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图论

深度优先遍历

双联通分量

割点

1. 树根是割点当且仅当它有两个及以上的孩子
2. 非根节点u是割点当且仅当u存在一个子节点v，v以及其子节点都没有反向边连向u的子节点

```
int tot, dfs_clock, bcc_cnt;
int to[M], nxt[M], head[N], dfn[N], low[N], iscut[N], bccno[N];

void init() {
    tot = dfs_clock = bcc_cnt = 0; memset(head, 0, sizeof(head));
    memset(dfn, 0, sizeof(dfn)); memset(iscut, 0, sizeof(iscut));
    memset(bccno, 0, sizeof(bccno));
}

void addEdge(int u, int v) {
    to[++tot] = v, nxt[tot] = head[u], head[u] = tot;
}

void dfs(int u, int fa = -1) {
    dfn[u] = low[u] = ++dfs_clock;
    int child = 0;
    for(int i = head[u]; i; i = nxt[i]) {
        int v = to[i];
        if(!dfn[v]) {
            child++;
            dfs(v, u);
            low[u] = min(low[u], low[v]);
            if(low[v] >= dfn[u]) iscut[u] = true;
        } else if(dfn[v] < dfn[u] && v != fa)
            low[u] = min(low[u], dfn[v]);
    }
    if(fa < 0 && child == 1) iscut[u] = 0;
}
```

点-双联通分量

不同双联通分量之间最多只有一个公共点，且一定是割点。计算点双联通分量的过程和计算割点类似，用一个栈来保存在当前BCC中的边。

```
int tot, dfs_clock, bcc_cnt;
int to[M], nxt[M], head[N], dfn[N], low[N], iscut[N], bccno[N];
stack<pair<int, int> >stk;
vector<int>bcc[N];

void init() {
    tot = dfs_clock = bcc_cnt = 0; memset(head, 0, sizeof(head));
    memset(dfn, 0, sizeof(dfn)); memset(iscut, 0, sizeof(iscut));
    memset(bccno, 0, sizeof(bccno));
}

void addEdge(int u, int v) {
    to[++tot] = v, nxt[tot] = head[u], head[u] = tot;
}

void dfs(int u, int fa = -1) {
    dfn[u] = low[u] = ++dfs_clock;
    int child = 0;
    for(int i = head[u]; i; i = nxt[i]) {
        int v = to[i];
        if(!dfn[v]) {
            stk.push(make_pair(u, v)), child++;
            dfs(v, u);
            low[u] = min(low[u], low[v]);
            if(low[v] >= dfn[u]) {
                iscut[u] = true; bcc_cnt++;
                bcc[bcc_cnt].clear();
                for(; ; ) {
                    int a = stk.top().first, b = stk.top().second;
stk.pop();
                    if(bccno[a] != bcc_cnt) {
                        bcc[bcc_cnt].push_back(a); bccno[a] =
bcc_cnt;
                    }
                    if(bccno[b] != bcc_cnt) {
                        bcc[bcc_cnt].push_back(b); bccno[b] =
bcc_cnt;
                    }
                    if(a == u && b == v) break ;
                }
            }
```

```
        } else if(dfn[v] < dfn[u] && v != fa) {
            stk.push(make_pair(u, v));
            low[u] = min(low[u], dfn[v]);
        }
    }
    if(fa < 0 && child == 1) iscut[u] = 0;
}
```

割边

对于一条边(u,v)，若v及其后代都不能到达u及其祖先，则(u,v)是割边

```
void addEdge(int u, int v, int id) {
    to[++tot] = v, nxt[tot] = head[u], index[tot] = id, head[u] =
tot;
}

void dfs(int u, int from = 0) {
    dfn[u] = low[u] = ++dfs_clock;
    for(int i = head[u]; i; i = nxt[i]) if(index[i] != from){
        int v = to[i];
        if(!dfn[v]) {
            dfs(v, index[i]);
            low[u] = min(low[u], low[v]);
            if(low[v] > dfn[u]) iscut[index[i]] = true;
        } else if(dfn[v] < dfn[u])
            low[u] = min(low[u], dfn[v]);
    }
}
```

边-双联通分量

求出割边之后，在原图中dfs，每个联通块为一个边-双联通分量

```
void getIdx(int u, int id) {
    bccno[u] = id;
    for(int i = head[u]; i; i = nxt[i])
        if(!iscut[index[i]] && !bccno[to[i]]) getIdx(to[i], id);
}
```

