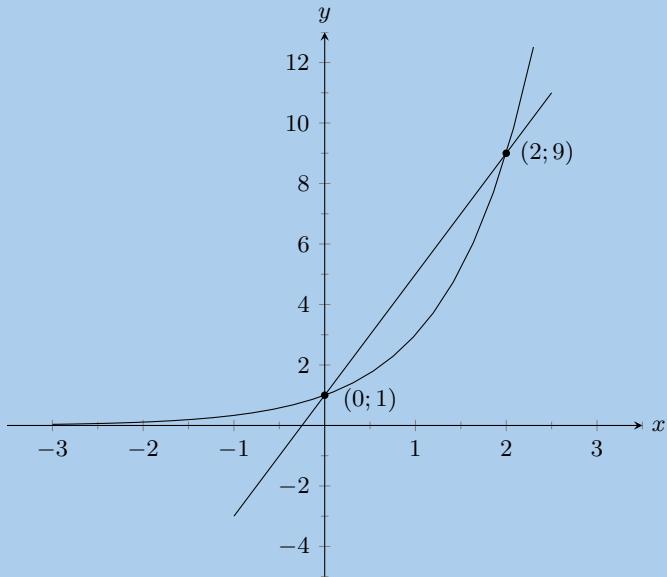


- a) Give the domain of  $g$ .
- b) What is the amplitude of  $f$ ?
- c) Determine for which values of  $\theta$ :
- $f(\theta) = 0 = g(\theta)$
  - $f(\theta) \times g(\theta) < 0$
  - $\frac{g(\theta)}{f(\theta)} > 0$
  - $f(\theta)$  is increasing
56. Determine the equations for the graphs given below.

a)



b)



57. Choose the correct answer:

- a) Which of the following does not have a gradient of 3?  
i.  $y = 3x + 6$       ii.  $3y = 9x - 1$       iii.  $\frac{1}{3}(y - 1) = x$       iv.  $\frac{1}{2}(y - 3) = 6x$
- b) The asymptote of  $xy = 3 + x$  is:  
i. 3      ii. 1      iii. -3      iv. -1

58. Sketch the following

- a)  $y = -1,5^x$   
b)  $xy = 5 + 2x$   
c)  $2y + 2x = 3$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1a. 2G2M	1b. 2G2N	2a. 2G2P	2b. 2G2Q	3a. 2G2R	3b. 2G2S
4a. 2G2T	4b. 2G2V	5. 2G2W	6. 2G2X	7a. 2G2Y	7b. 2G2Z
7c. 2G32	8a. 2G33	8b. 2G34	9a. 2G35	9b. 2G36	9c. 2G37
10. 2G38	11. 2G39	12. 2G3B	13. 2G3C	14. 2G3D	15. 2G3F
16a. 2G3G	16b. 2G3H	16c. 2G3J	17. 2G3K	18. 2G3M	19. 2G3N
20. 2G3P	21a. 2G3Q	21b. 2G3R	21c. 2G3S	22a. 2G3T	22b. 2G3V
22c. 2G3W	22d. 2G3X	23. 2G3Y	24a. 2G3Z	24b. 2G42	24c. 2G43
25. 2G44	26. 2G45	27. 2G46	28a. 2G47	28b. 2G48	28c. 2G49
28d. 2G4B	28e. 2G4C	28f. 2G4D	29. 2G4F	30. 2G4G	31. 2G4H
32. 2G4J	33. 2G4K	34. 2G4M	35. 2G4N	36. 2G4P	37. 2G4Q
38. 2G4R	39a. 2G4S	39b. 2G4T	39c. 2G4V	39d. 2G4W	39e. 2G4X
40a. 2G4Y	40b. 2G4Z	41. 2G52	42. 2G53	43. 2G54	44. 2G55
45. 2G56	46. 2G57	47. 2G58	48. 2G59	49a. 2G5B	49b. 2G5C
50. 2G5D	51. 2G5F	52. 2G5G	53. 2G5H	54. 2G5J	55. 2G5K
56a. 2G5M	56b. 2G5N	57a. 2G5P	57b. 2G5Q	58a. 2G5R	58b. 2G5S
58c. 2G5T					



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# CHAPTER



## *Euclidean geometry*

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## 7 Euclidean geometry

Geometry (from the Greek “geo” = earth and “metria” = measure) arose as the field of knowledge dealing with spatial relationships. Analytical geometry deals with space and shape using algebra and a coordinate system. Euclidean geometry deals with space and shape using a system of logical deductions.

Euclidean geometry was first used in surveying and is still used extensively for surveying today. Euclidean geometry is also used in architecture to design new buildings. Other uses of Euclidean geometry are in art and to determine the best packing arrangement for various types of objects.



Figure 7.1: A small piece of the original version of Euclid's elements. Euclid is considered to be the father of modern geometry. Euclid's elements was used for many years as the standard text for geometry.

### VISIT:

This video highlights some of the basic concepts used in geometry.

► See video: [2G5V](http://2G5V) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### DID YOU KNOW?

In Euclidean geometry we use two fundamental types of measurement: angles and distances.

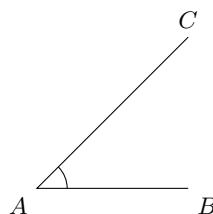
## 7.1 Introduction

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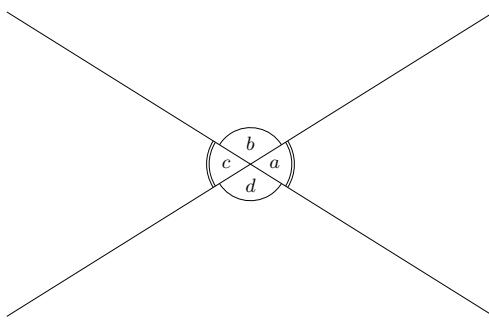
### Angles

EMA5N

An angle is formed when two straight lines meet at a point, also known as a vertex. Angles are labelled with a caret on a letter, for example,  $\hat{B}$ . Angles can also be labelled according to the line segments that make up the angle, for example  $C\hat{B}A$  or  $A\hat{B}C$ . The  $\angle$  symbol is a short method of writing angle in geometry and is often used in phrases such as “sum of  $\angle$ s in  $\triangle$ ”. Angles are measured in degrees which is denoted by  $^\circ$ , a small circle raised above the text, similar to an exponent.



In the diagram below two straight lines intersect at a point, forming the four angles  $\hat{a}$ ,  $\hat{b}$ ,  $\hat{c}$  and  $\hat{d}$ .



The following table summarises the different types of angles, with examples from the figure above.

Term	Property	Examples
Acute angle	$0^\circ < \text{angle} < 90^\circ$	$\hat{a}; \hat{c}$
Right angle	$\text{Angle} = 90^\circ$	
Obtuse angle	$90^\circ < \text{angle} < 180^\circ$	$\hat{b}; \hat{d}$
Straight angle	$\text{Angle} = 180^\circ$	$\hat{a} + \hat{b}; \hat{b} + \hat{c}$
Reflex angle	$180^\circ < \text{angle} < 360^\circ$	$\hat{a} + \hat{b} + \hat{c}$
Adjacent angles	Angles that share a vertex and a common side.	$\hat{a}$ and $\hat{d}; \hat{c}$ and $\hat{d}$
Vertically opposite angles	Angles opposite each other when two lines intersect. They share a vertex and are equal.	$\hat{a} = \hat{c}; \hat{b} = \hat{d}$
Supplementary angles	Two angles that add up to $180^\circ$	$\hat{a} + \hat{b} = 180^\circ; \hat{b} + \hat{c} = 180^\circ$
Complementary angles	Two angles that add up to $90^\circ$	
Revolution	The sum of all angles around a point.	$\hat{a} + \hat{b} + \hat{c} + \hat{d} = 360^\circ$

Note that adjacent angles on a straight line are supplementary.

#### VISIT:

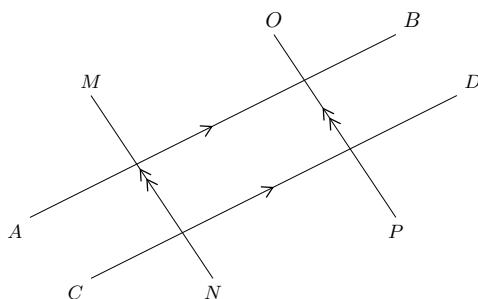
The following video provides a summary of the terms used to refer to angles.

► See video: [2G5W](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## Parallel lines and transversal lines

Two lines intersect if they cross each other at a point. For example, at a traffic intersection two or more streets intersect; the middle of the intersection is the common point between the streets.

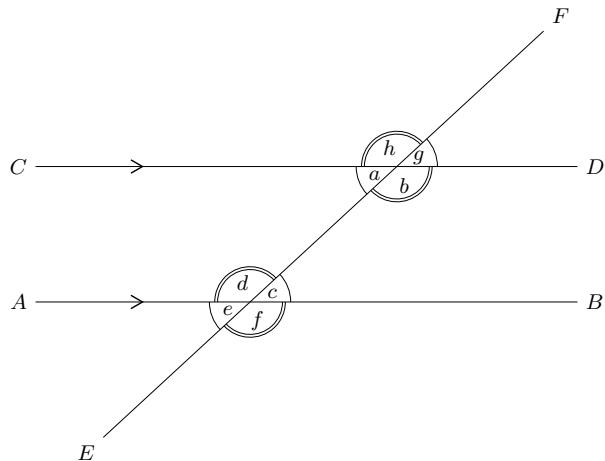
Parallel lines are always the same distance apart and they are denoted by arrow symbols as shown below.



In writing we use two vertical lines to indicate that two lines are parallel:

$$AB \parallel CD \text{ and } MN \parallel OP$$

A transversal line intersects two or more parallel lines. In the diagram below,  $AB \parallel CD$  and  $EF$  is a transversal line.



The properties of the angles formed by these intersecting lines are summarised in the following table:

Name of angle	Definition	Examples	Notes
Interior angles	Angles that lie in between the parallel lines.	$\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are interior angles.	Interior means inside.
Exterior angles	Angles that lie outside the parallel lines.	$\hat{e}, \hat{f}, \hat{g}$ and $\hat{h}$ are exterior angles.	Exterior means outside.
Corresponding angles	Angles on the same side of the lines and the same side of the transversal. If the lines are parallel, the corresponding angles will be equal.	$\hat{a}$ and $\hat{e}$ , $\hat{b}$ and $\hat{f}$ , $\hat{c}$ and $\hat{g}$ , $\hat{d}$ and $\hat{h}$ are pairs of corresponding angles. $\hat{a} = \hat{e}$ , $\hat{b} = \hat{f}$ , $\hat{c} = \hat{g}$ and $\hat{d} = \hat{h}$ .	F shape
Co-interior angles	Angles that lie in between the lines and on the same side of the transversal. If the lines are parallel, the angles are supplementary.	$\hat{a}$ and $\hat{d}$ , $\hat{b}$ and $\hat{c}$ are pairs of co-interior angles. $\hat{a} + \hat{d} = 180^\circ$ , $\hat{b} + \hat{c} = 180^\circ$ .	C shape
Alternate interior angles	Equal interior angles that lie inside the lines and on opposite sides of the transversal. If the lines are parallel, the interior angles will be equal.	$\hat{a}$ and $\hat{c}$ , $\hat{b}$ and $\hat{d}$ are pairs of alternate interior angles. $\hat{a} = \hat{c}$ , $\hat{b} = \hat{d}$	Z shape

### VISIT:

This video provides a short summary of some of the angles formed by intersecting lines.

► See video: [2G5X](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

If two lines are intersected by a transversal such that:

- corresponding angles are equal; or
- alternate interior angles are equal; or
- co-interior angles are supplementary

then the two lines are parallel.

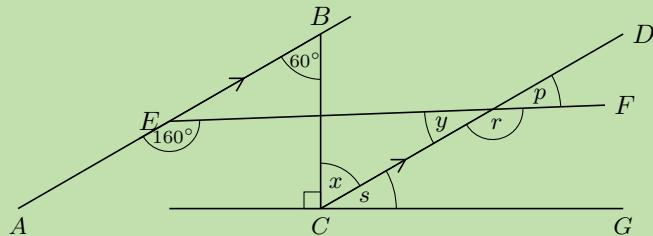
### NOTE:

When we refer to lines we can either write  $EF$  to mean the line through points  $E$  and  $F$  or  $\overline{EF}$  to mean the line segment from point  $E$  to point  $F$ .

## Worked example 1: Finding angles

### QUESTION

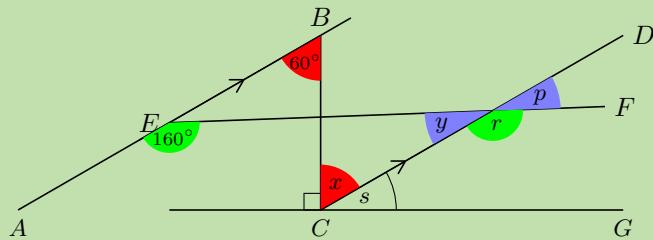
Find all the unknown angles. Is  $EF \parallel CG$ ? Explain your answer.



### SOLUTION

#### Step 1: Use the properties of parallel lines to find all equal angles on the diagram

Redraw the diagram and mark all the equal angles.



#### Step 2: Determine the unknown angles

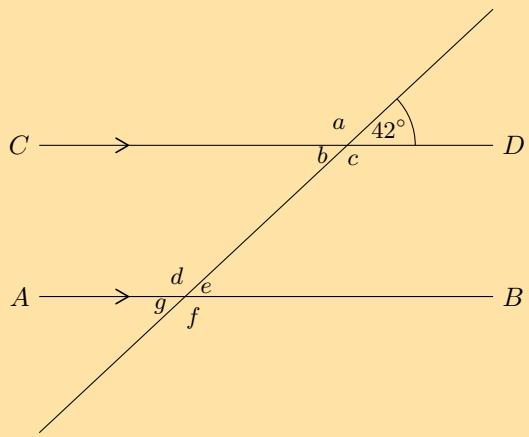
$$\begin{aligned} AB &\parallel CD && (\text{given}) \\ \therefore \hat{x} &= 60^\circ && (\text{alt } \angle s; AB \parallel CD) \\ \hat{y} + 160^\circ &= 180^\circ && (\text{co-int } \angle s; AB \parallel CD) \\ \therefore \hat{y} &= 20^\circ \\ \hat{p} &= \hat{y} && (\text{vert opp } \angle s =) \\ \therefore \hat{p} &= 20^\circ \\ \hat{r} &= 160^\circ && (\text{corresp } \angle s; AB \parallel CD) \\ \hat{s} + \hat{x} + 90^\circ &= 180^\circ && (\angle s \text{ on a str line}) \\ \hat{s} + 60^\circ &= 90^\circ \\ \therefore \hat{s} &= 30^\circ \end{aligned}$$

#### Step 3: Determine whether $EF \parallel CG$

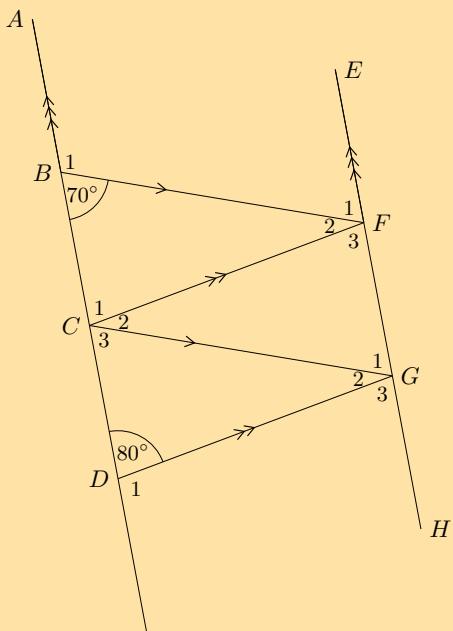
If  $EF \parallel CG$  then  $\hat{p}$  will be equal to corresponding angle  $\hat{s}$ , but  $\hat{p} = 20^\circ$  and  $\hat{s} = 30^\circ$ . Therefore  $EF$  is not parallel to  $CG$ .

### Exercise 7 – 1:

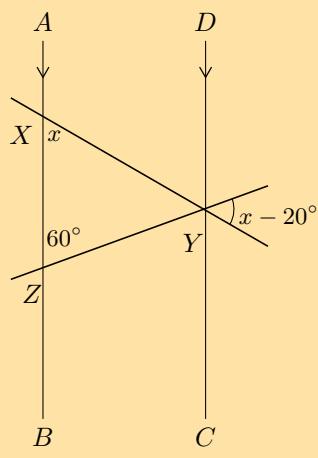
1. Use adjacent, corresponding, co-interior and alternate angles to fill in all the angles labelled with letters in the diagram:



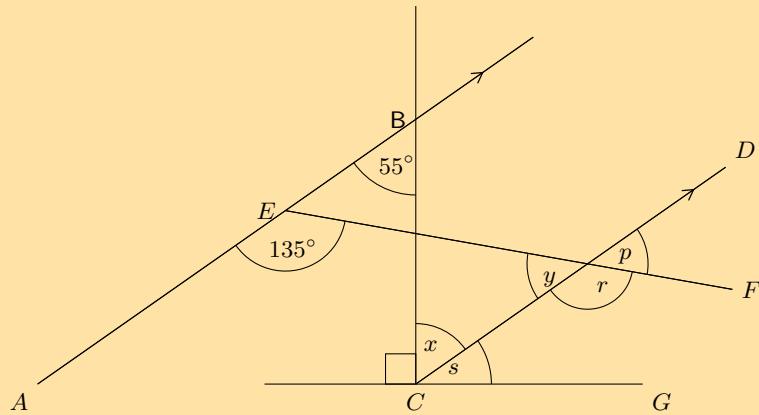
2. Find all the unknown angles in the figure:



3. Find the value of  $x$  in the figure:

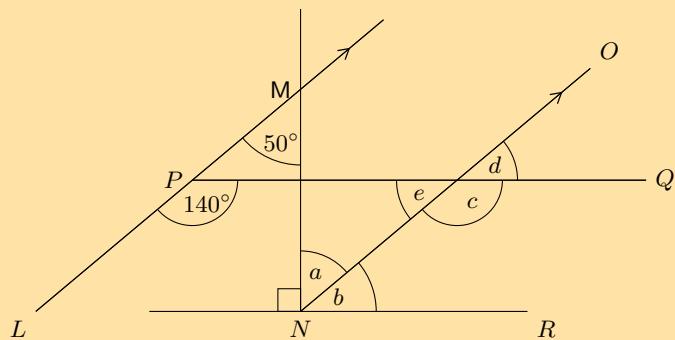


4. Given the figure below:



- Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.
- Based on the results for the angles above, is  $EF \parallel CG$ ?

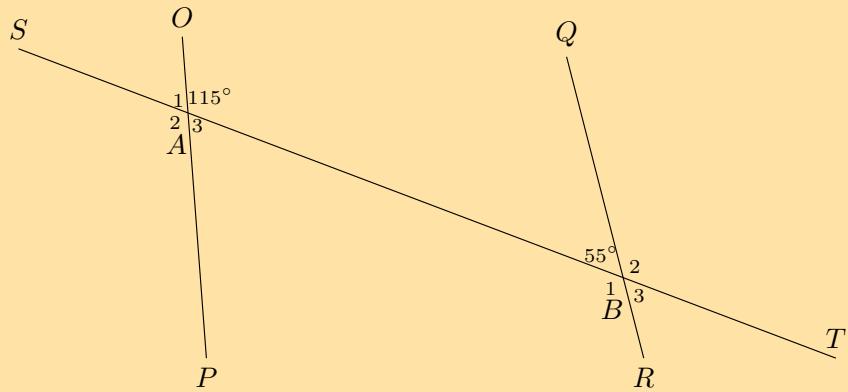
5. Given the figure below:



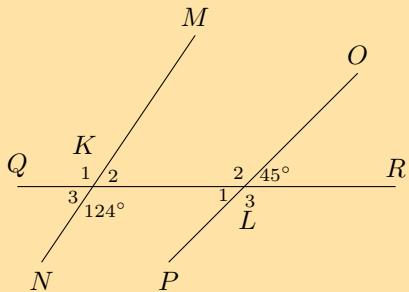
- Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.
- Based on the results for the angles above, is  $PQ \parallel NR$ ?

6. Determine whether the pairs of lines in the following figures are parallel:

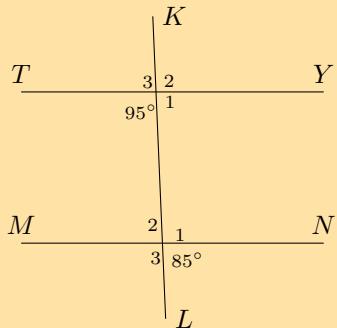
a)



b)



c)



7. If  $AB$  is parallel to  $CD$  and  $AB$  is parallel to  $EF$ , explain why  $CD$  must be parallel to  $EF$ .

$C$  —————  $D$

$A$  —————  $B$

$E$  —————  $F$

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2G5Y](#) 2. [2G5Z](#) 3. [2G62](#) 4. [2G63](#) 5. [2G64](#) 6a. [2G65](#) 6b. [2G66](#) 6c. [2G67](#) 7. [2G68](#)



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## 7.2 Triangles

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### Classification of triangles

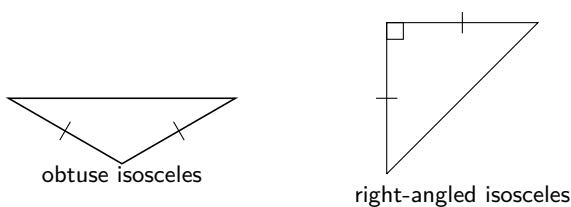
EMA5S

A triangle is a three-sided polygon. Triangles can be classified according to sides: equilateral, isosceles and scalene. Triangles can also be classified according to angles: acute-angled, obtuse-angled and right-angled.

We use the notation  $\triangle ABC$  to refer to a triangle with vertices labelled  $A$ ,  $B$  and  $C$ .

Name	Diagram	Properties
Scalene		All sides and angles are different.
Isosceles		Two sides are equal in length. The angles opposite the equal sides are also equal.
Equilateral		All three sides are equal in length and all three angles are equal.
Acute		Each of the three interior angles is less than 90°.
Obtuse		One interior angle is greater than 90°.
Right-angled		One interior angle is 90°.

Different combinations of these properties are also possible. For example, an obtuse isosceles triangle and a right-angled isosceles triangle are shown below:

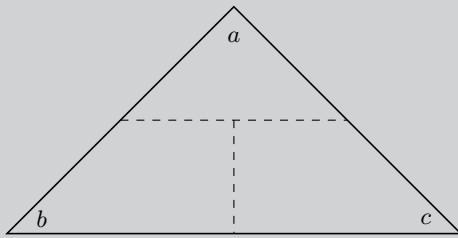


### VISIT:

This video shows the different ways to classify triangles.

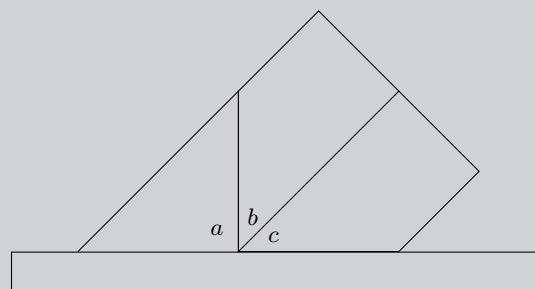
▶ See video: 2G69 at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Investigation: Interior angles of a triangle

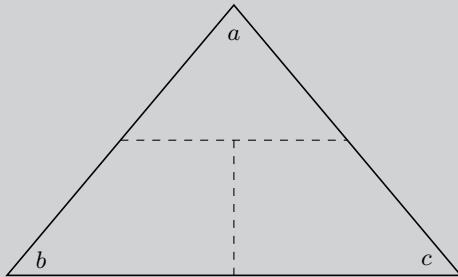


1. On a piece of paper draw a triangle of any size and shape.
2. Cut it out and label the angles  $\hat{a}$ ,  $\hat{b}$  and  $\hat{c}$  on both sides of the paper.
3. Draw dotted lines as shown and cut along these lines to get three pieces of paper.
4. Place them along your ruler as shown in the figure below.
5. What can we conclude?

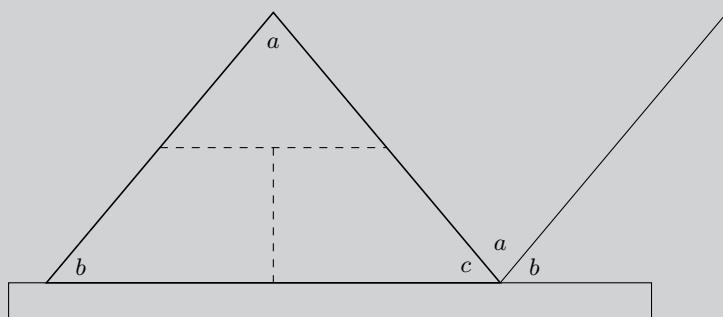
Hint: What is the sum of angles on a straight line?



### Investigation: Exterior angles of a triangle



1. On a piece of paper draw a triangle of any size and shape. On another piece of paper, make a copy of the triangle.
2. Cut both out and label the angles of both triangles  $\hat{a}$ ,  $\hat{b}$  and  $\hat{c}$  on both sides of the paper.
3. Draw dotted lines on **one** triangle as shown and cut along the lines.
4. Place the second triangle and the cut out pieces as shown in the figure below.
5. What can we can conclude?



**VISIT:**

We can use the fact that the angles in a triangle add up to  $180^\circ$  to work out the sum of the exterior angles in a pentagon. This video shows you how.

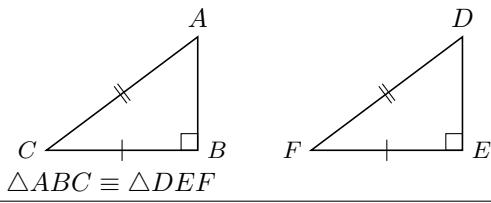
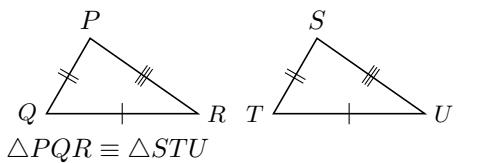
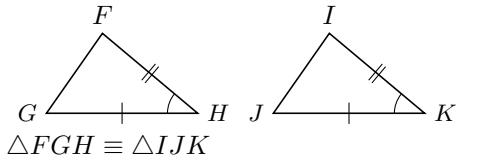
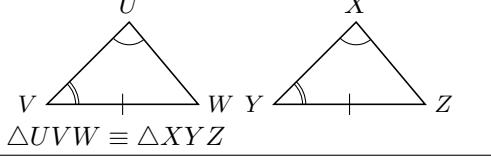
► See video: [2G6B](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## Congruency

EMA5T

Two triangles are congruent if one fits exactly over the other. This means that the triangles have equal corresponding angles and sides. To determine whether two triangles are congruent, it is not necessary to check every side and every angle. We indicate congruency using  $\equiv$ .

The following table describes the requirements for congruency:

Rule	Description	Diagram
RHS or $90^\circ$ HS ( $90^\circ$ , hypotenuse, side)	If the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and the corresponding side of another right-angled triangle, then the two triangles are congruent.	 $\triangle ABC \equiv \triangle DEF$
SSS (side, side, side)	If three sides of a triangle are equal in length to the corresponding sides of another triangle, then the two triangles are congruent.	 $\triangle PQR \equiv \triangle STU$
SAS or S $\angle$ S (side, angle, side)	If two sides and the <b>included</b> angle of a triangle are equal to the corresponding two sides and <b>included</b> angle of another triangle, then the two triangles are congruent.	 $\triangle FGH \equiv \triangle IJK$
AAS or $\angle$ $\angle$ S (angle, angle, side)	If one side and two angles of a triangle are equal to the corresponding one side and two angles of another triangle, then the two triangles are congruent.	 $\triangle UVW \equiv \triangle XYZ$

The order of letters when labelling congruent triangles is very important.

$$\triangle ABC \equiv \triangle DEF$$

This notation indicates the following properties of the two triangles:  $\hat{A} = \hat{D}$ ,  $\hat{B} = \hat{E}$ ,  $\hat{C} = \hat{F}$ ,  $AB = DE$ ,  $AC = DF$  and  $BC = EF$ .

**NOTE:**

You might see  $\cong$  used to show that two triangles are congruent. This is the internationally recognised symbol for congruency.

**VISIT:**

This video shows some practice examples of finding congruent triangles.

► See video: [2G6C](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

Two triangles are similar if one triangle is a scaled version of the other. This means that their corresponding angles are equal in measure and the ratio of their corresponding sides are in proportion. The two triangles have the same shape, but different scales. Congruent triangles are similar triangles, but not all similar triangles are congruent. We use  $\|\|$  to indicate that two triangles are similar.

The following table describes the requirements for similarity:

Rule	Description	Diagram
AAA (angle, angle, angle)	If all three pairs of corresponding angles of two triangles are equal, then the triangles are similar.	$\begin{aligned} \hat{A} &= \hat{D}, \hat{B} = \hat{E}, \hat{C} = \hat{F} \\ \therefore \triangle ABC &\ \  \triangle DEF \end{aligned}$
SSS (side, side, side)	If all three pairs of corresponding sides of two triangles are in proportion, then the triangles are similar.	$\begin{aligned} \frac{MN}{RS} &= \frac{ML}{RT} = \frac{NL}{ST} \\ \therefore \triangle MNL &\ \  \triangle RST \end{aligned}$

The order of letters for similar triangles is very important. Always label similar triangles in corresponding order. For example,

$$\begin{aligned} \triangle MNL &\|\| \triangle RST \text{ is correct; but} \\ \triangle MNL &\|\| \triangle RTS \text{ is incorrect.} \end{aligned}$$

**NOTE:**

You might see  $\sim$  used to show that two triangles are similar. This is the internationally recognised symbol for similarity.

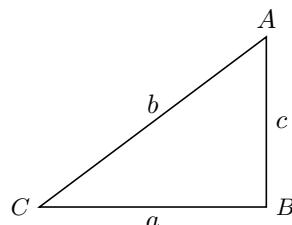
**VISIT:**

The following video explains similar triangles.

► See video: [2G6D](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

If  $\triangle ABC$  is right-angled with  $\hat{B} = 90^\circ$ , then  $b^2 = a^2 + c^2$ .

**Converse:** If  $b^2 = a^2 + c^2$ , then  $\triangle ABC$  is right-angled with  $\hat{B} = 90^\circ$ .



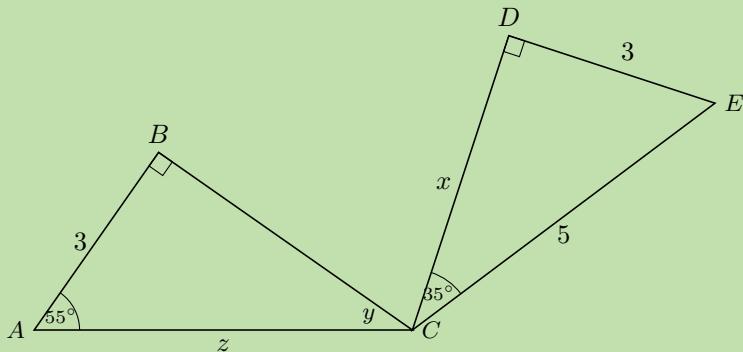
**VISIT:**

The following video explains the theorem of Pythagoras and shows some examples of working with the theorem of Pythagoras.

► See video: [2G6F](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

**Worked example 2: Triangles****QUESTION**

Determine if the two triangles are congruent. Use the result to find  $x$ ,  $y$  and  $z$ .

**SOLUTION**

**Step 1: Examine the information given for both triangles**

**Step 2: Determine whether  $\triangle CDE \equiv \triangle CBA$**

In  $\triangle CDE$ :

$$\begin{aligned}\hat{D} + \hat{C} + \hat{E} &= 180^\circ && (\text{sum of } \angle\text{s in } \triangle) \\ 90^\circ + 35^\circ + \hat{E} &= 180^\circ \\ \therefore \hat{E} &= 55^\circ\end{aligned}$$

In  $\triangle CDE$  and  $\triangle CBA$ :

$$\begin{aligned}\hat{DCE} &= \hat{BAC} = 55^\circ && (\text{proved}) \\ \hat{CDE} &= \hat{CBA} = 90^\circ && (\text{given}) \\ DE &= BA = 3 && (\text{given}) \\ \therefore \triangle CDE &\equiv \triangle CBA && (\text{AAS})\end{aligned}$$

**Step 3: Determine the unknown angles and sides**

In  $\triangle CDE$ :

$$\begin{aligned}CE^2 &= DE^2 + CD^2 && (\text{Pythagoras}) \\ 5^2 &= 3^2 + x^2 \\ x^2 &= 16 \\ \therefore x &= 4\end{aligned}$$

In  $\triangle CBA$ :

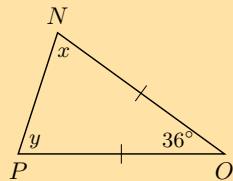
$$\begin{aligned}\hat{B} + \hat{A} + \hat{y} &= 180^\circ && \text{(sum of } \angle \text{s in } \triangle) \\ 90^\circ + 55^\circ + \hat{y} &= 180^\circ \\ \therefore \hat{y} &= 35^\circ\end{aligned}$$

$$\begin{aligned}\triangle CDE &\equiv \triangle CBA && \text{(proved)} \\ \therefore CE &= CA \\ \therefore z &= 5\end{aligned}$$

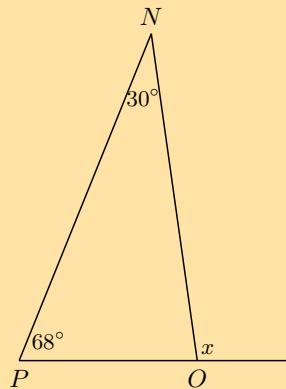
### Exercise 7 – 2:

1. Calculate the unknown variables in each of the following figures.

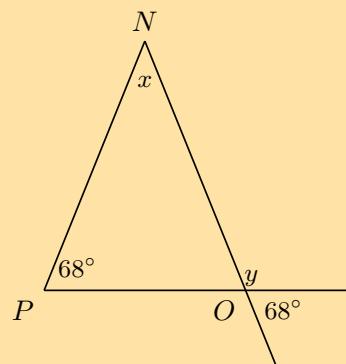
a)



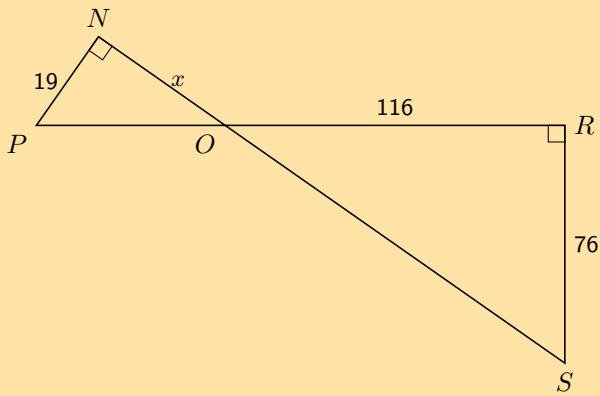
b)



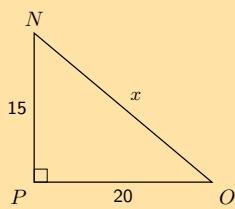
c)



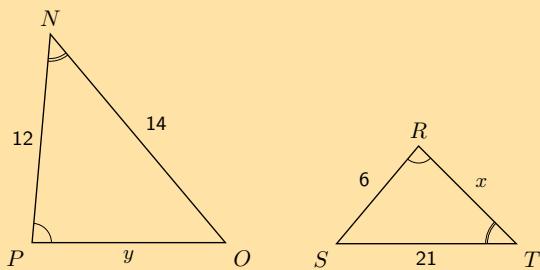
d)



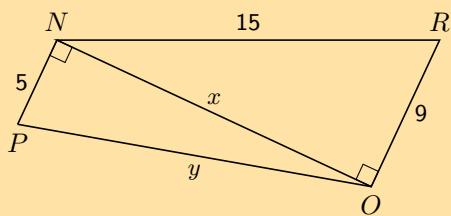
e)



f)



g)



2. Given the following diagrams:

Diagram A

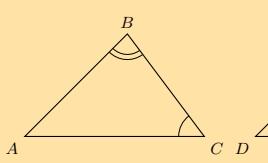
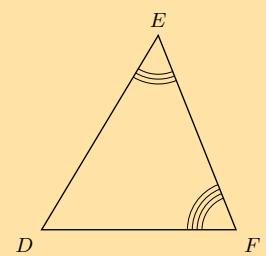
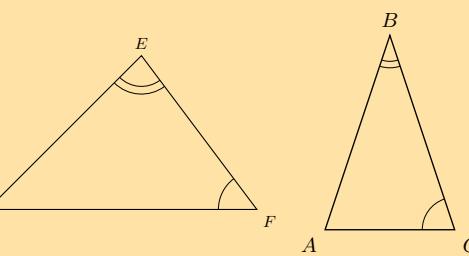


Diagram B



Which diagram correctly gives a pair of similar triangles?

3. Given the following diagrams:

Diagram A

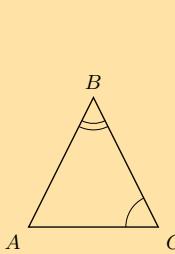
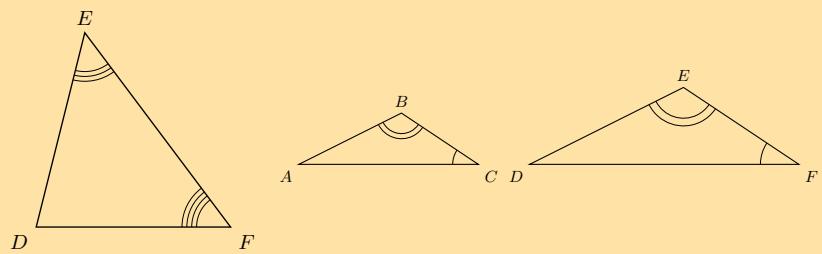
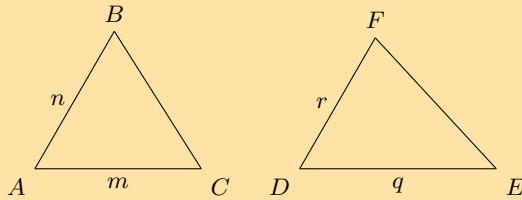


Diagram B



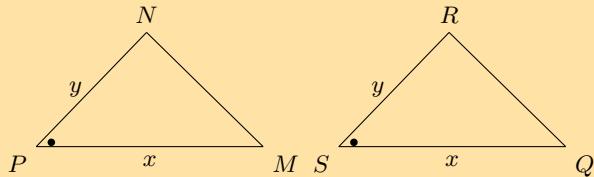
Which diagram correctly gives a pair of similar triangles?

4. Have a look at the following triangles, which are drawn to scale:



Are the two triangles congruent? If so state the reason and use the correct notation to state that they are congruent.

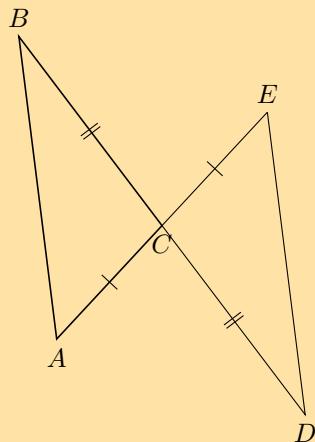
5. Have a look at the following triangles, which are drawn to scale:



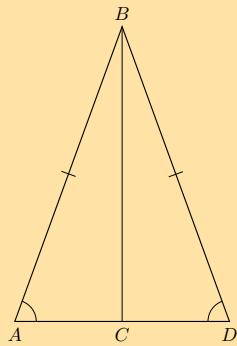
Are the two triangles congruent? If so state the reason and use the correct notation to state that they are congruent.

6. State whether the following pairs of triangles are congruent or not. Give reasons for your answers. If there is not enough information to make a decision, explain why.

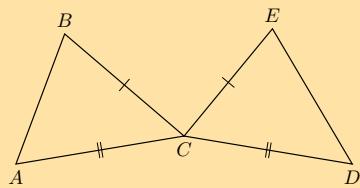
a)



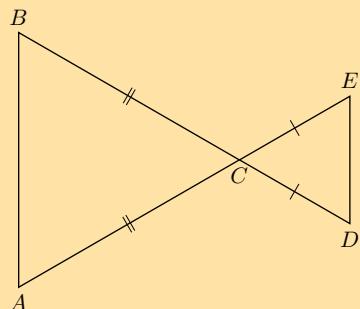
b)



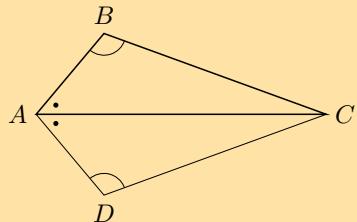
c)



d)



e)



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- |          |          |          |          |          |          |          |          |
|----------|----------|----------|----------|----------|----------|----------|----------|
| 1a. 2G6G | 1b. 2G6H | 1c. 2G6J | 1d. 2G6K | 1e. 2G6M | 1f. 2G6N | 1g. 2G6P | 2. 2G6Q  |
| 3. 2G6R  | 4. 2G6S  | 5. 2G6T  | 6a. 2G6V | 6b. 2G6W | 6c. 2G6X | 6d. 2G6Y | 6e. 2G6Z |



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**DEFINITION:** *Quadrilateral*

A quadrilateral is a closed shape consisting of four straight line segments.

**NOTE:**

The interior angles of a quadrilateral add up to  $360^\circ$ .

## Parallelogram

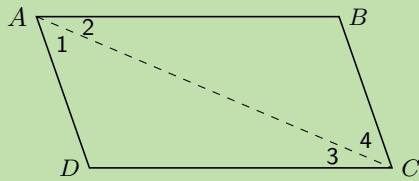
**DEFINITION:** *Parallelogram*

A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

**Worked example 3: Properties of a parallelogram****QUESTION**

$ABCD$  is a parallelogram with  $AB \parallel DC$  and  $AD \parallel BC$ . Show that:

1.  $AB = DC$  and  $AD = BC$
2.  $\hat{A} = \hat{C}$  and  $\hat{B} = \hat{D}$

**SOLUTION****Step 1: Connect  $AC$  to form  $\triangle ABC$  and  $\triangle CDA$** 

Redraw the diagram and draw line  $AC$ .

**Step 2: Use properties of parallel lines to indicate all equal angles on the diagram**

On your diagram mark all the equal angles.

**Step 3: Prove  $\triangle ABC \equiv \triangle CDA$** 

In  $\triangle ABC$  and  $\triangle CDA$ :

$$\begin{aligned}
 \hat{A}_2 &= \hat{C}_3 && (\text{alt } \angle\text{s}; AB \parallel DC) \\
 \hat{C}_4 &= \hat{A}_1 && (\text{alt } \angle\text{s}; BC \parallel AD) \\
 AC &&& (\text{common side}) \\
 \therefore \triangle ABC &\equiv \triangle CDA && (\text{AAS}) \\
 \therefore AB &= CD \quad \text{and} \quad BC = DA
 \end{aligned}$$

$\therefore$  Opposite sides of a parallelogram have equal length.

We have already shown  $\hat{A}_2 = \hat{C}_3$  and  $\hat{A}_1 = \hat{C}_4$ . Therefore,

$$\hat{A} = \hat{A}_1 + \hat{A}_2 = \hat{C}_3 + \hat{C}_4 = \hat{C}$$

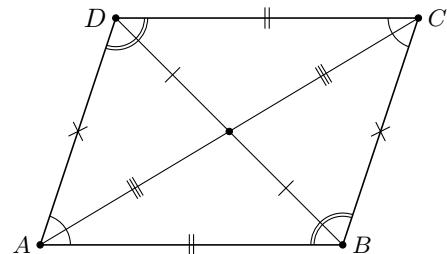
Furthermore,

$$\hat{B} = \hat{D} \quad (\triangle ABC \equiv \triangle CDA)$$

Therefore opposite angles of a parallelogram are equal.

Summary of the properties of a parallelogram:

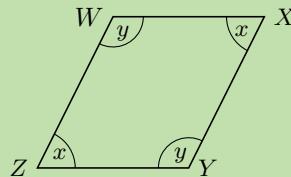
- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.



#### Worked example 4: Proving a quadrilateral is a parallelogram

##### QUESTION

Prove that if both pairs of opposite angles in a quadrilateral are equal, the quadrilateral is a parallelogram.



##### SOLUTION

###### Step 1: Find the relationship between $\hat{x}$ and $\hat{y}$

In  $WXYZ$ :

$$\begin{aligned}
 \hat{W} &= \hat{Y} &= \hat{y} && \text{(given)} \\
 \hat{Z} &= \hat{X} &= \hat{x} && \text{(given)} \\
 \hat{W} + \hat{X} + \hat{Y} + \hat{Z} &= 360^\circ && \text{(sum of } \angle\text{s in a quad)} \\
 \therefore 2\hat{x} + 2\hat{y} &= 360^\circ \\
 \therefore \hat{x} + \hat{y} &= 180^\circ \\
 \hat{W} + \hat{Z} &= \hat{x} + \hat{y} \\
 &= 180^\circ
 \end{aligned}$$

But these are co-interior angles between lines  $WX$  and  $ZY$ . Therefore  $WX \parallel ZY$ .

###### Step 2: Find parallel lines

Similarly  $\hat{W} + \hat{X} = 180^\circ$ . These are co-interior angles between lines  $XY$  and  $WZ$ . Therefore  $XY \parallel WZ$ .

Both pairs of opposite sides of the quadrilateral are parallel, therefore  $WXYZ$  is a parallelogram.

## Investigation: Proving a quadrilateral is a parallelogram

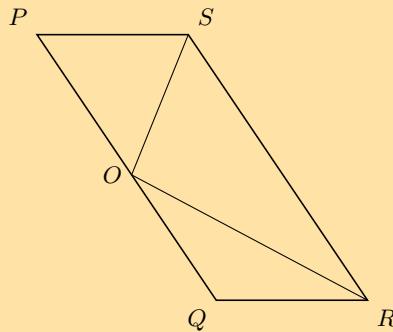
1. Prove that if both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
2. Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
3. Prove that if one pair of opposite sides of a quadrilateral are both equal and parallel, then the quadrilateral is a parallelogram.

A quadrilateral is a parallelogram if:

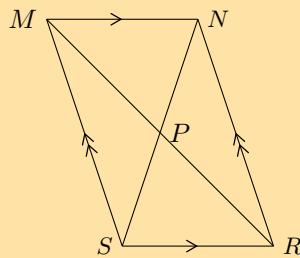
- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal.
- Both pairs of opposite angles are equal.
- The diagonals bisect each other.
- One pair of opposite sides are both equal and parallel.

### Exercise 7 – 3:

1.  $PQRS$  is a parallelogram.  $PS = OS$  and  $QO = QR$ .  $\hat{SOR} = 96^\circ$  and  $\hat{QOR} = x$ .



- a) Find with reasons, two other angles equal to  $x$ .
  - b) Write  $\hat{P}$  in terms of  $x$ .
  - c) Calculate the value of  $x$ .
2. Prove that the diagonals of parallelogram  $MNRS$  bisect one another at  $P$ .



Hint: Use congruency.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'. 1. [2G72](#) 2. [2G73](#)



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**DEFINITION:** Rectangle

A rectangle is a parallelogram that has all four angles equal to  $90^\circ$ .

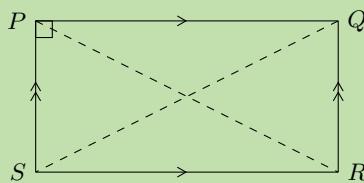
A rectangle has all the properties of a parallelogram:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.

It also has the following special property:

**Worked example 5: Special property of a rectangle****QUESTION**

$PQRS$  is a rectangle. Prove that the diagonals are of equal length.

**SOLUTION**

**Step 1: Connect  $P$  to  $R$  and  $Q$  to  $S$  to form  $\triangle PSR$  and  $\triangle QRS$**

**Step 2: Use the definition of a rectangle to fill in on the diagram all equal angles and sides**

**Step 3: Prove  $\triangle PSR \cong \triangle QRS$**

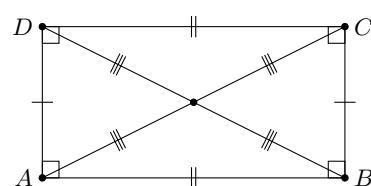
In  $\triangle PSR$  and  $\triangle QRS$ :

$$\begin{aligned}
 PS &= QR && \text{(opp sides of rectangle)} \\
 SR &\text{ and} && \text{amp; (common side)} \\
 \hat{P}SR &= \hat{Q}RS = 90^\circ && \text{(\angle s of rectangle)} \\
 \therefore \triangle PSR &\cong \triangle QRS && \text{(RHS)} \\
 \text{Therefore } PR &= QS
 \end{aligned}$$

The diagonals of a rectangle are of equal length.

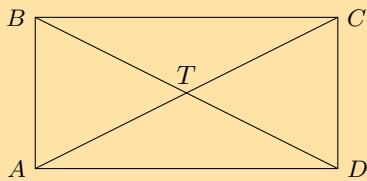
Summary of the properties of a rectangle:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are of equal length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- Diagonals are equal in length.
- All interior angles are equal to  $90^\circ$



### Exercise 7 – 4:

1.  $ABCD$  is a quadrilateral. Diagonals  $AC$  and  $BD$  intersect at  $T$ .  $AC = BD$ ,  $AT = TC$ ,  $DT = TB$ .



Prove that:

- $ABCD$  is a parallelogram
- $ABCD$  is a rectangle

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## Rhombus

EMA62

### DEFINITION: Rhombus

A rhombus is a parallelogram with all four sides of equal length.

A rhombus has all the properties of a parallelogram:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.

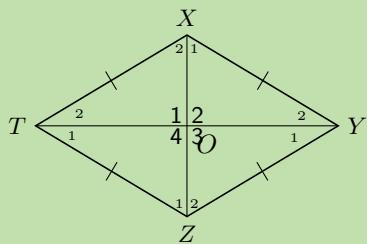
It also has two special properties:

### Worked example 6: Special properties of a rhombus

#### QUESTION

$XYZT$  is a rhombus. Prove that:

1. the diagonals bisect each other perpendicularly;
2. the diagonals bisect the interior angles.



#### SOLUTION

**Step 1:** Use the definition of a rhombus to fill in on the diagram all equal angles and sides

**Step 2:** Prove  $\triangle XTO \cong \triangle ZTO$

$$\begin{aligned}
 XT &= ZT && \text{(sides of rhombus)} \\
 TO &\text{ and} \\
 XO &= OZ && \text{(common side)} \\
 \therefore \triangle XTO &\equiv \triangle ZTO && \text{(diags of rhombus)} \\
 \therefore \hat{O}_1 &= \hat{O}_4 && \text{(SSS)} \\
 \text{But } \hat{O}_1 + \hat{O}_4 &= 180^\circ && (\angle s \text{ on a str line}) \\
 \therefore \hat{O}_1 &= \hat{O}_4 = 90^\circ
 \end{aligned}$$

We can further conclude that  $\hat{O}_1 = \hat{O}_2 = \hat{O}_3 = \hat{O}_4 = 90^\circ$ .

Therefore the diagonals bisect each other perpendicularly.

### Step 3: Use properties of congruent triangles to prove diagonals bisect interior angles

$$\begin{aligned}
 \hat{X}_2 &= \hat{Z}_1 && (\triangle XTO \equiv \triangle ZTO) \\
 \text{and } \hat{X}_2 &= \hat{Z}_2 && (\text{alt } \angle s; XT \parallel YZ) \\
 \therefore \hat{Z}_1 &= \hat{Z}_2
 \end{aligned}$$

Therefore diagonal  $XZ$  bisects  $\hat{Z}$ . Similarly, we can show that  $XZ$  also bisects  $\hat{X}$ ; and that diagonal  $TY$  bisects  $\hat{T}$  and  $\hat{Y}$ .

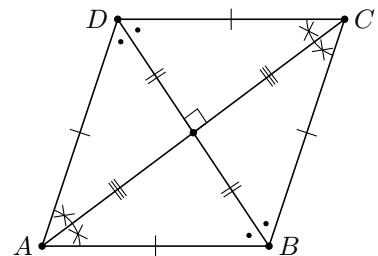
We conclude that the diagonals of a rhombus bisect the interior angles.

To prove a parallelogram is a rhombus, we need to show any one of the following:

- All sides are equal in length.
- Diagonals intersect at right angles.
- Diagonals bisect interior angles.

Summary of the properties of a rhombus:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- All sides are equal in length.
- The diagonals bisect each other at  $90^\circ$
- The diagonals bisect both pairs of opposite angles.



## Square

EMA63

### DEFINITION: Square

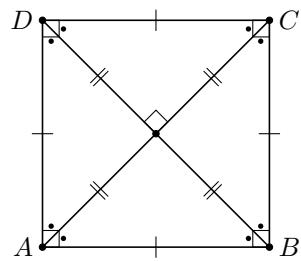
A square is a rhombus with all four interior angles equal to  $90^\circ$

OR

A square is a rectangle with all four sides equal in length.

A square has all the properties of a rhombus:

- Both pairs of opposite sides are parallel.
- Both pairs of opposite sides are equal in length.
- Both pairs of opposite angles are equal.
- Both diagonals bisect each other.
- All sides are equal in length.
- The diagonals bisect each other at  $90^\circ$
- The diagonals bisect both pairs of opposite angles.



It also has the following special properties:

- All interior angles equal  $90^\circ$ .
- Diagonals are equal in length.
- Diagonals bisect both pairs of interior opposite angles (i.e. all are  $45^\circ$ ).

To prove a parallelogram is a square, we need to show either one of the following:

- It is a rhombus (all four sides of equal length) with interior angles equal to  $90^\circ$ .
- It is a rectangle (interior angles equal to  $90^\circ$ ).

## Trapezium

EMA64

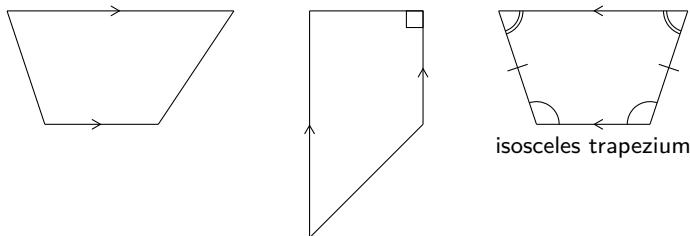
### **DEFINITION:** Trapezium

A trapezium is a quadrilateral with one pair of opposite sides parallel.

### **NOTE:**

A trapezium is sometimes called a trapezoid.

Some examples of trapeziums are given below:



## Kite

EMA65

### **DEFINITION:** Kite

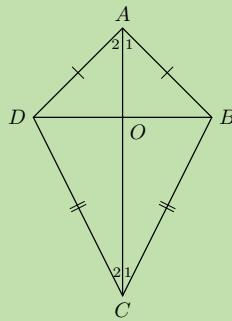
A kite is a quadrilateral with two pairs of adjacent sides equal.

### Worked example 7: Properties of a kite

### QUESTION

$ABCD$  is a kite with  $AD = AB$  and  $CD = CB$ . Prove that:

1.  $\hat{ADC} = \hat{ABC}$
2. Diagonal  $AC$  bisects  $\hat{A}$  and  $\hat{C}$



### SOLUTION

**Step 1: Prove  $\triangle ADC \cong \triangle ABC$**

In  $\triangle ADC$  and  $\triangle ABC$ :

$$\begin{aligned}
 AD &= AB && (\text{given}) \\
 CD &= CB && (\text{given}) \\
 AC & && (\text{common side}) \\
 \therefore \triangle ADC &\cong \triangle ABC && (\text{SSS}) \\
 \therefore \hat{A}DC &= \hat{A}BC
 \end{aligned}$$

Therefore one pair of opposite angles are equal in kite  $ABCD$ .

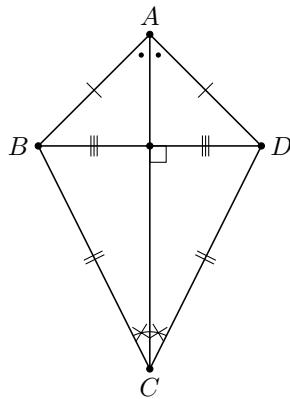
**Step 2: Use properties of congruent triangles to prove  $AC$  bisects  $\hat{A}$  and  $\hat{C}$**

$$\begin{aligned}
 \hat{A}_1 &= \hat{A}_2 && (\triangle ADC \cong \triangle ABC) \\
 \text{and } \hat{C}_1 &= \hat{C}_2 && (\triangle ADC \cong \triangle ABC)
 \end{aligned}$$

Therefore diagonal  $AC$  bisects  $\hat{A}$  and  $\hat{C}$ .

We conclude that the diagonal between the equal sides of a kite bisects the two interior angles and is an axis of symmetry.

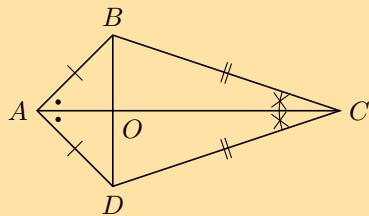
Summary of the properties of a kite:



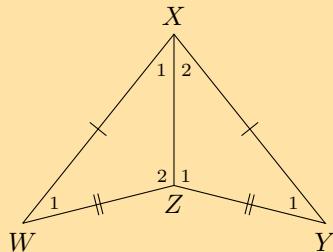
- Diagonal between equal sides bisects the other diagonal.
- One pair of opposite angles are equal (the angles between unequal sides).
- Diagonal between equal sides bisects the interior angles and is an axis of symmetry.
- Diagonals intersect at  $90^\circ$

### Exercise 7 – 5:

1. Use the sketch of quadrilateral  $ABCD$  to prove the diagonals of a kite are perpendicular to each other.



2. Explain why quadrilateral  $WXYZ$  is a kite. Write down all the properties of quadrilateral  $WXYZ$ .



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'. 1. [2G75](#) 2. [2G76](#)



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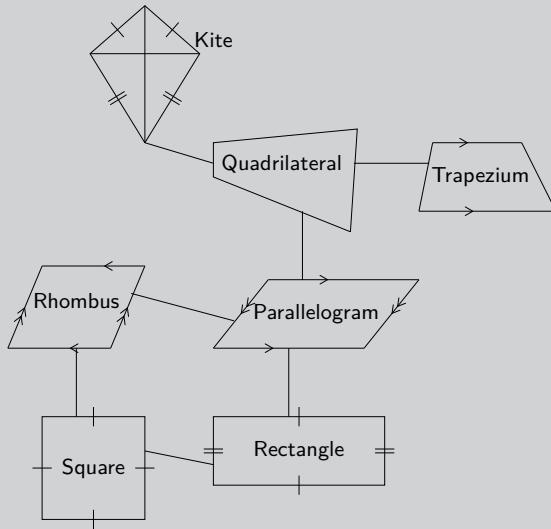
#### VISIT:

This video provides a summary of the different types of quadrilaterals and their properties.

► See video: [2G77](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

#### Investigation: Relationships between the different quadrilaterals

Heather has drawn the following diagram to illustrate her understanding of the relationships between the different quadrilaterals. The following diagram summarises the different types of special quadrilaterals.



1. Explain her possible reasoning for structuring the diagram as shown.
2. Design your own diagram to show the relationships between the different quadrilaterals and write a short explanation of your design.

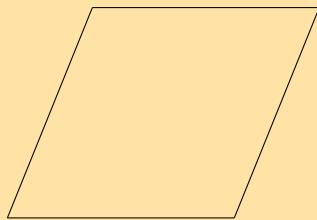
#### Exercise 7 – 6:

1. The following shape is drawn **to scale**:



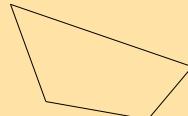
Give the most specific name for the shape.

2. The following shape is drawn **to scale**:

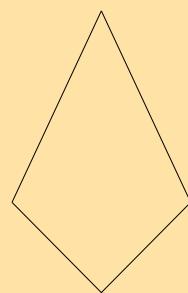


Give the most specific name for the shape.

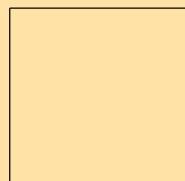
3. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale.



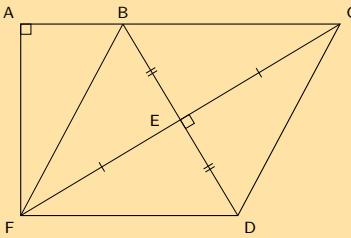
4. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale.



5. Based on the shape that you see list the all the names of the shape. The figure is drawn to scale.



6. Find the area of  $ACDF$  if  $AB = 8$ ,  $BF = 17$ ,  $FE = EC$ ,  $BE = ED$ ,  $\hat{A} = 90^\circ$ ,  $C\hat{E}D = 90^\circ$ .



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. 2G78   2. 2G79   3. 2G7B   4. 2G7C   5. 2G7D   6. 2G7F



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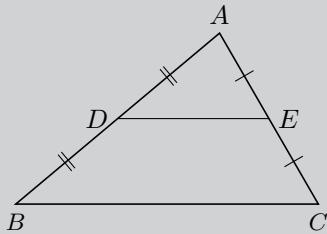


[m.everythingmaths.co.za](http://m.everythingmaths.co.za)

## 7.4 The mid-point theorem

EMA66

### Investigation: Proving the mid-point theorem

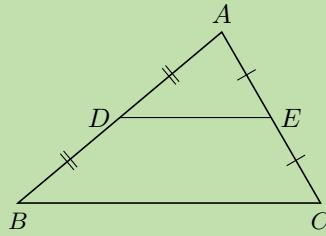


1. Draw a large scalene triangle on a sheet of paper.
2. Name the vertices  $A$ ,  $B$  and  $C$ . Find the mid-points ( $D$  and  $E$ ) of two sides and connect them.
3. Cut out  $\triangle ABC$  and cut along line  $DE$ .
4. Place  $\triangle ADE$  on quadrilateral  $BDEC$  with vertex  $E$  on vertex  $C$ . Write down your observations.
5. Shift  $\triangle ADE$  to place vertex  $D$  on vertex  $B$ . Write down your observations.
6. What do you notice about the lengths  $DE$  and  $BC$ ?
7. Make a conjecture regarding the line joining the mid-point of two sides of a triangle.

### Worked example 8: Mid-point theorem

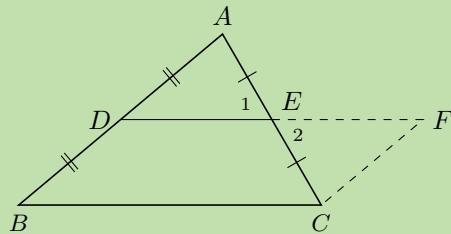
#### QUESTION

Prove that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.



### SOLUTION

**Step 1: Extend  $DE$  to  $F$  so that  $DE = EF$  and join  $FC$**



**Step 2: Prove  $BCFD$  is a parallelogram**

In  $\triangle EAD$  and  $\triangle ECF$ :

$$\begin{aligned} \hat{E}_1 &= \hat{E}_2 && (\text{vert opp } \angle s =) \\ AE &= CE && (\text{given}) \\ DE &= EF && (\text{by construction}) \\ \therefore \triangle EAD &\equiv \triangle ECF && (\text{SAS}) \\ \therefore A\hat{D}E &= C\hat{F}E \end{aligned}$$

But these are alternate interior angles, therefore  $BD \parallel FC$

$$\begin{aligned} BD &= DA && (\text{given}) \\ DA &= FC && (\triangle EAD \equiv \triangle ECF) \\ \therefore BD &= FC \\ \therefore BCFD \text{ is a parallelogram} && (\text{one pair opp. sides } = \text{ and } \parallel) \end{aligned}$$

Therefore  $DE \parallel BC$ .

We conclude that the line joining the two mid-points of two sides of a triangle is parallel to the third side.

**Step 3: Use properties of parallelogram  $BCFD$  to prove that  $DE = \frac{1}{2}BC$**

$$\begin{aligned} DF &= BC && (\text{opp sides } \parallel \text{m}) \\ \text{and } DF &= 2(DE) && (\text{by construction}) \\ \therefore 2DE &= BC \\ \therefore DE &= \frac{1}{2}BC \end{aligned}$$

We conclude that the line joining the mid-point of two sides of a triangle is equal to half the length of the third side.

## Converse

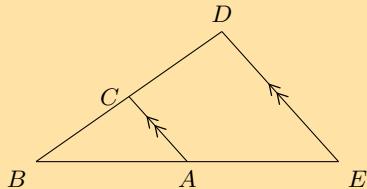
The converse of this theorem states: If a line is drawn through the mid-point of a side of a triangle parallel to the second side, it will bisect the third side.

### VISIT:

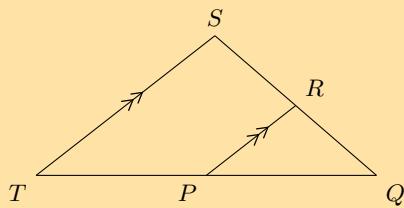
You can use [GeoGebra](#) to show that the converse of the mid-point theorem is true.

### Exercise 7 – 7:

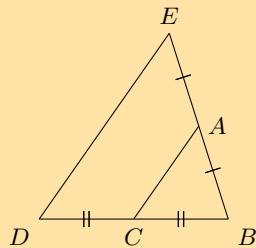
- Points  $C$  and  $A$  are the mid-points on lines  $BD$  and  $BE$ . Study  $\triangle EDB$  carefully. Identify the third side of this triangle, using the information as shown, together with what you know about the mid-point theorem. Name the third side by its endpoints, e.g.,  $FG$ .



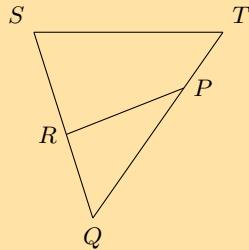
- Points  $R$  and  $P$  are the mid-points on lines  $QS$  and  $QT$ . Study  $\triangle TSQ$  carefully. Identify the third side of this triangle, using the information as shown, together with what you know about the mid-point theorem. Name the third side by its endpoints, e.g.,  $FG$ .



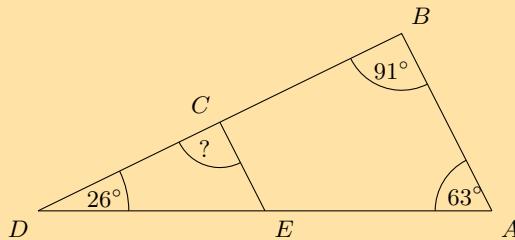
- Points  $C$  and  $A$  are given on the lines  $BD$  and  $BE$ . Study the triangle carefully, then identify and name the parallel lines.



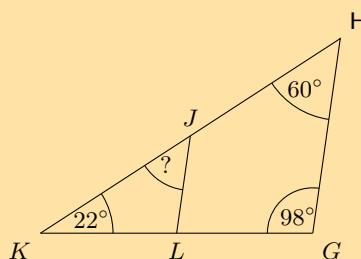
- Points  $R$  and  $P$  are given on the lines  $QS$  and  $QT$ . Study the triangle carefully, then identify and name the parallel lines.



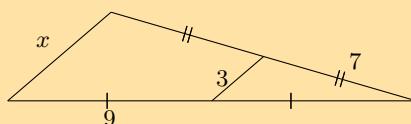
5. The figure below shows a large triangle with vertices  $A$ ,  $B$  and  $D$ , and a smaller triangle with vertices at  $C$ ,  $D$  and  $E$ . Point  $C$  is the mid-point of  $BD$  and point  $E$  is the mid-point of  $AD$ .



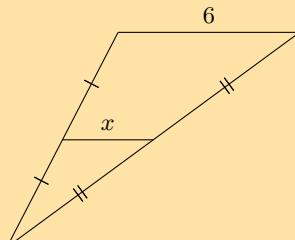
- a) Three angles are given:  $\hat{A} = 63^\circ$ ,  $\hat{B} = 91^\circ$  and  $\hat{D} = 26^\circ$ ; determine the value of  $D\hat{C}E$ .
- b) The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).
- $$\triangle DEC \parallel\!\!\!\parallel \triangle ?$$
6. The figure below shows a large triangle with vertices  $G$ ,  $H$  and  $K$ , and a smaller triangle with vertices at  $J$ ,  $K$  and  $L$ . Point  $J$  is the mid-point of  $HK$  and point  $L$  is the mid-point of  $GK$ .



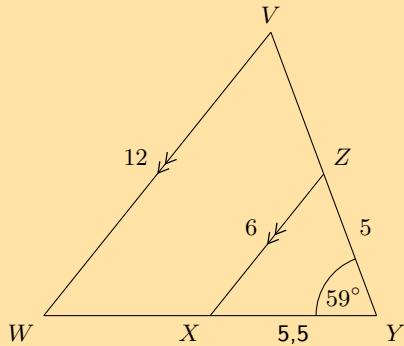
- a) Three angles are given:  $\hat{G} = 98^\circ$ ,  $\hat{H} = 60^\circ$ , and  $\hat{K} = 22^\circ$ ; determine the value of  $K\hat{J}L$ .
- b) The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).
- $$\triangle HKG \parallel\!\!\!\parallel \triangle ?$$
7. Consider the triangle in the diagram below. There is a line crossing through a large triangle. Notice that some lines in the figure are marked as equal to each other. One side of the triangle has a given length of 3. Determine the value of  $x$ .



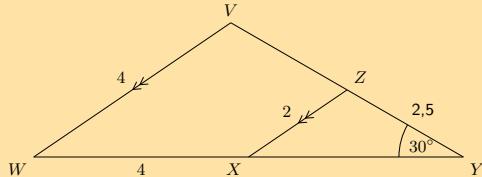
8. Consider the triangle in the diagram below. There is a line crossing through a large triangle. Notice that some lines in the figure are marked as equal to each other. One side of the triangle has a given length of 6. Determine the value of  $x$ .



9. In the figure below,  $VW \parallel ZX$ , as labelled. Furthermore, the following lengths and angles are given:  $VW = 12$ ;  $ZX = 6$ ;  $XY = 5,5$ ;  $YZ = 5$  and  $\hat{V} = 59^\circ$ . The figure is drawn to scale. Determine the length of  $WY$ .

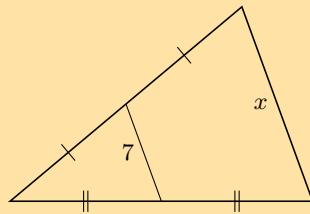


10. In the figure below,  $VW \parallel ZX$ , as labelled. Furthermore, the following lengths and angles are given:  $VW = 4$ ;  $ZX = 2$ ;  $WX = 4$ ;  $YZ = 3,5$  and  $\hat{Y} = 30^\circ$ . The figure is drawn to scale. Determine the length of  $XY$ .

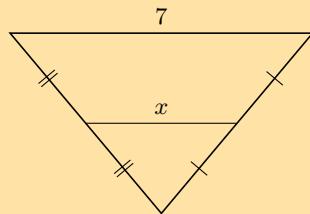


11. Find  $x$  and  $y$  in the following:

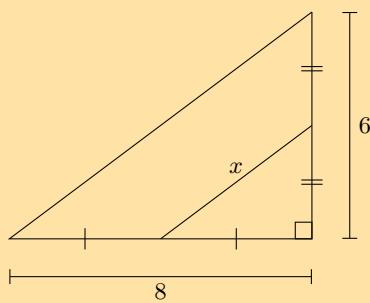
a)



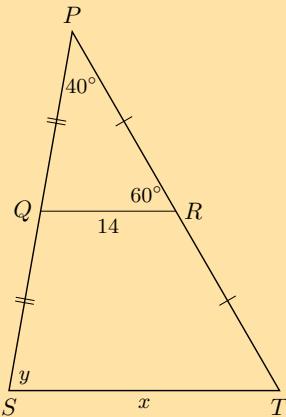
b)



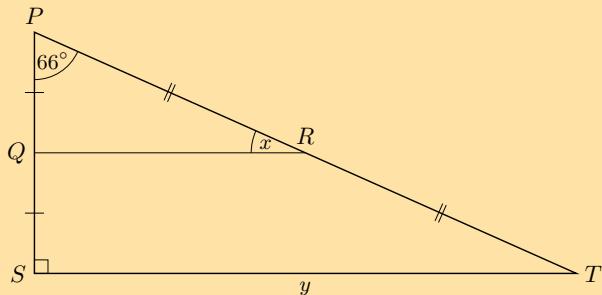
c)



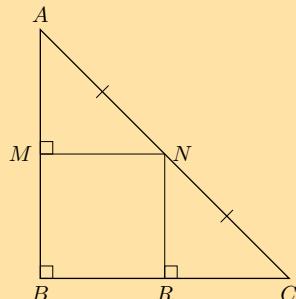
d)



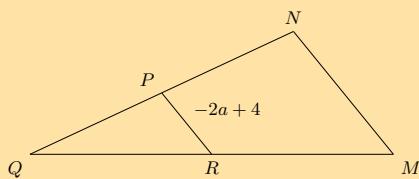
- e) In the following diagram  $PQ = 2,5$  and  $RT = 6,5$ .



12. Show that  $M$  is the mid-point of  $AB$  and that  $MN = RC$ .

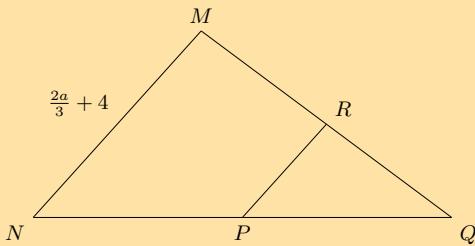


13. In the diagram below,  $P$  is the mid-point of  $NQ$  and  $R$  is the mid-point of  $MQ$ . The segment inside of the large triangle is labelled with a length of  $-2a + 4$ .

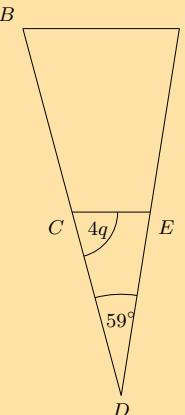


- a) Calculate the value of  $MN$  in terms of  $a$ .  
b) You are now told that  $MN$  has a length of 18. What is the value of  $a$ ? Give your answer as a fraction.

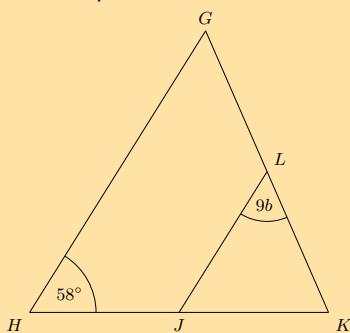
14. In the diagram below,  $P$  is the mid-point of  $NQ$  and  $R$  is the mid-point of  $MQ$ . One side of the triangle has a given length of  $\frac{2a}{3} + 4$ .



- a) Find the value of  $PR$  in terms of  $a$ .  
 b) You are now told that  $PR$  has a length of 8. What is the value of  $a$ ?
15. The figure below shows  $\triangle ABD$  crossed by  $EC$ . Points  $C$  and  $E$  bisect their respective sides of the triangle.



- a) The angles  $\hat{D} = 59^\circ$  and  $E\hat{C}D = 4q$  are given; determine the value of  $\hat{A}$  in terms of  $q$ .  
 b) You are now told that  $E\hat{C}D$  has a measure of  $72^\circ$ . Calculate for the value of  $q$ .
16. The figure below shows  $\triangle GHK$  crossed by  $LJ$ . Points  $J$  and  $L$  bisect their respective sides of the triangle.



- a) Given the angles  $\hat{H} = 58^\circ$  and  $K\hat{L}J = 9b$ , determine the value of  $\hat{K}$  in terms of  $b$ .  
 b) You are now told that  $\hat{K}$  has a measure of  $74^\circ$ . Solve for the value of  $b$ . Give your answer as a fraction.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |                          |                          |                           |                           |                           |                           |                           |                          |
|--------------------------|--------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|--------------------------|
| 1. <a href="#">2G7G</a>  | 2. <a href="#">2G7H</a>  | 3. <a href="#">2G7J</a>   | 4. <a href="#">2G7K</a>   | 5. <a href="#">2G7M</a>   | 6. <a href="#">2G7N</a>   | 7. <a href="#">2G7P</a>   | 8. <a href="#">2G7Q</a>  |
| 9. <a href="#">2G7R</a>  | 10. <a href="#">2G7S</a> | 11a. <a href="#">2G7T</a> | 11b. <a href="#">2G7V</a> | 11c. <a href="#">2G7W</a> | 11d. <a href="#">2G7X</a> | 11e. <a href="#">2G7Y</a> | 12. <a href="#">2G7Z</a> |
| 13. <a href="#">2G82</a> | 14. <a href="#">2G83</a> | 15. <a href="#">2G84</a>  | 16. <a href="#">2G85</a>  |                           |                           |                           |                          |



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► See presentation: [2G86](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

- A quadrilateral is a closed shape consisting of four straight line segments.
- A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
  - Both pairs of opposite sides are equal in length.
  - Both pairs of opposite angles are equal.
  - Both diagonals bisect each other.
- A rectangle is a parallelogram that has all four angles equal to  $90^\circ$ 
  - Both pairs of opposite sides are parallel.
  - Both pairs of opposite sides are equal in length.
  - The diagonals bisect each other.
  - The diagonals are equal in length.
  - All interior angles are equal to  $90^\circ$ .
- A rhombus is a parallelogram that has all four sides equal in length.
  - Both pairs of opposite sides are parallel.
  - All sides are equal in length.
  - Both pairs of opposite angles are equal.
  - The diagonals bisect each other at  $90^\circ$ .
  - The diagonals of a rhombus bisect both pairs of opposite angles.
- A square is a rhombus that has all four interior angles equal to  $90^\circ$ .
  - Both pairs of opposite sides are parallel.
  - The diagonals bisect each other at  $90^\circ$ .
  - All interior angles are equal to  $90^\circ$ .
  - The diagonals are equal in length.
  - The diagonals bisect both pairs of interior opposite angles (i.e. all are  $45^\circ$ )
- A trapezium is a quadrilateral with one pair of opposite sides parallel.
- A kite is a quadrilateral with two pairs of adjacent sides equal.
  - One pair of opposite angles are equal (the angles are between unequal sides).
  - The diagonal between equal sides bisects the other diagonal.
  - The diagonal between equal sides bisects the interior angles.
  - The diagonals intersect at  $90^\circ$ .
- The mid-point theorem states that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.

#### End of chapter Exercise 7 – 8:

1. Identify the types of angles shown below:

a)



b)



c)



d)



e)



f) An angle of  $91^\circ$

g) An angle of  $180^\circ$

h) An angle of  $210^\circ$

2. Assess whether the following statements are true or false. If the statement is false, explain why:

a) A trapezium is a quadrilateral with two pairs of opposite sides that are parallel.

b) Both diagonals of a parallelogram bisect each other.

c) A rectangle is a parallelogram that has all interior angles equal to  $90^\circ$ .

d) Two adjacent sides of a rhombus have different lengths.

e) The diagonals of a kite intersect at right angles.

f) All squares are parallelograms.

g) A rhombus is a kite with a pair of equal, opposite sides.

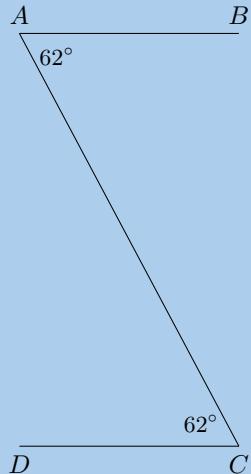
h) The diagonals of a parallelogram are axes of symmetry.

i) The diagonals of a rhombus are equal in length.

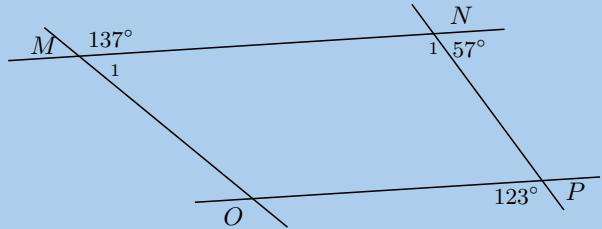
j) Both diagonals of a kite bisect the interior angles.

3. Find all pairs of parallel lines in the following figures, giving reasons in each case.

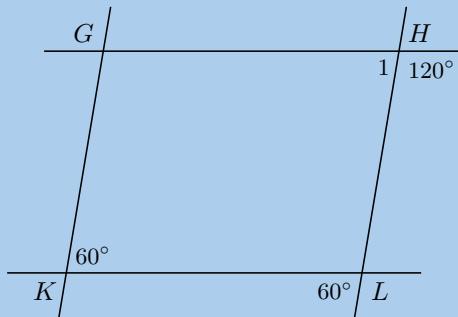
a)



b)

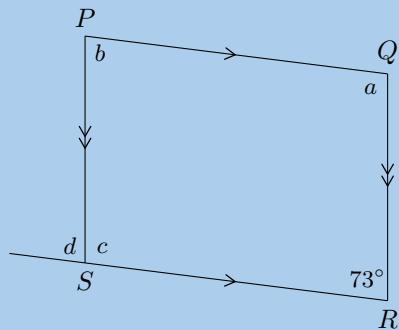


c)

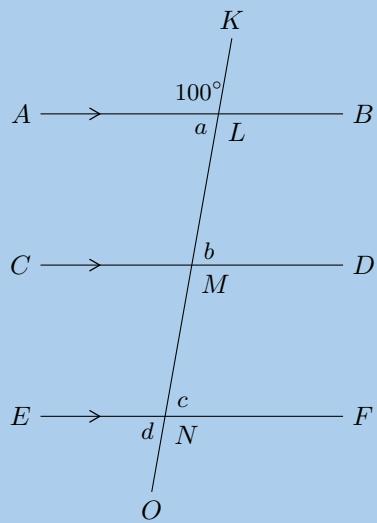


4. Find angles  $a$ ,  $b$ ,  $c$  and  $d$  in each case, giving reasons:

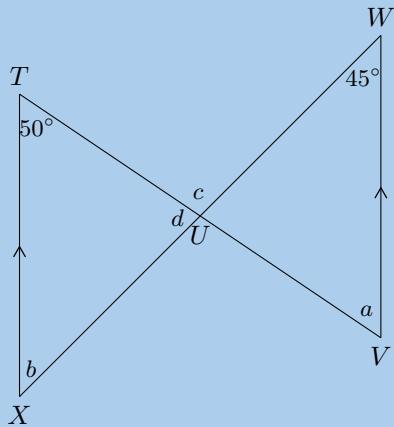
a)



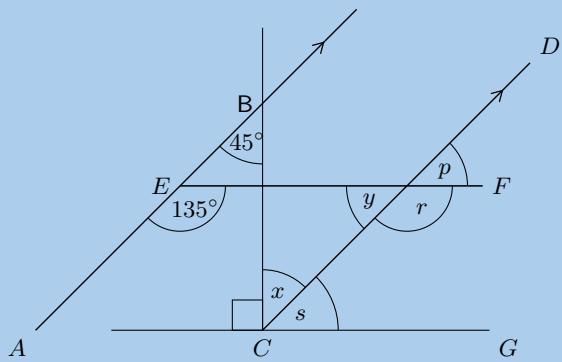
b)



c)



5. Given the figure below.



- Find each of the unknown angles marked in the figure below. Find a reason that leads to the answer in a single step.
- Based on the results for the angles above, is  $EF \parallel CG$ ?

6. Given the following diagrams:

Diagram A

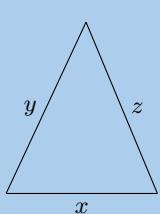
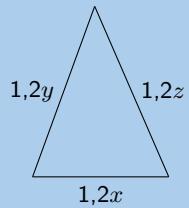
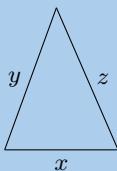


Diagram B



Which diagram correctly gives a pair of similar triangles?

