高等数学在高中数学中的推论与应用

Implications and Applications of Advanced Mathematics in High School Mathematics

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目录

前言 Preface

本教材旨在为读者提供系统的数学知识学习指导。
This textbook aims to provide readers with systematic guidance for learning mathematics.

Chapter 1

集合 Sets

1.1 集合中元素的特性 Properties of Set Elements

定义 **Definition 1.1.1** (集合 Set)**.** 集合是具有某种特定性质的事物的总体。组成集合的每个事物称为该集合的元素。
A set is a collection of objects that share certain specific properties. Each object that makes up the set is called an element of the set.

定义 **Definition 1.1.2** (属于关系 Membership)**.** 若元素 a 是集合 A 的元素，记作 $a \in A$；若元素 a 不是集合 A 的元素，记作 $a \notin A$。
If an object a is an element of set A, we write $a \in A$; if a is not an element of set A, we write $a \notin A$.

注 **Remark 1.1.3** (元素的特性 Element Properties)**.** 集合具有以下基本特性：
Sets have the following basic properties:

1. 确定性：任何事物对于一个集合来说，或者属于这个集合，或者不属于这个集合。
Definiteness: For any object and a given set, the object

either belongs to the set or does not belong to the set.

2. 无序性：集合中的元素没有顺序之分。例如，$\{1,2,3\}=\{3,1,2\}$。
   No order: Elements in a set have no particular order. For example, $\{1,2,3\}=\{3,1,2\}$.
3. 互异性：集合中的元素都是互不相同的。例如，$\{1,2,2,3\}=\{1,2,3\}$。
   Uniqueness: Elements in a set are distinct. For example, $\{1,2,2,3\}=\{1,2,3\}$.

1.2 集合的表示方法 Set Notation

定义 Definition 1.2.1 (集合的表示法 Set Representation). 集合主要有以下表示方法:
Sets can be represented in the following ways:

1. 列举法 (Listing Method)：直接列出所有元素。
   List all elements directly. 例 (Example): $A=\{1,2,3,4,5\}$
2. 描述法 (Description Method)：用谓词表达式描述元素的特征。
   Describe the characteristics of elements using predicate expressions. 例 (Example): $B=\{x|x\ \text{是小于6 的正整数}\}$
   $B=\{x|x$ is a positive integer less than $6\}$

例 Example 1.2.2 (集合表示的等价性 Equivalence of Set Representations). 以下集合表示是等价的:
The following set representations are equivalent:

$$A=\{1,2,3,4,5\}$$
$$A=\{x|x\ \text{是小于6 的正整数}\}$$
$$A=\{x|x\in\mathbb{N},1\le x\le 5\}$$

1.3 常用的集合性质 Common Set Properties

定义 Definition 1.3.1 (特殊集合 Special Sets). 1. 空集 (Empty Set)：不含任何元素的集合，记作 $\emptyset$ 或 $\{\}$。

A set that contains no elements, denoted as $\emptyset$ or $\{\}$.

2. 全集 (Universal Set)：在讨论问题时涉及的所有元素构成的集合，通常记作 U。
   The set of all elements under discussion, usually denoted as U.

定义 Definition 1.3.2 (子集 Subset). 若集合 A 的每个元素都是集合 B 的元素，则称 A 是 B 的子集，记作 $A \subseteq B$。
If every element of set A is also an element of set B, then A is called a subset of B, denoted as $A \subseteq B$.

定理 Theorem 1.3.3 (子集的基本性质 Basic Properties of Subsets). 对于任意集合 A, B, C:
For any sets A, B, C:

1. $\emptyset \subseteq A$ （空集是任意集合的子集）
   (The empty set is a subset of any set)
2. $A \subseteq A$ （任意集合是自身的子集）
   (Any set is a subset of itself)
3. 若 $A \subseteq B$ 且 $B \subseteq C$，则 $A \subseteq C$ （传递性）
   (If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ - transitivity)

1.4 有限集合的元素个数 Cardinality of Finite Sets

定义 Definition 1.4.1 (基数 Cardinality). 有限集合中元素的个数称为该集合的基数（或势），记作 $|A|$ 或 $\#A$。
The number of elements in a finite set is called its cardinality, denoted as $|A|$ or $\#A$.

定理 Theorem 1.4.2 (基数的性质 Properties of Cardinality). 对于有限集合 A 和 B:
For finite sets A and B:

1. $|A \cup B| = |A| + |B| - |A \cap B|$ （容斥原理）
   (Inclusion-Exclusion Principle)
2. 若 $A \subseteq B$，则 $|A| \leq |B|$
   (If $A \subseteq B$, then $|A| \leq |B|$)
3. $|\emptyset| = 0$
   (The cardinality of the empty set is 0)

1.5 集合间的运算 Set Operations

定义 Definition 1.5.1 (基本集合运算 Basic Set Operations). 设 A，B 是两个集合：
Let A, B be two sets:

1. 并集 (Union)：$A \cup B = \{x|x \in A \text{ 或} x \in B\}$
   $A \cup B = \{x|x \in A$ or $x \in B\}$
2. 交集 (Intersection)：$A \cap B = \{x|x \in A \text{ 且} x \in B\}$
   $A \cap B = \{x|x \in A$ and $x \in B\}$
3. 差集 (Difference)：$A \setminus B = \{x|x \in A \text{ 且} x \notin B\}$
   $A \setminus B = \{x|x \in A$ and $x \notin B\}$
4. 补集 (Complement)：$A^c = \{x|x \in U \text{ 且} x \notin A\}$
   $A^c = \{x|x \in U$ and $x \notin A\}$

定理 Theorem 1.5.2 (集合运算的基本性质 Basic Properties of Set Operations). 对任意集合 A，B，C：
For any sets A, B, C:

1. 交换律 (Commutativity)：

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2. 结合律 (Associativity):

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

3. 分配律 (Distributivity):

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. 德摩根律 (De Morgan's Laws):

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

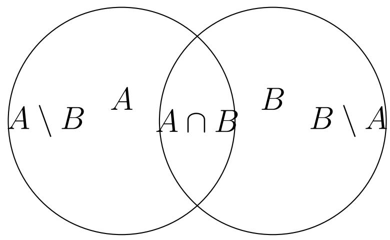

图 1.1: 集合运算的维恩图示意
Venn Diagram of Set Operations

Chapter 2

函数的性质 Properties of Functions

2.1 函数的相等 Equality of Functions

函数相等的定义是指两个函数的定义域相同，且对于定义域内的每一个值，两个函数的函数值都相等。这是一个非常严格的定义，需要同时满足两个条件。Two functions are equal if they have the same domain and yield the same values for each element in their domain. This is a strict definition that requires both conditions to be met simultaneously.

定义 Definition 2.1.1. 设函数 $f(x)$ 和 $g(x)$ 的定义域分别为 D_f 和 D_g，若:
Let functions $f(x)$ and $g(x)$ have domains D_f and D_g respectively. The functions are equal if:

1. $D_f = D_g$ （定义域相等 / Domains are equal）
2. $\forall x \in D_f$，都有 $f(x) = g(x)$ （函数值相等 / Function values are equal）

则称函数 $f(x)$ 与 $g(x)$ 相等，记作 $f(x) = g(x)$ 或 $f = g$。
Then we say $f(x)$ equals $g(x)$, denoted as $f(x) = g(x)$ or $f = g$.

例 Example 2.1.2 (函数相等的判断 Determining Function Equality). 考虑以下函数：
Consider the following functions:

$$f(x) = x^2, \quad D_f = [-1, 1]$$

$$g(x) = x^2, \quad D_g = [0, 2]$$

$$h(x) = x^2, \quad D_h = [-1, 1]$$

这里：
Here:

- f 与 g 不相等，因为它们的定义域不同。
  f and g are not equal because they have different domains.
- f 与 h 相等，因为它们的定义域相同且对应的函数值相同。
  f and h are equal because they have the same domain and corresponding function values.

注 Remark 2.1.3 (函数相等的重要性 Importance of Function Equality). 函数相等的概念在数学分析中起着重要作用：
The concept of function equality plays a crucial role in mathematical analysis:

1. 在研究函数的极限时，我们需要判断两个函数是否相等。
   When studying function limits, we need to determine if two functions are equal.
2. 在证明某些函数性质时，常常需要证明两个不同形式的函数实际上是相等的。
   When proving certain function properties, we often need to prove that two differently expressed functions are actually equal.

3. 在解微分方程时，验证解的正确性需要判断两个函数是否相等。
When solving differential equations, verifying solutions requires determining if two functions are equal.

2.2 函数的单调性 Monotonicity of Functions

函数的单调性是描述函数值变化趋势的重要特征。单调性不仅帮助我们理解函数的基本性质，还在求解方程、不等式等问题中发挥重要作用。
Monotonicity is a crucial characteristic that describes how function values change. It not only helps us understand the basic properties of functions but also plays an important role in solving equations and inequalities.

