Linear Algebra with Applications

Module 1.1

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About me...

Carlos Eduardo Chavez

- Former TA at UC3M (2009 - 2016)
  - Linear Algebra
  - Calculus I & II
  - Differential equations
- Visiting researcher at Inria (Bordeaux, France)
- Data Scientist at Bayes Forecast, Solutio and Wise Athena
- Currently working at Vodafone (Big Data and AI for Vodafone Group)

@CarlosEChavez3

What is Linear Algebra?

- Linear Algebra is the branch of mathematics associated to linear equations such as:

$$\boxed{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b}$$

In this module we are going to understand all the necessary tools to solve this type of equations.

For example:

- Most practical ways to visualize a system of equations
- Identify whether or not the equation has a solution
- Ways to solve the system of equations

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$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b$$

Name	Heigth	Weight	Age
Miguel	120	47	7
Laura	150	50	9
Juan	90	37	10
Pedro	87	40	8
María	50	27	6
Luisa	135	42	8
Jose	160	60	12

$$\Rightarrow \begin{cases} 120x + 47y = 7 \\ 150x + 50y = 9 \\ 90x + 37y = 10 \\ 87x + 40y = 8 \\ 50x + 27y = 6 \\ 135x + 42y = 8 \\ 160y + 60x = 12 \end{cases}$$

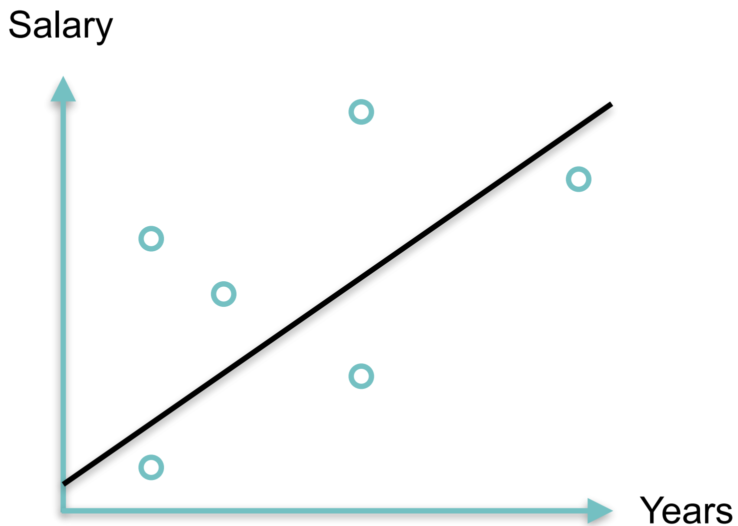
Does this system of equations have a solution? Is it unique?

What properties should the system have in order to have a solution?

Linear Algebra Applications

Another example:

Name	Years of Experience	Salary
Miguel	1	30
Laura	4	45
Juan	5	60
Pedro	10	65
María	2	28
Luisa	5	40
Jose	3	52



$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b$$

There is a solution? It is unique?

Outline

- General notations
    - Scalars, Vectors, Matrices and Tensors
- Matrix properties:
    - Multiplying Matrices and Vectors
    - Identity and Inverse Matrices
    - Determinant
    - Linear Dependence and Rank
- Norm
- Special kinds of matrices and vectors
- Pseudoinverse
- System of linear equations

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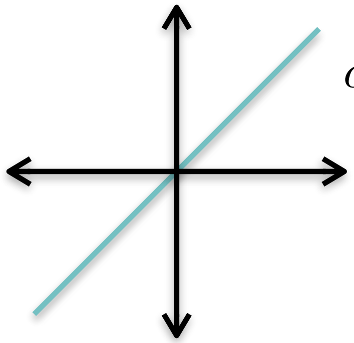
Scalar