7. Given the following diagrams:

Diagram A

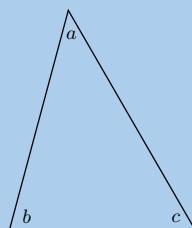
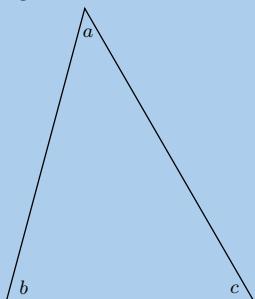
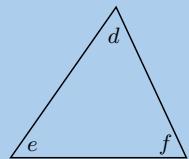
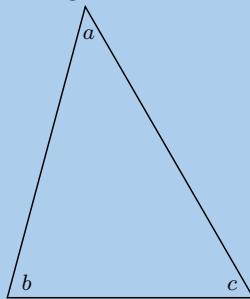
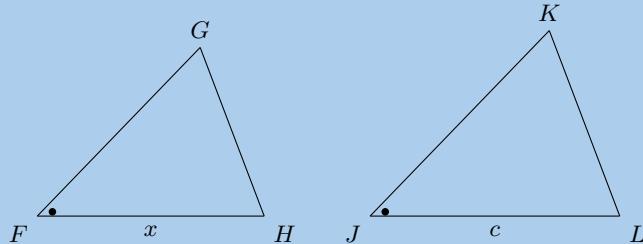


Diagram B



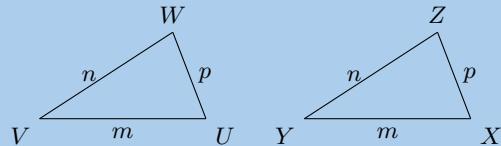
Which diagram correctly gives a pair of similar triangles?

8. Have a look at the following triangles, which are drawn to scale:



Are the triangles congruent? If so state the reason and use correct notation to state that they are congruent.

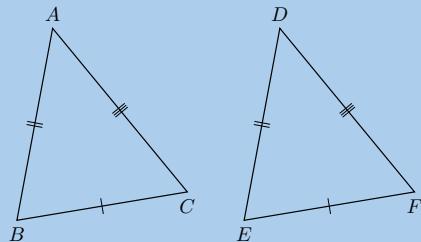
9. Have a look at the following triangles, which are drawn to scale:



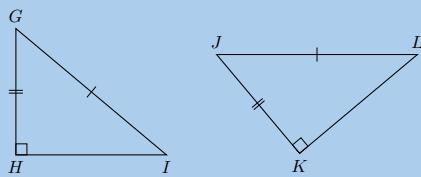
Are the triangles congruent? If so state the reason and use correct notation to state that they are congruent.

10. Say which of the following pairs of triangles are congruent with reasons.

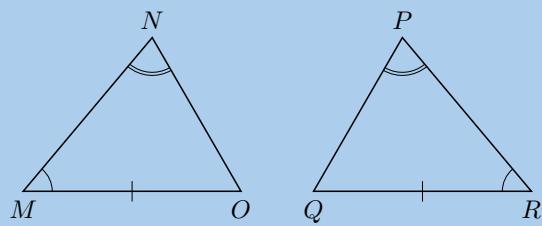
a)



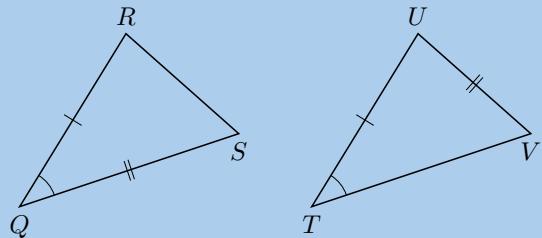
b)



c)

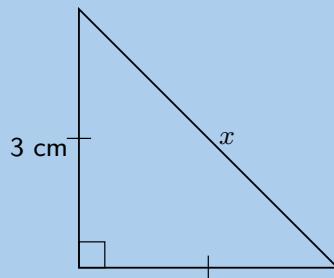


d)

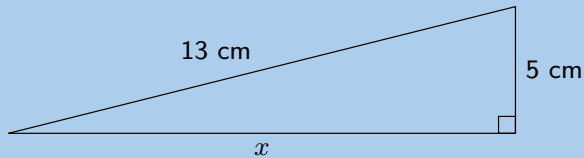


11. Using the theorem of Pythagoras, calculate the length  $x$ :

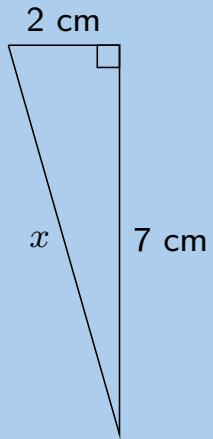
a)



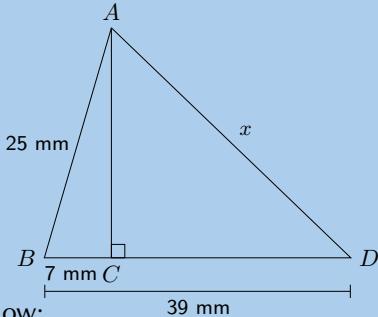
b)



c)

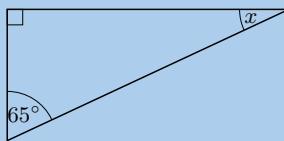


d)



12. Calculate  $x$  and  $y$  in the diagrams below:

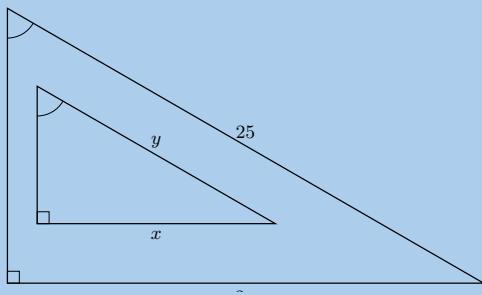
a)



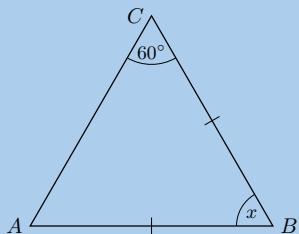
b)



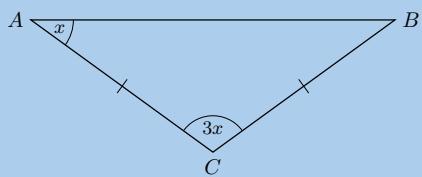
c)



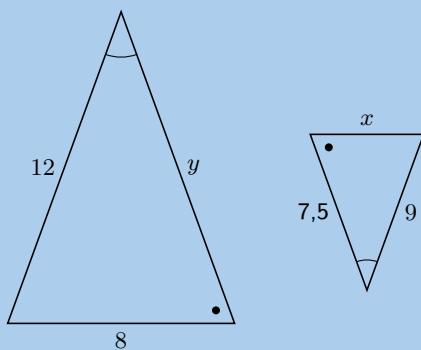
d)



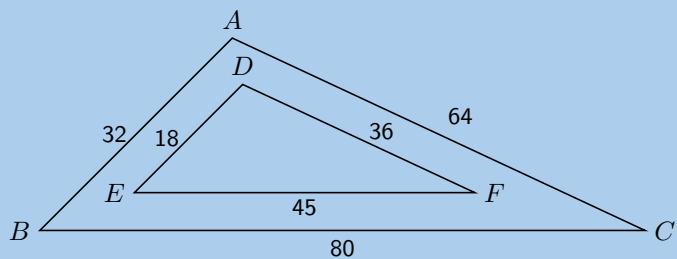
e)



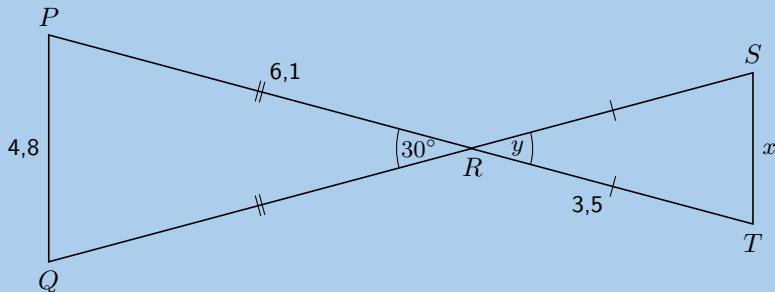
f)



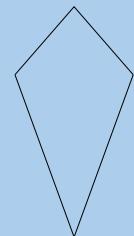
13. Consider the diagram below. Is  $\triangle ABC \sim \triangle DEF$ ? Give reasons for your answer.



14. Explain why  $\triangle PQR$  is similar to  $\triangle TSR$  and calculate the values of  $x$  and  $y$ .

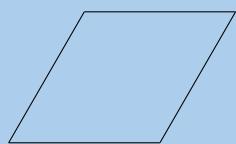


15. The following shape is drawn to scale:

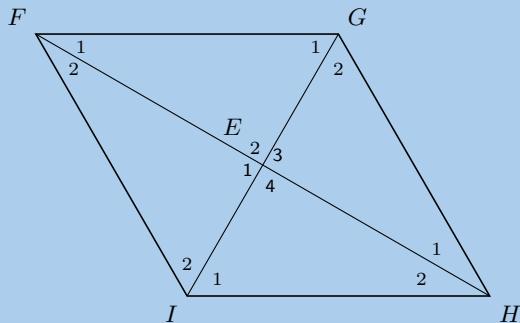


Give the most specific name for the shape.

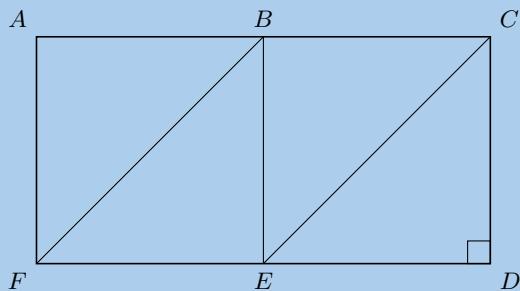
16. Based on the shape that you see list all the names of the shape. The figure is drawn to scale.



17.  $FGHI$  is a rhombus.  $\hat{F}_1 = 3x + 20^\circ$ ;  $\hat{G}_1 = x + 10^\circ$ . Determine the value of  $x$ .

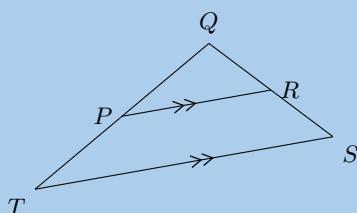


18. In the diagram below,  $AB = BC = CD = DE = EF = FA = BE$ .

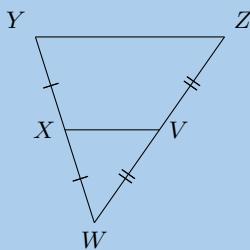


Name:

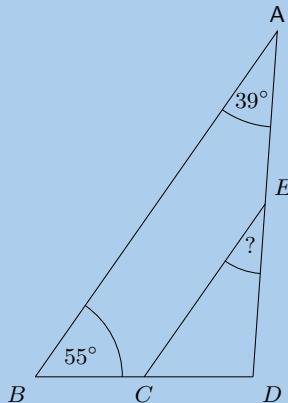
- a) 3 rectangles
  - b) 4 parallelograms
  - c) 2 trapeziums
  - d) 2 rhombi
19. Points  $R$  and  $P$  are the mid-points on lines  $QS$  and  $QT$ . Study  $\triangle TSQ$  carefully. Identify the third side of this triangle, using the information as shown, together with what you know about the mid-point theorem. (Name the third side by its endpoints, e.g.,  $FG$ .)



20. Points  $X$  and  $V$  are given on the segments  $WY$  and  $WZ$ . Study the triangle carefully, then identify and name the parallel line segments.

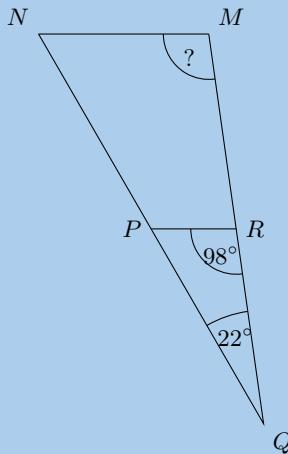


21. The figure below shows a large triangle with vertices  $A$ ,  $B$  and  $D$ , and a smaller triangle with vertices at  $C$ ,  $D$  and  $E$ . Point  $C$  is the mid-point of  $BD$  and point  $E$  is the mid-point of  $AD$ .



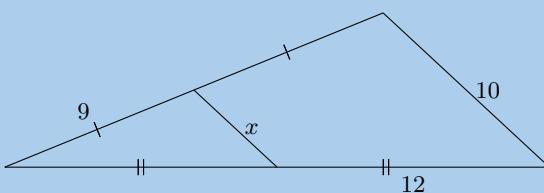
- The angles  $\hat{A} = 39^\circ$  and  $\hat{B} = 55^\circ$  are given; determine the value of  $D\hat{E}C$ .
- The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).  
 $\triangle DEC \parallel\!\!\!\parallel \triangle ?$

22. The figure below shows a large triangle with vertices  $M$ ,  $N$  and  $Q$ , and a smaller triangle with vertices at  $P$ ,  $Q$  and  $R$ . Point  $P$  is the mid-point of  $NQ$  and point  $R$  is the mid-point of  $MQ$ .

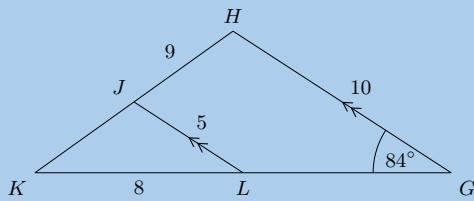


- With the two angles given,  $\hat{Q} = 22^\circ$  and  $Q\hat{R}P = 98^\circ$ , determine the value of  $\hat{M}$ .
- The two triangles in this question are similar triangles. Complete the following statement correctly by giving the three vertices in the correct order (there is only one correct answer).  
 $\triangle QMN \parallel\!\!\!\parallel \triangle ?$

23. Consider the triangle in the diagram below. There is a line segment crossing through a large triangle. Notice that some segments in the figure are marked as equal to each other. One side of the triangle has a given length of 10. Determine the value of  $x$ .

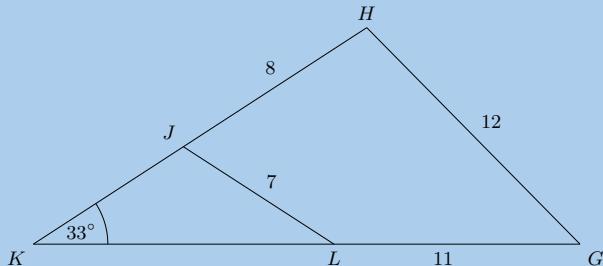


24. In the figure below,  $GH \parallel LJ$ , as labelled. Furthermore, the following lengths and angles are given:  $GH = 10$ ;  $LJ = 5$ ;  $HJ = 9$ ;  $KL = 8$  and  $\hat{G} = 84^\circ$ . The figure is drawn to scale.



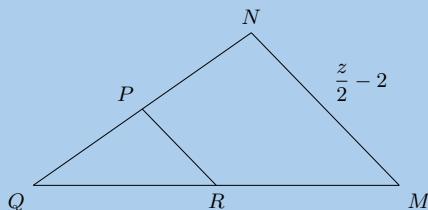
Calculate the length of  $JK$ .

25. The figure below shows triangle  $GHK$  with the smaller triangle  $JKL$  sitting inside of it. Furthermore, the following lengths and angles are given:  $GH = 12$ ;  $LJ = 7$ ;  $HJ = 8$ ;  $LG = 11$ ;  $\hat{K} = 33^\circ$ . The figure is drawn to scale.



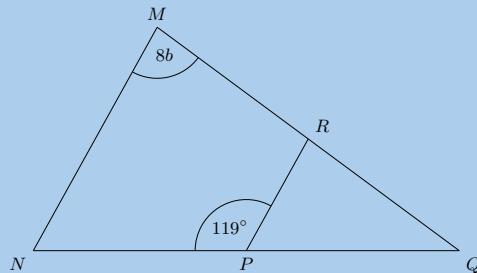
Find the length of  $KL$ .

26. In the diagram below,  $P$  is the mid-point of  $NQ$  and  $R$  is the mid-point of  $MQ$ . One side of the triangle has a given length of  $\frac{z}{2} - 2$ .



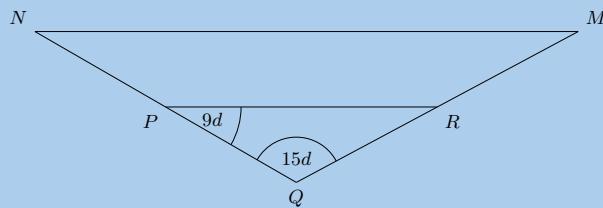
- Determine the value of  $PR$  in terms of  $z$ .
- You are now told that  $PR$  has a length of 2. What is the value of  $z$ ?

27. The figure below shows  $\triangle MNQ$  crossed by  $RP$ . Points  $P$  and  $R$  bisect their respective sides of the triangle.



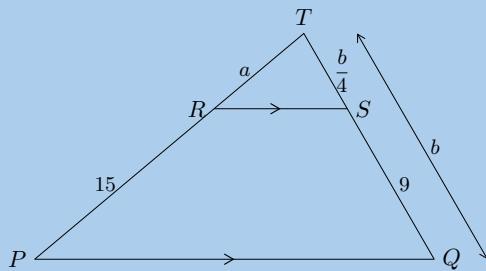
- With the two angles given,  $\hat{M} = 8b$  and  $\angle NPR = 119^\circ$ , determine the value of  $\hat{Q}$  in terms of  $b$ .
- You are now told that  $\hat{M}$  has a measure of  $76^\circ$ . Determine for the value of  $b$ . Give your answer as an exact fractional value.

28. The figure below shows  $\triangle MNQ$  crossed by  $RP$ . Points  $P$  and  $R$  bisect their respective sides of the triangle.

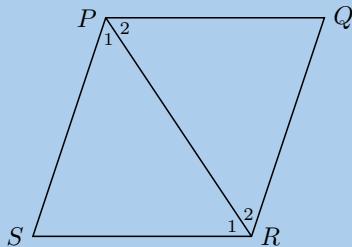


- The angles  $\hat{Q} = 15d$  and  $\hat{R}PQ = 9d$  are given in the large triangle; determine the value of  $\hat{M}$  in terms of  $d$ .
- You are now told that  $\hat{R}PQ$  has a measure of  $60^\circ$ . Solve for the value of  $d$ . Give your answer as an exact fractional value.

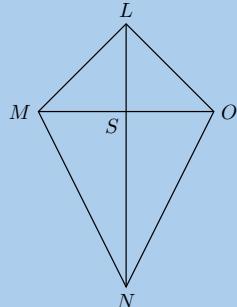
29. Calculate  $a$  and  $b$ :



30.  $\triangle PQR$  and  $\triangle PSR$  are equilateral triangles. Prove that  $PQRS$  is a rhombus.

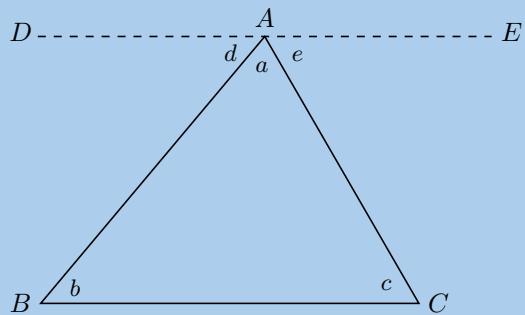


31.  $LMNO$  is a quadrilateral with  $LM = LO$  and diagonals that intersect at  $S$  such that  $MS = SO$ . Prove that:

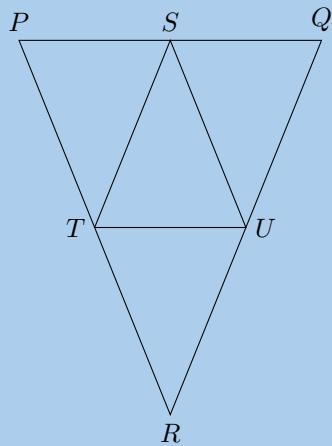


- $M\hat{L}S = S\hat{L}O$
- $\triangle LON \cong \triangle LMN$
- $MO \perp LN$

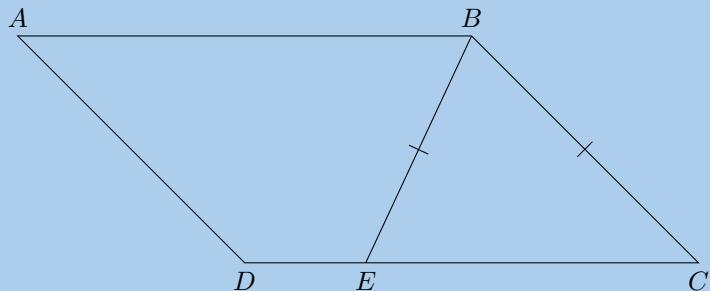
32. Using the figure below, show that the sum of the three angles in a triangle is  $180^\circ$ . Line  $DE$  is parallel to  $BC$ .



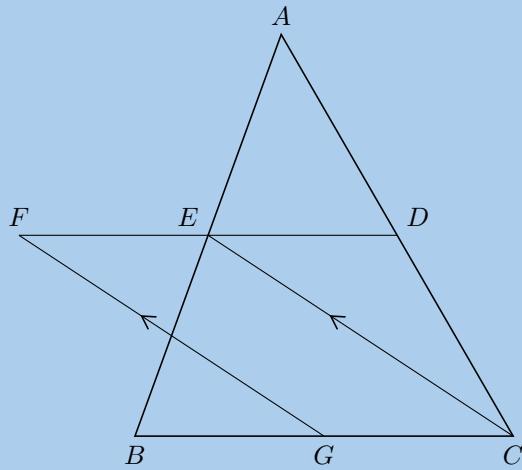
33.  $PQR$  is an isosceles triangle with  $PR = QR$ .  $S$  is the mid-point of  $PQ$ ,  $T$  is the mid-point of  $PR$  and  $U$  is the mid-point of  $RQ$ .



- a) Prove  $\triangle STU$  is also isosceles.
  - b) What type of quadrilateral is  $STRU$ ? Motivate your answer.
  - c) If  $R\hat{T}U = 68^\circ$  calculate, with reasons, the size of  $T\hat{S}U$ .
34.  $ABCD$  is a parallelogram.  $BE = BC$ . Prove that  $A\hat{B}E = B\hat{C}D$ .



35. In the diagram below,  $D$ ,  $E$  and  $G$  are the mid-points of  $AC$ ,  $AB$  and  $BC$  respectively.  $EC \parallel FG$ .



- a) Prove that  $FECG$  is a parallelogram.  
 b) Prove that  $FE = ED$ .

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1a. 2G87	1b. 2G88	1c. 2G89	1d. 2G8B	1e. 2G8C	1f. 2G8D
1g. 2G8F	1h. 2G8G	2a. 2G8H	2b. 2G8J	2c. 2G8K	2d. 2G8M
2e. 2G8N	2f. 2G8P	2g. 2G8Q	2h. 2G8R	2i. 2G8S	2j. 2G8T
3a. 2G8V	3b. 2G8W	3c. 2G8X	4a. 2G8Y	4b. 2G8Z	4c. 2G92
5. 2G93	6. 2G94	7. 2G95	8. 2G96	9. 2G97	10a. 2G98
10b. 2G99	10c. 2G9B	10d. 2G9C	11a. 2G9D	11b. 2G9F	11c. 2G9G
11d. 2G9H	12a. 2G9J	12b. 2G9K	12c. 2G9M	12d. 2G9N	12e. 2G9P
12f. 2G9Q	13. 2G9R	14. 2G9S	15. 2G9T	16. 2G9V	17. 2G9W
18. 2G9X	19. 2G9Y	20. 2G9Z	21. 2GB2	22. 2GB3	23. 2GB4
24. 2GB5	25. 2GB6	26. 2GB7	27. 2GB8	28. 2GB9	29. 2GBB
30. 2GBC	31. 2GBD	32. 2GBF	33. 2GBG	34. 2GBH	35. 2GBJ



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# CHAPTER



## *Analytical geometry*

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## 8 Analytical geometry

Analytical geometry is the study of geometric properties, relationships and measurement of points, lines and angles in the Cartesian plane. Geometrical shapes are defined using a coordinate system and algebraic principles. Some consider the introduction of analytical geometry, also called coordinate or Cartesian geometry, to be the beginning of modern mathematics.

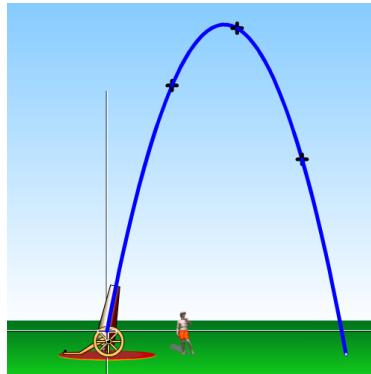


Figure 8.1: The motion of a projectile can be plotted on the Cartesian plane. Animators use this information to help them create animations.

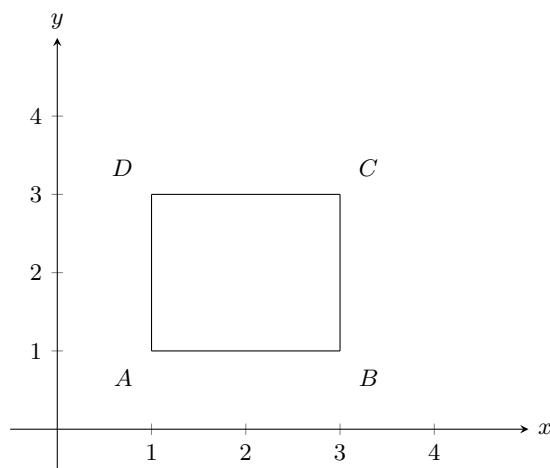
### 8.1 Drawing figures on the Cartesian plane

EMA68

If we are given the coordinates of the vertices of a figure, we can draw the figure on the Cartesian plane. For example, quadrilateral  $ABCD$  with coordinates  $A(1; 1)$ ,  $B(3; 1)$ ,  $C(3; 3)$  and  $D(1; 3)$ .

**NOTE:**

You might also see coordinates written as  $A(1, 1)$ .



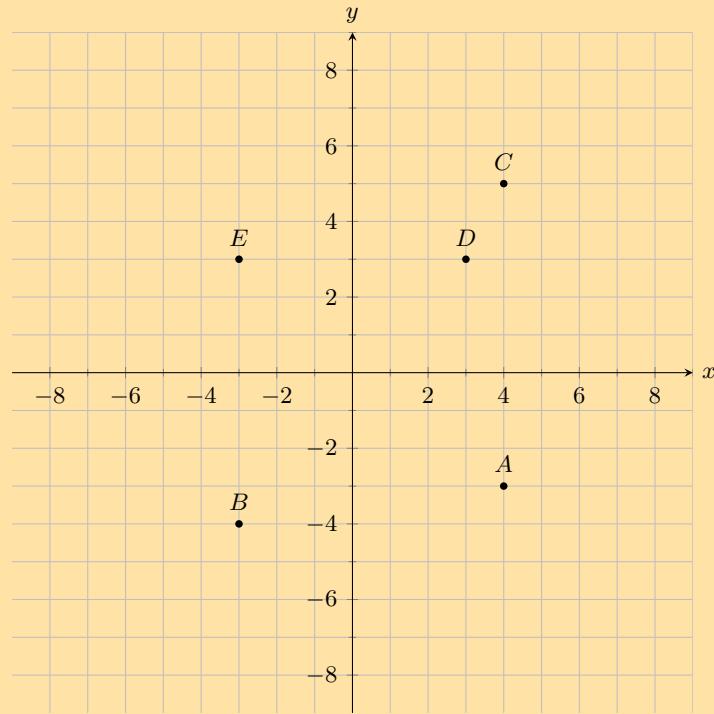
The order of the letters for naming a figure is important. It indicates the order in which points must be joined:  $A$  to  $B$ ,  $B$  to  $C$ ,  $C$  to  $D$  and  $D$  back to  $A$ . So the above quadrilateral can be referred to as quadrilateral  $ABCD$  or  $CBAD$  or  $BADC$ . However it is conventional to write the letters in alphabetical order and so we only refer to the quadrilateral as  $ABCD$ .

**VISIT:**

You can use an online tool to help you when plotting points on the Cartesian plane. Click here to try this one on [mathsisfun.com](http://mathsisfun.com).

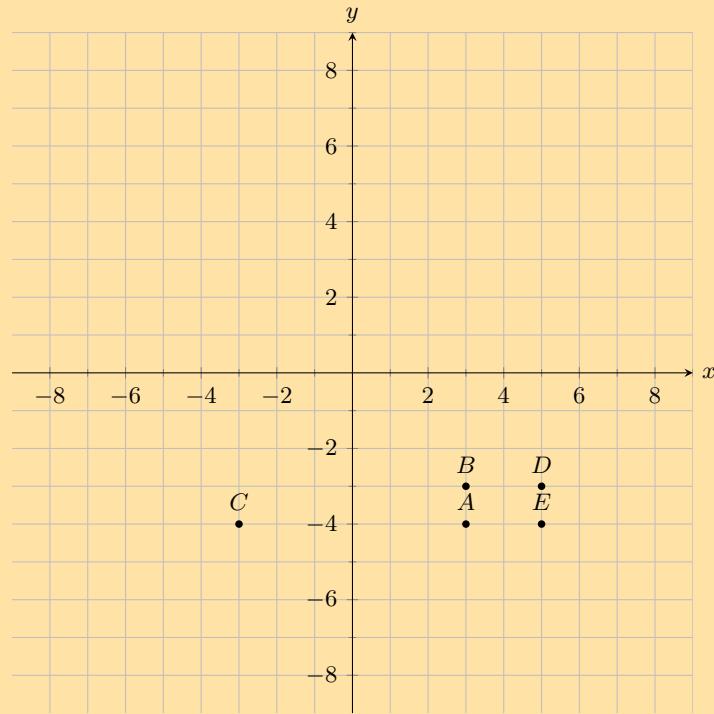
**Exercise 8 – 1:**

1. You are given the following diagram, with various points shown:



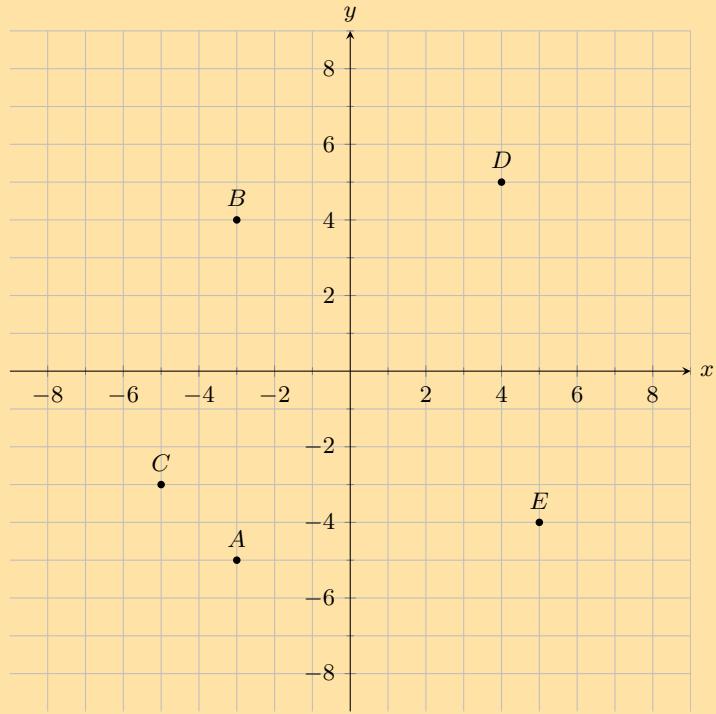
Find the coordinates of point  $D$ .

2. You are given the following diagram, with various points shown:



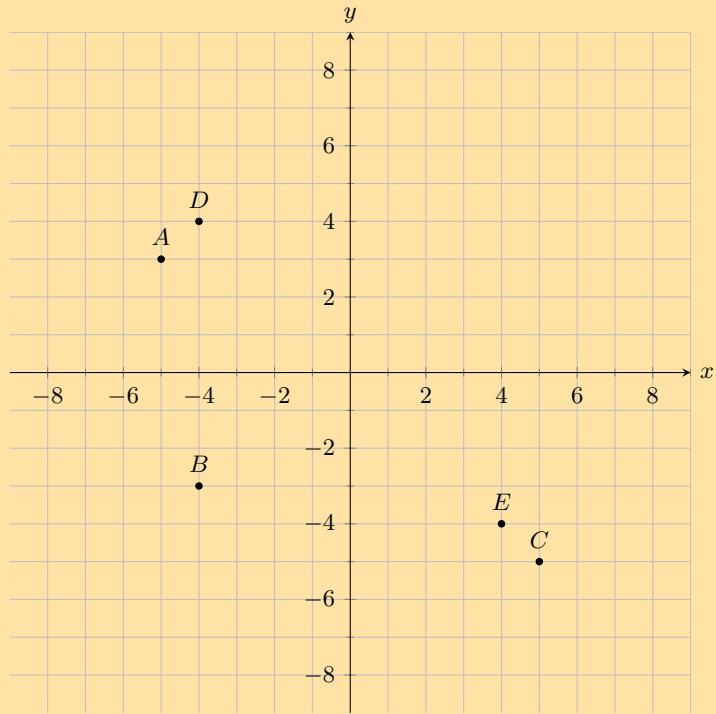
Find the coordinates of all the labelled points.

3. You are given the following diagram, with various points shown:



Which point lies at the coordinates  $(5; -4)$ ?

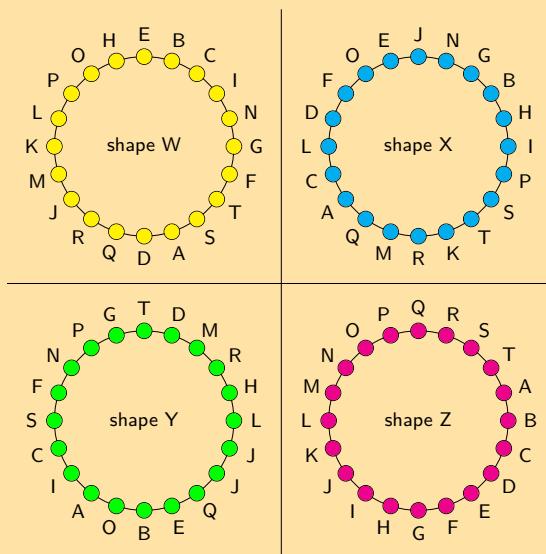
4. You are given the following diagram, with various points shown:



Which point lies at the coordinates  $(-4; -3)$ ?

5. You are given the following diagram, with 4 shapes drawn.

All the shapes are identical, but each shape uses a different naming convention:

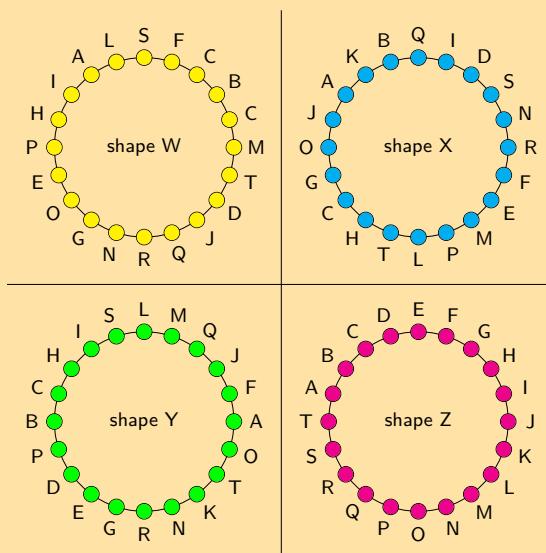


Which shape uses the correct naming convention?

6. You are given the following diagram, with 4 shapes drawn.

All the shapes are identical, but each shape uses a different naming convention:

Which shape uses the correct naming convention?



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2GBK](#) 2. [2GBM](#) 3. [2GBN](#) 4. [2GBP](#) 5. [2GBQ](#) 6. [2GBR](#)



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A point is a simple geometric object having location as its only property.

**DEFINITION:** *Point*

A point is an ordered pair of numbers written as  $(x; y)$ .

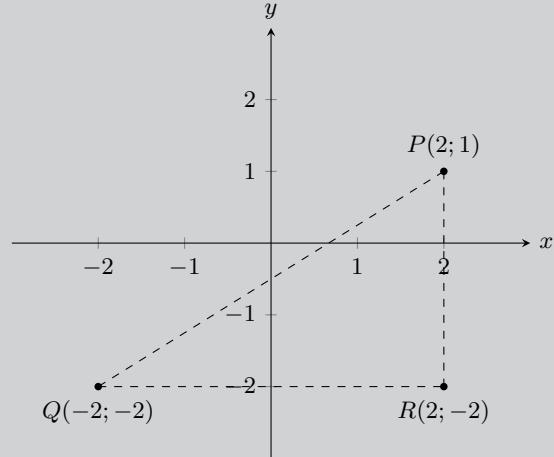
**DEFINITION:** *Distance*

Distance is a measure of the length between two points.

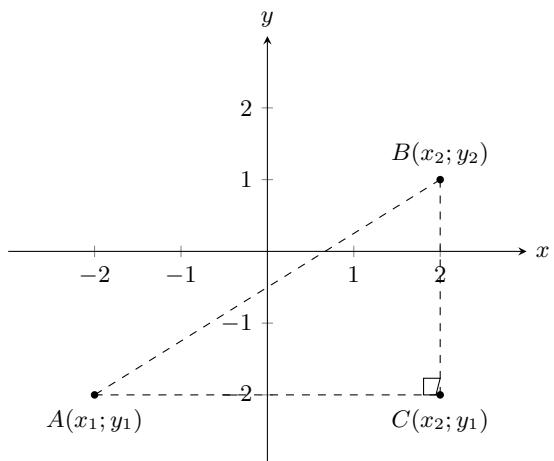
**Investigation: Distance between two points**

Points  $P(2; 1)$ ,  $Q(-2; -2)$  and  $R(2; -2)$  are given.

- Can we assume that  $\hat{R} = 90^\circ$ ? If so, why?
- Apply the theorem of Pythagoras in  $\triangle PQR$  to find the length of  $PQ$ .



To derive a general formula for the distance between two points  $A(x_1; y_1)$  and  $B(x_2; y_2)$  we use the theorem of Pythagoras.



$$AB^2 = AC^2 + BC^2$$

$$\therefore AB = \sqrt{AC^2 + BC^2}$$

And:

$$AC = x_2 - x_1$$

$$BC = y_2 - y_1$$

Therefore:

$$AB = \sqrt{AC^2 + BC^2}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Therefore to calculate the distance between any two points,  $(x_1; y_1)$  and  $(x_2; y_2)$ , we use:

$$\text{distance } (d) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Note that  $(x_1 - x_2)^2 = (x_2 - x_1)^2$ .

### VISIT:

The following video gives two examples of working with the distance formula and shows how to determine the distance formula.

► See video: [2GBS](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

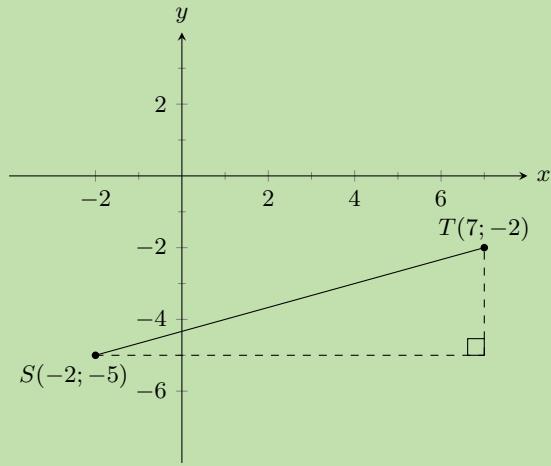
### Worked example 1: Using the distance formula

#### QUESTION

Find the distance between  $S(-2; -5)$  and  $T(7; -2)$ .

#### SOLUTION

##### Step 1: Draw a sketch



**Step 2: Assign values to  $(x_1; y_1)$  and  $(x_2; y_2)$** 

Let the coordinates of  $S$  be  $(x_1; y_1)$  and the coordinates of  $T$  be  $(x_2; y_2)$ .

$$x_1 = -2 \quad y_1 = -5 \quad x_2 = 7 \quad y_2 = -2$$

**Step 3: Write down the distance formula**

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

**Step 4: Substitute values**

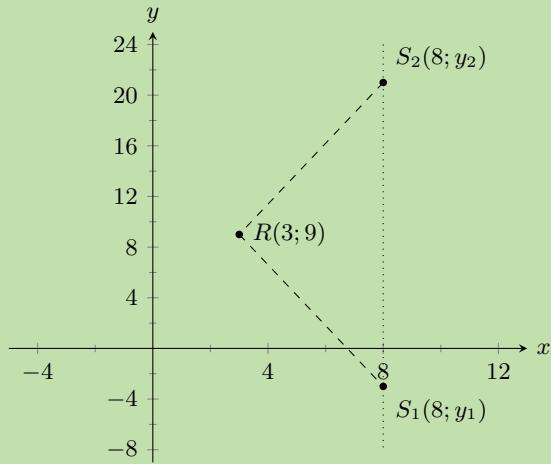
$$\begin{aligned} d_{ST} &= \sqrt{(-2 - 7)^2 + (-5 - (-2))^2} \\ &= \sqrt{(-9)^2 + (-3)^2} \\ &= \sqrt{81 + 9} \\ &= \sqrt{90} \\ &= 9,5 \end{aligned}$$

**Step 5: Write the final answer**

The distance between  $S$  and  $T$  is 9,5 units.

**Worked example 2: Using the distance formula****QUESTION**

Given  $RS = 13$ ,  $R(3; 9)$  and  $S(8; y)$ . Find  $y$ .

**SOLUTION****Step 1: Draw a sketch**

Note that we expect two possible values for  $y$ . This is because the distance formula includes the term  $(y_1 - y_2)^2$  which results in a quadratic equation when we substitute in the  $y$  coordinates.

**Step 2: Assign values to  $(x_1; y_1)$  and  $(x_2; y_2)$** 

Let the coordinates of  $R$  be  $(x_1; y_1)$  and the coordinates of  $S$  be  $(x_2; y_2)$ .

$$x_1 = 3 \quad y_1 = 9 \quad x_2 = 8 \quad y_2 = y$$

**Step 3: Write down the distance formula**

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

**Step 4: Substitute values and solve for  $y$** 

$$\begin{aligned} 13 &= \sqrt{(3 - 8)^2 + (9 - y)^2} \\ 13^2 &= (-5)^2 + (81 - 18y + y^2) \\ 0 &= y^2 - 18y - 63 \\ &= (y + 3)(y - 21) \\ \therefore y &= -3 \text{ or } y = 21 \end{aligned}$$

**Step 5: Check both values for  $y$** 

Check  $y = -3$ :

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(3 - 8)^2 + (9 + 3)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Check  $y = 21$ :

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(3 - 8)^2 + (9 - 21)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

**Step 6: Write the final answer**

$S$  is  $(8; -3)$  or  $(8; 21)$ .

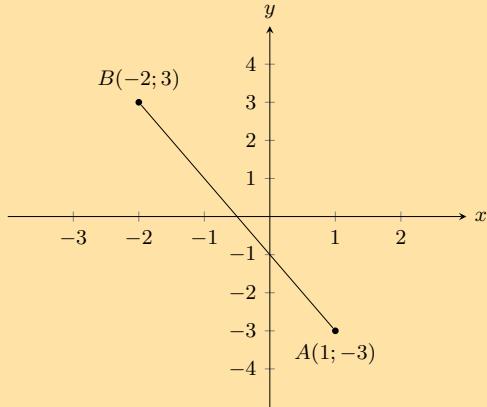
Therefore  $y = -3$  or  $y = 21$ .

**NOTE:**

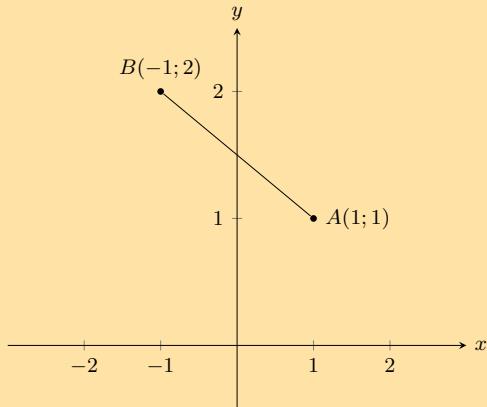
Drawing a sketch helps with your calculation and makes it easier to check if your answer is correct.

**Exercise 8 – 2:**

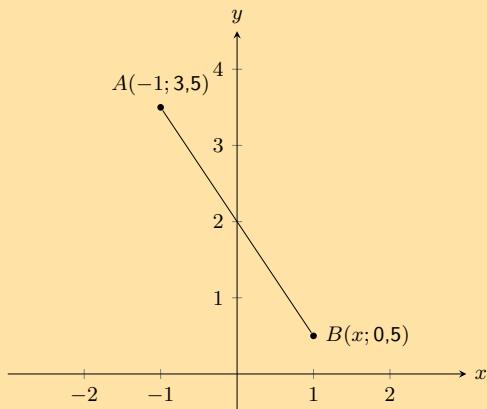
1. You are given the following diagram:



- Calculate the length of line  $AB$ , correct to 2 decimal places.  
2. You are given the following diagram:

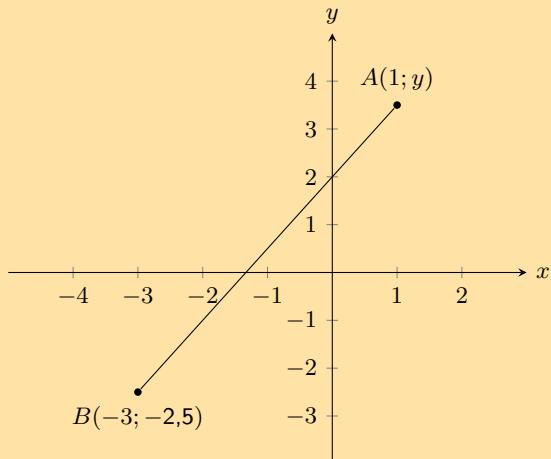


- Calculate the length of line  $AB$ , correct to 2 decimal places.  
3. The following picture shows two points on the Cartesian plane,  $A$  and  $B$ .



The distance between the points is 3,6056. Calculate the missing coordinate of point  $B$ .

4. The following picture shows two points on the Cartesian plane,  $A$  and  $B$ .



The line  $AB$  has a length of 7,2111. Calculate the missing coordinate of point  $B$ . Round your answer to one decimal place.

5. Find the length of  $AB$  for each of the following. Leave your answer in surd form.
- $A(2; 7)$  and  $B(-3; 5)$
  - $A(-3; 5)$  and  $B(-9; 1)$
  - $A(x; y)$  and  $B(x + 4; y - 1)$
6. The length of  $CD = 5$ . Find the missing coordinate if:
- $C(6; -2)$  and  $D(x; 2)$ .
  - $C(4; y)$  and  $D(1; -1)$ .
7. If the distance between  $C(0; -3)$  and  $F(8; p)$  is 10 units, find the possible values of  $p$ .

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2GBT](#)    2. [2GBV](#)    3. [2GBW](#)    4. [2GBX](#)    5a. [2GBY](#)    5b. [2GBZ](#)  
 5c. [2GC2](#)    6a. [2GC3](#)    6b. [2GC4](#)    7. [2GC5](#)



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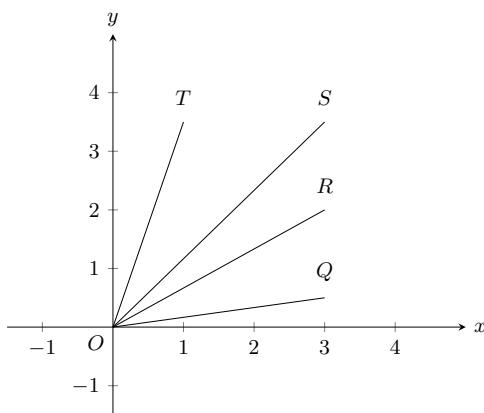
## 8.3 Gradient of a line

EMA6B

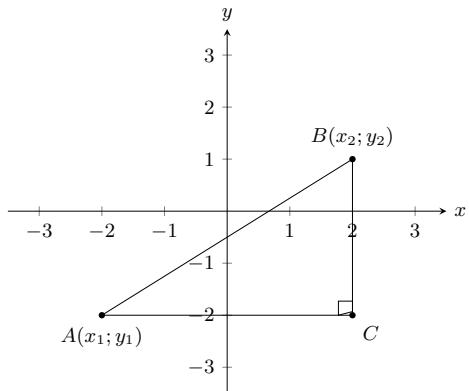
### DEFINITION: Gradient

The gradient of a line is determined by the ratio of vertical change to horizontal change.

Gradient ( $m$ ) describes the slope or steepness of the line joining two points. In the figure below, line  $OQ$  is the least steep and line  $OT$  is the steepest.



To derive the formula for gradient, we consider any right-angled triangle formed from  $A(x_1; y_1)$  and  $B(x_2; y_2)$  with hypotenuse  $AB$  as shown in the diagram alongside. The gradient is determined by the ratio of the length of the vertical side of the triangle to the length of the horizontal side of the triangle. The length of the vertical side of the triangle is the difference in  $y$ -values of points  $A$  and  $B$ . The length of the horizontal side of the triangle is the difference in  $x$ -values of points  $A$  and  $B$ .



Therefore, gradient is determined using the following formula:

$$\text{Gradient } (m) = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}$$

### IMPORTANT!

Remember to be consistent:  $m \neq \frac{y_1 - y_2}{x_2 - x_1}$

### VISIT:

To learn more about determining the gradient you can watch the following video.

► See video: [2GC6](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

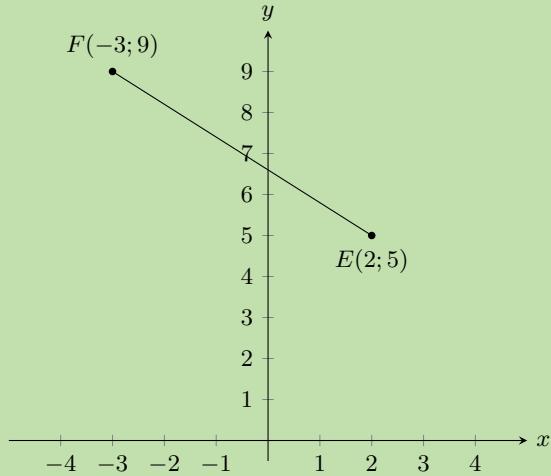
### Worked example 3: Gradient between two points

#### QUESTION

Find the gradient of the line between points  $E(2; 5)$  and  $F(-3; 9)$ .

#### SOLUTION

##### Step 1: Draw a sketch



**Step 2: Assign values to  $(x_1; y_1)$  and  $(x_2; y_2)$** 

Let the coordinates of  $E$  be  $(x_1; y_1)$  and the coordinates of  $F$  be  $(x_2; y_2)$ .

$$x_1 = 2 \quad y_1 = 5 \quad x_2 = -3 \quad y_2 = 9$$

**Step 3: Write down the formula for gradient**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

**Step 4: Substitute known values**

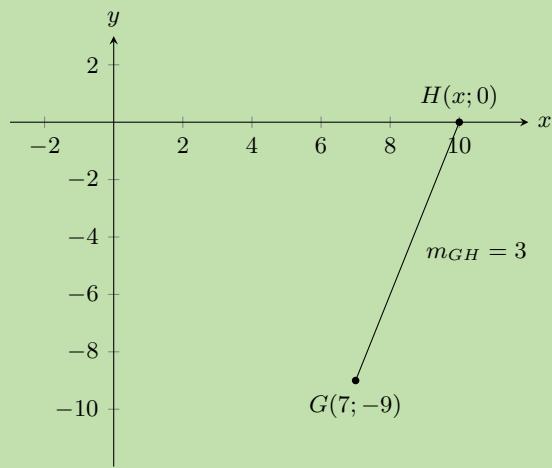
$$\begin{aligned} m_{EF} &= \frac{9 - 5}{-3 - 2} \\ &= \frac{4}{-5} \end{aligned}$$

**Step 5: Write the final answer**

The gradient of  $EF = -\frac{4}{5}$

**Worked example 4: Gradient between two points****QUESTION**

Given  $G(7; -9)$  and  $H(x; 0)$ , with  $m_{GH} = 3$ , find  $x$ .

**SOLUTION****Step 1: Draw a sketch**

**Step 2: Assign values to  $(x_1; y_1)$  and  $(x_2; y_2)$** 

Let the coordinates of  $G$  be  $(x_1; y_1)$  and the coordinates of  $H$  be  $(x_2; y_2)$

$$x_1 = 7 \quad y_1 = -9 \quad x_2 = x \quad y_2 = 0$$

**Step 3: Write down the formula for gradient**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

**Step 4: Substitute values and solve for  $x$** 

$$\begin{aligned} 3 &= \frac{0 - (-9)}{x - 7} \\ 3(x - 7) &= 9 \\ x - 7 &= \frac{9}{3} \\ x - 7 &= 3 \\ x &= 3 + 7 \\ &= 10 \end{aligned}$$

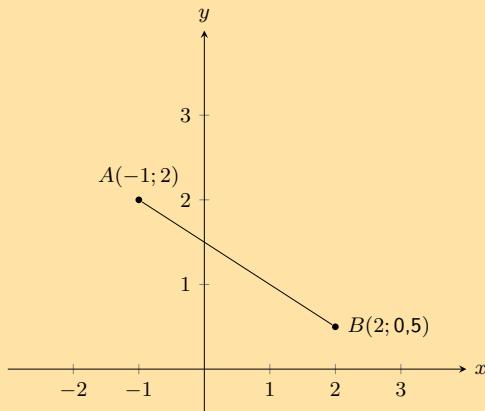
**Step 5: Write the final answer**

The coordinates of  $H$  are  $(10; 0)$ .

Therefore  $x = 10$ .

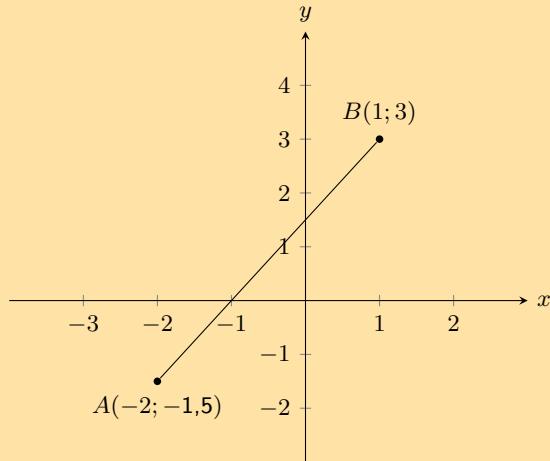
**Exercise 8 – 3:**

1. Find the gradient of  $AB$  if:
  - a)  $A(7; 10)$  and  $B(-4; 1)$
  - b)  $A(-5; -9)$  and  $B(3; 2)$
  - c)  $A(x - 3; y)$  and  $B(x; y + 4)$
2. You are given the following diagram:



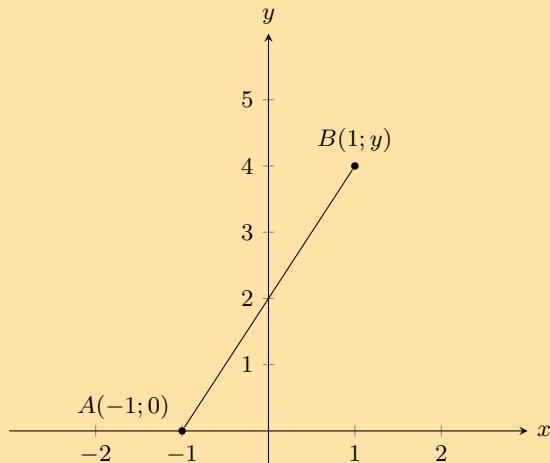
Calculate the gradient ( $m$ ) of line  $AB$ .

3. Calculate the gradient ( $m$ ) of line  $AB$  in the following diagram:

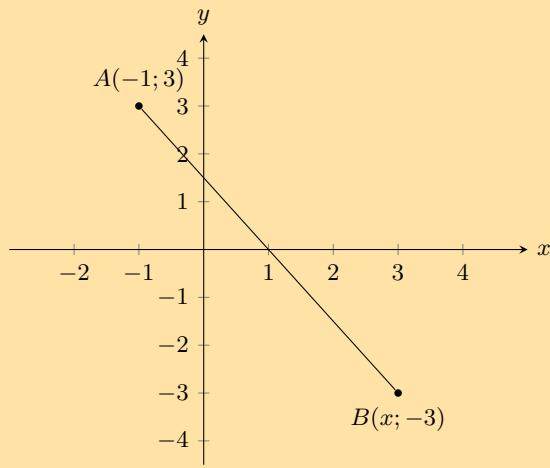


4. If the gradient of  $CD = \frac{2}{3}$ , find  $p$  given:

- a)  $C(16; 2)$  and  $D(8; p)$ .
  - b)  $C(3; 2p)$  and  $D(9; 14)$ .
5. In the following diagram line  $AB$  has a gradient ( $m$ ) of 2. Calculate the missing co-ordinate of point  $B$ .



6. You are given the following diagram:



You are also told that line  $AB$  has a gradient ( $m$ ) of  $-1,5$ .

Calculate the missing co-ordinate of point  $B$ .

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- 1a. [2GC7](#) 1b. [2GC8](#) 1c. [2GC9](#) 2. [2GCB](#) 3. [2GCC](#) 4a. [2GCD](#)  
4b. [2GCF](#) 5. [2GCG](#) 6. [2GCH](#)



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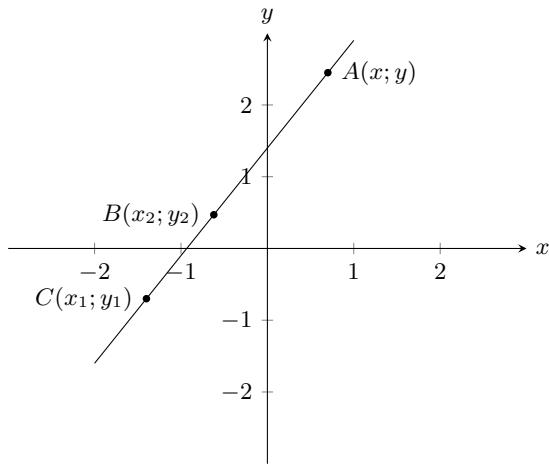
## Straight lines

EMA6C

**DEFINITION:** *Straight line*

A straight line is a set of points with a constant gradient between any two of the points.

Consider the diagram below with points  $A(x; y)$ ,  $B(x_2; y_2)$  and  $C(x_1; y_1)$ .



We have  $m_{AB} = m_{BC} = m_{AC}$  and  $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$

The general formula for finding the equation of a straight line is  $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$  where  $(x; y)$  is any point on the line.

This formula can also be written as  $y - y_1 = m(x - x_1)$ .

The standard form of the straight line equation is  $y = mx + c$  where  $m$  is the gradient and  $c$  is the  $y$ -intercept.

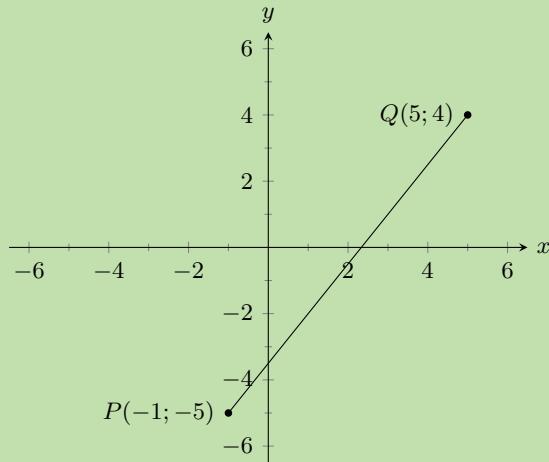
### Worked example 5: Finding the equation of a straight line

#### QUESTION

Find the equation of the straight line through  $P(-1; -5)$  and  $Q(5; 4)$ .

## **SOLUTION**

### **Step 1: Draw a sketch**



### **Step 2: Assign values**

Let the coordinates of  $P$  be  $(x_1; y_1)$  and the coordinates of  $Q$  be  $(x_2; y_2)$ .

$$x_1 = -1 \quad y_1 = -5 \quad x_2 = 5 \quad y_2 = 4$$

### **Step 3: Write down the general formula of the line**

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

### **Step 4: Substitute values and make $y$ the subject of the equation**

$$\begin{aligned}\frac{y - (-5)}{x - (-1)} &= \frac{4 - (-5)}{5 - (-1)} \\ \frac{y + 5}{x + 1} &= \frac{3}{2} \\ 2(y + 5) &= 3(x + 1) \\ 2y + 10 &= 3x + 3 \\ 2y &= 3x - 7 \\ y &= \frac{3}{2}x - \frac{7}{2}\end{aligned}$$

### **Step 5: Write the final answer**

The equation of the straight line is  $y = \frac{3}{2}x - \frac{7}{2}$ .

Two lines that run parallel to each other are always the same distance apart and have equal gradients.

If two lines intersect perpendicularly, then the product of their gradients is equal to  $-1$ .

If line  $WX \perp YZ$ , then  $m_{WX} \times m_{YZ} = -1$ . Perpendicular lines have gradients that are the negative inverses of each other.

### VISIT:

The following video shows some examples of calculating the slope of a line and determining if two lines are perpendicular or parallel.

► See video: [2GCJ](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

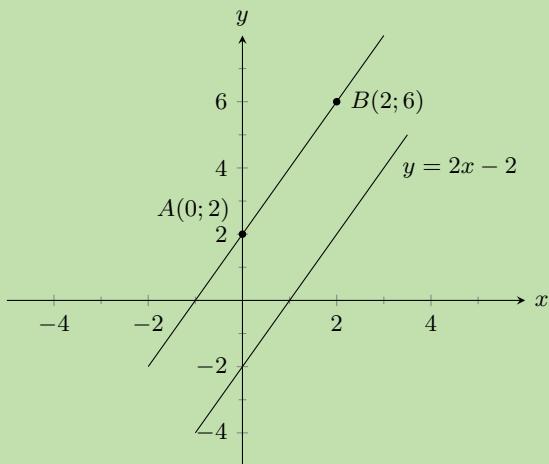
## Worked example 6: Parallel lines

### QUESTION

Prove that line  $AB$  with  $A(0; 2)$  and  $B(2; 6)$  is parallel to line  $CD$  with equation  $2x - y = 2$ .

### SOLUTION

#### Step 1: Draw a sketch



(Be careful - some lines may look parallel but are not!)

#### Step 2: Write down the formula for gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

#### Step 3: Substitute values to find the gradient for line $AB$

$$\begin{aligned} m_{AB} &= \frac{6 - 2}{2 - 0} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

**Step 4: Check that the equation of  $CD$  is in the standard form  $y = mx + c$**

$$\begin{aligned}2x - y &= 2 \\y &= 2x - 2 \\\therefore m_{CD} &= 2\end{aligned}$$

**Step 5: Write the final answer**

$$m_{AB} = m_{CD}$$

therefore line  $AB$  is parallel to line  $CD$ .

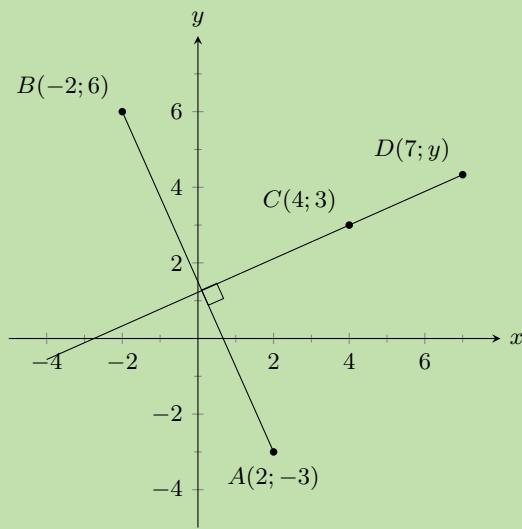
### Worked example 7: Perpendicular lines

#### QUESTION

Line  $AB$  is perpendicular to line  $CD$ . Find  $y$  given  $A(2; -3)$ ,  $B(-2; 6)$ ,  $C(4; 3)$  and  $D(7; y)$ .

#### SOLUTION

**Step 1: Draw a sketch**



**Step 2: Write down the relationship between the gradients of the perpendicular lines  $AB \perp CD$**

$$\begin{aligned}m_{AB} \times m_{CD} &= -1 \\ \frac{y_B - y_A}{x_B - x_A} \times \frac{y_D - y_C}{x_D - x_C} &= -1\end{aligned}$$

### Step 3: Substitute values and solve for $y$

$$\begin{aligned}\frac{6 - (-3)}{-2 - 2} \times \frac{y - 3}{7 - 4} &= -1 \\ \frac{9}{-4} \times \frac{y - 3}{3} &= -1 \\ \frac{y - 3}{3} &= -1 \times \frac{-4}{9} \\ \frac{y - 3}{3} &= \frac{4}{9} \\ y - 3 &= \frac{4}{9} \times 3 \\ y - 3 &= \frac{4}{3} \\ y &= \frac{4}{3} + 3 \\ &= \frac{4 + 9}{3} \\ &= \frac{13}{3} \\ &= 4\frac{1}{3}\end{aligned}$$

### Step 4: Write the final answer

Therefore the coordinates of  $D$  are  $(7; 4\frac{1}{3})$ .

## Horizontal and vertical lines

EMA6F

A line that runs parallel to the  $x$ -axis is called a horizontal line and has a gradient of zero. This is because there is no vertical change:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{0}{\text{change in } x} = 0$$

A line that runs parallel to the  $y$ -axis is called a vertical line and its gradient is undefined. This is because there is no horizontal change:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{change in } y}{0} = \text{undefined}$$

## Points on a line

EMA6G

A straight line is a set of points with a constant gradient between any of the two points. There are two methods to prove that points lie on the same line: the gradient method and a method using the distance formula.

#### NOTE:

If two points lie on the same line then the two points are said to be collinear.

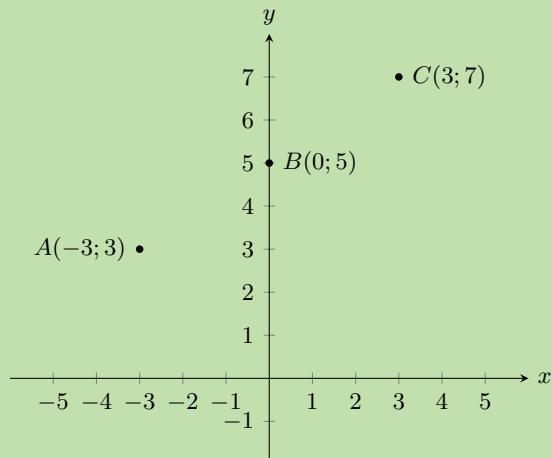
## Worked example 8: Points on a line

### QUESTION

Prove that  $A(-3; 3)$ ,  $B(0; 5)$  and  $C(3; 7)$  are on a straight line.

### SOLUTION

#### Step 1: Draw a sketch



#### Step 2: Calculate two gradients between any of the three points

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m_{AB} = \frac{5 - 3}{0 - (-3)} = \frac{2}{3}$$

and

$$m_{BC} = \frac{7 - 5}{3 - 0} = \frac{2}{3}$$

OR

$$m_{AC} = \frac{3 - 7}{3 - 0} = \frac{-4}{-6} = \frac{2}{3}$$

and

$$m_{BC} = \frac{7 - 5}{3 - 0} = \frac{2}{3}$$

#### Step 3: Explain your answer

$$m_{AB} = m_{BC} = m_{AC}$$

Therefore the points  $A$ ,  $B$  and  $C$  are on a straight line.

To prove that three points are on a straight line using the distance formula, we must calculate the distances between each pair of points and then prove that the sum of the two smaller distances equals the longest distance.

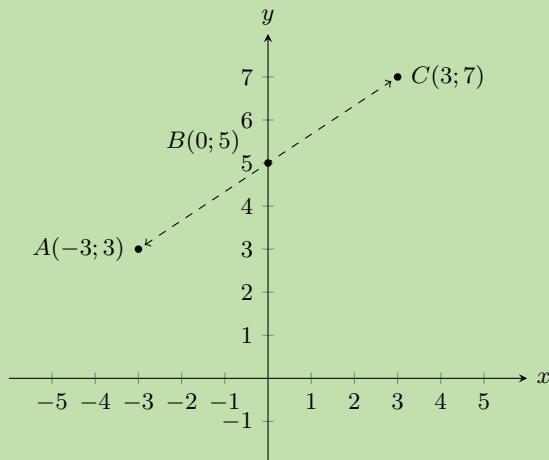
## Worked example 9: Points on a straight line

### QUESTION

Prove that  $A(-3; 3)$ ,  $B(0; 5)$  and  $C(3; 7)$  are on a straight line.

### SOLUTION

#### Step 1: Draw a sketch



#### Step 2: Calculate the three distances $AB$ , $BC$ and $AC$

$$\begin{aligned}d_{AB} &= \sqrt{(-3-0)^2 + (3-5)^2} & d_{BC} &= \sqrt{(0-3)^2 + (5-7)^2} & d_{AC} &= \sqrt{((-3)-3)^2 + (3-7)^2} \\&= \sqrt{(-3)^2 + (-2)^2} & &= \sqrt{(-3)^2 + (-2)^2} & &= \sqrt{(-6)^2 + (-4)^2} \\&= \sqrt{9+4} & &= \sqrt{9+4} & &= \sqrt{36+16} \\&= \sqrt{13} & &= \sqrt{13} & &= \sqrt{52}\end{aligned}$$

#### Step 3: Find the sum of the two shorter distances

$$d_{AB} + d_{BC} = \sqrt{13} + \sqrt{13} = 2\sqrt{13} = \sqrt{4 \times 13} = \sqrt{52}$$

#### Step 4: Explain your answer

$$d_{AB} + d_{BC} = d_{AC}$$

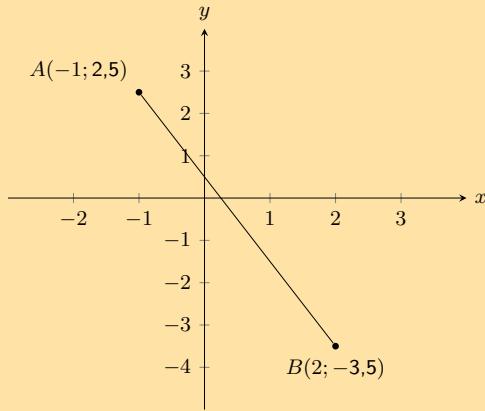
therefore points  $A$ ,  $B$  and  $C$  lie on the same straight line.

### Exercise 8 – 4:

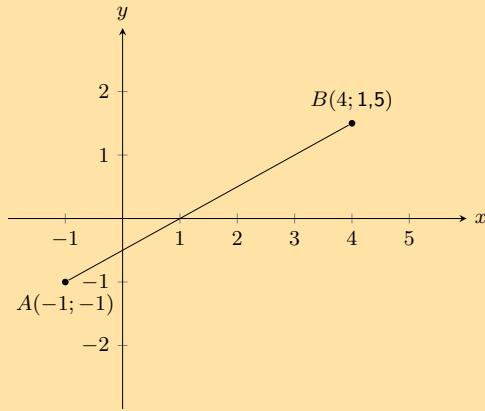
- Determine whether  $AB$  and  $CD$  are parallel, perpendicular or neither if:

- $A(3; -4)$ ,  $B(5; 2)$ ,  $C(-1; -1)$ ,  $D(7; 23)$
- $A(3; -4)$ ,  $B(5; 2)$ ,  $C(-1; -1)$ ,  $D(0; -4)$
- $A(3; -4)$ ,  $B(5; 2)$ ,  $C(-1; 3)$ ,  $D(-2; 2)$

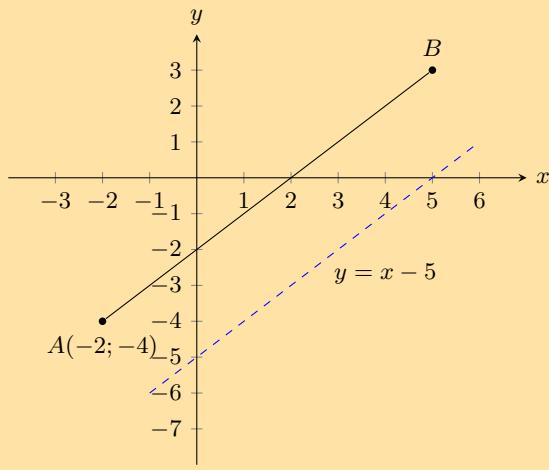
2. Determine whether the following points lie on the same straight line:
- $E(0; 3), F(-2; 5), G(2; 1)$
  - $H(-3; -5), I(0; 0), J(6; 10)$
  - $K(-6; 2), L(-3; 1), M(1; -1)$
3. Calculate the equation of the line  $AB$  in the following diagram:



4. Calculate the equation of the line  $AB$  in the following diagram:



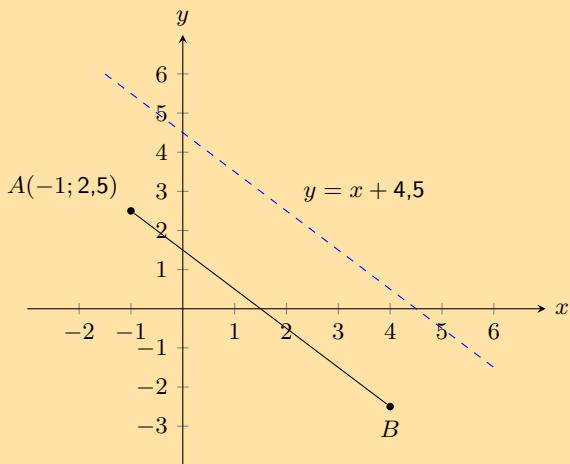
5. Points  $P(-6; 2), Q(2; -2)$  and  $R(-3; r)$  lie on a straight line. Find the value of  $r$ .
6. Line  $PQ$  with  $P(-1; -7)$  and  $Q(q; 0)$  has a gradient of 1. Find  $q$ .
7. You are given the following diagram:



You are also told that line  $AB$  runs parallel to the following line:  $y = x - 5$ . Point  $A$  is at  $(-2; -4)$ .

Find the equation of the line  $AB$ .

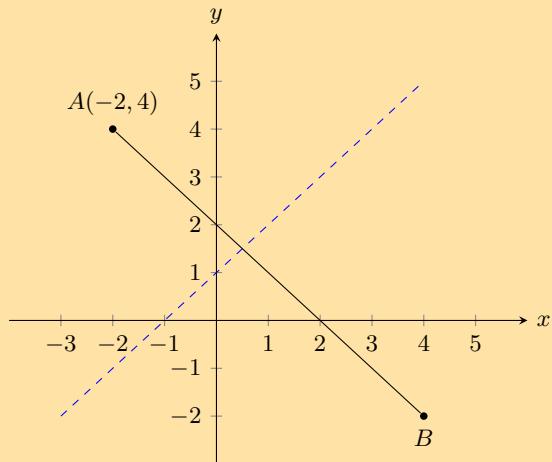
8. You are given the following diagram:



You are also told that line  $AB$  runs parallel to the following line:  $y = -x + 4.5$ . Point  $A$  is at  $(-1; 2.5)$ .

Find the equation of the line  $AB$ .

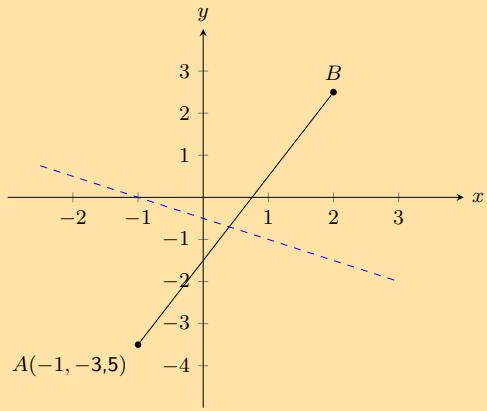
9. Given line  $AB$  which runs parallel to  $y = 0.5x - 6$ . Points  $A(-1; -2.5)$  and  $B(x; 0)$  are also given. Calculate the missing co-ordinate of point  $B$ .
10. Given line  $AB$  which runs parallel to  $y = -1.5x + 4$ . Points  $A(-2; 4)$  and  $B(2; y)$  are also given. Calculate the missing co-ordinate of point  $B(2; y)$ .
11. The graph here shows line  $AB$ . The blue dashed line is perpendicular to  $AB$ .



The equation of the blue dashed line is  $y = x + 1$ . Point  $A$  is at  $(-2; 4)$ .

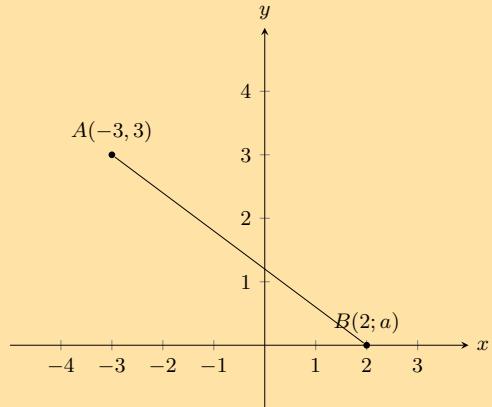
Determine the equation of line  $AB$ .

12. The graph here shows line  $AB$ . The blue dashed line is perpendicular to  $AB$ .



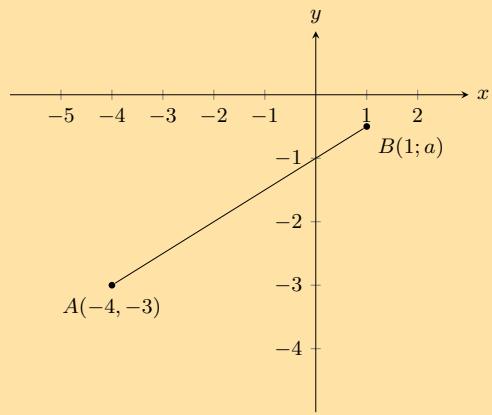
The equation of the blue dashed line is  $y = -0.5x - 0.5$ . Point  $A$  is at  $(-1; -3,5)$ . Determine the equation of line  $AB$ .

13. Given line  $AB$  which runs perpendicular to line  $CD$  with equation  $y = -2x + 1$ . Points  $A(-5; -1)$  and  $B(3; a)$  are also given. Calculate the missing co-ordinate of point  $B$ .
14. Given line  $AB$  which runs perpendicular to line  $CD$  with equation  $y = 2x - 0,75$ . Points  $A(-5; 1)$  and  $B(a; -2,5)$  are also given. Calculate the missing co-ordinate of point  $B$ .
15. You are given the following diagram:



You are also told that line  $AB$  has the following equation:  $y = -0,5x + 1,5$ . Calculate the missing co-ordinate of point  $B$ .

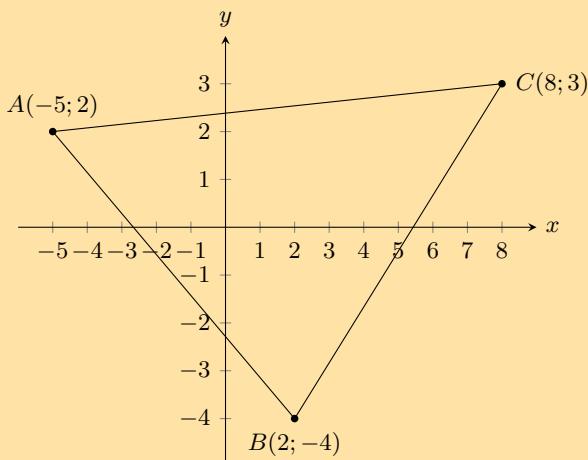
16. You are given the following diagram:



You are also told that line  $AB$  has the following equation:  $y = 0,5x - 1$ .

Calculate the missing coordinate of point  $B$ .

17.  $A$  is the point  $(-3; -5)$  and  $B$  is the point  $(n; -11)$ .  $AB$  is perpendicular to line  $CD$  with equation  $y = \frac{3}{2}x - 5$ . Find the value of  $n$ .
18. The points  $A(4; -3)$ ,  $B(-5; 0)$  and  $C(-3; p)$  are given. Determine the value of  $p$  if  $A$ ,  $B$  and  $C$  are collinear.
19. Refer to the diagram below:



- a) Show that  $\triangle ABC$  is right-angled. Show your working.
- b) Find the area of  $\triangle ABC$ .
20. The points  $A(-3; 1)$ ,  $B(3; -2)$  and  $C(9; 10)$  are given.
  - a) Prove that triangle  $ABC$  is a right-angled triangle.
  - b) Find the coordinates of  $D$ , if  $ABCD$  is a parallelogram.
  - c) Find the equation of a line parallel to the line  $BC$ , which passes through the point  $A$ .

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |                          |                          |                          |                          |                          |                          |                          |                          |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1a. <a href="#">2GCK</a> | 1b. <a href="#">2GCM</a> | 1c. <a href="#">2GCN</a> | 2a. <a href="#">2GCP</a> | 2b. <a href="#">2GCQ</a> | 2c. <a href="#">2GCR</a> | 3. <a href="#">2GCS</a>  | 4. <a href="#">2GCT</a>  |
| 5. <a href="#">2GCV</a>  | 6. <a href="#">2GCW</a>  | 7. <a href="#">2GCX</a>  | 8. <a href="#">2GCY</a>  | 9. <a href="#">2GCZ</a>  | 10. <a href="#">2GD2</a> | 11. <a href="#">2GD3</a> | 12. <a href="#">2GD4</a> |
| 13. <a href="#">2GD5</a> | 14. <a href="#">2GD6</a> | 15. <a href="#">2GD7</a> | 16. <a href="#">2GD8</a> | 17. <a href="#">2GD9</a> | 18. <a href="#">2GDB</a> | 19. <a href="#">2GDC</a> | 20. <a href="#">2GDD</a> |



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## 8.4 Mid-point of a line

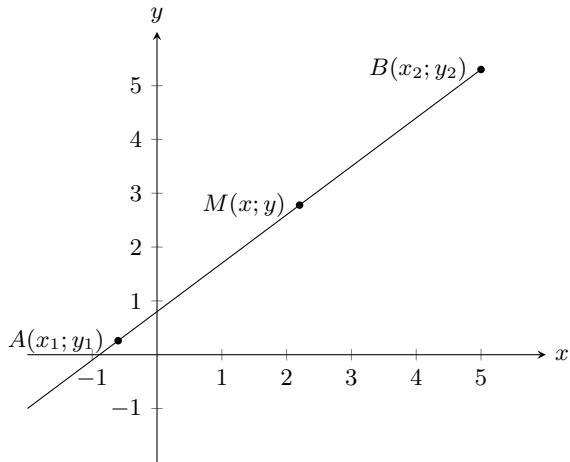
EMA6H

### Investigation: Finding the mid-point of a line

On graph paper, accurately plot the points  $P(2; 1)$  and  $Q(-2; 2)$  and draw the line  $PQ$ .

- Fold the piece of paper so that point  $P$  is exactly on top of point  $Q$ .
- Where the folded line intersects with line  $PQ$ , label point  $S$ .
- Count the blocks and find the exact position of  $S$ .
- Write down the coordinates of  $S$ .