2.2.1 单调性的定义 Definition of Monotonicity

定义 Definition 2.2.1 (严格单调性 Strict Monotonicity). 设函数 $f(x)$ 的定义域为 D，区间 $I \subseteq D$:
Let $f(x)$ be a function with domain D, and interval $I \subseteq D$:

1. 若对于区间 I 内的任意两点 $x_1 < x_2$，恒有 $f(x_1) < f(x_2)$，则称 $f(x)$ 在 I 上严格单调递增。
If for any $x_1 < x_2$ in I, we have $f(x_1) < f(x_2)$, then $f(x)$ is strictly increasing on I.
2. 若对于区间 I 内的任意两点 $x_1 < x_2$，恒有 $f(x_1) > f(x_2)$，则称 $f(x)$ 在 I 上严格单调递减。
If for any $x_1 < x_2$ in I, we have $f(x_1) > f(x_2)$, then $f(x)$ is strictly decreasing on I.

定义 Definition 2.2.2 (广义单调性 Non-strict Monotonicity). 设函数 $f(x)$ 的定义域为 D，区间 $I \subseteq D$:
Let $f(x)$ be a function with domain D, and interval $I \subseteq D$:

1. 若对于区间 I 内的任意两点 $x_1 < x_2$，恒有 $f(x_1) \leq f(x_2)$，则称 $f(x)$

在 I 上单调递增。
If for any $x_1 < x_2$ in I, we have $f(x_1) \leq f(x_2)$, then $f(x)$ is monotonically increasing on I.

2. 若对于区间 I 内的任意两点 $x_1 < x_2$，恒有 $f(x_1) \geq f(x_2)$，则称 $f(x)$ 在 I 上单调递减。
If for any $x_1 < x_2$ in I, we have $f(x_1) \geq f(x_2)$, then $f(x)$ is monotonically decreasing on I.

例 Example 2.2.3 (常见函数的单调性 Monotonicity of Common Functions). 1. $f(x) = x^3$ 在 $(-\infty, +\infty)$ 上严格单调递增。
$f(x) = x^3$ is strictly increasing on $(-\infty, +\infty)$.

2. $g(x) = \frac{1}{x}$ 在 $(0, +\infty)$ 上严格单调递减。
$g(x) = \frac{1}{x}$ is strictly decreasing on $(0, +\infty)$.
3. $h(x) = |x|$ 在 $(-\infty, 0]$ 上严格单调递减，在 $[0, +\infty)$ 上严格单调递增。
$h(x) = |x|$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$.
4. $k(x) = \sin x$ 在 $[-\frac{\pi}{2}, \frac{\pi}{2}]$ 上严格单调递增。
$k(x) = \sin x$ is strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

定理 Theorem 2.2.4 (单调函数的性质 Properties of Monotonic Functions). 设函数 $f(x)$ 在区间 I 上单调:
Let $f(x)$ be monotonic on interval I:

1. 若 $f(x)$ 在 I 上严格单调，则 $f(x)$ 在 I 上是一一对应的。
If $f(x)$ is strictly monotonic on I, then $f(x)$ is one-to-one on I.
2. 若 $f(x)$ 在 I 上严格单调递增，则其反函数 $f^{-1}(x)$ 在 $f(I)$ 上也严格单调递增。
If $f(x)$ is strictly increasing on I, then its inverse function $f^{-1}(x)$ is also strictly increasing on $f(I)$.
3. 若 $f(x)$ 在 I 上严格单调递减，则其反函数 $f^{-1}(x)$ 在 $f(I)$ 上也严格单调递减。
If $f(x)$ is strictly decreasing on I, then its inverse function $f^{-1}(x)$ is also strictly decreasing on $f(I)$.

2.3 极值与最值 Extrema and Extreme Values

2.3.1 局部极值 Local Extrema

定义 Definition 2.3.1 (局部极值 Local Extrema). 设函数 $f(x)$ 在点 x_0 的某邻域内有定义:
Let $f(x)$ be defined in a neighborhood of point x_0:

1. 若存在 x_0 的某个邻域 $U(x_0)$,使得对于该邻域内的任意点 $x \neq x_0$,恒有 $f(x) < f(x_0)$,则称 $f(x_0)$ 为 $f(x)$ 的局部极大值,点 x_0 称为局部极大值点。
   If there exists a neighborhood $U(x_0)$ of x_0 where $f(x) < f(x_0)$ for all $x \neq x_0$, then $f(x_0)$ is a local maximum, and x_0 is a local maximum point.
2. 若存在 x_0 的某个邻域 $U(x_0)$,使得对于该邻域内的任意点 $x \neq x_0$,恒有 $f(x) > f(x_0)$,则称 $f(x_0)$ 为 $f(x)$ 的局部极小值,点 x_0 称为局部极小值点。
   If there exists a neighborhood $U(x_0)$ of x_0 where $f(x) > f(x_0)$ for all $x \neq x_0$, then $f(x_0)$ is a local minimum, and x_0 is a local minimum point.

定理 Theorem 2.3.2 (可导函数的极值必要条件 Necessary Condition for Extrema of Differentiable Functions). 若函数 $f(x)$ 在点 x_0 处可导且取得极值,则必有:
If function $f(x)$ is differentiable at x_0 and has an extremum there, then:

$$f'(x_0) = 0$$

这样的点称为函数的驻点。
Such points are called stationary points.

例 Example 2.3.3 (极值点的判定 Determining Extremum Points). 考虑函数 $f(x) = x^3 - 3x^2 + 2$:
Consider the function $f(x) = x^3 - 3x^2 + 2$:

1. 求导: $f'(x) = 3x^2 - 6x = 3x(x-2)$
   Differentiate: $f'(x) = 3x^2 - 6x = 3x(x-2)$
2. 驻点: $x = 0$ 或 $x = 2$
   Stationary points: $x = 0$ or $x = 2$
3. 通过二阶导数 $f''(x) = 6x - 6$ 判断:
   Use second derivative $f''(x) = 6x - 6$ to determine:

   - 当 $x = 0$ 时, $f''(0) < 0$, 所以 $x = 0$ 是局部极大值点
     At $x = 0$, $f''(0) < 0$, so $x = 0$ is a local maximum point
   - 当 $x = 2$ 时, $f''(2) > 0$, 所以 $x = 2$ 是局部极小值点
     At $x = 2$, $f''(2) > 0$, so $x = 2$ is a local minimum point

2.3.2 全局最值 Global Extreme Values

定义 Definition 2.3.4 (全局最值 Global Extreme Values). 设函数 $f(x)$ 的定义域为 D:
Let $f(x)$ have domain D:

1. 若存在点 $x_M \in D$, 使得对于任意 $x \in D$, 都有 $f(x) \leq f(x_M)$, 则称 $f(x_M)$ 为 $f(x)$ 在 D 上的最大值。
   If there exists $x_M \in D$ such that $f(x) \leq f(x_M)$ for all $x \in D$, then $f(x_M)$ is the maximum value of $f(x)$ on D.
2. 若存在点 $x_m \in D$, 使得对于任意 $x \in D$, 都有 $f(x) \geq f(x_m)$, 则称 $f(x_m)$ 为 $f(x)$ 在 D 上的最小值。
   If there exists $x_m \in D$ such that $f(x) \geq f(x_m)$ for all $x \in D$, then $f(x_m)$ is the minimum value of $f(x)$ on D.

定理 Theorem 2.3.5 (最值存在定理 Existence Theorem for Extreme Values). 若函数 $f(x)$ 在闭区间 $[a, b]$ 上连续, 则:
If function $f(x)$ is continuous on closed interval $[a, b]$, then:

1. $f(x)$ 在 $[a, b]$ 上必有最大值和最小值。
   $f(x)$ must have both maximum and minimum values on $[a, b]$.

2. 最值点只可能在以下位置:
   The extreme points can only occur at:

   - 区间端点 a 或 b
     Endpoints a or b
   - 区间内部的驻点（若函数可导）
     Interior stationary points (if the function is differentiable)
   - 区间内部的不可导点
     Interior points where the function is not differentiable

2.4 函数的对称性 Symmetry of Functions

2.4.1 奇偶性 Odd and Even Functions

定义 Definition 2.4.1 (奇偶函数 Odd and Even Functions). 设函数 $f(x)$ 的定义域关于原点对称:
Let $f(x)$ have a domain symmetric about the origin:

1. 若对于任意 $x \in D_f$，恒有 $f(-x) = f(x)$，则称 $f(x)$ 为偶函数。
   If $f(-x) = f(x)$ for all $x \in D_f$, then $f(x)$ is an even function.
2. 若对于任意 $x \in D_f$，恒有 $f(-x) = -f(x)$，则称 $f(x)$ 为奇函数。
   If $f(-x) = -f(x)$ for all $x \in D_f$, then $f(x)$ is an odd function.

例 Example 2.4.2 (常见的奇偶函数 Common Odd and Even Functions). 1. 偶函数例子:
   Examples of even functions:

   - $f(x) = x^2$
   - $g(x) = \cos x$
   - $h(x) = |x|$

2. 奇函数例子:
   Examples of odd functions:

   - $f(x) = x^3$
   - $g(x) = \sin x$
   - $h(x) = \tan x$

定理 Theorem 2.4.3 (奇偶函数的性质 Properties of Odd and Even Functions). 两个偶函数的和、差、积都是偶函数。
The sum, difference, and product of two even functions are even functions.