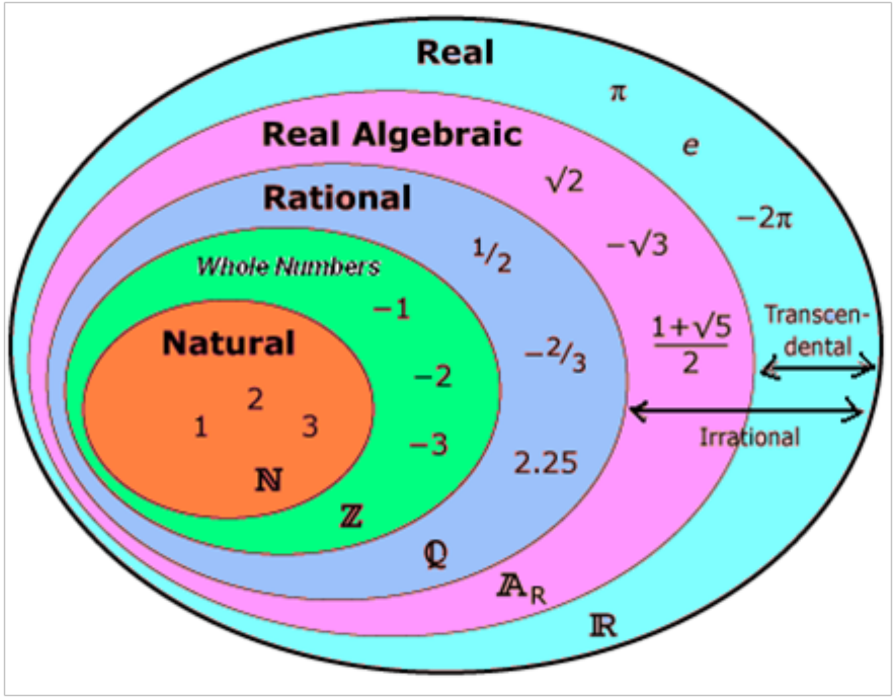
A scalar can be associated to any value (or field). It can be natural, integer, rational, real and complex

Represented in lower-case italic: α

Example: let $\alpha \in \mathbb{R}$ be the slope of the line



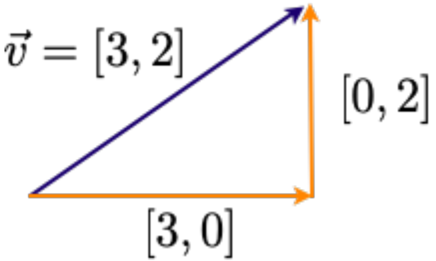
$$\alpha x + b = 0$$

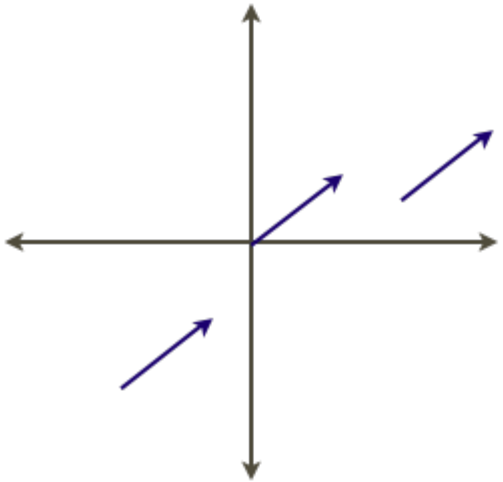


Vectors

Definition: A vector can be defined in general as an entity that directs a certain amount in a certain direction and direction.

For instance, the vector $\vec{v} = [3, 2]$ can be interpreted as being at any point, move 3 units to the right and then 2 units up.





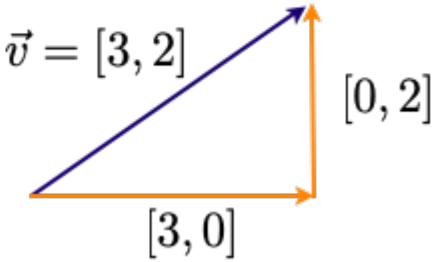
These coordinates are called vector components.

We can think of vector as points in space where each elements give coordinate along an axis

