1 Parabolic PDEs: Thermal Diffusion

In this problem, we will consider the fluctuations of the temperature of the Earth's crust with time due to daily and seasonal oscillations. The thermal diffusion equation is:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}, \tag{1}$$

where $T(z,t)$ is temperature, z is depth into the Earth, and α is the thermal diffusivity. We will model the effects of daily and seasonal variations with a time-dependent boundary condition at $z=0$:

$$\begin{aligned} T(z=0,t) &= T_0 + T_1 \sin(\omega_1 t), & (2)\\ T(z=H,t) &= T_0, & (3) \end{aligned}$$

where we assume that at some depth $z=H$ that the interior temperature is no longer affected by what is happening at the surface, and so the temperature is fixed.

This system of equations has an exact solution:

$$T(z,t) = T_0 + T_1 \sin\left[\left(\frac{\omega_1}{2\alpha}\right)^{1/2} z - \omega_1 t\right] \exp\left[\left(\frac{\omega_1}{2\alpha}\right)^{1/2} z\right], \tag{4}$$

in the case that $H \to \infty$. You can verify that this is indeed the solution by substituting into the PDE, and checking that it also satisfies the boundary conditions. The term $D \equiv \sqrt{2\alpha/\omega}$ is called the "skin depth". Temperature fluctuations propagate as a thermal wave into the Earth, but are damped and only penetrate a few skin depths. The thermal diffusivity for the Earth is approximately $\alpha \approx 10^{-6}$ m^2/s. The skin depth is frequency dependent. For daily fluctuations, $D_{daily} \approx 15$ cm, and for yearly fluctuations, $D_{yearly} \approx 3$ m. Thus, short-term variations don't penetrate deeply, while long-term variations do. If you go deep enough into the Earth, the temperature doesn't fluctuate much at all, no matter what is happening at the surface.

It is advisable to choose natural units for the problem. For time, we choose the inverse frequency of oscillations: $t = t^* t_u$, where $t_u = 1/\omega_1$. With this unit of time, one period corresponds to $t^* = 2\pi$. What should we use for a unit of length? Well, the units of α are m^2/s. Because we already have a unit of time, we can construct a unit of length from α and ω: $L_u = \sqrt{2\alpha/\omega}$. Hey, that turns out to be the skin-depth! Thus, we will measure z in units of $L_u = D$: $z = z^* D$. Finally, for temperature, let's use T_0 as our unit, so that $T = T^* T_0$. Substituting into the differential equations and boundary conditions yields:

$$\begin{aligned} \frac{\partial T^*}{\partial t^*} &= \frac{1}{2}\frac{\partial^2 T^*}{\partial z^{*2}}, & (5)\\ T^*(0,t^*) &= 1 + T_1^* \sin t, & (6)\\ T^*(H^*,t^*) &= 1, & (7)\\ T^*(z^*,0) &= 1. & (8) \end{aligned}$$

Your task is to solve this partial differential equation with an *implicit method*, specifically Crank-Nicholson. Choose $H^* = 10$ (that is, 10 skin depths into the Earth), and $T_1^* = 0.1$ (that is, a 10% fluctuation of temperature). You will need to choose sensible values for Δt and Δz. Simulate the temperature for a few periods of oscillation. Make a single space-time plot illustrating the evolution of temperature. For example, make a 3D plot with z going into page, t going to the right, and T up and down. Are the amplitudes of fluctuations consistent with depth in terms of skin depths? Are there any phase lags? How deep would you have to go such that the minimum temperature at depth occurs at maximum temperature at the surface, and vice versa? As a bonus, you may want to try to make an animation.

2 Hyperbolic PDEs: The Wave Equation

The 1 D hydrodynamic equations for an isentropic gas are:

$$\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{9}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v)}{\partial x}, \tag{10}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{11}$$

where v is the fluid velocity, p is the gas pressure, ρ is the gas density, and γ is the adiabatic index.

Consider small perturbations:

$$p = p_0 + \tilde{p}, \qquad |\tilde{p}| \ll p_0, \tag{12}$$

$$\rho = \rho_0 + \tilde{\rho}, \qquad |\tilde{\rho}| \ll \rho_0, \tag{13}$$

$$v = 0 + \tilde{v}, \qquad |\tilde{v}| \ll c_s, \tag{14}$$

where $c_s^2 \equiv (\partial p/\partial \rho)_s = \gamma p/\rho$.

Then the hydrodynamic equations can be approximated:

$$\frac{\partial \tilde{v}}{\partial t} = -c_{s0}^2 \frac{\partial(\tilde{\rho}/\rho_0)}{\partial x}, \tag{15}$$

$$\frac{\partial(\tilde{\rho}/\rho_0)}{\partial t} = -\frac{\partial \tilde{v}}{\partial x}. \tag{16}$$

We can set $\rho_0 = c_{s0} = 1$ (really, we are choosing units such that these values are unity).

Your task: Solve with an implicit Crank-Nicholson in time and FFTs in space. Experiment with different initial conditions (for example, an initial density bump with no initial velocity, and vice versa).