To calculate the coordinates of the mid-point  $M(x; y)$  of any line between the points  $A(x_1; y_1)$  and  $B(x_2; y_2)$ , we use the following formulae:



$$x = \frac{x_1 + x_2}{2}$$

$$y = \frac{y_1 + y_2}{2}$$

From this we obtain the mid-point of a line:

$$\text{Mid-point } M(x; y) = \left( \frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2} \right)$$

### VISIT:

This video shows some examples of finding the mid-point of a line.

► See video: [2GDF](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 10: Calculating the mid-point

#### QUESTION

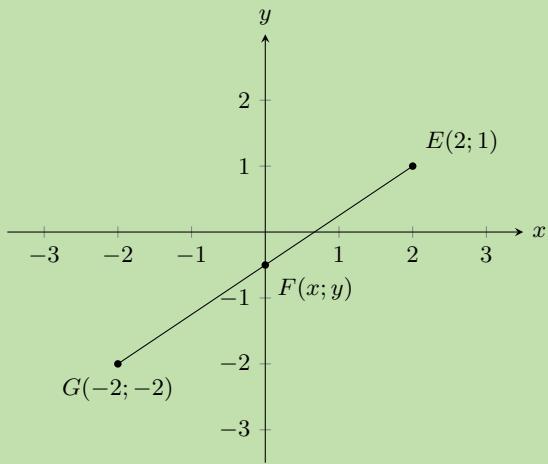
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Calculate the coordinates of the mid-point  $F(x; y)$  of the line between point  $E(2; 1)$  and point  $G(-2; -2)$ .

#### SOLUTION

---

##### Step 1: Draw a sketch



From the sketch, we can estimate that  $F$  will lie on the  $y$ -axis, with a negative  $y$ -coordinate.

**Step 2: Assign values to  $(x_1; y_1)$  and  $(x_2; y_2)$**

$$x_1 = -2 \quad y_1 = -2 \quad x_2 = 2 \quad y_2 = 1$$

**Step 3: Write down the mid-point formula**

$$F(x; y) = \left( \frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2} \right)$$

**Step 4: Substitute values into the mid-point formula**

$$\begin{aligned} x &= \frac{x_1 + x_2}{2} & y &= \frac{y_1 + y_2}{2} \\ &= \frac{-2 + 2}{2} & &= \frac{-2 + 1}{2} \\ &= 0 & &= -\frac{1}{2} \end{aligned}$$

**Step 5: Write the answer**

The mid-point is at  $F(0; -\frac{1}{2})$ .

Looking at the sketch we see that this is what we expect for the coordinates of  $F$ .

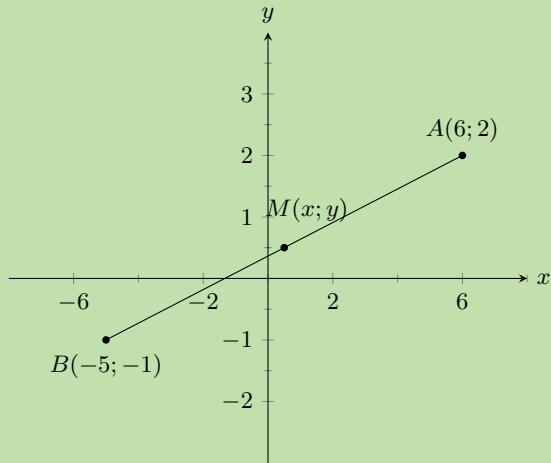
**Worked example 11: Calculating the mid-point**

**QUESTION**

Find the mid-point of line  $AB$ , given  $A(6; 2)$  and  $B(-5; -1)$ .

**SOLUTION**

**Step 1: Draw a sketch**



From the sketch, we can estimate that  $M$  will lie in quadrant I, with positive  $x$ - and  $y$ -coordinates.

**Step 2: Assign values to  $(x_1; y_1)$  and  $(x_2; y_2)$**

Let the mid-point be  $M(x; y)$

$$x_1 = 6 \quad y_1 = 2 \quad x_2 = -5 \quad y_2 = -1$$

**Step 3: Write down the mid-point formula**

$$M(x; y) = \left( \frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2} \right)$$

**Step 4: Substitute values and simplify**

$$M(x; y) = \left( \frac{6 - 5}{2}; \frac{2 - 1}{2} \right) = \left( \frac{1}{2}; \frac{1}{2} \right)$$

**Step 5: Write the final answer**

$M\left(\frac{1}{2}; \frac{1}{2}\right)$  is the mid-point of line  $AB$ .

We expected  $M$  to have a positive  $x$ - and  $y$ -coordinate and this is indeed what we have found by calculation.

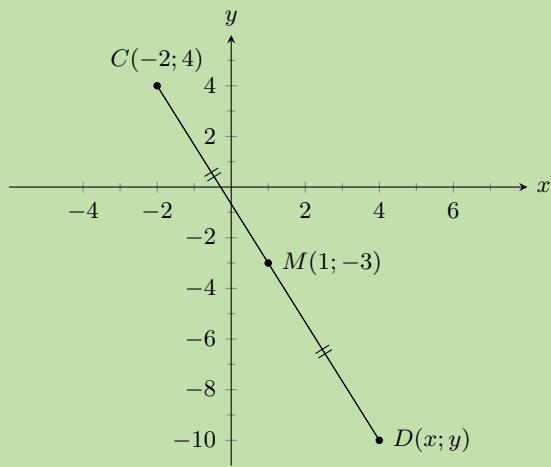
**Worked example 12: Using the mid-point formula**

**QUESTION**

The line joining  $C(-2; 4)$  and  $D(x; y)$  has the mid-point  $M(1; -3)$ . Find point  $D$ .

**SOLUTION**

**Step 1: Draw a sketch**



From the sketch, we can estimate that  $D$  will lie in Quadrant IV, with a positive  $x$ - and negative  $y$ -coordinate.

**Step 2: Assign values to  $(x_1; y_1)$  and  $(x_2; y_2)$** 

Let the coordinates of  $C$  be  $(x_1; y_1)$  and the coordinates of  $D$  be  $(x_2; y_2)$ .

$$x_1 = -2 \quad y_1 = 4 \quad x_2 = x \quad y_2 = y$$

**Step 3: Write down the mid-point formula**

$$M(x; y) = \left( \frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2} \right)$$

**Step 4: Substitute values and solve for  $x_2$  and  $y_2$** 

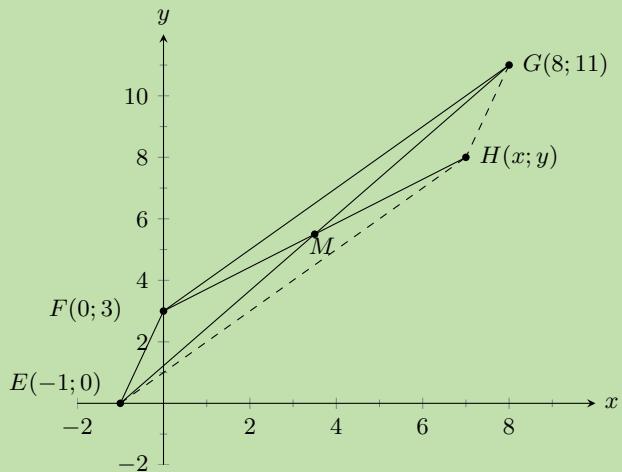
$$\begin{array}{rcl} 1 & = & \frac{-2 + x_2}{2} \\ 1 \times 2 & = & -2 + x_2 \\ 2 & = & -2 + x_2 \\ x_2 & = & 2 + 2 \\ x_2 & = & 4 \end{array} \qquad \begin{array}{rcl} -3 & = & \frac{4 + y_2}{2} \\ -3 \times 2 & = & 4 + y_2 \\ -6 & = & 4 + y_2 \\ y_2 & = & -6 - 4 \\ y_2 & = & -10 \end{array}$$

**Step 5: Write the final answer**

The coordinates of point  $D$  are  $(4; -10)$ .

**Worked example 13: Using the mid-point formula****QUESTION**

Points  $E(-1; 0)$ ,  $F(0; 3)$ ,  $G(8; 11)$  and  $H(x; y)$  are points on the Cartesian plane. Find  $H(x; y)$  if  $EFGH$  is a parallelogram.

**SOLUTION****Step 1: Draw a sketch**

Method: the diagonals of a parallelogram bisect each other, therefore the mid-point of  $EG$  will be the same as the mid-point of  $FH$ . We must first find the mid-point of  $EG$ . We can then use it to determine the coordinates of point  $H$ .

**Step 2: Assign values to  $(x_1; y_1)$  and  $(x_2; y_2)$** 

Let the mid-point of  $EG$  be  $M(x; y)$

$$x_1 = -1 \quad y_1 = 0 \quad x_2 = 8 \quad y_2 = 11$$

**Step 3: Write down the mid-point formula**

$$M(x; y) = \left( \frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2} \right)$$

**Step 4: Substitute values calculate the coordinates of  $M$** 

$$M(x; y) = \left( \frac{-1 + 8}{2}; \frac{0 + 11}{2} \right) = \left( \frac{7}{2}; \frac{11}{2} \right)$$

**Step 5: Use the coordinates of  $M$  to determine  $H$** 

$M$  is also the mid-point of  $FH$  so we use  $M\left(\frac{7}{2}; \frac{11}{2}\right)$  and  $F(0; 3)$  to solve for  $H(x; y)$ .

**Step 6: Substitute values and solve for  $x$  and  $y$** 

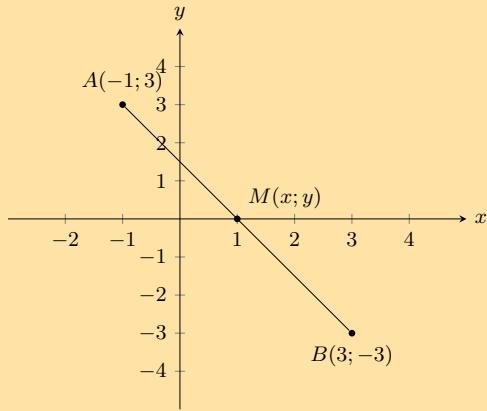
$$\begin{array}{rcl} \frac{7}{2} & = & \frac{0 + x}{2} \\ \frac{7}{2} & = & x + 0 \\ x & = & 7 \end{array} \qquad \begin{array}{rcl} \frac{11}{2} & = & \frac{3 + y}{2} \\ \frac{11}{2} & = & 3 + y \\ y & = & 8 \end{array}$$

**Step 7: Write the final answer**

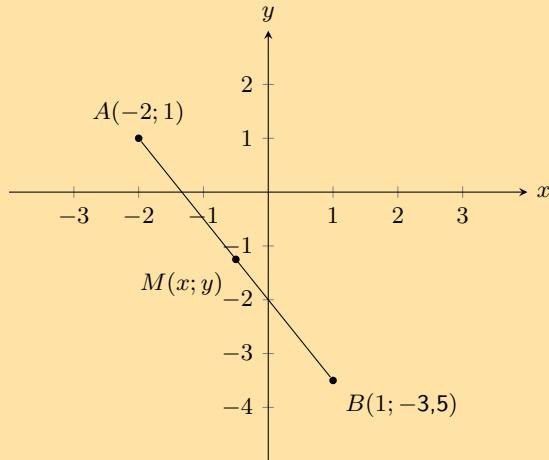
The coordinates of  $H$  are  $(7; 8)$ .

**Exercise 8 – 5:**

1. Calculate the coordinates of the mid-point ( $M$ ) between point  $A(-1; 3)$  and point  $B(3; -3)$  in the following diagram:



2. Calculate the coordinates of the mid-point ( $M$ ) between point  $A(-2; 1)$  and point  $B(1; -3,5)$  in the following diagram:



3. Find the mid-points of the following lines:  
 a)  $A(2; 5), B(-4; 7)$       b)  $C(5; 9), D(23; 55)$       c)  $E(x + 2; y - 1), F(x - 5; y - 4)$   
 4. The mid-point  $M$  of  $PQ$  is  $(3; 9)$ . Find  $P$  if  $Q$  is  $(-2; 5)$ .  
 5.  $PQRS$  is a parallelogram with the points  $P(5; 3), Q(2; 1)$  and  $R(7; -3)$ . Find point  $S$ .

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2GDG](#)   2. [2GDH](#)   3a. [2GDJ](#)   3b. [2GDK](#)   3c. [2GDM](#)   4. [2GDN](#)   5. [2GDP](#)



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## 8.5 Chapter summary

EMA6J

► See presentation: [2GDQ](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

- A point is an ordered pair of numbers written as  $(x; y)$ .
- Distance is a measure of the length between two points.
- The formula for finding the distance between any two points is:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- The gradient between two points is determined by the ratio of vertical change to horizontal change.
- The formula for finding the gradient of a line is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- A straight line is a set of points with a constant gradient between any two of the points.
- The standard form of the straight line equation is  $y = mx + c$ .
- The equation of a straight line can also be written as

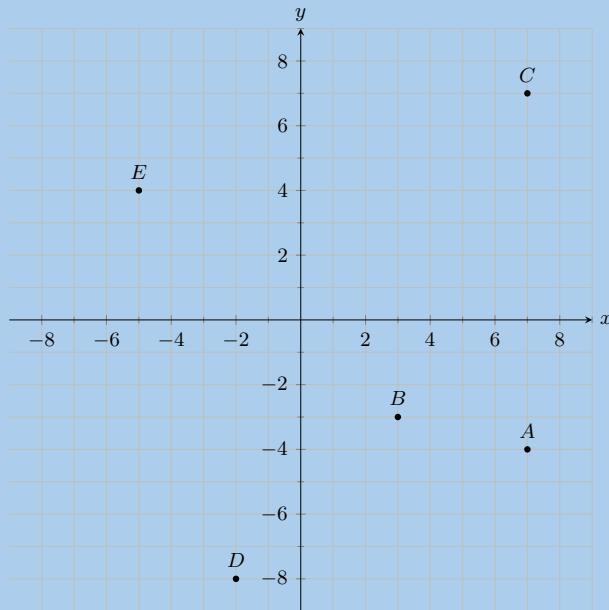
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

- If two lines are parallel, their gradients are equal.
- If two lines are perpendicular, the product of their gradients is equal to  $-1$ .
- For horizontal lines the gradient is equal to 0.
- For vertical lines the gradient is undefined.
- The formula for finding the mid-point between two points is:

$$M(x; y) = \left( \frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2} \right)$$

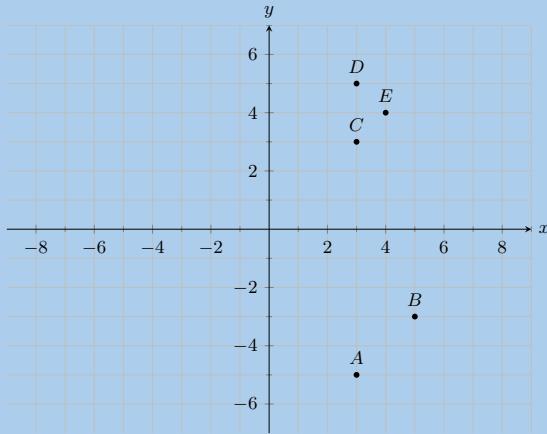
**End of chapter Exercise 8 – 6:**

1. You are given the following diagram, with various points shown:



Find the coordinates of points  $A$ ,  $B$ ,  $C$ ,  $D$  and  $E$ .

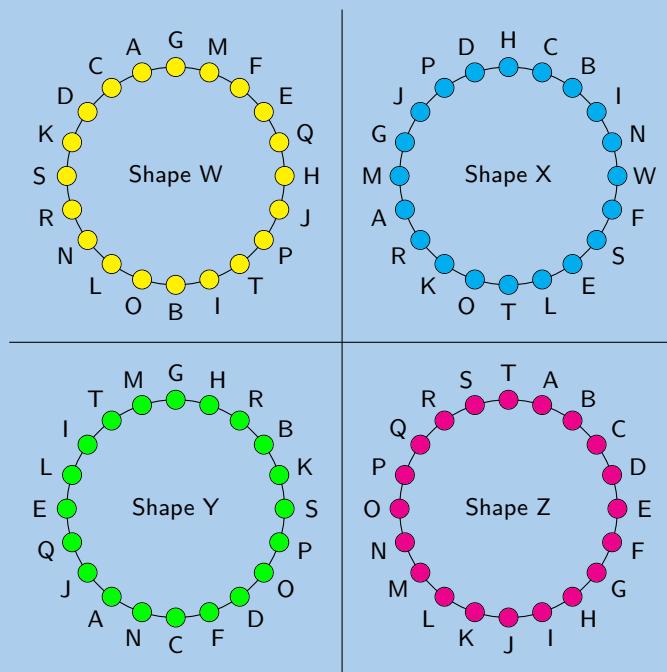
2. You are given the following diagram, with various points shown:



Which point lies at the coordinates  $(3; -5)$ ?

3. You are given the following diagram, with 4 shapes drawn.

All the shapes are identical, but each shape uses a different naming convention:

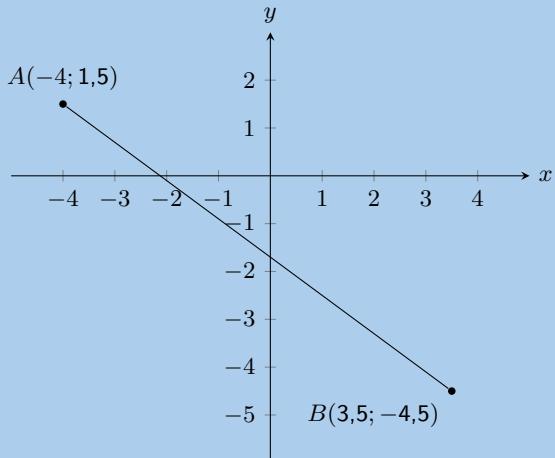


Which shape uses the correct naming convention?

4. Represent the following figures in the Cartesian plane:

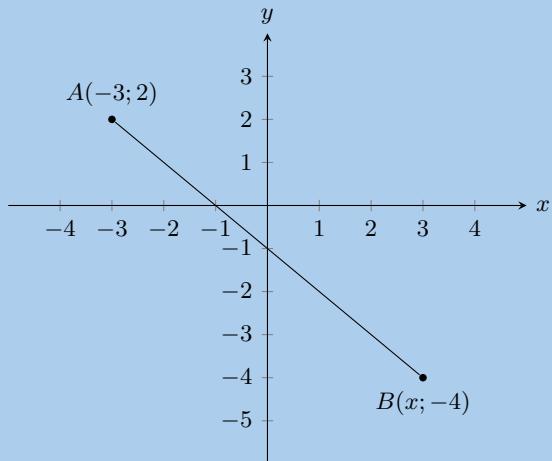
- Triangle  $DEF$  with  $D(1; 2)$ ,  $E(3; 2)$  and  $F(2; 4)$ .
- Quadrilateral  $GHIJ$  with  $G(2; -1)$ ,  $H(0; 2)$ ,  $I(-2; -2)$  and  $J(1; -3)$ .
- Quadrilateral  $MNOP$  with  $M(1; 1)$ ,  $N(-1; 3)$ ,  $O(-2; 3)$  and  $P(-4; 1)$ .
- Quadrilateral  $WXYZ$  with  $W(1; -2)$ ,  $X(-1; -3)$ ,  $Y(2; -4)$  and  $Z(3; -2)$ .

5. You are given the following diagram:



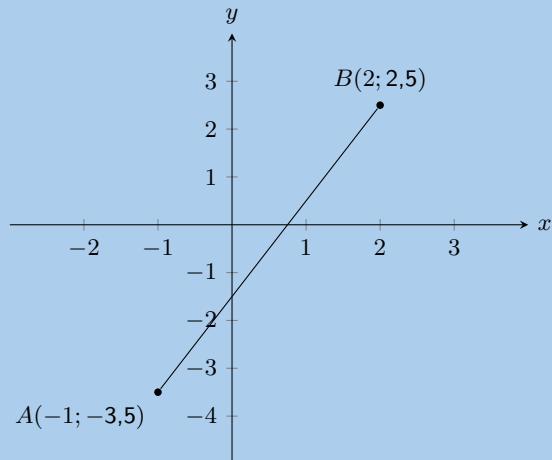
Calculate the length of line  $AB$ , correct to 2 decimal places.

6. The following picture shows two points on the Cartesian plane,  $A$  and  $B$ .



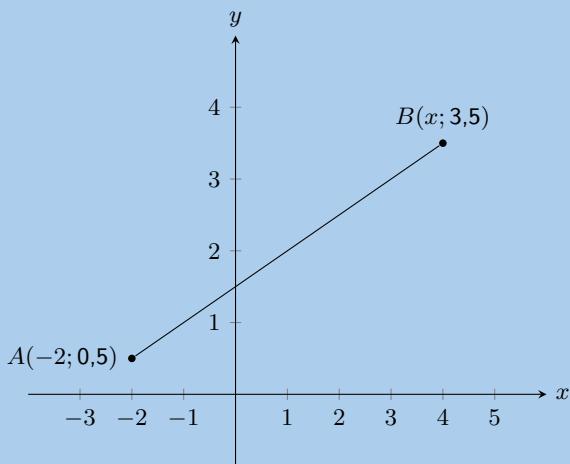
The distance between the points is 8,4853. Calculate the missing coordinate of point B.

7. You are given the following diagram:



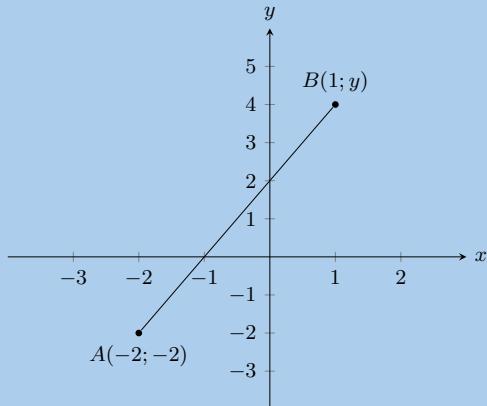
Calculate the gradient ( $m$ ) of line  $AB$ . The coordinates are  $A(-1; -3,5)$  and  $B(2; 2,5)$  respectively.

8. You are given the following diagram:



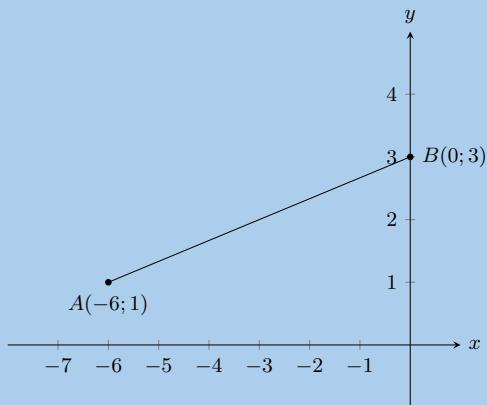
You are also told that line  $AB$  has a gradient,  $m$ , of 0,5. Calculate the missing co-ordinate of point B.

9. You are given the following diagram:



You are also told that line  $AB$  has a gradient,  $m$ , of 2. Calculate the missing co-ordinate of point  $B$ .

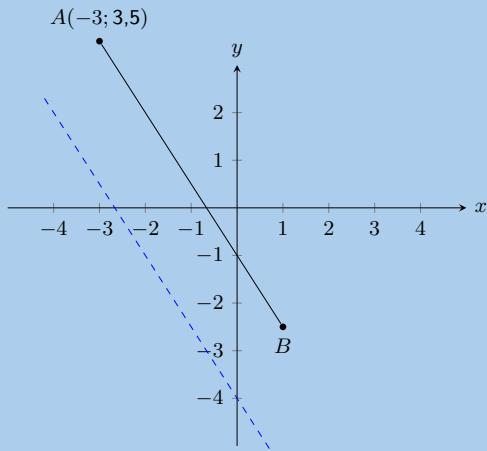
10. In the diagram,  $A$  is the point  $(-6; 1)$  and  $B$  is the point  $(0; 3)$ .



a) Find the equation of line  $AB$ .

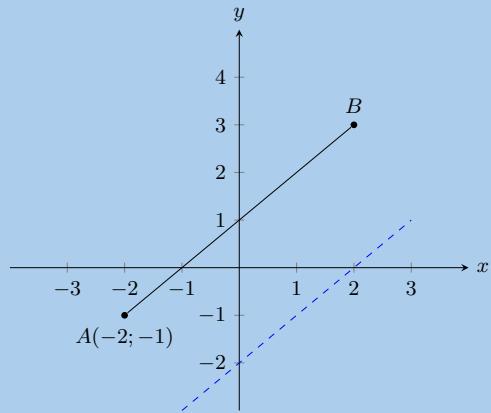
b) Calculate the length of  $AB$ .

11. You are given the following diagram:



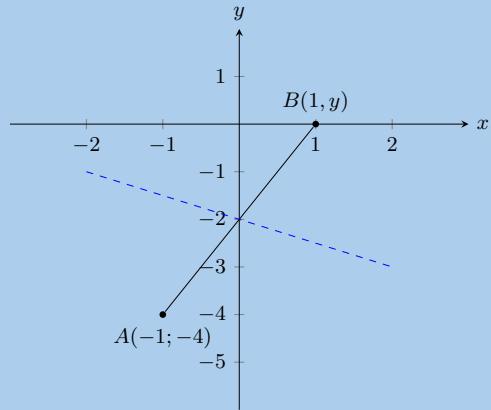
You are also told that line  $AB$  runs parallel to the following line:  $y = -1,5x - 4$ . Point  $A$  is at  $(-3; 3,5)$ . Find the equation of the line  $AB$ .

12. You are given the following diagram:



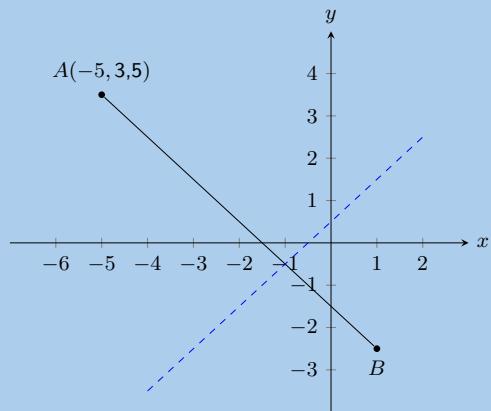
You are also told that line  $AB$  runs parallel to the following line:  $y = x - 2$ . Calculate the missing co-ordinate of point  $B(x; 3)$ .

13. You are given the following diagram:



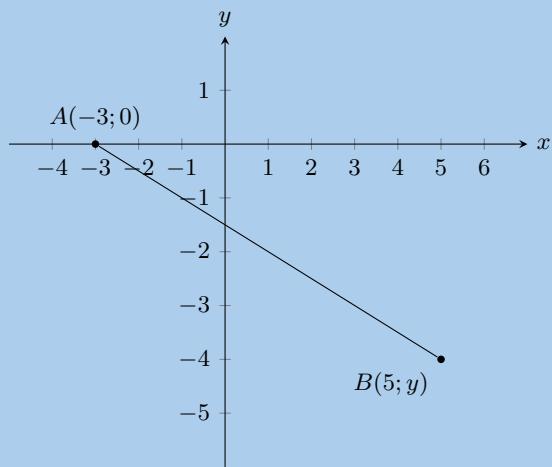
You are also told that line  $AB$  runs perpendicular to the following line:  $y = -0,5x - 2$ . Calculate the missing co-ordinate of point  $B$ .

14. The graph here shows line,  $AB$ . The blue dashed line is perpendicular to  $AB$ .



The equation of the blue dashed line is  $y = x + 0,5$ . Point  $A$  is at  $(-5; 3,5)$ . Determine the equation of line  $AB$ .

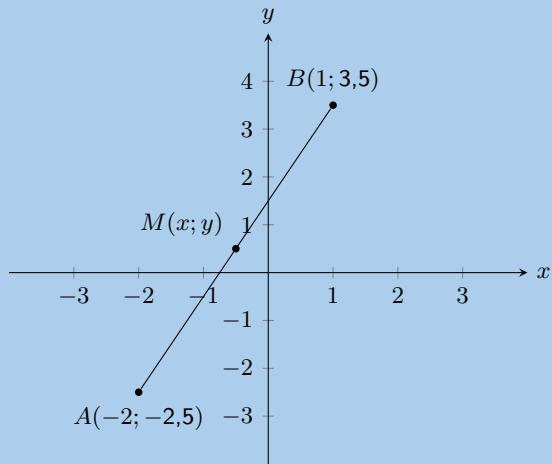
15. You are given the following diagram:



You are also told that line  $AB$  has the following equation:  $y = -0.5x - 1.5$ .

Calculate the missing co-ordinate of point  $B$ .

16. You are given the following diagram:

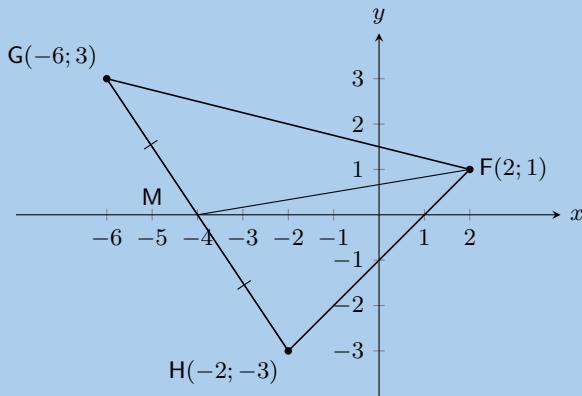


Calculate the coordinates of the mid-point ( $M$ ) between point  $A(-2; -2.5)$  and point  $B(1; 3.5)$  correct to 1 decimal place.

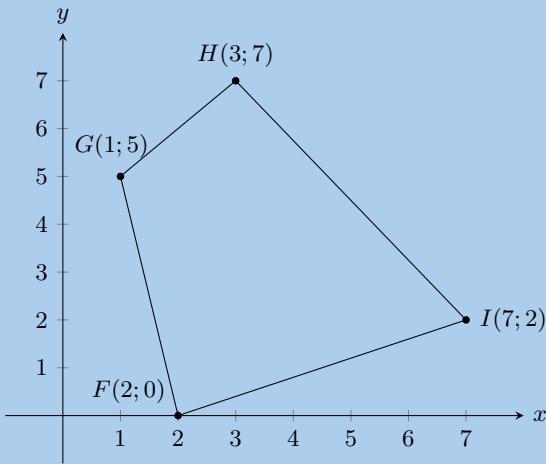
17.  $A(-2; 3)$  and  $B(2; 6)$  are points in the Cartesian plane.  $C(a; b)$  is the mid-point of  $AB$ . Find the values of  $a$  and  $b$ .

18. Determine the equations of the following straight lines:

- passing through  $P(5; 5)$  and  $Q(-2; 12)$ .
- parallel to  $y = 3x + 4$  and passing through  $(4; 0)$ .
- passing through  $F(2; 1)$  and the mid-point of  $GH$  where  $G(-6; 3)$  and  $H(-2; -3)$ .

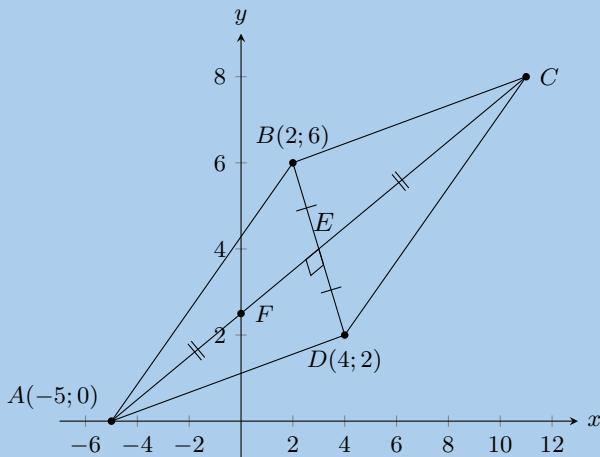


19. In the diagram below, the vertices of the quadrilateral are  $F(2; 0)$ ,  $G(1; 5)$ ,  $H(3; 7)$  and  $I(7; 2)$ .



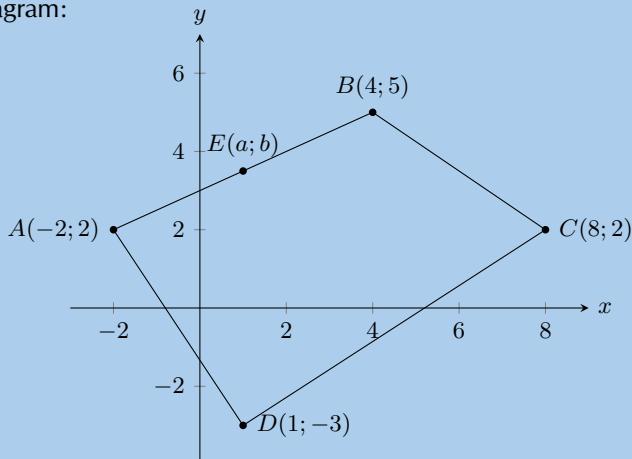
- a) Calculate the lengths of the sides of  $FGHI$ .
  - b) Are the opposite sides of  $FGHI$  parallel?
  - c) Do the diagonals of  $FGHI$  bisect each other?
  - d) Can you state what type of quadrilateral  $FGHI$  is? Give reasons for your answer.
20. Consider a quadrilateral  $ABCD$  with vertices  $A(3; 2)$ ,  $B(4; 5)$ ,  $C(1; 7)$  and  $D(1; 3)$ .
- a) Draw the quadrilateral.
  - b) Find the lengths of the sides of the quadrilateral.
21.  $ABCD$  is a quadrilateral with vertices  $A(0; 3)$ ,  $B(4; 3)$ ,  $C(5; -1)$  and  $D(-1; -1)$ .
- a) Show by calculation that:
    - i.  $AD = BC$
    - ii.  $AB \parallel DC$
  - b) What type of quadrilateral is  $ABCD$ ?
  - c) Show that the diagonals  $AC$  and  $BD$  do not bisect each other.
22.  $P$ ,  $Q$ ,  $R$  and  $S$  are the points  $(-2; 0)$ ,  $(2; 3)$ ,  $(5; 3)$  and  $(-3; -3)$  respectively.
- a) Show that:
    - i.  $SR = 2PQ$
    - ii.  $SR \parallel PQ$
  - b) Calculate:
    - i.  $PS$
    - ii.  $QR$
  - c) What kind of quadrilateral is  $PQRS$ ? Give reasons for your answer.

23.  $EFGH$  is a parallelogram with vertices  $E(-1; 2)$ ,  $F(-2; -1)$  and  $G(2; 0)$ . Find the coordinates of  $H$  by using the fact that the diagonals of a parallelogram bisect each other.
24.  $PQRS$  is a quadrilateral with points  $P(0; -3)$ ,  $Q(-2; 5)$ ,  $R(3; 2)$  and  $S(3; -2)$  in the Cartesian plane.
- Find the length of  $QR$ .
  - Find the gradient of  $PS$ .
  - Find the mid-point of  $PR$ .
  - Is  $PQRS$  a parallelogram? Give reasons for your answer.
25. Consider triangle  $ABC$  with vertices  $A(1; 3)$ ,  $B(4; 1)$  and  $C(6; 4)$ .
- Sketch triangle  $ABC$  on the Cartesian plane.
  - Show that  $ABC$  is an isosceles triangle.
  - Determine the coordinates of  $M$ , the mid-point of  $AC$ .
  - Determine the gradient of  $AB$ .
  - Show that  $D(7; -1)$  lies on the line that goes through  $A$  and  $B$ .
26.  $\triangle PQR$  has vertices  $P(1; 8)$ ,  $Q(8; 7)$  and  $R(7; 0)$ . Show through calculation that  $\triangle PQR$  is a right angled isosceles triangle.
27.  $\triangle ABC$  has vertices  $A(-3; 4)$ ,  $B(3; -2)$  and  $C(-5; -2)$ .  $M$  is the mid-point of  $AC$  and  $N$  is the mid-point of  $BC$ . Use  $\triangle ABC$  to prove the mid-point theorem using analytical geometry methods.
28. a) List two properties of a parallelogram.  
 b) The points  $A(-2; -4)$ ,  $B(-4; 1)$ ,  $C(2; 4)$  and  $D(4; -1)$  are the vertices of a quadrilateral. Show that the quadrilateral is a parallelogram.
29. The diagram shows a quadrilateral. The points  $B$  and  $D$  have the coordinates  $(2; 6)$  and  $(4; 2)$  respectively. The diagonals of  $ABCD$  bisect each other at right angles.  $F$  is the point of intersection of line  $AC$  with the  $y$ -axis.



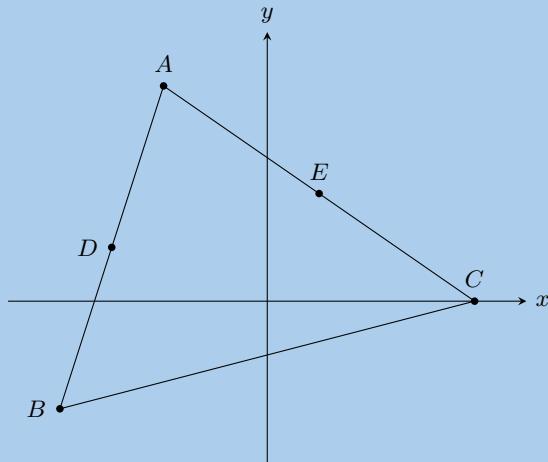
- Determine the gradient of  $AC$ .
  - Show that the equation of  $AC$  is given by  $2y = x + 5$ .
  - Determine the coordinates of  $C$ .
30.  $A(4; -1)$ ,  $B(-6; -3)$  and  $C(-2; 3)$  are the vertices of  $\triangle ABC$ .
- Find the length of  $BC$ , correct to 1 decimal place.
  - Calculate the gradient of  $AC$ .
  - If  $P$  has coordinates  $(-26; 19)$ , show that  $A$ ,  $C$  and  $P$  are collinear.
  - Determine the equation of line  $BC$ .
  - Show that  $\triangle ABC$  is right-angled.

31. Given the following diagram:

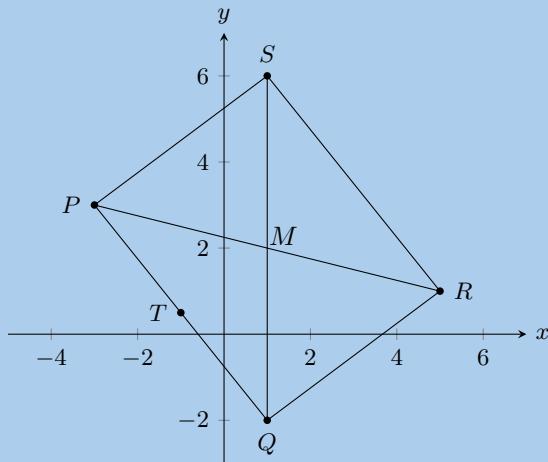


- a) If  $E$  is the mid-point of  $AB$ , find the values of  $a$  and  $b$ .
  - b) Find the equation of the line perpendicular to  $BC$ , which passes through the origin.
  - c) Find the coordinates of the mid-point of diagonal  $BD$ .
  - d) Hence show that  $ABCD$  is not a parallelogram.
  - e) If  $C$  could be moved, give its new coordinates so that  $ABCD$  would be a parallelogram.
32. A triangle has vertices  $A(-1; 7)$ ,  $B(8; 4)$  and  $C(5; -5)$ .
- a) Calculate the gradient of  $AB$ .
  - b) Prove that the triangle is right-angled at  $B$ .
  - c) Determine the length of  $AB$ .
  - d) Determine the equation of the line from  $A$  to the mid-point of  $BC$ .
  - e) Find the area of the triangle  $ABC$ .
33. A quadrilateral has vertices  $A(0; 5)$ ,  $B(-3; -4)$ ,  $C(0; -5)$  and  $D(4; k)$  where  $k \geq 0$ .
- a) What should  $k$  be so that  $AD$  is parallel to  $CD$ ?
  - b) What should  $k$  be so that  $CD = \sqrt{52}$ ?
34. On the Cartesian plane, the three points  $P(-3; 4)$ ,  $Q(7; -1)$  and  $R(3; b)$  are collinear.
- a) Find the length of  $PQ$ .
  - b) Find the gradient of  $PQ$ .
  - c) Find the equation of  $PQ$ .
  - d) Find the value of  $b$ .
35. Given  $A(4; 9)$  and  $B(-2; -3)$ .
- a) Find the mid-point  $M$  of  $AB$ .
  - b) Find the gradient of  $AB$ .
  - c) Find the gradient of the line perpendicular to  $AB$ .
  - d) Find the equation of the perpendicular bisector of  $AB$ .
  - e) Find the equation of the line parallel to  $AB$ , passing through  $(0; 6)$ .
36.  $L(-1; -1)$ ,  $M(-2; 4)$ ,  $N(x; y)$  and  $P(4; 0)$  are the vertices of parallelogram  $LMNP$ .
- a) Determine the coordinates of  $N$ .
  - b) Show that  $MP$  is perpendicular to  $LN$  and state what type of quadrilateral  $LMNP$  is, other than a parallelogram.
  - c) Show that  $LMNP$  is a square.

37.  $A(-2; 4)$ ,  $B(-4; -2)$  and  $C(4; 0)$  are the vertices of  $\triangle ABC$ .  $D$  and  $E(1; 2)$  are the mid-points of  $AB$  and  $AC$  respectively.

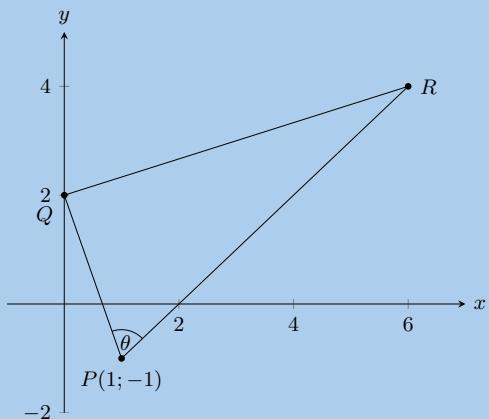


- a) Find the gradient of  $BC$ .
  - b) Show that the coordinates of  $D$ , the mid-point of  $AB$  are  $(-3; 1)$ .
  - c) Find the length of  $DE$ .
  - d) Find the gradient of  $DE$ . Make a conjecture regarding lines  $BC$  and  $DE$ .
  - e) Determine the equation of  $BC$ .
38. In the diagram points  $P(-3; 3)$ ,  $Q(1; -2)$ ,  $R(5; 1)$  and  $S(x; y)$  are the vertices of a parallelogram.

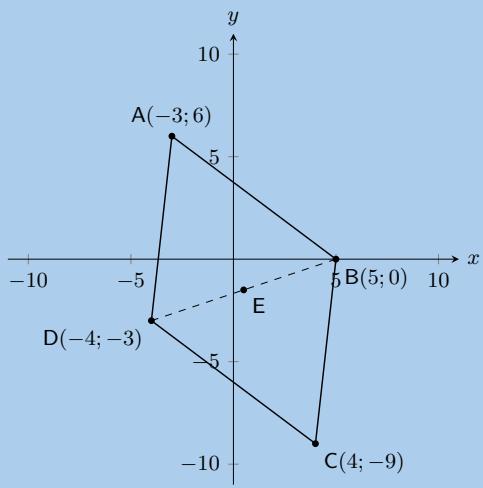


- a) Calculate the length of  $PQ$ .
  - b) Find the coordinates of  $M$  where the diagonals meet.
  - c) Find  $T$ , the mid-point of  $PQ$ .
  - d) Show that  $MT \parallel QR$ .
  - e) Calculate the coordinates of  $S$ .
39. The coordinates of  $\triangle PQR$  are as follows:  $P(5; 1)$ ,  $Q(1; 3)$  and  $R(1; -2)$ .
- a) Through calculation, determine whether the triangle is equilateral, isosceles or scalene. Be sure to show all your working.
  - b) Find the coordinates of points  $S$  and  $T$ , the mid-points of  $PQ$  and  $QR$ .
  - c) Determine the gradient of the line  $ST$ .
  - d) Prove that  $ST \parallel PR$ .

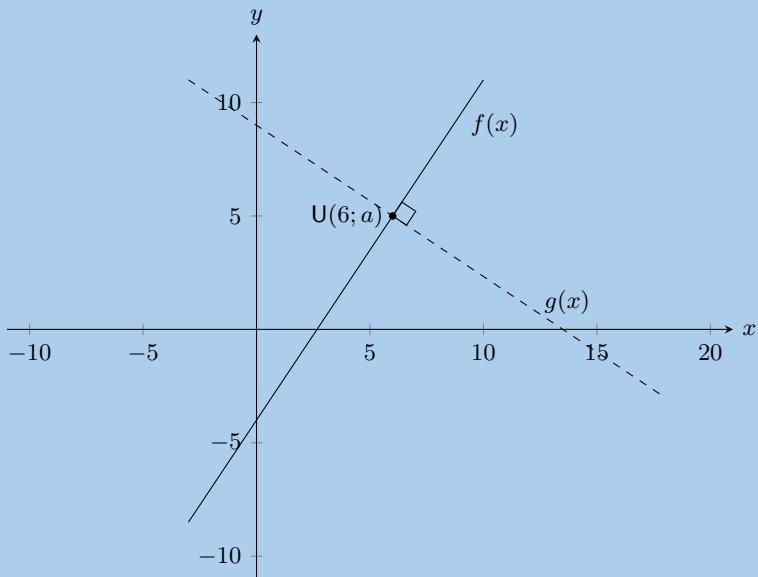
40. The following diagram shows  $\triangle PQR$  with  $P(-1; 1)$ . The equation of  $QR$  is  $x - 3y = -6$  and the equation of  $PR$  is  $x - y - 2 = 0$ .  $R\hat{P}Q = \theta$ .



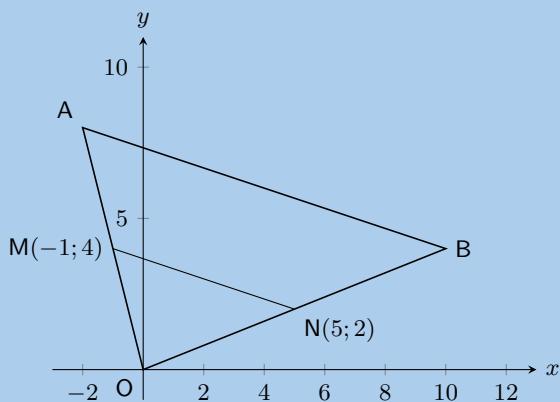
- Write down the coordinates of  $Q$ .
  - Prove that  $PQ \perp QR$ .
  - Write down the gradient of  $PR$ .
  - If the  $y$ -coordinate of  $R$  is 4, calculate the  $x$ -coordinate.
  - Find the equation of the line from  $P$  to  $S$  (the mid-point of  $QR$ ).
41. The points  $E(-3; 0)$ ,  $L(3; 5)$  and  $S(t + 1, 2, 5)$  are collinear.
- Determine the value of  $t$ .
  - Determine the values of  $a$  and  $b$  if the equation of the line passing through  $E$ ,  $L$  and  $S$  is  $\frac{x}{a} + \frac{y}{b} = 1$ .
42. Given:  $A(-3; -4)$ ,  $B(-1; -7)$ ,  $C(2; -5)$  and  $D(0; -2)$ .
- Calculate the distance  $AC$  and the distance  $BD$ . Leave your answers in surd form.
  - Determine the coordinates of  $M$ , the mid-point of  $BD$ .
  - Prove that  $AM \perp BD$ .
  - Prove that  $A$ ,  $M$  and  $C$  are collinear.
  - What type of quadrilateral is  $ABCD$ ?
43.  $M(2; -2)$  is the mid-point of  $AB$  with point  $A(3; 1)$ . Determine:
- the coordinates of  $B$ .
  - the gradient of  $AM$ .
  - the equation of the line  $AM$ .
  - the perpendicular bisector of  $AB$ .
44. ABCD is a quadrilateral with  $A(-3; 6)$ ,  $B(5; 0)$ ,  $C(4; -9)$ ,  $D(-4; -3)$ .



- a) Determine the coordinates of  $E$ , the mid-point of  $BD$ .
- b) Prove that  $ABCD$  is a parallelogram.
- c) Find the equation of diagonal  $BD$ .
- d) Determine the equation of the perpendicular bisector of  $BD$ .
- e) Determine the gradient of  $AC$ .
- f) Is  $ABCD$  a rhombus? Explain why or why not.
- g) Find the length of  $AB$ .
45. In the diagram below,  $f(x) = \frac{3}{2}x - 4$  is sketched with  $U(6; a)$  on  $f(x)$ .



- a) Determine the value of  $a$  in  $U(6; a)$ .
- b) A line,  $g(x)$ , passing through  $U$ , is perpendicular to  $f(x)$ .  $V(b; 4)$  lies on  $g(x)$ . Determine the value of  $b$ .
- c) If  $U(6; 5)$ ,  $V(7\frac{1}{2}, 4)$  and  $W(1; c)$  are collinear, determine the value of  $c$ .
46. In the diagram below,  $M$  and  $N$  are the mid-points of  $OA$  and  $OB$  respectively.



- a) Calculate the gradient of  $MN$ .
- b) Find the equation of the line through  $M$  and  $N$  in the form  $y = mx + c$ .
- c) Show that  $AB \parallel MN$ .

- d) Write down the value of the ratio:  $\frac{\text{area } \triangle OAB}{\text{area } \triangle OMN}$ .
- e) Write down the coordinates of  $P$  such that  $OAPB$  is a parallelogram.
47.  $A(6; -4)$ ,  $B(8; 2)$ ,  $C(3; a)$  and  $D(b; c)$  are points on the Cartesian plane. Determine the value of:
- $a$  if  $A$ ,  $B$  and  $C$  are collinear.
  - $b$  and  $c$  if  $B$  is the mid-point of  $A$  and  $D$ .

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1. <a href="#">2GDR</a>	2. <a href="#">2GDS</a>	3. <a href="#">2GDT</a>	4a. <a href="#">2GDV</a>	4b. <a href="#">2GDW</a>	4c. <a href="#">2GDX</a>
4d. <a href="#">2GDY</a>	5. <a href="#">2GDZ</a>	6. <a href="#">2GF2</a>	7. <a href="#">2GF3</a>	8. <a href="#">2GF4</a>	9. <a href="#">2GF5</a>
10. <a href="#">2GF6</a>	11. <a href="#">2GF7</a>	12. <a href="#">2GF8</a>	13. <a href="#">2GF9</a>	14. <a href="#">2GFB</a>	15. <a href="#">2GFC</a>
16. <a href="#">2GFD</a>	17. <a href="#">2GFF</a>	18. <a href="#">2GFG</a>	19. <a href="#">2GFH</a>	20. <a href="#">2GFJ</a>	21. <a href="#">2GFK</a>
22. <a href="#">2GFM</a>	23. <a href="#">2GFN</a>	24. <a href="#">2GFP</a>	25. <a href="#">2GFQ</a>	26. <a href="#">2GFR</a>	27. <a href="#">2GFS</a>
28. <a href="#">2GFT</a>	29. <a href="#">2GFV</a>	30. <a href="#">2GFW</a>	31. <a href="#">2GFX</a>	32. <a href="#">2GFY</a>	33. <a href="#">2GFZ</a>
34. <a href="#">2GG2</a>	35. <a href="#">2GG3</a>	36. <a href="#">2GG4</a>	37. <a href="#">2GG5</a>	38. <a href="#">2GG6</a>	39. <a href="#">2GG7</a>
40. <a href="#">2GG8</a>	41. <a href="#">2GG9</a>	42. <a href="#">2GGB</a>	43. <a href="#">2GGC</a>	44. <a href="#">2GGD</a>	45. <a href="#">2GGF</a>
46. <a href="#">2GGG</a>	47. <a href="#">2GGH</a>				



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# CHAPTER



## *Finance and growth*

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## 9.1 Introduction

EMA6K

In this chapter, we apply mathematical skills to everyday financial situations.

If you had R 1000, you could either keep it in your piggy bank, or deposit it into a bank account. If you deposit the money into a bank account, you are effectively lending money to the bank. Because you are lending the bank money, you can expect some extra money back. This is known as interest. Similarly, if you borrow money from a bank, then you can expect to pay interest on the loan. Interest is charged at a percentage of the money owed over the period of time it takes to pay back the loan. This means that the longer the loan exists, the more interest will have to be paid on it.



Figure 9.1: The entrance to the Johannesburg Stock Exchange (JSE) located in Sandton, Johannesburg - the financial centre of South Africa. The JSE is Africa's largest stock exchange and the 19<sup>th</sup> largest in the world. Each month, more than 60 billion rand worth of shares are traded on the JSE.

The concept is simple, yet it is core to the world of finance. Accountants, actuaries and bankers can spend their entire working career dealing with the effects of interest on financial matters.

**DEFINITION:** *Interest*

In finance, interest is the money charged for borrowing money. It is usually expressed as a percentage of the borrowed amount.

## 9.2 Simple interest

EMA6M

**DEFINITION:** *Simple interest*

Simple interest is interest calculated only on the initial amount that you invested.

As an easy example of simple interest, consider how much we will get by investing R 1000 for 1 year with a bank that pays 5% p.a. simple interest.

At the end of the year we have:

$$\begin{aligned}\text{Interest} &= \text{R } 1000 \times 5\% \\ &= \text{R } 1000 \times \frac{5}{100} \\ &= \text{R } 1000 \times 0,05 \\ &= \text{R } 50\end{aligned}$$

With an opening balance of R 1000 at the start of the year, the closing balance at the end of the year will therefore be:

$$\begin{aligned}\text{Closing balance} &= \text{Opening balance} + \text{Interest} \\ &= \text{R } 1000 + \text{R } 50 \\ &= \text{R } 1050\end{aligned}$$

The opening balance in financial calculations is often called the principal, denoted as  $P$  (R 1000 in the example). The interest rate is usually labelled  $i$  (5% p.a. in the example and “p.a.” means per annum or per year). The interest amount is labelled  $I$  (R 50 in the example).

So we can see that:

$$I = P \times i$$

and:

$$\begin{aligned}\text{Closing balance} &= \text{Opening balance} + \text{Interest} \\ &= P + I \\ &= P + P \times i \\ &= P(1 + i)\end{aligned}$$

The above calculations give a good idea of what the simple interest formula looks like. However, the example shows an investment that lasts for only one year. If the investment or loan is over a longer period, we need to take this into account. We use the symbol  $n$  to indicate time period, which must be given in years.

The general formula for calculating simple interest is

$$A = P(1 + in)$$

Where:

- $A$  = accumulated amount (final)
- $P$  = principal amount (initial)
- $i$  = interest written as decimal
- $n$  = number of years

### Worked example 1: Calculating interest on a deposit

#### QUESTION

Carine deposits R 1000 into a special bank account which pays a simple interest rate of 7% p.a. for 3 years. How much will be in her account at the end of the investment term?

#### SOLUTION

##### Step 1: Write down known values

$$\begin{aligned}P &= 1000 \\ i &= 0,07 \\ n &= 3\end{aligned}$$

**Step 2: Write down the formula**

$$A = P(1 + in)$$

**Step 3: Substitute the values**

$$\begin{aligned} A &= 1000(1 + 0,07 \times 3) \\ &= 1210 \end{aligned}$$

**Step 4: Write the final answer**

At the end of 3 years, Carine will have R 1210 in her bank account.

**Worked example 2: Calculating interest on a loan****QUESTION**

Sarah borrows R 5000 from her neighbour at an agreed simple interest rate of 12,5% p.a. She will pay back the loan in one lump sum at the end of 2 years. How much will she have to pay her neighbour?

**SOLUTION****Step 1: Write down the known variables**

$$\begin{aligned} P &= 5000 \\ i &= 0,125 \\ n &= 2 \end{aligned}$$

**Step 2: Write down the formula**

$$A = P(1 + in)$$

**Step 3: Substitute the values**

$$\begin{aligned} A &= 5000(1 + 0,125 \times 2) \\ &= 6250 \end{aligned}$$

**Step 4: Write the final answer**

At the end of 2 years, Sarah will pay her neighbour R 6250.

We can use the simple interest formula to find pieces of missing information. For example, if we have an amount of money that we want to invest for a set amount of time to achieve a goal amount, we can rearrange the variables to solve for the required interest rate. The same principles apply to finding the length of time we would need to invest the money, if we knew the principal and accumulated amounts and the interest rate.

**Important:** to get a more accurate answer, try to do all your calculations on the calculator in one go. This will prevent rounding off errors from influencing your final answer.

### Worked example 3: Determining the investment period to achieve a goal amount

#### QUESTION

Prashant deposits R 30 000 into a bank account that pays a simple interest rate of 7,5% p.a.. How many years must he invest for to generate R 45 000?

#### SOLUTION

##### Step 1: Write down the known variables

$$A = 45\ 000$$

$$P = 30\ 000$$

$$i = 0,075$$

##### Step 2: Write down the formula

$$A = P(1 + in)$$

##### Step 3: Substitute the values and solve for $n$

$$\begin{aligned} 45\ 000 &= 30\ 000(1 + 0,075 \times n) \\ \frac{45\ 000}{30\ 000} &= 1 + 0,075 \times n \\ \frac{45\ 000}{30\ 000} - 1 &= 0,075 \times n \\ \frac{\left(\frac{45\ 000}{30\ 000}\right) - 1}{0,075} &= n \\ n &= 6\frac{2}{3} \end{aligned}$$

##### Step 4: Write the final answer

It will take 6 years and 8 months to make R 45 000 from R 30 000 at a simple interest rate of 7,5% p.a.

### Worked example 4: Calculating the simple interest rate to achieve the desired growth

#### QUESTION

At what simple interest rate should Fritha invest if she wants to grow R 2500 to R 4000 in 5 years?

## SOLUTION

### Step 1: Write down the known variables

$$A = 4000$$

$$P = 2500$$

$$n = 5$$

### Step 2: Write down the formula

$$A = P(1 + in)$$

### Step 3: Substitute the values and solve for $i$

$$\begin{aligned} 4000 &= 2500(1 + i \times 5) \\ \frac{4000}{2500} &= 1 + i \times 5 \\ \frac{4000}{2500} - 1 &= i \times 5 \\ \frac{\left(\frac{4000}{2500}\right) - 1}{5} &= i \\ i &= 0,12 \end{aligned}$$

### Step 4: Write the final answer

A simple interest rate of 12% p.a. will be needed when investing R 2500 for 5 years to become R 4000.

### Exercise 9 – 1:

1. An amount of R 3500 is invested in a savings account which pays simple interest at a rate of 7,5% per annum. Calculate the balance accumulated by the end of 2 years.
2. An amount of R 4090 is invested in a savings account which pays simple interest at a rate of 8% per annum. Calculate the balance accumulated by the end of 4 years.
3. An amount of R 1250 is invested in a savings account which pays simple interest at a rate of 6% per annum. Calculate the balance accumulated by the end of 6 years.
4. An amount of R 5670 is invested in a savings account which pays simple interest at a rate of 8% per annum. Calculate the balance accumulated by the end of 3 years.
5. Calculate the accumulated amount in the following situations:
  - a) A loan of R 300 at a rate of 8% for 1 year.
  - b) An investment of R 2250 at a rate of 12,5% p.a. for 6 years.
6. A bank offers a savings account which pays simple interest at a rate of 6% per annum. If you want to accumulate R 15 000 in 5 years, how much should you invest now?

7. Sally wanted to calculate the number of years she needed to invest R 1000 for in order to accumulate R 2500. She has been offered a simple interest rate of 8,2% p.a. How many years will it take for the money to grow to R 2500?
8. Joseph deposited R 5000 into a savings account on his son's fifth birthday. When his son turned 21, the balance in the account had grown to R 18 000. If simple interest was used, calculate the rate at which the money was invested.
9. When his son was 6 years old, Methuli made a deposit of R 6610 in the bank. The investment grew at a simple interest rate and when Methuli's son was 18 years old, the value of the investment was R 11 131,24.  
At what rate was the money invested? Give the answer correct to one decimal place.
10. When his son was 6 years old, Phillip made a deposit of R 5040 in the bank. The investment grew at a simple interest rate and when Phillip's son was 18 years old, the value of the investment was R 7338,24.  
At what rate was the money invested? Give your answer correct to one decimal place.
11. When his son was 10 years old, Lefu made a deposit of R 2580 in the bank. The investment grew at a simple interest rate and when Lefu's son was 20 years old, the value of the investment was R 3689,40.  
At what rate was the money invested? Give your answer correct to one decimal place.
12. Abdoul wants to invest R 1080 at a simple interest rate of 10,9% p.a.  
How many years will it take for the money to grow to R 3348? Round **up** your answer to the nearest year.
13. Andrew wants to invest R 3010 at a simple interest rate of 11,9% p.a.  
How many years will it take for the money to grow to R 14 448? Round **up** your answer to the nearest year.

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1. [2GGJ](#)   2. [2GK](#)   3. [2GGM](#)   4. [2GGN](#)   5a. [2GGP](#)   5b. [2GQQ](#)   6. [2GGR](#)   7. [2GGS](#)  
 8. [2GGT](#)   9. [2GGV](#)   10. [2GGW](#)   11. [2GGX](#)   12. [2GGY](#)   13. [2GGZ](#)



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## 9.3 Compound interest

EMA6N

Compound interest allows interest to be earned on interest. With simple interest, only the original investment earns interest, but with compound interest, the original investment and the interest earned on it, both earn interest. Compound interest is advantageous for investing money but not for taking out a loan.

**DEFINITION:** *Compound interest*

Compound interest is the interest earned on the principal amount and on its accumulated interest.

Consider the example of R 1000 invested for 3 years with a bank that pays 5% p.a. compound interest.

At the end of the first year, the accumulated amount is

$$\begin{aligned} A_1 &= P(1 + i) \\ &= 1000(1 + 0,05) \\ &= 1050 \end{aligned}$$

The amount  $A_1$  becomes the new principal amount for calculating the accumulated amount at the end of the second year.

$$\begin{aligned}A_2 &= P(1+i) \\&= 1050(1+0,05) \\&= 1000(1+0,05)(1+0,05) \\&= 1000(1+0,05)^2\end{aligned}$$

Similarly, we use the amount  $A_2$  as the new principal amount for calculating the accumulated amount at the end of the third year.

$$\begin{aligned}A_3 &= P(1+i) \\&= 1000(1+0,05)^2(1+0,05) \\&= 1000(1+0,05)^3\end{aligned}$$

Do you see a pattern?

Using the formula for simple interest, we can develop a similar formula for compound interest.

With an opening balance  $P$  and an interest rate of  $i$ , the closing balanced at the end of the first year is:

$$\text{Closing balance after 1 year} = P(1+i)$$

This is the same as simple interest because it only covers a single year. This closing balance becomes the opening balance for the second year of investment.

$$\begin{aligned}\text{Closing balance after 2 years} &= [P(1+i)] \times (1+i) \\&= P(1+i)^2\end{aligned}$$

And similarly, for the third year

$$\begin{aligned}\text{Closing balance after 3 years} &= [P(1+i)^2] \times (1+i) \\&= P(1+i)^3\end{aligned}$$

We see that the power of the term  $(1+i)$  is the same as the number of years. Therefore the general formula for calculating compound interest is:

$$A = P(1+i)^n$$

Where:

$A$  = accumulated amount

$P$  = principal amount

$i$  = interest written as a decimal

$n$  = number of years

## Worked example 5: Compound interest

### QUESTION

Mpho wants to invest R 30 000 into an account that offers a compound interest rate of 6% p.a. How much money will be in the account at the end of 4 years?

### SOLUTION

#### Step 1: Write down the known variables

$$P = 30\ 000$$

$$i = 0,06$$

$$n = 4$$

#### Step 2: Write down the formula

$$A = P(1 + i)^n$$

#### Step 3: Substitute the values

$$\begin{aligned} A &= 30\ 000(1 + 0,06)^4 \\ &= 37\ 874,31 \end{aligned}$$

#### Step 4: Write the final answer

Mpho will have R 37 874,31 in the account at the end of 4 years.

## Worked example 6: Calculating the compound interest rate to achieve the desired growth

### QUESTION

Charlie has been given R 5000 for his sixteenth birthday. Rather than spending it, he has decided to invest it so that he can put down a deposit of R 10 000 on a car on his eighteenth birthday. What compound interest rate does he need to achieve this growth? Comment on your answer.

### SOLUTION

#### Step 1: Write down the known variables

$$A = 10\ 000$$

$$P = 5000$$

$$n = 2$$

### Step 2: Write down the formula

$$A = P(1 + i)^n$$

### Step 3: Substitute the values and solve for $i$

$$\begin{aligned}10\ 000 &= 5000(1 + i)^2 \\ \frac{10\ 000}{5000} &= (1 + i)^2 \\ \sqrt{\frac{10\ 000}{5000}} &= 1 + i \\ \sqrt{\frac{10\ 000}{5000}} - 1 &= i \\ i &= 0,4142\end{aligned}$$

### Step 4: Write the final answer and comment

Charlie needs to find an account that offers a compound interest rate of 41,42% p.a. to achieve the desired growth. A typical savings account gives a return of approximately 2% p.a. and an aggressive investment portfolio gives a return of approximately 13% p.a. It therefore seems unlikely that Charlie will be able to invest his money at an interest rate of 41,42% p.a.

## The power of compound interest

EMA6P

To illustrate how important “interest on interest” is, we compare the difference in closing balances for an investment earning simple interest and an investment earning compound interest. Consider an amount of R 10 000 invested for 10 years, at an interest rate of 9% p.a.

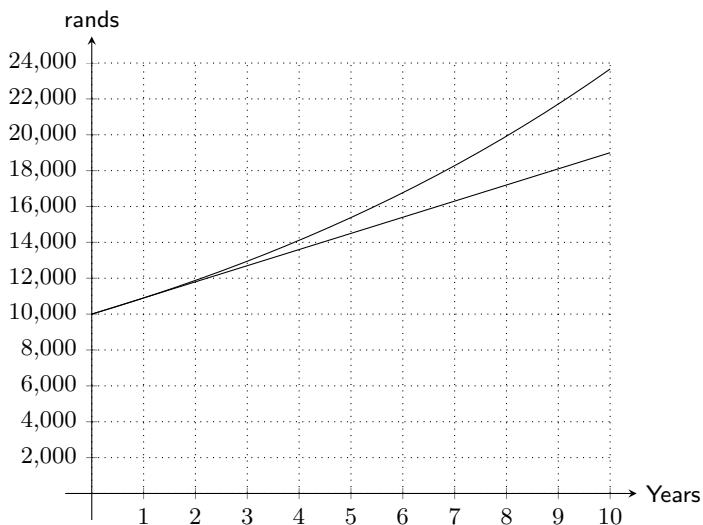
The closing balance for the investment earning simple interest is

$$\begin{aligned}A &= P(1 + in) \\ &= 10\ 000(1 + 0,09 \times 10) \\ &= \text{R } 19\ 000\end{aligned}$$

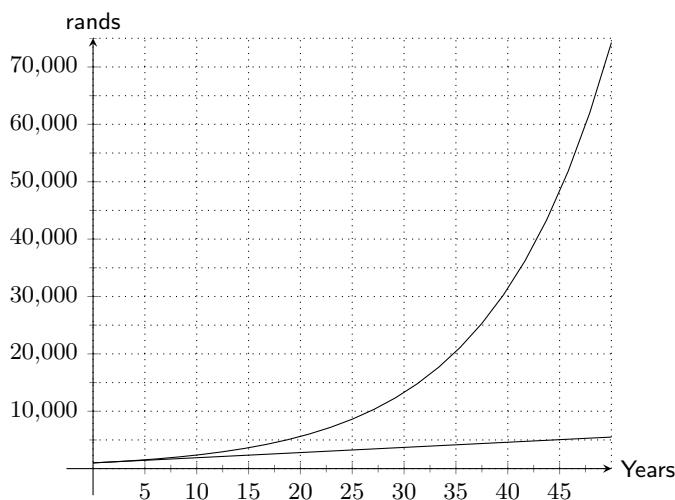
The closing balance for the investment earning compound interest is

$$\begin{aligned}A &= P(1 + i)^n \\ &= 10\ 000(1 + 0,09)^{10} \\ &= \text{R } 23\ 673,64\end{aligned}$$

We plot the growth of the two investments on the same set of axes and note the significant different in their rate of change: simple interest is a straight line graph and compound interest is an exponential graph.



It is easier to see the vast difference in growth if we extend the time period to 50 years:



Keep in mind that this is good news and bad news. When earning interest on money invested, compound interest helps that amount to grow exponentially. But if money is borrowed the accumulated amount of money owed will increase exponentially too.

#### VISIT:

This video explains the difference between simple and compound interest. Note that the video uses dollars but the calculation is the same for rands.

▶ See video: [2GH3](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

#### Exercise 9 – 2:

- An amount of R 3500 is invested in a savings account which pays a compound interest rate of 7,5% p.a. Calculate the balance accumulated by the end of 2 years.
- An amount of R 3070 is invested in a savings account which pays a compound interest rate of 11,6% p.a. Calculate the balance accumulated by the end of 6 years. As usual with financial calculations, round your answer to two decimal places, but do not round off until you have reached the solution.

3. An amount of R 6970 is invested in a savings account which pays a compound interest rate of 10,2% p.a. Calculate the balance accumulated by the end of 3 years. As usual with financial calculations, round your answer to two decimal places, but do not round off until you have reached the solution.
4. Nicola wants to invest some money at a compound interest rate of 11% p.a. How much money (to the nearest rand) should be invested if she wants to reach a sum of R 100 000 in five years time?
5. Thobeka wants to invest some money at a compound interest rate of 11,8% p.a. How much money should be invested if she wants to reach a sum of R 30 000 in 2 years' time? Round up your answer to the nearest rand.
6. Likengkeng wants to invest some money at a compound interest rate of 11,4% p.a. How much money should be invested if she wants to reach a sum of R 38 200 in 7 years' time? Round up your answer to the nearest rand.
7. Morgan invests R 5000 into an account which pays out a lump sum at the end of 5 years. If he gets R 7500 at the end of the period, what compound interest rate did the bank offer him?
8. Kabir invests R 1790 into an account which pays out a lump sum at the end of 9 years. If he gets R 2613,40 at the end of the period, what compound interest rate did the bank offer him? Give the answer correct to one decimal place.
9. Bongani invests R 6110 into an account which pays out a lump sum at the end of 7 years. If he gets R 6904,30 at the end of the period, what compound interest rate did the bank offer him? Give the answer correct to one decimal place.

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1. [2GH4](#) 2. [2GH5](#) 3. [2GH6](#) 4. [2GH7](#) 5. [2GH8](#) 6. [2GH9](#) 7. [2GHB](#) 8. [2GHC](#) 9. [2GHD](#)



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## 9.4 Calculations using simple and compound interest

EMA6Q

### Hire purchase

EMA6R

As a general rule, it is not wise to buy items on credit. When buying on credit you have to borrow money to pay for the object, meaning you will have to pay more for it due to the interest on the loan. That being said, occasionally there are appliances, such as a fridge, that are very difficult to live without. Most people don't have the cash up front to purchase such items, so they buy it on a hire purchase agreement.

A hire purchase agreement is a financial agreement between the shop and the customer about how the customer will pay for the desired product. The interest on a hire purchase loan is always charged at a simple interest rate and only charged on the amount owing. Most agreements require that a deposit is paid before the product can be taken by the customer. The principal amount of the loan is therefore the cash price minus the deposit. The accumulated loan will be worked out using the number of years the loan is needed for. The total loan amount is then divided into monthly payments over the period of the loan.

#### **IMPORTANT!**

Hire purchase is charged at a simple interest rate. When you are asked a hire purchase question, don't forget to always use the simple interest formula.

**VISIT:**

This video explains hire purchase and shows some examples of hire purchase calculations.

► See video: [2GHF](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

**Worked example 7: Hire purchase****QUESTION**

Troy wants to buy an additional screen for his computer which he saw advertised for R 2500 on the internet. There is an option of paying a 10% deposit and then making 24 monthly payments using a hire purchase agreement, where interest is calculated at 7,5% p.a. simple interest. Calculate what Troy's monthly payments will be.

**SOLUTION****Step 1: Write down the known variables**

A new opening balance is required, as the 10% deposit is paid in cash.

$$\begin{aligned}10\% \text{ of } 2500 &= 250 \\ \therefore P &= 2500 - 250 = 2250 \\ i &= 0,075 \\ n &= \frac{24}{12} = 2\end{aligned}$$

**Step 2: Write down the formula**

$$A = P(1 + in)$$

**Step 3: Substitute the values**

$$\begin{aligned}A &= 2250(1 + 0,075 \times 2) \\ &= 2587,50\end{aligned}$$

**Step 4: Calculate the monthly repayments on the hire purchase agreement**

$$\begin{aligned}\text{Monthly payment} &= \frac{2587,50}{24} \\ &= 107,81\end{aligned}$$

**Step 5: Write the final answer**

Troy's monthly payment is R 107,81.

A shop can also add a monthly insurance premium to the monthly instalments. This insurance premium will be an amount of money paid monthly and gives the customer more time between a missed payment and possible repossession of the product.

**NOTE:**

The monthly payment is also called the monthly instalment.

**Worked example 8: Hire purchase with extra conditions****QUESTION**

Cassidy wants to buy a TV and decides to buy one on a hire purchase agreement. The TV's cash price is R 5500. She will pay it off over 54 months at an interest rate of 21% p.a. An insurance premium of R 12,50 is added to every monthly payment. How much are her monthly payments?

**SOLUTION****Step 1: Write down the known variables**

$$P = 5500$$

$$i = 0,21$$

$$n = \frac{54}{12} = 4,5$$

The question does not mention a deposit, therefore we assume that Cassidy did not pay one.

**Step 2: Write down the formula**

$$A = P(1 + in)$$

**Step 3: Substitute the values**

$$\begin{aligned} A &= 5500(1 + 0,21 \times 4,5) \\ &= 10\,697,50 \end{aligned}$$

**Step 4: Calculate the monthly repayments on the hire purchase agreement**

$$\begin{aligned} \text{Monthly payment} &= \frac{10\,697,50}{54} \\ &= 198,10 \end{aligned}$$

**Step 5: Add the insurance premium**

$$198,10 + 12,50 = 210,60$$

**Step 6: Write the final answer**

Cassidy will pay R 210,60 per month for 54 months until her TV is paid off.

### Exercise 9 – 3:

1. Angelique wants to buy a microwave on a hire purchase agreement. The cash price of the microwave is R 4400. She is required to pay a deposit of 10% and pay the remaining loan amount off over 12 months at an interest rate of 9% p.a.
  - a) What is the principal loan amount?
  - b) What is the accumulated loan amount?
  - c) What are Angelique's monthly repayments?
  - d) What is the total amount she has paid for the microwave?
2. Nyakallo wants to buy a television on a hire purchase agreement. The cash price of the television is R 5600. She is required to pay a deposit of 15% and pay the remaining loan amount off over 24 months at an interest rate of 14% p.a.
  - a) What is the principal loan amount?
  - b) What is the accumulated loan amount?
  - c) What are Nyakallo's monthly repayments?
  - d) What is the total amount she has paid for the television?
3. A company wants to purchase a printer. The cash price of the printer is R 4500. A deposit of 15% is required on the printer. The remaining loan amount will be paid off over 24 months at an interest rate of 12% p.a.
  - a) What is the principal loan amount?
  - b) What is the accumulated loan amount?
  - c) How much will the company pay each month?
  - d) What is the total amount the company paid for the printer?
4. Sandile buys a dining room table costing R 8500 on a hire purchase agreement. He is charged an interest rate of 17,5% p.a. over 3 years.
  - a) How much will Sandile pay in total?
  - b) How much interest does he pay?
  - c) What is his monthly instalment?
5. Mike buys a table costing R 6400 on a hire purchase agreement. He is charged an interest rate of 15% p.a. over 4 years.
  - a) How much will Mike pay in total?
  - b) How much interest does he pay?
  - c) What is his monthly instalment?
6. Talwar buys a cupboard costing R 5100 on a hire purchase agreement. He is charged an interest rate of 12% p.a. over 2 years.
  - a) How much will Talwar pay in total?
  - b) How much interest does he pay?
  - c) What is his monthly instalment?
7. A lounge suite is advertised for sale on TV, to be paid off over 36 months at R 150 per month.
  - a) Assuming that no deposit is needed, how much will the buyer pay for the lounge suite once it has been paid off?
  - b) If the interest rate is 9% p.a., what is the cash price of the suite?
8. Two stores are offering a fridge and washing machine combo package. Store A offers a monthly payment of R 350 over 24 months. Store B offers a monthly payment of R 175 over 48 months.

If both stores offer 7,5% interest, which store should you purchase the fridge and washing machine from if you want to pay the least amount of interest?

9. Tlali wants to buy a new computer and decides to buy one on a hire purchase agreement. The computers cash price is R 4250. He will pay it off over 30 months at an interest rate of 9,5% p.a. An insurance premium of R 10,75 is added to every monthly payment. How much are his monthly payments?
10. Richard is planning to buy a new stove on hire purchase. The cash price of the stove is R 6420. He has to pay a 10% deposit and then pay the remaining amount off over 36 months at an interest rate of 8% p.a. An insurance premium of R 11,20 is added to every monthly payment. Calculate Richard's monthly payments.

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1. 2GHG    2. 2GHH    3. 2GHJ    4. 2GHK    5. 2GHM    6. 2GHN  
7. 2GHP    8. 2GHQ    9. 2GHR    10. 2GHS



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## Inflation

EMA6S

There are many factors that influence the change in price of an item, one of them is inflation. Inflation is the average increase in the price of goods each year and is given as a percentage. Since the rate of inflation increases year on year, it is calculated using the compound interest formula.

### Worked example 9: Calculating future cost based on inflation

#### QUESTION

Milk costs R 14 for two litres. How much will it cost in 4 years time if the inflation rate is 9% p.a.?

#### SOLUTION

##### Step 1: Write down the known variables

$$\begin{aligned}P &= 14 \\i &= 0,09 \\n &= 4\end{aligned}$$

##### Step 2: Write down the formula

$$A = P(1 + i)^n$$

##### Step 3: Substitute the values

$$\begin{aligned}A &= 14(1 + 0,09)^4 \\&= 19,76\end{aligned}$$

##### Step 4: Write the final answer

In four years time, two litres of milk will cost R 19,76.

## Worked example 10: Calculating past cost based on inflation

### QUESTION

A box of chocolates costs R 55 today. How much did it cost 3 years ago if the average rate of inflation was 11% p.a.?

### SOLUTION

#### Step 1: Write down the known variables

$$A = 55$$

$$i = 0,11$$

$$n = 3$$

#### Step 2: Write down the formula

$$A = P(1 + i)^n$$

#### Step 3: Substitute the values and solve for $P$

$$\begin{aligned} 55 &= P(1 + 0,11)^3 \\ \frac{55}{(1 + 0,11)^3} &= P \\ \therefore P &= 40,22 \end{aligned}$$

#### Step 4: Write the final answer

Three years ago, the box of chocolates would have cost R 40,22.

### Exercise 9 – 4:

1. The price of a bag of apples is R 12. How much will it cost in 9 years time if the inflation rate is 12% p.a.?
2. The price of a bag of potatoes is R 15. How much will it cost in 6 years time if the inflation rate is 12% p.a.?
3. The price of a box of popcorn is R 15. How much will it cost in 4 years time if the inflation rate is 11% p.a.?
4. A box of raisins costs R 24 today. How much did it cost 4 years ago if the average rate of inflation was 13% p.a.? Round your answer to 2 decimal places.
5. A box of biscuits costs R 24 today. How much did it cost 5 years ago if the average rate of inflation was 11% p.a.? Round your answer to 2 decimal places.
6. If the average rate of inflation for the past few years was 7,3% p.a. and your water and electricity account is R 1425 on average, what would you expect to pay in 6 years time?

7. The price of popcorn and a cooldrink at the movies is now R 60. If the average rate of inflation is 9,2% p.a. what was the price of popcorn and cooldrink 5 years ago?

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1. [2GHT](#) 2. [2GHV](#) 3. [2GHW](#) 4. [2GHX](#) 5. [2GHY](#) 6. [2GHZ](#) 7. [2GJ2](#)



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## Population growth

EMA6T

Family trees increase exponentially as every person born has the ability to start another family. For this reason we calculate population growth using the compound interest formula.

### Worked example 11: Population growth

#### QUESTION

If the current population of Johannesburg is 3 888 180, and the average rate of population growth in South Africa is 2,1% p.a., what can city planners expect the population of Johannesburg to be in 10 years?

#### SOLUTION

##### Step 1: Write down the known variables

$$P = 3\ 888\ 180$$

$$i = 0,021$$

$$n = 10$$

##### Step 2: Write down the formula

$$A = P(1 + i)^n$$

##### Step 3: Substitute the values

$$\begin{aligned} A &= 3\ 888\ 180(1 + 0,021)^{10} \\ &= 4\ 786\ 343 \end{aligned}$$

##### Step 4: Write the final answer

City planners can expect Johannesburg's population to be 4 786 343 in ten years time.

### Exercise 9 – 5:

1. The current population of Durban is 3 879 090 and the average rate of population growth in South Africa is 1,1% p.a.  
What can city planners expect the population of Durban to be in 6 years time? Round your answer to the nearest integer.
2. The current population of Polokwane is 3 878 970 and the average rate of population growth in South Africa is 0,7% p.a.  
What can city planners expect the population of Polokwane to be in 12 years time? Round your answer to the nearest integer.
3. A small town in Ohio, USA is experiencing a huge increase in births. If the average growth rate of the population is 16% p.a., how many babies will be born to the 1600 residents in the next 2 years?

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2GJ3](#)   2. [2GJ4](#)   3. [2GJ5](#)



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## 9.5 Foreign exchange rates

EMA6V

Different countries have their own currencies. In England, a Big Mac from McDonald's costs £ 4, in South Africa it costs R 20 and in Norway it costs 48 kr. The meal is the same in all three countries but in some places it costs more than in others. If £ 1 = R 12,41 and 1 kr = R 1,37, this means that a Big Mac in England costs R 49,64 and a Big Mac in Norway costs R 65,76.

Exchange rates affect a lot more than just the price of a Big Mac. The price of oil increases when the South African rand weakens. This is because when the rand is weaker, we can buy less of other currencies with the same amount of money.

A currency gets stronger when money is invested in the country. When we buy products that are made in South Africa, we are investing in South African business and keeping the money in the country. When we buy products imported from other countries, we are investing money in those countries and as a result, the rand will weaken. The more South African products we buy, the greater the demand for them will be and more jobs will become available for South Africans. Local is lekker!

#### NOTE:

The three currencies you are most likely to see are the British pound (£), the American dollar (\$) and the euro (€).

#### VISIT:

This video explains exchange rates and shows some examples of exchange rate calculations.

▶ See video: [2GJ6](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 12: Foreign exchange rates

#### QUESTION

Saba wants to travel to see her family in Spain. She has been given R 10 000 spending money. How many euros can she buy if the exchange rate is currently € 1 = R 10,68?

## SOLUTION

### Step 1: Write down the equation

Let the equivalent amount in euros be  $x$

$$\begin{aligned}x &= \frac{10\ 000}{10,68} \\&= 936,33\end{aligned}$$

### Step 2: Write the final answer

Saba can buy € 936,33 with R 10 000.