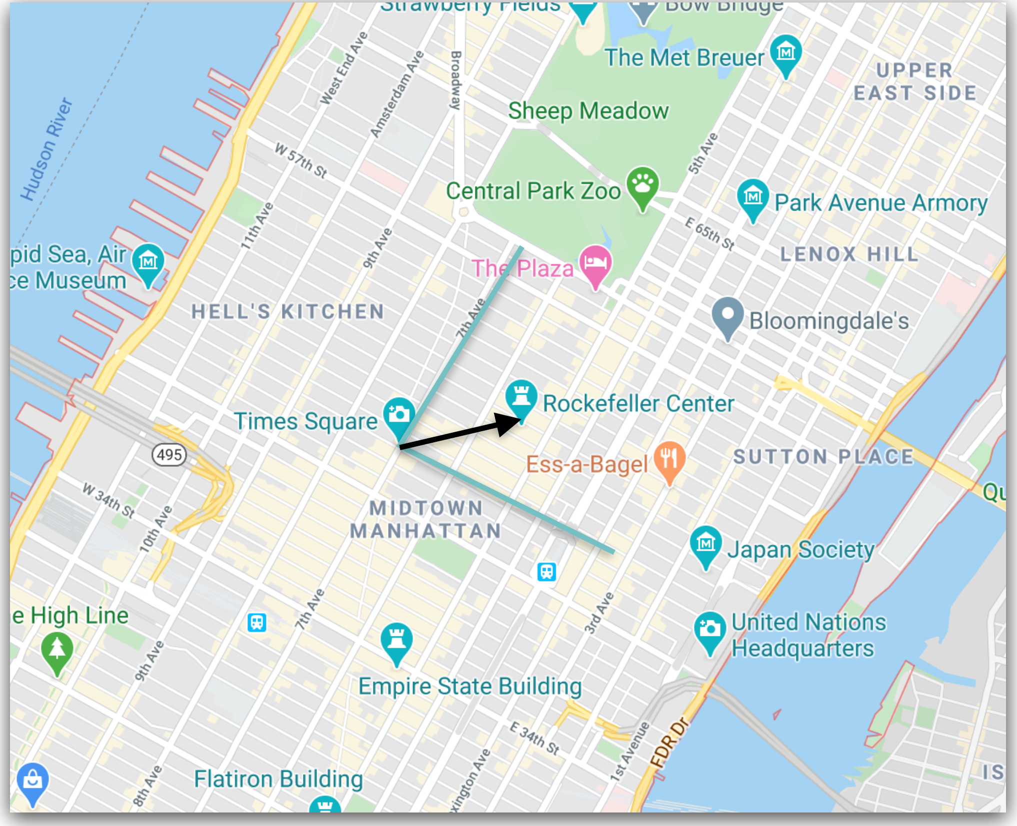
Vectors

Example:

City Map



Direction from Times Square
to Rockefeller Center



Vectors

Elementary vectors:

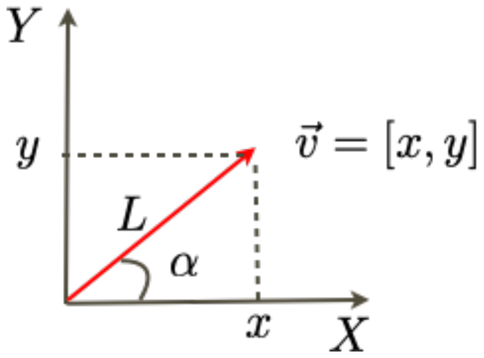
Null vector or zero: $\vec{0} = [0, 0, 0]_{1 \times 3}$

Row vector $\vec{v} = [x, y, z]_{1 \times 3}$

rows columns

Column vector $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3 \times 1}$

In addition to their coordinates, vectors can be characterized by their length and angle.



where

$$L = \sqrt{x^2 + y^2}$$

$$\begin{cases} \alpha = \arctan \frac{y}{x} & si \quad x \neq 0 \\ \alpha = \frac{\pi}{2} & si \quad x = 0 \rightarrow \begin{cases} y > 0 & \alpha = \pi/2 \\ y < 0 & \alpha = -\pi/2 \end{cases} \end{cases}$$

Vectors

Vectors properties

Sum: the sum of two vectors is made component by component. **You can only add vectors with the same number of components.**

Product: The product of a vector by a scalar is made by multiplying each component by the number.

Operations

Let $\vec{a}$ and $\vec{b}$ vectors and λ a real number:

- Commutative $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
- Associative $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$
- Null element $\vec{0}+\vec{a}=\vec{a}+\vec{0}=\vec{a}$
- Distributive $\lambda(\vec{a}+\vec{b})=\lambda\vec{a}+\lambda\vec{b}$
- Opposite element $(-\vec{a})+\vec{a}=\vec{a}+(-\vec{a})=\vec{0}$

Matrix

Definition: A matrix can be defined in general as an arrangement of numbers in rows and columns.