2. 两个奇函数的和、差是奇函数，积是偶函数。
   The sum and difference of two odd functions are odd functions, while their product is an even function.
3. 奇函数与偶函数的积是奇函数。
   The product of an odd function and an even function is an odd function.
4. 偶函数的导数是奇函数，奇函数的导数是偶函数。
   The derivative of an even function is odd, and the derivative of an odd function is even.
5. 偶函数的定积分在对称区间上的值:
   For even functions, definite integrals over symmetric intervals:
   $$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$
6. 奇函数的定积分在对称区间上的值:
   For odd functions, definite integrals over symmetric intervals:
   $$\int_{-a}^{a} f(x)dx = 0$$

2.5 周期函数 Periodic Functions

定义 Definition 2.5.1 (周期函数 Periodic Functions). 若存在一个正数 T，使得对于函数 $f(x)$ 的定义域内的任意 x，都有:
If there exists a positive number T such that for any x in

the domain of $f(x)$:

$$f(x+T)=f(x)$$

则称 $f(x)$ 为周期函数，T 称为 $f(x)$ 的一个周期。最小的正周期称为基本周期。

Then $f(x)$ is called a periodic function, and T is called a period of $f(x)$. The smallest positive period is called the fundamental period.

例 Example 2.5.2 (常见的周期函数 Common Periodic Functions). 1. 三角函数：
Trigonometric functions:

- $\sin x$ 和 $\cos x$ 的基本周期是 2π
  The fundamental period of $\sin x$ and $\cos x$ is 2π
- $\tan x$ 的基本周期是 π
  The fundamental period of $\tan x$ is π

2. 复合三角函数：
   Composite trigonometric functions:

   - $f(x)=\sin(ax)$ 的周期是 $\frac{2\pi}{|a|}$
     The period of $f(x)=\sin(ax)$ is $\frac{2\pi}{|a|}$
   - $g(x)=\sin^2 x$ 的周期是 π
     The period of $g(x)=\sin^2 x$ is π

定理 Theorem 2.5.3 (周期函数的基本性质 Basic Properties of Periodic Functi
若 T 是周期函数 $f(x)$ 的一个周期，则 nT（n 为非零整数）也是 $f(x)$ 的周期。
If T is a period of $f(x)$, then nT (n being a non-zero integer) is also a period.

2. 若 T_1、T_2 都是 $f(x)$ 的周期，则 $|mT_1+nT_2|$（m、n 为整数）也是 $f(x)$ 的周期。
   If T_1 and T_2 are periods of $f(x)$, then $|mT_1+nT_2|$ (m, n being integers) is also a period.
3. 周期函数的导数（如果存在）也是周期函数，且周期相同。
   The derivative of a periodic function (if it exists) is

also periodic with the same period.

4. 周期函数的定积分在一个周期上的值与积分区间的起点无关，只与区间长度有关。
   The definite integral of a periodic function over one period is independent of the starting point of the interval and only depends on the length of the interval.

2.6 组合函数的单调性 Monotonicity of Composite Functions

定理 Theorem 2.6.1 (复合函数的单调性 Monotonicity of Composite Functions). 设 $f(x)$ 的定义域为 D_f，值域为 R_f，$g(x)$ 的定义域为 D_g，且 $R_f \subseteq D_g$：
Let $f(x)$ have domain D_f and range R_f, and $g(x)$ have domain D_g where $R_f \subseteq D_g$:

1. 若 $f(x)$ 和 $g(x)$ 都在各自的定义域上单调递增，则 $g(f(x))$ 在 D_f 上单调递增。
   If both $f(x)$ and $g(x)$ are increasing on their domains, then $g(f(x))$ is increasing on D_f.
2. 若 $f(x)$ 和 $g(x)$ 都在各自的定义域上单调递减，则 $g(f(x))$ 在 D_f 上单调递增。
   If both $f(x)$ and $g(x)$ are decreasing on their domains, then $g(f(x))$ is increasing on D_f.
3. 若其中一个函数单调递增，另一个单调递减，则 $g(f(x))$ 在 D_f 上单调递减。
   If one function is increasing and the other is decreasing, then $g(f(x))$ is decreasing on D_f.

例 Example 2.6.2 (复合函数单调性的应用 Application of Composite Function Monotonicity). 考虑函数 $h(x) = \sin(\sqrt{x})$，$x \geq 0$：
Consider the function $h(x) = \sin(\sqrt{x})$, $x \geq 0$:

1. 令 $f(x)=\sqrt{x}$，$g(x)=\sin x$
   Let $f(x)=\sqrt{x}$, $g(x)=\sin x$
2. $f(x)$ 在 $[0,+\infty)$ 上单调递增
   $f(x)$ is increasing on $[0,+\infty)$
3. $g(x)=\sin x$ 在 $[0,\frac{\pi}{2}]$ 上单调递增
   $g(x)=\sin x$ is increasing on $[0,\frac{\pi}{2}]$
4. 因此，$h(x)=g(f(x))$ 在 $[0,\frac{\pi^2}{4}]$ 上单调递增
   Therefore, $h(x)=g(f(x))$ is increasing on $[0,\frac{\pi^2}{4}]$

2.7 特殊函数的性质 Properties of Special Functions

2.7.1 反函数 Inverse Functions

定理 Theorem 2.7.1 (反函数的性质 Properties of Inverse Functions). 若函数 $f(x)$ 在区间 I 上严格单调，则:
If function $f(x)$ is strictly monotonic on interval I, then:

1. $f(x)$ 在 I 上存在反函数 $f^{-1}(x)$。
   $f(x)$ has an inverse function $f^{-1}(x)$ on I.
2. 若 $f(x)$ 单调递增，则 $f^{-1}(x)$ 也单调递增。
   If $f(x)$ is increasing, then $f^{-1}(x)$ is also increasing.
3. 若 $f(x)$ 单调递减，则 $f^{-1}(x)$ 也单调递减。
   If $f(x)$ is decreasing, then $f^{-1}(x)$ is also decreasing.
4. 若 $f(x)$ 在 I 上连续，则 $f^{-1}(x)$ 在 $f(I)$ 上也连续。
   If $f(x)$ is continuous on I, then $f^{-1}(x)$ is also continuous on $f(I)$.

2.7.2 分段函数 Piecewise Functions

定义 Definition 2.7.2 (分段函数的连续性 Continuity of Piecewise Functions). 分段函数在分段点处连续的充要条件是各段函数在该点的函数值相等。

A piecewise function is continuous at a partition point if and only if the function values from different pieces are equal at that point.

例 Example 2.7.3 (分段函数的性质分析 Analysis of Piecewise Function Properties). 考虑函数:
Consider the function:

$$f(x) = \begin{cases} x^2, & x < 0 \\ \sin x, & 0 \leq x \leq \pi \\ x - \pi + 1, & x > \pi \end{cases}$$

分析其性质:
Analyze its properties:

1. 连续性: 需要检查 $x = 0$ 和 $x = \pi$ 处的函数值
   Continuity: need to check function values at $x = 0$ and $x = \pi$
2. 单调性: 需要分别分析每个区间的单调性
   Monotonicity: need to analyze monotonicity in each interval
3. 极值点: 可能出现在分段点或各段函数的极值点处
   Extreme points: may occur at partition points or at extreme points of each piece

Chapter 3

二次函数 Quadratic Functions

3.1 二次函数的表示方式 Representations of Quadratic Functions

二次函数有三种标准的表示方式，每种形式都有其特定的用途和优点。
Quadratic functions have three standard forms of representation, each with its specific uses and advantages.

3.1.1 一般式 General Form

定义 Definition 3.1.1 (一般式 General Form). 二次函数的一般式为：
The general form of a quadratic function is:

$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$

其中：
where:

- a 为二次项系数，决定了抛物线的开口方向和宽窄
  a is the coefficient of the quadratic term, determining the direction and width of the parabola
- b 为一次项系数，影响对称轴的位置
  b is the coefficient of the linear term, affecting the position of the axis of symmetry

- c 为常数项，表示函数图像与 y 轴的交点
  c is the constant term, representing the y-intercept

3.1.2 顶点式 Vertex Form

定义 Definition 3.1.2 (顶点式 Vertex Form). 二次函数的顶点式为:
The vertex form of a quadratic function is:

$$f(x) = a(x-h)^2 + k \quad (a \neq 0)$$

其中:
where:

- (h, k) 为抛物线的顶点
  (h, k) is the vertex of the parabola
- a 的符号决定开口方向，$|a|$ 决定开口的宽窄
  The sign of a determines the direction, and $|a|$ determines the width

例 Example 3.1.3 (一般式转顶点式 Converting General Form to Vertex Form). 将 $f(x) = 2x^2 + 4x - 1$ 转化为顶点式:
Convert $f(x) = 2x^2 + 4x - 1$ to vertex form:

1. 配方: $f(x) = 2(x^2 + 2x) - 1$
   Complete the square: $f(x) = 2(x^2 + 2x) - 1$
2. 继续配方: $f(x) = 2(x^2 + 2x + 1 - 1) - 1$
   Continue: $f(x) = 2(x^2 + 2x + 1 - 1) - 1$
3. 最终形式: $f(x) = 2(x+1)^2 - 3$
   Final form: $f(x) = 2(x+1)^2 - 3$
4. 顶点为 $(-1, -3)$
   The vertex is $(-1, -3)$

3.1.3 因式分解式 Factored Form

定义 Definition 3.1.4 (因式分解式 Factored Form). 当二次函数有实数根时，可以写成因式分解式：
When the quadratic function has real roots, it can be written in factored form:

$$f(x) = a(x - x_1)(x - x_2) \quad (a \neq 0)$$

其中：
where:

- x_1、x_2 为函数的根（零点）
  x_1 and x_2 are the roots (zeros) of the function
- 当 $x_1 = x_2$ 时，表示有重根
  When $x_1 = x_2$, it indicates a repeated root

3.2 一元二次方程的性质 Properties of Quadratic Equations

3.2.1 判别式 Discriminant

定义 Definition 3.2.1 (判别式 Discriminant). 对于一元二次方程 $ax^2 + bx + c = 0$ $(a \neq 0)$，其判别式为：
For the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$, its discriminant is:

$$\Delta = b^2 - 4ac$$

判别式决定了方程根的性质：
The discriminant determines the nature of the roots:

- $\Delta > 0$：方程有两个不同的实根
  $\Delta > 0$: equation has two distinct real roots
- $\Delta = 0$：方程有一个重根
  $\Delta = 0$: equation has one repeated root

- $\Delta < 0$：方程有两个共轭复根
  $\Delta < 0$: equation has two conjugate complex roots

3.2.2 韦达定理 Vieta's Formulas

定理 Theorem 3.2.2 (韦达定理 Vieta's Formulas). 设一元二次方程 $ax^2 + bx + c = 0 \ (a \neq 0)$ 的两个根为 x_1 和 x_2，则：
If x_1 and x_2 are the roots of the quadratic equation $ax^2+bx+c = 0 \ (a \neq 0)$, then:

$$x_1 + x_2 = -\frac{b}{a}$$
$$x_1x_2 = \frac{c}{a}$$

这两个公式反映了根与系数之间的关系。
These formulas reflect the relationship between roots and coefficients.

例 Example 3.2.3 (韦达定理的应用 Application of Vieta's Formulas). 求解方程 $x^2 - 5x + 6 = 0$ 的根：
Find the roots of the equation $x^2 - 5x + 6 = 0$:

1. 由韦达定理：$x_1 + x_2 = 5$，$x_1x_2 = 6$
   By Vieta's formulas: $x_1 + x_2 = 5$, $x_1x_2 = 6$
2. 因此 x_1 和 x_2 是方程 $t^2 - 5t + 6 = 0$ 的根
   Therefore, x_1 and x_2 are roots of $t^2 - 5t + 6 = 0$
3. 解得：$x_1 = 2$，$x_2 = 3$
   Solving: $x_1 = 2$, $x_2 = 3$

3.3 二次函数的图像特征 Graphical Features of Quadratic Functions

3.3.1 对称性 Symmetry

定理 Theorem 3.3.1 (对称轴 Axis of Symmetry). 二次函数 $f(x) = ax^2 + bx + c \ (a \neq 0)$ 的图像关于直线 $x = -\frac{b}{2a}$ 对称。

The graph of the quadratic function $f(x) = ax^2 + bx + c \ \ (a \neq 0)$ is symmetric about the line $x = -\frac{b}{2a}$.

3.3.2 顶点 Vertex

定理 Theorem 3.3.2 (顶点坐标 Vertex Coordinates). 二次函数 $f(x) = ax^2 + bx + c \ \ (a \neq 0)$ 的顶点坐标为:
The vertex coordinates of the quadratic function $f(x) = ax^2 + bx + c \ \ (a \neq 0)$ are:

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

或写作:
or written as:

$$\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$$

其中 $\Delta = b^2 - 4ac$ 为判别式。
where $\Delta = b^2 - 4ac$ is the discriminant.

3.4 一元二次不等式 Quadratic Inequalities

3.4.1 求解方法 Solution Methods

定理 Theorem 3.4.1 (一元二次不等式的求解步骤 Steps for Solving Quadratic Inequalities). 求解 $ax^2 + bx + c > 0 \ \ (a \neq 0)$ 的步骤:
Steps to solve $ax^2 + bx + c > 0 \ \ (a \neq 0)$:

1. 求出二次函数的根（如果存在）
   Find the roots of the quadratic function (if they exist)
2. 确定二次项系数 a 的符号
   Determine the sign of the coefficient a
3. 根据抛物线的开口方向和零点，写出解集
   Write the solution set based on the direction of the parabola and its zeros

例 Example 3.4.2 (一元二次不等式的求解 Solving Quadratic In-

equalities). 求解不等式 $2x^2-3x-2>0$:
Solve the inequality $2x^2-3x-2>0$:

1. 求根: $2x^2-3x-2=0$
   Find roots: $2x^2-3x-2=0$
2. 因式分解: $2x^2-3x-2=(2x+1)(x-2)$
   Factorize: $2x^2-3x-2=(2x+1)(x-2)$
3. 得到根: $x=-\frac{1}{2}$ 或 $x=2$
   Get roots: $x=-\frac{1}{2}$ or $x=2$
4. 由于 $a>0$，抛物线开口向上
   Since $a>0$, the parabola opens upward
5. 解集为: $(-\infty,-\frac{1}{2})\cup(2,+\infty)$
   Solution set: $(-\infty,-\frac{1}{2})\cup(2,+\infty)$

3.5 不等式恒成立问题 Inequality Identity Problems

3.5.1 基本方法 Basic Methods

定理 Theorem 3.5.1 (不等式恒成立的判定方法 Methods for Determining Inequality Identities). 判断二次不等式 $ax^2+bx+c>0$ 恒成立的方法:
Methods to determine if a quadratic inequality $ax^2+bx+c>0$ is always true:

1. 判别式法: 若 $a>0$ 且 $\Delta<0$，则不等式恒成立
   Discriminant method: if $a>0$ and $\Delta<0$, the inequality always holds
2. 配方法: 将左边化为完全平方式判断
   Complete square method: transform the left side into perfect square form
3. 函数图像法: 判断函数图像是否始终在 x 轴上方
   Graphical method: determine if the function graph is

always above the x-axis

例 Example 3.5.2 (不等式恒成立的判断 Determining Inequality Identities). 判断对于任意实数 x，不等式 $2x^2+2x+1>0$ 是否恒成立：
Determine if the inequality $2x^2+2x+1>0$ holds for all real numbers x:

1. 计算判别式：$\Delta=4-8=-4<0$
   Calculate discriminant: $\Delta=4-8=-4<0$
2. 由于 $a=2>0$ 且 $\Delta<0$
   Since $a=2>0$ and $\Delta<0$
3. 配方：$2x^2+2x+1=2(x^2+x)+1=2(x+\frac{1}{2})^2+\frac{1}{2}$
   Complete square: $2x^2+2x+1=2(x^2+x)+1=2(x+\frac{1}{2})^2+\frac{1}{2}$
4. 右边恒大于 $\frac{1}{2}>0$，所以不等式恒成立
   The right side is always greater than $\frac{1}{2}>0$, so the inequality always holds

3.5.2 参数问题 Parameter Problems

定理 Theorem 3.5.3 (含参数不等式恒成立的条件 Conditions for Parametric Inequality Identities). 对于含参数 k 的二次不等式 $ax^2+bx+c>0$，要使其恒成立：
For a quadratic inequality $ax^2+bx+c>0$ with parameter k, to make it always true:

1. 首先确保 $a>0$
   First ensure $a>0$
2. 判别式必须小于零：$b^2-4ac<0$
   The discriminant must be negative: $b^2-4ac<0$
3. 解出参数 k 的取值范围
   Solve for the range of parameter k

例 Example 3.5.4 (含参数不等式的讨论 Discussion of Parametric Inequalities). 讨论什么条件下不等式 $x^2 + 2kx + k^2 + 1 > 0$ 对任意实数 x 恒成立:
Discuss under what conditions the inequality $x^2+2kx+k^2+1 > 0$ holds for all real numbers x:

1. 二次项系数 $a = 1 > 0$，满足第一个条件
   Quadratic coefficient $a = 1 > 0$, satisfying the first condition
2. 计算判别式: $\Delta = 4k^2 - 4(1)(k^2 + 1) = -4$
   Calculate discriminant: $\Delta = 4k^2 - 4(1)(k^2 + 1) = -4$
3. 由于 $\Delta = -4 < 0$，对任意 k 都成立
   Since $\Delta = -4 < 0$, it holds for any k
4. 也可以配方: $(x + k)^2 + 1$，显然恒大于零
   Can also complete square: $(x + k)^2 + 1$, obviously always greater than zero

Chapter 4

指数函数 Exponential Functions

4.1 指数函数的概念 Concept of Exponential Functions

4.1.1 定义与基本性质 Definition and Basic Properties

定义 Definition 4.1.1 (指数函数 Exponential Function). 设 a 为正实数且 $a \neq 1$，定义指数函数为:
Let a be a positive real number and $a \neq 1$, the exponential function is defined as:

$$f(x) = a^x \quad (a > 0, a \neq 1)$$

其中:
where:

- a 称为底数（$a > 0, a \neq 1$）
  a is called the base ($a > 0, a \neq 1$)
- x 为自变量，定义域为全体实数
  x is the variable, with domain being all real numbers

定理 Theorem 4.1.2 (指数函数的基本性质 Basic Properties of

Exponential Functions). 对于指数函数 $f(x)=a^x \ \ (a>0, a\neq 1)$:
For exponential function $f(x)=a^x \ \ (a>0, a\neq 1)$:

1. 定义域为 $(-\infty, +\infty)$
   The domain is $(-\infty, +\infty)$
2. 值域为 $(0, +\infty)$
   The range is $(0, +\infty)$
3. 在定义域内连续
   Continuous in its domain
4. 当 $a>1$ 时，函数单调递增；当 $0<a<1$ 时，函数单调递减
   When $a>1$, the function is strictly increasing; when $0<a<1$, the function is strictly decreasing

4.2 常用指数公式 Common Exponential Formulas

4.2.1 基本运算法则 Basic Operation Rules

定理 Theorem 4.2.1 (指数运算法则 Laws of Exponents). 设 $a>0$, m、n 为实数，则:
Let $a>0$, and m, n be real numbers, then:

1. $a^m \cdot a^n = a^{m+n}$
   Product rule: multiply powers with the same base
2. $\frac{a^m}{a^n} = a^{m-n}$
   Quotient rule: divide powers with the same base
3. $(a^m)^n = a^{mn}$
   Power rule: power of a power
4. $(ab)^n = a^n b^n$
   Product raised to a power
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
   Quotient raised to a power
6. $a^0 = 1 \ \ (a\neq 0)$

Zero exponent rule

7. $a^{-n} = \frac{1}{a^n}$
   Negative exponent rule

例 Example 4.2.2 (指数运算 Exponential Operations). 化简表达式：$\frac{2^{x+2} \cdot 2^{x-1}}{2^x}$
Simplify the expression: $\frac{2^{x+2} \cdot 2^{x-1}}{2^x}$

$$\begin{aligned}\frac{2^{x+2} \cdot 2^{x-1}}{2^x} &= \frac{2^{x+2+x-1}}{2^x} \\ &= \frac{2^{2x+1}}{2^x} \\ &= 2^{2x+1-x} \\ &= 2^{x+1} \\ &= 2 \cdot 2^x\end{aligned}$$

4.3 指数函数图像间的关系 Relationships Between Exponential Graphs

4.3.1 平移与伸缩 Translation and Scaling

定理 Theorem 4.3.1 (指数函数的变换 Transformations of Exponential Functions). 对于基本指数函数 $f(x) = a^x$，有以下变换关系：
For the basic exponential function $f(x) = a^x$, we have the following transformations:

1. $y = a^{x+h}$ 为向左平移 h 个单位
   $y = a^{x+h}$ is a shift h units to the left
2. $y = a^x + k$ 为向上平移 k 个单位
   $y = a^x + k$ is a shift k units up
3. $y = ka^x$ $(k > 0)$ 为纵向伸缩
   $y = ka^x$ $(k > 0)$ is vertical scaling
4. $y = a^{kx}$ $(k \neq 0)$ 为横向伸缩
   $y = a^{kx}$ $(k \neq 0)$ is horizontal scaling

例 Example 4.3.2 (图像变换 Graph Transformations). 描述函数 $f(x) = 2^{x-1} + 3$ 的图像特征:
Describe the characteristics of the function $f(x) = 2^{x-1} + 3$:

1. 基础函数是 $y = 2^x$
   The base function is $y = 2^x$
2. 向右平移 1 个单位
   Shift 1 unit to the right
3. 向上平移 3 个单位
   Shift 3 units up
4. 渐近线为 $y = 3$
   The asymptote is $y = 3$

4.4 被开方数的取值范围 Domain Restrictions for Roots

4.4.1 基本原则 Basic Principles

定理 Theorem 4.4.1 (开方的定义域 Domain of Root Functions). 在实数范围内:
In the real number system:

1. 偶次方根的被开方数必须非负
   Even roots require non-negative radicands
2. 奇次方根的被开方数可以是任意实数
   Odd roots allow any real number as radicand

例 Example 4.4.2 (取值范围问题 Domain Range Problems). 求解不等式 $\sqrt{2^x - 1} > 3$ 的解集:
Solve the inequality $\sqrt{2^x - 1} > 3$:

1. 由于开偶次方，被开方数必须非负：$2^x - 1 \geq 0$
   Since it's an even root, the radicand must be non-negative: $2^x - 1 \geq 0$
2. 解得：$2^x \geq 1$，即 $x \geq 0$
   Solving: $2^x \geq 1$, thus $x \geq 0$
3. 原不等式化为：$2^x - 1 > 9$
   The original inequality becomes: $2^x - 1 > 9$
4. 解得：$2^x > 10$，即 $x > \log_2 10$
   Solving: $2^x > 10$, thus $x > \log_2 10$
5. 最终解集：$x \in (\log_2 10, +\infty)$
   Final solution set: $x \in (\log_2 10, +\infty)$

4.5 指数方程的解法 Solving Exponential Equations

4.5.1 基本方法 Basic Methods

定理 Theorem 4.5.1 (指数方程的基本解法 Basic Methods for Solving Exponential Equations). 解指数方程的主要方法：
Main methods for solving exponential equations:

1. 利用指数函数的单调性
   Using the monotonicity of exponential functions
2. 利用对数函数求解
   Using logarithmic functions
3. 换元法
   Substitution method
4. 分类讨论法
   Case analysis

例 Example 4.5.2 (指数方程求解 Solving Exponential Equations).
求解方程 $2^{x+1} + 2^x = 12$：
Solve the equation $2^{x+1} + 2^x = 12$:

1. 提取公因式：$2^x(2+1) = 12$
   Factor out: $2^x(2+1) = 12$
2. 化简：$3 \cdot 2^x = 12$
   Simplify: $3 \cdot 2^x = 12$
3. 继续化简：$2^x = 4$
   Continue: $2^x = 4$
4. 两边取对数：$x = 2$
   Take logarithm of both sides: $x = 2$
5. 验证：代入原方程成立
   Verify: substitution confirms the solution

4.5.2 实际应用 Practical Applications

例 Example 4.5.3 (复利计算 Compound Interest). 某笔资金以年利率 4% 复利计息，求多少年后本息之和为原来的 2 倍：
A sum of money is invested at 4% annual compound interest. Find how many years it takes to double the initial amount:

1. 设初始金额为 P，x 年后金额为 $2P$
   Let initial amount be P, amount after x years be $2P$
2. 列方程：$P(1+0.04)^x = 2P$
   Set up equation: $P(1+0.04)^x = 2P$
3. 化简：$(1.04)^x = 2$
   Simplify: $(1.04)^x = 2$
4. 两边取对数：$x \ln(1.04) = \ln(2)$
   Take natural logarithm: $x \ln(1.04) = \ln(2)$
5. 解得：$x = \frac{\ln(2)}{\ln(1.04)} \approx 17.7$ 年
   Solve: $x = \frac{\ln(2)}{\ln(1.04)} \approx 17.7$ years

例 Example 4.5.4 (人口增长模型 Population Growth Model). 某地区人口以每年 2% 的速率增长，若初始人口为 100 万，求 10 年后的人口数：
A region's population grows at 2% annually. If the initial population is 1 million, find the population after 10 years:

1. 设 $P(t)$ 为 t 年后的人口数
   Let $P(t)$ be the population after t years
2. 建立模型：$P(t) = 100(1.02)^t$ 万人
   Establish model: $P(t) = 100(1.02)^t$ (in 10,000s)
3. 代入 $t = 10$：$P(10) = 100(1.02)^{10}$
   Substitute $t = 10$: $P(10) = 100(1.02)^{10}$
4. 计算得：$P(10) \approx 122$ 万人
   Calculate: $P(10) \approx 1.22$ million

Chapter 5

对数函数 Logarithmic Functions

5.1 对数函数的概念 Concept of Logarithmic Functions

5.1.1 定义与基本性质 Definition and Basic Properties

定义 Definition 5.1.1 (对数函数 Logarithmic Function). 设 a 为正实数且 $a \neq 1$，定义对数函数为:
Let a be a positive real number and $a \neq 1$, the logarithmic function is defined as:

$$y = \log_a x \Leftrightarrow a^y = x \quad (a > 0, a \neq 1, x > 0)$$

其中:
where:

- a 称为底数（$a > 0, a \neq 1$）
  a is called the base ($a > 0, a \neq 1$)
- x 为真数，必须大于零
  x is the argument, must be positive
- 特别地，当 $a = e$ 时称为自然对数，记作 $\ln x$
  Specifically, when $a = e$, it's called natural logarithm, denoted as $\ln x$

定理 Theorem 5.1.2 (对数函数的基本性质 Basic Properties of Logarithmic Functions). 对于对数函数 $y = \log_a x \ (a > 0, a \neq 1)$:
For logarithmic function $y = \log_a x \ (a > 0, a \neq 1)$:

1. 定义域为 $(0, +\infty)$
   The domain is $(0, +\infty)$
2. 值域为 $(-\infty, +\infty)$
   The range is $(-\infty, +\infty)$
3. 在定义域内连续
   Continuous in its domain
4. 当 $a > 1$ 时，函数单调递增；当 $0 < a < 1$ 时，函数单调递减
   When $a > 1$, the function is strictly increasing; when $0 < a < 1$, the function is strictly decreasing

5.2 常用对数公式 Common Logarithmic Formulas

5.2.1 基本运算法则 Basic Operation Rules

定理 Theorem 5.2.1 (对数运算法则 Laws of Logarithms). 设 $a > 0$，$a \neq 1$，$M > 0$，$N > 0$，则:
Let $a > 0$, $a \neq 1$, $M > 0$, $N > 0$, then:

1. $\log_a(M \cdot N) = \log_a M + \log_a N$
   Product rule: logarithm of a product
2. $\log_a(\frac{M}{N}) = \log_a M - \log_a N$
   Quotient rule: logarithm of a quotient
3. $\log_a(M^n) = n \log_a M$
   Power rule: logarithm of a power
4. $\log_a a = 1$
   Base identity
5. $\log_a 1 = 0$
   Zero identity

6. $a^{\log_a x} = x$
   Exponential-logarithmic identity

5.2.2 其他重要公式 Other Important Formulas

定理 Theorem 5.2.2 (换底公式 Change of Base Formula). 对于任意正数 M 和任意两个不同的正数 a、b（$a \neq 1$，$b \neq 1$）:
For any positive number M and any two different positive numbers a, b ($a \neq 1$, $b \neq 1$):

$$\log_a M = \frac{\log_b M}{\log_b a}$$

特别地:
Specifically:

$$\log_a M = \frac{\ln M}{\ln a}$$

5.3 指数函数与对数函数的转化 Conversion Between Exponential and Logarithmic Functions

5.3.1 基本转化方法 Basic Conversion Methods

定理 Theorem 5.3.1 (指对转化 Exponential-Logarithmic Conversion). 以下三个等式等价:
The following three equations are equivalent:

1. $y = \log_a x$
   Logarithmic form
2. $a^y = x$
   Exponential form
3. $x = a^y$
   Alternative exponential form

这种转化在解方程时特别有用。

This conversion is particularly useful in solving equations.

例 Example 5.3.2 (指对转化应用 Application of Exponential-Logarithmic Conversion). 解方程 $\log_2(x+1) = 3$:
Solve the equation $\log_2(x+1) = 3$:

1. 转化为指数形式: $2^3 = x + 1$
   Convert to exponential form: $2^3 = x + 1$
2. 计算: $8 = x + 1$
   Calculate: $8 = x + 1$
3. 解得: $x = 7$
   Solve: $x = 7$
4. 验证: 代入原方程成立
   Verify: substitution confirms the solution

5.4 立方和与立方差 Cube Sums and Differences

5.4.1 基本公式 Basic Formulas

定理 Theorem 5.4.1 (立方和公式 Cube Sum Formula). 对于任意实数 a、b:
For any real numbers a, b:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

这个公式可以用来因式分解立方和。
This formula can be used to factorize the sum of cubes.

定理 Theorem 5.4.2 (立方差公式 Cube Difference Formula). 对于任意实数 a、b:
For any real numbers a, b:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

这个公式可以用来因式分解立方差。

This formula can be used to factorize the difference of cubes.

5.5 和的立方与差的立方 Cube of Sum and Difference

5.5.1 展开公式 Expansion Formulas

定理 Theorem 5.5.1 (和的立方 Cube of Sum). 对于任意实数 a、b:
For any real numbers a, b:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

定理 Theorem 5.5.2 (差的立方 Cube of Difference). 对于任意实数 a、b:
For any real numbers a, b:

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

例 Example 5.5.3 (立方公式应用 Application of Cube Formulas). 计算 $(2x+1)^3$:
Calculate $(2x+1)^3$:

$$\begin{aligned}(2x+1)^3 &= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3 \\ &= 8x^3 + 12x^2 + 6x + 1\end{aligned}$$

5.6 函数图像的平移变换 Translation Transformations of Functions

5.6.1 基本平移变换 Basic Translation Transformations

定理 Theorem 5.6.1 (平移变换规律 Laws of Translation Transformations). 对于基本函数 $y = f(x)$:
For the basic function $y = f(x)$:

1. $y = f(x+h)$ 为向左平移 h 个单位
   $y = f(x+h)$ is a shift h units to the left
2. $y = f(x-h)$ 为向右平移 h 个单位
   $y = f(x-h)$ is a shift h units to the right
3. $y = f(x)+k$ 为向上平移 k 个单位
   $y = f(x)+k$ is a shift k units up
4. $y = f(x)-k$ 为向下平移 k 个单位
   $y = f(x)-k$ is a shift k units down

例 Example 5.6.2 (对数函数的平移 Translation of Logarithmic Functions). 描述函数 $y = \log_2(x-3)+2$ 的图像特征:
Describe the characteristics of the function $y = \log_2(x-3)+2$:

1. 基础函数是 $y = \log_2 x$
   The base function is $y = \log_2 x$
2. 向右平移 3 个单位
   Shift 3 units to the right
3. 向上平移 2 个单位
   Shift 2 units up
4. 渐近线为 $x = 3$
   The asymptote is $x = 3$

5.6.2 复合变换 Composite Transformations

例 Example 5.6.3 (复合变换问题 Composite Transformation Problems). 描述函数 $y = \ln(2x+1)-3$ 的变换过程:
Describe the transformation process of the function $y = \ln(2x+1)-3$:

1. 先将 x 变为 $2x$（横向压缩为原来的 $\frac{1}{2}$）
   First change x to $2x$ (horizontal compression by factor $\frac{1}{2}$)
2. 再加 1（向左平移 $\frac{1}{2}$ 个单位）

Then add 1 (shift $\frac{1}{2}$ units to the left)

3. 最后减 3（向下平移 3 个单位）
   Finally subtract 3 (shift 3 units down)
4. 渐近线为 $x = -\frac{1}{2}$
   The asymptote is $x = -\frac{1}{2}$

Chapter 6

Derivatives 导数

6.1 Basic Concepts of Derivatives 导数的基本概念

6.1.1 Definition of Derivatives 导数的定义

The derivative represents the instantaneous rate of change of a function at any given point. 导数表示函数在任意给定点处的瞬时变化率。

定义 **Definition 6.1.1 (Derivative 导数).** For a function $f(x)$, the derivative at point x is defined as: 对于函数 $f(x)$，在点 x 处的导数定义为:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

This can also be written as: y' or $(f(x))'$ 也可记作: y' 或 $(f(x))'$

6.1.2 Geometric Interpretation 几何意义

The derivative $f'(x)$ represents the slope of the tangent line to the curve $y = f(x)$ at point $(x, f(x))$. 导数 $f'(x)$ 表示曲线 $y = f(x)$ 在点 $(x, f(x))$ 处的切线斜率。

That is to say, the derivative $f'(x_0)$ equals the slope k of the tangent line at point $(x_0, f(x_0))$. 也就是说，导数 $f'(x_0)$ 等于曲线在点 $(x_0, f(x_0))$ 处切线的斜率 k。

6.2 Properties and Applications of Derivatives 导数的性质与应用

6.2.1 Extreme Values 极值

Conditions for function $f(x)$ to have a maximum value at $x = a$: 函数 $f(x)$ 在 $x = a$ 处取得极大值的条件：

- Necessary condition: $f'(a) = 0$ or $f'(a)$ does not exist
- 必要条件：$f'(a) = 0$ 或 $f'(a)$ 不存在
- Sufficient condition: In some neighborhood of $x = a$, $f'(x) > 0$ when $x < a$, and $f'(x) < 0$ when $x > a$
- 充分条件：在 $x = a$ 的某邻域内，当 $x < a$ 时，$f'(x) > 0$；当 $x > a$ 时，$f'(x) < 0$

Conditions for function $f(x)$ to have a minimum value at $x = a$: 函数 $f(x)$ 在 $x = a$ 处取得极小值的条件：

- Necessary condition: $f'(a) = 0$ or $f'(a)$ does not exist
- 必要条件：$f'(a) = 0$ 或 $f'(a)$ 不存在
- Sufficient condition: In some neighborhood of $x = a$, $f'(x) < 0$ when $x < a$, and $f'(x) > 0$ when $x > a$
- 充分条件：在 $x = a$ 的某邻域内，当 $x < a$ 时，$f'(x) < 0$；当 $x > a$ 时，$f'(x) > 0$

6.3 Basic Differentiation Rules 基本求导法则

6.3.1 Basic Derivative Formulas 基本求导公式

- $(x^n)' = nx^{n-1}$ (Power Function Derivative 幂函数求导)
- $(e^x)' = e^x$ (Exponential Function Derivative 指数函数求导)
- $(\ln x)' = \frac{1}{x}$ (Natural Logarithm Derivative 对数函数求导)
- $(\sin x)' = \cos x$ (Sine Function Derivative 正弦函数求导)
- $(\cos x)' = -\sin x$ (Cosine Function Derivative 余弦函数求导)
- $(a^x)' = a^x \ln a$ (General Exponential Function Derivative 一般指数函数求导)
- $(\log_a x)' = \frac{1}{x \ln a}$ (General Logarithm Derivative 一般对数函数求导)

6.3.2 Differentiation Rules 求导规则

- Sum and Difference Rule 和差的导数：$(u \pm v)' = u' \pm v'$
- Product Rule 积的导数：$(uv)' = u'v + uv'$
- Quotient Rule 商的导数：$(\frac{u}{v})' = \frac{u'v - uv'}{v^2}$ $(v \neq 0)$
- Chain Rule 复合函数的导数：If 若 $y = f(u)$，$u = g(x)$，then 则 $y' = (f(g(x)))' = f'(g(x)) \cdot g'(x)$

6.4 Integration 积分

6.4.1 Basic Integration Formulas 基本积分公式

Antiderivative, also known as indefinite integral, is denoted as $\int f(x)dx$. 原函数也称为不定积分，记作 $\int f(x)dx$。