强连通分量

```
void init() {
    tot = dfs_clock = scc_cnt = 0;
    memset(head, 0, sizeof(head)); memset(dfn, 0, sizeof(dfn));
    memset(sccno, 0, sizeof(sccno));
}

void addEdge(int u, int v) {
    to[++tot] = v, nxt[tot] = head[u], head[u] = tot;
}

void tarjan(int u) {
    dfn[u] = low[u] = ++dfs_clock;
    S.push(u);
    for(int i = head[u]; i; i = nxt[i]) {
        int v = to[i];
        if(!dfn[v]) tarjan(v), low[u] = min(low[u], low[v]);
        else if(!sccno[v]) low[u] = min(low[u], dfn[v]);
    }
    if(low[u] == dfn[u]) {
        scc_cnt++;
        for(; ; ) {
            int x = S.top(); S.pop();
            sccno[x] = scc_cnt;
            if(x == u) break ;
        }
    }
}
```

二分图

匈牙利算法

```
bool hungary(int u) {
    vis[u] = clk;
    int v;
    for(int i = head[u]; i; i = nxt[i]) if(vis[v = to[i]] != clk) {
        vis[v] = clk;
        if(!xlink[v] || hungary(ylink[v])) {
            xlink[u] = v, ylink[v] = u;
            return true;
        }
    }
    return false;
}
```

Hopcroft-Karp

```
int n, m, w;
int vis[N], xlink[N], ylink[N], dis[N];
vector<int>G[N];

bool BFS() {
    queue<int>q;
    memset(dis, -1, sizeof(dis));
    for(int i = 1; i <= m; i++)
        if(xlink[i] == -1) {
            q.push(i); dis[i] = 0;
        }
    w = INT_INF;
    while(!q.empty()) {
        int p = q.front(); q.pop();
        if(dis[p] > w) break;
        for(int i = 0; i < G[p].size(); i++) {
            int l = G[p][i];
            if(dis[l] == -1) {
                dis[l] = dis[p] + 1;
                if(ylink[l] == -1) w = dis[l];
                else {
                    dis[ylink[l]] = dis[l] + 1;
```

```
                    q.push(ylink[l]);
                }
            }
        }
    }
    return w != INT_INF;
}

int Find(int u) {
    for(int i = 0; i < G[u].size(); i++) {
        int l = G[u][i];
        if(!vis[l] && dis[l] == dis[u] + 1) {
            vis[l] = 1;
            if(ylink[l] != -1 && dis[l] == w) continue;
            if(ylink[l] == -1 || Find(ylink[l])) {
                xlink[u] = l, ylink[l] = u;
                return 1;
            }
        }
    }
    return 0;
}

int main() {
    scanf("%d%d", &n, &m);
    int u, v;
    while(scanf("%d%d", &u, &v) != EOF) G[u].push_back(v);
    memset(xlink, -1, sizeof(xlink));
    memset(ylink, -1, sizeof(ylink));
    int ans = 0;
    while(BFS()) {
        memset(vis, 0, sizeof(vis));
        for(int i = 1; i <= m; i++)
            if(xlink[i] == -1) ans += Find(i);
    }
    printf("%d\n", ans);
}
```

最佳完美匹配

```
bool match(int u) {
    S[u] = true;
    for(int v = 1; v <= n; v++) if(!T[v]) {
        int gap = lx[u]+ly[v]-w[u][v];
        if(gap == 0) {
            T[v] = true;
            if(!ylink[v] || match(ylink[v])) {
                xlink[u] = v, ylink[v] = u;
                return true;
            }
        } else {
            slack[v] = min(slack[v], gap);
        }
    }
    return false;
}

void update() {
    int a = inf;
    for(int i = 1; i <= n; i++) if(!T[i]) a = min(a, slack[i]);
    for(int i = 1; i <= n; i++) {
        if(S[i]) lx[i] -= a;
        if(T[i]) ly[i] += a;
        else slack[i] -= a;
    }
}

void KM() {
    for(int i = 1; i <= n; i++) {
        xlink[i] = ylink[i] = lx[i] = ly[i] = 0;
        for(int j = 1; j <= n; j++) lx[i] = max(lx[i], w[i][j]);
    }
    for(int i = 1; i <= n; i++) {
        for(int j = 1; j <= n; j++) slack[j] = inf;
        while(true) {
            for(int j = 1; j <= n; j++) S[j] = T[j] = 0;
            if(match(i)) break; else update();
        }
```

```
    }
}
```