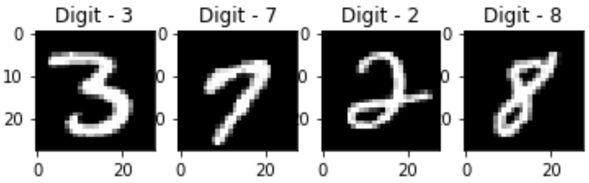
Elements indicated by name in italic but not bold

$$A_{m,n} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

A matrix ***A*** of ***m*** rows and ***n*** columns is represented by $A_{m\,\mathrm{x}\,n}$

Examples:

Handwritten letter recognition



Spreadsheets, CSV, etc

	A	B	C	D	E	F	G
1	**Expense**	**Jan-18**	**Feb-18**	**Mar-18**	**Apr-18**	**May-18**	
2	Phone	$ 46.0	$ 47.0	$ 56.0	$ 65.0	$ 58.0	
3	Insurance	$ 80.0	$ 80.0	$ 80.0	$ 80.0	$ 80.0	
4	Rent	$ 900.0	$ 900.0	$ 900.0	$ 900.0	$ 900.0	
5	Medicine	$ 120.0	$ 60.0	$ 87.0	$ 90.0	$ 55.0	
6	Electric Bill	$ 200.0	$ 180.0	$ 145.0	$ 170.0	$ 140.0	
7	Water Bill	$ 120.0	$ 100.0	$ 99.0	$ 110.0	$ 120.0	
8	**Total**	**$ 1,466.0**	**$ 1,367.0**	**$ 1,367.0**	**$ 1,415.0**	**$ 1,353.0**	
9							
10							

Matrix properties

Sum: the sum of two Matrices is made component by component. **You can only add vectors with the same number of components.**

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & \ddots & & a_{2,m} \\ \vdots & & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix} + \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\ b_{2,1} & \ddots & & b_{2,m} \\ \vdots & & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,m} \end{pmatrix} = \begin{pmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} & \cdots & a_{1,m}+b_{1,m} \\ a_{2,1}+b_{2,1} & \ddots & & a_{2,m}+b_{2,m} \\ \vdots & & \ddots & \vdots \\ a_{n,1}+b_{n,1} & a_{n,2}+b_{n,2} & \cdots & a_{n,m}+b_{n,m} \end{pmatrix}$$

Like vectors, matrices meet the following properties for addition:

- Commutative
- Associative
- Null element
- Distributive
- Opposite element

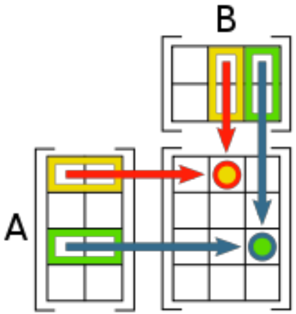
Matrix product

<u>Product:</u> The product of two Matrices is possible **if and only if the number of columns of the first is equal to the number of rows of the second matrix**. And the result is a size matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & \ddots & & a_{2,m} \\ \vdots & & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix}_{n,m} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,p} \\ b_{2,1} & \ddots & & b_{2,p} \\ \vdots & & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,p} \end{pmatrix}_{m,p} = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,p} \\ c_{2,1} & \ddots & & c_{2,p} \\ \vdots & & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,p} \end{pmatrix}_{n,p}$$

where the component i, j of the resulting matrix is obtained as follows:

$$c_{i,j} = \sum_{r=1}^{n} a_{i,r}\ b_{r,j} = a_{i,1} \cdot b_{1,i} + a_{i,2} \cdot b_{2,i} + \cdots + a_{i,m} \cdot b_{m,i}$$



Matrix product

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & \ddots & & a_{2,m} \\ \vdots & & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix}_{n,m} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,p} \\ b_{2,1} & \ddots & & b_{2,p} \\ \vdots & & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,p} \end{pmatrix}_{m,p} = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,p} \\ c_{2,1} & \ddots & & c_{2,p} \\ \vdots & & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,p} \end{pmatrix}_{n,p}$$

Example 1:

$$\begin{pmatrix} 8 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} * \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 8 * a + c * 2 \\ 3 * a + 4 * c \\ 5 * a + 6 * c \end{pmatrix} = \begin{pmatrix} 8a + 2c \\ 3a + 4c \\ 5a + 6c \end{pmatrix}$$