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ (Power Function Integration 幂函数积分)
- $\int e^x dx = e^x + C$ (Exponential Function Integration 指数函数积分)

- $\int \frac{1}{x} dx = \ln|x| + C$ (Natural Logarithm Integration 对数函数积分)
- $\int \sin x dx = -\cos x + C$ (Sine Function Integration 正弦函数积分)
- $\int \cos x dx = \sin x + C$ (Cosine Function Integration 余弦函数积分)
- $\int a^x dx = \frac{a^x}{\ln a} + C$ (General Exponential Function Integration 一般指数函数积分)

6.4.2 Integration Methods 积分方法

1. First Substitution Method (Let $u = g(x)$) 第一换元法(设 $u = g(x)$):
$$\int f(g(x))g'(x)dx = \int f(u)du$$
2. Second Substitution Method (Let $x = \varphi(t)$) 第二换元法(设 $x = \varphi(t)$):
$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt$$
3. Integration by Parts 分部积分法:
$$\int udv = uv - \int vdu$$

6.5 Important Theorems 重要定理

6.5.1 Fundamental Theorem of Calculus 微积分基本定理

If $F(x)$ is an antiderivative of $f(x)$, then: 设 $F(x)$ 是 $f(x)$ 的一个原函数,则:

$$\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b$$

6.5.2 Mean Value Theorem 中值定理

If function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one point ξ in (a,b) such that: 如果函数 $f(x)$ 在闭区间 $[a,b]$ 上连续，在开区间 (a,b) 内可导，则在 (a,b) 内至少存在一点 ξ，使得:

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}$$

6.6 Derivatives of Composite Functions 复合函数的导数

Chain Rule for Composite Functions: 复合函数求导的链式法则: If 如果 $y=f(u)$，$u=g(x)$，then 则:

$$y'=(f(g(x)))'=f'(g(x))\cdot g'(x)$$

For example 例如: Let 设 $y=\sin(x^2)$，then 则:

$$y'=\cos(x^2)\cdot 2x$$

6.6.1 Applications 应用

- Related Rates Problems 相关变化率问题
- Maximum and Minimum Problems 最值问题
- Tangent and Normal Lines to Function Graphs 函数图像的切线与法线
- Monotonicity and Extreme Values of Functions 函数的单调性与极值

Chapter 7

Trigonometric Functions 三角函数

7.1 Basic Concepts 基本概念

7.1.1 Radian Measure 弧度制

The radian is the standard unit of angular measure. 弧度是角度测量的标准单位。

定义 Definition 7.1.1 (Radian 弧度). One radian is the angle subtended at the center of a circle by an arc whose length is equal to the radius of the circle. 一弧度是圆心角对应的弧长等于半径时的角度。

7.1.2 Conversion between Radians and Degrees 弧度与角度的换算

- $180^\circ = \pi$ radians 弧度
- 1 radian 弧度 $= \frac{180^\circ}{\pi} \approx 57.3^\circ$
- $1^\circ = \frac{\pi}{180}$ radians 弧度

7.1.3 Arc Length and Area Formulas 弧长和面积公式

For a circle with radius r: 对于半径为 r 的圆:

- Arc Length 弧长: $s = r\theta$ (where θ is in radians 其中 θ 以弧度表示)
- Sector Area 扇形面积: $A = \frac{1}{2}r^2\theta$ (where θ is in radians 其中 θ 以弧度表示)

7.2 Special Angles and Their Values 特殊角的三角函数值

7.2.1 Common Angles 常见角度

角度 Angle 弧度 Radian	$0°$ 0	$30°$ $\frac{\pi}{6}$	$45°$ $\frac{\pi}{4}$	$60°$ $\frac{\pi}{3}$	$90°$ $\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

7.3 Basic Trigonometric Relations 三角函数的基本关系

7.3.1 Fundamental Relations 基本关系式

- $\sin^2\theta + \cos^2\theta = 1$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$

7.4 Addition and Subtraction Formulas 和角差角公式

7.4.1 Sum and Difference Formulas 和差公式

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

7.5 Power Reduction and Increase Formulas 降幂升幂公式

7.5.1 Power Reduction Formulas 降幂公式

$$\begin{aligned}
\sin^2\theta &= \frac{1 - \cos 2\theta}{2} \\
\cos^2\theta &= \frac{1 + \cos 2\theta}{2}
\end{aligned}$$

7.5.2 Power Increase Formulas 升幂公式

$$\begin{aligned}
\sin^3\theta &= \frac{3\sin\theta - \sin 3\theta}{4} \\
\cos^3\theta &= \frac{3\cos\theta + \cos 3\theta}{4}
\end{aligned}$$

7.6 Half-Angle and Multiple-Angle Formulas 半角和多倍角公式

7.6.1 Half-Angle Formulas 半角公式

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta}$$

7.6.2 Double-Angle Formulas 二倍角公式

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

7.6.3 Triple-Angle Formulas 三倍角公式

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

7.7 Product and Sum Transformations 积和变换公式

7.7.1 Product-to-Sum Formulas 积化和差公式

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$
$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

7.8 Area Theorems and Universal Formulas 面积定理和万能公式

7.8.1 Area of a Triangle 三角形面积

- $S = \frac{1}{2}ab\sin C$ (where C is the included angle 其中 C 为夹角)
- $S = \sqrt{p(p-a)(p-b)(p-c)}$ (Heron's formula 海伦公式，其中 $p = \frac{a+b+c}{2}$)

7.8.2 Universal Substitution 万能代换

Let $t = \tan\frac{x}{2}$, then: 令 $t = \tan\frac{x}{2}$，则:

$$\sin x = \frac{2t}{1+t^2}$$
$$\cos x = \frac{1-t^2}{1+t^2}$$
$$\tan x = \frac{2t}{1-t^2}$$

7.9 Inverse Trigonometric Functions 反三角函数

7.9.1 Basic Definitions 基本定义

- $y = \arcsin x$ if and only if $\sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- $y = \arccos x$ if and only if $\cos y = x$ and $0 \leq y \leq \pi$
- $y = \arctan x$ if and only if $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$

7.10 Transformations and Properties 变换与性质

7.10.1 Graph Transformations 图像变换

For function $y = A\sin(B(x + C)) + D$: 对于函数 $y = A\sin(B(x + C)) + D$:

- $|A|$ - amplitude 振幅
- B - period is $\frac{2\pi}{|B|}$ 周期为 $\frac{2\pi}{|B|}$
- C - phase shift 相位移动
- D - vertical shift 垂直平移

7.10.2 Symmetry Properties 对称性质

- Center of Symmetry 对称中心: (a, b) if $f(a + x) + f(a - x) = 2b$
- Axis of Symmetry 对称轴: $x = a$ if $f(a + x) = f(a - x)$

7.10.3 Periodicity 周期性

- $\sin x$, $\cos x$: period 周期 2π
- $\tan x$, $\cot x$: period 周期 π
- $\sec x$, $\csc x$: period 周期 2π

Chapter 8

Solving Triangles 解三角形

8.1 Basic Concepts 基本概念

In triangle ABC, we denote: 在三角形 ABC 中，我们记：

- Three angles: $\angle A$, $\angle B$, $\angle C$ (三个角)
- Three sides: a (opposite to $\angle A$), b (opposite to $\angle B$), c (opposite to $\angle C$) (三条边：a 对 $\angle A$，b 对 $\angle B$，c 对 $\angle C$)
- Area: S (面积)

8.2 The Law of Sines 正弦定理

定理 Theorem 8.2.1 (Law of Sines 正弦定理). In any triangle ABC: 在任意三角形 ABC 中：

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the radius of the circumscribed circle of the triangle. 其中 R 是三角形外接圆的半径。

注 Remark 8.2.2 (Applications 应用). The Law of Sines is particularly useful when solving triangles in these cases:

正弦定理在以下情况求解三角形时特别有用：

- Given two angles and one side (AAS or ASA) （已知两角一边）
- Given two sides and an angle opposite to one of them (SSA) （已知两边及其中一边所对的角）

8.3 The Law of Cosines 余弦定理

定理 Theorem 8.3.1 (Law of Cosines 余弦定理). In any triangle ABC: 在任意三角形 ABC 中：

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

注 Remark 8.3.2 (Applications 应用). The Law of Cosines is particularly useful when: 余弦定理在以下情况特别有用：

- Given three sides (SSS) to find angles （已知三边求角）
- Given two sides and the included angle (SAS) to find the third side （已知两边及其夹角求第三边）

8.4 Triangle Angle Sum Theorem 三角形内角和定理

定理 Theorem 8.4.1 (Triangle Angle Sum 三角形内角和). The sum of the measures of the three interior angles of any triangle is 180°: 任意三角形的三个内角的度数和为 180°：

$$A + B + C = 180^\circ$$

推论 Corollary 8.4.2 (Exterior Angle 外角). The measure of an exterior angle of a triangle equals the sum of the two non-adjacent interior angles: 三角形的外角等于两个非相邻内角的和:

$$\text{exterior angle} = \text{sum of two non-adjacent interior angles}$$

外角 = 两个非相邻内角的和

8.5 Tangent Identities in Triangles 三角形中的正切恒等式

定理 Theorem 8.5.1 (Tangent of Half Angles 半角正切公式). In any triangle ABC: 在任意三角形 ABC 中:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

where $s = \frac{a+b+c}{2}$ is the semi-perimeter. 其中 $s = \frac{a+b+c}{2}$ 是半周长。