生成树计数

Kirchhoff矩阵：对于无向图G，它的Kirchhoff矩阵为G对应的度数矩阵减去它的邻接矩阵

Matrix-Tree定理：对于无向图G，它的生成树个数等于其Kirchhoff矩阵任何一个$n-1$阶主子式的行列式的绝对值

```
LL det(LL A[N][N], int n) {
    int t = 0;
    for(int i = 1; i <= n; i++) {
        for(int j = i+1; j <= n; j++) while(A[j][i]) {
            LL x = A[i][i]/A[j][i];
            t++;
            for(int k = 1; k <= n; k++)
                A[i][k] = (A[i][k]-x*A[j][k]%MOD+MOD)%MOD;
            for(int k = 1; k <= n; k++)
                swap(A[i][k], A[j][k]);
        }
    }
    LL ans = 1;
    for(int i = 1; i <= n; i++) ans = (ans*A[i][i])%MOD;
    if(t&1) ans = -ans;
    return (ans%MOD+MOD)%MOD;
}
```

网络流

最大流

```
const int N = 4100, M = 200010;
int tot = 1, src, sink;
int to[M], _next[M], cap[M], head[N], pre[N], vis[N], cur[N],
used[N];

void addEdge(int u, int v, int c) {
    to[++tot] = v, _next[tot] = head[u], cap[tot] = c, head[u] =
tot;
}

void init() {
    tot = 1; memset(head, 0, sizeof(head));
}

bool BFS() {
    memset(vis, false, sizeof(vis));
    queue<int>q; q.push(src);
    vis[src] = true;
    while(!q.empty()) {
        int u = q.front(); q.pop();
        for(int i = head[u]; i; i = _next[i]) {
            int v = to[i];
            if(!cap[i] || vis[v]) continue ;
            vis[v] = true, pre[v] = pre[u]+1;
            q.push(v);
        }
    }
    return vis[sink];
}

int DFS(int u, int c) {
    if(u==sink || c==0) return c;
    int flow = 0, f;
    for(int &i = cur[u]; i; i = _next[i]) {
        int v = to[i];
        if(pre[v]==pre[u]+1 && (f=DFS(v, min(c, cap[i])))>0) {
            flow += f, c -= f, cap[i] -= f, cap[i^1] += f;
        }
```

```
    }
    return flow;
}

int maxFlow() {
    int flow = 0;
    while(BFS()) {
        memcpy(cur, head, sizeof(cur));
        flow += DFS(src, INT_INF);
    }
    return flow;
}
```

有上下界网络流

无源汇有上下界可行流

如果存在可行流，那么每条边的流量都大于等于流量的下界，因此可以令每条边的初始流量为流量的下界，在此基础上建立残量网络。初始流不一定满足流量守恒，考虑在残量网络上求出另一个不守恒的附加流。

1. 对于原图中每条(u, v, l, r)的边，在新图中建立$(u, v, r-l)$的边
2. $ du[i] $表示初始流中节点$i$的流入流量与流出流量之差
3. 对于新图，建立超级源点$source$与汇点$sink$
4. 对于满足$du[i] > 0$的节点i，在新图中建立$(source, i, du[i])$的边
5. 对于满足$du[i] < 0$的节点i，在信徒中建立$(i, sink, -du[i])$的边

若新图可以满流，则存在一个可行流，否则不存在。

```
bool lowbound_flow(int n, vector<int>U, vector<int>V, vector<int>L,
vector<int>R) {
    dinic::init();
    memset(du, 0, sizeof(du));
    int ln = U.size();
    for(int i = 0; i < ln; i++) {
        if(R[i] < L[i]) return 0;
        dinic::addEdge(U[i], V[i], R[i]-L[i], i+1);
        dinic::addEdge(V[i], U[i], 0, 0);
        du[U[i]] -= L[i], du[V[i]] += L[i];
    }
    dinic::src = n+1, dinic::sink = n+2;
    int sum = 0;
    for(int i = 0; i <= n; i++) {
        if(du[i] > 0) {
            dinic::addEdge(dinic::src, i, du[i], 0);
            dinic::addEdge(i, dinic::src, 0, 0);
            sum += du[i];
        } else if(du[i] < 0) {
            dinic::addEdge(i, dinic::sink, -du[i], 0);
            dinic::addEdge(dinic::sink, i, 0, 0);
        }
    }
    return dinic::maxFlow() == sum;
}
```