Matrix product

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & \ddots & & a_{2,m} \\ \vdots & & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix}_{n,m} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,p} \\ b_{2,1} & \ddots & & b_{2,p} \\ \vdots & & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,p} \end{pmatrix}_{m,p} = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,p} \\ c_{2,1} & \ddots & & c_{2,p} \\ \vdots & & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,p} \end{pmatrix}_{n,p}$$

Example 2:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1*a+2*c & 1*b+2*d \\ 3*a+4*c & 3*b+4*d \\ 5*a+6*c & 5*b+6*d \end{pmatrix} = \begin{pmatrix} a+2c & 1b+2d \\ 3c+4d & 3*b+4d \\ 5a+6c & 5b+6d \end{pmatrix}$$

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Matrix properties

Product: Unlike vectors, in the product of matrices, it does not meet all the properties, that is:

- It is not commutative: $AB \neq BA$

Example:

$$\boxed{\begin{pmatrix} 3 & 4 \\ 1 & 8 \end{pmatrix} * \begin{pmatrix} -5 & 2 \\ 4 & 13 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} -5 & 2 \\ 4 & 13 \end{pmatrix} * \begin{pmatrix} 3 & 4 \\ 1 & 8 \end{pmatrix}}$$

- The neutral element of the product is the identity matrix: $AI = A$ where $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$
- Associative: $A(BC) = (AB)C$
- Distributive: $A(B + C) = AB + AC$

Transpose of a Matrix

It is consider one of the most important operation on matrices.

The transpose of a matrix A is denoted as A^T and is defined as: $(A^T)_{i,j} = A_{j,i}$

The mirror image across a diagonal line

$$A_{3,3} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \rightarrow A_{3,3}^T = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,2} & a_{2,2} & a_{3,2} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 1 & 14 & 4 & 15 \\ 8 & 11 & 5 & 10 \\ 13 & 2 & 16 & 3 \\ 12 & 7 & 6 & 6 \end{pmatrix}$$

Transpose of a Matrix

Let's code!

It is consider one of the most important operation on matrices.

The transpose of a matrix A is denoted as A^T and is defined as: $(A^T)_{i,j} = A_{j,i}$

The mirror image across a diagonal line

$$A_{3,3} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \rightarrow A^T_{3,3} = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,2} & a_{2,2} & a_{3,2} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{pmatrix}$$

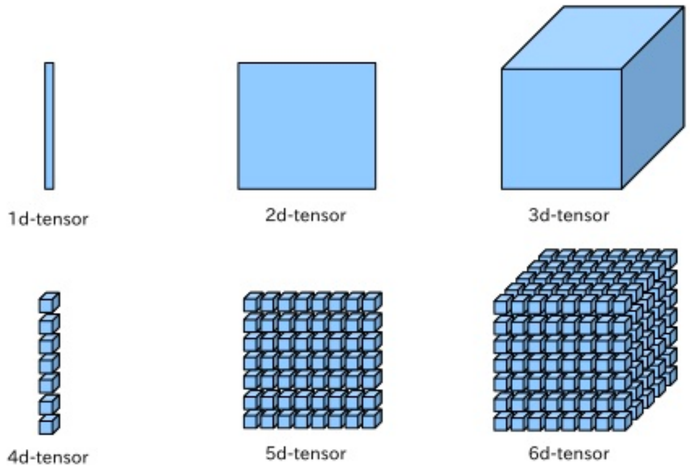
Example

$$A = \begin{pmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 1 & 14 & 4 & 15 \\ 8 & 11 & 5 & 10 \\ 13 & 2 & 16 & 3 \\ 12 & 7 & 6 & 6 \end{pmatrix}$$

Tensor

Definition: is a k-D array of numbers. For instance, an RGB color image is a tensor with three axes.