### Exercise 9 – 6:

1. Bridget wants to buy an iPod that costs £ 100, with the exchange rate currently at £ 1 = R 14. She estimates that the exchange rate will drop to R 12 in a month.
  - a) How much will the iPod cost in rands, if she buys it now?
  - b) How much will she save if the exchange rate drops to R 12?
  - c) How much will she lose if the exchange rate moves to R 15?
2. Mthuli wants to buy a television that costs £ 130, with the exchange rate currently at £ 1 = R 11. He estimates that the exchange rate will drop to R 9 in a month.
  - a) How much will the television cost in rands, if he buys it now?
  - b) How much will he save if the exchange rate drops to R 9?
  - c) How much will he lose if the exchange rate moves to R 19?
3. Nthabiseng wants to buy an iPad that costs £ 120, with the exchange rate currently at £ 1 = R 14. She estimates that the exchange rate will drop to R 9 in a month.
  - a) How much will the iPad cost, in rands, if she buys it now?
  - b) How much will she save if the exchange rate drops to R 9?
  - c) How much will she lose if the exchange rate moves to R 18?
4. Study the following exchange rate table:

Country	Currency	Exchange Rate
United Kingdom (UK)	Pounds (£)	R 14,13
United States (USA)	Dollars (\$)	R 7,04

- a) In South Africa the cost of a new Honda Civic is R 173 400. In England the same vehicle costs £ 12 200 and in the USA \$21 900. In which country is the car the cheapest?
  - b) Sollie and Arinda are waiters in a South African restaurant attracting many tourists from abroad. Sollie gets a £ 6 tip from a tourist and Arinda gets \$12. Who got the better tip?
5. Yaseen wants to buy a book online. He finds a publisher in London selling the book for £ 7,19. This publisher is offering free shipping on the product.  
He then finds the same book from a publisher in New York for \$8,49 with a shipping fee of \$2. Next he looks up the exchange rates to see which publisher has the better deal. If \$1 = R 11,48 and £ 1 = R 17,36, which publisher should he buy the book from?
6. Mathe is saving up to go visit her friend in Germany. She estimates the total cost of her trip to be R 50 000. The exchange rate is currently € 1 = R 13,22.  
Her friend decides to help Mathe out by giving her € 1000. How much (in rand) does Mathe now need to save up?

7. Lulamile and Jacob give tours over the weekends. They do not charge for these tours but instead accept tips from the group. The table below shows the total amount of tips they receive from various tour groups.

*The current exchange rates are:*

Group	Total tips
British tourists	£ 5,50
Japanese tourists	¥ 85,50
American tourists	\$ 7,00
Dutch tourists	€ 9,70
Brazilian tourists	40,50 BRL
Australian tourists	9,20 AUD
South African tourists	R 55,00

$$\begin{aligned} \text{£ } 1 &= \text{R } 17,12 \\ \text{¥ } 1 &= \text{R } 0,10 \\ \$ 1 &= \text{R } 11,42 \\ \text{€ } 1 &= \text{R } 12,97 \\ 1 \text{ BRL} &= \text{R } 4,43 \\ 1 \text{ AUD} &= \text{R } 9,12 \end{aligned}$$

- a) Which group of tourists tipped the most? How much did they tip (give your answer in rand)?  
b) Which group of tourists tipped the least? How much did they tip (give your answer in rand)?
8. Kayla is planning a trip to visit her family in Malawi followed by spending some time in Tanzania at the Serengeti. She will first need to convert her South African rands into the Malawian kwacha. After that she will convert her remaining Malawian kwacha into Tanzanian shilling.  
She looks up the current exchange rates and finds the following information:

$$\begin{aligned} \text{R } 1 &= 39,46 \text{ MWK} \\ 1 \text{ MWK} &= 4,01 \text{ TZS} \end{aligned}$$

She starts off with R 5000 in South Africa. In Malawi she spends 65 000 MWK. When she converts the remaining Malawian kwacha to Tanzanian shilling, how much money does she have (in Tanzanian shilling)?

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1. [2GJ7](#)   2. [2GJ8](#)   3. [2GJ9](#)   4. [2GJB](#)   5. [2GJC](#)   6. [2GJD](#)   7. [2GJF](#)   8. [2GJG](#)



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## 9.6 Chapter summary

EMA6W

⦿ See presentation: [2GJH](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

- There are two types of interest rates:

Simple interest	Compound interest
$A = P(1 + in)$	$A = P(1 + i)^n$

Where:

$A$  = accumulated amount

$P$  = principal amount

$i$  = interest written as decimal

$n$  = number of years

- Hire purchase loan repayments are calculated using the simple interest formula on the cash price minus the deposit. Monthly repayments are calculated by dividing the accumulated amount by the number of months for the repayment.
- Population growth and inflation are calculated using the compound interest formula.
- Foreign exchange rate is the price of one currency in terms of another.

1. An amount of R 6330 is invested in a savings account which pays simple interest at a rate of 11% p.a.. Calculate the balance accumulated by the end of 7 years.
2. An amount of R 1740 is invested in a savings account which pays simple interest at a rate of 7% p.a.. Calculate the balance accumulated by the end of 6 years.
3. Adam opens a savings account when he is 13. He would like to have R 50 000 by the time he is 18. If the savings account offers simple interest at a rate of 8,5% per annum, how much money should he invest now to reach his goal?
4. When his son was 4 years old, Dumile made a deposit of R 6700 in the bank. The investment grew at a simple interest rate and when Dumile's son was 24 years old, the value of the investment was R 11 524. At what rate was the money invested? Give your answer correct to one decimal place.
5. When his son was 7 years old, Jared made a deposit of R 5850 in the bank. The investment grew at a simple interest rate and when Jared's son was 35 years old, the value of the investment was R 11 746,80. At what rate was the money invested? Give your answer correct to one decimal place.
6. Sehlolo wants to invest R 6360 at a simple interest rate of 12,4% p.a.  
How many years will it take for the money to grow to R 26 075? Round **up** your answer to the nearest year.
7. Mphikeleli wants to invest R 5540 at a simple interest rate of 9,1% p.a.  
How many years will it take for the money to grow to R 16 620? Round **up** your answer to the nearest year.
8. An amount of R 3500 is invested in an account which pays simple interest at a rate of 6,7% per annum. Calculate the amount of interest accumulated at the end of 4 years.
9. An amount of R 3270 is invested in a savings account which pays a compound interest rate of 12,2% p.a. Calculate the balance accumulated by the end of 7 years. As usual with financial calculations, round your answer to two decimal places, but do not round off until you have reached the solution.
10. An amount of R 2380 is invested in a savings account which pays a compound interest rate of 8,3% p.a. Calculate the balance accumulated by the end of 7 years. As usual with financial calculations, round your answer to two decimal places, but do not round off until you have reached the solution.
11. Emma wants to invest some money at a compound interest rate of 8,2% p.a.  
How much money should be invested if she wants to reach a sum of R 61 500 in 4 years' time? Round up your answer to the nearest rand.
12. Limpopo wants to invest some money at a compound interest rate of 13,9% p.a.  
How much money should be invested if she wants to reach a sum of R 24 300 in 2 years' time? Round up your answer to the nearest rand.
13. Calculate the compound interest for the following problems.
  - a) A R 2000 loan for 2 years at 5% p.a.
  - b) A R 1500 investment for 3 years at 6% p.a.
  - c) A R 800 loan for 1 year at 16% p.a.
14. Ali invests R 1110 into an account which pays out a lump sum at the end of 12 years.  
If he gets R 1642,80 at the end of the period, what compound interest rate did the bank offer him? Give your answer correct to one decimal place.
15. Christopher invests R 4480 into an account which pays out a lump sum at the end of 7 years.  
If he gets R 6496,00 at the end of the period, what compound interest rate did the bank offer him? Give your answer correct to one decimal place.
16. Calculate how much you will earn if you invested R 500 for 1 year at the following interest rates:
  - a) 6,85% simple interest
  - b) 4,00% compound interest

17. Bianca has R 1450 to invest for 3 years. Bank A offers a savings account which pays simple interest at a rate of 11% per annum, whereas Bank B offers a savings account paying compound interest at a rate of 10,5% per annum. Which account would leave Bianca with the highest accumulated balance at the end of the 3 year period?
18. Given:  
A loan of R 2000 for a year at an interest rate of 10% p.a.
- How much simple interest is payable on the loan?
  - How much compound interest is payable on the loan?
19. R 2250 is invested at an interest rate of 5,25% per annum.  
Complete the following table.
- | Number of years | Simple interest | Compound interest |
|-----------------|-----------------|-------------------|
| 1               |                 |                   |
| 2               |                 |                   |
| 3               |                 |                   |
| 4               |                 |                   |
| 20              |                 |                   |
20. Discuss:
- Which type of interest would you like to use if you are the borrower?
  - Which type of interest would you like to use if you were the banker?
21. Portia wants to buy a television on a hire purchase agreement. The cash price of the television is R 6000. She is required to pay a deposit of 20% and pay the remaining loan amount off over 12 months at an interest rate of 9% p.a.
- What is the principal loan amount?
  - What is the accumulated loan amount?
  - What are Portia's monthly repayments?
  - What is the total amount she has paid for the television?
22. Gabisile wants to buy a heater on a hire purchase agreement. The cash price of the heater is R 4800. She is required to pay a deposit of 10% and pay the remaining loan amount off over 12 months at an interest rate of 12% p.a.
- What is the principal loan amount?
  - What is the accumulated loan amount?
  - What are Gabisile's monthly repayments?
  - What is the total amount she has paid for the heater?
23. Khayalethu buys a couch costing R 8000 on a hire purchase agreement. He is charged an interest rate of 12% p.a. over 3 years.
- How much will Khayalethu pay in total?
  - How much interest does he pay?
  - What is his monthly instalment?
24. Jwayelani buys a sofa costing R 7700 on a hire purchase agreement. He is charged an interest rate of 16% p.a. over 5 years.
- How much will Jwayelani pay in total?
  - How much interest does he pay?
  - What is his monthly instalment?

25. Bonnie bought a stove for R 3750. After 3 years she had finished paying for it and the R 956,25 interest that was charged for hire purchase. Determine the rate of simple interest that was charged.
26. A new furniture store has just opened in town and is offering the following special:  
Purchase a lounge suite, a bedroom suite and kitchen appliances (fridge, stove, washing machine) for just R 50 000 and receive a free microwave. No deposit required, 5 year payment plan available. Interest charged at just 6,5% p.a.  
Babelwa purchases all the items on hire purchase. She decides to pay a R 1500 deposit. The store adds in an insurance premium of R 35,00 per month.  
What is Babelwa's monthly payment on the items?
27. The price of 2 litres of milk is R 17. How much will it cost in 3 years time if the inflation rate is 13% p.a.?
28. The price of a 2 l bottle of juice is R 16. How much will the juice cost in 8 years time if the inflation rate is 7% p.a.?
29. A box of fruity-chews costs R 27 today. How much did it cost 8 years ago if the average rate of inflation was 10% p.a.? Round your answer to 2 decimal places.
30. A box of smarties costs R 23 today. How much did the same box cost 8 years ago if the average rate of inflation was 14% p.a.? Round your answer to 2 decimal places.
31. According to the latest census, South Africa currently has a population of 57 000 000.
- If the annual growth rate is expected to be 0,9%, calculate how many South Africans there will be in 10 years time (correct to the nearest hundred thousand).
  - If it is found after 10 years that the population has actually increased by 10 million to 67 million, what was the growth rate?
32. The current population of Cape Town is 3 875 190 and the average rate of population growth in South Africa is 0,4% p.a.  
What can city planners expect the population of Cape Town to be in 12 years time?  
Note: Round your answer to the nearest integer.
33. The current population of Pretoria is 3 888 420 and the average rate of population growth in South Africa is 0,7% p.a.  
What will the population of Pretoria be in 7 years time?  
Note: Round your answer to the nearest integer.
34. Monique wants to buy an iPad that costs £ 140, with the exchange rate currently at £ 1 = R 15. She estimates that the exchange rate will drop to R 9 in a month.
- How much will the iPad cost in rands, if she buys it now?
  - How much will she save if the exchange rate drops to R 9?
  - How much will she lose if the exchange rate moves to R 20?
35. Xolile wants to buy a CD player that costs £ 140, with the exchange rate currently at £ 1 = R 14. She estimates that the exchange rate will drop to R 10 in a month.
- How much will the CD player cost in rands, if she buys it now?
  - How much will she save if the exchange rate drops to R 10?
  - How much will she lose if the exchange rate moves to R 20?
36. Alison is going on holiday to Europe. Her hotel will cost € 200 per night. How much will she need, in rands, to cover her hotel bill, if the exchange rate is € 1 = R 9,20?
37. Jennifer is buying some books online. She finds a publisher in the UK selling the books for £ 16,99. She then finds the same books from a publisher in the USA for \$23,50.  
Next she looks up the exchange rates to see which publisher has the better deal. If \$1 = R 12,43 and £ 1 = R 16,89, which publisher should she buy the books from?
38. Bonani won a trip to see Machu Picchu in Peru followed by a trip to Brazil for the carnival. He is given R 25 000 to spend while on the trip.

He then looks up the current exchange rates and finds the following information:

$$\begin{aligned}R 1 &= 0,26 \text{ PEN} \\1 \text{ BRL} &= 1,17 \text{ PEN}\end{aligned}$$

In Peru he spends 2380 PEN. When he converts the remaining Peruvian sol to Brazilian real, how much money does he have (in Brazilian real)?

39. If the exchange rate to the rand for the Japanese yen is ¥ 100 = R 6,23 and for the Australian dollar is 1 AUD = R 5,11, determine the exchange rate between the Australian dollar and the Japanese yen.
40. Khetang has just been to Europe to work for a few months. He returns to South Africa with € 2850 to invest in a savings account.

His bank offers him a savings account which pays 5,3% compound interest per annum. The bank converts Khetang's Euros to rands at an exchange rate of € 1 = R 12,89.

If Khetang invests his money for 6 years, how much interest does he earn on his investment?

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1. <a href="#">2GJJ</a>	2. <a href="#">2GJK</a>	3. <a href="#">2GJM</a>	4. <a href="#">2GJN</a>	5. <a href="#">2GJP</a>	6. <a href="#">2GJQ</a>
7. <a href="#">2GJR</a>	8. <a href="#">2GJS</a>	9. <a href="#">2GJT</a>	10. <a href="#">2GJV</a>	11. <a href="#">2GJW</a>	12. <a href="#">2GJX</a>
13a. <a href="#">2GJY</a>	13b. <a href="#">2GJZ</a>	13c. <a href="#">2GK2</a>	14. <a href="#">2GK3</a>	15. <a href="#">2GK4</a>	16. <a href="#">2GK5</a>
17. <a href="#">2GK6</a>	18. <a href="#">2GK7</a>	19. <a href="#">2GK8</a>	20. <a href="#">2GK9</a>	21. <a href="#">2GKB</a>	22. <a href="#">2GKC</a>
23. <a href="#">2GKD</a>	24. <a href="#">2GKF</a>	25. <a href="#">2GKG</a>	26. <a href="#">2GKH</a>	27. <a href="#">2GKJ</a>	28. <a href="#">2GKK</a>
29. <a href="#">2GKM</a>	30. <a href="#">2GKN</a>	31. <a href="#">2GKP</a>	32. <a href="#">2GKQ</a>	33. <a href="#">2GKR</a>	34. <a href="#">2GKS</a>
35. <a href="#">2GKT</a>	36. <a href="#">2GKV</a>	37. <a href="#">2GKW</a>	38. <a href="#">2GKX</a>	39. <a href="#">2GKY</a>	40. <a href="#">2GKZ</a>



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# CHAPTER



## *Statistics*

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# 10 Statistics

When running an experiment or conducting a survey we can potentially end up with many hundreds, thousands or even millions of values in the resulting data set. Too much data can be overwhelming and we need to reduce them or represent them in a way that is easier to understand and communicate.

Statistics is about summarising data. The methods of statistics allow us to represent the essential information in a data set while disregarding the unimportant information. We have to be careful to make sure that we do not accidentally throw away some of the important aspects of a data set.

By applying statistics properly we can highlight the important aspects of data and make the data easier to interpret. By applying statistics poorly or dishonestly we can also hide important information and let people draw the wrong conclusions.

In this chapter we will look at a few numerical and graphical ways in which data sets can be represented, to make them easier to interpret.

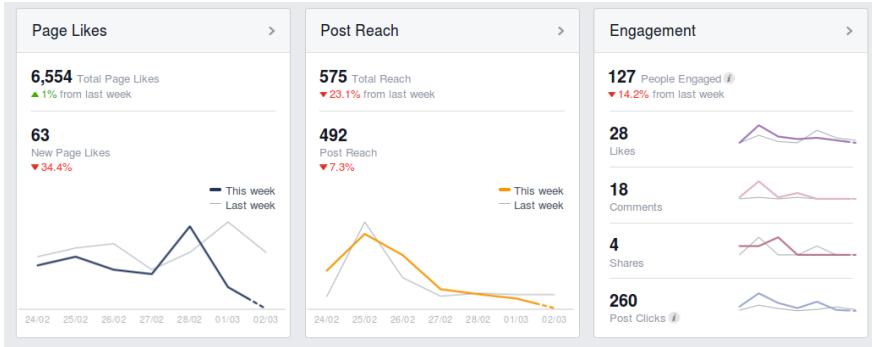


Figure 10.1: Statistics is used by various websites to show users who is viewing their content.

## 10.1 Collecting data

EMA6X

### DEFINITION: Data

Data refers to the pieces of information that have been observed and recorded, from an experiment or a survey.

### NOTE:

The word **data** is the plural of the word **datum**, and therefore one should say, “the data are” and not “the data is”.

We distinguish between two main types of data: quantitative and qualitative.

### DEFINITION: Quantitative data

Quantitative data are data that can be written as numbers.

Quantitative data can be discrete or continuous.

Discrete quantitative data can be represented by integers and usually occur when we count things, for example, the number of learners in a class, the number of molecules in a chemical solution, or the number of SMS messages sent in one day.

Continuous quantitative data can be represented by real numbers, for example, the height or mass of a person,

the distance travelled by a car, or the duration of a phone call.

### **DEFINITION:** Qualitative data

Qualitative data are data that cannot be written as numbers.

Two common types of qualitative data are categorical and anecdotal data. Categorical data can come from one of a limited number of possibilities, for example, your favourite cooldrink, the colour of your cell phone, or the language that you learnt to speak at home.

Anecdotal data take the form of an interview or a story, for example, when you ask someone what their personal experience was when using a product, or what they think of someone else's behaviour.

Categorical qualitative data are sometimes turned into quantitative data by counting the number of times that each category appears. For example, in a class with 30 learners, we ask everyone what the colours of their cell phones are and get the following responses:

black	black	black	white	purple	red	red	black	black	black
white	white	black	black	black	black	purple	black	black	white
purple	black	red	red	white	black	orange	orange	black	white

This is a categorical qualitative data set since each of the responses comes from one of a small number of possible colours.

We can represent exactly the same data in a different way, by counting how many times each colour appears.

Colour	Count
black	15
white	6
red	4
purple	3
orange	2

This is a discrete quantitative data set since each count is an integer.

### **Worked example 1: Qualitative and quantitative data**

#### **QUESTION**

Thembisile is interested in becoming an airtime reseller to his classmates. He would like to know how much business he can expect from them. He asked each of his 20 classmates how many SMS messages they sent during the previous day. The results were:

20	3	0	14	30	9	11	13	13	15
9	13	16	12	13	7	17	14	9	13

Is this data set qualitative or quantitative? Explain your answer.

#### **SOLUTION**

The number of SMS messages is a count represented by an integer, which means that it is quantitative and discrete.

## Worked example 2: Qualitative and quantitative data

### QUESTION

Thembisile would like to know who the most popular cellular provider is among learners in his school. This time Thembisile randomly selects 20 learners from the entire school and asks them which cellular provider they currently use. The results were:

Cell C	Vodacom	Vodacom	MTN	Vodacom
MTN	MTN	Virgin Mobile	Cell C	8-ta
Vodacom	MTN	Vodacom	Vodacom	MTN
Vodacom	Vodacom	Vodacom	Virgin Mobile	MTN

Is this data set qualitative or quantitative? Explain your answer.

### SOLUTION

Since each response is not a number, but one of a small number of possibilities, these are categorical qualitative data.

### Exercise 10 – 1:

1. The following data set of dreams that learners have was collected from Grade 12 learners just after their final exams: {"I want to build a bridge!"; "I want to help the sick."; "I want running water!"} Categorise the data set.
2. Categorise the following data set of sweets in a packet that was collected from visitors to a sweet shop. {23; 25; 22; 26; 27; 25; 21; 28}
3. Categorise the following data set of questions answered correctly that was collected from a class of maths learners.  
{3; 5; 2; 6; 7; 5; 1; 2}

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1. [2GM2](#)   2. [2GM3](#)   3. [2GM4](#)



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## 10.2 Measures of central tendency

EMA6Y

### Mean

EMA6Z

#### DEFINITION: Mean

The mean is the sum of a set of values, divided by the number of values in the set. The notation for the mean of a set of values is a horizontal bar over the variable used to represent the set, for example  $\bar{x}$ . The formula for the mean of a data set  $\{x_1; x_2; \dots; x_n\}$  is:

$$\begin{aligned}\bar{x} &= \frac{1}{n} \sum_{i=1}^n x_i \\ &= \frac{x_1 + x_2 + \dots + x_n}{n}\end{aligned}$$

The mean is sometimes also called the average or the arithmetic mean.

### Worked example 3: Calculating the mean

#### QUESTION

What is the mean of the data set {10; 20; 30; 40; 50}?

#### SOLUTION

##### Step 1: Calculate the sum of the data

$$10 + 20 + 30 + 40 + 50 = 150$$

##### Step 2: Divide by the number of values in the data set to get the mean

Since there are 5 values in the data set, the mean is:

$$\text{mean} = \frac{150}{5} = 30$$

## Median

EMA72

#### DEFINITION: Median

The median of a data set is the value in the central position, when the data set has been arranged from the lowest to the highest value.

Note that exactly half of the values from the data set are less than the median and the other half are greater than the median.

To calculate the median of a quantitative data set, first sort the data from the smallest to the largest value and then find the value in the middle. If there is an odd number of values in the data set, the median will be equal to one of the values in the data set. If there is an even number of values in the data set, the median will lie halfway between two values in the data set.

### Worked example 4: Median for an odd number of values

#### QUESTION

What is the median of {10; 14; 86; 2; 68; 99; 1}?

#### SOLUTION

##### Step 1: Sort the values

The values in the data set, arranged from the smallest to the largest, are

$$1; 2; 10; 14; 68; 86; 99$$

## Step 2: Find the number in the middle

There are 7 values in the data set. Since there are an odd number of values, the median will be equal to the value in the middle, namely, in the fourth position. Therefore the median of the data set is 14.

## Worked example 5: Median for an even number of values

### QUESTION

What is the median of {11; 10; 14; 86; 2; 68; 99; 1}?

### SOLUTION

#### Step 1: Sort the values

The values in the data set, arranged from the smallest to the largest, are

$$1; 2; 10; 11; 14; 68; 86; 99$$

#### Step 2: Find the number in the middle

There are 8 values in the data set. Since there are an even number of values, the median will be halfway between the two values in the middle, namely, between the fourth and fifth positions. The value in the fourth position is 11 and the value in the fifth position is 14. The median lies halfway between these two values and is therefore

$$\text{median} = \frac{11 + 14}{2} = 12,5$$

## Mode

EMA73

### DEFINITION: Mode

The mode of a data set is the value that occurs most often in the set. The mode can also be described as the most frequent or most common value in the data set.

To calculate the mode, we simply count the number of times that each value appears in the data set and then find the value that appears most often.

A data set can have more than one mode if there is more than one value with the highest count. For example, both 2 and 3 are modes in the data set {1; 2; 2; 3; 3}. If all points in a data set occur with equal frequency, it is equally accurate to describe the data set as having many modes or no mode.

### VISIT:

The following video explains how to calculate the mean, median and mode of a data set.

⌚ See video: 2GM5 at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## Worked example 6: Finding the mode

### QUESTION

Find the mode of the data set  $\{2; 2; 3; 4; 4; 4; 6; 6; 7; 8; 8; 10; 10\}$ .

### SOLUTION

**Step 1:** Count the number of times that each value appears in the data set

Value	Count
2	2
3	1
4	3
6	2
7	1
8	2
10	2

**Step 2:** Find the value that appears most often

From the table above we can see that 4 is the only value that appears 3 times. All the other values appear less than 3 times. Therefore the mode of the data set is 4.

One problem with using the mode as a measure of central tendency is that we can usually not compute the mode of a continuous data set. Since continuous values can lie anywhere on the real line, any particular value will almost never repeat. This means that the frequency of each value in the data set will be 1 and that there will be no mode. We will look at one way of addressing this problem in the section on grouping data.

## Worked example 7: Comparison of measures of central tendency

### QUESTION

There are regulations in South Africa related to bread production to protect consumers. By law, if a loaf of bread is not labelled, it must weigh 800 g, with the leeway of 5 percent under or 10 percent over. Vishnu is interested in how a well-known, national retailer measures up to this standard. He visited his local branch of the supplier and recorded the masses of 10 different loaves of bread for one week. The results, in grams, are given below:

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
802,4	787,8	815,7	807,4	801,5	786,6	799,0
796,8	798,9	809,7	798,7	818,3	789,1	806,0
802,5	793,6	785,4	809,3	787,7	801,5	799,4
819,6	812,6	809,1	791,1	805,3	817,8	801,0
801,2	795,9	795,2	820,4	806,6	819,5	796,7
789,0	796,3	787,9	799,8	789,5	802,1	802,2
789,0	797,7	776,7	790,7	803,2	801,2	807,3
808,8	780,4	812,6	801,8	784,7	792,2	809,8
802,4	790,8	792,4	789,2	815,6	799,4	791,2
796,2	817,6	799,1	826,0	807,9	806,7	780,2

1. Is this data set qualitative or quantitative? Explain your answer.
2. Determine the mean, median and mode of the mass of a loaf of bread for each day of the week. Give your answer correct to 1 decimal place.
3. Based on the data, do you think that this supplier is providing bread within the South African regulations?

## SOLUTION

### Step 1: Qualitative or quantitative?

Since each mass can be represented by a number, the data set is quantitative. Furthermore, since a mass can be any real number, the data are continuous.

### Step 2: Calculate the mean

In each column (for each day of the week), we add up the measurements and divide by the number of measurements, 10.

For Monday, the sum of the measured values is 8007,9 and so the mean for Monday is

$$\frac{8007,9}{10} = 800,8 \text{ g}$$

In the same way, we can compute the mean for each day of the week. See the table below for the results.

### Step 3: Calculate the median

In each column we sort the numbers from lowest to highest and find the value in the middle. Since there are an even number of measurements (10), the median is halfway between the two numbers in the middle.

For Monday, the sorted list of numbers is

$$789,0; 789,0; 796,2; 796,7; 801,2; \\ 802,3; 802,3; 802,5; 808,7; 819,6$$

The two numbers in the middle are 801,2 and 802,3 and so the median is

$$\frac{801,2 + 802,3}{2} = 801,8 \text{ g}$$

In the same way, we can compute the median for each day of the week:

Day	Mean	Median
Monday	800,8 g	801,8 g
Tuesday	797,2 g	796,1 g
Wednesday	798,4 g	797,2 g
Thursday	803,4 g	800,8 g
Friday	802,0 g	804,3 g
Saturday	801,6 g	801,4 g
Sunday	799,3 g	800,2 g

From the above calculations we can see that the means and medians are close to one another, but not quite equal. In the next worked example we will see that the mean and median are not always close to each other.

### Step 4: Determine the mode

Since the data are continuous we cannot compute the mode. In the next section we will see how we can group data in order to make it possible to compute an approximation for the mode.

### Step 5: Conclusion: Is the supplier reliable?

From the question, the requirements are that the mass of a loaf of bread be between 800 g minus 5%, which is 760 g, and plus 10%, which is 880 g. Since every one of the measurements made by Vishnu lies within this range and since the means and medians are all close to 800 g, we can conclude that the supplier is reliable.

**DEFINITION:** *Outlier*

An outlier is a value in the data set that is not typical of the rest of the set. It is usually a value that is much greater or much less than all the other values in the data set.

**Worked example 8: Effect of outliers on mean and median****QUESTION**

The heights of 10 learners are measured in centimetres to obtain the following data set:

$$\{150; 172; 153; 156; 146; 157; 157; 143; 168; 157\}$$

Afterwards, we include one more learner in the group, who is exceptionally tall at 181 cm.

Compare the mean and median of the heights of the learners before and after the eleventh learner was included.

**SOLUTION****Step 1: Calculate the mean of the first 10 learners**

$$\begin{aligned}\text{mean} &= \frac{150 + 172 + 153 + 156 + 146 + 157 + 157 + 143 + 168 + 157}{10} \\ &= 155,9 \text{ cm}\end{aligned}$$

**Step 2: Calculate the mean of all 11 learners**

$$\begin{aligned}\text{mean} &= \frac{150 + 172 + 153 + 156 + 146 + 157 + 157 + 143 + 168 + 157 + 181}{11} \\ &= 158,2 \text{ cm}\end{aligned}$$

From this we see that the average height changes by  $158,2 - 155,9 = 2,3$  cm when we introduce the outlier value (the tall person) to the data set.

**Step 3: Calculate the median of the first 10 learners**

To find the median, we need to sort the data set:

$$\{143; 146; 150; 153; 156; 157; 157; 157; 168; 172\}$$

Since there are an even number of values, 10, the median lies halfway between the fifth and sixth values:

$$\text{median} = \frac{156 + 157}{2} = 156,5 \text{ cm}$$

**Step 4: Calculate the median of all 11 learners**

After adding the tall learner, the sorted data set is

$$\{143; 146; 150; 153; 156; 157; 157; 157; 168; 172; 181\}$$

Now, with 11 values, the median is the sixth value: 157 cm. So, the median changes by only 0,5 cm when we add the outlier value to the data set.

In general, the median is less affected by the addition of outliers to a data set than the mean is. This is important because it is quite common that outliers are measured during an experiment, because of problems with the equipment or unexpected interference.

### Exercise 10 – 2:

1. Calculate the **mean** of the following data set: {9; 14; 9; 14; 8; 8; 9; 8; 9; 9}. Round your answer to 1 decimal place.
2. Calculate the **median** of the following data set: {4; 13; 10; 13; 13; 4; 2; 13; 13; 13}.
3. Calculate the **mode** of the following data set: {6; 10; 6; 6; 13; 12; 12; 7; 13; 6}
4. Calculate the mean, median and mode of the following data sets:
  - a) {2; 5; 8; 8; 11; 13; 22; 23; 27}
  - b) {15; 17; 24; 24; 26; 28; 31; 43}
  - c) {4; 11; 3; 15; 11; 13; 25; 17; 2; 11}
  - d) {24; 35; 28; 41; 31; 49; 31}
5. The ages of 15 runners of the Comrades Marathon were recorded:

$$\{31; 42; 28; 38; 45; 51; 33; 29; 42; 26; 34; 56; 33; 46; 41\}$$

Calculate the mean, median and modal age.

6. A group of 10 friends each have some stones. They work out that the **mean** number of stones they have is 6. Then 7 friends leave with an unknown number ( $x$ ) of stones. The remaining 3 friends work out that the **mean** number of stones they have left is 12,33.

When the 7 friends left, how many stones did they take with them?

7. A group of 9 friends each have some coins. They work out that the **mean** number of coins they have is 4. Then 5 friends leave with an unknown number ( $x$ ) of coins. The remaining 4 friends work out that the **mean** number of coins they have left is 2,5.

When the 5 friends left, how many coins did they take with them?

8. A group of 9 friends each have some marbles. They work out that the **mean** number of marbles they have is 3. Then 3 friends leave with an unknown number ( $x$ ) of marbles. The remaining 6 friends work out that the **mean** number of marbles they have left is 1,17.

When the 3 friends left, how many marbles did they take with them?

9. In the first of a series of jars, there is 1 sweet. In the second jar, there are 3 sweets. The mean number of sweets in the first two jars is 2.

- a) If the mean number of sweets in the first three jars is 3, how many sweets are there in the third jar?
- b) If the mean number of sweets in the first four jars is 4, how many sweets are there in the fourth jar?

10. Find a set of five ages for which the mean age is 5, the modal age is 2 and the median age is 3 years.

11. Four friends each have some marbles. They work out that the mean number of marbles they have is 10. One friend leaves with 4 marbles. How many marbles do the remaining friends have together?

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2GM6](#)   2. [2GM7](#)   3. [2GM8](#)   4a. [2GM9](#)   4b. [2GMB](#)   4c. [2GMC](#)   4d. [2GMD](#)  
5. [2GMF](#)   6. [2GMG](#)   7. [2GMH](#)   8. [2GMJ](#)   9. [2GMK](#)   10. [2GMM](#)   11. [2GMN](#)



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A common way of handling continuous quantitative data is to subdivide the full range of values into a few sub-ranges. By assigning each continuous value to the sub-range or class within which it falls, the data set changes from continuous to discrete.

Grouping is done by defining a set of ranges and then counting how many of the data fall inside each range. The sub-ranges must not overlap and must cover the entire range of the data set.

One way of visualising grouped data is as a histogram. A histogram is a collection of rectangles, where the base of a rectangle (on the  $x$ -axis) covers the values in the range associated with it, and the height of a rectangle corresponds to the number of values in its range.

#### VISIT:

The following video explains how to group data.

► See video: [2GMP](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

#### Worked example 9: Groups and histograms

#### QUESTION

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The heights in centimetres of 30 learners are given below.

142	163	169	132	139	140	152	168	139	150
161	132	162	172	146	152	150	132	157	133
141	170	156	155	169	138	142	160	164	168

Group the data into the following ranges and draw a histogram of the grouped data:

$$130 \leq h < 140$$

$$140 \leq h < 150$$

$$150 \leq h < 160$$

$$160 \leq h < 170$$

$$170 \leq h < 180$$

(Note that the ranges do not overlap since each one starts where the previous one ended.)

#### SOLUTION

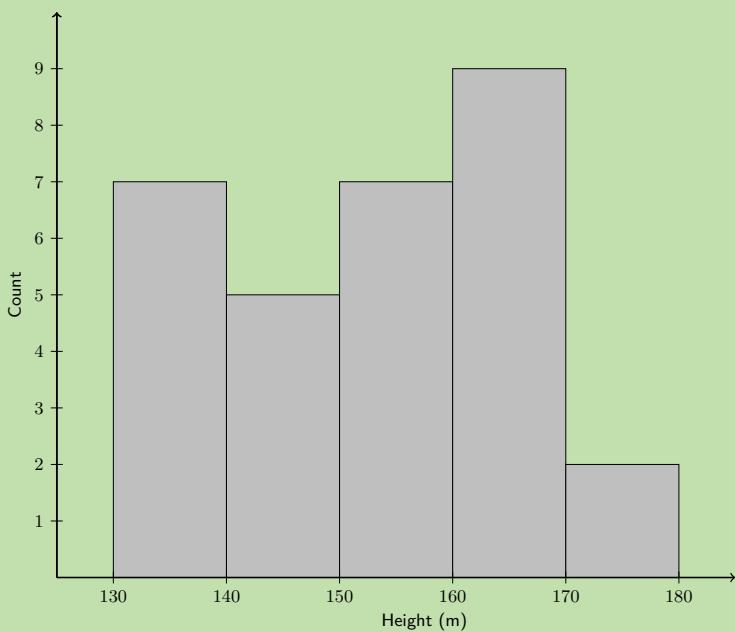
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#### Step 1: Count the number of values in each range

Range	Count
$130 \leq h < 140$	7
$140 \leq h < 150$	5
$150 \leq h < 160$	7
$160 \leq h < 170$	9
$170 \leq h < 180$	2

#### Step 2: Draw the histogram

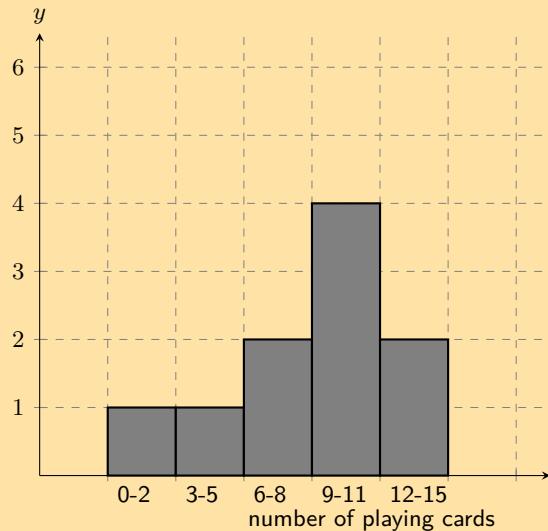
Since there are 5 ranges, the histogram will have 5 rectangles. The base of each rectangle is defined by its range. The height of each rectangle is determined by the count in its range.



The histogram makes it easy to see in which range most of the heights are located and provides an overview of the distribution of the values in the data set.

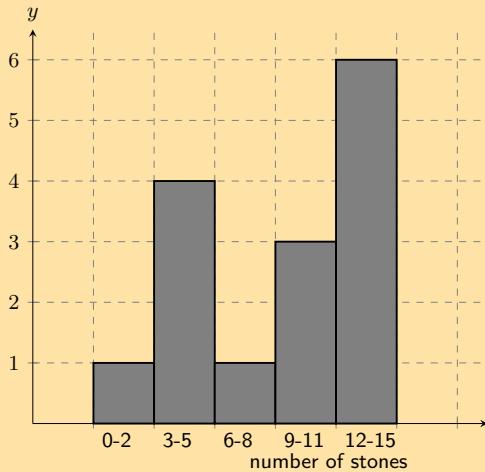
#### Exercise 10 – 3:

1. A group of 10 learners count the number of playing cards they each have. This is a histogram describing the data they collected:



Count the number of playing cards in the following range:  $0 \leq \text{number of playing cards} \leq 2$

2. A group of 15 learners count the number of stones they each have. This is a histogram describing the data they collected:



Count the number of stones in the following range:  $0 \leq \text{number of stones} \leq 2$

3. A group of **20** learners count the number of playing cards they each have. This is the data they collect:

14	9	11	8	13	2	3	4	16	17
9	19	10	14	4	16	16	11	2	17

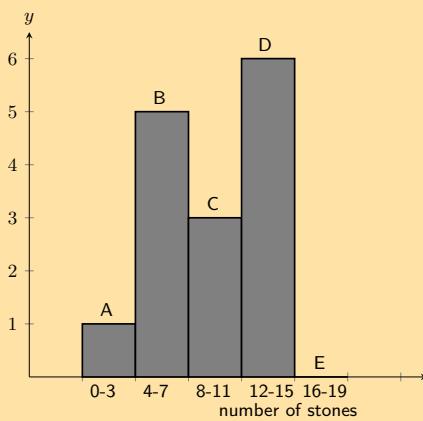
Count the number of learners who have from 12 up to 15 playing cards. In other words, how many learners have playing cards in the following range:  $12 \leq \text{number of playing cards} \leq 15$ ? It may be helpful for you to draw a histogram in order to answer the question.

4. A group of **20** learners count the number of stones they each have. This is the data they collect:

16	6	11	19	20	17	13	1	5	12
5	2	16	11	16	6	10	13	6	17

Count the number of learners who have from 4 up to 7 stones. In other words, how many learners have stones in the following range:  $4 \leq \text{number of stones} \leq 7$ ? It may be helpful for you to draw a histogram in order to answer the question.

5. A group of 20 learners count the number of stones they each have. The learners draw a histogram describing the data they collected. However, they have made a mistake in drawing the histogram.

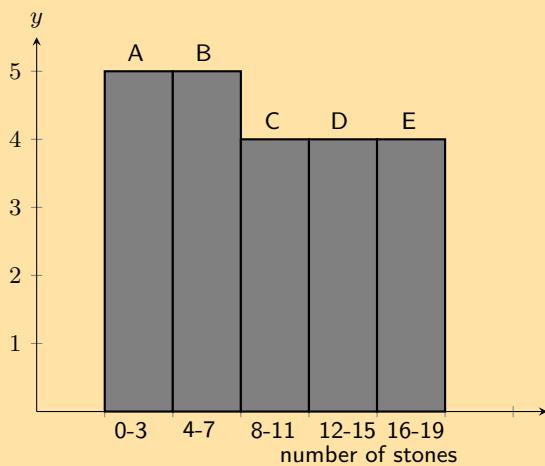


The data set below shows the correct information for the number of stones the learners have. Each value represents the number of stones for one learner.

$$\{4; 12; 15; 14; 18; 12; 17; 15; 1; 6; 6; 12; 6; 8; 6; 8; 17; 19; 16; 8\}$$

Help them figure out which **column** in the histogram is **incorrect**.

6. A group of 20 learners count the number of stones they each have. The learners draw a histogram describing the data they collected. However, they have made a mistake in drawing the histogram.

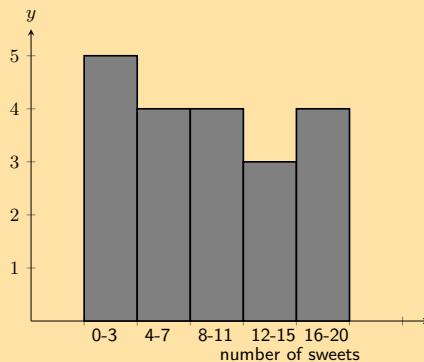


The data set below shows the correct information for the number of stones the learners have. Each value represents the number of stones for one learner.

$$\{19; 11; 5; 2; 3; 4; 14; 2; 12; 19; 11; 14; 2; 19; 11; 5; 17; 10; 1; 12\}$$

Help them figure out which **column** in the histogram is **incorrect**.

7. A group of learners count the number of sweets they each have. This is a histogram describing the data they collected:



A cat jumps onto the table, and all their notes land on the floor, mixed up, by accident!

Help them find which of the following data sets match the above histogram:

Data Set A

$$\begin{array}{cccccccccc} 2 & 1 & 20 & 10 & 5 & 3 & 10 & 2 & 6 & 1 \\ 2 & 2 & 17 & 3 & 18 & 3 & 7 & 10 & 8 & 18 \end{array}$$

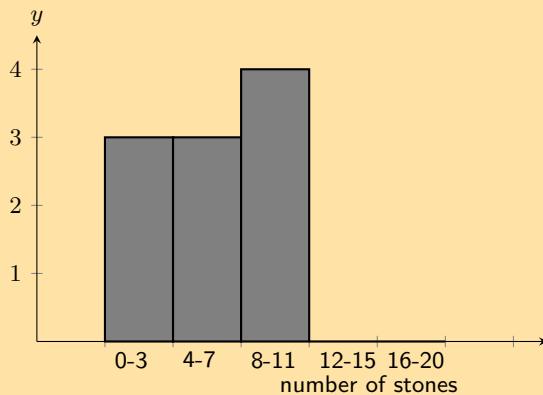
Data Set B

$$\begin{array}{cccccccccc} 2 & 9 & 12 & 10 & 5 & 9 & 9 & 10 \\ 13 & 6 & 5 & 11 & 10 & 7 & 7 \end{array}$$

Data Set C

$$\begin{array}{cccccccccc} 3 & 12 & 16 & 10 & 15 & 17 & 18 & 2 & 3 & 7 \\ 11 & 12 & 8 & 2 & 7 & 17 & 3 & 11 & 4 & 4 \end{array}$$

8. A group of learners count the number of stones they each have. This is a histogram describing the data they collected:



A cleaner knocks over their table, and all their notes land on the floor, mixed up, by accident!  
Help them find which of the following data sets match the above histogram:

Data Set A

12	4	2	15	10	18	10	16	16	19
1	2	9	10	16	10	11	9	2	13

Data Set B

7	10	4	5	8	7	12	10
14	5	1	9	2	13	3	

Data Set C

9	3	8	5	8
5	8	1	4	3

9. A class experiment was conducted and 50 learners were asked to guess the number of sweets in a jar.  
The following guesses were recorded:

56	49	40	11	33	33	37	29	30	59
21	16	38	44	38	52	22	24	30	34
42	15	48	33	51	44	33	17	19	44
47	23	27	47	13	25	53	57	28	23
36	35	40	23	45	39	32	58	22	40

- Draw up a grouped frequency table using the intervals  $10 < x \leq 20$ ,  $20 < x \leq 30$ ,  $30 < x \leq 40$ ,  $40 < x \leq 50$  and  $50 < x \leq 60$ .
- Draw the histogram corresponding to the frequency table of the grouped data.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2GMQ](#)   2. [2GMR](#)   3. [2GMS](#)   4. [2GMT](#)   5. [2GMV](#)   6. [2GMW](#)   7. [2GMX](#)   8. [2GMY](#)   9. [2GMZ](#)



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With grouped data our estimates of central tendency will change because we lose some information when we place each value in a range. If all we have to work with is the grouped data, we do not know the measured values to the same accuracy as before. The best we can do is to assume that values are grouped at the centre of each range.

Looking back to the previous worked example, we started with this data set of learners' heights.

$$\{132; 132; 132; 133; 138; 139; 139; 140; 141; 142; 142; 146; 150; 150; 152; 152; 155; 156; 157; 160; 161; 162; 163; 164; 168; 168; 169; 169; 170; 172\}$$

Note that the data are sorted.

The mean of these data is 151,8 and the median is 152. The mode is 132, but remember that there are problems with computing the mode of continuous quantitative data.

After grouping the data, we now have the data set shown below. Note that each value is placed at the centre of its range and that the number of times that each value is repeated corresponds exactly to the counts in each range.

$$\{135; 135; 135; 135; 135; 135; 135; 145; 145; 145; 145; 145; 155; 155; 155; 155; 155; 155; 155; 165; 165; 165; 165; 165; 165; 165; 165; 165; 175; 175\}$$

The grouping changes the measures of central tendency since each datum is treated as if it occurred at the centre of the range in which it was placed.

The mean is now 153, the median 155 and the mode is 165. This is actually a better estimate of the mode, since the grouping showed in which range the learners' heights were clustered.

#### **NOTE:**

We can also just give the modal group and the median group for grouped data. The modal group is the group that has the highest number of data values. The median group is the central group when the groups are arranged in order.

#### **Exercise 10 – 4:**

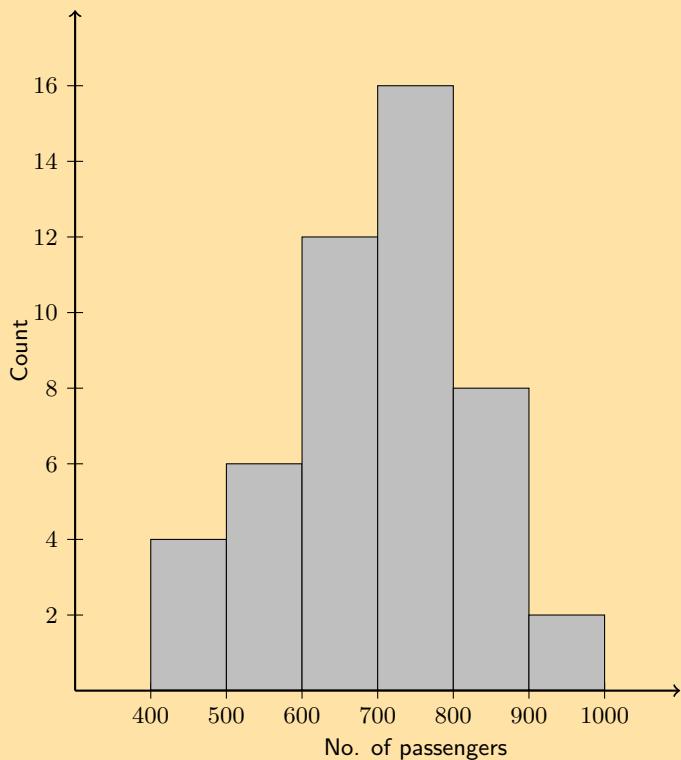
1. Consider the following grouped data and calculate the mean, the modal group and the median group.

Mass (kg)	Count
$40 < m \leq 45$	7
$45 < m \leq 50$	10
$50 < m \leq 55$	15
$55 < m \leq 60$	12
$60 < m \leq 65$	6

2. Find the mean, the modal group and the median group in this data set of how much time people needed to complete a game.

Time (s)	Count
$35 < t \leq 45$	5
$45 < t \leq 55$	11
$55 < t \leq 65$	15
$65 < t \leq 75$	26
$75 < t \leq 85$	19
$85 < t \leq 95$	13
$95 < t \leq 105$	6

3. The histogram below shows the number of passengers that travel in Alfred's minibus taxi per week.



Calculate:

- the modal interval
- the total number of passengers to travel in Alfred's taxi
- an estimate of the mean
- an estimate of the median
- if it is estimated that every passenger travelled an average distance of 5 km, how much money would Alfred have made if he charged R 3,50 per km?

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1. [2GN2](#)   2. [2GN3](#)   3. [2GN4](#)

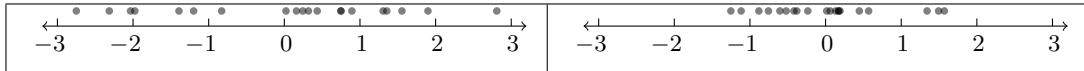


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The central tendency is not the only interesting or useful information about a data set. The two data sets illustrated below have the same mean (0), but have different spreads around the mean. Each circle represents one value from the data set (or one datum).



Dispersion is a general term for different statistics that describe how values are distributed around the centre. In this section we will look at measures of dispersion.

**DEFINITION:** *Range*

The range of a data set is the difference between the maximum and minimum values in the set.

The most straightforward measure of dispersion is the range. The range simply tells us how far apart the largest and smallest values in a data set are. The range is very sensitive to outliers.

**Worked example 10: Range**
**QUESTION**


---

Find the range of the following data set:

$$\{1; 4; 5; 8; 6; 7; 5; 6; 7; 4; 10; 9; 10\}$$

What would happen if we removed the first value from the set?

**SOLUTION**


---

**Step 1: Determine the range**

The smallest value in the data set is 1 and the largest value is 10.

The range is  $10 - 1 = 9$

**Step 2: Remove the first value**

If the first value, 1, were to be removed from the set, the minimum value would be 4. This means that the range would change to  $10 - 4 = 6$ . 1 is not typical of the other values. It is an outlier and has a big influence on the range.

**DEFINITION:** *Percentile*

The  $p^{\text{th}}$  percentile is the value,  $v$ , that divides a data set into two parts, such that  $p$  percent of the values in the data set are less than  $v$  and  $100 - p$  percent of the values are greater than  $v$ . Percentiles can lie in the range  $0 \leq p \leq 100$ .

To understand percentiles properly, we need to distinguish between 3 different aspects of a datum: its value, its rank and its percentile:

- The value of a datum is what we measured and recorded during an experiment or survey.
- The rank of a datum is its position in the sorted data set (for example, first, second, third, and so on).
- The percentile at which a particular datum is, tells us what percentage of the values in the full data set are less than this datum.

The table below summarises the value, rank and percentile of the data set:

$$\{14,2; 13,9; 19,8; 10,3; 13,0; 11,1\}$$

Value	Rank	Percentile
10,3	1	0
11,1	2	20
13,0	3	40
13,9	4	60
14,2	5	80
19,8	6	100

As an example, 13,0 is at the  $40^{\text{th}}$  percentile since there are 2 values less than 13,0 and 3 values greater than 13,0.

$$\frac{2}{2+3} = 0,4 = 40\%$$

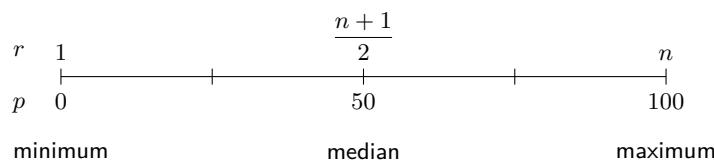
In general, the formula for finding the  $p^{\text{th}}$  percentile in an ordered data set with  $n$  values is

$$r = \frac{p}{100} (n - 1) + 1$$

This gives us the rank,  $r$ , of the  $p^{\text{th}}$  percentile. To find the value of the  $p^{\text{th}}$  percentile, we have to count from the first value in the ordered data set up to the  $r^{\text{th}}$  value.

Sometimes the rank will not be an integer. This means that the percentile lies between two values in the data set. The convention is to take the value halfway between the two values indicated by the rank.

The figure below shows the relationship between rank and percentile graphically. We have already encountered three percentiles in this chapter: the median ( $50^{\text{th}}$  percentile), the minimum ( $0^{\text{th}}$  percentile) and the maximum ( $100^{\text{th}}$ ). The median is defined as the value halfway in a sorted data set.



## Worked example 11: Using the percentile formula

### QUESTION

Determine the minimum, maximum and median values of the following data set using the percentile formula.

$$\{14; 17; 45; 20; 19; 36; 7; 30; 8\}$$

### SOLUTION

#### Step 1: Sort the values in the data set

Before we can use the rank to find values in the data set, we always have to order the values from the smallest to the largest. The sorted data set is:

$$\{7; 8; 14; 17; 19; 20; 30; 36; 45\}$$

#### Step 2: Find the minimum

We already know that the minimum value is the first value in the ordered data set. We will now confirm that the percentile formula gives the same answer. The minimum is equivalent to the 0<sup>th</sup> percentile. According to the percentile formula the rank,  $r$ , of the  $p = 0^{\text{th}}$  percentile in a data set with  $n = 9$  values is:

$$\begin{aligned} r &= \frac{p}{100} (n - 1) + 1 \\ &= \frac{0}{100} (9 - 1) + 1 \\ &= 1 \end{aligned}$$

This confirms that the minimum value is the first value in the list, namely 7.

#### Step 3: Find the maximum

We already know that the maximum value is the last value in the ordered data set. The maximum is also equivalent to the 100<sup>th</sup> percentile. Using the percentile formula with  $p = 100$  and  $n = 9$ , we find the rank of the maximum value is:

$$\begin{aligned} r &= \frac{p}{100} (n - 1) + 1 \\ &= \frac{100}{100} (9 - 1) + 1 \\ &= 9 \end{aligned}$$

This confirms that the maximum value is the last (the ninth) value in the list, namely 45.

#### Step 4: Find the median

The median is equivalent to the 50<sup>th</sup> percentile. Using the percentile formula with  $p = 50$  and  $n = 9$ , we find the rank of the median value is:

$$\begin{aligned} r &= \frac{50}{100} (n - 1) + 1 \\ &= \frac{50}{100} (9 - 1) + 1 \\ &= \frac{1}{2} (8) + 1 \\ &= 5 \end{aligned}$$

This shows that the median is in the middle (at the fifth position) of the ordered data set. Therefore the median value is 19.

## **DEFINITION:** Quartiles

The quartiles are the three data values that divide an ordered data set into four groups, where each group contains an equal number of data values. The median (50<sup>th</sup> percentile) is the second quartile ( $Q_2$ ). The 25<sup>th</sup> percentile is also called the first or lower quartile ( $Q_1$ ). The 75<sup>th</sup> percentile is also called the third or upper quartile ( $Q_3$ ).

### **Worked example 12: Quartiles**

#### **QUESTION**

Determine the quartiles of the following data set:

$$\{7; 45; 11; 3; 9; 35; 31; 7; 16; 40; 12; 6\}$$

#### **SOLUTION**

##### **Step 1: Sort the data set**

$$\{3; 6; 7; 7; 9; 11; 12; 16; 31; 35; 40; 45\}$$

##### **Step 2: Find the ranks of the quartiles**

Using the percentile formula with  $n = 12$ , we can find the rank of the 25<sup>th</sup>, 50<sup>th</sup> and 75<sup>th</sup> percentiles:

$$\begin{aligned}r_{25} &= \frac{25}{100} (12 - 1) + 1 \\&= 3,75 \\r_{50} &= \frac{50}{100} (12 - 1) + 1 \\&= 6,5 \\r_{75} &= \frac{75}{100} (12 - 1) + 1 \\&= 9,25\end{aligned}$$

##### **Step 3: Find the values of the quartiles**

Note that each of these ranks is a fraction, meaning that the value for each percentile is somewhere in between two values from the data set.

For the 25<sup>th</sup> percentile the rank is 3,75, which is between the third and fourth values. Since both these values are equal to 7, the 25<sup>th</sup> percentile is 7.

For the 50<sup>th</sup> percentile (the median) the rank is 6,5, meaning halfway between the sixth and seventh values. The sixth value is 11 and the seventh value is 12, which means that the median is  $\frac{11+12}{2} = 11,5$ . For the 75<sup>th</sup> percentile the rank is 9,25, meaning between the ninth and tenth values. Therefore the 75<sup>th</sup> percentile is  $\frac{31+35}{2} = 33$ .

## Deciles

The deciles are the nine data values that divide an ordered data set into ten groups, where each group contains an equal number of data values.

For example, consider the ordered data set:

28; 33; 35; 45; 57; 59; 61; 68; 69; 72; 75; 78; 80; 83; 86; 91;  
92; 95; 101; 105; 111; 117; 118; 125; 127; 131; 137; 139; 141

The nine deciles are: 35; 59; 69; 78; 86; 95; 111; 125; 137.

## Percentiles for grouped data

EMA79

In grouped data, the percentiles will lie somewhere inside a range, rather than at a specific value. To find the range in which a percentile lies, we still use the percentile formula to determine the rank of the percentile and then find the range within which that rank is.

### Worked example 13: Percentiles in grouped data

#### QUESTION

The mathematics marks of 100 grade 10 learners at a school have been collected. The data are presented in the following table:

Percentage mark	Number of learners
$0 \leq x < 20$	2
$20 \leq x < 30$	5
$30 \leq x < 40$	18
$40 \leq x < 50$	22
$50 \leq x < 60$	18
$60 \leq x < 70$	13
$70 \leq x < 80$	12
$80 \leq x < 100$	10

1. Calculate the mean of this grouped data set.
2. In which intervals are the quartiles of the data set?
3. In which interval is the 30<sup>th</sup> percentile of the data set?

#### SOLUTION

##### Step 1: Calculate the mean

Since we are given grouped data rather than the original ungrouped data, the best we can do is approximate the mean as if all the learners in each interval were located at the central value of the interval.

$$\begin{aligned}\text{Mean} &= \frac{2(10) + 5(25) + 18(35) + 22(45) + 18(55) + 13(65) + 12(75) + 10(90)}{100} \\ &= 54\%\end{aligned}$$

## Step 2: Find the quartiles

Since the data have been grouped, they have also already been sorted. Using the percentile formula and the fact that there are 100 learners, we can find the rank of the 25<sup>th</sup>, 50<sup>th</sup> and 75<sup>th</sup> percentiles as

$$\begin{aligned}r_{25} &= \frac{25}{100} (100 - 1) + 1 \\&= 24,75 \\r_{50} &= \frac{50}{100} (100 - 1) + 1 \\&= 50,5 \\r_{75} &= \frac{75}{100} (100 - 1) + 1 \\&= 75,25\end{aligned}$$

Now we need to find in which ranges each of these ranks lie.

- For the lower quartile, we have that there are  $2 + 5 = 7$  learners in the first two ranges combined and  $2 + 5 + 18 = 25$  learners in the first three ranges combined. Since  $7 < r_{25} < 25$ , this means the lower quartile lies somewhere in the third range:  $30 \leq x < 40$ .
- For the second quartile (the median), we have that there are  $2 + 5 + 18 + 22 = 47$  learners in the first four ranges combined. Since  $47 < r_{50} < 65$ , this means that the median lies somewhere in the fifth range:  $50 \leq x < 60$ .
- For the upper quartile, we have that there are 65 learners in the first five ranges combined and  $65 + 13 = 78$  learners in the first six ranges combined. Since  $65 < r_{75} < 78$ , this means that the upper quartile lies somewhere in the sixth range:  $60 \leq x < 70$ .

## Step 3: Find the 30<sup>th</sup> percentile

Using the same method as for the quartiles, we first find the rank of the 30<sup>th</sup> percentile.

$$\begin{aligned}r &= \frac{30}{100} (100 - 1) + 1 \\&= 30,7\end{aligned}$$

Now we have to find the range in which this rank lies. Since there are 25 learners in the first 3 ranges combined and 47 learners in the first 4 ranges combined, the 30<sup>th</sup> percentile lies in the fourth range:  $40 \leq x < 50$

## Ranges

EMA7B

We define data ranges in terms of percentiles. We have already encountered the full data range, which is simply the difference between the 100<sup>th</sup> and the 0<sup>th</sup> percentile (that is, between the maximum and minimum values in the data set).

**DEFINITION:** *Interquartile range*

The interquartile range is a measure of dispersion, which is calculated by subtracting the first quartile ( $Q_1$ ) from the third quartile ( $Q_3$ ). This gives the range of the middle half of the data set.

**DEFINITION:** *Semi interquartile range*

The semi interquartile range is half of the interquartile range.

### Exercise 10 – 5:

1. A group of **15** learners count the number of sweets they each have. This is the data they collect:

4	11	14	7	14	5	8	7
12	12	5	13	10	6	7	

Calculate the **range** of values in the data set.

2. A group of **10** learners count the number of playing cards they each have. This is the data they collect:

5	1	3	1	4	10	1	3	3	4
---	---	---	---	---	----	---	---	---	---

Calculate the **range** of values in the data set.

3. Find the range of the following data set:

$$\{1; 2; 3; 4; 4; 5; 6; 7; 8; 8; 9; 10; 10\}$$

4. What are the quartiles of this data set?

$$\{3; 5; 1; 8; 9; 12; 25; 28; 24; 30; 41; 50\}$$

5. A class of 12 learners writes a test and the results are as follows:

$$\{20; 39; 40; 43; 43; 46; 53; 58; 63; 70; 75; 91\}$$

Find the range, quartiles and the interquartile range.

6. Three sets of data are given:

**Data set 1:** {9; 12; 12; 14; 16; 22; 24}

**Data set 2:** {7; 7; 8; 11; 13; 15; 16}

**Data set 3:** {11; 15; 16; 17; 19; 22; 24}

For each data set find:

- |                       |                            |                                 |
|-----------------------|----------------------------|---------------------------------|
| a) the range          | b) the lower quartile      | c) the median                   |
| d) the upper quartile | e) the interquartile range | f) the semi-interquartile range |

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

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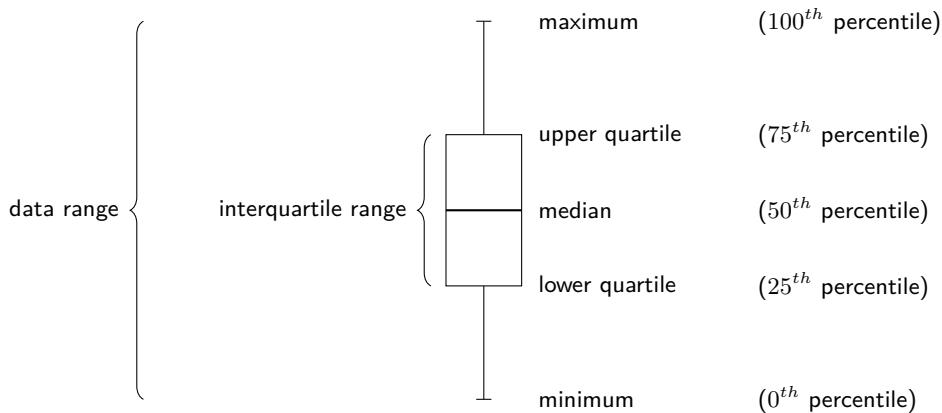
## 10.5 Five number summary

EMA7C

A common way of summarising the overall data set is with the five number summary and the box-and-whisker plot. These two represent exactly the same information, numerically in the case of the five number summary and graphically in the case of the box-and-whisker plot.

The five number summary consists of the minimum value, the maximum value and the three quartiles. Another way of saying this is that the five number summary consists of the following percentiles: 0<sup>th</sup>, 25<sup>th</sup>, 50<sup>th</sup>, 75<sup>th</sup>, 100<sup>th</sup>.

The box-and-whisker plot shows these five percentiles as in the figure below. The box shows the interquartile range (the distance between  $Q_1$  and  $Q_3$ ). A line inside the box shows the median. The lines extending outside the box (the whiskers) show where the minimum and maximum values lie. This graph can also be drawn horizontally.



## VISIT:

This video explains how to draw a box-and-whisker plot for a data set.

► See video: 2GNC at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## Worked example 14: Five number summary

## **QUESTION**

Draw a box and whisker diagram for the following data set:

$$\{1,25; 1,5; 2,5; 2,5; 3,1; 3,2; 4,1; 4,25; 4,75; 4,8; 4,95; 5,1\}$$

### **SOLUTION**

#### **Step 1: Determine the minimum and maximum**

Since the data set is already sorted, we can read off the minimum as the first value (1,25) and the maximum as the last value (5,1).

### **Step 2: Determine the quartiles**

There are 12 values in the data set. Using the percentile formula, we can determine that the median lies between the sixth and seventh values, making it:

$$\frac{3,2 + 4,1}{2} = 3,65$$

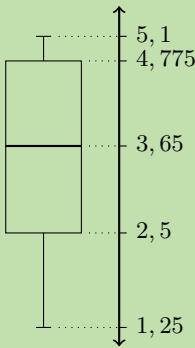
The first quartile lies between the third and fourth values, making it:

$$\frac{2,5 + 2,5}{2} = 2,5$$

The third quartile lies between the ninth and tenth values, making it:

$$\frac{4,75 + 4,8}{2} = 4,775$$

This provides the five number summary of the data set and allows us to draw the following box-and-whisker plot.



#### Exercise 10 – 6:

1. Lisa is working in a computer store. She sells the following number of computers each month:

$$\{27; 39; 3; 15; 43; 27; 19; 54; 65; 23; 45; 16\}$$

Give the five number summary and box-and-whisker plot of Lisa's sales.

2. Zithulele works as a telesales person. He keeps a record of the number of sales he makes each month. The data below show how much he sells each month.

$$\{49; 12; 22; 35; 2; 45; 60; 48; 19; 1; 43; 12\}$$

Give the five number summary and box-and-whisker plot of Zithulele's sales.

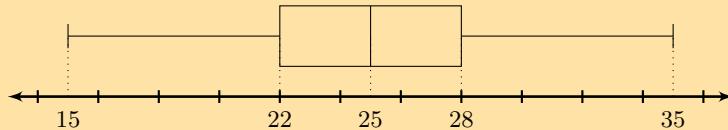
3. Nombusa has worked as a florist for nine months. She sold the following number of wedding bouquets:

$$\{16; 14; 8; 12; 6; 5; 3; 5; 7\}$$

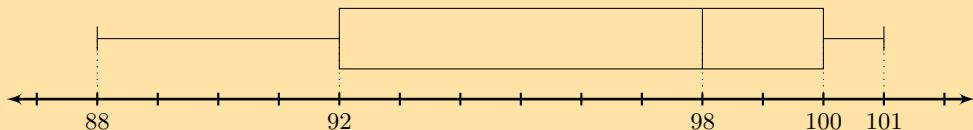
Give the five number summary of Nombusa's sales.

4. Determine the five number summary for each of the box-and-whisker plots below.

a)



b)



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- Data refer to the pieces of information that have been observed and recorded, from an experiment or a survey.
- Quantitative data are data that can be written as numbers. Quantitative data can be discrete or continuous.
- Qualitative data are data that cannot be written as numbers. There are two common types of qualitative data: categorical and anecdotal data.
- The mean is the sum of a set of values divided by the number of values in the set.

$$\begin{aligned}\bar{x} &= \frac{1}{n} \sum_{i=1}^n x_i \\ &= \frac{x_1 + x_2 + \dots + x_n}{n}\end{aligned}$$

- The median of a data set is the value in the central position, when the data set has been arranged from the lowest to the highest value. If there are an odd number of data, the median will be equal to one of the values in the data set. If there are an even number of data, the median will lie half way between two values in the data set.
- The mode of a data set is the value that occurs most often in the set.
- An outlier is a value in the data set that is not typical of the rest of the set. It is usually a value that is much greater or much less than all the other values in the data set.
- Continuous quantitative data can be grouped by dividing the full range of values into a few sub-ranges. By assigning each continuous value to the sub-range or class within which it falls, the data set changes from continuous to discrete.
- Dispersion is a general term for different statistics that describe how values are distributed around the centre.
- The range of a data set is the difference between the maximum and minimum values in the set.
- The  $p^{\text{th}}$  percentile is the value,  $v$ , that divides a data set into two parts, such that  $p\%$  of the values in the data set are less than  $v$  and  $100 - p\%$  of the values are greater than  $v$ . The general formula for finding the  $p^{\text{th}}$  percentile in an ordered data set of  $n$  values is

$$r = \frac{p}{100} (n - 1) + 1$$

- The quartiles are the three data values that divide an ordered data set into four groups, where each group contains an equal number of data values. The lower quartile is denoted  $Q_1$ , the median is  $Q_2$  and the upper quartile is  $Q_3$ .
- The interquartile range is a measure of dispersion, which is calculated by subtracting the lower (first) quartile from the upper (third) quartile. This gives the range of the middle half of the data set.
- The semi interquartile range is half of the interquartile range.
- The five number summary consists of the minimum value, the maximum value and the three quartiles ( $Q_1$ ,  $Q_2$  and  $Q_3$ ).
- The box-and-whisker plot is a graphical representation of the five number summary.

- The following data set of heights was collected from a class of learners.  
 $\{1,70 \text{ m}; 1,41 \text{ m}; 1,60 \text{ m}; 1,32 \text{ m}; 1,80 \text{ m}; 1,40 \text{ m}\}$   
 Categorise the data set.
- The following data set of sandwich spreads was collected from learners at lunch.  
 $\{\text{cheese; peanut butter; jam; cheese; honey}\}$   
 Categorise the data set.
- Calculate the **mode** of the following data set:  
 $\{10; 10; 10; 18; 7; 10; 3; 10; 7; 10; 7\}$
- Calculate the **median** of the following data set:  
 $\{5; 5; 10; 7; 10; 2; 16; 10; 10; 10; 7\}$
- In a park, the tallest 7 trees have heights (in metres):

$$\{41; 60; 47; 42; 44; 42; 47\}$$

Find the median of their heights.

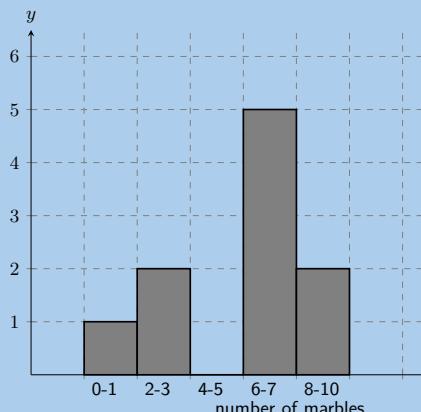
- The learners in Ndeme's class have the following ages:

$$\{5; 6; 7; 5; 4; 6; 6; 6; 7; 4\}$$

Find the mode of their ages.

- A group of 7 friends each have some sweets. They work out that the **mean** number of sweets they have is 6. Then 4 friends leave with an unknown number ( $x$ ) of sweets. The remaining 3 friends work out that the **mean** number of sweets they have left is 10,67.  
 When the 4 friends left, how many sweets did they take with them?
- A group of 10 friends each have some sweets. They work out that the **mean** number of sweets they have is 3. Then 5 friends leave with an unknown number ( $x$ ) of sweets. The remaining 5 friends work out that the **mean** number of sweets they have left is 3.  
 When the 5 friends left, how many sweets did they take with them?

- Five data values are represented as follows:  $3x; x + 2; x - 3; x + 4; 2x - 5$ , with a mean of 30. Solve for  $x$ .
- Five data values are represented as follows:  $p + 1; p + 2; p + 9$ . Find the mean in terms of  $p$ .
- A group of 10 learners count the number of marbles they each have. This is a histogram describing the data they collected:



Count the number of marbles in the following range:  $0 \leq \text{number of marbles} \leq 1$

12. A group of **20** learners count the number of playing cards they each have. This is the data they collect:

12	1	5	4	17
14	7	5	1	3
9	4	12	17	5
19	1	19	7	15

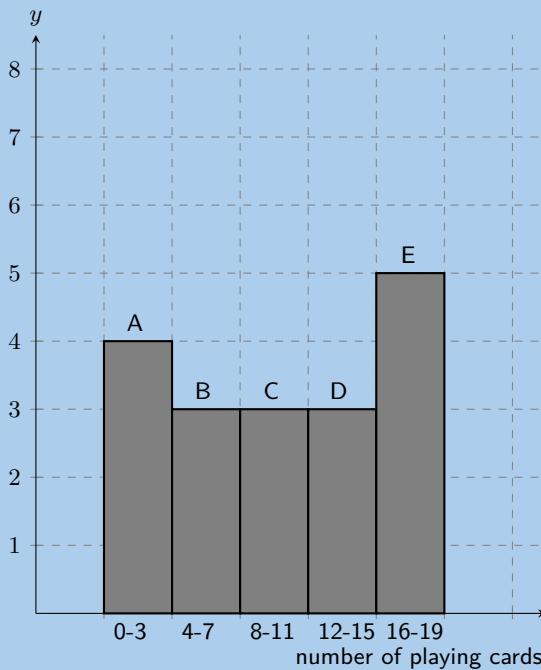
Count the number of learners who have from 0 up to 3 playing cards. In other words, how many learners have playing cards in the following range:  $0 \leq \text{number of playing cards} \leq 3$ ? It may be helpful for you to draw a histogram in order to answer the question.

13. A group of **20** learners count the number of coins they each have. This is the data they collect:

17	11	1	15	14
3	4	18	5	14
18	19	4	18	15
16	13	20	8	18

Count the number of learners who have from 4 up to 7 coins. In other words, how many learners have coins in the following range:  $4 \leq \text{number of coins} \leq 7$ ? It may be helpful for you to draw a histogram in order to answer the question.

14. A group of 20 learners count the number of playing cards they each have. The learners draw a histogram describing the data they collected. However, they have made a mistake in drawing the histogram.

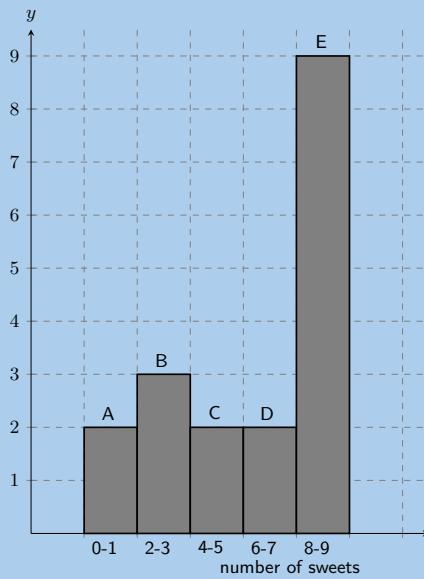


The data set below shows the correct information for the number of playing cards the learners have. Each value represents the number of playing cards for one learner.

{18; 10; 3; 2; 19; 15; 2; 13; 11; 14; 10; 3; 5; 9; 4; 18; 11; 18; 16; 5}

Help them figure out which **column** in the histogram is **incorrect**.

15. A group of 10 learners count the number of sweets they each have. The learners draw a histogram describing the data they collected. However, they have made a mistake in drawing the histogram.

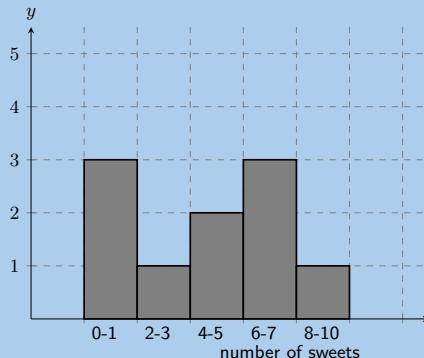


The data set below shows the correct information for the number of sweets the learners have. Each value represents the number of sweets for one learner.

$$\{1; 3; 7; 4; 5; 8; 2; 2; 1; 7\}$$

Help them figure out which **column** in the histogram is **incorrect**.

16. A group of learners count the number of sweets they each have. This is a histogram describing the data they collected:



A cleaner knocks over their table, and all their notes land on the floor, mixed up, by accident!

Help them find which of the following data sets match the above histogram:

Data set A

$$1 \quad 8 \quad 4 \quad 8 \quad 8 \quad 6 \quad 1 \quad 5 \quad 7 \quad 5$$

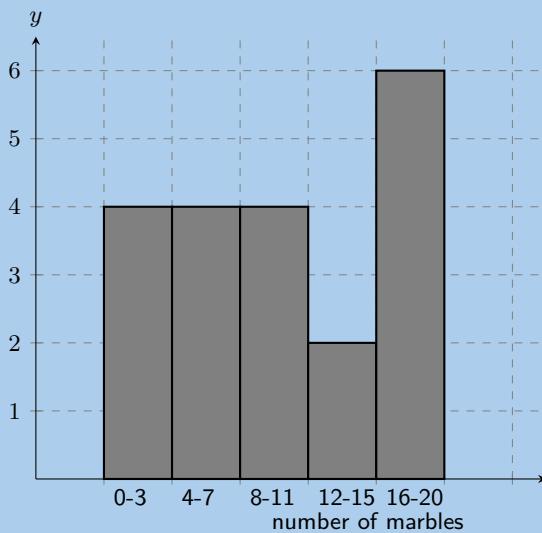
Data set B

$$5 \quad 6 \quad 9 \quad 2 \quad 1 \quad 6 \quad 6 \quad 4 \quad 4 \quad 6$$

Data set C

$$7 \quad 2 \quad 4 \quad 1 \quad 5 \quad 1 \quad 1 \quad 7 \quad 8 \quad 6$$

17. A group of learners count the number of marbles they each have. This is a histogram describing the data they collected:



A cat jumps onto the table, and all their notes land on the floor, mixed up, by accident!

Help them find which of the following data sets match the above histogram:

Data set A

7	13	15	13	12	13	8	14
3	15	1	7	4	11	1	

Data set B

17	1	5	4	11	13	6	19	6	20
19	1	14	9	17	3	16	3	10	10

Data set C

10	3	5	5	6	5	2	1	4	3
----	---	---	---	---	---	---	---	---	---

18. A group of **20** learners count the number of marbles they each have. This is the data they collect:

11	8	17	13	9	12	2	6	15	7
14	15	1	6	6	13	19	9	6	19

Calculate the **range** of values in the data set.

19. A group of **15** learners count the number of sweets they each have. This is the data they collect:

5	13	4	15	5	6	1	3
13	13	15	14	7	2	4	

Calculate the **range** of values in the data set.

20. An engineering company has designed two different types of engines for motorbikes. The two different motorbikes are tested for the time (in seconds) it takes for them to accelerate from  $0 \text{ km}\cdot\text{h}^{-1}$  to  $60 \text{ km}\cdot\text{h}^{-1}$ .

Test	1	2	3	4	5	6	7	8	9	10
<b>Bike 1</b>	1,55	1,00	0,92	0,80	1,49	0,71	1,06	0,68	0,87	1,09
<b>Bike 2</b>	0,9	1,0	1,1	1,0	1,0	0,9	0,9	1,0	0,9	1,1

- a) Which measure of central tendency should be used for this information?
- b) Calculate the measure of central tendency that you chose in the previous question, for each motorbike.
- c) Which motorbike would you choose based on this information? Take note of the accuracy of the numbers from each set of tests.
21. In a traffic survey, a random sample of 50 motorists were asked the distance they drove to work daily. This information is shown in the table below.
- | Distance (km)    | Count |
|------------------|-------|
| $0 < d \leq 5$   | 4     |
| $5 < d \leq 10$  | 5     |
| $10 < d \leq 15$ | 9     |
| $15 < d \leq 20$ | 10    |
| $20 < d \leq 25$ | 7     |
| $25 < d \leq 30$ | 8     |
| $30 < d \leq 35$ | 3     |
| $35 < d \leq 40$ | 2     |
| $40 < d \leq 45$ | 2     |
- a) Find the approximate mean of the data.
- b) What percentage of drivers had a distance of
- less than or equal to 15 km?
  - more than 30 km?
  - between 16 km and 30 km?
- c) Draw a histogram to represent the data.
22. A company wanted to evaluate the training programme in its factory. They gave the same task to trained and untrained employees and timed each one in seconds.
- |                  | 121 | 137 | 131 | 135 | 130 |
|------------------|-----|-----|-----|-----|-----|
| <b>Trained</b>   | 128 | 130 | 126 | 132 | 127 |
|                  | 129 | 120 | 118 | 125 | 134 |
|                  | 135 | 142 | 126 | 148 | 145 |
| <b>Untrained</b> | 156 | 152 | 153 | 149 | 145 |
|                  | 144 | 134 | 139 | 140 | 142 |
- a) Find the medians and quartiles for both sets of data.
- b) Find the interquartile range for both sets of data.
- c) Comment on the results.
- d) Draw a box-and-whisker diagram for each data set to illustrate the five number summary.
23. A small firm employs 9 people. The annual salaries of the employees are:

R 600 000	R 250 000	R 200 000
R 120 000	R 100 000	R 100 000
R 100 000	R 90 000	R 80 000

- a) Find the mean of these salaries.
- b) Find the mode.
- c) Find the median.
- d) Of these three figures, which would you use for negotiating salary increases if you were a trade union official? Why?

24. The stem-and-leaf diagram below indicates the pulse rate per minute of ten Grade 10 learners.

7	8
8	1    3    5    5
9	0    1    1
10	3    5

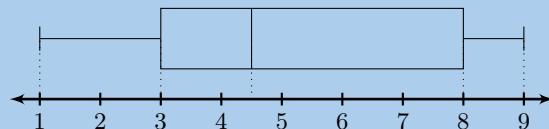
**Key:** 7|8 = 78

- a) Determine the mean and the range of the data.
- b) Give the five-number summary and create a box-and-whiskers plot for the data.

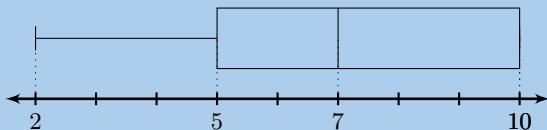
25. The following is a list of data: 3; 8; 8; 5; 9; 1; 4;  $x$

In each separate case, determine the value of  $x$  if the:

- a) range = 16
- b) mode = 8
- c) median = 6
- d) mean = 6
- e) box-and whiskers plot

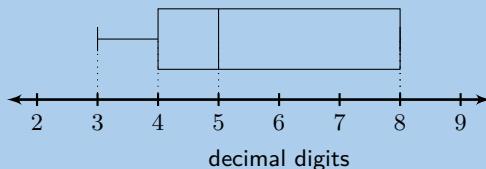


26. Write down one list of numbers that satisfies the box-and-whisker plot below:



27. Given  $\phi$  (which represents the golden ratio) to 20 decimal places: 1,61803398874989484820

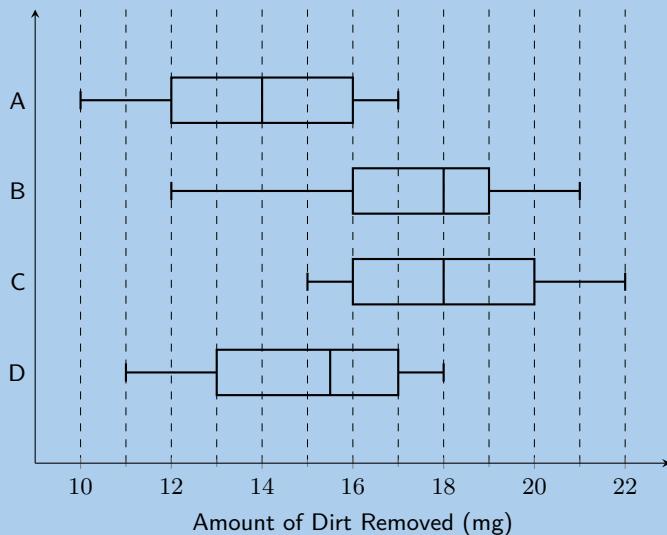
- a) For the first 20 decimal digits of ( $\Phi$ ), determine the:
  - i. median
  - ii. mode
  - iii. mean
- b) If the mean of the first 21 decimal digits of ( $\Phi$ ) is 5,38095 determine the 21<sup>st</sup> decimal digit.
- c) Below is a box-and-whisker plot of the 21<sup>st</sup> - 27<sup>th</sup> decimal digits. Write down one list of numbers that satisfies this box-and-whisker plot.