有源汇有上下界可行流

连接一条$(sink, src, inf)$的边，问题转化为无源汇有上下界可行流

有源汇有上下界最大流

先求一个有源汇有上下界可行流，然后再在原来的残量网络上面进行增广，最后的最大流即为可行流（原图中$sink$到src的流量）$+$ 残量网络最大流

```
int lowboundMaxflow(int s, int t, int n, vector<int>U, vector<int>V,
vector<int>L, vector<int>R) {
    memset(du, 0, sizeof(du)); dinic::init();
    int m = U.size();
    dinic::src = n+1, dinic::sink = n+2;
    for(int i = 0; i < m; i++) {
        if(L[i] > R[i]) return -1;
        dinic::addEdge(U[i], V[i], R[i]-L[i]);
        dinic::addEdge(V[i], U[i], 0);
        du[U[i]] -= L[i], du[V[i]] += L[i];
    }
    int sum = 0;
    for(int i = 0; i <= n; i++) {
        if(du[i] > 0) {
            dinic::addEdge(dinic::src, i, du[i]);
            dinic::addEdge(i, dinic::src, 0);
            sum += du[i];
        } else if(du[i] < 0) {
            dinic::addEdge(i, dinic::sink, -du[i]);
            dinic::addEdge(dinic::sink, i, 0);
        }
    }
    dinic::addEdge(t, s, INF), dinic::addEdge(s, t, 0);
    if(dinic::maxFlow() < sum) return -1;
    else {
        int flow = dinic::cap[dinic::tot];
        dinic::cap[dinic::tot] = dinic::cap[dinic::tot-1] = 0;
        for(int i = dinic::head[dinic::src]; i; i = dinic::nxt[i])
            dinic::cap[i] = dinic::cap[i^1] = 0;
        for(int i = dinic::head[dinic::sink]; i; i = dinic::nxt[i])
            dinic::cap[i] = dinic::cap[i^1] = 0;
        dinic::src = s, dinic::sink = t;
        return flow + dinic::maxFlow();
```

```
    }
}
```

有源汇有上下界最小流

求完可行流之后在残量网络上进行从t到s的最大流，可行流减去t到s的最大流即为最小流

最小费用最大流

```
void addEdge(int u, int v, int w, int c) {
    to[++tot] = v, nxt[tot] = head[u], cap[tot] = w, cost[tot] = c,
head[u] = tot;
}

bool spfaMin(int &c, int &f) {
    for(int i = src; i <= sink; i++) d[i] = INT_INF, inq[i]=0;
    queue<int>q; q.push(src);
    d[src] = 0, inq[src] = 1, pre[src] = 0, a[src] = INT_INF;
    while(!q.empty()) {
        int p = q.front(); q.pop();
        inq[p] = 0;
        for(int i = head[p]; i; i = nxt[i]) {
            int l = to[i];
            if(cap[i] && d[l] > d[p] + cost[i]) {
                d[l] = d[p] + cost[i]; pre[l] = i; a[l] = min(a[p],
cap[i]);
                if(!inq[l]) q.push(l); inq[l] = 1;
            }
        }
    }
    if(d[sink] == INT_INF) return false;
    c += d[sink]*a[sink], f += a[sink];
    int u = sink;
    while(u != src) {
        cap[pre[u]] -= a[sink], cap[pre[u]^1] += a[sink];
        u = to[pre[u]^1];
    }
    return true;
}
```

最大费用最大流

```
int tot = 1, src, sink;
int to[M], nxt[M], cap[M], cost[M], head[N], d[N], a[N], inq[N],
pre[N];
```

```
void addEdge(int u, int v, int w, int c) {
    to[++tot] = v, nxt[tot] = head[u], cap[tot] = w, cost[tot] = c,
head[u] = tot;
}

bool spfaMax(int &c, int &f) {
    for(int i = src; i <= sink; i++) d[i] = -INT_INF, inq[i]=0;
    queue<int>q; q.push(src);
    d[src] = 0, inq[src] = 1, pre[src] = 0, a[src] = INT_INF;
    while(!q.empty()) {
        int p = q.front(); q.pop();
        inq[p] = 0;
        for(int i = head[p]; i; i = nxt[i]) {
            int l = to[i];
            if(cap[i] && d[l] < d[p] + cost[i]) {
                d[l] = d[p] + cost[i]; pre[l] = i; a[l] = min(a[p],
cap[i]);
                if(!inq[l]) q.push(l); inq[l] = 1;
            }
        }
    }
    if(d[sink] == -INT_INF) return false;
    c += d[sink]*a[sink], f += a[sink];
    int u = sink;
    while(u != src) {
        cap[pre[u]] -= a[sink], cap[pre[u]^1] += a[sink];
        u = to[pre[u]^1];
    }
    return true;
}
```