Denote a tensor with bold typeface $\mathbf{A}$ and the elements of tensor can be denoted by $\mathbf{a}_{i,j,k}$



Thereby, we can deduce that a 3-D tensor is a cube, 4-D tensor is a vector of cubes, 5-D is a matrix of cubes, etc

Photo credit to knoldus

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Real-Word Examples of Tensors

- **Vector Data:** 2D tensor of shape with sample and features (as the first example)

- **Time-Series Data or Sequence Data:** a **3D tensor** of shape with samples, timesteps and features

Example:

A datataset of stoke prices. Every minute, we store the current price of the stock, the highest price in the past minute, and the lowest price in the past minute

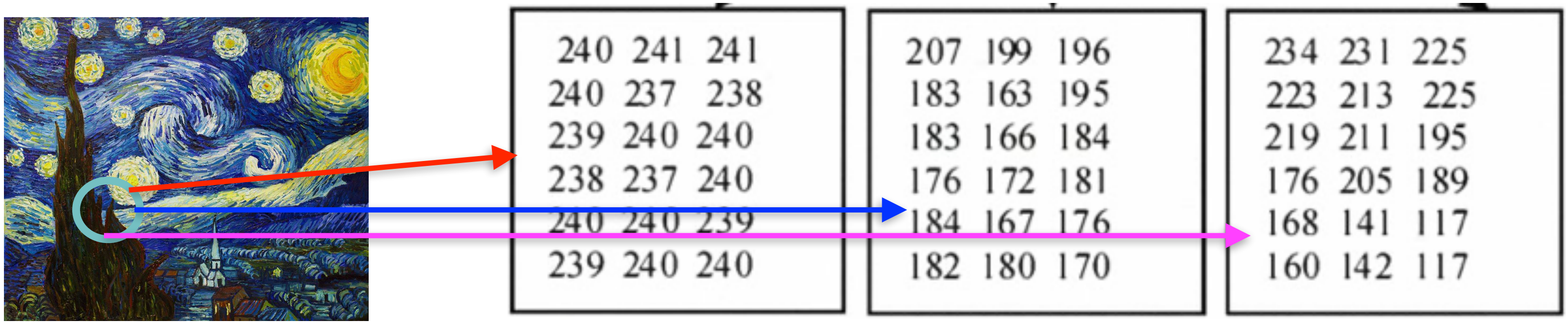
Real-Word Examples of Tensors

- **Image Data:** 4D or 5D tensor of shape with samples, height, width and channels

  - Images typically have three dimensions: height, width and color depth
  - Grayscale Images contains only one channel
  - RGB Images has three color channels

Example:

A batch of 128 grayscale images of size 256 × 256 could thus be stored in a tensor of shape (128, 256, 256, 1), and a batch of 128 color images could be stored in a tensor of shape (128, 256, 256, 3)



Linear Algebra Applications (II)

Let's go back to our initial problem

$$\begin{cases} 120x + 47y = 7 \\ 150x + 50y = 9 \\ 90x + 37y = 10 \\ 87x + 40y = 8 \\ 50x + 27y = 6 \\ 135x + 42y = 8 \\ 160y + 60x = 12 \end{cases} \Rightarrow \begin{pmatrix} 120 & 47 \\ 150 & 50 \\ 90 & 37 \\ 87 & 40 \\ 50 & 27 \\ 135 & 42 \\ 160 & 60 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 10 \\ 8 \\ 6 \\ 8 \\ 12 \end{pmatrix}$$

$$\boxed{Ax = b}$$

Let’s check this notation,
it is possible to obtain the equation system again?

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Linear Transformation

$$\boxed{Ax = b}$$

where $A \in \mathbb{R}^{n\,x\,m}$ and $b \in \mathbb{R}^{m}$

Can view A as a linear transformation of vector x to vector b

Sometimes we wish to solve for the unknowns $x = (x_1, x_2, \cdots, x_n)$ when A and b provide constrains

$$\boxed{x = A^{-1}b} \longrightarrow \quad A^{-1} \;\; ??$$

Matrix Inverse

The inverse of a square matrix A is defined as: $\boxed{A^{-1}A = I_n}$

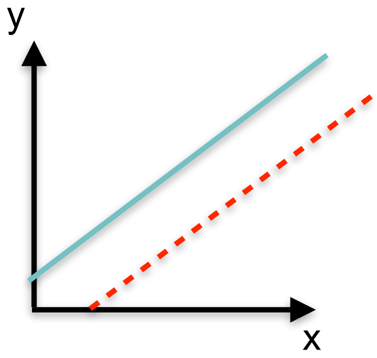
We can solve $\boxed{Ax = b}$ as follows:

$$\boxed{\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ I_n x &= A^{-1}b \\ x &= A^{-1}b \end{aligned}}$$

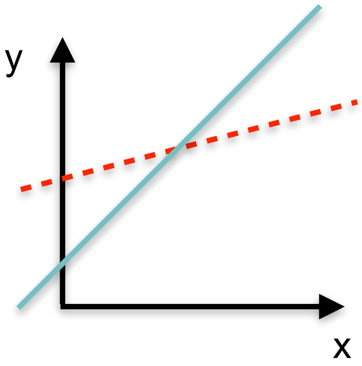
If A^{-1} exist, there are several methods for finding it

How many solutions for $\boxed{Ax = b}$ exist?