28. There are 14 men working in a factory. Their ages are : 22; 25; 33; 35; 38; 48; 53; 55; 55; 55; 55; 56; 59; 64

- a) Write down the five number summary.
- b) If 3 men had to be retrenched, but the median had to stay the same, show the ages of the 3 men you would retrench.

- c) Find the mean age of the men in the factory using the original data.
29. The example below shows a comparison of the amount of dirt removed by four brands of detergents (brands A to D).



- Which brand has the biggest range, and what is this range?
- For brand C, what does the number 18 mg represent?
- Give the interquartile range for brand B.
- Which brand of detergent would you buy? Explain your answer.

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- |          |          |          |          |          |          |
|----------|----------|----------|----------|----------|----------|
| 1. 2GNM  | 2. 2GNN  | 3. 2GNP  | 4. 2GNQ  | 5. 2GNR  | 6. 2GNS  |
| 7. 2GNT  | 8. 2GNV  | 9. 2GNW  | 10. 2GNX | 11. 2GNY | 12. 2GNZ |
| 13. 2GP2 | 14. 2GP3 | 15. 2GP4 | 16. 2GP5 | 17. 2GP6 | 18. 2GP7 |
| 19. 2GP8 | 20. 2GP9 | 21. 2GPB | 22. 2GPC | 23. 2GPD | 24. 2GPF |
| 25. 2GPG | 26. 2GPH | 27. 2GPJ | 28. 2GPK | 29. 2GPM |          |



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# CHAPTER



## *Trigonometry*

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# 11 Trigonometry

Trigonometry was developed in ancient civilisations to solve practical problems such as building construction and navigating by the stars. We will show that trigonometry can also be used to solve some other practical problems. We can use the trigonometric ratios to solve problems in two dimensions that involve right-angled triangles.

As revision the three trigonometric ratios can be defined for right-angled triangles as:

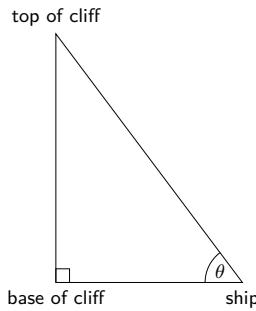
$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

We will use these three ratios and the theorem of Pythagoras to help us solve two-dimensional problems.

## 11.1 Two-dimensional problems

EMA7F

In two-dimensional problems we will often refer to the angle of elevation and the angle of depression. To understand these two angles let us consider a person sailing alongside some cliffs. The person looks up and sees the top of the cliffs as shown below:



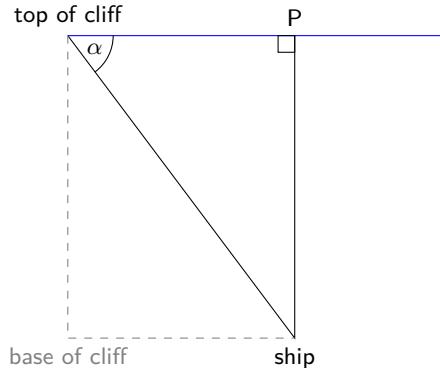
In this diagram  $\theta$  is the angle of elevation.

**DEFINITION:** *Angle of elevation*

The angle of elevation is the angle formed by the line of sight and the horizontal plane for an object above the horizontal plane.

In our diagram the line of sight is from the ship to the top of the cliffs. The horizontal plane is from the ship to the base of the cliffs. Also note the we can consider the cliffs to be a straight vertical line and so we have a right-angled triangle.

To understand the angle of depression let us now consider the same situation as above but instead our observer is standing on top of the cliffs looking down at the ship.



In this diagram  $\alpha$  is the angle of depression.

**DEFINITION:** *Angle of depression*

The angle of depression is the angle formed by the line of sight and the horizontal plane for an object below the horizontal plane.

In our diagram the line of sight is from the top of the cliffs to the ship. The horizontal plane is from the top of the cliffs through  $P$ . Note that this is parallel to the line between the base of the cliffs and the ship.  $P$  lies directly above the ship. We can construct a vertical, perpendicular line to the horizontal plane at the point  $P$ .

Finally we can compare the angle of elevation and the angle of depression. In the following diagram the line from the base of the cliffs to the ship is parallel to the line from the top of the cliffs to  $P$ . The angle of elevation and the angle of depression are indicated. Notice that  $\alpha = \theta$ .

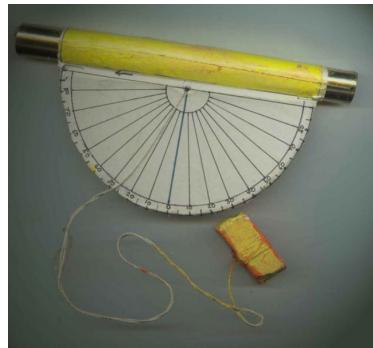
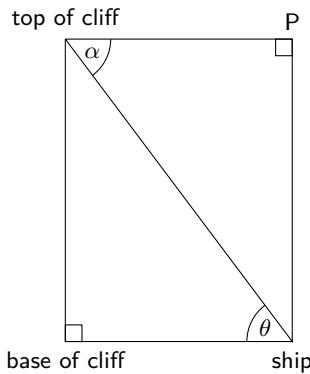


Figure 11.1: An inclinometer. Inclinometers can be used to measure angles of inclination and so can be used to determine the height of an object.

**VISIT:**

You can make an [inclinometer](#) to measure the angle of inclination of a tall building or tree. Once you have the angle of inclination you can determine the height of the building or tree.

**NOTE:**

In trigonometry the angle of inclination is the same as the angle of elevation.

## Worked example 1: Flying a kite

### QUESTION

Mandla flies a kite on a 17 m string at an inclination of  $63^\circ$ .

- What is the height,  $h$ , of the kite above the ground?
- If Mandla's friend Sipho stands directly below the kite, calculate the distance,  $d$ , between the two friends.

### SOLUTION

#### Step 1: Make a sketch and identify the opposite and adjacent sides and the hypotenuse



#### Step 2: Use given information and appropriate ratio to solve for $h$ and $d$

1.

$$\begin{aligned}\sin 63^\circ &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \sin 63^\circ &= \frac{h}{17} \\ \therefore h &= 17 \sin 63^\circ \\ &= 15,14711\dots \\ &\approx 15,15 \text{ m}\end{aligned}$$

2.

$$\begin{aligned}\cos 63^\circ &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ \cos 63^\circ &= \frac{d}{17} \\ \therefore d &= 17 \cos 63^\circ \\ &= 7,7178\dots \\ &\approx 7,72 \text{ m}\end{aligned}$$

Note that the third side of the triangle can also be calculated using the theorem of Pythagoras:  $d^2 = 17^2 - h^2$ .

#### Step 3: Write the final answers

- The kite is 15,15 m above the ground.
- Mandla and Sipho are 7,72 m apart.

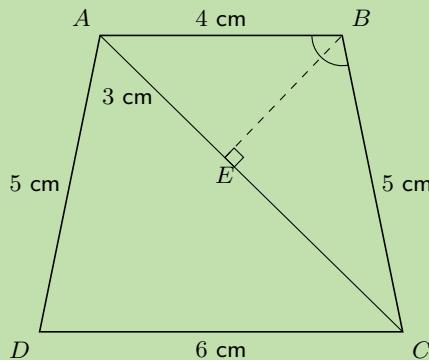
## Worked example 2: Calculating angles

### QUESTION

$ABCD$  is a trapezium with  $AB = 4 \text{ cm}$ ,  $CD = 6 \text{ cm}$ ,  $BC = 5 \text{ cm}$  and  $AD = 5 \text{ cm}$ . Point  $E$  on diagonal  $AC$  divides the diagonal such that  $AE = 3 \text{ cm}$ .  $\hat{BEC} = 90^\circ$ . Find  $\hat{ABC}$ .

### SOLUTION

**Step 1: Draw the trapezium and label all given lengths on diagram. Indicate that  $\hat{BEC} = 90^\circ$**



We will use  $\triangle ABE$  and  $\triangle CBE$  to find  $\hat{A}BE$  and  $\hat{C}BE$ . We can then add these two angles together to find  $\hat{ABC}$ .

**Step 2: Find the first angle,  $\hat{A}BE$**

The hypotenuse and opposite side are given for both triangles, therefore we use the sin function.

In  $\triangle ABE$ :

$$\sin \hat{A}BE = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$= \frac{3}{4}$$

$$\hat{A}BE = 48,5903\dots$$

$$\approx 48,6^\circ$$

**Step 3: Use the theorem of Pythagoras to determine  $BE$**

In  $\triangle ABE$ :

$$BE^2 = AB^2 - AE^2$$

$$= 4^2 - 3^2$$

$$= 7$$

$$\therefore BE = \sqrt{7} \text{ cm}$$

**Step 4: Find the second angle  $\hat{C}BE$**

In  $\triangle CBE$ :

$$\cos \hat{C}BE = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{\sqrt{7}}{5}$$

$$= 0,52915\dots$$

$$\hat{C}BE = 58,0519\dots$$

$$\approx 58,1^\circ$$

### Step 5: Calculate the sum of the angles

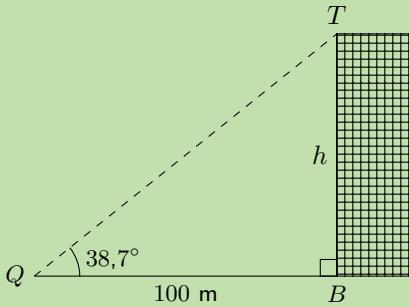
$$A\hat{B}C = 48,6^\circ + 58,1^\circ = 106,7^\circ$$

Another application is using trigonometry to find the height of a building. We could use a tape measure lowered from the roof, but this is impractical (and dangerous) for tall buildings. It is much more sensible to use trigonometry.

### Worked example 3: Finding the height of a building

#### QUESTION

The given diagram shows a building of unknown height  $h$ . We start at point  $B$  and walk 100 m away from the building to point  $Q$ . Next we measure the angle of elevation from the ground to the top of the building,  $T$ , and find that the angle is  $38,7^\circ$ . Calculate the height of the building, correct to the nearest metre.



#### SOLUTION

##### Step 1: Identify the opposite and adjacent sides and the hypotenuse

We have a right-angled triangle and know the length of one side and an angle. We can therefore calculate the height of the building.

##### Step 2:

In  $\triangle QTB$ :

$$\begin{aligned}\tan 38,7^\circ &= \frac{\text{opposite}}{\text{adjacent}} \\ &= \frac{h}{100}\end{aligned}$$

##### Step 3: Rearrange and solve for $h$

$$\begin{aligned}h &= 100 \times \tan 38,7^\circ \\ &= 80,1151... \\ &\approx 80\end{aligned}$$

##### Step 4: Write final answer

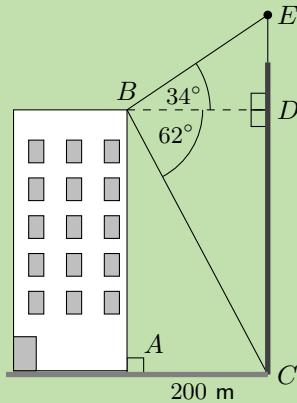
The height of the building is 80 m.

## Worked example 4: Angles of elevation and depression

### QUESTION

A block of flats is 200 m away from a cellphone tower. Someone stands at  $B$ . They measure the angle from  $B$  to the top of the tower ( $E$ ) to be  $34^\circ$  (the angle of elevation). They then measure the angle from  $B$  to the bottom of the tower ( $C$ ) to be  $62^\circ$  (the angle of depression).

What is the height of the cellphone tower (correct to the nearest metre)?



Note: diagram not drawn to scale

### SOLUTION

#### Step 1: To determine height $CE$ , first calculate lengths $DE$ and $CD$

$\triangle BDE$  and  $\triangle BDC$  are both right-angled triangles. In each of the triangles, the length  $BD$  is known. Therefore we can calculate the sides of the triangles.

#### Step 2: Calculate $CD$

The length  $AC$  is given.  $CABD$  is a rectangle so  $BD = AC = 200$  m.

In  $\triangle CBD$ :

$$\begin{aligned}\tan C\hat{B}D &= \frac{CD}{BD} \\ \therefore CD &= BD \times \tan C\hat{B}D \\ &= 200 \times \tan 62^\circ \\ &= 376,1452... \\ &\approx 376 \text{ m}\end{aligned}$$

#### Step 3: Calculate $DE$

In  $\triangle DBE$ :

$$\begin{aligned}\tan D\hat{B}E &= \frac{DE}{BD} \\ \therefore DE &= BD \times \tan D\hat{B}E \\ &= 200 \times \tan 34^\circ \\ &= 134,9017... \\ &\approx 135 \text{ m}\end{aligned}$$

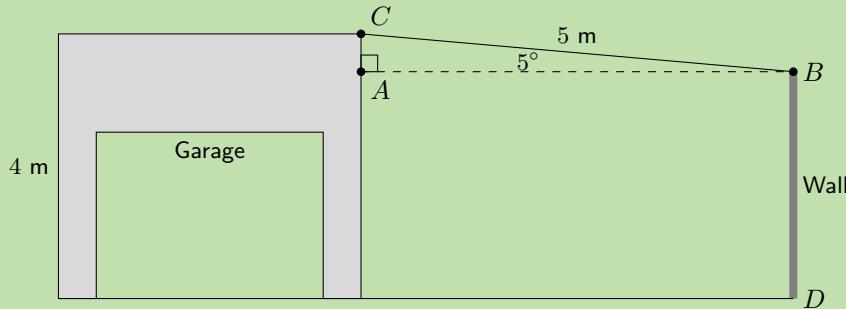
#### Step 4: Add the two heights to get the final answer

The height of the tower is:  $CE = CD + DE = 135 \text{ m} + 376 \text{ m} = 511 \text{ m}$ .

## Worked example 5: Building plan

### QUESTION

Mr Nkosi has a garage at his house and he decides to add a corrugated iron roof to the side of the garage. The garage is 4 m high, and his sheet for the roof is 5 m long. If the angle of the roof is  $5^\circ$ , how high must he build the wall  $BD$ ? Give the answer correct to 1 decimal place.



### SOLUTION

#### Step 1: Identify opposite and adjacent sides and hypotenuse

$\triangle ABC$  is right-angled. The hypotenuse and an angle are known therefore we can calculate  $AC$ . The height of the wall  $BD$  is then the height of the garage minus  $AC$ .

$$\begin{aligned}\sin A\hat{B}C &= \frac{AC}{BC} \\ \therefore AC &= BC \times \sin A\hat{B}C \\ &= 5 \sin 5^\circ \\ &= 0,43577\dots \\ &\approx 0,4 \text{ m}\end{aligned}$$

$$\begin{aligned}\therefore BD &= 4 \text{ m} - 0,4 \text{ m} \\ &= 3,6 \text{ m}\end{aligned}$$

#### Step 2: Write the final answer

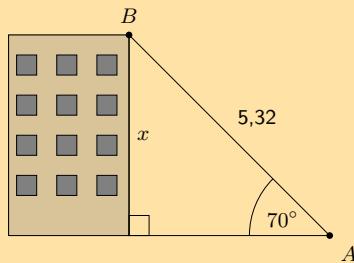
Mr Nkosi must build his wall to be 3,6 m high.

### Exercise 11 – 1:

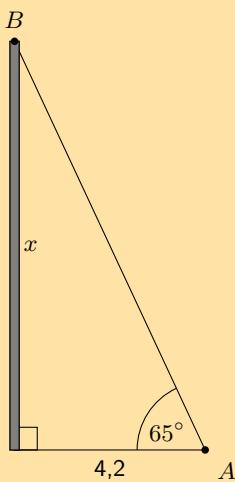
1. A person stands at point  $A$ , looking up at a bird sitting on the top of a building, point  $B$ .

The height of the building is  $x$  meters, the line of sight distance from point  $A$  to the top of the building (point  $B$ ) is 5,32 meters, and the angle of elevation to the top of the building is  $70^\circ$ .

Calculate the height of the building ( $x$ ) as shown in the diagram below:



2. A person stands at point  $A$ , looking up at a bird sitting on the top of a pole (point  $B$ ).  
The height of the pole is  $x$  meters, point  $A$  is 4,2 meters away from the foot of the pole, and the angle of elevation to the top of the pole is  $65^\circ$ .  
Calculate the height of the pole ( $x$ ), to the nearest metre.



3. A boy flying a kite is standing 30 m from a point directly under the kite. If the kite's string is 50 m long, find the angle of elevation of the kite.  
4. What is the angle of elevation of the sun when a tree 7,15 m tall casts a shadow 10,1 m long?  
5. From a distance of 300 m, Susan looks up at the top of a lighthouse. The angle of elevation is  $5^\circ$ . Determine the height of the lighthouse to the nearest metre.  
6. A ladder of length 25 m is resting against a wall, the ladder makes an angle  $37^\circ$  to the wall. Find the distance between the wall and the base of the ladder to the nearest metre.

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1. [2GPN](#) 2. [2GPP](#) 3. [2GPQ](#) 4. [2GPR](#) 5. [2GPS](#) 6. [2GPT](#)



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## 11.2 Chapter summary

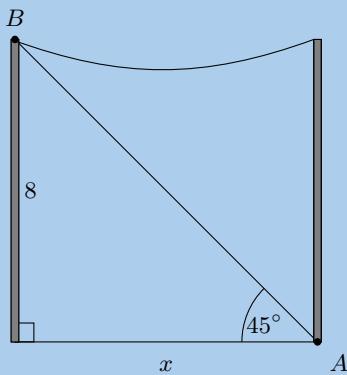
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► See presentation: [2GPV](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

- We can define three trigonometric ratios for right-angled triangles: sine (sin), cosine (cos) and tangent (tan).
- Trigonometry is used to help us solve problems in two dimensions that involve right-angled triangles, such as finding the height of a building.
- The angle of elevation is the angle formed by the line of sight and the horizontal plane for an object above the horizontal plane.
- The angle of depression is the angle formed by the line of sight and the horizontal plane for an object below the horizontal plane.

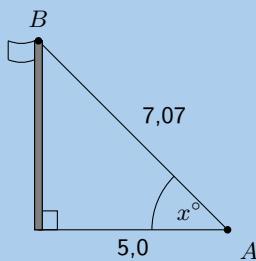
1. A ladder of length 15 m is resting against a wall, the base of the ladder is 5 m from the wall. Find the angle between the wall and the ladder.
2. Captain Jack is sailing towards a cliff with a height of 10 m.
  - a) The distance from the boat to the bottom of the cliff is 30 m. Calculate the angle of elevation from the boat to the top of the cliff (correct to the nearest degree).
  - b) If the boat sails 7 m closer to the cliff, what is the new angle of elevation from the boat to the top of the cliff?
3. Jim stands at point  $A$  at the base of a telephone pole, looking up at a bird sitting on the top of another telephone pole (point  $B$ ). The height of each of the telephone poles is 8 meters, and the angle of elevation from  $A$  to the top of  $B$  is  $45^\circ$ .

Calculate the distance between the telephone poles ( $x$ ) as shown in the diagram below:

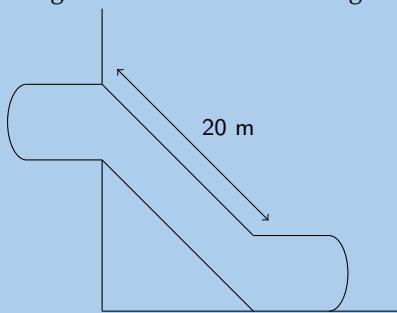


4. Alfred stands at point  $A$ , looking up at a flag on a pole (point  $B$ ). Point  $A$  is 5,0 meters away from the bottom of the flag pole, the line of sight distance from point  $A$  to the top of the flag pole (point  $B$ ) is 7,07 meters, and the angle of elevation to the top of the flag pole is  $x^\circ$ .

Calculate the angle of elevation to the top of the flag pole ( $x$ ) as shown in the diagram below:

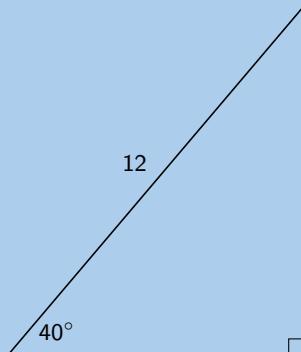


5. A rugby player is trying to kick a ball through the poles. The rugby crossbar is 3,4 m high. The ball is placed 24 m from the poles. What is the minimum angle he needs to launch the ball to get it over the bar?
6. The escalator at a mall slopes at an angle of  $30^\circ$  and is 20 m long.



Through what height would a person be lifted by travelling on the escalator?

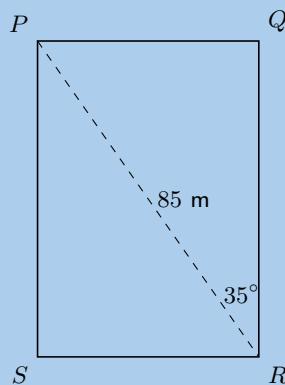
7. A ladder is 8 metres long. It is leaning against the wall of a house and reaches 6 metres up the wall.
- Draw a sketch of the situation.
  - Calculate the angle which the ladder makes with the flat (level) ground.
8. Nandi is standing on level ground 70 metres away from a tall tower. From her position, the angle of elevation of the top of the tower is  $38^\circ$ .
- Draw a sketch of the situation.
  - What is the height of the tower?
9. The top of a pole is anchored by a 12 m cable which makes an angle of  $40^\circ$  with the horizontal. What is the height of the pole?



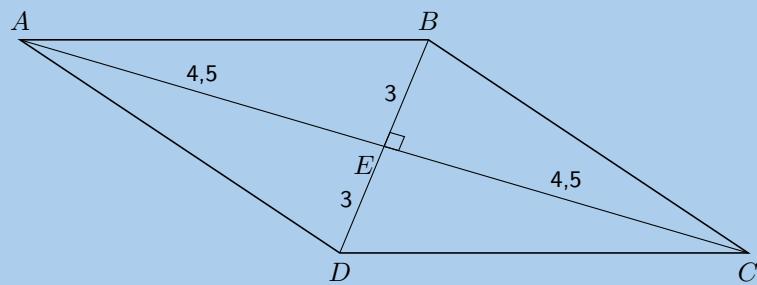
10. A ship's navigator observes a lighthouse on a cliff. According to the navigational charts the top of the lighthouse is 35 metres above sea level. She measures the angle of elevation of the top of the lighthouse to be  $0,7^\circ$ .

Ships have been advised to stay at least 4 km away from the shore. Is the ship safe?

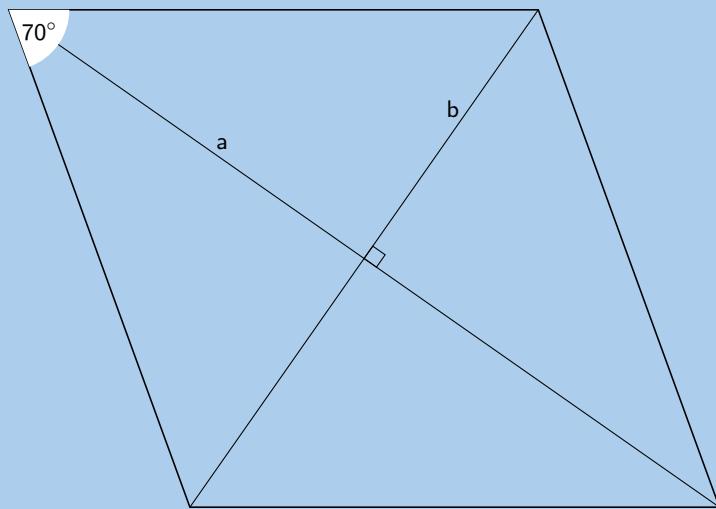
11. Determine the perimeter of rectangle  $PQRS$ :



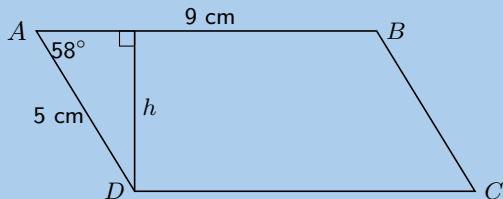
12. A rhombus has diagonals of lengths 6 cm and 9 cm. Calculate the sizes of its interior vertex angles.



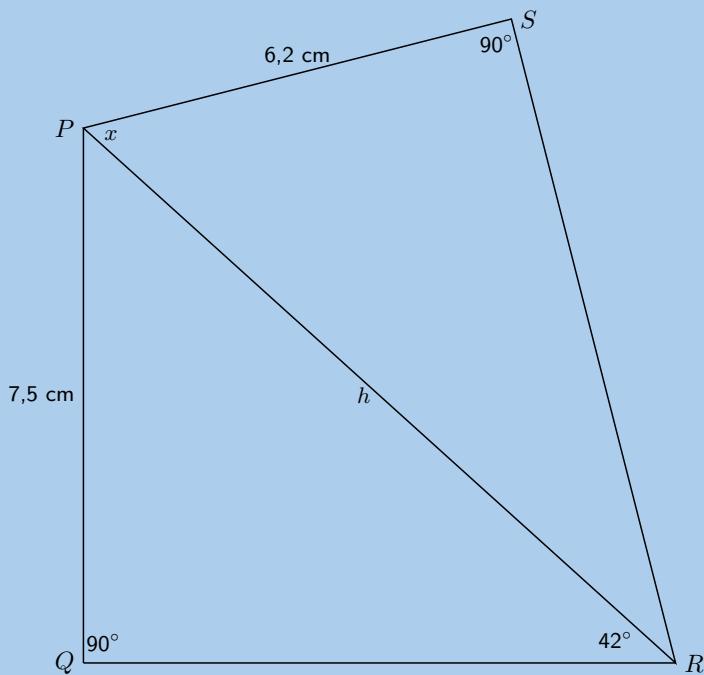
13. A rhombus has edge lengths of 7 cm. Its acute interior vertex angles are both  $70^\circ$ . Calculate the lengths of both of its diagonals.



14. A parallelogram has edge-lengths of 5 cm and 9 cm respectively, and an angle of  $58^\circ$  between them. Calculate the perpendicular distance between the two longer edges.



15. One of the angles of a rhombus with perimeter 20 cm is  $30^\circ$ .
- Find the lengths the sides of the rhombus.
  - Find the length of both diagonals.
16. Upright sticks and the shadows they cast can be used to judge the sun's altitude in the sky (the angle the sun makes with the horizontal) and the heights of objects.
- An upright stick, 1 metre tall, casts a shadow which is 1.35 metres long. What is the altitude of the sun?
  - At the same time, the shadow of a building is found to be 47 metres long. What is the height of the building?
17. The angle of elevation of a hot air balloon, climbing vertically, changes from 25 degrees at 11:00 am to 60 degrees at 11:02 am. The point of observation of the angle of elevation is situated 300 metres away from the take off point.
- Draw a sketch of the situation.
  - Calculate the increase in height between 11:00 am and 11:02 am.
18. When the top,  $T$ , of a mountain is viewed from point  $A$ , 2000 m from the ground, the angle of depression ( $a$ ) is equal to  $15^\circ$ . When it is viewed from point  $B$  on the ground, the angle of elevation ( $b$ ) is equal to  $10^\circ$ . If the points  $A$  and  $B$  are on the same vertical line, find the height,  $h$ , of the mountain. Round your answer to one decimal place.
19. The diagram below shows quadrilateral  $PQRS$ , with  $PQ = 7.5$  cm,  $PS = 6.2$  cm, angle  $R = 42^\circ$  and angles  $S$  and  $Q$  are right angles.



- a) Find  $PR$ , correct to 2 decimal places.
- b) Find the size of the angle marked  $x$ , correct to one decimal place.
20. From a boat at sea ( $S$ ), the angle of elevation of the top of a lighthouse  $PQ$ , on a cliff  $QR$ , is  $27^\circ$ .  
The lighthouse is 10 m high and the cliff top is 75 m above sea level.  
How far is the boat from the base of the cliff, to the nearest metre?

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- |                          |                          |                          |                          |                          |                          |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1. <a href="#">2GPW</a>  | 2. <a href="#">2GPX</a>  | 3. <a href="#">2GPY</a>  | 4. <a href="#">2GPZ</a>  | 5. <a href="#">2GQ2</a>  | 6. <a href="#">2GQ3</a>  |
| 7. <a href="#">2GQ4</a>  | 8. <a href="#">2GQ5</a>  | 9. <a href="#">2GQ6</a>  | 10. <a href="#">2GQ7</a> | 11. <a href="#">2GQ8</a> | 12. <a href="#">2GQ9</a> |
| 13. <a href="#">2GQB</a> | 14. <a href="#">2GQC</a> | 15. <a href="#">2GQD</a> | 16. <a href="#">2GQF</a> | 17. <a href="#">2GQG</a> | 18. <a href="#">2GQH</a> |
| 19. <a href="#">2GQJ</a> | 20. <a href="#">2GQK</a> |                          |                          |                          |                          |



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# CHAPTER



## *Euclidean geometry*

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12.2	<i>Chapter summary</i>	409

## 12 Euclidean geometry

Geometry (from the Greek “geo” = earth and “metria” = measure) arose as the field of knowledge dealing with spatial relationships. Geometry can be split into Euclidean geometry and analytical geometry. Analytical geometry deals with space and shape using algebra and a coordinate system. Euclidean geometry deals with space and shape using a system of logical deductions.

### 12.1 Proofs and conjectures

EMA7H

We will now apply what we have learnt about geometry and the properties of polygons (in particular triangles and quadrilaterals) to prove some of these properties. We will also look at how we can prove a particular quadrilateral is one of the special quadrilaterals.

#### VISIT:

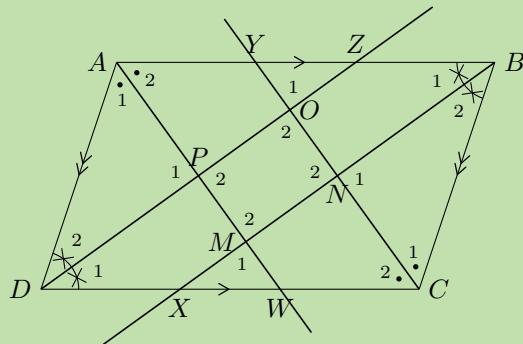
This video shows how to prove that the diagonals of a rhombus are perpendicular.

► See video: [2GQM](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

#### Worked example 1: Proving a quadrilateral is a parallelogram

#### QUESTION

In parallelogram  $ABCD$ , the bisectors of the angles ( $AW$ ,  $BX$ ,  $CY$  and  $DZ$ ) have been constructed. You are also given  $AB = CD$ ,  $AD = BC$ ,  $AB \parallel CD$ ,  $AD \parallel BC$ ,  $\hat{A} = \hat{C}$ ,  $\hat{B} = \hat{D}$ . Prove that  $MNOP$  is a parallelogram.



#### SOLUTION

**Step 1: Use properties of the parallelogram  $ABCD$  to fill in on the diagram all equal sides and angles.**

**Step 2: Prove that  $\hat{M}_2 = \hat{O}_2$**

In  $\triangle CDZ$  and  $\triangle ABX$ ,

$$\begin{aligned} D\hat{C}Z &= B\hat{A}X && \text{(given)} \\ \hat{D}_1 &= \hat{B}_1 && \text{(given)} \\ DC &= AB && \text{(given)} \\ \therefore \triangle CDZ &\cong \triangle ABX && \text{(AAS)} \\ \therefore CZ &= AX \\ \text{and } C\hat{Z}D &= A\hat{X}B \end{aligned}$$

In  $\triangle XAM$  and  $\triangle ZCO$

$$\begin{aligned}
 X\hat{A}M &= Z\hat{C}O && (\text{given: } \triangle CDZ \equiv \triangle ABX) \\
 A\hat{X}M &= C\hat{Z}O && (\text{proved above}) \\
 AX &= CZ && (\text{proved above}) \\
 \therefore \triangle XAM &\equiv \triangle ZCO && (\text{AAS}) \\
 \therefore \hat{M}_1 &= \hat{O}_1 \\
 \text{but } \hat{M}_1 &= \hat{M}_2 && (\text{vert opp } \angle s =) \\
 \text{and } \hat{O}_1 &= \hat{O}_2 && (\text{vert opp } \angle s =) \\
 \therefore \hat{M}_2 &= \hat{O}_2
 \end{aligned}$$

**Step 3:** Similarly, we can show that  $\hat{N}_2 = \hat{P}_2$

First show  $\triangle ADW \equiv \triangle CBY$ . Then show  $\triangle PDW \equiv \triangle NBY$ .

**Step 4: Conclusion**

Both pairs of opposite angles of  $MNOP$  are equal. Therefore  $MNOP$  is a parallelogram.

### VISIT:

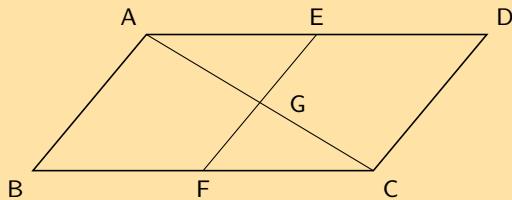
This video shows how to prove that the opposite angles of a parallelogram are equal.

► See video: 2GQN at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Exercise 12 – 1:

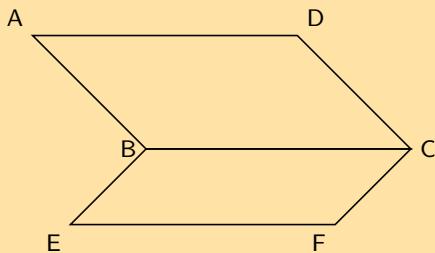
1. In the diagram below,  $AC$  and  $EF$  bisect each other at  $G$ .  $E$  is the midpoint of  $AD$ , and  $F$  is the midpoint of  $BC$ .

- a) Prove  $AECF$  is a parallelogram.

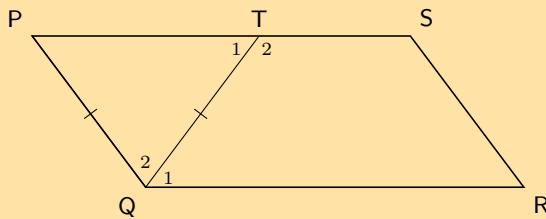


- b) Prove  $ABCD$  is a parallelogram.

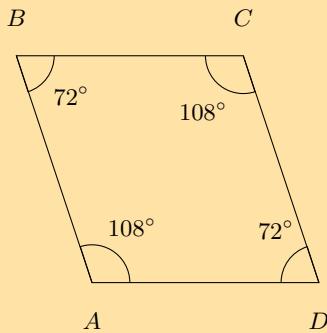
2. Parallelogram  $ABCD$  and  $BEFC$  are shown below. Prove  $AD = EF$ .



3. In the diagram below  $PQRS$  is a parallelogram with  $PQ = TQ$ . Prove  $\hat{Q}_1 = \hat{R}$

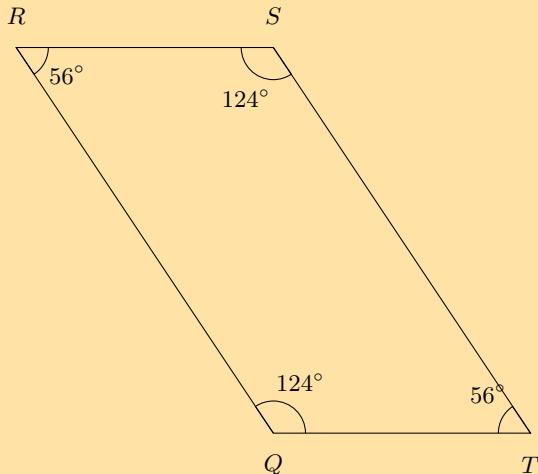


4. Study the quadrilateral  $ABCD$  with opposite angles  $\hat{A} = \hat{C} = 108^\circ$  and angles  $\hat{B} = \hat{D} = 72^\circ$  carefully.  
Fill in the missing reasons and steps to prove that the quadrilateral  $ABCD$  is a parallelogram.



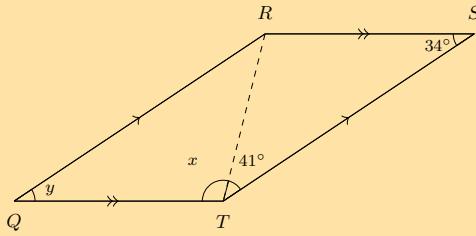
Steps	Reasons
?	given both $\angle$ 's = $108^\circ$
$A\hat{B}C = A\hat{D}C$	given both $\angle$ 's = $72^\circ$
$\hat{A} + \hat{B} + \hat{C} + \hat{D} = 360^\circ$	sum of $\angle$ s in quad
$B\hat{A}D + A\hat{D}C = 180^\circ$	given $108^\circ + 72^\circ = 180^\circ$
$\therefore AB \parallel DC$	?
$\therefore BC \parallel AD$	?
$\therefore ABCD$ is a parallelogram	opp sides of quad $\parallel$

5. Study the quadrilateral  $QRST$  with opposite angles  $Q = S = 124^\circ$  and angles  $R = T = 56^\circ$  carefully.  
Fill in the missing reasons and steps to prove that the quadrilateral  $QRST$  is a parallelogram.

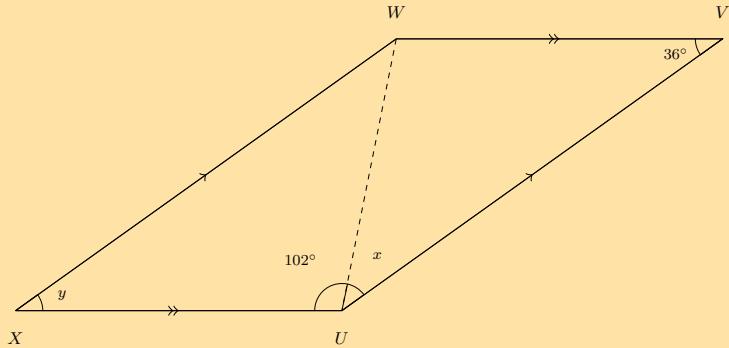


Steps	Reasons
$R\hat{Q}T = R\hat{S}T$	given both $\angle$ s = $124^\circ$
?	given both $\angle$ s = $56^\circ$
$\hat{Q} + \hat{R} + \hat{S} + \hat{T} = 360^\circ$	?
$R\hat{Q}T + Q\hat{T}S = 180^\circ$	?
$\therefore QR \parallel TS$	co-int $\angle$ s; $QR \parallel TS$
$\therefore RS \parallel QT$	co-int $\angle$ s; $RS \parallel QT$
?	opp sides of quad $\parallel$

6. a) Quadrilateral  $QRST$  with sides  $QR \parallel TS$  and  $QT \parallel RS$  is given. You are also given that:  $\hat{Q} = y$  and  $\hat{S} = 34^\circ$ ;  $Q\hat{T}R = x$  and  $R\hat{T}S = 41^\circ$ . Prove that  $QRST$  is a parallelogram.



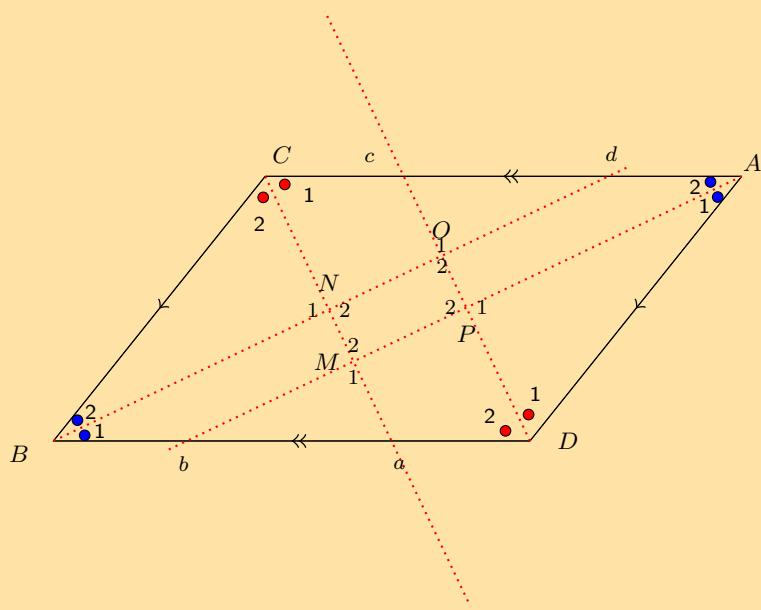
- b) Find the value of  $y$ .  
 c) Find the value of  $x$ .
7. a) Quadrilateral  $XWVU$  with sides  $XW \parallel UV$  and  $XU \parallel WV$  is given. Also given is  $\hat{X} = y$  and  $\hat{V} = 36^\circ$ ;  $X\hat{U}W = 102^\circ$  and  $W\hat{U}V = x$ . Prove that  $XWVU$  is a parallelogram.



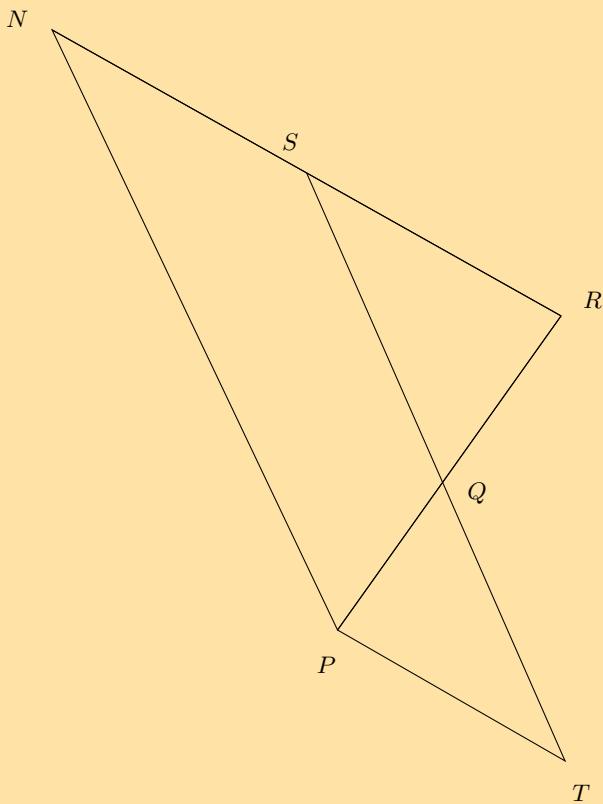
- b) Determine the value of  $y$ .  
 c) Determine the value of  $x$ .
8. In parallelogram  $ADBC$ , the bisectors of the angles ( $A, D, B, C$ ) have been constructed, indicated with the red lines below. You are also given  $AD = CB$ ,  $DB = AC$ ,  $AD \parallel CB$ ,  $DB \parallel AC$ ,  $\hat{A} = \hat{B}$  and  $\hat{D} = \hat{C}$ .

Prove that the quadrilateral  $MNOP$  is a parallelogram.

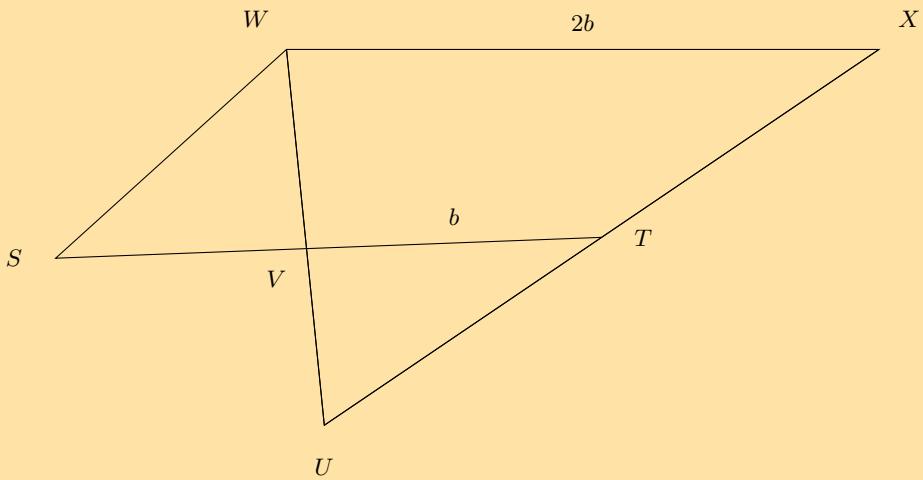
Note the diagram is drawn to scale.



9. Study the diagram below; it is not necessarily drawn to scale. Two triangles in the figure are congruent:  $\triangle QRS \cong \triangle QPT$ . Additionally,  $SN = SR$ . You need to prove that  $NPTS$  is a parallelogram.



10. Study the diagram below; it is not necessarily drawn to scale. Quadrilateral  $XWST$  is a parallelogram and  $TV$  and  $XW$  have lengths  $b$  and  $2b$ , respectively, as shown. You need to prove that  $\triangle TVU \cong \triangle SVW$ .



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- 1. [2GQP](#)
- 2. [2GQQ](#)
- 3. [2GQR](#)
- 4. [2GQS](#)
- 5. [2GQT](#)
- 6. [2GQV](#)
- 7. [2GQW](#)
- 8. [2GQX](#)
- 9. [2GQY](#)
- 10. [2GQZ](#)



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► See presentation: [2GR2](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

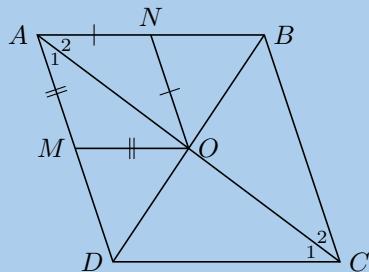
- A quadrilateral is a closed shape consisting of four straight line segments.
- A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
  - Both pairs of opposite sides are equal in length.
  - Both pairs of opposite angles are equal.
  - Both diagonals bisect each other.
- A rectangle is a parallelogram that has all four angles equal to  $90^\circ$ 
  - Both pairs of opposite sides are parallel.
  - Both pairs of opposite sides are equal in length.
  - The diagonals bisect each other.
  - The diagonals are equal in length.
  - All interior angles are equal to  $90^\circ$
- A rhombus is a parallelogram that has all four sides equal in length.
  - Both pairs of opposite sides are parallel.
  - All sides are equal in length.
  - Both pairs of opposite angles are equal.
  - The diagonals bisect each other at  $90^\circ$
  - The diagonals of a rhombus bisect both pairs of opposite angles.
- A square is a rhombus that has all four interior angles equal to  $90^\circ$ 
  - Both pairs of opposite sides are parallel.
  - The diagonals bisect each other at  $90^\circ$
  - All interior angles are equal to  $90^\circ$
  - The diagonals are equal in length.
  - The diagonals bisect both pairs of interior opposite angles (i.e. all are  $45^\circ$ )
- A trapezium is a quadrilateral with one pair of opposite sides parallel.
- A kite is a quadrilateral with two pairs of adjacent sides equal.
  - One pair of opposite angles are equal (the angles are between unequal sides).
  - The diagonal between equal sides bisects the other diagonal.
  - The diagonal between equal sides bisects the interior angles.
  - The diagonals intersect at  $90^\circ$
- The mid-point theorem: The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.

**VISIT:**

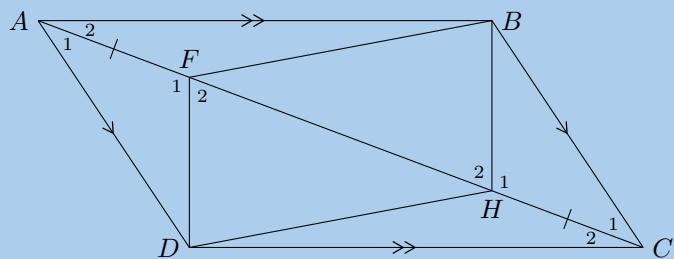
This video shows how to prove that the opposite sides of a parallelogram are equal.

► See video: [2GR3](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

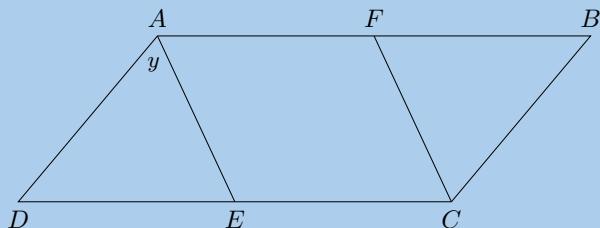
1.  $ABCD$  is a rhombus with  $AM = MO$  and  $AN = NO$ . Prove  $ANOM$  is also a rhombus.



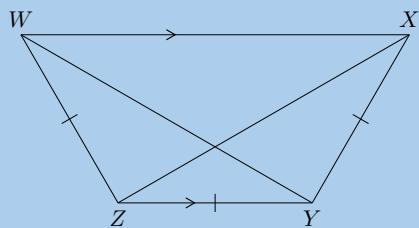
2.  $ABCD$  is a parallelogram with diagonal  $AC$ . Given that  $AF = HC$ , show that:



- a)  $\triangle AFD \equiv \triangle CHB$
  - b)  $DF \parallel HB$
  - c)  $DFBH$  is a parallelogram
3. Given parallelogram  $ABCD$  with  $AE$  bisecting  $\hat{A}$  and  $FC$  bisecting  $\hat{C}$ .

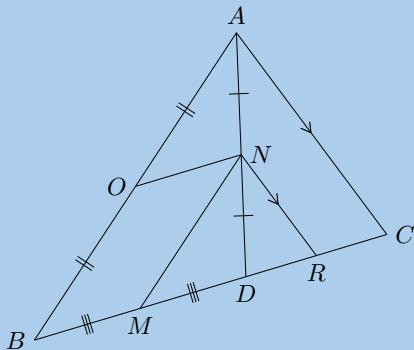


- a) Write all interior angles in terms of  $y$ .
  - b) Prove that  $AFCE$  is a parallelogram.
4. Given that  $WZ = ZY = YX$ ,  $\hat{W} = \hat{X}$  and  $WX \parallel ZY$ , prove that:

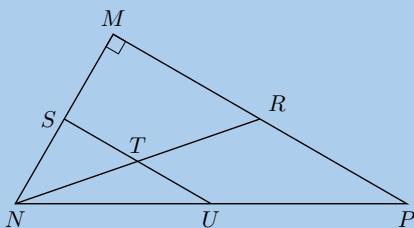


- a)  $XZ$  bisects  $\hat{X}$
- b)  $WY = XZ$

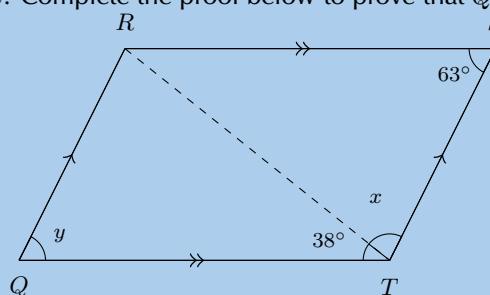
5.  $D$  is a point on  $BC$ , in  $\triangle ABC$ .  $N$  is the mid-point of  $AD$ .  $O$  is the mid-point of  $AB$  and  $M$  is the mid-point of  $BD$ .  $NR \parallel AC$ .



- a) Prove that  $OBMN$  is a parallelogram.  
 b) Prove that  $BC = 2MR$ .
6. In  $\triangle MNP$ ,  $\hat{M} = 90^\circ$ ,  $S$  is the mid-point of  $MN$  and  $T$  is the mid-point of  $NR$ .



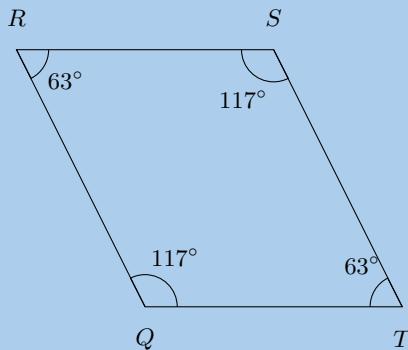
- a) Prove  $U$  is the mid-point of  $NP$ .  
 b) If  $ST = 4$  cm and the area of  $\triangle SNT$  is  $6 \text{ cm}^2$ , calculate the area of  $\triangle MNR$ .  
 c) Prove that the area of  $\triangle MNR$  will always be four times the area of  $\triangle SNT$ , let  $ST = x$  units and  $SN = y$  units.
7. a) Given quadrilateral  $QRST$  with sides  $QR \parallel TS$  and  $QT \parallel RS$ . Also given:  $\hat{Q} = y$  and  $\hat{S} = 63^\circ$ ;  $Q\hat{T}R = 38^\circ$  and  $R\hat{T}S = x$ . Complete the proof below to prove that  $QRST$  is a parallelogram.



Steps	Reasons
$Q\hat{T}R = T\hat{R}S$	alt $\angle$ s; $QT \parallel RS$
$S\hat{T}R = Q\hat{R}T$	alt $\angle$ s; $QR \parallel TS$
?	?
$\therefore \triangle QRT \cong \triangle STR$	(AAS)
?	congruent triangles
$\hat{Q} = \hat{S}$	congruent triangles
$\therefore QRST$ is a parallelogram	?

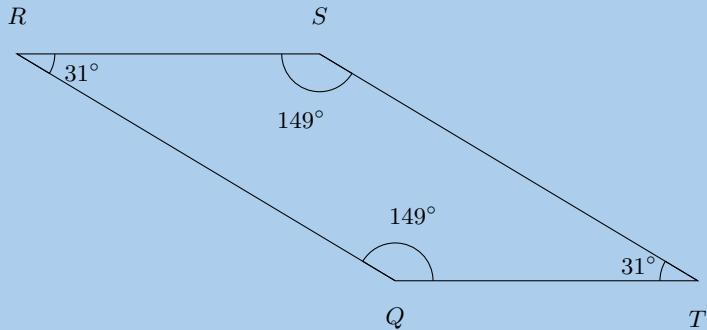
- b) Calculate the value of  $y$ .  
 c) Calculate the value of  $x$ .

8. Study the quadrilateral  $QRST$  with opposite angles  $\hat{Q} = \hat{S} = 117^\circ$  and angles  $\hat{R} = \hat{T} = 63^\circ$  carefully. Fill in the correct reasons or steps to prove that the quadrilateral  $QRST$  is a parallelogram.



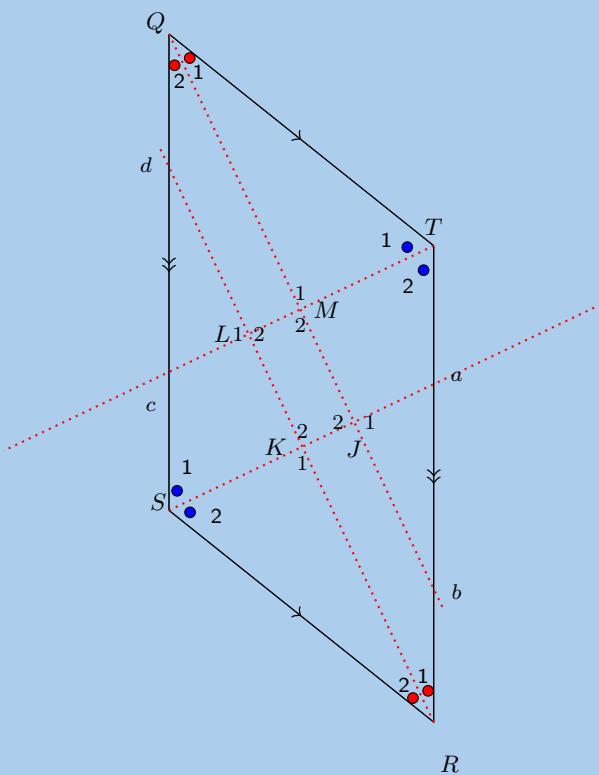
Steps	Reasons
?	given both $\angle s = 117^\circ$
$Q\hat{R}S = Q\hat{T}S$	given both $\angle s = 63^\circ$
?	sum of $\angle s$ in quad
$R\hat{Q}T + Q\hat{T}S = 180^\circ$	$117^\circ + 63^\circ = 180^\circ$
$\therefore QR \parallel TS$	co-int $\angle s$ ; $QR \parallel TS$
$\therefore RS \parallel QT$	?
$\therefore QRST$ is a parallelogram	?

9. Study the quadrilateral  $QRST$  with  $\hat{Q} = \hat{S} = 149^\circ$  and  $\hat{R} = \hat{T} = 31^\circ$  carefully. Fill in the correct reasons or steps to prove that the quadrilateral  $QRST$  is a parallelogram.

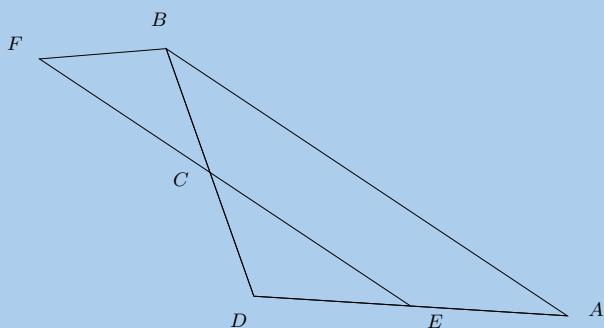


Steps	Reasons
$R\hat{Q}T = R\hat{S}T$	given both $\angle s = 149^\circ$
$Q\hat{R}S = Q\hat{T}S$	?
$\hat{Q} + \hat{R} + \hat{S} + \hat{T} = 360^\circ$	sum of $\angle s$ in quad
$R\hat{Q}T + Q\hat{T}S = 180^\circ$	?
?	co-int $\angle s$ ; $QR \parallel TS$
?	co-int $\angle s$ ; $RS \parallel QT$
$\therefore QRST$ is a parallelogram	opp. sides are parallel

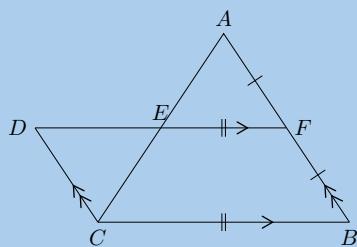
10. In parallelogram  $QTRS$ , the bisectors of the angles have been constructed, indicated with the red lines below. You are also given  $QT = SR$ ,  $TR = QS$ ,  $QT \parallel SR$ ,  $TR \parallel QS$ ,  $\hat{Q} = \hat{R}$  and  $\hat{T} = \hat{S}$ . Prove that the quadrilateral  $JKLM$  is a parallelogram.  
Note the diagram is drawn to scale.



11. Study the diagram below; it is not necessarily drawn to scale. Two triangles in the figure are congruent:  $\triangle CDE \cong \triangle CBF$ . Additionally,  $EA = ED$ . You need to prove that  $ABFE$  is a parallelogram.

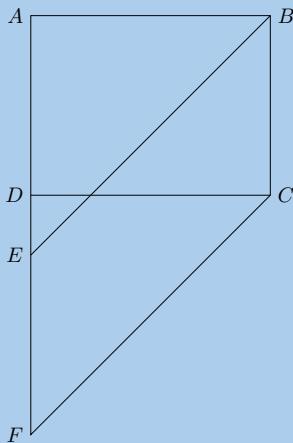


12. Given the following diagram:

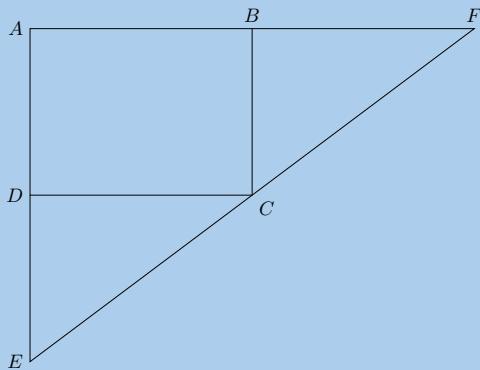


- a) Show that  $BCDF$  is a parallelogram.
  - b) Show that  $ADCF$  is a parallelogram.
  - c) Prove that  $AE = EC$ .

13.  $ABCD$  is a parallelogram.  $BEFC$  is a parallelogram.  $ADEF$  is a straight line. Prove that  $AE = DF$ .



14. In the figure below  $AB = BF$ ,  $AD = DE$ .  $ABCD$  is a parallelogram. Prove  $EF$  is a straight line.



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- |         |         |          |          |          |          |          |
|---------|---------|----------|----------|----------|----------|----------|
| 1. 2GR4 | 2. 2GR5 | 3. 2GR6  | 4. 2GR7  | 5. 2GR8  | 6. 2GR9  | 7. 2GRB  |
| 8. 2GRC | 9. 2GRD | 10. 2GRF | 11. 2GRG | 12. 2GRH | 13. 2GRJ | 14. 2GRK |



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# CHAPTER



## Measurements

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# 13 Measurements

Knowing how to calculate the surface area and volume of an object can be useful in many contexts, particularly when we need to know how much a task is going to cost or how much material is needed to create an object. Some examples of this are calculating the surface area of a container, to help us work out the cost of the material, or calculating the volume of a dam, so we know how much water the dam can hold.

This chapter examines the surface areas and volumes of three dimensional objects, otherwise known as solids.

In order to work with these objects, you need to know how to calculate the surface area and perimeter of two dimensional shapes.



Figure 13.1: A tennis court. The position of each of the lines is carefully calculated to ensure that the area of the rectangles is the same anywhere in the world.

## VISIT:

To revise how to calculate the area and perimeter of squares and rectangles you can watch the video below.

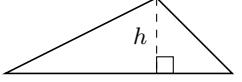
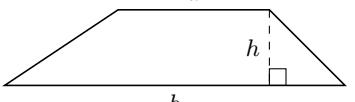
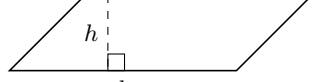
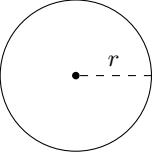
► See video: [2GRM](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

## 13.1 Area of a polygon

EMA7K

### DEFINITION: Area

Area is the two dimensional space inside the boundary of a flat object. It is measured in square units.

Name	Shape	Formula
Square		area (A) = $s^2$
Rectangle		area (A) = $b \times h$
Triangle		area (A) = $\frac{1}{2}b \times h$
Trapezium		area (A) = $\frac{1}{2}(a + b) \times h$
Parallelogram		area (A) = $b \times h$
Circle		area (A) = $\pi r^2$ (circumference = $2\pi r$ )

### DID YOU KNOW?

The acre and the hectare are two common measurements used for the area of land. One hectare is about 0,01 square kilometres and one acre is about 0,004 square kilometres.

### VISIT:

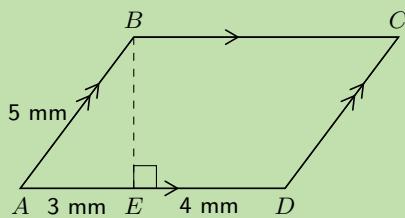
The video below shows some examples of calculations involving the area of a circle.

► See video: [2GRN](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 1: Finding the area of a polygon

#### QUESTION

Find the area of the following parallelogram:



#### SOLUTION

##### Step 1: Find the height $BE$

$$\begin{aligned} AB^2 &= BE^2 + AE^2 && \text{Pythagoras} \\ \therefore BE^2 &= AB^2 - AE^2 \\ &= 5^2 - 3^2 \\ &= 16 \\ \therefore BE &= 4 \text{ mm} \end{aligned}$$

##### Step 2: Find the area using the formula for a parallelogram

$$\begin{aligned} \text{area} &= b \times h \\ &= AD \times BE \\ &= 7 \times 4 \\ &= 28 \text{ mm}^2 \end{aligned}$$

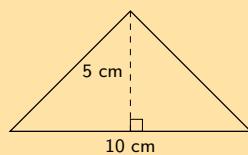
### VISIT:

The following Phet simulation allows you to build different shapes and calculate the area and perimeter for the shapes: [Phet: area builder](#).

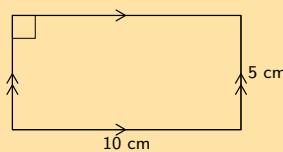
**Exercise 13 – 1:**

1. Find the area of each of the polygons below:

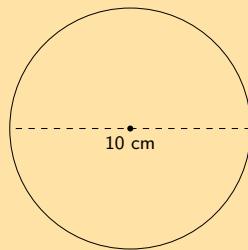
a)



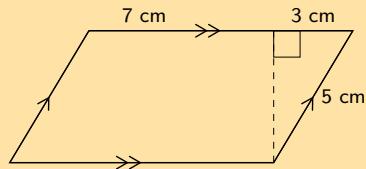
b)



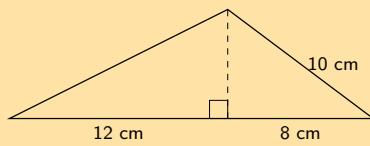
c)



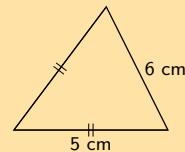
d)



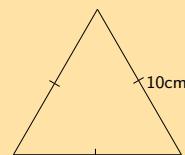
e)



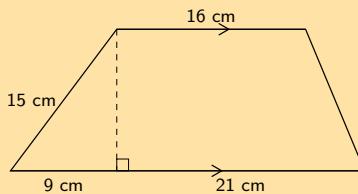
f)



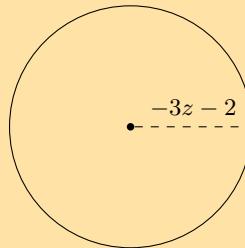
g)



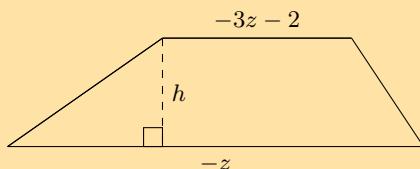
h)



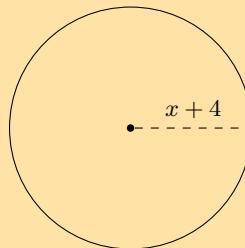
2. a) Find an expression for the area of this figure in terms of  $z$  and  $\pi$ . The circle has a radius of  $-3z - 2$ . Write your answer in expanded form (not factorised).



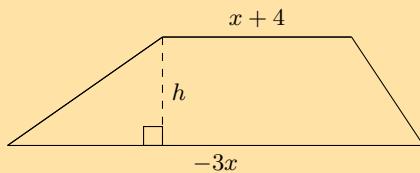
- b) Find an expression for the area of this figure in terms of  $z$  and  $h$ . The height of the figure is  $h$ , and two sides are labelled as  $-3z - 2$  and  $-z$ . Write your answer in expanded form (not factorised).



3. a) Find an expression for the area of this figure in terms of  $x$  and  $\pi$ . The circle has a radius of  $x + 4$ . Write your answer in expanded form (not factorised).



- b) Find an expression for the area of this figure in terms of  $x$  and  $h$ . The height of the figure is  $h$ , and two sides are labelled as  $x + 4$  and  $-3x$ . Write your answer in expanded form (not factorised).



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- 1a. [2GRP](#) 1b. [2GRQ](#) 1c. [2GRR](#) 1d. [2GRS](#) 1e. [2GRT](#) 1f. [2GRV](#)  
1g. [2GRW](#) 1h. [2GRX](#) 2. [2GRY](#) 3. [2GRZ](#)



[www.everythingmaths.co.za](http://www.everythingmaths.co.za)

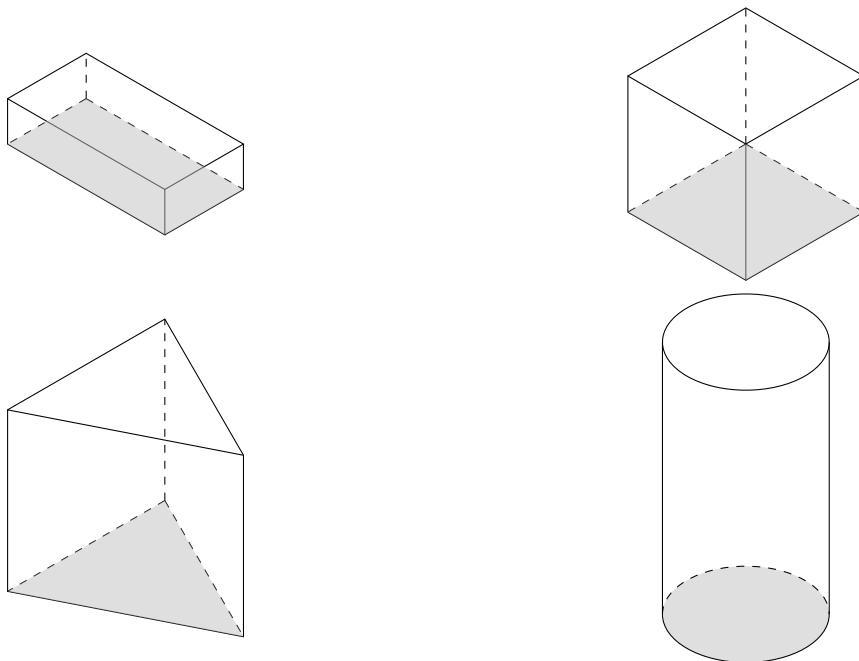


[m.everythingmaths.co.za](http://m.everythingmaths.co.za)

**DEFINITION:** *Right prism*

A right prism is a geometric solid that has a polygon as its base and vertical faces perpendicular to the base. The base and top surface are the same shape and size. It is called a “right” prism because the angles between the base and faces are right angles.

A triangular prism has a triangle as its base, a rectangular prism has a rectangle as its base, and a cube is a rectangular prism with all its sides of equal length. A cylinder has a circle as its base. Examples of right prisms and a cylinder are given below: a rectangular prism, a cube and a triangular prism.

**DEFINITION:** *Surface area*

Surface area is the total area of the exposed or outer surfaces of a prism.

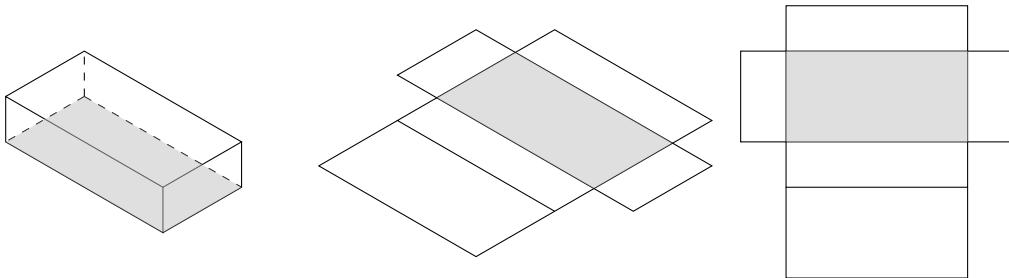
This is easier to understand if we imagine the prism to be a cardboard box that we can unfold. A solid that is unfolded like this is called a net. When a prism is unfolded into a net, we can clearly see each of its faces. In order to calculate the surface area of the prism, we can then simply calculate the area of each face, and add them all together.

For example, when a triangular prism is unfolded into a net, we can see that it has two faces that are triangles and three faces that are rectangles. To calculate the surface area of the prism, we find the area of each triangle and each rectangle, and add them together.

In the case of a cylinder the top and bottom faces are circles and the curved surface flattens into a rectangle with a length that is equal to the circumference of the circular base. To calculate the surface area we therefore find the area of the two circles and the rectangle and add them together.

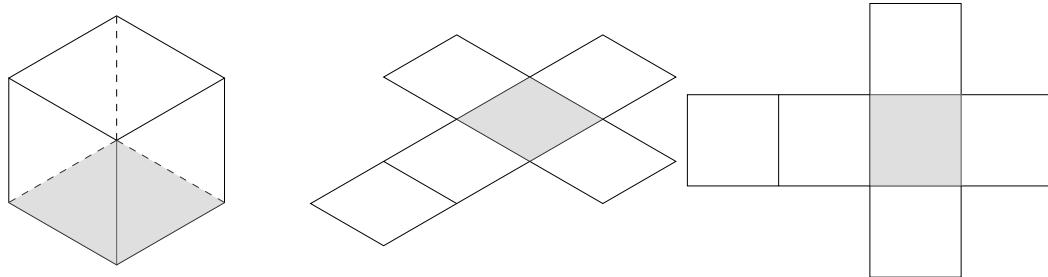
Below are examples of right prisms and a cylinder that have been unfolded into nets:

### Rectangular prism



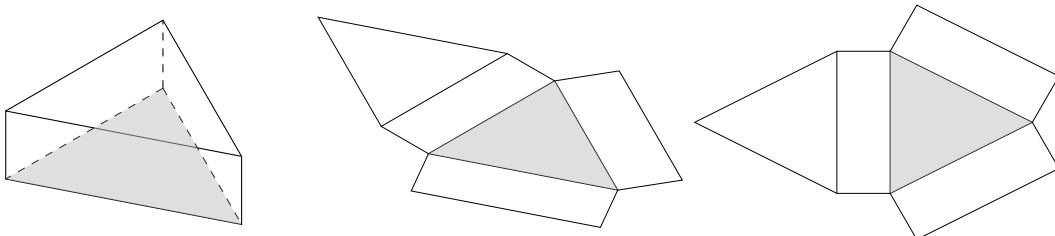
A rectangular prism unfolded into a net is made up of six rectangles.

### Cube



A cube unfolded into a net is made up of six identical squares.

### Triangular prism



A triangular prism unfolded into a net is made up of two triangles and three rectangles. The sum of the lengths of the rectangles is equal to the perimeter of the triangles.

### Cylinder



A cylinder unfolded into a net is made up of two identical circles and a rectangle with a length equal to the circumference of the circles.

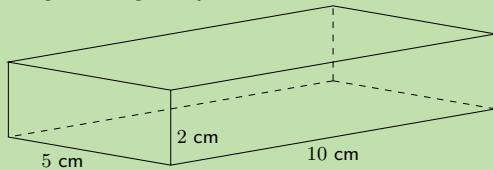
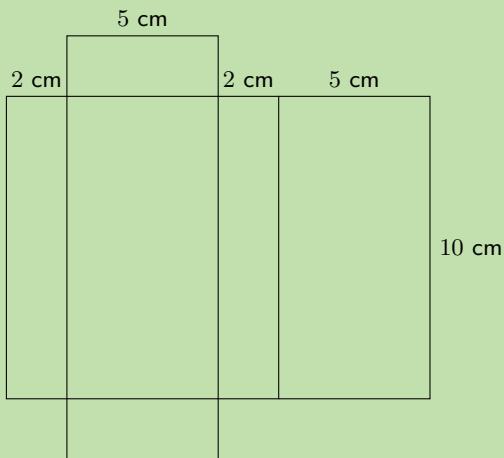
**VISIT:**

This video explains how we can unfold solids into nets.

► See video: [2GS2](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

**Worked example 2: Finding the surface area of a rectangular prism****QUESTION**

Find the surface area of the following rectangular prism:

**SOLUTION****Step 1: Sketch and label the net of the prism****Step 2: Find the areas of the different shapes in the net**

$$\begin{aligned}\text{large rectangle} &= \text{perimeter of small rectangle} \times \text{length} \\ &= (2 + 5 + 2 + 5) \times 10 \\ &= 14 \times 10 \\ &= 140 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}2 \times \text{small rectangle} &= 2(5 \times 2) \\ &= 2(10) \\ &= 20 \text{ cm}^2\end{aligned}$$

**Step 3: Find the sum of the areas of the faces**

$$\text{large rectangle} + 2 \times \text{small rectangle} = 140 + 20 = 160$$

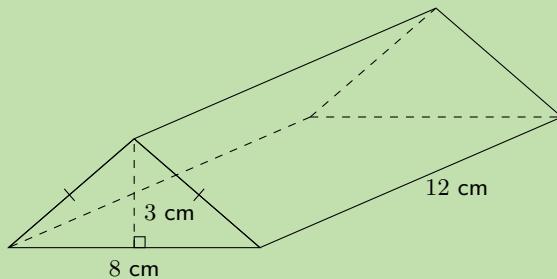
**Step 4: Write the final answer**

The surface area of the rectangular prism is 160 cm<sup>2</sup>.

### Worked example 3: Finding the surface area of a triangular prism

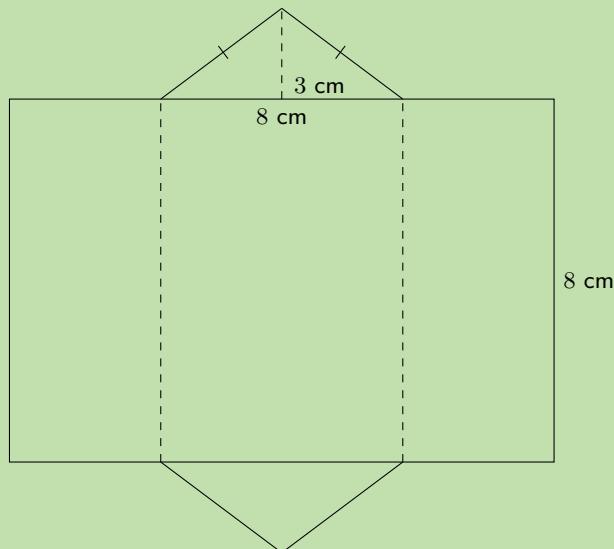
#### QUESTION

Find the surface area of the following triangular prism:



#### SOLUTION

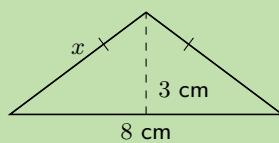
**Step 1: Sketch and label the net of the prism**



**Step 2: Find the area of the different shapes in the net**

To find the area of the rectangle, we need to calculate its length, which is equal to the perimeter of the triangles.

To find the perimeter of the triangle, we have to first find the length of its sides using the theorem of Pythagoras:



$$\begin{aligned}
 x^2 &= 3^2 + \left(\frac{8}{2}\right)^2 \\
 x^2 &= 3^2 + 4^2 \\
 &= 25 \\
 \therefore x &= 5 \text{ cm} \\
 \therefore \text{perimeter of triangle} &= 5 + 5 + 8 \\
 &= 18 \text{ cm}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{area of large rectangle} &= \text{perimeter of triangle} \times \text{length} \\
 &= 18 \times 12 \\
 &= 216 \text{ cm}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{area of triangle} &= \frac{1}{2} b \times h \\
 &= \frac{1}{2} \times 8 \times 3 \\
 &= 12 \text{ cm}^2
 \end{aligned}$$

### Step 3: Find the sum of the areas of the faces

$$\begin{aligned}
 \text{surface area} &= \text{area large rectangle} + (2 \times \text{area of triangle}) \\
 &= 216 + 2(12) \\
 &= 240 \text{ cm}^2
 \end{aligned}$$

### Step 4: Write the final answer

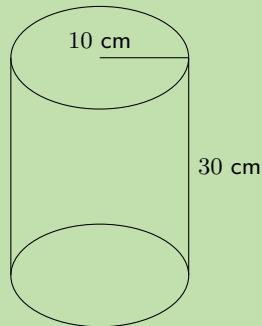
The surface area of the triangular prism is  $240 \text{ cm}^2$ .

### Worked example 4: Finding the surface area of a cylinder

#### **QUESTION**

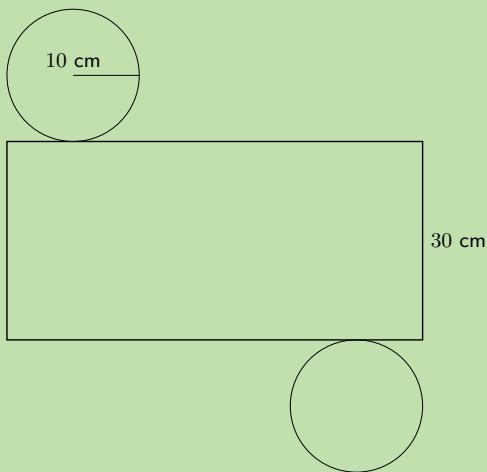
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Find the surface area of the following cylinder (correct to 1 decimal place):



## SOLUTION

### Step 1: Sketch and label the net of the cylinder



### Step 2: Find the area of the different shapes in the net

$$\begin{aligned}\text{area of large rectangle} &= \text{circumference of circle} \times \text{length} \\ &= 2\pi r \times l \\ &= 2\pi(10) \times 30 \\ &= 1884.9555\dots \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{area of circle} &= \pi r^2 \\ &= \pi(10)^2 \\ &= 314.1592\dots \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{surface area} &= \text{area large rectangle} + (2 \times \text{area of circle}) \\ &= 1884.9555\dots + 2(314.1592\dots) \\ &= 2513.3 \text{ cm}^2\end{aligned}$$

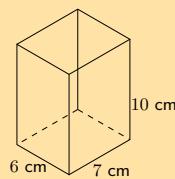
### Step 3: Write the final answer

The surface area of the cylinder is 2513,3 cm<sup>2</sup>.

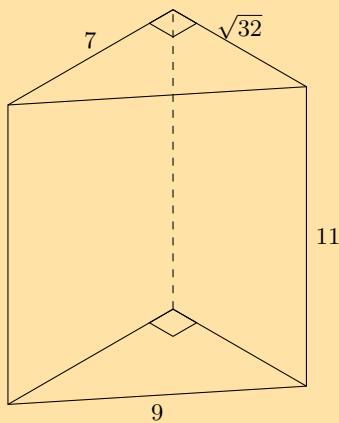
## Exercise 13 – 2:

- Calculate the surface area of the following prisms:

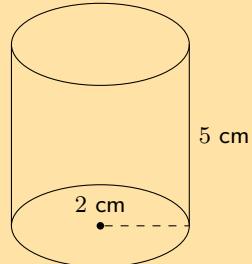
a)



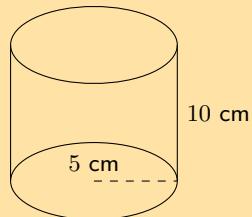
b)



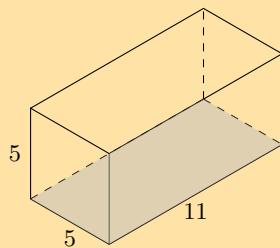
c)



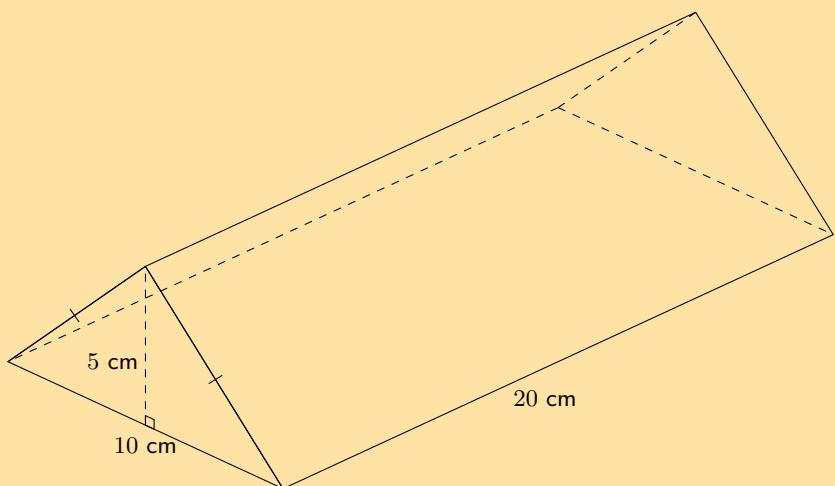
d)



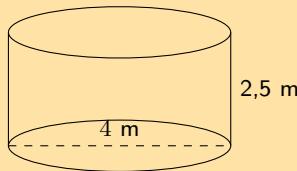
e)



f)



2. If a litre of paint covers an area of  $2 \text{ m}^2$ , how much paint does a painter need to cover:
- a rectangular swimming pool with dimensions  $4 \text{ m} \times 3 \text{ m} \times 2,5 \text{ m}$  (the inside walls and floor only);
  - the inside walls and floor of a circular reservoir with diameter  $4 \text{ m}$  and height  $2,5 \text{ m}$ .



