SFSU Physics 440/740: Computational Physics, Assignment #3

0.) Read Chapter 15: Continuous Nonlinear Dynamics in the e-textbook. You may also want to read the Wikipedia page on "Chaos Theory" for a quick overview.

1.) **Hamiltonian Dynamics of a Nonlinear Pendulum:** Consider a simple pendulum of length ℓ in gravitational field g. The frequency in the limit of small angles is $\Omega_0 \equiv \sqrt{g/\ell}$, but do not assume the limit of small angles for the following calculations.

(a) Start with the Hamiltonian and develop two first order equations for the angle θ and its conjugate momentum p_θ.

(b) Use a second-order leapfrog algorithm to compute the motion of the pendulum. If we choose a computational unit of time $[T] = \Omega_0^{-1}$, then 2π computational time units equals one period in the limit of small oscillations. Another way to think about it is that we can choose a set of units such that $\Omega_0 = 1$. Make a graph of phase space trajectories for a variety of initial conditions.

(c) Liouville's Theorem states that the phase-space volume of a infinitesimally close ensemble of states is conserved. Demonstrate Liouville's Theorem by considering an ensemble of closely spaced initial conditions.

2.) **Chaos in the Damped Driven Pendulum (DDP):** Consider a simple pendulum that is subject to linear dissipation and external forcing:

$$m\ell^2\ddot{\theta} = -b\dot{\theta} - mg\ell\sin\theta + \Gamma\cos\omega_f t. \tag{1}$$

Rewrite the equation using units $[M] = m$, $[L] = \ell$, $[T] = \Omega_0^{-1} = \sqrt{\ell/g}$. The rewritten differential equation should depend on the following nondimensional variables and parameters: θ, t^*, b^*, ω_f^*, and Γ^*. [To check your work, this is equivalent to working in a set of units such that $m = \ell = g = 1$.] Once you are convinced you have the correct nondimensional equation, you can drop the superscript asterisks.

Let's consider a case of weak damping by setting $b = 0.050$. Let's also fix the driving frequency at some value below the resonant frequency: $\omega_f = 0.70$.

Choose $\Gamma = 0.40$. Integrate the equations of motion for long enough so that the transient solution has decayed (say 10 times the unforced, undamped period). Construct 3 plots: (i) angular velocity $\dot{\theta}$ vs. time t, (ii) angular velocity $\dot{\theta}$ vs. angle θ (this is a phase-space diagram), and (iii) a phase-space diagram, but plotting points only at times corresponding to the period of the driving torque (this kind of plot in which you strobe the motion at a particular frequency is called a Poincaré section). For this value of Γ, you should find that the motion is well-behaved (the phase-space plot should be an ellipse and the Poincaré section should just be a point). Now repeat, increasing Γ by $\Delta\Gamma = 0.10$ up to $\Gamma = 1.0$. You should have 21 plots; place them on a page in 7 rows, 3 columns.

Comment on your results. For what values of Γ does the system appear chaotic? Do you see any "period doubling" before transitions to chaos? Do you see any "topological mixing"?

3.) **Physics 740: More Chaos:** Read up on chaotic dynamical systems online, and numerically investigate its chaotic dynamics. For example, you may want to investigate the strange attractor in the Lorenz equations (a super simple model for weather/climate), or you may want to look at the Chua's circuit, a simple electrical circuit that you can build in the lab that exhibits chaotic behavior.