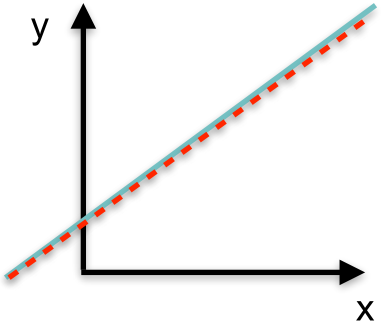
No solution



1 solution



Infinite number of solutions



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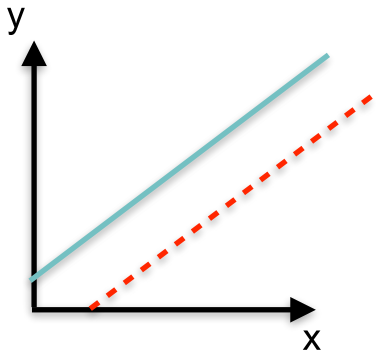
How many solutions for $\boxed{Ax = b}$ exist?

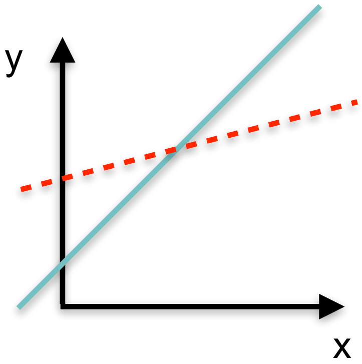
No solution



$$2x - y = 3$$

$$2x - y = 1$$

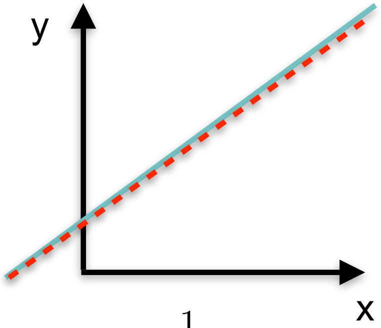
1 solution



$$3x - 2y = 1$$

$$x - 3y = 4$$

Infinite number of solutions

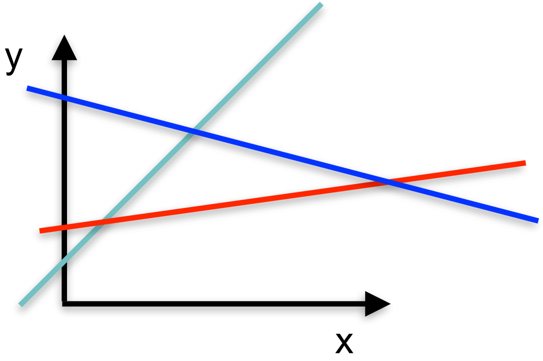


$$2x - \frac{1}{2}y = 4$$

$$4x - y = 8$$

Overdetermined and Undetermined systems $\boxed{Ax = b}$

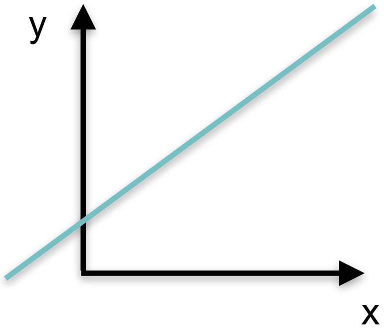
Overdetermined



No solution

If there are more equations than unknowns

Underdetermined



$$y = 2x + 1$$

Infinite number of solutions

If there are more unknown than equations

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How many solutions for $\boxed{Ax = b}$ exist?