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1a. [2GS3](#) 1b. [2GS4](#) 1c. [2GS5](#) 1d. [2GS6](#) 1e. [2GS7](#) 1f. [2GS8](#) 2. [2GS9](#)



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## Volume of prisms and cylinders

EMA7P

### DEFINITION: Volume

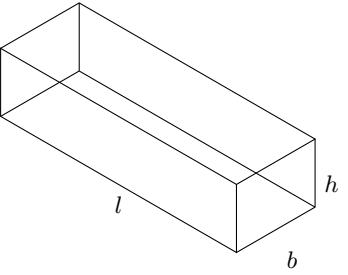
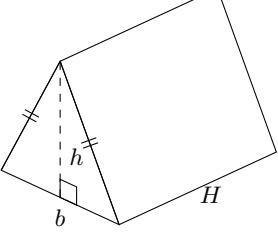
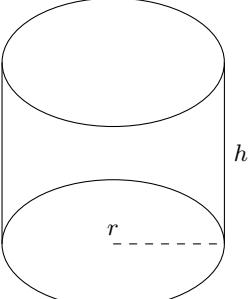
Volume is the three dimensional space occupied by an object, or the contents of an object. It is measured in cubic units.

The volume of right prisms and cylinders is simply calculated by multiplying the area of the base of the solid by the height of the solid.

### VISIT:

The video below shows several examples of calculating the volume of a right prism.

► See video: [2GSB](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

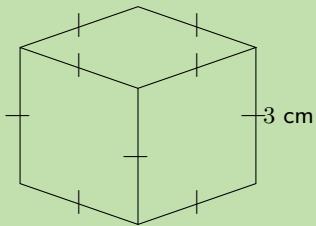
<b>Rectangular prism</b>		Volume = area of base × height = area of rectangle × height = $l \times b \times h$
<b>Triangular prism</b>		Volume = area of base × height = area of triangle × height = $(\frac{1}{2}b \times h) \times H$
<b>Cylinder</b>		Volume = area of base × height = area of circle × height = $\pi r^2 \times h$

### Worked example 5: Finding the volume of a cube

#### QUESTION

---

Find the volume of the following cube:



#### SOLUTION

---

##### Step 1: Find the area of the base

$$\begin{aligned}\text{area of square} &= s^2 \\ &= 3^2 \\ &= 9 \text{ cm}^2\end{aligned}$$

**Step 2: Multiply the area of the base by the height of the solid to find the volume**

$$\begin{aligned}\text{volume} &= \text{area of base} \times \text{height} \\ &= 9 \times 3 \\ &= 27 \text{ cm}^3\end{aligned}$$

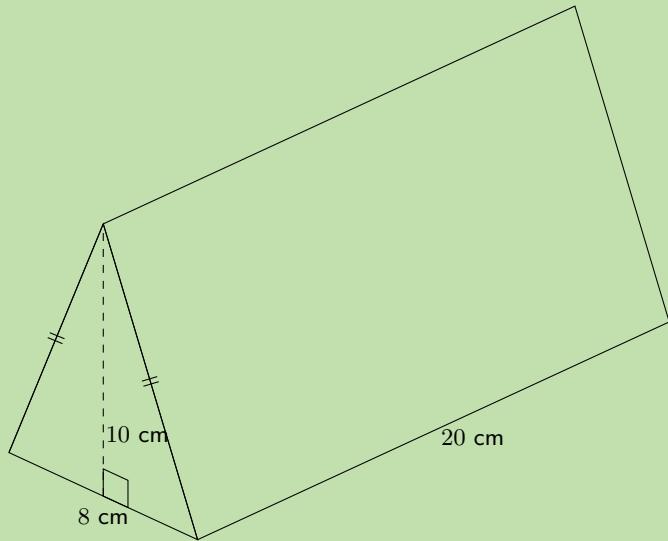
**Step 3: Write the final answer**

The volume of the cube is 27 cm<sup>3</sup>.

#### Worked example 6: Finding the volume of a triangular prism

##### **QUESTION**

Find the volume of the triangular prism:



##### **SOLUTION**

**Step 1: Find the area of the base**

$$\begin{aligned}\text{area of triangle} &= \frac{1}{2} b \times h \\ &= \left(\frac{1}{2} \times 8\right) \times 10 \\ &= 40 \text{ cm}^2\end{aligned}$$

**Step 2: Multiply the area of the base by the height of the solid to find the volume**

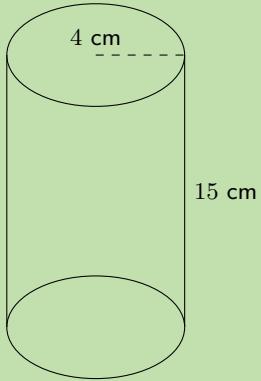
$$\begin{aligned}\text{volume} &= \text{area of base} \times \text{height} \\ &= \frac{1}{2}b \times h \times H \\ &= 40 \times 20 \\ &= 800 \text{ cm}^3\end{aligned}$$

**Step 3: Write the final answer**

The volume of the triangular prism is 800 cm<sup>3</sup>.

**Worked example 7: Finding the volume of a cylinder****QUESTION**

Find the volume of the following cylinder (correct to 1 decimal place):

**SOLUTION****Step 1: Find the area of the base**

$$\begin{aligned}\text{area of circle} &= \pi r^2 \\ &= \pi(4)^2 \\ &= 16\pi \text{ cm}^2\end{aligned}$$

**Step 2: Multiply the area of the base by the height of the solid to find the volume**

$$\begin{aligned}\text{volume} &= \text{area of base} \times \text{height} \\ &= \pi r^2 \times h \\ &= 16\pi \times 15 \\ &\approx 754,0 \text{ cm}^3\end{aligned}$$

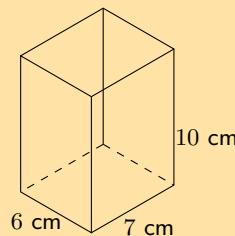
**Step 3: Write the final answer**

The volume of the cylinder is 754,0 cm<sup>3</sup>.

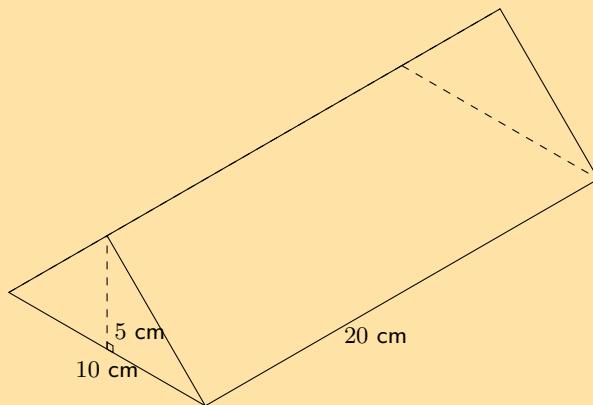
**Exercise 13 – 3:**

1. Calculate the volumes of the following prisms (correct to 1 decimal place):

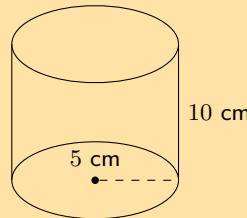
a)



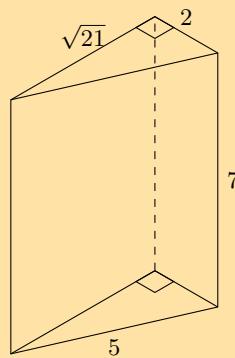
b)



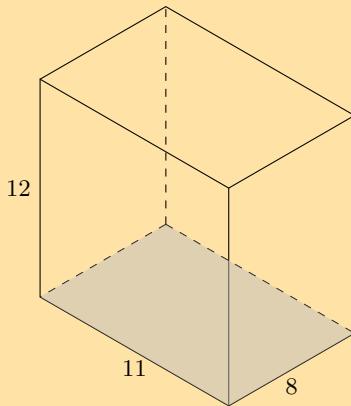
c)



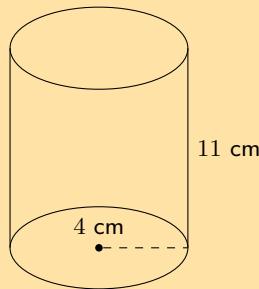
2. The figure here is a triangular prism. The height of the prism is 7 units; the triangles, which both contain right angles, have sides which are 2,  $\sqrt{21}$  and 5 units long. Calculate the volume of the figure. Round to two decimal places if necessary.



3. The figure here is a rectangular prism. The height of the prism is 12 units; the other dimensions of the prism are 11 and 8 units. Find the volume of the figure.



4. The picture below shows a cylinder. The height of the cylinder is 11 units; the radius of the cylinder is  $r = 4$  units. Determine the volume of the figure. Round your answer to two decimal places.



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- 1a. [2GSC](#) 1b. [2GSD](#) 1c. [2GSF](#) 2. [2GSG](#) 3. [2GSH](#) 4. [2GSJ](#)



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[m.everythingmaths.co.za](http://m.everythingmaths.co.za)

### 13.3 Right pyramids, right cones and spheres

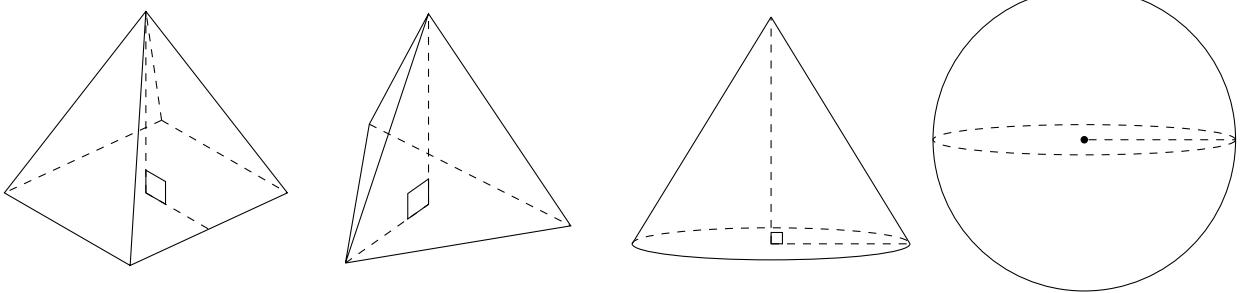
EMA7Q

#### DEFINITION: Pyramid

A pyramid is a geometric solid that has a polygon as its base and faces that converge at a point called the apex. In other words the faces are **not** perpendicular to the base.

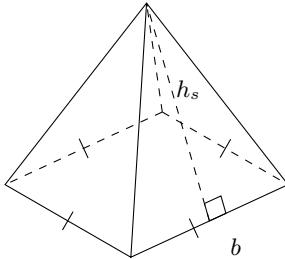
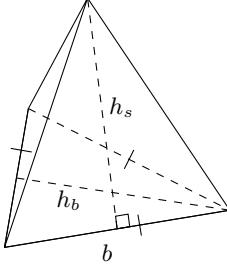
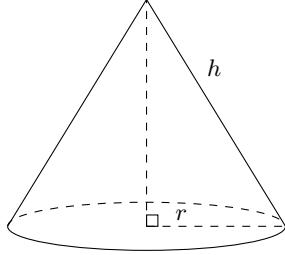
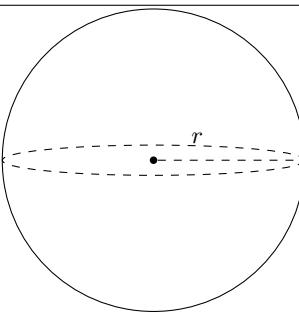
The triangular pyramid and square pyramid take their names from the shape of their base. We call a pyramid a "right pyramid" if the line between the apex and the centre of the base is perpendicular to the base. Cones are similar to pyramids except that their bases are circles instead of polygons. Spheres are solids that are perfectly round and look the same from any direction.

Examples of a square pyramid, a triangular pyramid, a cone and a sphere:



## Surface area of pyramids, cones and spheres

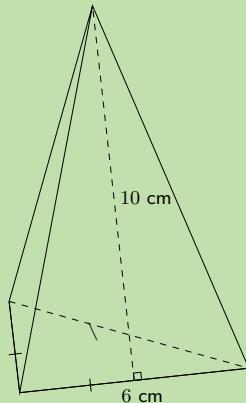
EMA7R

<b>Square pyramid</b>		$\begin{aligned}\text{Surface area} &= \text{area of base} + \\ &\quad \text{area of triangular sides} \\ &= b^2 + 4\left(\frac{1}{2}bh_s\right) \\ &= b(b + 2h_s)\end{aligned}$
<b>Triangular pyramid</b>		$\begin{aligned}\text{Surface area} &= \text{area of base} + \\ &\quad \text{area of triangular sides} \\ &= \left(\frac{1}{2}b \times h_b\right) + 3\left(\frac{1}{2}b \times h_s\right) \\ &= \frac{1}{2}b(h_b + 3h_s)\end{aligned}$
<b>Right cone</b>		$\begin{aligned}\text{Surface area} &= \text{area of base} + \\ &\quad \text{area of walls} \\ &= \pi r^2 + \frac{1}{2} \times 2\pi r h \\ &= \pi r(r + h)\end{aligned}$
<b>Sphere</b>		$\text{Surface area} = 4\pi r^2$

## Worked example 8: Finding the surface area of a triangular pyramid

### QUESTION

Find the surface area of the following triangular pyramid (correct to one decimal place):

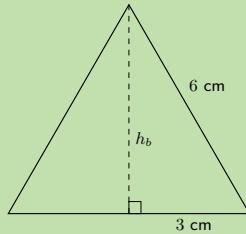


### SOLUTION

#### Step 1: Find the area of the base

$$\text{area of base triangle} = \frac{1}{2}bh_b$$

To find the height of the base triangle ( $h_b$ ) we use the theorem of Pythagoras:



$$\begin{aligned} 6^2 &= 3^2 + h_b^2 \\ \therefore h_b &= \sqrt{6^2 - 3^2} \\ &= 3\sqrt{3} \\ \therefore \text{area of base triangle} &= \frac{1}{2} \times 6 \times 3\sqrt{3} \\ &= 9\sqrt{3} \text{ cm}^2 \end{aligned}$$

#### Step 2: Find the area of the sides

$$\begin{aligned} \text{area of sides} &= 3 \left( \frac{1}{2} \times b \times h_s \right) \\ &= 3 \left( \frac{1}{2} \times 6 \times 10 \right) \\ &= 90 \text{ cm}^2 \end{aligned}$$

**Step 3: Find the sum of the areas**

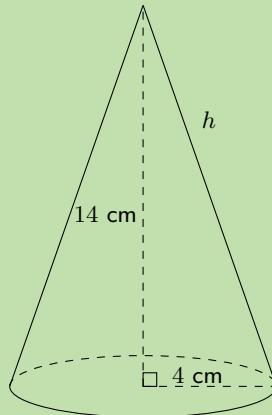
$$9\sqrt{3} + 90 = 105,6 \text{ cm}^2$$

**Step 4: Write the final answer**

The surface area of the triangular pyramid is  $105,6 \text{ cm}^2$ .

**Worked example 9: Finding the surface area of a cone****QUESTION**

Find the surface area of the following cone (correct to 1 decimal place):

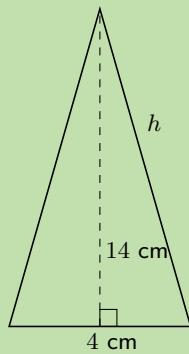
**SOLUTION****Step 1: Find the area of the base**

$$\begin{aligned}\text{area of base circle} &= \pi r^2 \\ &= \pi \times 4^2 \\ &= 16\pi\end{aligned}$$

**Step 2: Find the area of the walls**

$$\text{area of sides} = \pi r h$$

To find the slant height,  $h$ , we use the theorem of Pythagoras:



$$\begin{aligned} h^2 &= 4^2 + 14^2 \\ \therefore h &= \sqrt{4^2 + 14^2} \\ &= 2\sqrt{53} \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{area of walls} &= \frac{1}{2} 2\pi r h \\ &= \pi (4) (2\sqrt{53}) \\ &= 8\pi\sqrt{53} \text{ cm}^2 \end{aligned}$$

### Step 3: Find the sum of the areas

$$\begin{aligned} \text{total surface area} &= 16\pi + 8\pi\sqrt{53} \\ &= 233,2 \text{ cm}^2 \end{aligned}$$

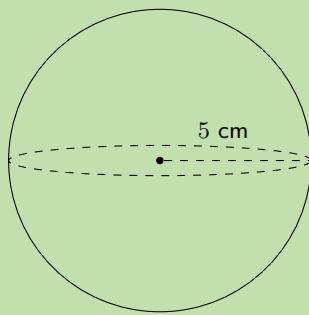
### Step 4: Write the final answer

The surface area of the cone is 233,2 cm<sup>2</sup>.

### Worked example 10: Finding the surface area of a sphere

#### **QUESTION**

Find the surface area of the following sphere (correct to 1 decimal place):



## SOLUTION

$$\begin{aligned}\text{surface area of sphere} &= 4\pi r^2 \\ &= 4\pi(5)^2 \\ &= 100\pi \\ &= 314,2 \text{ cm}^2\end{aligned}$$

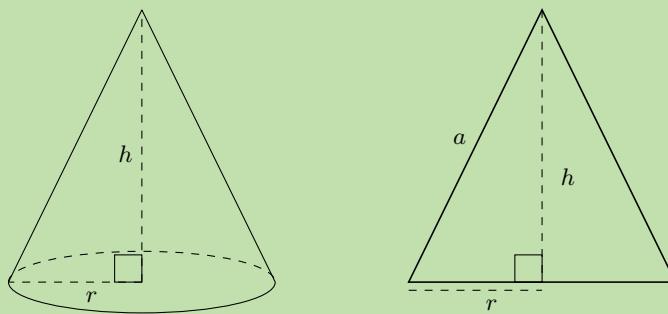
## Worked example 11: Examining the surface area of a cone

### QUESTION

If a cone has a height of  $h$  and a base of radius  $r$ , show that the surface area is:  $\pi r^2 + \pi r \sqrt{r^2 + h^2}$

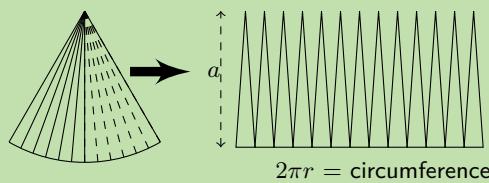
### SOLUTION

#### Step 1: Sketch and label the cone



#### Step 2: Identify the faces that make up the cone

The cone has two faces: the base and the walls. The base is a circle of radius  $r$  and the walls can be opened out to a sector of a circle:



This curved surface can be cut into many thin triangles with height close to  $a$  (where  $a$  is the slant height). The area of these triangles or sectors can be summed as follows:

$$\begin{aligned}\text{Area of sector} &= \frac{1}{2} \times \text{base} \times \text{height} \text{ (of a small triangle)} \\ &= \frac{1}{2} \times 2\pi r \times a \\ &= \pi r a\end{aligned}$$

**Step 3: Calculate  $a$** 

$a$  can be calculated using the theorem of Pythagoras:

$$a = \sqrt{r^2 + h^2}$$

**Step 4: Calculate the area of the circular base ( $A_b$ )**

$$A_b = \pi r^2$$

**Step 5: Calculate the area of the curved walls ( $A_w$ )**

$$\begin{aligned} A_w &= \pi r a \\ &= \pi r \sqrt{r^2 + h^2} \end{aligned}$$

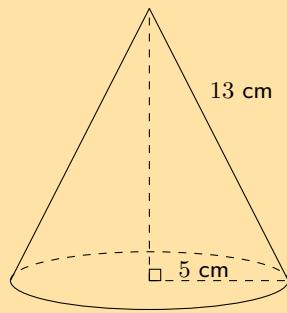
**Step 6: Find the sum of the areas  $A$** 

$$\begin{aligned} A &= A_b + A_w \\ &= \pi r^2 + \pi r \sqrt{r^2 + h^2} \\ &= \pi r \left( r + \sqrt{r^2 + h^2} \right) \end{aligned}$$

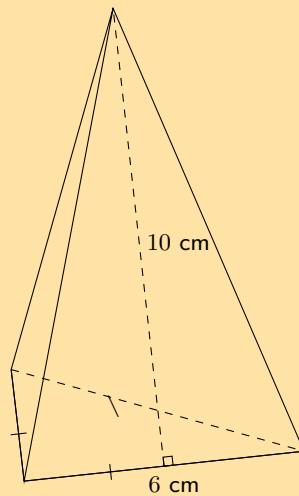
**Exercise 13 – 4:**

1. Find the total surface area of the following objects (correct to 1 decimal place if necessary):

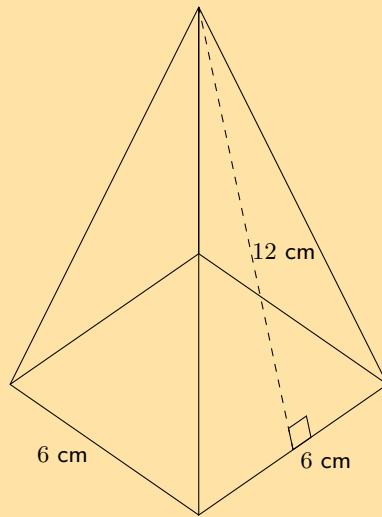
a)



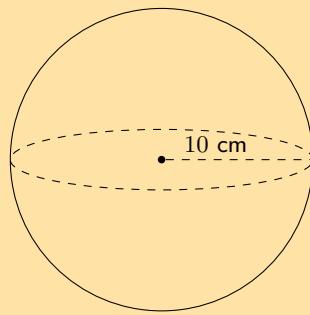
b)



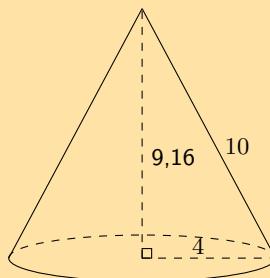
c)



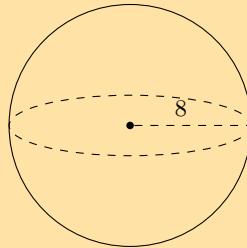
d)



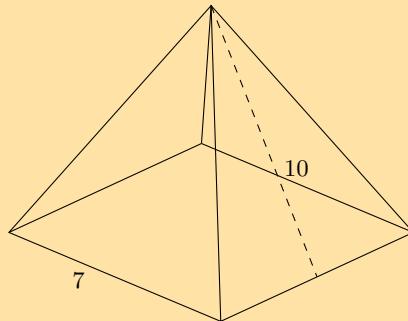
2. The figure here is a cone. The vertical height of the cone is  $H = 9.16$  units and the slant height of the cone is  $h = 10$  units; the radius of the cone is shown,  $r = 4$  units. Calculate the surface area of the figure. Round your answer to two decimal places.



3. The figure here is a sphere. The radius of the sphere is  $r = 8$  units. Calculate the surface area of the figure. Round your answer to two decimal places.



4. The figure here shows a pyramid with a square base. The sides of the base are each 7 units long. The vertical height of the pyramid is 9,36 units, and the slant height of the pyramid is 10 units. Determine the surface area of the pyramid.



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

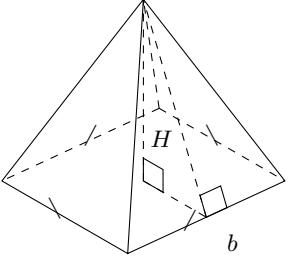
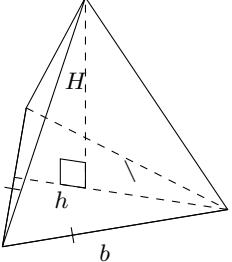
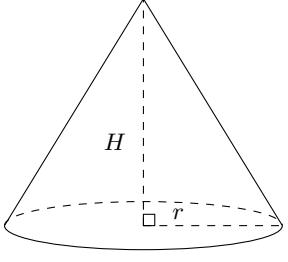
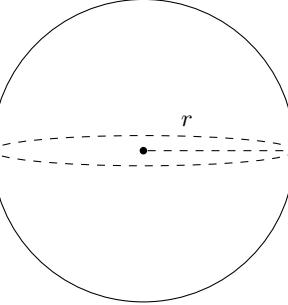
- 1a. [2GSK](#) 1b. [2GSM](#) 1c. [2GSN](#) 1d. [2GSP](#) 2. [2GSQ](#) 3. [2GSR](#) 4. [2GSS](#)



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<b>Square pyramid</b>		Volume $= \frac{1}{3} \times \text{area of base} \times \text{height of pyramid}$ $= \frac{1}{3} \times b^2 \times H$
<b>Triangular pyramid</b>		Volume $= \frac{1}{3} \times \text{area of base} \times \text{height of pyramid}$ $= \frac{1}{3} \times \frac{1}{2}bh \times H$
<b>Right cone</b>		Volume $= \frac{1}{3} \times \text{area of base} \times \text{height of cone}$ $= \frac{1}{3} \times \pi r^2 \times H$
<b>Sphere</b>		Volume $= \frac{4}{3}\pi r^3$

**VISIT:**

This video gives an example of calculating the volume of a sphere.

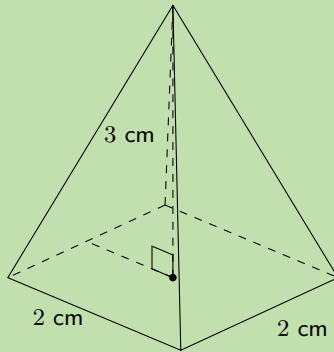
► See video: [2GST](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

**Worked example 12: Finding the volume of a square pyramid****QUESTION**

Find the volume of a square pyramid with a height of 3 cm and a side length of 2 cm.

## SOLUTION

**Step 1: Sketch and label the pyramid**



**Step 2: Select the correct formula and substitute the given values**

We are given  $b = 2$  and  $H = 3$ , therefore:

$$\begin{aligned}V &= \frac{1}{3} \times b^2 \times H \\V &= \frac{1}{3} \times 2^2 \times 3 \\&= \frac{1}{3} \times 12 \\&= 4 \text{ cm}^3\end{aligned}$$

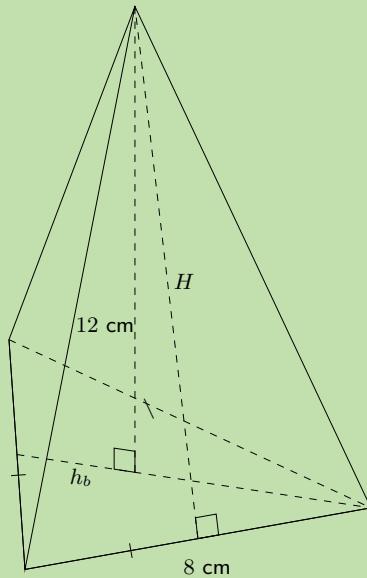
**Step 3: Write the final answer**

The volume of the square pyramid is  $4 \text{ cm}^3$ .

## Worked example 13: Finding the volume of a triangular pyramid

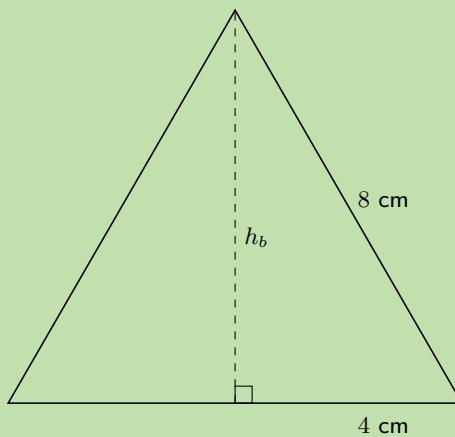
### QUESTION

Find the volume of the following triangular pyramid (correct to 1 decimal place):



## SOLUTION

**Step 1:** Sketch the base triangle and calculate its area



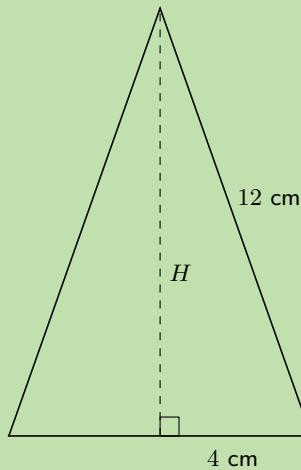
The height of the base triangle ( $h_b$ ) is:

$$\begin{aligned} 8^2 &= 4^2 + h_b^2 \\ \therefore h_b &= \sqrt{8^2 - 4^2} \\ &= 4\sqrt{3} \text{ cm} \end{aligned}$$

The area of the base triangle is:

$$\begin{aligned} \text{area of base triangle} &= \frac{1}{2} b \times h_b \\ &= \frac{1}{2} \times 8 \times 4\sqrt{3} \\ &= 16\sqrt{3} \text{ cm}^2 \end{aligned}$$

**Step 2:** Sketch the side triangle and calculate pyramid height  $H$



$$12^2 = 2\sqrt{3}^2 + H^2$$

$$H^2 = 130$$

$$\therefore H = \sqrt{130} \text{ cm}$$

### Step 3: Calculate the volume of the pyramid

$$\begin{aligned}V &= \frac{1}{3} \times \frac{1}{2} b h_b \times H \\&= \frac{1}{3} \times 16\sqrt{3} \times \sqrt{130} \\&= 105,3 \text{ cm}^3\end{aligned}$$

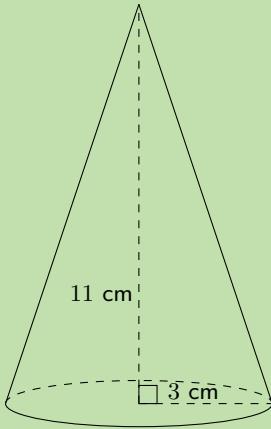
### Step 4: Write the final answer

The volume of the triangular pyramid is 105,3 cm<sup>3</sup>.

### Worked example 14: Finding the volume of a cone

#### QUESTION

Find the volume of the following cone (correct to 1 decimal place):



#### SOLUTION

##### Step 1: Find the area of the base

$$\begin{aligned}\text{area of circle} &= \pi r^2 \\&= \pi \times 3^2 \\&= 9\pi \text{ cm}^2\end{aligned}$$

### Step 2: Calculate the volume

$$\begin{aligned}V &= \frac{1}{3} \times \pi r^2 \times H \\&= \frac{1}{3} \times 9\pi \times 11 \\&= 103,7 \text{ cm}^3\end{aligned}$$

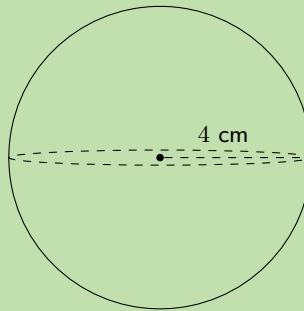
### Step 3: Write the final answer

The volume of the cone is 103,7 cm<sup>3</sup>.

### Worked example 15: Finding the volume of a sphere

#### QUESTION

Find the volume of the following sphere (correct to 1 decimal place):



#### SOLUTION

### Step 1: Use the formula to find the volume

$$\begin{aligned}\text{volume} &= \frac{4}{3} \pi r^3 \\&= \frac{4}{3} \pi (4)^3 \\&= 268,1 \text{ cm}^3\end{aligned}$$

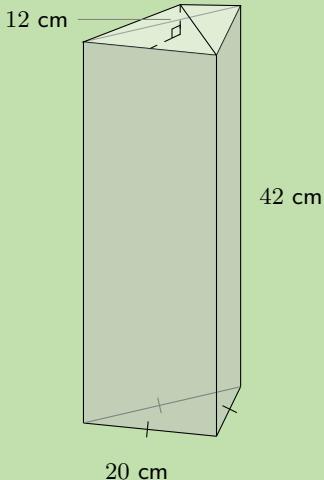
### Step 2: Write the final answer

The volume of the sphere is 268,1 cm<sup>3</sup>.

## Worked example 16: Finding the volume of a complex object

### QUESTION

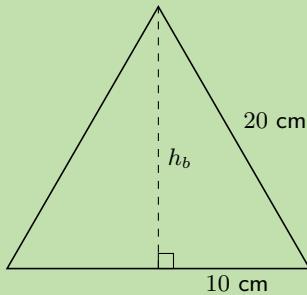
A triangular pyramid is placed on top of a triangular prism, as shown below. The base of the prism is an equilateral triangle of side length 20 cm and the height of the prism is 42 cm. The pyramid has a height of 12 cm. Calculate the total volume of the object.



### SOLUTION

#### Step 1: Calculate the volume of the prism

First find the height of the base triangle using the theorem of Pythagoras:



$$\begin{aligned} 20^2 &= 10^2 + h_b^2 \\ \therefore h_b &= \sqrt{20^2 - 10^2} \\ &= 10\sqrt{3} \text{ cm} \end{aligned}$$

Next find the area of the base triangle:

$$\begin{aligned} \text{area of base triangle} &= \frac{1}{2} \times 20 \times 10\sqrt{3} \\ &= 100\sqrt{3} \text{ cm}^2 \end{aligned}$$

Now we can find the volume of the prism:

$$\begin{aligned}\therefore \text{volume of prism} &= \text{area of base triangle} \times \text{height of prism} \\ &= 100\sqrt{3} \times 42 \\ &= 4200\sqrt{3} \text{ cm}^3\end{aligned}$$

### Step 2: Calculate the volume of the pyramid

The area of the base triangle is equal to the area of the base of the pyramid.

$$\begin{aligned}\therefore \text{volume of pyramid} &= \frac{1}{3} (\text{area of base}) \times H \\ &= \frac{1}{3} \times 100\sqrt{3} \times 12 \\ &= 400\sqrt{3} \text{ cm}^3\end{aligned}$$

### Step 3: Calculate the total volume

$$\begin{aligned}\text{total volume} &= 4200\sqrt{3} + 400\sqrt{3} \\ &= 4600\sqrt{3} \\ &= 7967,4 \text{ cm}^3\end{aligned}$$

Therefore the total volume of the object is 7967,4 cm<sup>3</sup>.

## Worked example 17: Finding the surface area of a complex object

### QUESTION

With the same complex object as in the previous example, you are given the additional information that the slant height  $h_s = 13,3$  cm. Now calculate the total surface area of the object.

### SOLUTION

#### Step 1: Calculate the surface area of each exposed face of the pyramid

$$\begin{aligned}\text{area of one pyramid face} &= \frac{1}{2} b \times h_s \\ &= \frac{1}{2} \times 20 \times 13,3 \\ &= 133 \text{ cm}^2\end{aligned}$$

Because the base triangle is equilateral, each face has the same base, and therefore the same surface area. Therefore the surface area for each face of the pyramid is 133 cm<sup>2</sup>.

#### Step 2: Calculate the surface area of each side of the prism

Each side of the prism is a rectangle with base  $b = 20$  cm and height  $h_p = 42$  cm.

$$\begin{aligned}\text{area of one prism side} &= b \times h_p \\ &= 20 \times 42 \\ &= 840 \text{ cm}^2\end{aligned}$$

Because the base triangle is equilateral, each side of the prism has the same area. Therefore the surface area for each side of the prism is  $840 \text{ cm}^2$ .

### Step 3: Calculate the total surface area of the object

$$\begin{aligned}\text{total surface area} &= \text{area of base of prism} + \text{area of sides of prism} + \text{area of exposed faces of pyramid} \\ &= (100\sqrt{3}) + 3(840) + 3(133) \\ &= 3092,2 \text{ cm}^2\end{aligned}$$

Therefore the total surface area (of the exposed faces) of the object is  $3092,2 \text{ cm}^2$ .

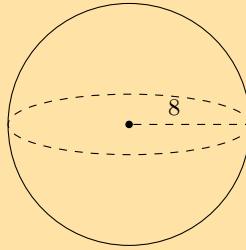
#### VISIT:

This video shows an example of calculating the volume of a complex object.

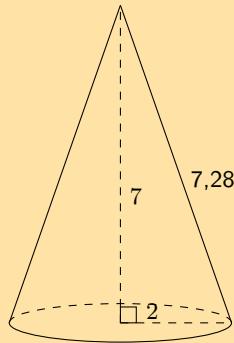
See video: [2GSV](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

#### Exercise 13 – 5:

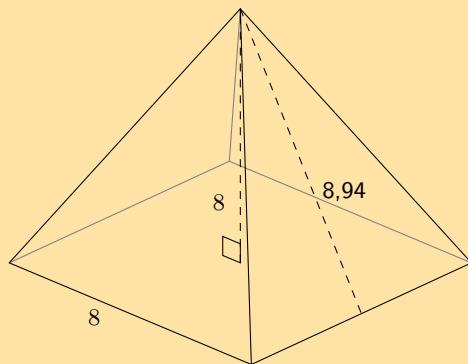
- The figure below shows a sphere. The radius of the sphere is  $r = 8$  units. Determine the volume of the figure. Round your answer to two decimal places.



- The figure here is a cone. The vertical height of the cone is  $H = 7$  units and the slant height is  $h = 7,28$  units; the radius of the cone is shown,  $r = 2$  units. Calculate the volume of the figure. Round your answer to two decimal places.

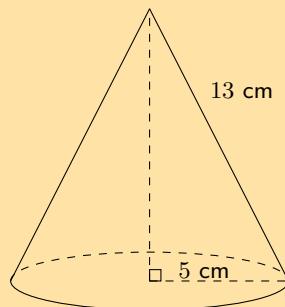


3. The figure here is a pyramid with a square base. The vertical height of the pyramid is  $H = 8$  units and the slant height is  $h = 8.94$  units; each side of the base of the pyramid is  $b = 8$  units. Round your answer to two decimal places.

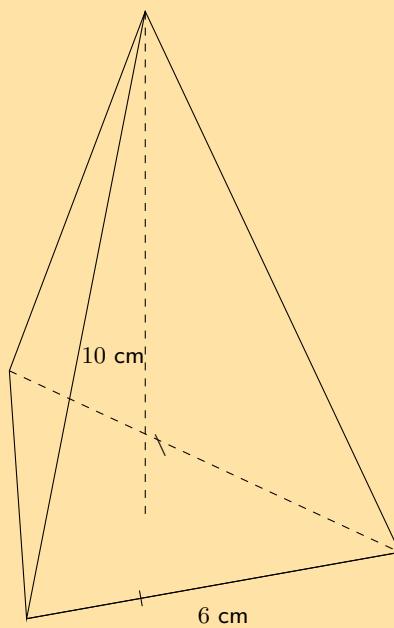


4. Find the volume of the following objects (round off to 1 decimal place if needed):

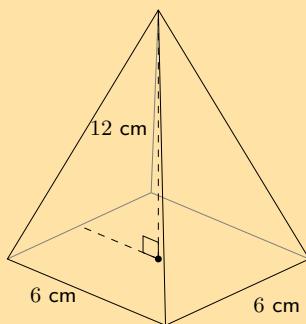
a)



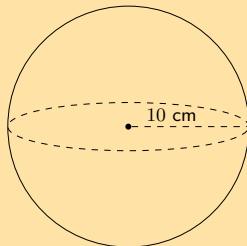
b)



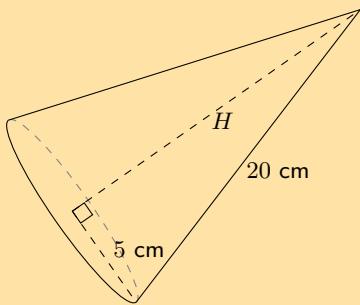
c)



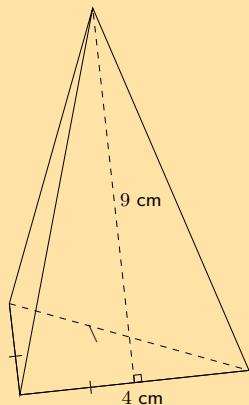
d)



5. Find the surface area and volume of the cone shown here. Round your answers to the nearest integer.

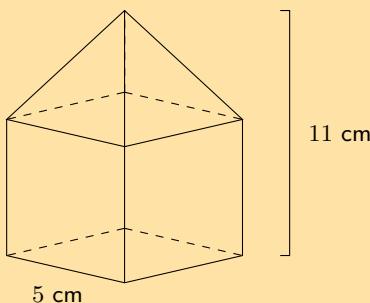


6. Calculate the following properties for the pyramid shown below. Round your answers to two decimal places.



- a) Surface area  
b) Volume

7. The solid below is made up of a cube and a square pyramid. Find its volume and surface area (correct to 1 decimal place):



For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2GSW](#) 2. [2GSX](#) 3. [2GSY](#) 4a. [2GSZ](#) 4b. [2GT2](#) 4c. [2GT3](#)  
4d. [2GT4](#) 5. [2GT5](#) 6. [2GT6](#) 7. [2GT7](#)



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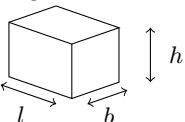
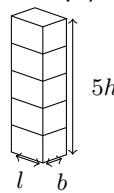
## 13.4 The effect of multiplying a dimension by a factor of $k$

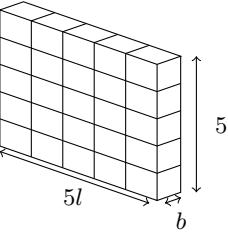
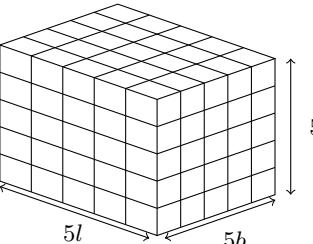
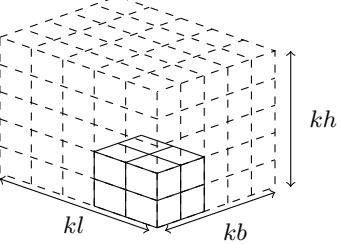
EMA7T

When one or more of the dimensions of a prism or cylinder is multiplied by a constant, the surface area and volume will change. The new surface area and volume can be calculated by using the formulae from the preceding section.

It is possible to see a relationship between the change in dimensions and the resulting change in surface area and volume. These relationships make it simpler to calculate the new volume or surface area of an object when its dimensions are scaled up or down.

Consider a rectangular prism of dimensions  $l$ ,  $b$  and  $h$ . Below we multiply one, two and three of its dimensions by a constant factor of 5 and calculate the new volume and surface area.

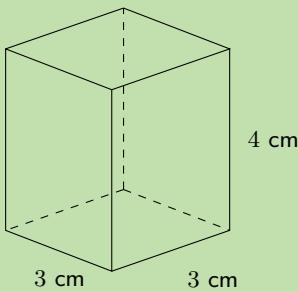
Dimensions	Volume	Surface
Original dimensions 	$\begin{aligned}V &= l \times b \times h \\&= lbh\end{aligned}$	$\begin{aligned}A &= 2[(l \times h) + (l \times b) + (b \times h)] \\&= 2(lh + lb + bh)\end{aligned}$
Multiply one dimension by 5 	$\begin{aligned}V_1 &= l \times b \times 5h \\&= 5(lbh) \\&= 5V\end{aligned}$	$\begin{aligned}A_1 &= 2[(l \times 5h) + (l \times b) + (b \times 5h)] \\&= 2(5lh + lb + 5bh)\end{aligned}$

Dimensions	Volume	Surface
Multiply two dimensions by 5 	$V_2 = 5l \times b \times 5h$ $= 5.5(lbh)$ $= 5^2 \times V$	$A_2 = 2[(5l \times 5h) + (5l \times b) + (b \times 5h)]$ $= 2 \times 5(5lh + lb + bh)$
Multiply all three dimensions by 5 	$V_3 = 5l \times 5b \times 5h$ $= 5^3(lbh)$ $= 5^3V$	$A_3 = 2[(5l \times 5h) + (5l \times 5b) + (5b \times 5h)]$ $= 2(5^2lh + 5^2lb + 5^2bh)$ $= 5^2 \times 2(lh + lb + bh)$ $= 5^2A$
Multiply all three dimensions by k 	$V_k = kl \times kb \times kh$ $= k^3(lbh)$ $= k^3V$	$A_k = 2[(kl \times kh) + (kl \times kb) + (kb \times kh)]$ $= k^2 \times 2(lh + lb + bh)$ $= k^2A$

### Worked example 18: Calculating the new dimensions of a rectangular prism

#### QUESTION

Consider a rectangular prism with a height of 4 cm and base lengths of 3 cm.



- Calculate the surface area and volume.
- Calculate the new surface area ( $A_n$ ) and volume ( $V_n$ ) if the base lengths are multiplied by a constant factor of 3.
- Express the new surface area and volume as a factor of the original surface area and volume.

## **SOLUTION**

### **Step 1: Calculate the original volume and surface area**

$$\begin{aligned}V &= l \times b \times h \\&= 3 \times 3 \times 4 \\&= 36 \text{ cm}^3\end{aligned}$$

$$\begin{aligned}A &= 2[(l \times h) + (l \times b) + (b \times h)] \\&= 2[(3 \times 4) + (3 \times 3) + (3 \times 4)] \\&= 66 \text{ cm}^2\end{aligned}$$

### **Step 2: Calculate the new volume and surface area**

Two of the dimensions are multiplied by a factor of 3.

$$\begin{aligned}V_n &= 3l \times 3b \times h \\&= 3(3) \times 3(3) \times 4 \\&= 324 \text{ cm}^3\end{aligned}$$

$$\begin{aligned}A_n &= 2[(3l \times h) + (3l \times 3b) + (3b \times h)] \\&= 2[(3(3) \times 4) + (3(3) \times 3(3)) + (3(3) \times 4)] \\&= 306 \text{ cm}^2\end{aligned}$$

### **Step 3: Express the new dimensions as a factor of the original dimensions**

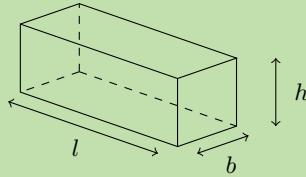
$$\begin{aligned}V &= 36 \\V_n &= 324 \\ \frac{V_n}{V} &= \frac{324}{36} \\&= 9 \\ \therefore V_n &= 9V \\&= 3^2 V\end{aligned}$$

$$\begin{aligned}A &= 66 \\A_n &= 306 \\ \frac{A_n}{A} &= \frac{306}{66} \\ \therefore A_n &= \frac{306}{66} A \\&= \frac{51}{11} A\end{aligned}$$

### Worked example 19: Multiplying the dimensions of a rectangular prism by $k$

#### QUESTION

Prove that if the height of a rectangular prism with dimensions  $l$ ,  $b$  and  $h$  is multiplied by a constant value of  $k$ , the volume will also increase by a factor  $k$ .



#### SOLUTION

##### Step 1: Calculate the original volume

We are given the original dimensions  $l$ ,  $b$  and  $h$  and so the original volume is  $V = l \times b \times h$ .

##### Step 2: Calculate the new volume

The new dimensions are  $l$ ,  $b$ , and  $kh$  and so the new volume is:

$$\begin{aligned}V_n &= l \times b \times (kh) \\&= k(lbh) \\&= kV\end{aligned}$$

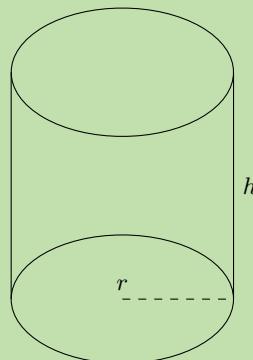
##### Step 3: Write the final answer

If the height of a rectangular prism is multiplied by a constant  $k$ , then the volume also increases by a factor of  $k$ .

### Worked example 20: Multiplying the dimensions of a cylinder by $k$

#### QUESTION

Consider a cylinder with a radius of  $r$  and a height of  $h$ . Calculate the new volume and surface area (expressed in terms of  $r$  and  $h$ ) if the radius is multiplied by a constant factor of  $k$ .



## SOLUTION

### Step 1: Calculate the original volume and surface area

$$V = \pi r^2 \times h$$
$$A = \pi r^2 + 2\pi r h$$

### Step 2: Calculate the new volume and surface area

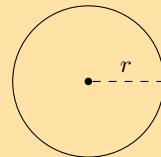
The new dimensions are  $kr$  and  $h$ .

$$\begin{aligned} V_n &= \pi(kr)^2 \times h \\ &= \pi k^2 r^2 \times h \\ &= k^2 \times \pi r^2 h \\ &= k^2 V \end{aligned}$$

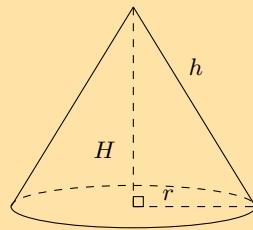
$$\begin{aligned} A_n &= \pi(kr)^2 + 2\pi(kr)h \\ &= \pi k^2 r^2 + 2\pi krh \\ &= k^2(\pi r^2) + k(2\pi rh) \end{aligned}$$

### Exercise 13 – 6:

1. If the length of the radius of a circle is a third of its original size, what will the area of the new circle be?



2. If the length of the base's radius and height of a cone is doubled, what will the surface area of the new cone be?

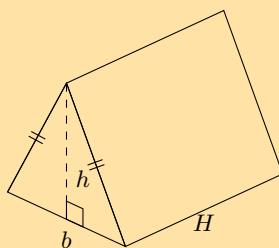


3. If the height of a prism is doubled, how much will its volume increase by?

4. Describe the change in the volume of a rectangular prism if the:

- length and breadth increase by a constant factor of 3.
- length, breadth and height are multiplied by a constant factor of 3.

5. If the length of each side of a triangular prism is quadrupled, what will the volume of the new triangular prism be?



6. Given a prism with a volume of  $493 \text{ cm}^3$  and a surface area of  $6007 \text{ cm}^2$ , find the new surface area and volume for a prism if all dimensions are increased by a constant factor of 4.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2GT8](#)   2. [2GT9](#)   3. [2GTB](#)   4a. [2GTC](#)   4b. [2GTD](#)   5. [2GTF](#)   6. [2GTG](#)



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## 13.5 Chapter summary

EMA7V

► See presentation: [2GTH](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

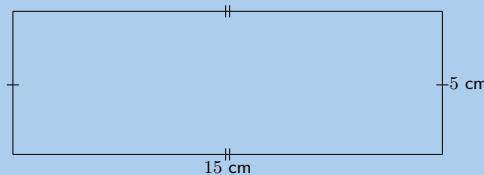
- Area is the two dimensional space inside the boundary of a flat object. It is measured in square units.
- Area formulae:
  - square:  $s^2$
  - rectangle:  $b \times h$
  - triangle:  $\frac{1}{2}b \times h$
  - trapezium:  $\frac{1}{2}(a + b) \times h$
  - parallelogram:  $b \times h$
  - circle:  $\pi r^2$
- A right prism is a geometric solid that has a polygon as its base and vertical sides perpendicular to the base. The base and top surface are the same shape and size. It is called a "right" prism because the angles between the base and sides are right angles.
- A triangular prism has a triangle as its base, a rectangular prism has a rectangle as its base, and a cube is a rectangular prism with all its sides of equal length. A cylinder is another type of right prism which has a circle as its base.
- Surface area is the total area of the exposed or outer surfaces of a prism.
- A net is the unfolded "plan" of a solid.
- Volume is the three dimensional space occupied by an object, or the contents of an object. It is measured in cubic units.
- Volume formulae for prisms and cylinders:
  - Volume of a rectangular prism:  $l \times b \times h$
  - Volume of a triangular prism:  $(\frac{1}{2}b \times h) \times H$
  - Volume of a square prism or cube:  $s^3$
  - Volume of a cylinder:  $\pi r^2 \times h$

- A pyramid is a geometric solid that has a polygon as its base and sides that converge at a point called the apex. The sides are not perpendicular to the base.
- The triangular pyramid and square pyramid take their names from the shape of their base. We call a pyramid a “right pyramid” if the line between the apex and the centre of the base is perpendicular to the base. Cones are similar to pyramids except that their bases are circles instead of polygons. Spheres are solids that are perfectly round and look the same from any direction.
- Surface area formulae for right pyramids, right cones and spheres:
  - square pyramid:  $b(b + 2h)$
  - triangular pyramid:  $\frac{1}{2}b(h_b + 3h_s)$
  - right cone:  $\pi r(r + h_s)$
  - sphere:  $4\pi r^2$
- Volume formulae for right pyramids, right cones and spheres:
  - square pyramid:  $\frac{1}{3} \times b^2 \times H$
  - triangular pyramid:  $\frac{1}{3} \times \frac{1}{2}bh \times H$
  - right cone:  $\frac{1}{3} \times \pi r^2 \times H$
  - sphere:  $\frac{4}{3}\pi r^3$
- Multiplying one or more dimensions of a prism or cylinder by a constant  $k$  affects the surface area and volume.

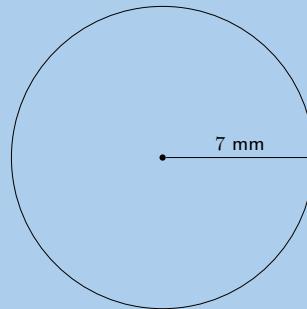
**End of chapter Exercise 13 – 7:**

1. Find the area of each of the shapes shown. Round your answer to two decimal places if necessary.

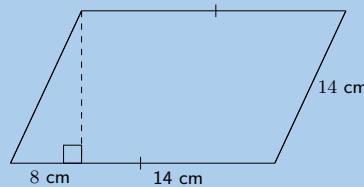
a)



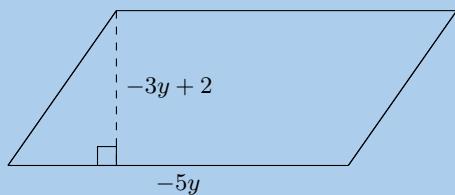
b)



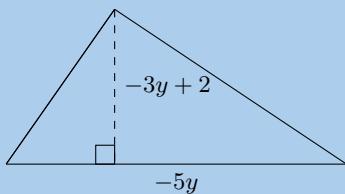
c)



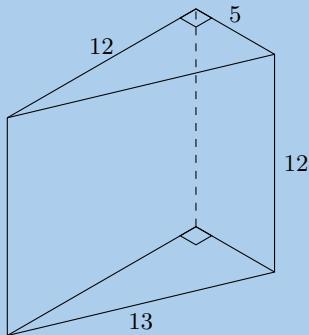
2. a) Find an expression for the area of this figure in terms of  $y$ . The dimensions of the figure are labelled  $-5y$  and  $-3y + 2$ . Write your answer in expanded form (not factorised).



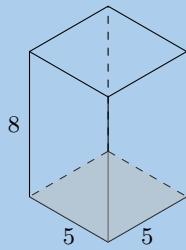
- b) Find an expression for the area of this figure in terms of  $y$ . The figure has dimensions of  $-5y$  and  $-3y + 2$ , as labelled. Write your answer in expanded form (not factorised).



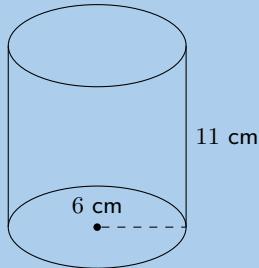
3. The figure here is a triangular prism. The height of the prism is 12 units; the triangles, which are both right triangles, have sides which are 5, 12 and 13 units long. Find the surface area of the figure.



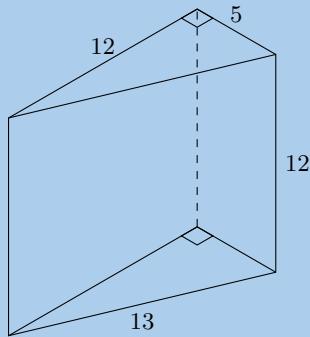
4. The figure here is a rectangular prism. The height of the prism is 5 units; the other dimensions of the prism are 8 and 5 units. Find the surface area of the figure.



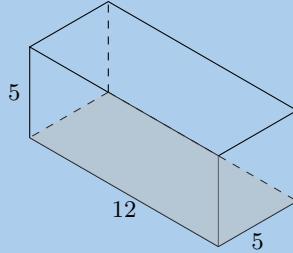
5. A cylinder is shown below. The height of the cylinder is 11 cm; the radius of the cylinder is  $r = 6$  cm, as shown. Find the surface area of the figure. Round your answer to two decimal places.



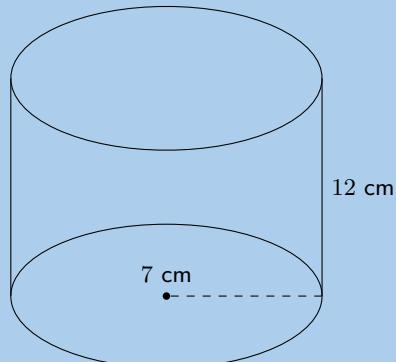
6. The figure here is a triangular prism. The height of the prism is 12 units; the triangles, which both contain right angles, have sides which are 5, 12 and 13 units long. Determine the volume of the figure.



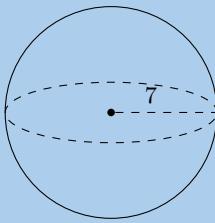
7. The figure here is a rectangular prism. The height of the prism is 5 units; the other dimensions of the prism are 12 and 5 units. Calculate the volume of the figure.



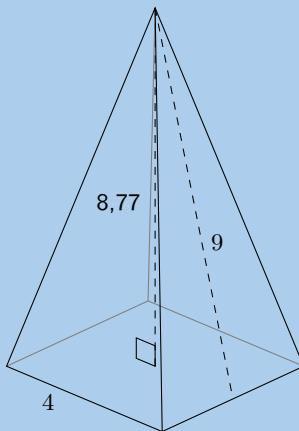
8. The picture below shows a cylinder. The height of the cylinder is 12 cm; the radius of the cylinder is  $r = 7$  cm. Calculate the volume of the figure. Round your answer two decimal places.



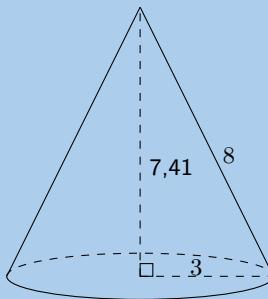
9. The figure here is a sphere. The radius of the sphere is  $r = 7$  units. Find the surface area of the figure. Round your answer two decimal places.



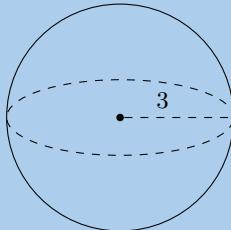
10. The figure here shows a pyramid with a square base. The sides of the base are each 4 units long. The vertical height of the pyramid is 8.77 units, and the slant height of the pyramid is 9 units. Determine the surface area of the pyramid.



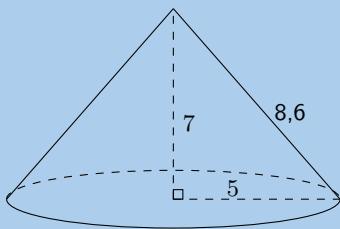
11. The figure here is a cone. The vertical height of the cone is  $H = 7.41$  units and the slant height of the cone is  $h = 8$  units; the radius of the cone is shown,  $r = 3$  units. Find the surface area of the figure. Round your answer two decimal places.



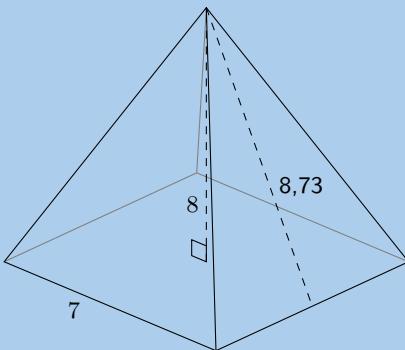
12. The figure below shows a sphere. The radius of the sphere is  $r = 3$  units. Determine the volume of the figure. Round your answer to two decimal places.



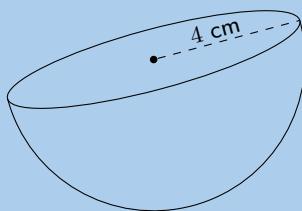
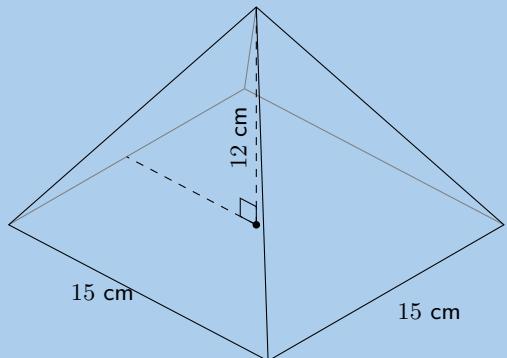
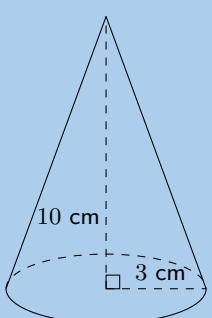
13. The figure here is a cone. The vertical height of the cone is  $H = 7$  units and the slant height is  $h = 8.60$  units; the radius of the cone is shown,  $r = 5$  units. Find the volume of the figure. Round your answer to two decimal places.



14. The figure here is a pyramid with a square base. The vertical height of the pyramid is  $H = 8$  units and the slant height is  $h = 8.73$  units; each side of the base of the pyramid is  $b = 7$  units. Find the volume of the figure. Round your answer to two decimal places.



15. Consider the solids below:

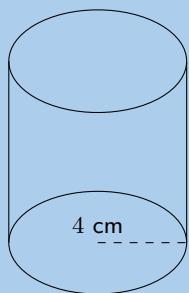


a) Calculate the surface area of each solid.

b) Calculate the volume of each solid.

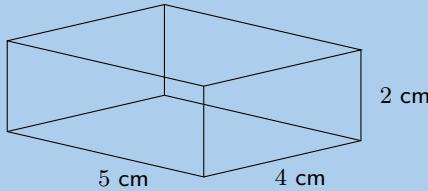
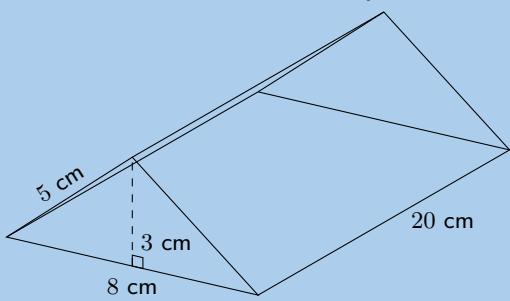
16. If the length of each side of a square is a quarter of its original size, what will the area of the new square be?
17. If the length of each side of a square pyramid is a third of its original size, what will the surface area of the new square pyramid be?
18. If the length of the base's radius and the height of a cylinder is halved, what will the volume of the new cylinder be?

19. Consider the solids below and answer the questions that follow (correct to 1 decimal place, if necessary):



10 cm

4 cm

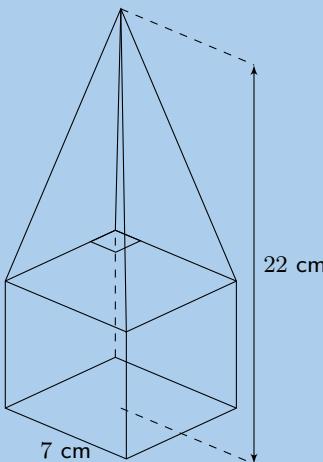


2 cm

5 cm      4 cm

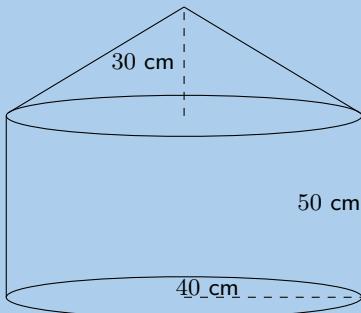
- Calculate the surface area of each solid.
- Calculate volume of each solid.
- If each dimension of the solids is increased by a factor of 3, calculate the new surface area of each solid.
- If each dimension of the solids is increased by a factor of 3, calculate the new volume of each solid.

20. The solid below is made of a cube and a square pyramid. Answer the following:



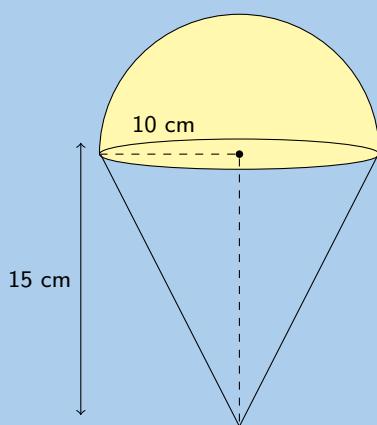
- Find the surface area of the solid shown. Give your answers to two decimal places.
- Now determine the volume of the shape. Give your answer to the nearest integer value.

21. Calculate the volume and surface area of the solid below (correct to 1 decimal place):

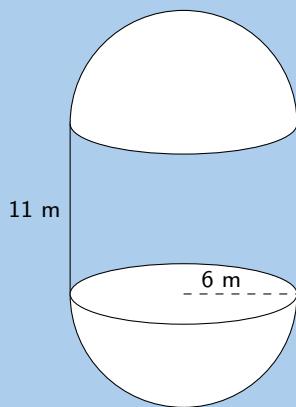


22. Find the volume and surface areas of the following composite shapes.

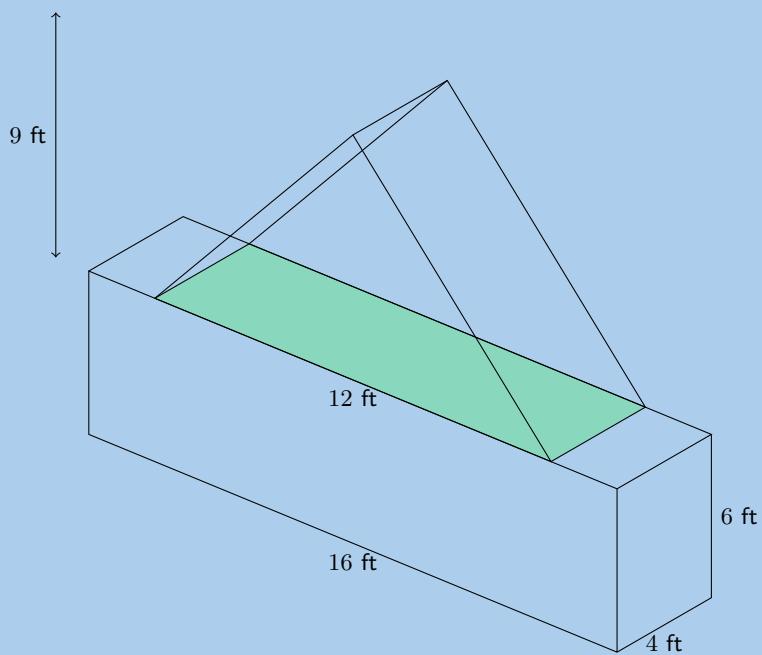
a)



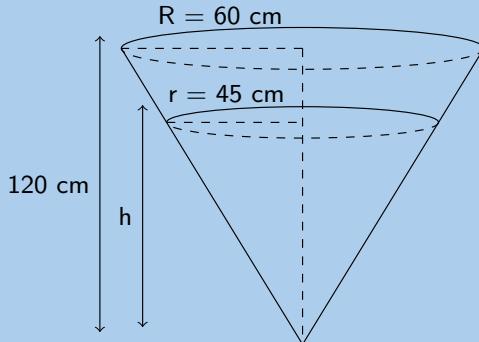
b)



c)

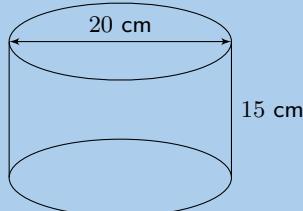


23. An ice-cream cone (right cone) has a radius of 3 cm and a height of 12 cm. A half scoop of ice-cream (hemisphere) is placed on top of the cone. If the ice-cream melts, will it fit into the cone? Show all your working.
24. A receptacle filled with petrol has the shape of an inverted right circular cone of height 120 cm and base radius of 60 cm. A certain amount of fuel is siphoned out of the receptacle leaving a depth of  $h$  cm.

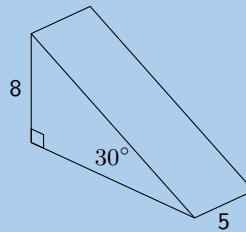


- a) Show that  $h = 90$  cm.
- b) Determine the volume of fuel that has been siphoned out. Express your answer in litres if  $1 \text{ l} = 1000 \text{ cm}^3$
25. Find the **volume** and **surface area** of the following prisms.

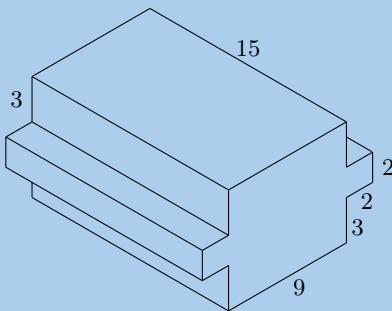
a)



b)

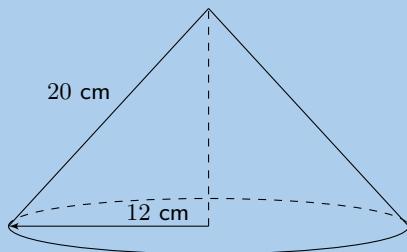


c)

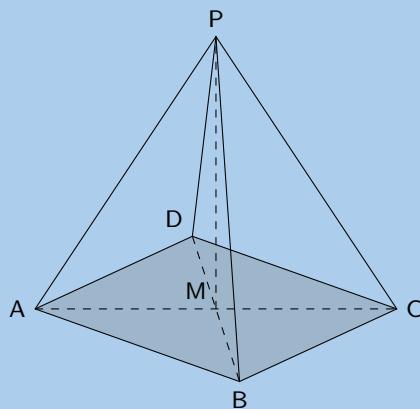


26. Determine the volume of the following:

a)

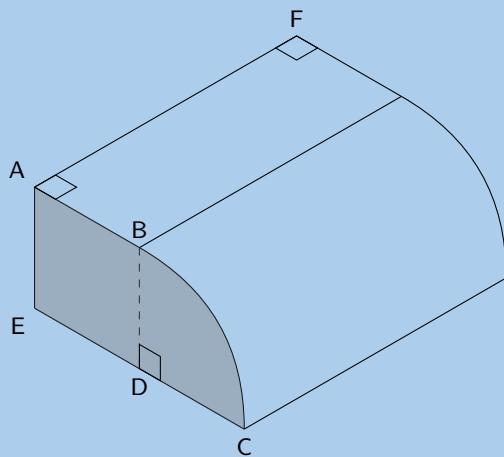


- b)  $ABCD$  is a square,  $AC = 12 \text{ cm}$ ,  $AP = 10 \text{ cm}$ .



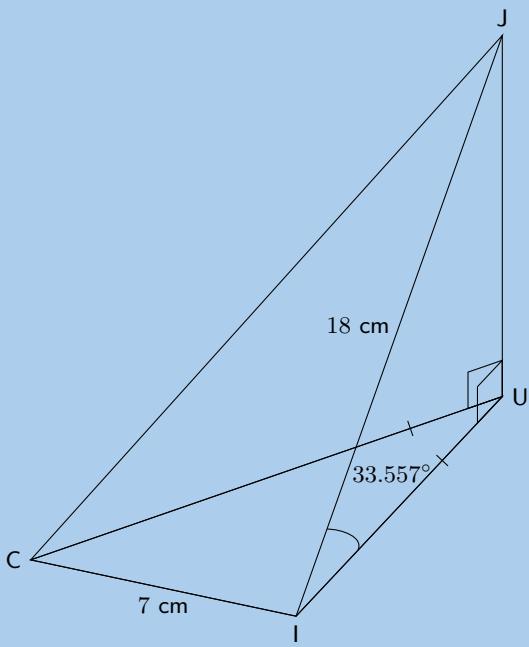
27. The prism alongside has the following dimensions:

$AB = 4 \text{ units}$ ,  $EC = 8 \text{ units}$ ,  $AF = 10 \text{ units}$ .  $BC$  is an arc of a circle with centre  $D$ .  $AB \parallel EC$ .



- a) Explain why  $BD$ , the radius of the arc  $BC$ , is 4 units.  
b) Calculate the area of the shaded surface.  
c) Find the volume of the prism.

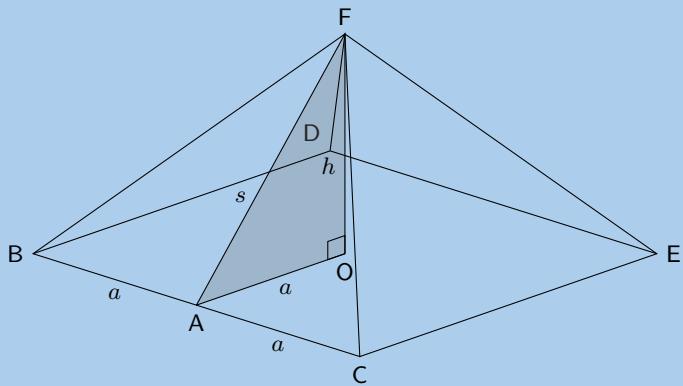
28. A cooldrink container is made in the shape of a pyramid with an isosceles triangular base. This is known as a tetrahedron. The angle of elevation of the top of the container is  $33,557^\circ$ .  $CI = 7 \text{ cm}$ ;  $JI = 18 \text{ cm}$ .



- a) i. Show that the length  $UI$  is 15 cm.  
 ii. Find the height  $JU$  (to the nearest unit).  
 iii. Calculate the area of  $\triangle CUI$ .  
 Hint: construct a perpendicular line from  $U$  to  $CI$   
 iv. Find the volume of the container
- b) The container is filled with the juice such that an 11,85% gap of air is left. Determine the volume of the juice.

29. Below is a diagram of The Great Pyramid.

This is a square-based pyramid and  $O$  is the centre of the square.



$BA = AC = a$  and  $OF = h = \text{height of the pyramid}$ . The length of the side of the pyramid  $BC = 755,79$  feet and the height of the pyramid is 481,4 feet.

- a) Determine the area of the base of the pyramid in terms of  $a$ .  
 b) Calculate  $AF (= s)$  to 5 decimal places.  
 c) From your calculation in question (b) determine  $\frac{s}{a}$ .  
 d) Determine the volume and surface area of the pyramid.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |                           |                           |                           |                           |                           |                          |
|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|--------------------------|
| 1. <a href="#">2GTJ</a>   | 2. <a href="#">2GTK</a>   | 3. <a href="#">2GTM</a>   | 4. <a href="#">2GTN</a>   | 5. <a href="#">2GTP</a>   | 6. <a href="#">2GTQ</a>  |
| 7. <a href="#">2GTR</a>   | 8. <a href="#">2GTS</a>   | 9. <a href="#">2GTT</a>   | 10. <a href="#">2GTV</a>  | 11. <a href="#">2GTW</a>  | 12. <a href="#">2GTX</a> |
| 13. <a href="#">2GTY</a>  | 14. <a href="#">2GTZ</a>  | 15a. <a href="#">2GV2</a> | 15b. <a href="#">2GV3</a> | 16. <a href="#">2GV4</a>  | 17. <a href="#">2GV5</a> |
| 18. <a href="#">2GV6</a>  | 19a. <a href="#">2GV7</a> | 19b. <a href="#">2GV8</a> | 19c. <a href="#">2GV9</a> | 19d. <a href="#">2GVB</a> | 20. <a href="#">2GVC</a> |
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| 25a. <a href="#">2GVM</a> | 25b. <a href="#">2GVN</a> | 25c. <a href="#">2GVP</a> | 26a. <a href="#">2GVQ</a> | 26b. <a href="#">2GVR</a> | 27. <a href="#">2GVS</a> |
| 28. <a href="#">2GVT</a>  | 29. <a href="#">2GVV</a>  |                           |                           |                           |                          |



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# CHAPTER



## Probability

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## 14 Probability

We use probability to describe uncertain events. When you accidentally drop a slice of bread, you don't know if it's going to fall with the buttered side facing upwards or downwards. When your favourite sports team plays a game, you don't know whether they will win or not. When the weatherman says that there is a 40% chance of rain tomorrow, you may or may not end up getting wet. Uncertainty presents itself to some degree in every event that occurs around us and in every decision that we make.

We will see in this chapter that all of these uncertainties can be described using the rules of probability theory and that we can make definite conclusions about uncertain processes.



Figure 14.1: Tracking a superstorm. Meteorologists use computer software to help them track storms and predict the weather.

We'll use three examples of uncertain processes to help you understand the meanings of the different words used in probability theory: tossing a coin, rolling dice, and a soccer match.

### VISIT:

The following video introduces the concepts used in probability.

► See video: [2GVW](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### DEFINITION: Experiment

An experiment refers to an uncertain process.

### DEFINITION: Outcome

An outcome of an experiment is a single result of that experiment.

**Experiment 1:** A coin is tossed and it lands with either heads (H) or tails (T) facing upwards. An example outcome of tossing a coin is that it lands with heads facing up:



**Experiment 2:** Two dice are rolled and the total number of dots added up. An example outcome of rolling two dice:



**Experiment 3:** Two teams play a soccer match and we are interested in the final score. An example outcome of a soccer match:

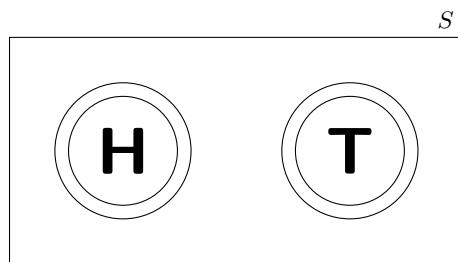


**DEFINITION:** *Sample space*

The sample space of an experiment is the set of all possible outcomes of that experiment. The sample space is denoted with the symbol  $S$  and the size of the sample space (the total number of possible outcomes) is denoted with  $n(S)$

Even though we are usually interested in the outcome of an experiment, we also need to know what the other outcomes could have been. Let's have a look at the sample spaces of each of our three experiments.

**Experiment 1:** Since a coin can land in one of only two ways (we will ignore the possibility that the coin lands on its edge), the sample space is the set  $S = \{H; T\}$ . The size of the sample space of the coin toss is  $n(S) = 2$ :



**Experiment 2:** Each of the dice can land on a number from 1 to 6. In this experiment the sample space of all possible outcomes is every possible combination of the 6 numbers on the first die with the 6 numbers on the second die. This gives a total of  $n(S) = 6 \times 6 = 36$  possible outcomes. The figure below shows all of the outcomes in the sample space of rolling two dice:

S					

**Experiment 3:** Each soccer team can get an integer score from 0 upwards. Usually we don't expect a score to go much higher than 5 goals, but there is no reason why this cannot happen. So the sample space of this experiment consists of all possible combinations of two non-negative integers. The figure below shows all of the possibilities. Since we do not limit the score of a team, this sample space is infinitely large:

<i>S</i>				
0 – 0	1 – 0	2 – 0	3 – 0	...
0 – 1	1 – 1	2 – 1	3 – 1	...
0 – 2	1 – 2	2 – 2	3 – 2	...
0 – 3	1 – 3	2 – 3	3 – 3	
:	:	:		⋮

**NOTE:**

When we represent a sample space containing real numbers we can either write out all the outcomes in the sample space: {1; 2; 3; 4; 5; 6; 7; 8; 9; 10} or we can represent the sample space as:  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 10\}$ .

**DEFINITION:** *Event*

An event is a specific set of outcomes of an experiment that you are interested in. An event is denoted with the letter  $E$  and the number of outcomes in the event with  $n(E)$ .

**Experiment 1:** Let us say that we would like the coin to land heads up. Here the event contains a single outcome:  $E = \{\text{H}\}$ . The size of the event set is  $n(E) = 1$ .

**Experiment 2:** Let us say that we are interested in the sum of the dice being 8. In this case the event set is:

$$E = \{(\square \blacksquare \blacksquare); (\blacksquare \square \blacksquare); (\blacksquare \blacksquare \square); (\blacksquare \blacksquare \blacksquare); (\blacksquare \blacksquare \square \square)\}$$

since it contains all of the possible ways to get 8 dots with 2 dice. The size of the event set is  $n(E) = 5$ .

**Experiment 3:** We would like to know whether the first team will win. For this event to happen the first score must be greater than the second.

$E = \{(1; 0); (2; 0); (2; 1); (3; 0); (3; 1); (3; 2); \dots\}$ . This event set is infinitely large.

## 14.1 Theoretical probability

EMA7W

**DEFINITION:** *Probability*

A probability is a real number between 0 and 1 that describes how likely it is that an event will occur.

We can describe probabilities in three ways:

1. As a real number between 0 and 1. For example 0,75.
2. As a percentage. For example 0,75 can be written as 75%.
3. As a fraction. For example 0,75 can also be written as  $\frac{3}{4}$ .

We note the following about probabilities:

- A probability of 0 means that an event will never occur.
- A probability of 1 means that an event will always occur.
- A probability of 0,5 means that an event will occur half the time, or 1 time out of every 2.

When all of the possible outcomes of an experiment have an equal chance of occurring, we can compute the exact theoretical probability of an event. The probability of an event is the ratio between the number of outcomes in the event set and the number of possible outcomes in the sample space.

$$P(E) = \frac{n(E)}{n(S)}$$

### VISIT:

The following video shows an example of calculating the theoretical probabilities of an event.

► See video: [2GVX](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 1: Theoretical probabilities

#### QUESTION

---

What is the theoretical probability of each of the events in the first two of our three experiments?

#### SOLUTION

---

##### **Step 1: Write down the value of $n(S)$**

Experiment 1 (coin):  $n(S) = 2$

Experiment 2 (dice):  $n(S) = 36$

##### **Step 2: Write down the size of the event set**

Experiment 1:  $n(E) = 1$

Experiment 2:  $n(E) = 5$

##### **Step 3: Compute the theoretical probability**

Experiment 1:

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{2} = 0,5$$

Experiment 2:

$$P(E) = \frac{n(E)}{n(S)} = \frac{5}{36} = 0,13\dot{8}$$

Note that we do not consider the theoretical probability of the third experiment. The third experiment is different from the first two in an important way, namely that all possible outcomes (all final scores) are not

equally likely. For example, we know that a soccer score of 1–1 is quite common, while a score of 11–15 is very, very rare. Because all outcomes are not equally likely, we cannot use the ratio between  $n(E)$  and  $n(S)$  to compute the theoretical probability of a team winning.

#### Exercise 14 – 1:

1. A learner wants to understand the term “event”. So the learner rolls 2 dice hoping to get a total of 8. Which of the following is the most appropriate example of the term “event”?
  - event set =  $\{(4; 4)\}$
  - event set =  $\{(2; 6); (3; 5); (4; 4); (5; 3); (6; 2)\}$
  - event set =  $\{(2; 6); (6; 2)\}$
2. A learner wants to understand the term “sample space”. So the learner rolls a die. Which of the following is the most appropriate example of the term “sample space”?
  - $\{1; 2; 3; 4; 5; 6\}$
  - $\{H; T\}$
  - $\{1; 3; 5\}$
3. A learner finds a 6 sided die and then rolls the die once on a table. What is the probability that the die lands on either 1 or 2?  
Write your answer as a simplified fraction.
4. A learner finds a textbook that has 100 pages. He then selects one page from the textbook. What is the probability that the page has an odd page number?  
Write your answer as a decimal (correct to 2 decimal places).
5. Even numbers from 2 to 100 are written on cards. What is the probability of selecting a multiple of 5, if a card is drawn at random?
6. A bag contains 6 red balls, 3 blue balls, 2 green balls and 1 white ball. A ball is picked at random. Determine the probability that it is:
  - a) red
  - b) blue or white
  - c) not green
  - d) not green or red
7. A playing card is selected randomly from a pack of 52 cards. Determine the probability that it is:
  - a) the 2 of hearts
  - b) a red card
  - c) a picture card
  - d) an ace
  - e) a number less than 4

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on ‘Practise Maths’.

1. [2GVY](#) 2. [2GVZ](#) 3. [2GW2](#) 4. [2GW3](#) 5. [2GW4](#) 6. [2GW5](#) 7. [2GW6](#)



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**DEFINITION:** *Relative frequency*

The relative frequency of an event is defined as the number of times that the event occurs during experimental trials, divided by the total number of trials conducted.

The relative frequency is not a theoretical quantity, but an experimental one. We have to repeat an experiment a number of times and count how many times the outcome of the trial is in the event set. Because it is experimental, it is possible to get a different relative frequency every time that we repeat an experiment.

**VISIT:**

The following video explains the concept of relative frequency using the throw of a dice.

► See video: [2GW7](http://www.everythingmaths.co.za) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

**Worked example 2: Relative frequency and theoretical probability****QUESTION**

We toss a coin 30 times and observe the outcomes. The results of the trials are shown in the table below.

<b>trial</b>	1	2	3	4	5	6	7	8	9	10
<b>outcome</b>	H	T	T	T	H	T	H	H	H	T
<b>trial</b>	11	12	13	14	15	16	17	18	19	20
<b>outcome</b>	H	T	T	H	T	T	T	H	T	T
<b>trial</b>	21	22	23	24	25	26	27	28	29	30
<b>outcome</b>	H	H	H	T	H	T	H	T	T	T

What is the relative frequency of observing heads after each trial and how does it compare to the theoretical probability of observing heads?

**SOLUTION****Step 1: Count the number of positive outcomes**

A positive outcome is when the outcome is in our event set. The table below shows a running count (after each trial  $t$ ) of the number of positive outcomes  $p$  we have observed. For example, after  $t = 20$  trials we have observed heads 8 times and tails 12 times and so the positive outcome count is  $p = 8$ .

$t$	1	2	3	4	5	6	7	8	9	10
$p$	1	1	1	1	2	2	3	4	5	5
$t$	11	12	13	14	15	16	17	18	19	20
$p$	6	6	6	7	7	7	7	8	8	8
$t$	21	22	23	24	25	26	27	28	29	30
$p$	9	10	11	11	12	12	13	13	13	13

**Step 2: Compute the relative frequency**

Since the relative frequency is defined as the ratio between the number of positive trials and the total number of trials,

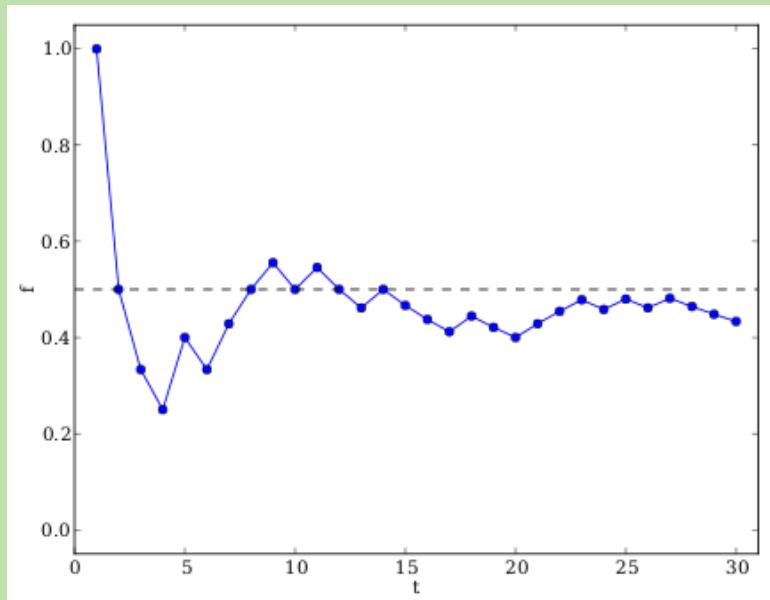
$$f = \frac{p}{t}$$

The relative frequency of observing heads,  $f$ , after having completed  $t$  coin tosses is:

<i>t</i>	1	2	3	4	5	6	7	8	9	10
<i>f</i>	1,00	0,50	0,33	0,25	0,40	0,33	0,43	0,50	0,56	0,50
<i>t</i>	11	12	13	14	15	16	17	18	19	20
<i>f</i>	0,55	0,50	0,46	0,50	0,47	0,44	0,41	0,44	0,42	0,40
<i>t</i>	21	22	23	24	25	26	27	28	29	30
<i>f</i>	0,43	0,45	0,48	0,46	0,48	0,46	0,48	0,46	0,45	0,43

From the last entry in this table we can now easily read the relative frequency after 30 trials, namely  $\frac{13}{30} = 0,4\dot{3}$ . The relative frequency is close to the theoretical probability of 0,5. In general, the relative frequency of an event tends to get closer to the theoretical probability of the event as we perform more trials.

A much better way to summarise the table of relative frequencies is in a graph:



The graph above is the plot of the relative frequency of observing heads, *f*, after having completed *t* coin tosses. It was generated from the table of numbers above by plotting the number of trials that have been completed, *t*, on the *x*-axis and the relative frequency, *f*, on the *y*-axis. In the beginning (after a small number of trials) the relative frequency fluctuates a lot around the theoretical probability at 0,5, which is shown with a dashed line. As the number of trials increases, the relative frequency fluctuates less and gets closer to the theoretical probability.

### Worked example 3: Relative frequency and theoretical probability

#### QUESTION

While watching 10 soccer games where Team 1 plays against Team 2, we record the following final scores:

Trial	1	2	3	4	5	6	7	8	9	10
Team 1	2	0	1	1	1	1	1	0	5	3
Team 2	0	2	2	2	2	1	1	0	0	0

What is the relative frequency of Team 1 winning?

## SOLUTION

### Step 1:

In this experiment, each trial takes the form of Team 1 playing a soccer match against Team 2.

### Step 2: Count the number of positive outcomes

We are interested in the event where Team 1 wins. From the table above we see that this happens 3 times.

### Step 3: Compute the relative frequency

The total number of trials is 10. This means that the relative frequency of the event is

$$\frac{3}{10} = 0,3$$

It is important to understand the difference between the theoretical probability of an event and the observed relative frequency of the event in experimental trials. The theoretical probability is a number that we can compute if we have enough information about the experiment. If each possible outcome in the sample space is equally likely, we can count the number of outcomes in the event set and the number of outcomes in the sample space to compute the theoretical probability.

The relative frequency depends on the sequence of outcomes that we observe while doing a statistical experiment. The relative frequency can be different every time we redo the experiment. The more trials we run during an experiment, the closer the observed relative frequency of an event will get to the theoretical probability of the event.

So why do we need statistical experiments if we have theoretical probabilities? In some cases, like our soccer experiment, it is difficult or impossible to compute the theoretical probability of an event. Since we do not know exactly how likely it is that one soccer team will score goals against another, we can never compute the theoretical probability of events in soccer. In such cases we can still use the relative frequency to estimate the theoretical probability, by running experiments and counting the number of positive outcomes.

#### VISIT:

You can use this Phet simulation on [probability](#) to do some experiments with dropping a ball through a triangular grid.

### Exercise 14 – 2:

1. A die is tossed 44 times and lands 5 times on the number 3.

What is the relative frequency of observing the die land on the number 3? Write your answer correct to 2 decimal places.

2. A coin is tossed 30 times and lands 17 times on heads.

What is the relative frequency of observing the coin land on heads? Write your answer correct to 2 decimal places.

3. A die is tossed 27 times and lands 6 times on the number 6.

What is the relative frequency of observing the die land on the number 6? Write your answer correct to 2 decimal places.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2GW8](#)
2. [2GW9](#)
3. [2GWB](#)



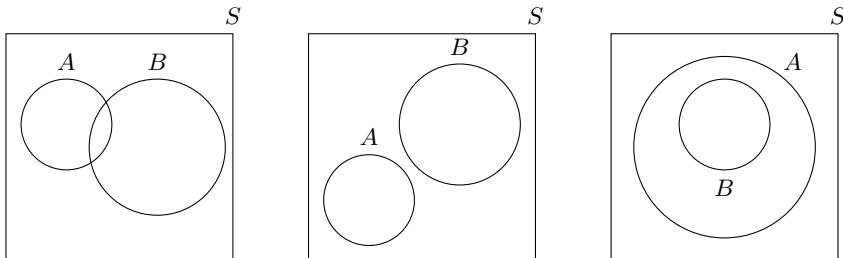
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A Venn diagram is a graphical way of representing the relationships between sets. In each Venn diagram a set is represented by a closed curve. The region inside the curve represents the elements that belong to the set, while the region outside the curve represents the elements that are excluded from the set.

Venn diagrams are helpful for thinking about probability since we deal with different sets. Consider two events,  $A$  and  $B$ , in a sample space  $S$ . The diagram below shows the possible ways in which the event sets can overlap, represented using Venn diagrams:



The sets are represented using a rectangle for  $S$  and circles for each of  $A$  and  $B$ . In the first diagram the two events overlap partially. In the second diagram the two events do not overlap at all. In the third diagram one event is fully contained in the other. Note that events will always appear inside the sample space since the sample space contains all possible outcomes of the experiment.

**VISIT:**

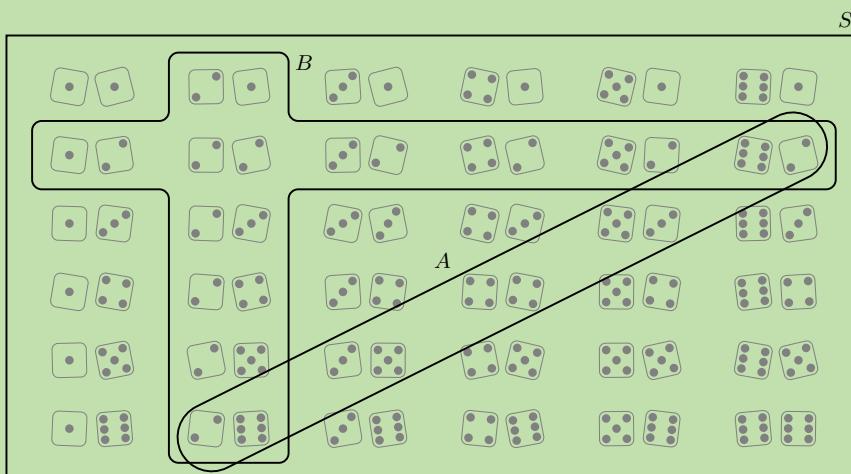
This video shows how to draw a Venn diagram using a deck of cards as the sample space.

► See video: [2GWC](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

**Worked example 4: Venn diagrams**
**QUESTION**

Represent the sample space of two rolled dice and the following two events using a Venn diagram:

- Event A: the sum of the dice equals 8
- Event B: at least one of the dice shows a 2

**SOLUTION**


## Worked example 5: Venn diagrams

### QUESTION

Consider the set of diamonds removed from a deck of cards. A random card is selected from the set of diamonds.

- Write down the sample space,  $S$ , for the experiment.
- What is the value of  $n(S)$ ?
- Consider the following two events:
  - $P$ : An even diamond is chosen
  - $R$ : A royal diamond is chosen

Represent the sample space  $S$  and events  $P$  and  $R$  using a Venn diagram.

### SOLUTION

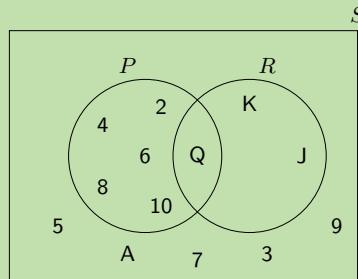
#### Step 1: Write down the sample space $S$

$$S = \{A; 2; 3; 4; 5; 6; 7; 8; 9; 10; J; Q; K\}$$

#### Step 2: Write down the value of $n(S)$

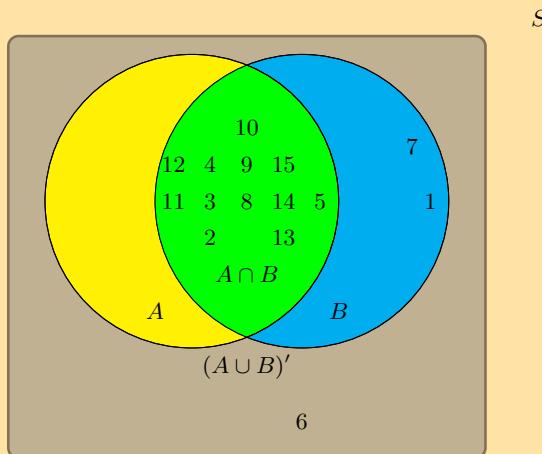
$$n(S) = 13$$

#### Step 3: Draw the Venn diagram



### Exercise 14 – 3:

1. A group of learners are given the following Venn diagram:

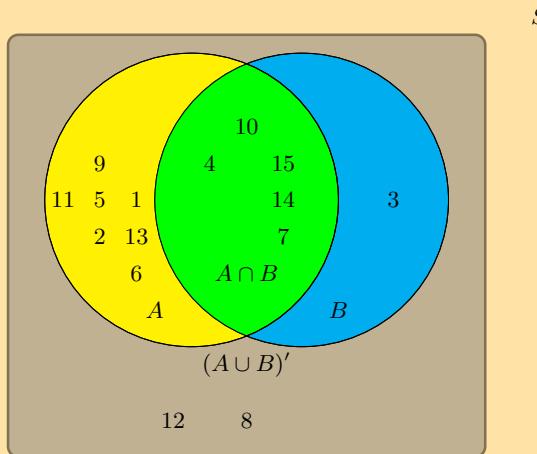


The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ .

They are asked to identify the event set of  $B$ . They get stuck, and you offer to help them find it. Which of the following sets best describes the event set of  $B$ ?

- $\{2; 3; 4; 5; 8; 9; 10; 11; 12; 13; 14; 15\}$
- $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 14; 15\}$
- $\{1; 6; 7\}$
- $\{6\}$

2. A group of learners are given the following Venn diagram:



The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ .

They are asked to identify the event set of  $A$ . They get stuck, and you offer to help them find it. Which of the following sets best describes the event set of  $A$ ?

- $\{1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; 14; 15\}$
- $\{3; 8; 12\}$
- $\{3; 4; 7; 10; 14; 15\}$
- $\{1; 2; 4; 5; 6; 7; 9; 10; 11; 13; 14; 15\}$
- $\{4; 7; 10; 14; 15\}$

3. Pieces of paper labelled with the numbers 1 to 12 are placed in a box and the box is shaken. One piece of paper is taken out and then replaced.

- What is the sample space,  $S$ ?
  - Write down the set  $A$ , representing the event of taking a piece of paper labelled with a factor of 12.
  - Write down the set  $B$ , representing the event of taking a piece of paper labelled with a prime number.
  - Represent  $A$ ,  $B$  and  $S$  by means of a Venn diagram.
  - Find:
    - $n(S)$
    - $n(A)$
    - $n(B)$
4. Let  $S$  denote the set of whole numbers from 1 to 16,  $X$  denote the set of even numbers from 1 to 16 and  $Y$  denote the set of prime numbers from 1 to 16. Draw a Venn diagram depicting  $S$ ,  $X$  and  $Y$ .
5. There are 71 Grade 10 learners at school. All of these take some combination of Maths, Geography and History. The number who take Geography is 41, those who take History is 36, and 30 take Maths. The number who take Maths and History is 16; the number who take Geography and History is 6, and there are 8 who take Maths only and 16 who take History only.

- Draw a Venn diagram to illustrate all this information.
- How many learners take Maths and Geography but not History?
- How many learners take Geography only?
- How many learners take all three subjects?

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- [2GWD](#)
- [2GWF](#)
- [2GWG](#)
- [2GWH](#)
- [2GWJ](#)



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## 14.4 Union and intersection

EMA7Z

### DEFINITION: Union

The union of two sets is a new set that contains all of the elements that are in at least one of the two sets. The union is written as  $A \cup B$  or "A or B".

### DEFINITION: Intersection

The intersection of two sets is a new set that contains all of the elements that are in both sets. The intersection is written as  $A \cap B$  or "A and B".

The figure below shows the union and intersection for different configurations of two events in a sample space, using Venn diagrams.

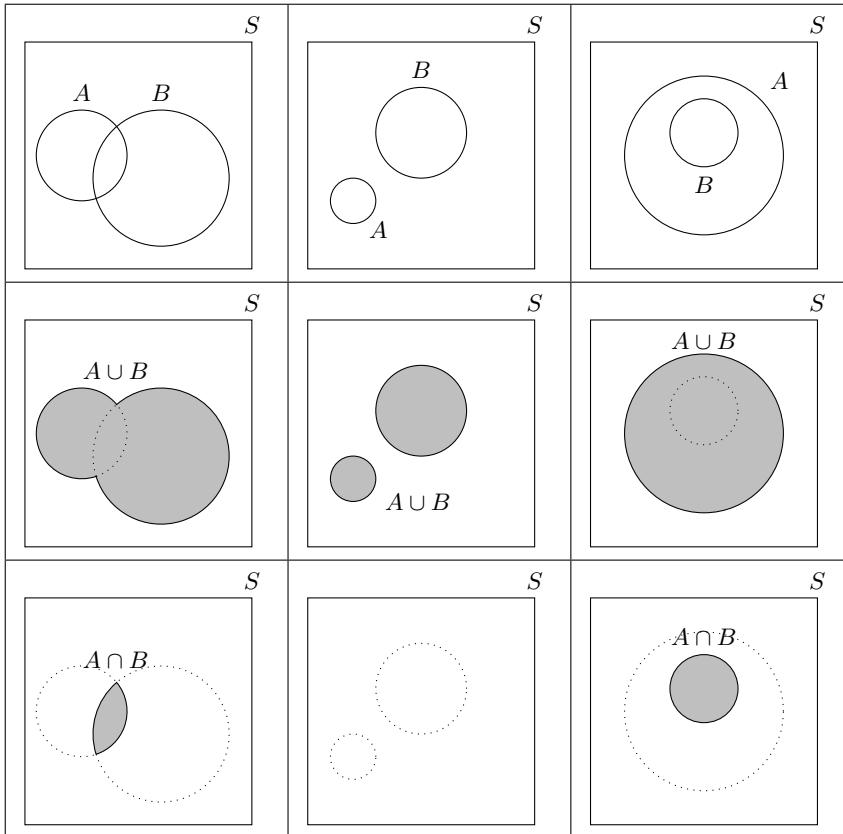
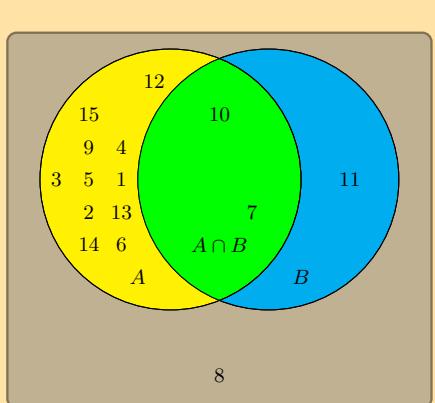


Figure 14.2: The unions and intersections of different events. Note that in the middle column the intersection,  $A \cap B$ , is empty since the two sets do not overlap. In the final column the union,  $A \cup B$ , is equal to  $A$  and the intersection,  $A \cap B$ , is equal to  $B$  since  $B$  is fully contained in  $A$ .

#### Exercise 14 – 4:

1. A group of learners are given the following Venn diagram:



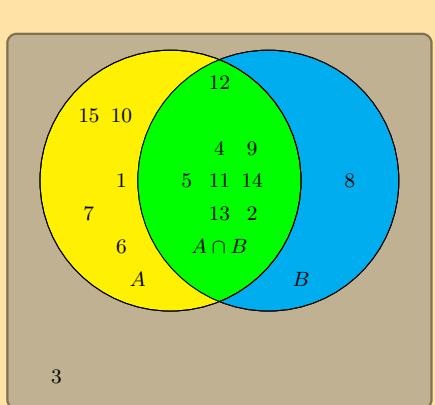
The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ .

They are asked to identify the event set of the intersection between event set A and event set B, also written as  $A \cap B$ . They get stuck, and you offer to help them find it.

Which set best describes the event set of  $A \cap B$ ?

- {7; 10; 11}
- {1; 2; 3; 4; 5; 6; 7; 9; 10; 11}
- {1; 2; 3; 4; 5; 6; 7; 9; 10}
- {7; 10}

2. A group of learners are given the following Venn diagram:



The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ .

They are asked to identify the event set of the union between event set A and event set B, also written as  $A \cup B$ . They get stuck, and you offer to help them find it.

Which set best describes the event set of  $A \cup B$ ?

- {1; 6; 7; 10; 15}
- {1; 2; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15}
- {2; 4; 5; 9; 10; 11; 12; 13; 14}
- {3}

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By definition, the sample space contains all possible outcomes of an experiment. So we know that the probability of observing an outcome from the sample space is 1.

$$P(S) = 1$$

We can calculate the probability of the union of two events using:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We will prove this identity using the Venn diagrams given above.

For each of the 4 terms in the union and intersection identity, we can draw the Venn diagram and then add and subtract the different diagrams. The area of a region represents its probability.

We will do this for the first column of the Venn diagram figure given previously. You should also try it for the other columns.

$$\begin{aligned}
 & P(A) + P(B) - P(A \cap B) \\
 = & \boxed{\text{Diagram A}} + \boxed{\text{Diagram B}} - \boxed{\text{Diagram C}} \\
 = & \boxed{\text{Diagram A}} + \boxed{\text{Diagram D}} \\
 = & \boxed{\text{Diagram E}} \\
 = & P(A \cup B)
 \end{aligned}$$

### VISIT:

This video gives an example of how we can add probabilities together.

▶ See video: [2GWN](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

### Worked example 6: Union and intersection of events

#### QUESTION

Relate the probabilities of events  $A$  and  $B$  from Example 4 (two rolled dice) and show that they satisfy the identity:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

#### SOLUTION

##### Step 1: Write down the probabilities of the two events, their union and their intersection

From the Venn diagram in Example 4, we can count the number of outcomes in each event. To get the probability of an event, we divide the size of the event by the size of the sample space, which is  $n(S) = 36$ .

$$\begin{aligned}
 P(A) &= \frac{n(A)}{n(S)} = \frac{5}{36} \\
 P(B) &= \frac{n(B)}{n(S)} = \frac{11}{36} \\
 P(A \cap B) &= \frac{n(A \cap B)}{n(S)} = \frac{2}{36} \\
 P(A \cup B) &= \frac{n(A \cup B)}{n(S)} = \frac{14}{36}
 \end{aligned}$$

**Step 2: Write down and check the identity**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\text{RHS} = \frac{14}{36}$$

$$\begin{aligned}
 \text{LHS} &= \frac{5}{36} + \frac{11}{36} - \frac{2}{36} \\
 &= \frac{5}{36} + \frac{9}{36} \\
 &= \frac{14}{36}
 \end{aligned}$$

$$\therefore \text{RHS} = \text{LHS}$$

**Exercise 14 – 5:**

1. A group of learners is given the following event sets:

Event Set A	1	2	5	6
-------------	---	---	---	---

Event Set B	3
-------------	---

Event Set $A \cap B$	empty
----------------------	-------

The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 6\}$ .

They are asked to calculate the value of  $P(A \cup B)$ . They get stuck, and you offer to calculate it for them. Give your answer as a decimal number, rounded to two decimal places.

2. A group of learners is given the following event sets:

Event Set A	1	2	6
-------------	---	---	---

Event Set B	1	5
-------------	---	---

Event Set $A \cup B$	1	2	5	6
----------------------	---	---	---	---

The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 6\}$ .

They are asked to calculate the value of  $P(A \cap B)$ . They get stuck, and you offer to calculate it for them. Give your answer as a decimal number, rounded to two decimal value.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'. 1. [2GWP](#) 2. [2GWQ](#)



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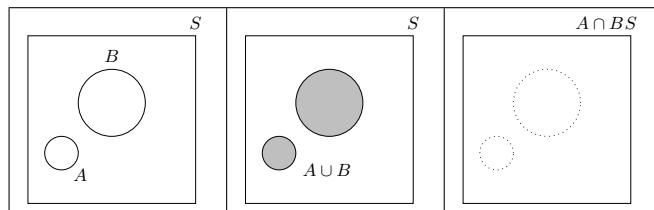


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**DEFINITION:** *Mutually exclusive events*

Two events are called mutually exclusive if they cannot occur at the same time. Whenever an outcome of an experiment is in the first event it cannot also be in the second event, and vice versa.

Another way of saying this is that the two event sets,  $A$  and  $B$ , cannot have any elements in common, or  $P(A \cap B) = \emptyset$  (where  $\emptyset$  denotes the empty set). We have already seen the Venn diagrams of mutually exclusive events in the middle column of the Venn diagrams provided earlier.



From this figure you can see that the intersection has no elements. You can also see that the probability of the union is the sum of the probabilities of the events.

$$P(A \cup B) = P(A) + P(B)$$

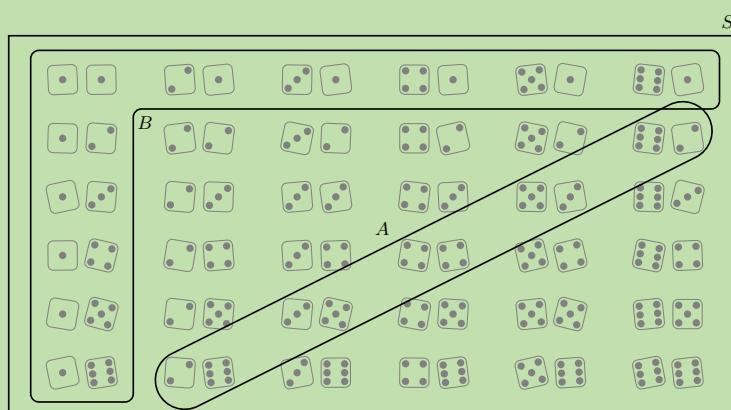
This relationship is true for mutually exclusive events only.

**Worked example 7: Mutually exclusive events****QUESTION**

We roll two dice and are interested in the following two events:

- $A$  : The sum of the dice equals 8
- $B$  : At least one of the dice shows a 1

Show that the events are mutually exclusive.

**SOLUTION****Step 1: Draw the sample space and the two events****Step 2: Determine the intersection**

From the above figure we notice that there are no elements in common in  $A$  and  $B$ . Therefore the events are mutually exclusive.

### Exercise 14 – 6:

State whether the following events are mutually exclusive or not.

1. A fridge contains orange juice, apple juice and grape juice. A cooldrink is chosen at random from the fridge. Event A: the cooldrink is orange juice. Event B: the cooldrink is apple juice.
2. A packet of cupcakes contains chocolate cupcakes, vanilla cupcakes and red velvet cupcakes. A cupcake is chosen at random from the packet. Event A: the cupcake is red velvet. Event B: the cupcake is vanilla.
3. A card is chosen at random from a deck of cards. Event A: the card is a red card. Event B: the card is a picture card.
4. A cricket team plays a game. Event A: they win the game. Event B: they lose the game.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. 2GWR    2. 2GWS    3. 2GWT    4. 2GWV



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## 14.7 Complementary events

EMA84

**DEFINITION:** *Complementary set*

The complement of a set,  $A$ , is a new set that contains all of the elements that are not in  $A$ . We write the complement of  $A$  as  $A'$ , or sometimes  $\text{not } (A)$ .

For an experiment with sample space  $S$  and an event  $A$  we can derive some identities for complementary events. Since every element in  $A$  is not in  $A'$ , we know that complementary events are mutually exclusive.

$$A \cap A' = \emptyset$$

Since every element in the sample space is either in  $A$  or in  $A'$ , the union of complementary events covers the sample space.

$$A \cup A' = S$$

From the previous two identities, we also know that the probabilities of complementary events sum to 1.

$$P(A) + P(A') = P(A \cup A') = P(S) = 1$$

### Worked example 8: Reasoning with Venn diagrams

#### QUESTION

In a survey 70 people were questioned about which product they use: A or B or both. The report of the survey shows that 25 people use product A, 35 people use product B and 15 people use neither. Use a Venn diagram to work out how many people:

1. use product A only
2. use product B only
3. use both product A and product B

## SOLUTION

### Step 1: Summarise the sizes of the sample space, the event sets, their union and their intersection

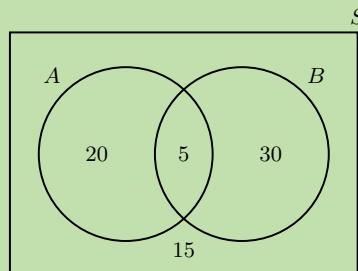
- We are told that 70 people were questioned, so the size of the sample space is  $n(S) = 70$ .
- We are told that 25 people use product A, so  $n(A) = 25$ .
- We are told that 35 people use product B, so  $n(B) = 35$ .
- We are told that 15 people use neither product. This means that  $70 - 15 = 55$  people use at least one of the two products, so  $n(A \cup B) = 55$ .
- We are not told how many people use both products, so we have to work out the size of the intersection,  $A \cap B$ , by using the identity for the union of two events:

$$\begin{aligned}P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \frac{n(A \cup B)}{n(S)} &= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} \\ \frac{55}{70} &= \frac{25}{70} + \frac{35}{70} - \frac{n(A \cap B)}{70} \\ \therefore n(A \cap B) &= 25 + 35 - 55 \\ &= 5\end{aligned}$$

### Step 2: Determine whether the events are mutually exclusive

Since the intersection of the events,  $A \cap B$ , is not empty, the events are not mutually exclusive. This means that their circles should overlap in the Venn diagram.

### Step 3: Draw the Venn diagram and fill in the numbers

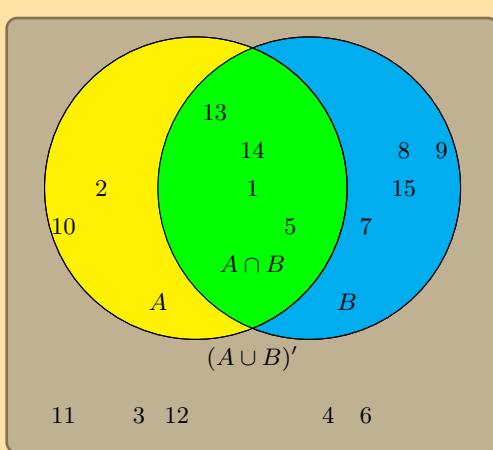


### Step 4: Read off the answers

1. 20 people use product A only.
2. 30 people use product B only.
3. 5 people use both products.

### Exercise 14 – 7:

1. A group of learners are given the following Venn diagram:



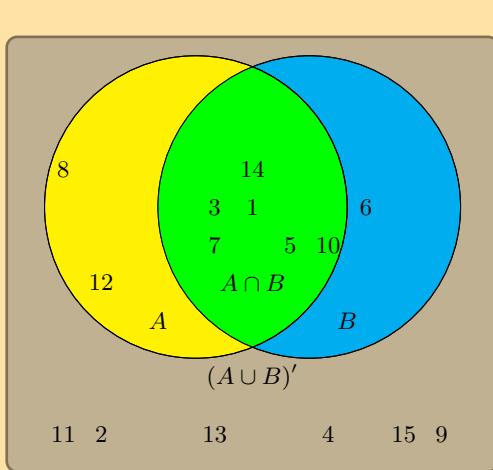
The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ .

They are asked to identify the complementary event set of  $B$ , also known as  $B'$ . They get stuck, and you offer to help them find it.

Which of the following sets best describes the event set of  $B'$ ?

- {1; 5; 13; 14}
- {2; 3; 4; 6; 10; 11; 12}
- {3; 4; 6; 11; 12}

2. A group of learners are given the following Venn diagram:



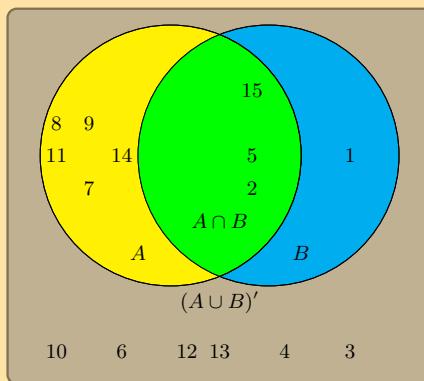
The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ .

They are asked to identify the complementary event set of  $(A ∪ B)$ , also known as  $(A ∪ B)'$ . They get stuck, and you offer to help them find it.

Which of the following sets best describes the event set of  $(A ∪ B)'$ ?

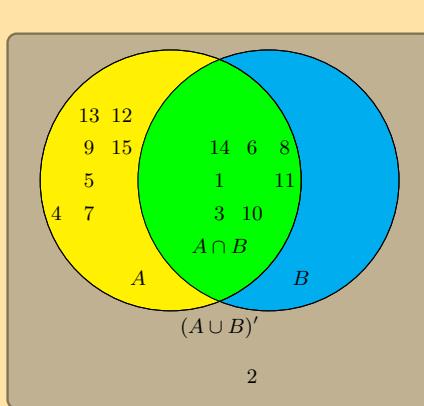
- {2; 4; 9; 11; 13; 15}
- {1; 3; 5; 6; 7; 8; 10; 12; 14}
- {6; 8; 12}

3. Given the following Venn diagram:



The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ . Are  $(A \cup B)'$  and  $A \cup B$  mutually exclusive?

4. Given the following Venn diagram:



The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ . Are  $A'$  and  $B'$  mutually exclusive?

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

1. [2GWW](#) 2. [2GXW](#) 3. [2GWY](#) 4. [2GWZ](#)



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## 14.8 Chapter summary

EMA85

⌚ See presentation: [2GX2](#) at [www.everythingmaths.co.za](http://www.everythingmaths.co.za)

- An experiment refers to an uncertain process.
- An outcome of an experiment is a single result of that experiment.
- The sample space of an experiment is the set of all possible outcomes of that experiment. The sample space is denoted with the symbol  $S$  and the size of the sample space (the total number of possible outcomes) is denoted with  $n(S)$ .
- An event is a specific set of outcomes of an experiment that you are interested in. An event is denoted with the letter  $E$  and the number of outcomes in the event with  $n(E)$ .
- A probability is a real number between 0 and 1 that describes how likely it is that an event will occur.

- A probability of 0 means that an event will never occur.
- A probability of 1 means that an event will always occur.
- A probability of 0,5 means that an event will occur half the time, or 1 time out of every 2.
- A probability can also be written as a percentage or as a fraction.
- When all of the possible outcomes of an experiment have an equal chance of occurring, we can compute the exact theoretical probability of an event. The probability of an event is the ratio between the number of outcomes in the event set and the number of possible outcomes in the sample space.

$$P(E) = \frac{n(E)}{n(S)}$$

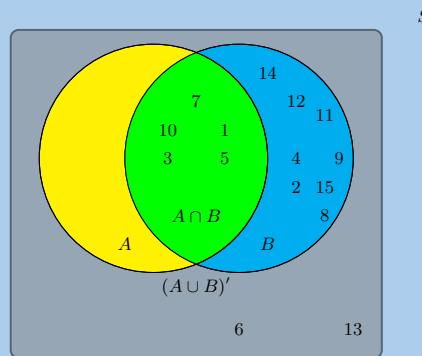
- The relative frequency of an event is defined as the number of times that the event occurs during experimental trials, divided by the total number of trials conducted.

$$f = \frac{\text{number of positive trials}}{\text{number of trials}} = \frac{p}{n}$$

- The union of two sets is a new set that contains all of the elements that are in at least one of the two sets. The union is written as  $A \cup B$  or  $A$  or  $B$ .
- The intersection of two sets is a new set that contains all of the elements that are in both sets. The intersection is written as  $A \cap B$  or  $A$  and  $B$ .
- The probability of observing an outcome from the sample space is 1:  $P(S) = 1$ .
- The probability of the union of two events is calculated using:  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .
- Mutually exclusive events are two events that cannot occur at the same time. Whenever an outcome of an experiment is in the first event, it can not also be in the second event.
- The complement of a set,  $A$ , is a different set that contains all of the elements that are not in  $A$ . We write the complement of  $A$  as  $A'$  or "not ( $A$ )".
- Complementary events are mutually exclusive:  $A \cap A' = \emptyset$ .
- Complementary events cover the sample space:  $A \cup A' = S$
- Probabilities of complementary events sum to 1:  $P(A) + P(A') = P(A \cup A') = P(S) = 1$ .

#### End of chapter Exercise 14 – 8:

1. A learner wants to understand the term "outcome". So the learner rolls a die. Which of the following is the most appropriate example of the term "outcome"?
  - A teacher walks into the class room.
  - The die lands on the number 5.
  - The clock strikes 3 pm.
2. A group of learners are given the following Venn diagram:

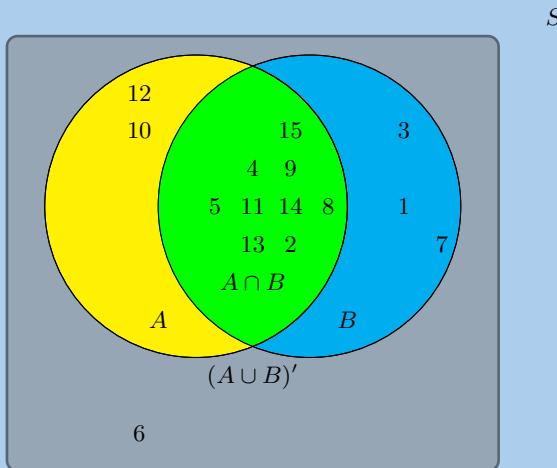


The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ .

They are asked to identify the event set of  $B$ . They get stuck, and you offer to help them find it. Which of the following sets best describes the event set of  $B$ ?

- $\{6; 13\}$
- $\{1; 3; 5; 7; 10\}$
- $\{2; 4; 6; 8; 9; 11; 12; 13; 14; 15\}$
- $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 14; 15\}$

3. A group of learners are given the following Venn diagram:

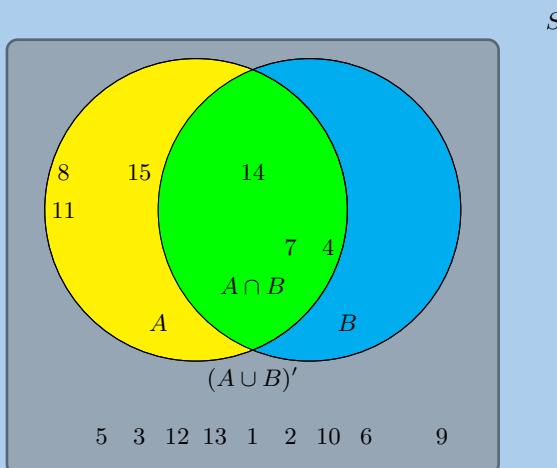


The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ .

They are asked to identify the event set of the union between event set  $A$  and event set  $B$ , also written as  $A \cup B$ . They get stuck, and you offer to help them find it.

Write down the event set that best describes  $A \cup B$ .

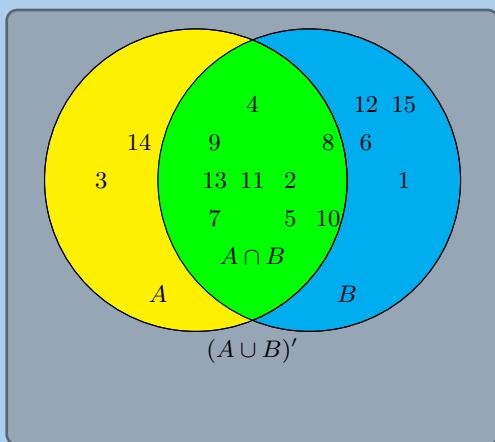
4. Given the following Venn diagram:



The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ .

Are  $A \cup B$  and  $(A \cup B)'$  mutually exclusive?

5. A group of learners are given the following Venn diagram:



The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 15\}$ .

They are asked to identify the complementary event set of  $(A \cap B)$ , also known as  $(A \cap B)'$ . They get stuck, and you offer to help them find it.

Write down the set that best describes the event set of  $(A \cap B)'$ .

6. A learner finds a deck of 52 cards and then takes one card from the deck. What is the probability that the card is a king?

Write your answer as a decimal (correct to 2 decimal places).

7. A die is tossed 21 times and lands 2 times on the number 3.

What is the relative frequency of observing the die land on the number 3? Write your answer correct to 2 decimal places.

8. A coin is tossed 44 times and lands 22 times on heads.

What is the relative frequency of observing the coin land on heads? Write your answer correct to 2 decimal places.

9. A group of 45 children were asked if they eat Frosties, Strawberry Pops or both. 31 children said they eat both and 6 said they only eat Frosties. What is the probability that a child chosen at random will eat only Strawberry Pops?

10. In a group of 42 learners, all but 3 had a packet of chips or a cooldrink or both. If 23 had a packet of chips and 7 of these also had a cooldrink, what is the probability that one learner chosen at random has:

- a) both chips and cooldrink
- b) only cooldrink

11. A box contains coloured blocks. The number of each colour is given in the following table.

Colour	Purple	Orange	White	Pink
Number of blocks	24	32	41	19

A block is selected randomly. What is the probability that the block will be:

- a) purple
- b) purple or white
- c) pink and orange
- d) not orange

12. A small nursery school has a class with children of various ages. The table gives the number of children of each age in the class.

Age	3 years old	4 years old	5 years old
Male	2	7	6
Female	6	5	4

If a child is selected at random what is the probability that the child will be:

- a) a female
  - b) a 4 year old male
  - c) aged 3 or 4
  - d) aged 3 and 4
  - e) not 5
  - f) either 3 or female
13. Fiona has 85 labelled discs, which are numbered from 1 to 85. If a disc is selected at random what is the probability that the disc number:
- a) ends with 5
  - b) is a multiple of 3
  - c) is a multiple of 6
  - d) is number 65
  - e) is not a multiple of 5
  - f) is a multiple of 3 or 4
  - g) is a multiple of 2 and 6
  - h) is number 1
14. Use a Venn diagram to work out the following probabilities for a die being rolled:
- a) a multiple of 5 and an odd number
  - b) a number that is neither a multiple of 5 nor an odd number
  - c) a number which is not a multiple of 5, but is odd
15. A packet has yellow sweets and pink sweets. The probability of taking out a pink sweet is  $\frac{7}{12}$ . What is the probability of taking out a yellow sweet?
16. In a car park with 300 cars, there are 190 Opels. What is the probability that the first car to leave the car park is:
- a) an Opel
  - b) not an Opel
17. Nezi has 18 loose socks in a drawer. Eight of these are plain orange and two are plain pink. The remaining socks are neither orange nor pink. Calculate the probability that the first sock taken out at random is:
- a) orange
  - b) not orange
  - c) pink
  - d) not pink
  - e) orange or pink
  - f) neither orange nor pink
18. A plate contains 9 shortbread cookies, 4 ginger biscuits, 11 chocolate chip cookies and 18 Jambos. If a biscuit is selected at random, what is the probability that:
- a) it is either a ginger biscuit or a Jambo
  - b) it is not a shortbread cookie

19. 280 tickets were sold at a raffle. Jabulile bought 15 tickets. What is the probability that Jabulile:
- wins the prize
  - does not win the prize
20. A group of children were surveyed to see how many had red hair and brown eyes. 44 children had red hair but not brown eyes, 14 children had brown eyes and red hair, 5 children had brown eyes but not red hair and 40 children did not have brown eyes or red hair.
- How many children were in the school?
  - What is the probability that a child chosen at random has:
    - brown eyes
    - red hair
  - A child with brown eyes is chosen randomly. What is the probability that this child will have red hair?
21. A jar has purple sweets, blue sweets and green sweets in it. The probability that a sweet chosen at random will be purple is  $\frac{1}{7}$  and the probability that it will be green is  $\frac{3}{5}$ .
- If I choose a sweet at random what is the probability that it will be:
    - purple or blue
    - green
    - purple
  - If there are 70 sweets in the jar how many purple ones are there?
  - $\frac{2}{5}$  of the purple sweets in (b) have streaks on them and the rest do not. How many purple sweets have streaks?
22. Box A contains 3 cards numbered 1, 2 and 3.  
 Box B contains 2 cards numbered 1 and 2.  
 One card is removed at random from each box.  
 Find the probability that:
- the sum of the numbers is 4.
  - the sum of the two numbers is a prime number.
  - the product of the two numbers is at least 3.
  - the sum is equal to the product.
23. A card is drawn at random from an ordinary pack of 52 playing cards.
- Find the probability that the card drawn is:
    - the three of diamonds
    - the three of diamonds or any heart
    - a diamond or a three
  - The card drawn is the three of diamonds. It is placed on the table and a second card is drawn. What is the probability that the second card drawn is not a diamond.
24. A group of learners is given the following event sets:

Event Set A	3	4
-------------	---	---

Event Set B	2	4	5
-------------	---	---	---

Event Set $A \cup B$	2	3	4	5
----------------------	---	---	---	---

The sample space can be described as  $\{n : n \in \mathbb{Z}, 1 \leq n \leq 6\}$

They are asked to calculate the value of  $P(A \cap B)$ . They get stuck, and you offer to calculate it for them. Give your answer as a decimal number, rounded to two decimal value.

25. For each of the following, draw a Venn diagram to represent the situation and find an example to illustrate the situation.

- a) a sample space in which there are two events that are not mutually exclusive
- b) a sample space in which there are two events that are complementary

26. Use a Venn diagram to prove that the probability of either event A or B occurring (A and B are not mutually exclusive) is given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

27. All the clubs are taken out of a pack of cards. The remaining cards are then shuffled and one card chosen. After being chosen, the card is replaced before the next card is chosen.

- a) What is the sample space?
- b) Find a set to represent the event,  $P$ , of drawing a picture card.
- c) Find a set for the event,  $N$ , of drawing a numbered card.
- d) Represent the above events in a Venn diagram.
- e) What description of the sets  $P$  and  $N$  is suitable? (Hint: Find any elements of  $P$  in  $N$  and of  $N$  in  $P$ .)

28. A survey was conducted at Mutende Primary School to establish how many of the 650 learners buy vetkoek and how many buy sweets during break. The following was found:

- 50 learners bought nothing
- 400 learners bought vetkoek
- 300 learners bought sweets

- a) Represent this information with a Venn diagram.
- b) If a learner is chosen randomly, calculate the probability that this learner buys:
  - i. sweets only
  - ii. vetkoek only
  - iii. neither vetkoek nor sweets
  - iv. vetkoek and sweets
  - v. vetkoek or sweets

29. In a survey at Lwandani Secondary School, 80 people were questioned to find out how many read the Sowetan, how many read the Daily Sun and how many read both. The survey revealed that 45 read the Daily Sun, 30 read the Sowetan and 10 read neither. Use a Venn diagram to find the percentage of people that read:

- a) only the Daily Sun
- b) only the Sowetan
- c) both the Daily Sun and the Sowetan

30. In a class there are

- 8 learners who play football and hockey
- 7 learners who do not play football or hockey
- 13 learners who play hockey
- 19 learners who play football

How many learners are there in the class?

31. Of 36 people, 17 have an interest in reading magazines and 12 have an interest in reading books, 6 have an interest in reading both magazines and books.

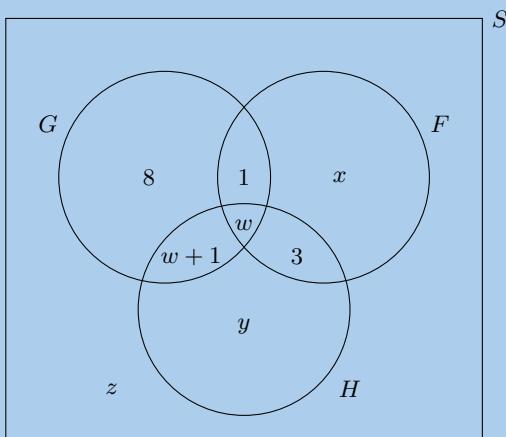
- a) Represent the information in a Venn diagram.
- b) How many people have no interest in reading magazines or books?
- c) If a person is chosen at random from the group, find the probability that the person will:
- have an interest in reading magazines and books.
  - have an interest in reading books only.
  - not have any interest in reading books.
32. 30 learners were surveyed and the following information was revealed from this group:
- 18 learners take Geography
  - 10 learners take French
  - 6 learners take History, but take neither Geography nor French.

In addition the following Venn Diagram has been filled in below:

Let  $G$  be the event that a learner takes Geography.

Let  $F$  be the event that a learner takes French.

Let  $H$  be the event that a learner takes History.



- a) From the information above, determine the values of  $w$ ,  $x$ ,  $y$  and  $z$ .
- b) Determine the probability that a learner chosen at random from this group:
- takes only Geography.
  - takes French and History, but not Geography.

For more exercises, visit [www.everythingmaths.co.za](http://www.everythingmaths.co.za) and click on 'Practise Maths'.

- |                           |                           |                          |                          |                          |                          |                          |                          |
|---------------------------|---------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 1. <a href="#">2GX3</a>   | 2. <a href="#">2GX4</a>   | 3. <a href="#">2GX5</a>  | 4. <a href="#">2GX6</a>  | 5. <a href="#">2GX7</a>  | 6. <a href="#">2GX8</a>  | 7. <a href="#">2GX9</a>  | 8. <a href="#">2GXB</a>  |
| 9. <a href="#">2GXC</a>   | 10. <a href="#">2GXD</a>  | 11. <a href="#">2GXF</a> | 12. <a href="#">2GXG</a> | 13. <a href="#">2GXH</a> | 14. <a href="#">2GXJ</a> | 15. <a href="#">2GXK</a> | 16. <a href="#">2GXM</a> |
| 17. <a href="#">2GXN</a>  | 18. <a href="#">2GXP</a>  | 19. <a href="#">2GXQ</a> | 20. <a href="#">2GXR</a> | 21. <a href="#">2GXS</a> | 22. <a href="#">2GXT</a> | 23. <a href="#">2GXV</a> | 24. <a href="#">2GXW</a> |
| 25a. <a href="#">2GXX</a> | 25b. <a href="#">2GXY</a> | 26. <a href="#">2GXZ</a> | 27. <a href="#">2GY2</a> | 28. <a href="#">2GY3</a> | 29. <a href="#">2GY4</a> | 30. <a href="#">2GY5</a> | 31. <a href="#">2GY6</a> |
| 32. <a href="#">2GY7</a>  |                           |                          |                          |                          |                          |                          |                          |



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# Solutions to exercises

## 1 Algebraic expressions

### Exercise 1 – 1:

- |    |  |   |                           |
|----|--|---|---------------------------|
| 1. | a) $\mathbb{Z}$  | b) rational   | b) $\frac{3}{25}$         |
|    | b) (ii)  | c) rational   | c) $\frac{29}{50}$        |
| 2. | a) between the rectangle and $\mathbb{Z}$                    | d) irrational   | d) $\frac{2589}{10\,000}$ |
|    | b) (ii)  |   |                           |
| 3. | a) real  | 6. a) rational  | 10. a) $0,\dot{1}$        |
|    | b) real  | b) rational   | b) $0,\overline{12}$      |
|    | c) non-real  | c) irrational   | c) $0,\overline{123}$     |
|    | d) undefined   | d) rational   | d) $0,114145$             |
|    | e) non-real  | 7. a) 7 and 11  | 11. a) $0,\dot{5}$        |
|    | f) real  | b) $-\sqrt{8}$ ; $3,3231089\dots$ ; $3 + \sqrt{2}$ ; $\pi$  | b) $0,\dot{5}$            |
| 4. | a) rational  | c) $\sqrt{-1}$  | c) $0,\dot{2}\dot{1}$     |
|    | b) irrational  | d) $-3; 0; -8\frac{4}{5}; \frac{22}{7}; 7; 1,\overline{34}$ | d) $0,\dot{6}$            |
|    | c) rational, an integer, a whole number and a natural number | e) $-3; 7; 11$  | e) $1,\overline{27}$      |
|    | d) irrational  | f) $\frac{14}{0}$   | f) $4,8\dot{3}$           |
|    | e) irrational  | 8. a) (i) 555 (ii) rational                                 | g) $2,\dot{1}$            |
|    | f) irrational  | b) (i) any three digits (ii) irrational                     |                           |
|    | g) rational, an integer, a whole number and a natural number | c) (i) any three digits (ii) irrational                     |                           |
|    | h) irrational  | d) (i) any three digits (ii) irrational                     |                           |
|    | i) irrational  | e) (i) 545 (ii) rational                                    |                           |
| 5. | a) rational  | 9. a) $\frac{1}{10}$  |                           |
|    |  |   | 12. a) $\frac{5}{9}$      |
|    |  |   | b) $\frac{57}{90}$        |
|    |  |   | c) $\frac{4}{9}$          |
|    |  |   | d) $\frac{526}{99}$       |
|    |  |   | e) $\frac{163}{33}$       |
|    |  |   | f) $\frac{130}{33}$       |

### Exercise 1 – 2:

- |    |           |    |               |    |               |               |
|----|-----------|----|---------------|----|---------------|---------------|
| 1. | a) 12,566 | 2. | a) 345,0440   | 3. | a) 9,87       | c) 648 768,22 |
|    | b) 3,317  |    | b) 1361,73    |    | b) 4,93       | 5. 11         |
|    | c) 0,267  |    | c) 728,009052 |    | c) 14,80      | 6. R 5,03     |
|    | d) 1,913  |    | d) 0,0370     |    | d) 14,8044... |               |
|    | e) 6,325  |    | e) 0,45455    | 4. | a) 0,01       |               |
|    | f) 0,056  |    | f) 0,08       |    | b) 519 854,59 |               |

### Exercise 1 – 3:

- |    |            |              |            |   |
|----|------------|--------------|------------|---|
| 1. | a) 4 and 5 | e) 12 and 13 | i) 4 and 5 | c) 3,9  |
|    | b) 5 and 6 | f) 7 and 8   | j) 4 and 5 | d) 3,5  |
|    | c) 1 and 2 | g) 8 and 9   | 2. a) 3,1  |   |
|    | d) 4 and 5 | h) 4 and 5   | b) 9,1     | 3. $-\sqrt{8}; -\sqrt{\frac{9}{4}}; 0,45; 0,45;$<br>$\frac{27}{7}; \sqrt{19}; 6; 2\pi; \sqrt{51}$ |

### Exercise 1 – 4:

- |    |                        |                                 |   |
|----|------------------------|---------------------------------|---|
| 1. | a) $2y^2 + 8y$         | t) $36d^2 - 49$                 | m) $-20y^3 - 116y^2 - 13y + 6$            |
|    | b) $y^2 + 7y + 10$     | u) $25z^2 - 1$                  | n) $-24y^3 + 126y^2 - 3y - 9$             |
|    | c) $2t^2 - 5t + 2$     | v) $1 - 9h^2$                   | o) $-20y^2 - 80y - 30$                    |
|    | d) $x^2 - 16$          | w) $4p^2 + 10p + 6$             | p) $49y^3 + 42y^2 + 79y + 30$             |
|    | e) $x^2 - 16$          | x) $8a^2 + 60a + 28$            | q) $a^3 + 4a^2b + 5ab^2 + 2b^3$           |
|    | f) $a^2 - b^2$         | y) $10r^2 + 28r + 16$           | r) $x^3 + y^3$                            |
|    | g) $6p^2 + 29p + 9$    | z) $w^2 - 1$                    | s) $52m^3 + 6m + 30m^2$                   |
|    | h) $3k^2 + 16k - 12$   | 2. a) $g^2 - 121$               | t) $48x^5 + 16x^2 + 24x^3$                |
|    | i) $s^2 + 12s + 36$    | b) $8b^2 - 20b + 8$             | u) $15k^5 + 9k^3 + 14k^2$                 |
|    | j) $x^2 - 49$          | c) $8b^2 - 10b + 3$             | v) $81x^4 - 72x + 16$                     |
|    | k) $9x^2 - 1$          | d) $18x^2 + 24x - 24$           | w) $-6y^6 + 107y^4 + 3y^3 - 176y^2 - 48y$ |
|    | l) $-14k^2 + 17k + 6$  | e) $6w^2 + 17w - 14$            | x) $x^4 + x^3 - 11x^2 - 9x + 18$          |
|    | m) $16x^2 - 8x + 1$    | f) $4t^2 - 12t + 9$             | y) $-3a^2 + 20a - 12$                     |
|    | n) $y^2 - 2y - 15$     | g) $25p^2 - 80p + 64$           | 3. a) $3x^2 + 16x + 5$                    |
|    | o) $-x^2 + 64$         | h) $16y^2 + 40y + 25$           | b) $2a^4 + 5a^3 - 5a - 2$                 |
|    | p) $x^2 + 18x + 81$    | i) $-10y^7 - 39y^6 - 36y^5$     | c) $-y^4 - 8y^3 - 14y^2 + 8y - 1$         |
|    | q) $84y^2 - 153y + 33$ | j) $72y^2 - 18y + 27$           | d) $2x^3 - 2x^2y - 2xy^2 - 2y^3$          |
|    | r) $g^2 - 10g + 25$    | k) $-10y^3 + 4y^2 + 103y - 132$ | e) $3a^3 - 30ab^2 + 9b^3$                 |
|    | s) $d^2 + 18d + 81$    | l) $-14y^3 + 26y^2 + 4y - 16$   |   |

- f)  $8a^4 - 12a^3b + 2a^2b^2 + 3ab^3 - b^4$   
 g)  $9x^2 + 6xy - 3y^2$   
 h)  $5x^2 - 6xy - 2y^2$   
 i)  $\frac{x^2}{12} + \frac{7}{12} + \frac{3}{x^2}$   
 j)  $\frac{x^2}{3} + \frac{10}{3} - \frac{8}{x^2}$   
 k)  $10x - 12y$   
 l)  $2a^2 + 3ab + 2a + 3b$   
 4. 4
5. -4  
 6. a) 5  
     b)  $k = -3$  or  $k = -1$   
     c)  $k < 0$   
     d)  $k > 4$   
 7. a)  $x^2 + 8 + \frac{16}{x^2}$   
     b) 6  
 8. a)  $a^2 + 2 + \frac{1}{a^2}$
- b) 9  
 c) 13  
 9. a)  $9y^2 + 3 + \frac{1}{4y^2}$   
     b) 16  
 10. a)  $a^2 + \frac{2}{3} + \frac{1}{9a^2}$   
     b)  $a^3 + \frac{1}{27a^3}$   
     c) 6

### Exercise 1 – 5:

1.  $4(3x + 8y)$   
 2.  $-2ab(b + 2a)$   
 3.  $3b(6a - c)$   
 4.  $6k(2j + 3q)$   
 5.  $12a(-1 + 2a^2)$
6.  $-2a(b + 4)$   
 7.  $8kj(3 - 2k)$   
 8.  $-ab(a + b)$   
 9.  $18b^2q(4 - bq)$   
 10.  $5(5x^3 - y)(5x^3 + y)$

11.  $2x(3x + 1 + 5x^2)$   
 12.  $xy(2y + yz + 3)$   
 13.  $12k^2j(1 + 2j)$   
 14.  $3(a^2 + 2a - 6)$   
 15.  $7a + 4$

### Exercise 1 – 6:

1.  $(y - 3)(4 - k)$   
 2.  $(a - 1)(a - 5)(a + 5)$   
 3.  $(b + 4)(m)(b - 6)$   
 4.  $(a + 7)(a^2 + 9)$   
 5.  $(b - 4)(3b + 7)$   
 6.  $(z + 6)(3g + 2)$   
 7.  $(y + 2)(4b + 5)$
8.  $(r + 5)(3d + 14)$   
 9.  $(6x + y - 3)(6x + y + 3)$   
 10.  $3(2x - y)(3y - 2x)$   
 11.  $(4a + 3b + 4c)(4a - 3b - 4c)$   
 12.  $(-2b + 11)(4b - 19)$   
 13.  $(29 - 26a)(30a - 41)$   
 14.  $(4k - 2)(4k + 2)$

15.  $(abc - 1)(abc + 1)$   
 16.  $(\frac{1}{3}a - 2b)(\frac{1}{3}a + 2b)$   
 17.  $2(\frac{1}{2}x + 1)(\frac{1}{2}x - 1)$   
 18.  $(y - \sqrt{8})(y + \sqrt{8})$   
 19.  $(y - \sqrt{13})(y + \sqrt{13})$   
 20.  $(a - 3b)(a - 5b)(a + 3b)^2$

### Exercise 1 – 7:

1.  $(2d - 3r)(3 + t^5)$   
 2.  $(z - 2m)(9 + b^3)$   
 3.  $(7z - 2y)(5 + c^5)$   
 4.  $(3 + a)(2x + 1)$   
 5.  $(x + 5)(x - 6)$   
 6.  $(5 - a)(x + 2y)$
7.  $(a - x)(a - 2)$   
 8.  $(y + 2)(5x - 3)$   
 9.  $(-a + b)(a + 1)$   
 10.  $(7m - 2n)(2 + j)$   
 11.  $(7r - 5x)(4 + g)$   
 12.  $(5d - 3m)(5 + y)$

13.  $(5q - 2z)(9 + c)$   
 14.  $(2j - 5v)(3 + y)$   
 15.  $(2a - 5k)(8 + z)$   
 16.  $(a - b)(x + y + 2)$   
 17.  $(3a + b)(x - y - 3)$

### Exercise 1 – 8:

1.  $(x + 5)(x + 3)$   
 2.  $(x + 8)(x + 1)$   
 3.  $(x + 6)^2$   
 4.  $(h + 3)(2h - 1)$   
 5.  $(x + 1)(3x + 1)$   
 6.  $(s + 2)(3s - 5)$   
 7.  $(x + 3)(x - 5)$   
 8.  $(x + 3)(x - 1)$   
 9.  $(x + 5)(x - 4)$
10.  $(x - 5)(x + 4)$   
 11.  $2(x + 1)(x + 10)$   
 12.  $2(a + 1)(3a + 4)$   
 13.  $3(2v - 3)(v - 3)$   
 14.  $3(g - 3)(2g + 1)$   
 15.  $(3x + 1)(x + 6)$   
 16.  $(3x - 1)(x + 6)$   
 17.  $(7x + 1)(x - 1)$   
 18.  $3(2x + 1)(x - 3)$

19.  $(a - 4b)(a - 3b)$   
 20.  $(3a - 4b)(a + 3b)$   
 21.  $2((7x + 2)(7x - 1))$   
 22.  $(x - 6)(x - 5)$   
 23.  $(a - 7)(a - 1)$   
 24.  $(y - 3)(y + 6)$   
 25.  $3(b + 4)(b + 1)$   
 26.  $6(a + 7)(a - 4)$

### Exercise 1 – 9:

1.  $(w - 2)(w^2 + 2w + 4)$   
 2.  $(g + 4)(g^2 - 4g + 16)$   
 3.  $(h + 1)(h^2 - h + 1)$   
 4.  $(x + 2)(x^2 - 2x + 4)$   
 5.  $(3 - m)(9 + 3m + m^2)$   
 6.  $2(x - y)(x^2 + xy + y^2)$   
 7.  $3(k + 3q)(k^2 - 3kq + 9q^2)$   
 8.  $(4t - 1)(16t^2 + 4t + 1)$   
 9.  $(8x - 1)(8x + 1)$   
 10.  $(5x + 1)(25x^2 - 5x + 1)$
11.  $(\sqrt[3]{25}x + 1)((\sqrt[3]{25})^2x^2 - \sqrt[3]{25}x + 1)$   
 12.  $(z)(1 - 5z)(1 + 5z + 25z^2)$   
 13.  $(2m^2 + n^3)(4m^4 - 2m^2n^3 + n^6)$   
 14.  $(6n - k)(36n^2 + 6nk + k^2)$   
 15.  $(5s + d)(25s^2 - 5sd + d^2)$   
 16.  $(2k + r)(4k^2 - 2kr + r^2)$   
 17.  $(2jkl - b)(4j^2k^2l^2 + 2jklabc + b^2)$   
 18.  $(3xy + w)(9x^2y^2 - 3xyw + w^2)$   
 19.  $2(4m + f)(16m^2 - 4mf + f^2)$   
 20.  $(p^5 - \frac{1}{2}y^4)(p^{10} + \frac{1}{2}p^5y^4 + \frac{1}{4}y^8)$

21.  $(\frac{3}{t} - s)(\frac{9}{t^2} + \frac{3s}{t} + s^2)$   
 22.  $(\frac{1}{4q} - h)(\frac{1}{16q^2} + \frac{h}{4q} + h^2)$   
 23.  $\frac{1}{3}(6g + v)(36g^2 - 6gv + v^2)$   
 24.  $(1 - x + y)(1 - x + y + x^2 - 2xy + y^2)$   
 25.  $(h - 1)(h + 1)(h^2 + 1)(2g^2 + h)(4g^4 - 2g^2h + h^2)$   
 26.  $(x + y)(5w - h)(25w^2 + 5wh + h^2)$   
 27.  $(x - 6)(x + 1)(3p + w)(9p^2 - 3pw + w^2)$

### Exercise 1 – 10:

1. a)  $\frac{a}{5}$   
b)  $\frac{a+5}{2}$   
c) 5  
d)  $a$   
e)  $\frac{3a}{2}$   
f)  $\frac{a+3}{a+2}$   
g)  $\frac{a(3b+1)}{b}$   
h)  $\frac{4xy}{3}$   
i)  $\frac{p(y-2)}{3y}$   
j)  $\frac{3x+4}{2}$   
k)  $\frac{-b+9}{2}$   
l)  $\frac{t+s}{t-s}$   
m)  $\frac{x+3}{5}$   
n)  $\frac{x-3}{x+3}$   
o)  $\frac{x+2}{x^2+3x+9}$   
p)  $\frac{a+8}{a^2+2a+4}$

q)  $\frac{a-6b}{a+2b}$   
r)  $\frac{2a-3}{b}$   
s)  $\frac{2x+1}{x(x+1)}$   
t)  $q + 16$   
u)  $p + 5$   
v)  $h + 13$   
w)  $af$

2. a) 5  
b) 3  
c) 4  
d)  $\frac{3(a+3)}{98}$   
e)  $\frac{4a^2(a-5)}{6(a+5)^2}$   
f)  $\frac{(3x+4)^2}{96p^2}$   
g)  $\frac{4(3a-1)}{3(a-1)}$   
h)  $2a$   
i)  $\frac{3q}{8}$   
j)  $\frac{30b^3}{4(a+b)}$

3. a)  $\frac{x-27}{12}$   
b)  $\frac{47-x}{36}$   
c)  $\frac{5x-8}{12}$   
d)  $\frac{19a}{(a+11)(a-8)}$   
e)  $\frac{6x}{(x-12)(x-6)}$   
f)  $\frac{20r}{(r+12)(r-8)}$   
g)  $\frac{2z+4y+3x}{xyz}$   
h)  $\frac{4t-12}{(t-2)(t-3)}$

4. a)  $x \neq 2$   
b)  $x \neq -1$   
c)  $x \neq 0$  and  $x \neq \pm 1$

### Exercise 1 – 11:

1. a) between the rectangle and  $\mathbb{Z}$   
b) (i)
2. a) non-real  
b) undefined  
c) real  
d) real  
e) non-real  
f) real
3. a) Irrational  
b) Irrational  
c) Rational  
d) Rational
4. a) rational  
b) rational  
c) irrational  
d) irrational
5. a)  $\sqrt{-24}$   
b)  $-\sqrt{24} ; \frac{3}{2} ; \frac{\pi}{2} ; \sqrt[3]{26} ; \pi ; \sqrt{39} ; 7,1\bar{1} ; 7,12 ; \sqrt{78} ; 9 ; 3\pi ; \pi^2$   
c)  $-\sqrt{24} ; \frac{\pi}{2} ; \sqrt[3]{26} ; \pi ; \sqrt{39} ; \sqrt{78} ; 3\pi ; \pi^2$   
d)  $\frac{3}{2} ; 7,1\bar{1} ; 7,12 ; 9$   
e) 9  
f)  $\frac{\sqrt{2}}{0}$
6. a)  $\frac{3}{25}$   
b)  $\frac{3}{500}$   
c)  $\frac{410}{99}$   
d)  $1\frac{59}{100}$   
e)  $\frac{221}{18}$   
f)  $\frac{37}{45}$   
g)  $\frac{81}{11}$
8. a) 0,0̄5  
b) 1,5
9.  $\frac{78}{99}$
10. a) (i) any three numbers (ii) irrational  
b) (i) 111 (ii) rational
11. a) 0,50  
b) 1,00  
c) 0,11
12. a) 3,142  
b) 1,618  
c) 1,414  
d) 2,718
13. 1523,0020
14. 1982,940290
15. 101,5238
16. a) 1,414  
b) 1,732  
c) 2,236  
d) 2,449
17. a) Irrational number.  
b) Irrational number.  
c) Rational number.  
d) Rational number.  
e) Rational number.  
f) Rational number.  
g) Irrational number.  
h) Rational number.  
i) Rational number.  
j) Irrational number.
18. a) 0,71  
b) 3,742
20. a) 2 and 3  
b) 3 and 4  
c) 4 and 5  
d) 5 and 6  
e) 1 and 2  
f) 2 and 3  
g) 2 and 3  
h) 3 and 4  
i) 9 and 10  
j) 8 and 9
21. a) 3,7  
b) 10,5  
c) 6,9  
d) 4,5
22. a)  $a^2 + 10a + 25$   
b)  $n^2 + 24n + 144$   
c)  $d^2 - 8d + 16$   
d)  $49w^2 - 4$   
e)  $144q^2 - 1$   
f)  $x^2 + 4x + 4$   
g)  $25k^2 - 16$   
h)  $10f^2 + 18f + 8$   
i)  $18n^2 + 51n + 30$   
j)  $2g^2 + 18g + 36$   
k)  $16y^2 + 36y + 8$   
l)  $7d^2 - 19d - 6$   
m)  $6z^2 - 16z + 8$   
n)  $25w^2 - 110w + 121$   
o)  $25s^2 - 10s + 1$   
p)  $9d^2 - 48d + 64$   
q)  $36f^3 + 25f^2 + 49f$   
r)  $74d^4 + 16d + 24d^2$   
s)  $59x^3 + 10x^2 + 49x$
23. a)  $y^6 - y^5 + y^4 - 2y^3 - 7y^2 - 2y$   
b)  $4x$   
c)  $x^4 - 2x^2 + 1$   
d)  $64a^3 - 27b^3$   
e)  $2x^3 + 4x^2y - 8xy^2 - 6y^3$   
f)  $9a^4 + 9a^3 - 34a^2b^2 + 25ab^3 - 25b^4$   
g)  $y^2 - \frac{1}{y^2}$   
h)  $\frac{a^2}{9} - \frac{3}{a^2}$   
i)  $6x$   
j)  $-4x - 8$
24. 4
25. a) -1 ; 0 ; 1 ; 6  
b) 0 ; 1 ; 6  
c)  $k > 0$   
d)  $k > -2$
26. a)  $9a^2 + 3 + \frac{1}{4a^2}$   
b)  $27a^3 - \frac{1}{8a^3}$   
c)  $374\frac{1}{2}$   
d) 64

- b) 25  
c) 2400800  
d) 100  
28. a)  $11 \times 13$   
b)  $2^3 \times 3 \times 7$   
c)  $29 \times 31$   
d)  $3^2 \times 11$   
e)  $3 \times 13 \times 41$   
29. a)  $(a - 3)(a + 3)$   
b)  $9(b - 3)(b + 3)$   
c)  $(m - \frac{1}{3})(m + \frac{1}{3})$   
d)  $5(1 - ab^3)(1 + ab^3)$   
e)  $b(2a - 3)(2a + 3)(4a^2 + 9)$   
f)  $(a - 5)(a - 5)$   
g)  $(4b + 7)(4b + 7)$   
h)  $-4b^2(6b^3 - 1)^2$   
i)  $(4 + x^2)(2 + x)(2 - x)$   
j)  $7(x - 2)(x + y)$   
k)  $(y - 10)(y + 3)$   
l)  $(1 - x)^2(1 + x)$   
m)  $(1 + p)(-2 + 3p)$   
n)  $x(x - 2) + (1 + y)(1 - y)(1 + y^2)$   
o)  $(x - 1)(x^2 + x + 1)(4b - x)$   
p)  $(v - 7)(3m + 19)$   
q)  $(3f + 19)(z + 3)$   
r)  $3(p - \frac{1}{3})(p^2 + \frac{p}{3} + \frac{1}{9})$   
s)  $(2x^2 - 5y^3)(4x^4 + 10x^2y^3 + 25y^6)$   
t)  $(-p)(12 + 18p + 7p^2)$   
u)  $\frac{(a+3b)^2(a-3b)}{3}$   
v)  $(2a - 5)(3a - 1)$   
w)  $(s - 3)(s + 5)$   
x)  $(2v + 3h)(8 + j^5)$
30. a)  $2(a + 1)(3a + 4)$   
b)  $3(g - 3)(2g + 1)$   
c)  $(5g - r)(25g^2 + 5gr + r^2)$   
d)  $(2r + z)(4r^2 - 2rz + z^2)$   
e)  $(7m - 2n)(2 + j)$   
f)  $(5d - 3m)(5 + y)$   
g)  $(g - 3)(g^2 + 3g + 9)$   
h)  $(z + 5)(z^2 - 5z + 25)$   
i)  $3(b - a)(3a - b)$   
j)  $(4x + 5y)(y - 4x)$   
k)  $4(4x^3 - 3y^4)(4x^3 + 3y^4)$   
l)  $\frac{1}{6}(a - 12b^2)(a + 12b^2)$   
m)  $(a - 3)(2 - 9x)(2 + 9x)$   
n)  $(b + 3)(b - 10)$   
o)  $(2x + 5y)(x + y)$   
p)  $(x - 5y)(x + 3y)$   
q)  $(4x^2 + 3)(x^2 + 2)$   
r)  $2(3x^2 - 4)(x^2 - 5)$   
s)  $(3a + b)(3a - b)(x + y + 3)$   
t)  $2(2y + 3)(y - 4)$   
u)  $\frac{(x-4)(x-3)(x+3)}{2}$   
v)  $(3rs - 1)(9r^2s^2 + 3rs + 1)$   
w)  $\left(\frac{1}{5h} + r\right)\left(\frac{1}{25h^2} - \frac{r}{5h} + r^2\right)$   
x)  $(j + k)(4n - b)(16n^2 + 4nb + b^2)$   
31. a)  $-8a + 4$   
b)  $125a^3 - 64b^3$   
c)  $16m^4 - 81$   
d)  $a^2 + 4ab + 4b^2 - c^2$   
e) 2
- f) 5  
g)  $-\frac{2+b}{3}$   
h)  $\frac{1}{x-2}$   
i)  $\frac{x-7}{3}$   
j) 5  
k)  $\frac{a-5}{(a+2)^2}$   
l)  $\frac{1}{a^2+6a+36}$   
m)  $-2(2a + b)$   
n)  $s + 31$   
o)  $n + 8$   
p)  $p^2 - 2pq + q^2$   
q)  $\frac{12-x^2}{6x}$   
r)  $\frac{-14}{(a+7)(a-7)}$   
s)  $\frac{32x^3+x+2}{2x^3}$   
t)  $\frac{4a-1}{(2a+1)(2a-1)(a-1)}$   
u)  $\frac{5x+20}{6}$   
v)  $\frac{4x^3+11x^2+7x+3}{x(x+2)(x+3)}$   
w)  $\frac{2(b^2+9)}{(b-3)(b+3)}$   
x)  $\frac{x(x+1)}{x^2+x+6}$   
y)  $\frac{17z}{(z+12)(z-5)}$   
z)  $\frac{7w}{(w-11)(w-4)}$
32.  $(3x - 4)(x + 2)$   
33.  $5x$   
34.  $8,85$   
35.  $a^2 + 2ab + 4b^2$   
36.  $9x^2 - 3x + 1$   
37. a)  $x \neq \frac{1}{3}$  and  $x \neq -1$   
b)  $a \neq b$  and  $a \neq -3$

## 2 Exponents

### Exercise 2 – 1:

1. 1  
2.  $16$   
3.  $11^{11}x$   
4.  $10^8x$   
5.  $216c^3$   
6.  $125n^3$   
7.  $\frac{1}{36}$   
8. 40  
9.  $\frac{27}{8}$   
10.  $a^3$
11.  $\frac{1}{x^3y^4}$   
12.  $x^{3t+3}$   
13.  $3^{2a+3}$   
14. 1  
15. 2  
16.  $6a^5b^2$   
17.  $56m^{10}n^9$   
18.  $-72a^{16}b^{16}$   
19.  $-\frac{1}{5}x^{14}y^{19}$   
20.  $a^{2x}$
21.  $5ax$   
22.  $2c^4p^3$   
23.  $3a^5m^5$   
24. 27  
25.  $7a^2$   
26.  $9a^5b^{27}$   
27.  $\frac{1}{9}$   
28.  $\frac{a^{30}}{b^{35}}$   
29.  $8t^{12}$   
30.  $3^{2n+6}$
31.  $\frac{1}{27}$   
32. 85  
33.  $3^{9x+3}$   
34.  $4^x + 12^b$   
35.  $5^{7y-8} \times 2^{9y-1}$   
36.  $\frac{2^{17}}{3^25^3}$   
37. 12  
38.  $\frac{51}{49}$   
39.  $\frac{4y-25y}{3}$

### Exercise 2 – 2:

1.  $3t^2$   
2.  $8x$
3.  $\frac{1}{2}$   
4.  $\frac{1}{3}$
5.  $3p^3$   
6.  $96a^3b^5$   
7.  $4a^3b$
8.  $\frac{b^3}{a}$   
9.  $2 \cdot 2^{\frac{1}{3}}a^4b^2$

### Exercise 2 – 3:

1. a) 0  
b)  $-\frac{5}{2}$   
c)  $-7$   
d)  $\frac{5}{9}$
- e) 6  
f)  $-2$   
g) 4  
h)  $-\frac{1}{2}$   
i)  $\frac{4}{5}$
- j) 3  
k)  $x = 2$  or  $x = 3$   
l)  $x = 3$  or  $x = 1$   
m)  $x = 0$  or  $x = 4$   
n) 1  
o)  $\frac{3}{2}$  or 0  
p) 1  
q)  $-\frac{7}{2}$   
r)  $-3$   
s)  $-3$

t) 3 or -1  
u) 3

v) -2  
2. 7

3.  $x \approx 2,81$   
4.  $x \approx 1,49$

### Exercise 2 – 4:

- |                               |                                  |  |   |
|-------------------------------|----------------------------------|--|---|
| 1. a) $512x^3$                | y) $-27$                         | c) 1                                     | 6. a) The sum of two powers of the same degree is not the power of the sum of the bases                                       |
| b) $2t^3$                     | z) $\frac{6}{5}$                 | d) $\frac{2}{3}$                         | b) The sum of two powers of the same degree is not the power of the sum of the bases  |
| c) $5^{5x+y+3z}$              | 2. a) $\frac{1}{3}$              | e) -9                                    | c) A negative sign is missing, when a power is moved from the denominator to the numerator, the sign of the exponent changes. |
| d) $15^{15x}$                 | b) $\frac{1}{8}$                 | f) -5                                    | d) We cannot multiply bases unless they are raised to the same power  |
| e) 7                          | c) 14                            | g) $\frac{1}{2}$                         | e) The sign of a base is not changed when an exponent is moved from the denominator to the numerator in a fraction            |
| f) $21d^7$                    | d) $2^{2p} - 2^p + 1$            | h) -1                                    | f) The power of a product is the product of all the bases raised to the same power  |
| g) $-\frac{1}{7}a^{15}b^{12}$ | e) $a^5b^3$                      | i) 16                                    | 7. $25 \times 10^{2013}$ 2015 digits.   |
| h) $b^{k^2+k}$                | f) $3x^4y^2$                     | j) $\frac{1}{16}$                        |   |
| i) $4c^6m^2$                  | g) 34                            | k) -1                                    |   |
| j) 2                          | h) $3^{20z+2}$                   | l) 1 or 16                               |   |
| k) $\frac{1}{7a^{10}b}$       | i) $11^b - 4^p$                  | m) 2                                     |   |
| l) $\frac{a^{14}}{b^8}$       | j) $11^{2c-2} \times 2^{14c-4}$  | n) $k = \frac{1}{81}$                    |   |
| m) $2^{5p} \cdot 3^{3p}$      | k) $\frac{3^4}{2^{13} \times 5}$ | o) $x = 81$                              |   |
| n) $27m^t$                    | l) $\frac{81}{50}$               | p) $\frac{1}{2}$                         |   |
| o) $\frac{1}{3x^5}$           | m) $\frac{9}{10}$                | q) $x = 3 \text{ or } x = 2$             |   |
| p) $\frac{1}{625}$            | n) $108a^6b^9$                   | r) $x = 2$                               |   |
| q) $\frac{27}{4}$             | o) $22p^{10}q^6$                 | s) $x = 1 \text{ or } x = -1$            |   |
| r) $\frac{1}{27}$             | p) $-(ab)^{-1}$                  | t) $x = -8$                              |   |
| s) $\frac{1}{27}$             | q) $x^6$                         | u) $x = 1$                               |   |
| t) $\frac{y^2}{x^5}$          | r) $(\frac{2}{3})^{2y}$          | v) $x = -1$                              |   |
| u) -8                         | s) 4                             | w) $x = -\frac{1}{3} \text{ or } x = -1$ |   |
| v) $8x^{6a}y^{3b}$            | 3. a) -3                         | x) no solution                           |   |
| w) $4^{x+3}$                  | b) 3                             | 4. $x \approx 2,73$                      |   |
| x) 22                         |                                  | 5. $x \approx 3,10$                      |   |

### 3 Number patterns

#### Exercise 3 – 1:

- |  |                                      |   |
|--|--------------------------------------|---|
| 2. $T_{n-1} = -1$                          | g) $T_4 = 42b; T_5 = 46b; T_6 = 50b$ | c) = 23   |
| 3. $T_{n-4} = C$                           | 6. $T_6 = 28 \text{ and } T_9 = 43$  | 13. a) 76 ; 106 ; 136                                       |
| 4. a) no common difference                 | 7. $T_5 = G \text{ and } T_8 = J$    | b) $30n - 74$   |
| b) $d = 7$                                 | 8. $T_5 = 23 \text{ and } T_8 = 35$  | c) = 16   |
| c) no common difference                    | 9. a) 8                              | 14. a) No common difference                                 |
| d) $d = -0,65$                             | b) -1                                | b) $T_1 = -3 \text{ and } T_2 = 3$                          |
| 5. a) 35, 45 and 55                        | c) -9 and -5                         | 15. a) $d = -2n - 3$  |
| b) 7, 12 and 17                            | d) 28                                | b) $T_1 = 2 \text{ and } T_3 = 0$                           |
| c) 21, 18 and 15                           | e) 27                                | 16. a) $-\frac{5}{3}$                                       |
| d) $T_4 = -28,1; T_5 = -33,1; T_6 = -38,1$ | 11. a) 13 triangles                  | b) $-\frac{14}{9}, \frac{43}{9} \text{ and } \frac{100}{9}$ |
| e) $T_4 = -39x; T_5 = -49x; T_6 = -59x$    | b) $2n + 3$                          | 17. a) $-\frac{1}{2}$                                       |
| f) $T_4 = 44,2; T_5 = 64,2; T_6 = 84,2$    | c) 53                                | b) -2, -3 and -4  |
|  | 12. a) 47 ; 55 ; 63                  | 18. E   |
|  | b) $8n + 7$                          |   |

#### Exercise 3 – 2:

- |  |   |                         |
|--|---|-------------------------|
| 2. $d = -3$                                    | b) $T_6 = -11$  | b) $T_n = 3n - 5$       |
| 3. no common difference                        | c) $T_6 = 18,9$   | c) $T_n = 4n + 7$       |
| 4. $T_{n-3} = 7$                               | 9. a) $T_n = -4n - 14, T_{10} = -54, T_{15} = -74, T_{30} = -134$ | d) $T_n = \frac{1}{3}n$ |
| 5. a) $T_4 = -67,2; T_5 = -86,2; T_6 = -105,2$ | b) $T_n = -7n + 8, T_{10} = -62, T_{15} = -97, T_{30} = -202$     | 12. a) -49 ; -63 ; 77   |
| b) $T_4 = 38r; T_5 = 34r; T_6 = 30r$           | 10. a) 12 and 16  | b) $= -14n + 7$         |
| 6. $T_6 = 31 \text{ and } T_8 = 41$            | b) -10  | c) = 66                 |
| 7. $T_6 = F \text{ and } T_{10} = J$           | c) 23 and 53  | 13. M                   |
| 8. a) $T_6 = 49$                               | 11. a) $T_n = 4n - 1$   | 14. C                   |
|  |   | 15. 111 seats           |
|  |   | 16. a) 17 boxes         |