System of equation with: ***n*** variables and ***m*** equations

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}}$$

It is possible for the system of equations to have **no solutions, a unique solution or an infinite number of solutions**

- Matrix must be square, i.e., ***m = n*** and all columns must be **linearly independent**

- Necessary condition is $n \geq m$

- Sufficient condition: at least one set of ***m linearly independent columns***

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Linear Dependency and Independency

If there is a scalar α such that

$$\boxed{\alpha \vec{v_1} = \vec{v_2}}$$

Then we can say that $\vec{v_1}$ and $\vec{v_2}$ are **linearly dependent**

Examples:

1)
$$2x - \frac{1}{2}y = 4$$
$$4x - y = 8$$

2)
$$\vec{v_1} = [12, 6]$$
$$\vec{v_2} = [36, 18]$$

3)
$$\vec{a} = [1, 5, 10]$$
$$\vec{b} = [5, 15, 50]$$

Linear Dependency and Independency

If there is a scalar α such that

$$\boxed{\alpha \vec{v_1} = \vec{v_2}}$$

Then we can say that $\vec{v_1}$ and $\vec{v_2}$ are **linearly dependent**

Examples:

linearly dependent

1)
$$2x - \frac{1}{2}y = 4$$
$$4x - y = 8$$

linearly dependent

2)
$$\vec{v_1} = [12, 6]$$
$$\vec{v_2} = [36, 18]$$

linearly independent

3)
$$\vec{a} = [1, 5, 10]$$
$$\vec{b} = [5, 15, 50]$$

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Matrix Determinant

<u>**Determinant**</u>**:** The determinant of a matrix is a scalar or a polynomial that results when performing a multiplication algorithm with its components, in 2nd and 3rd grade matrices usually calculated using the Sarrus Rule

$$det(A) = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$

$$det(A) = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

$$= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - (a_{1,3}a_{2,2}a_{3,1} + a_{1,2}a_{2,1}a_{3,3} + a_{1,1}a_{2,3}a_{3,2})$$

Note: to calculate the determinant of a matrix, **it must be square.**

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Matrix Determinant

<u>Determinant Properties</u>:

- The determinant of A is equal to the determinant of its transposition. $|A| = |A^T|$
- It is lineal $$\begin{vmatrix} a_1 & tb_1 + c_1 & d_1 \\ a_2 & tb_2 + c_2 & d_2 \\ a_3 & tb_3 + c_3 & d_3 \end{vmatrix} = t \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$$
- If two columns are equal $|A| = 0$
- It does not change if one column is added a multiple of the other
- If two columns are exchanged, if change sign
- **Column vectors are linearly independent if and only if the determinant is nonzero**

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Special kind of matrices and vectors

Diagonal Matrix:

Has mostly zeros, with non zeros entries only in diagonal

Examples

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 21 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -9 \end{pmatrix}$$

Triangular Matrix:

All entries *above* or *bellow* the diagonal are zero

Examples

$$\begin{pmatrix} 9 & 15 & 8 & -7 \\ 0 & 21 & -2 & 5 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -9 \end{pmatrix} \qquad \begin{pmatrix} 9 & 0 & 0 & 0 \\ 3 & 21 & 0 & 0 \\ 5 & 4 & 3 & 0 \\ -2 & -10 & -1 & -9 \end{pmatrix}$$

Norms

- Used for measuring the size of a vector
- Norms map vector to non-negative values
- Norms of a vector $x = [x_1, x_2, \cdots, x_n]^T$ is distance from origin to x
- It is any function f that satisfies:

$$f(x) = 0 \rightarrow x = 0$$

$$f(x + y) \leq f(x) + f(y)$$

$$\forall \ \alpha \in \mathbb{R} \quad f(\alpha x) = |\alpha| f(x)$$

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Norms

$$\boxed{||x||_p = \left(\sum_i |x_i|^p \right)^{\frac{1}{p}}}$$

- L^2 **Norm**

Called Euclidean norm:
- Simply the Euclidean distance between the origin and the point x

- L^1 **Norm**

Useful when 0 and non-zero have to be distinguished

- L^∞ **Norm**

$$\boxed{||x||_\infty = max_i |x_i|}$$

Called max norm

Pseudoinverse

Given a matrix $A \in \Re^{m \ x \ n}$ with ***linearly independent columns***, it is defined the pseudo inverse of A as:

$$A^+ = (A^T A)^{-1} A^T \qquad A^+ \in \Re^{n \ x \ m}$$

Properties

$$AA^+A = A$$

$$A^+AA^+ = A^+$$

AA^+ and A^+A are simetrics

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