- b)  $T_n = 3n - 1$   
c) 89  
17. a)  $T_1 = 4$   
b) 3  
c)  $T_n = 3n + 1$   
d) 76 matchsticks  
18. 10
19. a)  $d = 3y + 3$   
b)  $T_1 = -7$  and  $T_2 = -1$   
20. a)  $-\frac{1}{3}$   
b)  $-\frac{1}{6}, \frac{3}{2}$  and  $\frac{19}{6}$   
21.  $T_{85} = 7227$  blocks  
22. a) 29
- b)  $T_n = (n + 1)^2 + n$   
c)  $T_{14} = 239$  blocks  
23. 77 parts  
24. a) 7  
b) 0  
c) 4

## 4 Equations and inequalities

### Exercise 4 – 1:

- |                   |                      |                     |                      |                       |                      |
|-------------------|----------------------|---------------------|----------------------|-----------------------|----------------------|
| 1. 5              | 7. 12                | 13. 1               | 19. 5                | 25. 10                | 31. $\frac{223}{45}$ |
| 2. -8             | 8. 11                | 14. 3               | 20. 4                | 26. $-\frac{22}{6}$   | 32. 12               |
| 3. -1             | 9. $\frac{217}{20}$  | 15. 6               | 21. $-\frac{36}{11}$ | 27. 6                 | 33. $-\frac{17}{13}$ |
| 4. -3             | 10. 15               | 16. $\frac{17}{12}$ | 22. 8                | 28. -2                | 34. -19              |
| 5. 0              | 11. $\frac{9}{10}$   | 17. $\frac{20}{3}$  | 23. 5                | 29. -12,5             | 35. $-\frac{1}{6}$   |
| 6. $-\frac{7}{8}$ | 12. $-\frac{50}{13}$ | 18. 26              | 24. 24               | 30. $\frac{280}{451}$ |                      |

### Exercise 4 – 2:

- |  |  |  |
|--|--|--|
| 1. a) $5r^2 + 16r = 0$                     | l) $x = -\frac{3}{2}$ or $x = 4$           | 3. a) $y = 3$ or $y = -3$                  |
| b) $6r^2 - 25r + 39 = 0$                   | m) $x = \frac{6}{5}$ or $x = \frac{8}{3}$  | b) $z = -\frac{1}{5}$ or $z = \frac{1}{2}$ |
| c) $2d^2 + 15d + 17 = 0$                   | n) $x = 4$ or $x = 11$                     | c) $x = 3$ or $x = -6$                     |
| 2. a) $x = -5$ or $x = 3$                  | o) 2                                       | d) $y = \frac{1}{4}$ or $y = 4$            |
| b) $p = -2$ or $p = 9$                     | p) $x = \frac{3}{5}$ or $x = 3$            | e) $b = -\frac{1}{3}b$ or $b = 4$          |
| c) $x = -\frac{2}{3}$ or $x = \frac{4}{3}$ | q) $t = 0$ or $t = 3$                      | f) $y = -\frac{4}{3}$ or $y = 1$           |
| d) $x = \frac{9}{5}$ or $x = -6$           | r) 5                                       | g) $x = 4$ or $x = -4$                     |
| e) $z = -2$ or $z = -1$                    | s) $x = \sqrt{18}$ or $x = -\sqrt{18}$     | h) $z = 1$ or $z = -1$                     |
| f) $b = 3$ or $b = 4$                      | t) $p = 7$ or $p = -1$                     | i) $b = \pm 2$ or $b = \pm 3$              |
| g) $a = 3$ or $a = 6$                      | u) $x = -\frac{11}{4}$ or $x = 7$          | j) $a = -\frac{43}{8}$ or $a = 1$          |
| h) $y = \frac{3}{2}$ or $y = -\frac{3}{2}$ | v) $x = -\frac{6}{7}$ or $x = \frac{1}{2}$ | k) $x = 3$                                 |
| i) $x = \frac{1}{2}$ or $x = -\frac{9}{2}$ | w) $x = 3$ or $x = -2$                     | l) $x = 4$                                 |
| j) $\frac{3}{2}$                           | x) $a = -\frac{1}{2}$ or $a = 3,5$         | m) $a = -2$                                |
| k) $m = 0$ or $m = -\frac{4}{5}$           | y) $x = 11$ or $x = 1$                     | n) $a = 0$ or $a = \frac{2}{3}$            |

### Exercise 4 – 3:

- |   |   |   |
|---|---|---|
| 1. $x = -2$ and $y = -3$                          | f) $x = 11$ and $y = 5$                           | $y < 2$   |
| 2. there is no solution to $x$ and $y$            | g) $x = 9$ and $y = 2$                            | m) $a$ and $b$ can be any real number except for 0. |
| 3. $x = 2$ and $y = -3$                           | h) $a = \frac{16}{3}$ and $b = \frac{4}{3}$       | 5. a) $x = -1$ and $y = 2$                          |
| 4. a) $x = \frac{1}{10}$ and $y = -\frac{43}{50}$ | i) $x = -\frac{1}{10}$ and $y = -2$               | b) $x = 9$ and $y = -4$                             |
| b) $x = -7$ and $y = -\frac{3}{2}$                | j) $x = -\frac{7}{5}$ and $y = \frac{8}{5}$       | c) $x = -1$ and $y = -4$                            |
| c) $x = 5$ and $y = 3$                            | k) $x = \frac{1}{7}$ and $y = -\frac{1}{4}$       | d) $x = 2$ and $y = 1$                              |
| d) $x = -1$ and $y = -1$                          | l) $x$ can be any real number, $\frac{1}{2} \leq$ | e) $x = 1,5$ and $y = 3,5$                          |
| e) $x = 13$ and $y = -1$                          |   |   |

### Exercise 4 – 4:

- |  |                                    |  |
|--|------------------------------------|--|
| 1. 2 hours   | 8. $30^\circ$ and $60^\circ$       | 15. 8 years old                                      |
| 2. 1 hour  | 9. $b = 8$ cm and $l = 2b = 16$ cm | 16. 7 and 35 years old.                              |
| 3. Zwelibanzi achieved 80 marks and Jessica achieved 68 marks. | 10. 7 or -3                        | 17. $x = -\frac{1}{2}$                               |
| 4. 18 large shirts and 2 small shirts                          | 11. length: 6 cm, width: 4 cm      | 18. $x = 2$ or $x = -3$                              |
| 6. -34   | 12. $\frac{19}{50}$ litres         | 19. 15   |
| 7. a milkshake costs R 34 and a wrap costs R 27.               | 13. 9 and 11                       | 20. 34   |
|  | 14. $\frac{1}{2}$                  | 21. 48 blue beads, 96 red beads and 36 purple beads, |

### Exercise 4 – 5:

- |                              |                                     |                                      |
|------------------------------|-------------------------------------|--------------------------------------|
| 1. $x = 1 - 2y$              | 4. $x = \frac{-2b^2}{1-2b}$         | 7. $\frac{A-2\pi r}{2\pi r} = h$     |
| 2. $\frac{2(s-ut)}{t^2} = a$ | 5. $\pm \sqrt{\frac{V}{\pi h}} = r$ | 8. $\lambda = \frac{D}{tf}$          |
| 3. $\frac{pV}{RT} = n$       | 6. $\frac{E\lambda}{c} = h$         | 9. $\frac{E}{gh+\frac{1}{2}v^2} = m$ |

10.  $x = -a$  or  $x = -b$

11.  $b = \pm\sqrt{c^2 - a^2}$

12.  $U = \frac{VW}{W-V}$

13.  $r = \pm\sqrt{\frac{A-\pi R^2}{\pi}}$

14.  $C = \frac{5}{9}(F - 32^\circ)$

15.  $r = \sqrt[3]{\frac{3V}{4\pi}}$

16.  $x = a + 4$  or  $x = -1$

17.  $x = 2$  or  $x = -2$

18.  $x = 3,91$

19.  $u = 2,6$

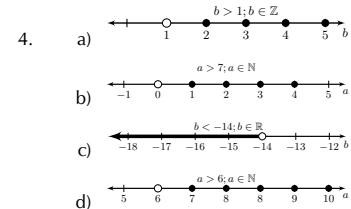
20.  $h = \pm 16$

21. 6

**Exercise 4 – 6:**

1. a)  $x < -1$  and  $x \geq 6$ ;  $x \in \mathbb{R}$   
b)  $3 < x < 6$ ;  $x \in \mathbb{R}$   
c)  $x \neq 3$ ;  $x \neq 6$ ;  $x \in \mathbb{R}$   
d)  $x > -10$ ;  $x \in \mathbb{R}$
- 2.
3. a)  $(-34; \infty)$   
b)  $(-\infty; 5)$

- c)  $x \in [\frac{29}{13}; \infty)$ .  
d)  $x \in (-\infty; \frac{6}{5}]$ .  
e)  $(-\infty; -\frac{55}{13})$   
f)  $(-\infty; -\frac{21}{11}]$   
g)  $(-\infty; -\frac{3}{2}) \cup (\frac{1}{2}; \infty)$   
h)  $[-\frac{5}{3}; -\frac{1}{2}]$



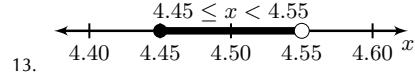
**Exercise 4 – 7:**

1. a) 1  
b) -1  
c) 1  
d) -8  
e)  $\frac{1}{2}$   
f)  $\frac{11}{54}$   
g) -2  
h) -2  
i) 10  
j)  $-\frac{8}{3}$   
k) 42  
l) 5  
m)  $\frac{1}{3}$
2. a)  $b = -9$  or  $b = 3$   
b)  $x = -1$  or  $x = 6$   
c)  $b = -5$  or  $b = 2$   
d)  $\frac{9}{7}$   
e)  $x = -\frac{1}{5}$  or  $x = 3$   
f) 1  
g) -32  
h) -23  
i)  $x = 2$  or  $x = 1$   
j)  $y = -3$  or  $y = 2$   
k)  $x = -\frac{9}{2}$  or  $x = 2$   
l)  $d = 4$
3.  $x = -1$  and  $y = -1$
4. there is no solution to  $x$  and  $y$ .
5.  $x = -1$  and  $y = 3$
6. a)  $x = 1$  and  $y = 2$   
b)  $x = 4$  and  $y = 2$   
c)  $x = 5$  and  $y = -2$   
d)  $x = 6$  and  $y = -5$   
e)  $x = -6$  and  $y = 4$   
f)  $x = 3$  and  $y = 6$   
g)  $x = -7$  and  $y = 5$   
h)  $x = 14$  and  $y = 2$   
i)  $x = -8$  and  $y = -5$   
j)  $x = 1$  and  $y = -1$   
k)  $x = \frac{5}{12}$  and  $y = \frac{31}{12}$   
l)  $x = -\frac{31}{38}$  and  $y = \frac{41}{38}$   
m) no solution  
n)  $a$  and  $b$  can be any real number except for 0  
o)  $x$  and  $y$  can be any real number except for 0  
p)  $x = 48$  and  $y = 7$

- q)  $x = -3$  and  $y = 4$   
r) a)  $\frac{120}{13}$   
b) each ruler costs R 5 and each pen costs R 3  
c) The price of the hotdog is R 31 while a milkshake costs R 25.  
d) Lefu has 77 marks and Monique has 89 marks  
e) 0,56 km  
f) The trucks will meet after 1  
g) 8 km  
h) R 20  
i) 33  
j) 12  
k)  $x = 5$
  8. a)  $x = \frac{a-c}{b}, b \neq 0$   
b)  $\frac{P}{V} = I$   
c)  $\frac{E}{c^2} = m$   
d)  $\frac{v-u}{a} = t$   
e)  $f = \frac{uv}{v+u}$   
f)  $mx + c = y$   
g) If  $(a+b+c) \neq 0$  then  $x = 4 - b$  If  $a+b+c = 0$ ,  $x \in \mathbb{R}$   
h)  $r = \frac{13}{15}$   
i)  $b = \pm 8$
  9. a)  $x < -1$  and  $x \geq 4$ ;  $x \in \mathbb{R}$   
b)  $x \geq -2$ ;  $x \in \mathbb{R}$   
c)  $-1 < x \leq -2$ ;  $x \in \mathbb{R}$
  10. a)  $(-\infty; -\frac{29}{2})$   
b)  $x \in (-\infty; -\frac{8}{5}]$ .  
c)  $[\frac{80}{31}; \infty)$   
d)  $(-\infty; \frac{1}{3}) \cup (\frac{17}{3}; \infty)$
- 
- 
- 
11. a)  $x < 4; x \in \mathbb{N}$   
b)  $b > 4; b \in \mathbb{Z}$   
c)  $x < 0; x \in \mathbb{R}$
  12. a)  $x = -\frac{317}{76}$   
b)  $x = -2$  or  $x = 10$   
c)  $a = 5$  or  $a = -11$   
d)  $x = 2$  or  $x = -2$   
e)  $a = 4$  or  $a = -3$   
f)  $a = 2$  or  $a = -3$   
g)  $a = 2$  or  $a = -9$   
h)  $b = \pm\sqrt{3}$  or  $b = \pm 1$   
i)  $y = \pm\frac{2}{3}$  or  $y = \pm 1$
- Solutions
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- j)  $b = 4$   
k)  $a = 1$   
l)  $x \geq 2$   
m)  $x < -1$   
n)  $x > \frac{588}{10}$   
o)  $a \leq -7$   
p)  $-3 \leq k < 2$   
q)  $x = 7$  or  $x = -6$   
r)  $x = -1$  or  $x = -2$   
s)  $x = \frac{3}{8}$

- t)  $x = \frac{a}{(a-b)}$  for  $a, b \neq 0$  and  $a \neq b$   
u)  $x = -\frac{2}{3}y$  or  $x = y$   
v)  $x = \frac{1}{2}y$  or  $x = -1$   
w) no solution since  $x \neq 4$   
x) no solution since  $\sqrt{-1}$   
y)  $x > -13$



13.

## 5 Trigonometry

### Exercise 5 – 1:

$$\begin{array}{ll} 3. \quad \text{a) } \cos \hat{O} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{m}{o} & \text{d) } \cos \hat{M} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{o}{n} \\ \text{b) } \tan \hat{M} = \frac{\text{opposite}}{\text{adjacent}} = \frac{m}{o} & 5. \quad \cos \hat{P} = \frac{r}{q} \\ \text{c) } \sin \hat{O} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{o}{n} & 6. \quad \sin \alpha = \frac{4}{5} \end{array}$$

### Exercise 5 – 2:

- |            |          |          |              |             |          |
|------------|----------|----------|--------------|-------------|----------|
| 1. a) 2,14 | g) -0,82 | m) 1,56  | s) 0,67      | x) 0,88     | c) true  |
| b) 0,62    | h) -1,48 | n) -1,19 | t) 5,01      | y) non-real | d) false |
| c) 0,28    | i) 2,37  | o) -1,59 | u) 0,91      | z) 0,21     | 3. 71,57 |
| d) 0,21    | j) 1,02  | p) -1,39 | v) undefined | 2. a) true  |          |
| e) 0,90    | k) 1,49  | q) 0,23  | w) 2,13      | b) false    |          |
| f) 1,15    | l) 1,60  | r) 2,52  |              |             |          |

### Exercise 5 – 3:

$$\begin{array}{llllll} 1. \quad \text{a) } \frac{1}{\sqrt{2}} & \text{d) } \frac{\sqrt{3}}{1} & \text{h) } \frac{1}{2} & \text{b) } \frac{3}{2} & \text{b) } \frac{-1}{2\sqrt{3}} & \text{f) } \frac{-\sqrt{3}}{2} \\ \text{b) } \frac{1}{\sqrt{2}} & \text{e) } \frac{1}{\sqrt{2}} & 2. \quad \frac{1}{\sqrt{2}} & \text{c) } \frac{\sqrt{3}-1}{2} & \text{c) } -1 & \text{g) } \frac{-\sqrt{3}}{2} \\ \text{c) } \frac{1}{\sqrt{3}} & \text{f) } \frac{1}{\sqrt{3}} & 3. \quad \frac{1}{\sqrt{3}} & 5. \quad \text{a) } \frac{2}{\sqrt{3}} & \text{d) } \frac{\sqrt{2}}{1} & 6. \\ \text{g) } \frac{1}{\sqrt{3}} & \text{4. a) } \frac{1}{2} & & \text{b) } \frac{2}{\sqrt{3}} & \text{e) } 2 & \text{7.} \end{array}$$

### Exercise 5 – 4:

- |             |          |                                       |                                    |
|-------------|----------|---------------------------------------|------------------------------------|
| 1. a) 36,11 | e) 3,51  | i) 2,87                               | c) $\frac{AC}{BC} = \frac{AD}{AB}$ |
| b) 8,91     | f) 33,43 | j) 9,06                               | 3. $MN = 12,86$ and $NP = 15,32$   |
| c) 10,90    | g) 29,46 | 2. a) $\frac{AC}{AB} = \frac{AD}{BD}$ | 4. $x = 29,82$ and $y = 31,98$     |
| d) 21,65    | h) 10    | b) $\frac{AD}{BD} = \frac{CD}{AD}$    |                                    |

### Exercise 5 – 5:

1.  $23,96^\circ$       2.  $35,23^\circ$       3.  $39,40^\circ$       4.  $26,31^\circ$       5.  $36,87^\circ$       6.  $45^\circ$       7.  $30^\circ$

### Exercise 5 – 6:

- |                    |                  |              |                     |
|--------------------|------------------|--------------|---------------------|
| 1. a) $59,5^\circ$ | i) $18,1^\circ$  | 2. a) -0,342 | 3. a) $54,49^\circ$ |
| b) $53,1^\circ$    | j) $40,5^\circ$  | b) 0,827     | b) $90^\circ$       |
| c) $71,3^\circ$    | k) no solution   | c) -0,440    | c) $36,12^\circ$    |
| d) $76,6^\circ$    | l) $18,4^\circ$  | d) 1,770     | d) $63,07^\circ$    |
| e) $80,1^\circ$    | m) no solution   | e) 0,242     | e) $-18,97^\circ$   |
| f) $41,8^\circ$    | n) $109,9^\circ$ | f) -2,924    | f) $43^\circ$       |
| g) no solution     | o) $26,6^\circ$  | g) 1,614     |                     |
| h) $41,4^\circ$    | p) $17,7^\circ$  | h) 0,625     |                     |

### Exercise 5 – 7:

- |                           |                              |                                |
|---------------------------|------------------------------|--------------------------------|
| 1. a) $\sqrt{10}$         | d) -3                        | 3. a) $\frac{\sqrt{t^2+4}}{2}$ |
| b) $\frac{1}{\sqrt{10}}$  | 2. a) $\frac{-\sqrt{21}}{5}$ | b) $\frac{2}{t}$               |
| c) $\frac{\sqrt{10}}{-3}$ | b) -2                        | c) $\frac{4}{t^2+4}$           |

4. a)  $\frac{-4}{5}$   
b)  $\frac{-82}{25}$
5. a)  $\frac{-8}{17}$   
b)  $\frac{15}{8}$   
c) 1
6.  $\frac{7}{13}$
7. a)  $\frac{8}{15}$   
b) -45
8. a)  $\sqrt{89}$   
b)  $\frac{8}{\sqrt{89}}$   
c)  $\frac{5}{\sqrt{89}}$   
d)  $\frac{8}{5}$   
e)  $\frac{\sqrt{89}}{8}$
- f)  $\frac{\sqrt{89}}{5}$   
g)  $\frac{5}{8}$   
h) 1
9. a)  $(-24; -7)$  and  $(-48; -14)$   
b)  $a = 4(-24) = -96$  and  $b = 4(-7) = -28$   
c)  $\frac{-7}{25}$
10. -13

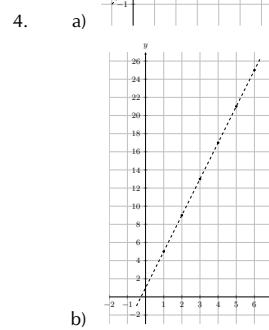
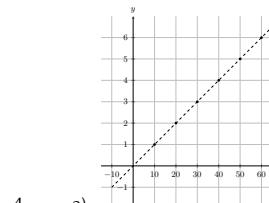
### Exercise 5 – 8:

1. a) not been written correctly  
b) been written correctly  
c) not been written correctly
2. a) 5,67  
b) 0,29  
c) 0,29  
d) -1,07  
e) -0,74  
f) 1,79  
g) 3,08  
h) 1,66  
i) -4,45  
j) 1,79  
k) 2,33  
l) 7,73  
m) -0,23  
n) 0,99  
o) 1,31  
p) 0,05  
q) 10,46  
r) 2,12
3. a)  $\frac{\sqrt{3}}{2}$   
b)  $\frac{1}{2}$   
c)  $\sqrt{3}$   
d)  $\frac{1}{2}$   
e)  $\frac{\sqrt{3}}{2}$   
f)  $\frac{1}{\sqrt{3}}$
4. a)  $\frac{1}{\sqrt{2}}$   
b)  $\frac{1}{\sqrt{2}}$   
c) 1
5. a)  $-\frac{\sqrt{3}}{2}$   
b)  $-\frac{1}{2\sqrt{3}}$   
c)  $-\frac{\sqrt{3}}{2}$   
d)  $\frac{1}{8}$   
e)  $\frac{\sqrt{3}}{\sqrt{2}}$   
f)  $\frac{\sqrt{3}}{2\sqrt{2}}$   
g)  $\frac{\sqrt{3}}{2}$   
h)  $\frac{3}{8}$
6.  $\frac{3}{2}$
7.  $\frac{3}{2\sqrt{3}}$
8.  $\sqrt{3}$
9. 9,96 mm and 8,35 mm
10. a)  $35^\circ$   
b) 4  
c) 7  
d) 6  
e) 3  
f)  $45^\circ$   
g)  $55^\circ$   
h)  $65^\circ$   
i) 5
11. a)  $10 \text{ cm}$   
b)  $25,08^\circ$
12. a)  $44,44^\circ$   
b)  $11,88^\circ$   
c)  $\sqrt{55}$   
d)  $\frac{3}{\sqrt{55}}$   
e)  $\theta = 67,976^\circ$
13.  $27,46 \text{ cm}$
14. a)  $42,07^\circ$   
b)  $63,43^\circ$   
c)  $25^\circ$   
d)  $56,25^\circ$   
e)  $14,32^\circ$   
f)  $30^\circ$   
g)  $59,26^\circ$   
h)  $45^\circ$
15. a)  $4,01$   
b)  $6,39$   
c)  $0,91$   
d)  $1,85$
16. a)  $\frac{-3}{\sqrt{34}}$   
b) -1
17. a)  $33,69^\circ$   
b)  $23,96^\circ$   
c)  $97,12^\circ$
18. a)  $8,7 \text{ cm}, 5,65 \text{ cm}$  and  $5,65 \text{ cm}$
19. a) Area of  $\triangle ABC = 16944 \text{ units}^2$

## 6 Functions

### Exercise 6 – 1:

1. a)  $\{x : x \in \mathbb{R}, x \leq 7\}$   
b)  $\{x : x \in \mathbb{R}, -13 \leq x < 4\}$   
c)  $\{x : x \in \mathbb{R}, x > 35\}$   
d)  $\{x : x \in \mathbb{R}, \frac{3}{4} \leq x < 21\}$   
e)  $\{x : x \in \mathbb{R}, -\frac{1}{2} \leq x \leq \frac{1}{2}\}$   
f)  $\{x : x \in \mathbb{R}, x > -\sqrt{3}\}$
2. a)  $(-\infty; 6]$   
b)  $(-5; 5)$   
c)  $(\frac{1}{5}; \infty)$   
d)  $[21; 41)$
3. a)  $y = 5x$   
b)  $y = 5$   
c)  $y = \frac{1}{2}x$



6. a) 2

- b) -11  
c) -2  
d) 3  
e) 8  
f) 1  
g) 10  
h) 3
7. a) R 108,88  
b) R 199,36
- c) 22,043 L  
d) 22,071 L  
e) 1,15V
8. a) 7 m  
b)  $0 \text{ m} \leq s(t) \leq 10 \text{ m}$   
c) The domain is  $0 \text{ s} \leq t \leq 20 \text{ s}$ . It represents the total time taken to reach the bottom of the slope.
9. a) 5 m  
b) 10 m  
c) 5 m

#### Exercise 6 – 2:

1. a)  $x$ -intercept = 1 and  $y$ -intercept = -1  
b)  $x$ -intercept = -2 and  $y$ -intercept = 2  
c)  $x$ -intercept = 3 and  $y$ -intercept = -3
2.  $m = 3$  and  $c = -1$ .
3.  $m = 1$  and  $c = -1$ .
4. a)  $(0; 1)$  and  $(-1; 0)$ . The graph is increasing  
b)  $(0; -1)$  and  $(1; 0)$ . The graph is increasing  
c)  $(0; -1)$  and  $(\frac{1}{2}; 0)$ . The graph is increasing  
d)  $(0; 1)$  and  $(\frac{1}{3}; 0)$ . The graph is decreasing  
e)  $(0; 2)$  and  $(-3; 0)$ . The graph is increasing  
f)  $(0; 3)$ . The graph is horizontal.  
g)  $(0; 0)$ . The graph is increasing  
h)  $(0; -3)$  and  $(2; 0)$ . The graph is increasing
5. a) False  
b) True  
c) True

- d) False  
e) False
6. a)  $y = -\frac{3}{2}x + \frac{1}{2}$   
b)  $y = -3x + 5$   
c)  $y = \frac{1}{3}x + \frac{4}{3}$   
d)  $y = -2x + 4$
7. a) E  
b) A  
c) B  
d) F  
e) D  
f) C
8. a)  $a(x) = -\frac{3}{4}x + 3$   
b)  $b(x) = \frac{3}{2}x - 6$   
c)  $c(x) = 3$   
d)  $d(x) = -\frac{3}{4}x$

#### Exercise 6 – 3:

1.  $a = -\frac{1}{3}$ ;  $q = 4$   
2.  $a = 2$ ;  $q = -3$   
3. a) -2  
b)  $(-0,63; 0)$  and  $(0,63; 0)$   
4. a) 1  
b)  $(-0,71; 0)$  and  $(0,71; 0)$

5. a)  $h(x)$   
b)  $g(x)$   
c)  $f(x)$   
d)  $k(x)$
6. a)  $h(x)$   
b)  $f(x)$

- c)  $g(x)$   
7. a)  $a = 1$ ;  $p = -9$   
b)  $b = -1$ ;  $q = 23$   
c)  $x \leq -4$  or  $x \geq 4$   
d)  $x \geq 0$

#### Exercise 6 – 4:

1.  $a = -1$  and  $q = 2$   
2.  $a = -2$  and  $q = 3$   
3. a) There is no  $y$ -intercept.  
b)  $-\frac{3}{2}$   
4. a) There is no  $y$ -intercept.

- b) -1  
5. a)  $h(x)$   
b)  $g(x)$   
c)  $k(x)$   
d)  $f(x)$

6. b) the point does lie on the graph  
c) -24  
e)  $y = 0$  and  $x = 0$   
f)  $(-3; 2)$

#### Exercise 6 – 5:

1. a)  $(0; 0,33)$   
b)  $(0,4; 0)$   
2.  $a = -1$  and  $q = 4$   
3.  $a = -1$  and  $q = 5$   
4. a)  $(0; -0,75)$   
b)  $(1; 0)$   
5. a)  $g(x)$
- b)  $k(x)$   
c)  $f(x)$   
d)  $h(x)$
6. b) asymptote  
c)  $y = (\frac{1}{2})^x$   
d)  $(0; 1)$
7. a)  $f(x) = 3^x$

- b)  $h(x) = -3^x$   
c)  $(-\infty; 0)$   
d)  $g(x) = 3^{-x}$   
e)  $j(x) = 2 \cdot 3^x$   
f)  $k(x) = 3^x - 3$

#### Exercise 6 – 6:

1.  $a = \frac{3}{2}$  and  $q = \frac{3}{2}$   
2.  $a = 4$  and  $q = -2$   
3.  $a = \frac{1}{2}$ , and  $q = -1$   
4.  $a = \frac{1}{2}$ , and  $q = 0$   
5.  $a = 3$  and  $q = \frac{1}{3}$

6.  $a = 1$  and  $q = 0$   
7. a)  $h(x)$   
b)  $k(x)$   
c)  $f(x)$   
d)  $g(x)$

8.  $g(x) = 4 \sin \theta$   
9. a)  $E(180^\circ; -2)$  and  $-2 \leq y \leq 0$   
b)  $E(360^\circ; 2)$  and  $0 \leq y \leq 4$   
c)  $E(90^\circ; -0,5)$  and  $-2,5 \leq y \leq -0,5$   
d)  $E(180^\circ; 0,5)$  and  $0,5 \leq y \leq$

- 4,5
11.  $E(180^\circ; 1)$ , range  $y \in \mathbb{R}$  and domain  $0 \leq \theta \leq 360$ ,  $x \neq 90$ ,  $x \neq 270$
13. a)  $y = -2 \cos \theta$   
b)  $y = \sin \theta + 1$
14.  $90^\circ < \theta < 270^\circ$
15.  $0^\circ < \theta < 210^\circ$  and  $330^\circ < \theta < 360^\circ$
16.  $60^\circ < \theta < 300^\circ$
17. a)  $A = (90^\circ; 4)$ ,  $B = (90^\circ; -2)$ ,  $C = (180^\circ; 4)$  and  $D = (180^\circ; -2)$   
b) 2  
c) 2  
d) 2
18. a)  $A = (90^\circ; 1)$ ,  $B = (180^\circ; -3)$ ,  $C = (270^\circ; -1)$  and  $D = (360^\circ; 3)$   
b) 2  
c) 2  
d) 2
19. a)  $A = (90^\circ; 3)$ ,  $B = (90^\circ; 2)$ ,  $C = (180^\circ; -4)$  and  $D = (270^\circ; 2)$   
b) 3  
c) 2  
d) -1

### Exercise 6 – 7:

2. a)  $y = x + 2$  and  $y = -\frac{1}{2}x^2 + 2$   
b)  $y = 8x$  and  $y = \frac{8}{x}$
3. a) (iii)  
b) (iv)
- c) (iii)  
d) (i)

### Exercise 6 – 8:

1. a)  $y = 3x$   
b)  $y = x - 4$
4. a)  $x$ -intercept  $= -\frac{5}{3}$  and  $y$ -intercept  $= -5$   
b)  $x$ -intercept  $= -2$  and  $y$ -intercept  $= 4$
5.  $m = 1$ , and  $y = x - 4$
6. a)  $E$   
b)  $B$   
c)  $A$   
d)  $C$   
e)  $F$   
f)  $D$
7. a) True  
b) False  
c) False
8. a)  $y = \frac{5}{2}x + 3$   
b)  $y = \frac{4}{3}x + \frac{1}{6}$
10. a) 120 mL  
b) 600 mL  
c) 33,33s  
d) 240 mL
11. a) 50km  
b) The domain is  $0 \leq t \leq 120$  min.  
c) The range is  $0 \leq s \leq 100$  km, it represents the total distance travelled.
12.  $a = -\frac{1}{3}$ ;  $q = 6$
13. a) 3  
b)  $(-0,77; 0)$  and  $(0,77; 0)$
14. a)  $k(x)$   
b)  $g(x)$   
c)  $h(x)$   
d)  $f(x)$
15. a)  $g(x)$   
b)  $f(x)$   
c)  $h(x)$
17. a) 8 m  
b) 6 m  
c) 3 m  
d) 1 m
18.  $a = 2$  and  $q = -3$
19. a) no solution  
b)  $\frac{3}{4}$
20. a)  $k(x)$   
b)  $f(x)$   
c)  $g(x)$   
d)  $h(x)$
22. a) Translation by 2 in the positive  $y$ -direction.
16.  $60^\circ < \theta < 300^\circ$
17. a)  $A = (90^\circ; 4)$ ,  $B = (90^\circ; -2)$ ,  $C = (180^\circ; 4)$  and  $D = (180^\circ; -2)$   
b) 2  
c) 3  
d) 1
18. a)  $A = (90^\circ; 1)$ ,  $B = (180^\circ; -3)$ ,  $C = (270^\circ; -1)$  and  $D = (360^\circ; 3)$   
b) 2  
c) 2  
d) -1
19. a)  $A = (90^\circ; 3)$ ,  $B = (90^\circ; 2)$ ,  $C = (180^\circ; -4)$  and  $D = (270^\circ; 2)$   
b) 3  
c) 2  
d) -1
25.  $a = 3$  and  $q = -4$
26. a)  $h(x)$   
b)  $f(x)$   
c)  $k(x)$   
d)  $g(x)$
27. a) -4  
b) -2  
c)  $-\frac{323}{81}$   
d)  $-\frac{7}{10}$   
e)  $\frac{1}{27}$   
f) 77  
g)  $-\frac{3}{2}$
28. a) False  
b) False  
c) True  
d) False  
e) False  
f) True
29. b)  $(-3; 12)$  and  $(2; 2)$   
d)  $y = -2x^2 + 6$
30. 1,6 units
31. a)  $x + y = 15$ ;  $y = x + 3$   
b)  
c) 6 R 5 coins and 9 R 2 coins
32.  $a = \frac{3}{2}$  and  $q = 3$
33.  $a = 3$  and  $q = 0$
34.  $a = 1$  and  $q = -3$
35. a)  $g(x)$   
b)  $f(x)$   
c)  $k(x)$   
d)  $h(x)$
36.  $f(x) = -3,5 \cos \theta$
40. a)  $E = (270^\circ; -2)$  and  $-2 \leq y \leq 2$   
b)  $E = (360^\circ; 2)$  and  $-2 \leq y \leq 2$
41.  $E = (45^\circ; 0)$ , range  $y \in \mathbb{R}$  and domain  $0 \leq x \leq 360$ ,  $x \neq 90$ ,  $\theta \neq 270$
42.  $0^\circ < \theta < 180^\circ$
43.  $0^\circ < \theta < 90^\circ$  and  $270^\circ < \theta < 360^\circ$
44.  $0^\circ < \theta < 360^\circ$
45. a)  $y = 3x$   
b)  $y = -2x^2 + 3$

- c)  $y = \frac{-3}{x}$   
d)  $y = x + 2$   
e)  $y = 5 \sin \theta + 1$   
f)  $y = 2 \times 2^x + 1$   
g)  $y = -\tan \theta - 2$
46. a)  $A(90^\circ; 1)$ ,  $B(90^\circ; -1)$ ,  $C(180^\circ; 2)$  and  $D(360^\circ; 1)$   
b) 0  
c) 1  
d) 5
47. a)  $A(90^\circ; 1)$ ,  $B(270^\circ; 3)$ ,  $C(270^\circ; 1)$  and  $D(360^\circ; 2)$   
b) 2  
c) 3  
d) 2
48. a)  $M(0; 1)$  and  $N(0; -1)$   
b) 2  
c) 1  
d)  $y = 2^{-x}$   
e) Range  $y = 2^x: (0; \infty)$   
Range  $y = -2^x: (-\infty; 0)$
50. a)  $q = 1$
51. b)  $BC = 3 + 4 = 7$  units  
c)  $y = -4^x$   
d)  $y = 4^x + 1$   
e)  $y = 0$   
f) Range  $f(x): (0; \infty)$ , Range  $g(x): (-\infty; 1]$   
g)  $y = x^2 + 4$   
h) Domain  $h: (-\infty; \infty)$ , Range  $h: [-4; \infty)$   
i)  $A(\sqrt{8}; \sqrt{8})$  and  $B(-\sqrt{8}; -\sqrt{8})$   
j)  $2\sqrt{8}$   
k) 8  
l) 2
53. a)  $A(-1; 0)$ ,  $B(1; 0)$ ,  $C(0; 3)$   
b)  $D(2; -9)$   
c)  $y = -6x + 3$   
d)  $\{\theta : 0^\circ \leq \theta \leq 360^\circ, \theta \neq 90^\circ; 270^\circ\}$   
e) 3
54. a)  $y = -2x$  en  $y = x^2 - 3$   
b)  $y = 4x + 1$  and  $y = 3^x$   
c) (iv)  
d) (ii)
55. 56. 57.

## 7 Euclidean geometry

### Exercise 7 – 1:

3.  $70^\circ$   
4. a)  $\hat{x} = 55^\circ$   
b)  $35^\circ$   
c)  $\hat{r} = 135^\circ$   
d)  $45^\circ$   
e)  $\hat{p} = 45^\circ$
5. f)  $EF$  is not parallel to  $CG$   
a)  $\hat{a} = 50^\circ$   
b)  $40^\circ$   
c)  $\hat{c} = 140^\circ$   
d)  $40^\circ$   
e)  $\hat{d} = 40^\circ$
6. f)  $PQ \parallel NR$   
a) no parallel lines  
b) not parallel  
c)  $\therefore TY \parallel MN$   
d) (ii)

### Exercise 7 – 2:

1. a)  $72^\circ$   
b)  $98^\circ$   
c)  $y = 112^\circ$  and  $x = 44^\circ$   
d) 29  
e) 25
2. diagram A  
3. diagram B  
5.  $\triangle PNM \equiv \triangle QSR$ , reason: SAS
6. a)  $\triangle ABC \equiv \triangle EDC$   
b) not congruent  
c) not enough information  
d) not enough information  
e)  $\triangle ABC \equiv \triangle ADC$

### Exercise 7 – 3:

1. a)  $S\hat{R}O$  and  $O\hat{R}Q$   
b)  $2x$   
c)  $28^\circ$

### Exercise 7 – 4:

### Exercise 7 – 5:

### Exercise 7 – 6:

1. rectangle  
2. rhombus  
3. quadrilateral  
4. kite and quadrilateral  
5. square, rectangle, rhombus, parallelogram, kite, trapezium and quadrilateral  
6. 315

### Exercise 7 – 7:

1.  $ED$   
2.  $TS$   
3.  $ED$  and  $AC$   
4. no parallel lines  
5. a)  $91^\circ$   
b)  $\triangle DEC \parallel\!\!\!\parallel \triangle DAB$   
6. a)  $60^\circ$
- b)  $\triangle HKG \parallel\!\!\!\parallel \triangle JKL$   
7. 6  
8. 3  
9. 11  
10. 4  
11. a) 14  
b) 3,5
- c) 5  
d)  $x = 28$  units and  $y = 100^\circ$   
e)  $x = 24^\circ$  and  $y = 13,93$   
13. a)  $MN = -4a + 8$   
b)  $-\frac{5}{2}$   
14. a)  $\frac{a}{3} + 2$   
b) 18

15. a)  $\hat{A} = -4q + 121^\circ$   
b)  $18^\circ$

16. a)  $\hat{K} = -9b + 122^\circ$   
b)  $\frac{16}{3}$

#### Exercise 7 – 8:

1. a) straight angle  
b) obtuse angle  
c) acute angle  
d) right angle  
e) Reflex angle  
f) obtuse angle  
g) straight angle  
h) reflex angle
2. a) False  
b) True  
c) True  
d) False  
e) True  
f) True  
g) True  
h) True  
i) False  
j) False
3. a)  $AB \parallel CD$   
b)  $MN \parallel OP$   
c)  $GH \parallel KL$  and  $GK \parallel HL$
4. a)  $a = 107^\circ, b = 73^\circ, c = 107^\circ, d = 73^\circ$   
b)  $a = b = c = d = 80^\circ$   
c)  $a = 50^\circ, b = 45^\circ, c = 95^\circ, d = 85^\circ$
5. a)  $\hat{x} = 45^\circ$   
b)  $45^\circ$   
c)  $\hat{r} = 135^\circ$   
d)  $45^\circ$   
e)  $\hat{p} = 45^\circ$   
f) parallel
6. diagram B  
7. diagram A  
8. not enough information  
9.  $(\triangle VWU \cong \triangle YZX)$ , reason: SSS  
10. a)  $\triangle ABC \cong \triangle DEF$   
b)  $\triangle GHI \cong \triangle JKL$   
c)  $\triangle MNO \cong \triangle RQP$   
d)  $\triangle QRS$  not congruent  $\triangle TUV$
11. a) 4,24 cm  
b) 12 cm  
c) 7,28 cm  
d) 40 mm
12. a)  $25^\circ$   
b)  $145^\circ$   
c)  $x = 10$  and  $y = 12,5$   
d)  $60^\circ$   
e)  $36^\circ$   
f)  $x = 6$  and  $y = 10$
13.  $x = 2,75$  and  $y = 30^\circ$   
14.  $x = 2,75$  and  $y = 30^\circ$   
15. kite
16. rhombus, parallelogram, kite, trapezium and quadrilateral  
17.  $x = 15^\circ$   
18. a)  $ACDF, ABEF$  and  $BCDE$   
b)  $ACDF, ABEF, BCDE$  and  $BCEF$   
c)  $ACEF$  and  $BCDF$   
d)  $ABEF$  and  $BCDE$   
19.  $TS$   
20.  $ZY$  and  $VX$   
21. a)  $39^\circ$   
b)  $\triangle DEC \parallel \triangle DAB$   
22. a)  $98^\circ$   
b)  $\triangle QMN \parallel \triangle QRP$   
23. 5  
24. 9  
25. no solution  
26. a)  $PR = \frac{z}{4} - 1$   
b)  $z = 12$   
27. a)  $-8b + 119^\circ$   
b)  $\frac{19}{2}$   
28. a)  $-24d + 180^\circ$   
b)  $\frac{20}{3}$   
29. a) 5 and b = 12  
33. b) rhombus  
c)  $44^\circ$

## 8 Analytical geometry

#### Exercise 8 – 1:

1. (3; 3)
2.  $A(3; -4), B(3; -3), C(-3; -4), D(5; -3)$  and  $E(5; -4)$ .
3. E
4. B
5. shape Z
6. shape Z

#### Exercise 8 – 2:

1. 6,71
2. 2,24
3.  $x = 1$
4.  $y = 3,5$
5. a)  $\sqrt{29}$   
b)  $\sqrt{52}$   
c)  $\sqrt{17}$
6. a)  $x = 3$  or  $x = 9$
7.  $p = 3$  or  $p = -9$

#### Exercise 8 – 3:

1. a)  $\frac{9}{11}$   
b)  $\frac{11}{8}$
2.  $-0,5$
3. 1,5
- c)  $\frac{4}{3}$
4. a)  $\frac{-10}{3}$   
b) 5
5. 4
6. 3

#### Exercise 8 – 4:

1. a) parallel  
b) neither  
c) perpendicular
2. a) collinear  
b) collinear  
c) not collinear
3.  $y = -2x + 0,5$
4.  $y = 0,5x - 0,5$
5.  $\frac{1}{2}$
6. 6
7.  $y = x - 2$
8.  $y = -x + 1,5$
9. 4
10.  $-2$
11.  $y = -x + 2$
12.  $y = 2x - 1,5$
13. 3
14.  $-2$
15. 0,5
16.  $-0,5$
17. 6
18.  $\frac{-2}{3}$
19. b) 42,5 units
20. b) (3; 13)  
c)  $y = 2x + 12$

#### Exercise 8 – 5:

1.  $M(1; 0)$

2.  $M(-0,5; -1,25)$

3. a)  $(-1; 6)$

b)  $(14; 32)$

c)  $\left(\frac{2x-3}{2}; \frac{2y-5}{2}\right)$

4.  $(8; 13)$

5.  $S(4; -5)$

**Exercise 8 – 6:**

1.  $A(7; -4), B(3; -3), C(7; 7), D(-2; -8)$  and  $E(-5; 4)$

2.  $A$

3. Shape Z

5. 9,81

6. 3

7. 2

8. 4

9. 4

10. a)  $y = \frac{1}{3}x + 3$   
b)  $\sqrt{40}$

11.  $y = -1,5x - 1$

12. 2

13. 0

14.  $y = -x - 1,5$

15. -4

16.  $(-0,5; 0,5)$

18. a)  $y = -x + 10$   
b)  $y = 3x + 12$   
c)  $y = \frac{1}{6}x + \frac{2}{3}$

19. b) opposite sides are not parallel  
c) do not bisect  
d) ordinary quadrilateral

21. b) isosceles trapezium

22. b) i.  $\sqrt{10}$

ii. 3

c) Trapezium

23.  $H(3; 3)$

24. a)  $\sqrt{34}$   
b)  $\frac{1}{3}$

c)  $\left(\frac{3}{2}; \frac{-1}{2}\right)$

25. c)  $(\frac{7}{2}; \frac{7}{2})$   
d)  $\frac{-2}{3}$

3. a)  $(-1; 6)$

b)  $(14; 32)$

c)  $\left(\frac{2x-3}{2}; \frac{2y-5}{2}\right)$

4.  $(8; 13)$

5.  $S(4; -5)$

29. a)  $\frac{1}{2}$   
c)  $C(11; 8)$

30. a) 9,8  
b)  $\frac{-2}{3}$

31. a)  $a = 1$  and  $b = \frac{7}{2}$   
b)  $y = \frac{4}{3}x$

c)  $(\frac{7}{2}; 1)$   
e)  $C(7; 0)$

32. a)  $-\frac{1}{3}$   
c) 9,49

d)  $y = -x + 6$   
e) 45

33. a) no value of  $k$   
b)  $k = 1$

34. a)  $5\sqrt{5}$   
b)  $-\frac{1}{2}$

c)  $y = -\frac{1}{2}x + \frac{5}{2}$   
d) 1

35. a)  $M(1; 3)$   
b) 2

c)  $-\frac{1}{2}$   
d)  $y = -\frac{1}{2}x + \frac{7}{2}$

e)  $y = 2x + 6$

36. a)  $N(3; 5)$   
b) rhombus

c)  $\frac{1}{4}$   
d)  $-\frac{1}{4}$

e)  $y = -\frac{1}{2}x - 4$

37. a)  $\frac{1}{4}$   
c) 4,1

d)  $\frac{1}{4}$   
e)  $y = \sqrt{41}$

38. a)  $\sqrt{41}$   
b)  $M(1; 2)$

39. a) isosceles  
c)  $\frac{3}{4}$

40. a)  $Q(0; 2)$   
c)  $m_{PR} = 1$

d) 6  
e)  $y = 2x - 3$

41. a) -1  
b)  $(-0,5; -4,5)$

c) square  
d)  $y = 3x - 8$

e)  $y = -\frac{1}{3}x - \frac{4}{3}$

44. a)  $E\left(\frac{1}{2}; -\frac{3}{2}\right)$   
b) ABCD is a parallelogram

c)  $y = \frac{1}{3}x - \frac{5}{3}$   
d)  $y = -3x$

e)  $m_{AC} = -\frac{15}{7}$   
f) not a rhombus

g) 10

45. a) 5  
b)  $7\frac{1}{2}$

c)  $8\frac{1}{3}$

46. a)  $m_{MN} = -\frac{1}{3}$   
b)  $y = -\frac{1}{3}x + 3\frac{2}{3}$

c)  $AB \parallel MN$   
d) 2

e)  $P(x; y) = (8; 12)$

47. a) -13

b)  $(8; 12)$

**9 Finance and growth****Exercise 9 – 1:**

1. R 4025

2. R 5398,80

3. R 1700,00

4. R 7030,80

5. a) R 324

b) R 3937,50

6. R 11 538,46

7. 19

8. 16,25%

9. 5,7% per annum

10. 3,8% per annum

11. 4,3% per annum

12. 20 years

13. 32 years

**Exercise 9 – 2:**

1. R 4044,69

2. R 5930,94

3. R 9327,76

4. R 59 345,13

5. R 24 002,00

6. R 17 942,00

7. 8,45% p.a

8. 4,3% per annum

9. 1,8% per annum

**Exercise 9 – 3:**

1. a) R 3960,00

b) R 4316,40

c) R 359,70

d) R 4756,40

2. a) R 4760,00

b) R 6092,80

c) R 253,87

d) R 6932,80

3. a) R 3825

b) R 4743

c) R 197,63

d) R 5418

4. a) R 12 962,50

b) R 4462,50

c) R 360,07

5. a) R 10 240

b) R 3840

c) R 213,33

6. a) R 6324  
b) R 1224  
c) R 263,50
7. a) R 5400  
b) R 4251,97
8. store A
9. R 156,84  
10. R 210,22

**Exercise 9 – 4:**

1. R 33,28  
2. R 29,61
3. R 22,77  
4. R 14,72
5. R 14,24  
6. R 2174,77
7. R 38,64

**Exercise 9 – 5:**

1. 4 142 255  
2. 4 217 645  
3. 553

**Exercise 9 – 6:**

- |                                       |                                      |  |
|---------------------------------------|--------------------------------------|--|
| 1. a) R 1400<br>b) R 200<br>c) R 100  | 3. a) R 1680<br>b) R 600<br>c) R 480 | 6. R 36 780                                      |
| 2. a) R 1430<br>b) R 260<br>c) R 1040 | 4. a) USA<br>b) Sollie               | 7. a) Brazilian tourists<br>b) Japanese tourists |
|                                       | 5. New York publisher                | 8. 530 523 TZS                                   |

**Exercise 9 – 7:**

- |  |   |  |
|--|---|--|
| 1. R 11 204,10<br>2. R 2470,80<br>3. R 35 087,72<br>4. 3,6% per annum<br>5. 3,6% per annum<br>6. 25 years<br>7. 22 years<br>8. R 938<br>9. R 7319,78<br>10. R 4158,88<br>11. R 44 872<br>12. R 18 731,00<br>13. a) R 205<br>b) R 286,52<br>c) R 128<br>14. 3,3% per annum<br>15. 5,5% per annum<br>16. a) R 534,25<br>b) R 520 | 17. Bank B<br>18. a) R 200<br>b) R 200<br>21. a) R 4800,00<br>b) R 5232,00<br>c) R 436,00<br>d) R 6432,00 | 27. R 24,53<br>28. R 27,49<br>29. R 12,60<br>30. R 8,06<br>31. a) 62,3 million people<br>b) 1,7<br>32. 4 065 346<br>33. 4 083 001<br>34. a) R 2100<br>b) R 840<br>c) R 700 |
|  | 22. a) R 4320<br>b) R 4838,40<br>c) R 403,20<br>d) R 5318,40  | 35. a) R 1960<br>b) R 560<br>c) R 840  |
|  | 23. a) R 10 880<br>b) R 2880<br>c) R 302,22   | 36. R 1840<br>37. UK publisher   |
|  | 24. a) R 13 860<br>b) R 6160<br>c) R 231,00   | 38. 3521,37 BRL<br>40. R 13 343,92   |
|  | 25. 8,5%<br>26. R 1106,04   |  |

## 10 Statistics

**Exercise 10 – 1:**

1. qualitative anecdotal  
2. quantitative discrete  
3. quantitative discrete

**Exercise 10 – 2:**

- |  |   |                                   |
|--|---|-----------------------------------|
| 1. 9,7<br>2. 13<br>3. 6<br>4. a) mean: 13,2; median: 11; mode:<br>8<br>b) mean: 26; median: 25; mode: 24 | c) mean: 11,2; median: 11; mode:<br>11<br>d) mean: 34,29; median: 31; mode:<br>none | 7. 26<br>8. 20<br>9. a) 5<br>b) 7 |
|  | 5. mean: 38,3; median 38; mode 33 and 42.<br>6. 23                                  | 10.<br>11. 36                     |

**Exercise 10 – 3:**

1. 1  
2. 1  
3. 3  
4. 5  
5. E  
6. B  
7. Data Set C  
8. Data Set C

**Exercise 10 – 4:**

- Mean: 52; Modal group:  $50 < m \leq 55$ ; Median group:  $50 < m \leq 55$
  - Mean: 70,66; Modal group:  $65 < t \leq 75$ ; Median group:  $65 < t \leq 75$
  - a)  $700 < x \leq 800$
- b) 33 600  
c) 700  
d) 750  
e) R 588 000

#### Exercise 10 – 5:

- 10
- 9
- 9
- $Q_1 = 6,5; Q_2 = 18;$   
 $Q_3 = 29$

#### Exercise 10 – 6:

#### Exercise 10 – 7:

- quantitative continuous
  - qualitative categorical
  - 10
  - 10
  - 44
  - 6
  - 10
  - 15
  - $x = 19$
  - $\bar{x} = p + 4$
  - 1
  - 4
  - 3
  - C
  - E
  - data set C
  - data set B
  - 18
  - 14
  - a) Mean and mode
  - a) 19,9
  - a) R 182 222,22
  - a) R 100 000
  - c) R 100 000
  - a) The mean and range are 89,2 and 27 respectively.
  - a)  $x = 17$
  - b)  $x$  is any integer with  $x \neq \{1; 3; 4; 5; 9\}$
  - c)  $x = 7$
- d)  $x = 10$   
e)  $x = 4$   
26.  $\{2; 5; 7; 10; 10\}$   
27. a) median = 6,5; mode = 8; mean = 5,45  
b)  $\phi_{21} = 4$   
c)  $\{3; 3; 5; 5; 8; 8; 8\}$  or  $\{3; 4; 4; 5; 8; 8; 8\}$   
28. a) 22; 36,5; 50; 55; 64  
b) no men can be retrenched  
c)  $\bar{x} = 42,643$   
29. a) B  
b) median  
c) 3  
d) brand C

## 11 Trigonometry

#### Exercise 11 – 1:

- 5
- 9
- $53,13^\circ$
- $35,30^\circ$
- 26 m
- 15 m

#### Exercise 11 – 2:

- $19,47^\circ$
- a)  $18^\circ$   
b)  $23^\circ$
- 8
- $44,99^\circ$
- $8^\circ$
- 10 m
- b)  $48,59^\circ$
- b)  $54,69 \text{ m}$
- 7,71 m
- the ship is not safe
- 473,52 m
- $106,62^\circ$  and  $67,38^\circ$
- $11,47 \text{ cm}$  and  $8,03 \text{ cm}$
- 4,24 cm
- a) 5 cm
- a)  $36,53^\circ$   
b)  $34,82 \text{ m}$
- b)  $379,73 \text{ m}$
- $793,77 \text{ m}$
- a)  $11,21 \text{ cm}$   
b)  $56,4^\circ$
- 167 m

## 12 Euclidean geometry

#### Exercise 12 – 1:

- a)  $AECF$  is a parm  
b)  $ABCD$  is a parm
- $AD = EF$
- $34^\circ$
- $105^\circ$
- $42^\circ$

#### Exercise 12 – 2:

- b)  $24 \text{ cm}^2$
  - a) (i) In  $\triangle QRT$  and  $\triangle RST$  side  $RT = RT$ , (ii) common side, (iii)  $QR = TS$  and  $QT = RS$  and (iv) opp. sides
- are equal  
b)  $63^\circ$   
c)  $79^\circ$

## 13 Measurements

#### Exercise 13 – 1:

- a)  $25 \text{ cm}^2$   
b)  $50 \text{ cm}^2$   
c)  $78,54 \text{ cm}^2$   
d)  $40 \text{ cm}^2$
- e)  $60 \text{ cm}^2$   
f)  $12 \text{ cm}^2$   
g)  $43,30 \text{ cm}^2$   
h)  $276 \text{ cm}^2$
- a)  $9\pi z^2 + 12\pi z + 4\pi$   
b)  $-2hz - h$   
c)  $\pi x^2 + 8\pi x + 16\pi$   
d)  $-hx + 2h$

**Exercise 13 – 2:**

1. a)  $344 \text{ cm}^2$   
     b)  $277,82 \text{ cm}^2$   
     c)  $87,96 \text{ cm}^2$
- d)  $471,24 \text{ cm}^2$   
     e)  $270$   
     f)  $532,84 \text{ cm}^2$

2. a)  $24 \text{ L}$   
     b)  $22 \text{ L}$

**Exercise 13 – 3:**

1.  $420 \text{ cm}^3$   
     2.  $500 \text{ cm}^3$
3.  $785,4 \text{ cm}^3$   
     4.  $32,06$

5.  $1056$   
     6.  $552,92$

**Exercise 13 – 4:**

1. a)  $282,7 \text{ cm}^2$   
     b)  $45,6 \text{ cm}^2$   
     c)  $180 \text{ cm}^2$
- d)  $1256,6 \text{ cm}^2$   
     2.  $175,93$   
     3.  $804,25 \text{ square units}$

4.  $189 \text{ square units}$

**Exercise 13 – 5:**

1.  $2144,66 \text{ units}^3$   
     2.  $29,32 \text{ units}^3$   
     3.  $170,67 \text{ units}^3$   
     4. a)  $314,16 \text{ cm}^3$
- b)  $52,0 \text{ cm}^3$   
     c)  $144 \text{ cm}^3$   
     d)  $4188,8 \text{ cm}^3$
5. surface area is  $393 \text{ cm}^2$  and the volume is  $507 \text{ cm}^3$

6. a)  $91,39 \text{ cm}^2$   
     b)  $29,39 \text{ cm}^3$   
     7. surface area is  $190 \text{ cm}^2$  and the volume is  $175 \text{ cm}^3$

**Exercise 13 – 6:**

1.  $\frac{1}{9}$   
     2. 4  
     3. doubles  
     4. a) increases by a factor of 9
- b) increases by a factor of 27  
     5. 64 times  
     6. volume is  $31\ 552 \text{ cm}^3$  and the surface area is  $96\ 112 \text{ cm}^2$ .

**Exercise 13 – 7:**

1. a)  $75 \text{ cm}^2$   
     b)  $153,94 \text{ mm}^2$   
     c)  $252,76 \text{ cm}^2$
2. a)  $15y^2 - 10y$   
     b)  $\frac{15y^2}{2} - 5y$
3. 420  
     4. 210  
     5.  $640,88$   
     6. 360  
     7. 300  
     8.  $1847,26$   
     9.  $615,75$
10. 88 square units  
     11.  $103,67 \text{ square units}$   
     12.  $113,1 \text{ units}^3$   
     13.  $183,26 \text{ units}^3$   
     14.  $130,67 \text{ units cubed}$
15. a)  $A_{\text{cone}} = 126,67 \text{ cm}^2$       $A_{\text{square pyramid}} = 437,26 \text{ cm}^2$       $A_{\text{half sphere}} = 150,80 \text{ cm}^2$   
     b)  $V_{\text{cone}} = 94,25 \text{ cm}^3$       $V_{\text{square pyramid}} = 900 \text{ cm}^3$       $V_{\text{half sphere}} = 134,04 \text{ cm}^3$
16.  $\frac{1}{16}$   
     17.  $\frac{1}{9}$   
     18.  $\frac{1}{8}$
19. a)  $A_{\text{cylinder}} = 351,9 \text{ cm}^2$       $A_{\text{triangular prism}} = 384 \text{ cm}^2$       $A_{\text{rectangular prism}} = 76 \text{ cm}^2$
- b)  $V_{\text{cylinder}} = 502,7 \text{ cm}^3$       $V_{\text{rectangular prism}} = 40 \text{ cm}^3$   
     c)  $A_{\text{cylinder}} = 3166,7 \text{ cm}^2$       $A_{\text{rectangular prism}} = 684 \text{ cm}^2$   
     d)  $V_{\text{cylinder}} = 13\ 571,9 \text{ cm}^3$       $V_{\text{rectangular prism}} = 1080 \text{ cm}^3$
20. a)  $460,64 \text{ cm}^2$   
     b)  $588 \text{ cm}^3$   
     21.  $30\ 159,52 \text{ cm}^2$  and  $301\ 592,88 \text{ cm}^3$   
     22. a)  $3665,19 \text{ cm}^3$  and  $1194,67 \text{ cm}^2$   
     b)  $2148,85 \text{ m}^3$  and  $867,08 \text{ m}^2$   
     c)  $600 \text{ ft}^3$  and  $600,37 \text{ ft}^2$
23. Yes  
     24. a)  $90 \text{ cm}$   
         b)  $261,5 \text{ l}$   
     25. a)  $47\ 123,89 \text{ cm}^3$  and  $1570,80 \text{ cm}^2$   
         b)  $92,38$  and  $180,04$   
         c)  $1200$  and  $886$   
     26. a)  $2412,743$   
         b)  $645,07 \text{ cm}^3$   
     27. b)  $28,57 \text{ units}^2$   
         c)  $285,664 \text{ units}^3$   
     28. b)  $74,626 \text{ cm}^3$   
         a)  $4a^2$   
         b)  $612,00538 \text{ feet}$   
         c)  $1,620$   
         d)  $91\ 661\ 532,5 \text{ feet}^3$  and  $1\ 551\ 425,432 \text{ feet}^2$

**14 Probability****Exercise 14 – 1:**

1. event set =  $\{(2; 6); (3; 5); (4; 4); (5; 3); (6; 2)\}$ . 6. a)  $\frac{1}{2}$   
 2.  $\{1; 2; 3; 4; 5; 6\}$  b)  $\frac{1}{3}$   
 3.  $= \frac{1}{3}$  c)  $\frac{5}{6}$   
 4. 0,50 d)  $\frac{1}{3}$   
 5.  $\frac{1}{5}$  e)  $\frac{3}{13}$   
 7. a)  $\frac{1}{52}$

**Exercise 14 – 2:**

1. 0,11 2. 0,57 3. 0,22

**Exercise 14 – 3:**

1.  $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 13; 14; 15\}$  b)  $\{1; 2; 3; 4; 6; 12\}$  c) 29  
 2.  $\{1; 2; 4; 5; 6; 7; 9; 10; 11; 13; 14; 15\}$  c)  $\{2; 3; 5; 7; 11\}$  d) 2  
 3. a)  $\{1; 2; \dots; 12\}$  5. b) 6

**Exercise 14 – 4:**

1.  $\{7; 10\}$  2.  $\{1; 2; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15\}$

**Exercise 14 – 5:**

1. 0,83 2. 0,17

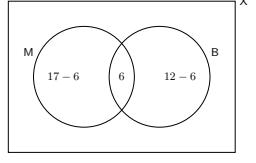
**Exercise 14 – 6:**

1. mutually exclusive 2. mutually exclusive 3. not mutually exclusive 4. mutually exclusive

**Exercise 14 – 7:**

1.  $\{2; 3; 4; 6; 10; 11; 12\}$  2.  $\{2; 4; 9; 11; 13; 15\}$  3. yes 4. no

**Exercise 14 – 8:**

1. the die lands on the number 5 e) 0,80  
 2.  $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 14; 15\}$  f) 0,55  
 3.  $\{1; 2; 3; 4; 5; 7; 8; 9; 10; 11; 12; 13; 14; 15\}$  g) 0,16  
 4. yes h) 0,01  
 5.  $\{1; 3; 6; 12; 14; 15\}$  14. a)  $\frac{1}{6}$   
 6. 0,08 b)  $\frac{1}{2}$   
 7. 0,1 c)  $\frac{1}{3}$   
 8. 0,50 15.  $\frac{5}{12}$   
 9. 0,18 16. a)  $\frac{19}{30}$   
 10. a)  $\frac{1}{6}$  b)  $\frac{8}{21}$   
 11. a) 0,21 b) 0,56  
 c) 0 c)  $\frac{1}{9}$   
 d) 0,72 d)  $\frac{8}{9}$   
 12. a) 0,5 e)  $\frac{5}{9}$   
 b) 0,23 f)  $\frac{4}{9}$   
 c) 0,67 18. a)  $\frac{11}{21}$   
 d) 0 b)  $\frac{11}{14}$   
 e) 0,67 19. a)  $\frac{3}{56}$   
 f) 0,56 b)  $\frac{53}{56}$   
 13. a) 0,11 20. a) 103  
 b) 0,33 c)  $\frac{14}{19}$   
 c) 0,16 21. b) 10  
 d) 0,01
22. a)  $P = \frac{1}{3}$   
 b)  $P = \frac{2}{3}$   
 c)  $P = \frac{1}{2}$   
 d)  $P = \frac{1}{6}$   
 23. a) (i)  $P = \frac{1}{52}$ ; (ii)  $P = \frac{7}{26}$  (iii)  
 $P = \frac{4}{13}$  b)  $\frac{13}{17}$   
 24. 0,17  
 27. e) Mutually exclusive and complementary.  
 29. a) 50%  
 b) 31,25%  
 c) 6,25%  
 30. 31
31. a)   
 b) 13  
 c) (i)  $P = \frac{1}{6}$ ; (ii)  $P = \frac{1}{6}$ ; (iii)  $P = \frac{2}{3}$   
 32. a)  $w = 4$ ,  $x = 2$ ,  $y = 6$  and  $z = 1$   
 b) (i)  $P = \frac{4}{15}$ ; (ii)  $P = \frac{1}{10}$

# Past exam papers

## 1 Mathematics, Paper 1, Exemplar 2012

MARKS: 100

TIME: 2 hours

### Instructions and information

Read the following instructions carefully before answering the questions.

1. This question paper consists of 7 questions.
2. Answer ALL the questions.
3. Clearly show ALL calculations, diagrams, et cetera that you have used in determining your answers.
4. Answers only will not necessarily be awarded full marks.
5. You may use an approved scientific calculator (non-programmable and non-graphical), unless stated otherwise.
6. If necessary, round off answers to TWO decimal places, unless stated otherwise.
7. Diagrams are NOT necessarily drawn to scale.
8. Number the answers correctly according to the numbering system used in this question paper.
9. Write neatly and legibly.

### Exercise 1 – 1:

1. a) Simplify the following expressions fully:

i.  $(m - 2n)(m^2 - 6mn - n^2)$  (3 marks)

ii.  $\frac{x^3 + 1}{x^2 - x + 1} - \frac{4x^2 - 3x - 1}{4x + 1}$  (5 marks)

- b) Factorise the following expressions fully:

i.  $6x^2 - 7x - 20$  (2 marks)

ii.  $a^2 + a - 2ab - 2b$  (3 marks)

- c) Determine, **without the use of a calculator**, between which two consecutive integers  $\sqrt{51}$  lies. (2 marks)

- d) Prove that  $0.\dot{2}\dot{4}\dot{5}$  is rational. (4 marks)

**[TOTAL: 19 marks]**

2. a) Determine, **without the use of a calculator**, the value of  $x$  in each of the following:

i.  $x^2 - 4x = 21$  (3 marks)

ii.  $96 = 3x^{\frac{5}{4}}$  (3 marks)

iii.  $R = \frac{2\sqrt{x}}{3S}$  (2 marks)

- b) Solve for  $p$  and  $q$  simultaneously if:

$$6q + 7p = 3$$

$$2q + p = 5$$

(5 marks)

**[TOTAL: 13 marks]**

3. a)  $3x + 1 ; 2x ; 3x - 7 ; \dots$  are the first three terms of a linear number pattern.

- i. If the value of  $x$  is three, write down the FIRST THREE terms. (3 marks)

- ii. Determine the formula for  $T_n$ , the general term of the sequence. (2 marks)

- iii. Which term in the sequence is the first to be less than  $-31$ ? (3 marks)

- b) The multiples of three form the number pattern:  $3 ; 6 ; 9 ; 12 ; \dots$

Determine the  $13^{\text{th}}$  number in this pattern that is even. (3 marks)

**[TOTAL: 11 marks]**

4. a) Thando has R 4500 in his savings account. The bank pays him a compound interest rate of 4,25% p.a. Calculate the amount Thando will receive if he decides to withdraw the money after 30 months. (3 marks)
- b) The following advertisement appeared with regard to buying a bicycle on a hire-purchase agreement loan:

Purchase price R 5999

Required deposit R 600

Loan term only 18 months, at 8% p.a. simple interest

- i. Calculate the monthly amount that a person has to budget for in order to pay for the bicycle. (6 marks)
- ii. How much interest does one have to pay over the full term of the loan? (1 marks)

- c) The following information is given:

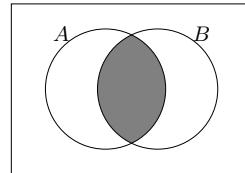
$$1 \text{ ounce} = 28,35 \text{ g}$$

$$\$1 = R\ 8,79$$

Calculate the rand value of a 1 kg gold bar, if 1 ounce of gold is worth \$978,34

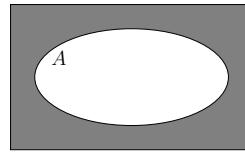
**[TOTAL: 14 marks]**

5. a) What expression BEST represents the shaded area of the following Venn diagrams?



i.

(1 mark)



ii.

(1 mark)

- b) State which of the following sets of events is mutually exclusive:

A	Event 1: The learners in Grade 10 in the swimming team
B	Event 1: The learners in Grade 8
C	Event 1: The learners who take Mathematics in Grade 10
Event 2: The learners in Grade 10 in the debating team	Event 2: The learners in Grade 12
Event 2: The learners who take Physical Sciences in Grade 10	

(1 mark)

- c) In a class of 40 learners the following information is TRUE:

- 7 learners are left-handed
- 18 learners play soccer
- 4 learners play soccer and are left-handed
- All 40 learners are either right-handed or left-handed

Let  $L$  be the set of all left-handed people and  $S$  be the set of all learners who play soccer.

- How many learners in the class are right-handed and do NOT play soccer? (1 mark)
- Draw a Venn diagram to represent the above information. (4 marks)
- Determine the probability that a learner is:
  - Left-handed or plays soccer (3 marks)
  - Right-handed and plays soccer (2 marks)

**[TOTAL: 13 marks]**

6. Given:  $f(x) = \frac{3}{x} + 1$  and  $g(x) = -2x - 4$

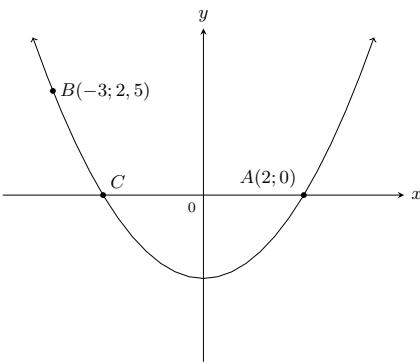
- Sketch the graphs of  $f$  and  $g$  on the same set of axes. (4 marks)
- Write down the equations of the asymptotes of  $f$ . (2 marks)
- Write down the domain of  $f$ . (2 marks)
- Solve for  $x$  if  $f(x) = g(x)$ . (5 marks)
- Determine the values of  $x$  for which  $-1 \leq g(x) \leq 3$ . (3 marks)
- Determine the  $y$ -intercept of  $k$  if  $k(x) = 2g(x)$ . (2 marks)
- Write down the coordinates of the  $x$ - and  $y$ -intercepts of  $h$  if  $h$  is the graph of  $g$  reflected about the  $y$ -axis. (2 marks)

**[TOTAL: 20 marks]**

7. The graph of  $f(x) = ax^2 + q$  is sketched below.

Points  $A(2; 0)$  and  $B(-3; 2,5)$  lie on the graph of  $f$ .

Points  $A$  and  $C$  are  $x$ -intercepts of  $f$ .



- Write down the coordinates of  $C$ . (1 mark)
- Determine the equation of  $f$ . (3 marks)
- Write down the range of  $f$ . (1 mark)
- Write down the range of  $h$ , where  $h(x) = -f(x) - 2$ . (2 marks)
- Determine the equation of an exponential function,  $g(x) = b^x + q$ , with range  $y > -4$  and which passes through the point  $A$ . (3 marks)

**[TOTAL: 10 marks]**

## 2 Mathematics, Paper 2, Exemplar 2012

MARKS: 100

TIME: 2 hours

### Instructions

- This question paper consists of 9 questions.
- Answer ALL the questions.
- Clearly show ALL calculations, diagrams, graphs, et cetera which you have used in determining the answers.
- Answers only will NOT necessarily be awarded full marks.
- You may use an approved scientific calculator (non-programmable and non-graphical), unless stated otherwise.
- If necessary, round off answers to TWO decimal places, unless stated otherwise.
- Diagrams are NOT necessarily drawn to scale.
- ONE diagram sheet for QUESTION 6.1.1 and QUESTION 9 is attached at the end of this question paper. Write your centre number and examination number on this sheet in the spaces provided and insert the sheet inside the back cover of your ANSWER BOOK.
- Number the answers correctly according to the numbering system used in this question paper.
- Write neatly and legibly.

### Exercise 2 – 1:

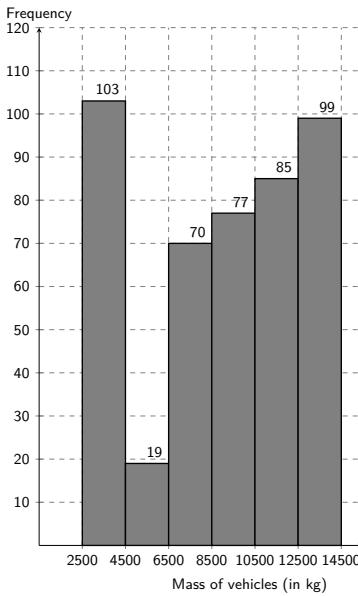
- A baker keeps a record of the number of scones that he sells each day. The data for 19 days is shown below.

31	36	62	74	65	63	60	34	46	56
37	46	40	52	48	39	43	31	66	

- Determine the mean of the given data.  
(2 marks)
- Rearrange the data in ascending order and then determine the median.  
(2 marks)
- Determine the lower and upper quartiles for the data.  
(2 marks)
- Draw a box and whisker diagram to represent the data.  
(2 marks)

**[TOTAL: 8 marks]**

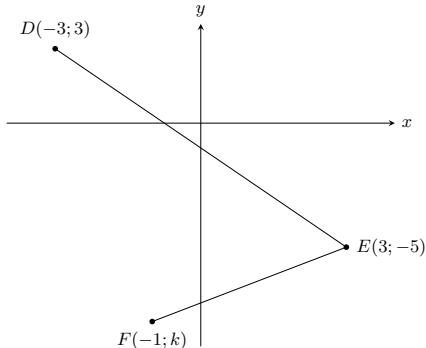
- Traffic authorities are concerned that heavy vehicles (trucks) are often overloaded. In order to deal with this problem, a number of weighbridges have been set up along the major routes in South Africa. The gross (total) vehicle mass is measured at these weighbridges. The histogram below shows the data collected at a weighbridge over a month.



- a) Write down the modal class of the data.  
(1 mark)
- b) Estimate the mean gross vehicle mass for the month.  
(5 marks)
- c) Which of the measures of central tendency, the modal class or the estimated mean, will be most appropriate to describe the data set? Explain your choice.  
(1 mark)

**[TOTAL: 7 marks]**

3. a) In the diagram below,  $D(-3; 3)$ ,  $E(-3; -5)$  and  $F(-1; k)$  are three points in the Cartesian plane.

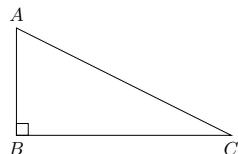


- Calculate the length of  $DE$ .  
(2 marks)
- Calculate the gradient of  $DE$ .  
(2 marks)
- Determine the value of  $k$  if  $D \hat{E} F = 90^\circ$ .  
(4 marks)
- If  $k = -8$ , determine the coordinates of  $M$ , the midpoint of  $DF$ .  
(2 marks)
- Determine the coordinates of a point  $G$  such that the quadrilateral  $DEFG$  is a rectangle.  
(4 marks)

- b)  $C$  is the point  $(1; -2)$ . The point  $D$  lies in the second quadrant and has coordinates  $(x; 5)$ . If the length of  $CD$  is  $\sqrt{53}$  units, calculate the value of  $x$ .  
(4 marks)

**[TOTAL: 18 marks]**

4. a) In the diagram below,  $\triangle ABC$  is right-angled at  $B$ .



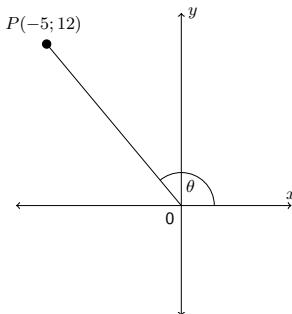
Complete the following statements:

i.  $\sin C = \frac{AB}{?}$   
(1 mark)

ii.  $?A = \frac{AB}{BC}$   
(1 mark)

- b) Without using a calculator, determine the value of  $\frac{\sin 60^\circ \cdot \tan 30^\circ}{\sec 45^\circ}$   
(4 marks)

- c) In the diagram,  $P(-5; 12)$  is a point in the Cartesian plane and  $R\hat{O}P = \theta$ .



Determine the value of:

i.  $\cos \theta$   
(3 marks)

ii.  $\operatorname{cosec}^2 \theta + 1$   
(3 marks)

[TOTAL: 12 marks]

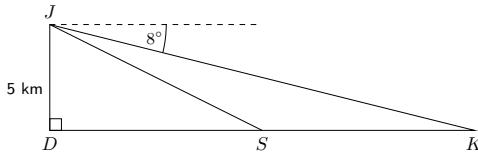
5. a) Solve for  $x$ , correct to ONE decimal place, in each of the following equations where  $0^\circ \leq x \leq 90^\circ$ .

i.  $5 \cos x = 3$   
(2 marks)

ii.  $\tan 2x = 1.19$   
(3 marks)

iii.  $4 \sec x - 3 = 5$   
(4 marks)

- b) An aeroplane at  $J$  is flying directly over a point  $D$  on the ground at a height of 5 kilometres. It is heading to land at point  $K$ . The angle of depression from  $J$  to  $K$  is  $8^\circ$ .  $S$  is a point along the route from  $D$  to  $K$ .



- i. Write down the size of  $J\hat{K}D$ .  
(1 mark)

- ii. Calculate the distance  $DK$ , correct to the nearest metre.  
(3 marks)

- iii. If the distance  $SK$  is 8 kilometres, calculate the distance  $DS$ .  
(1 mark)

- iv. Calculate the angle of elevation from point  $S$  to  $J$ , correct to ONE decimal place.  
(2 marks)

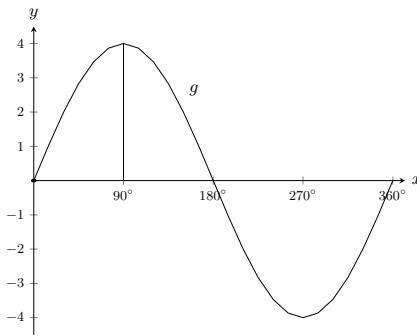
[TOTAL: 16 marks]

6. a) Consider the function  $y = 2 \tan x$ .

- i. Make a neat sketch of  $y = 2 \tan x$  for  $0^\circ \leq x \leq 360^\circ$  on the axes provided on DIAGRAM SHEET 1. Clearly indicate on your sketch the intercepts with the axes and the asymptotes.  
(4 marks)

- ii. If the graph of  $y = 2 \tan x$  is reflected about the  $x$ -axis, write down the equation of the new graph obtained by this reflection.  
(1 mark)

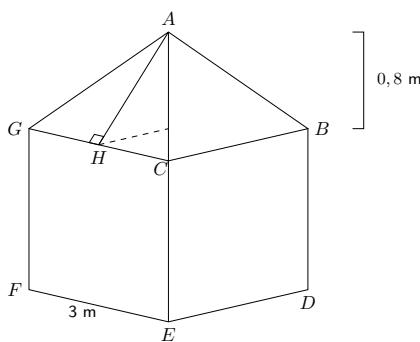
- b) The diagram below shows the graph of  $g(x) = a \sin x$  for  $0^\circ \leq x \leq 360^\circ$ .



- Determine the value of  $a$ .  
(1 mark)
- If the graph of  $g$  is translated 2 units upwards to obtain a new graph  $h$ , write down the range of  $h$ .  
(2 marks)

**[TOTAL: 8 marks]**

7. a) The roof of a canvas tent is in the shape of a right pyramid having a perpendicular height of 0,8 metres on a square base. The length of one side of the base is 3 metres.

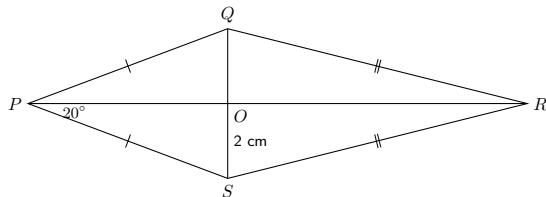


- Calculate the length of  $AH$ .  
(2 marks)
  - Calculate the surface area of the roof.  
(2 marks)
  - If the height of the walls of the tent is 2,1 metres, calculate the total amount of canvas required to make the tent if the floor is excluded.  
(2 marks)
- b) A metal ball has a radius 8 millimetres.
- Calculate the volume of metal used to make this ball, correct to TWO decimal places.  
(2 marks)
  - If the radius of the ball is doubled, write down the ratio of the new volume : the original volume.  
(2 marks)
  - You would like this ball to be silver plated to a thickness of 1 millimetre. What is the volume of silver required? Give your answer correct to TWO decimal places.  
(2 marks)

**[TOTAL: 12 marks]**

8.  $PQRS$  is a kite such that the diagonals intersect in  $O$ .

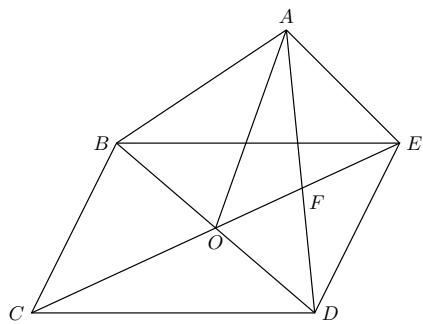
$OS = 2 \text{ cm}$  and  $O\hat{P}S = 20^\circ$ .



- Write down the length of  $OQ$ .  
(2 marks)
- Write down the size of  $P\hat{O}Q$ .  
(2 marks)
- Write down the size of  $Q\hat{P}S$ .  
(2 marks)

**[TOTAL: 6 marks]**

9. In the diagram,  $BCDE$  and  $AODE$  are parallelograms.



- a) Prove that  $OF \parallel AB$   
(4 marks)
- b) Prove that  $ABOE$  is a parallelogram.  
(4 marks)
- c) Prove that  $\triangle ABO \cong \triangle EOD$ .  
(5 marks)

[TOTAL: 13 marks]

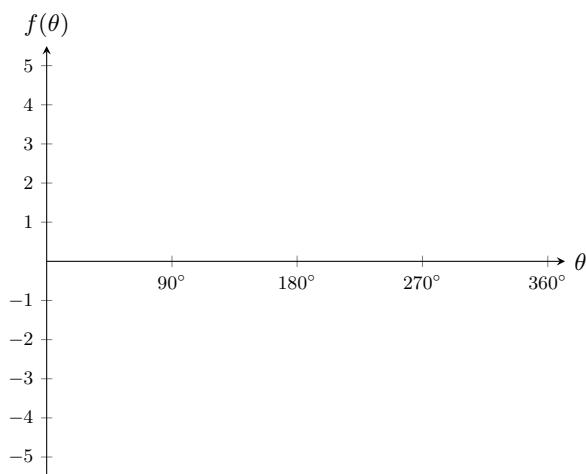
## 2.0 Diagram sheet 1

EMA85

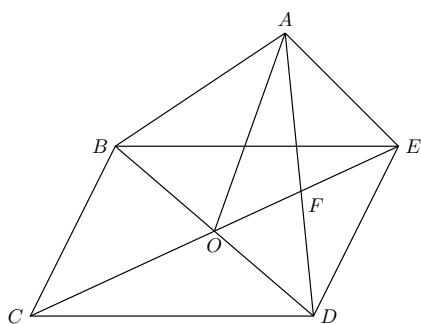
Centre number:

Examination number:

### Question 6.1.1



### Question 9



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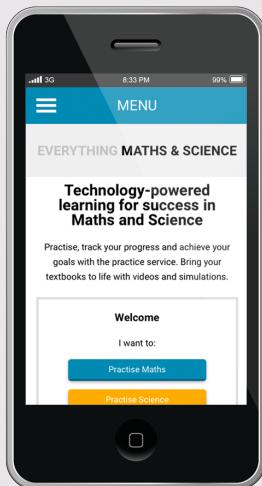
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