8.6 Determining Triangle Shape 三角形形状的判定

8.6.1 By Three Sides 用三边判定

For triangle ABC with sides a, b, c: 对于边长为 a, b, c 的三角形 ABC:

- Acute triangle 锐角三角形: $a^2 + b^2 > c^2$ (longest side) (最长边)
- Right triangle 直角三角形: $a^2 + b^2 = c^2$ (longest side) (最长边)

- Obtuse triangle 钝角三角形: $a^2 + b^2 < c^2$ (longest side) (最长边)

8.6.2 Triangle Inequality 三角形不等式

The sum of any two sides of a triangle must be greater than the third side: 三角形任意两边之和大于第三边:

$$a + b > c$$

$$b + c > a$$

$$c + a > b$$

8.7 Triangle Transformation Problems 三角形的转化问题

8.7.1 Area Formulas 面积公式

- $S = \frac{1}{2}ah$ (h is the height to side a) (h 是底边 a 上的高)
- $S = \frac{1}{2}bc \sin A$ (using two sides and included angle) (用两边及其夹角)
- $S = \frac{abc}{4R}$ (R is the circumradius) (R 是外接圆半径)
- $S = rs$ (r is the inradius, s is the semi-perimeter) (r 是内切圆半径, s 是半周长)
- Heron's formula 海伦公式: $S = \sqrt{s(s-a)(s-b)(s-c)}$

8.7.2 Common Transformation Techniques 常见转化技巧

1. Auxiliary Line Method 辅助线法

   - Drawing height 作高
   - Drawing median 作中线
   - Drawing angle bisector 作角平分线

2. Area Method 面积法

    - Comparing areas of different parts 比较不同部分的面积
    - Using area relationships 利用面积关系

3. Similar Triangle Method 相似三角形法

    - Finding similar triangles 寻找相似三角形
    - Using proportional relationships 利用比例关系

注 Remark 8.7.1 (Problem-Solving Strategy 解题策略). When solving triangle problems: 解三角形问题时:

1. First analyze known conditions 先分析已知条件
2. Choose appropriate theorem(s) 选择合适的定理
3. Consider possible transformations 考虑可能的转化
4. Verify solution reasonableness 验证解的合理性

Chapter 9

Plane and Space Vectors 平面向量与空间向量

9.1 Basic Concepts of Vectors 向量的基本概念

9.1.1 Vector Representation 向量的表示方法

A vector can be represented in several ways: 向量可以用以下几种方式表示:

- Geometric representation: $\overrightarrow{AB}$ (几何表示)
- Algebraic representation: $\boldsymbol{a}$ or $\vec{a}$ (代数表示)
- Component form: (x, y) or (x, y, z) (分量表示)
- Direction cosines: $(\cos\alpha, \cos\beta, \cos\gamma)$ (方向余弦)

9.1.2 Equal and Opposite Vectors 相等向量与相反向量

- Equal vectors: Same direction and magnitude (相等向量: 方向和大小都相同)

$$\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \text{平移后完全重合}$$

- Opposite vectors: Same magnitude but opposite directions (相反向量：大小相等但方向相反)

$$\overrightarrow{AB} = -\overrightarrow{BA}$$

9.1.3 Parallel Vectors 平行向量

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if: 向量 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 平行当且仅当:

$$\exists k \in \mathbb{R}, \boldsymbol{a} = k\boldsymbol{b}$$

9.1.4 Perpendicular Vectors 向量垂直

Two vectors are perpendicular if their dot product is zero: 两个向量垂直当且仅当它们的数量积为零:

$$\boldsymbol{a} \perp \boldsymbol{b} \Leftrightarrow \boldsymbol{a} \cdot \boldsymbol{b} = 0$$

9.2 Vector Operations 向量的运算

9.2.1 Basic Operations 基本运算律

- Addition (Triangle rule) 加法（三角形法则）:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

- Scalar multiplication 数乘:

$$k\boldsymbol{a} = (kx, ky) \text{ or } (kx, ky, kz)$$

- Distributive law 分配律:

$$k(\boldsymbol{a} + \boldsymbol{b}) = k\boldsymbol{a} + k\boldsymbol{b}$$

- Commutative law 交换律:

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$

- Associative law 结合律:

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$$

9.2.2 Vector Formulas 向量公式

- Midpoint formula 中点公式: $M = \frac{A+B}{2}$
- Centroid formula 重心公式: $G = \frac{A+B+C}{3}$
- Vector decomposition 向量分解: $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$
- Direction angles 方向角: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

9.3 Vector Products and Angles 向量的积与角度

9.3.1 Dot Product 数量积

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

where θ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. 其中 θ 是向量 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 的夹角。

9.3.2 Angle Between Vectors 向量夹角

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

9.3.3 Projection and Component Vectors 投影与分解向量

- Scalar projection 标量投影:

$$\text{proj}_{\boldsymbol{b}}\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$$

- Vector projection 向量投影:

$$\text{proj}_{\boldsymbol{b}}\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b}$$

- Orthogonal decomposition 正交分解:

$$\boldsymbol{a} = \boldsymbol{a}_{\parallel} + \boldsymbol{a}_{\perp}$$

9.4 Vector Inequalities and Theorems 向量不等式与定理

9.4.1 Triangle Inequality 三角形不等式

$$|\boldsymbol{a}+\boldsymbol{b}| \leq |\boldsymbol{a}|+|\boldsymbol{b}|$$

Equality holds if and only if vectors are parallel and in the same direction. 等号成立当且仅当向量平行且同向。

9.4.2 Translation Formula 平移公式

For any point P and vector $\boldsymbol{a}$: 对于任意点 P 和向量 $\boldsymbol{a}$:

$$P' = P + \boldsymbol{a}$$

where P' is the image of P under translation by $\boldsymbol{a}$. 其中 P' 是点 P 沿向量 $\boldsymbol{a}$ 平移后的像点。

9.5 Space Vectors 空间向量

9.5.1 Operations in Space 空间向量的运算

For vectors $\boldsymbol{a} = (x_1, y_1, z_1)$ and $\boldsymbol{b} = (x_2, y_2, z_2)$: 对于向量 $\boldsymbol{a} = (x_1, y_1, z_1)$ 和 $\boldsymbol{b} = (x_2, y_2, z_2)$:

- Addition 加法: $(x_1+x_2, y_1+y_2, z_1+z_2)$
- Dot product 数量积: $x_1x_2 + y_1y_2 + z_1z_2$
- Magnitude 模长: $|\boldsymbol{a}| = \sqrt{x_1^2+y_1^2+z_1^2}$
- Direction cosines 方向余弦: $\cos\alpha = \frac{x}{|\boldsymbol{a}|}$, $\cos\beta = \frac{y}{|\boldsymbol{a}|}$, $\cos\gamma = \frac{z}{|\boldsymbol{a}|}$

9.5.2 Distance Formula 距离公式

Distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$: 点 $A(x_1, y_1, z_1)$ 和 $B(x_2, y_2, z_2)$ 之间的距离:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

9.5.3 Division Point Formula 分点公式

- Internal division 内分点: $P = \frac{\mu A + \lambda B}{\lambda + \mu}$ $(\lambda, \mu > 0)$
- External division 外分点: $P = \frac{\mu A + \lambda B}{\lambda + \mu}$ $(\lambda\mu < 0)$
- Section formula 分点公式: $\overrightarrow{AP} = \frac{\lambda}{\lambda + \mu}\overrightarrow{AB}$

9.6 Fundamental Theorems 基本定理

9.6.1 Collinear Vectors 共线向量

定理 Theorem 9.6.1 (Collinear Vector Theorem 共线向量基本定理). Three points A, B, C are collinear if and only if: 三点 A、B、C 共线当且仅当:

$$\overrightarrow{AB} = k\overrightarrow{AC} \text{ or } \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$$

9.6.2 Coplanar Vectors 共面向量

定理 Theorem 9.6.2 (Coplanar Vector Theorem 共面向量基本定理). Four points A, B, C, D are coplanar if and only if: 四点 A、B、C、D 共面当且仅当:

$$\overrightarrow{AD} = \alpha\overrightarrow{AB} + \beta\overrightarrow{AC}$$

where α and β are scalars not both zero. 其中 α 和 β 是不全为零的实数。

9.7 Applications in Geometry 几何应用

9.7.1 Five Centers of Triangle 三角形的五心

- Centroid 重心：$G = \frac{A+B+C}{3}$
- Circumcenter 外心：三条垂直平分线的交点
- Incenter 内心：三条角平分线的交点
- Orthocenter 垂心：三条高线的交点
- Center of the Nine-Point Circle 九点圆心：垂足三角形的外心

9.7.2 Important Theorems about Triangle Centers 三角形五心定理

- Euler line theorem 欧拉线定理：重心、外心、垂心三点共线
- Centroid division theorem 重心分线定理：$AG : GM = 2 : 1$
- Nine-point circle theorem 九点圆定理：九点共圆

9.7.3 Position Relations 位置关系

Using vectors to prove: 用向量证明：

- Parallel lines 平行线：$\boldsymbol{a} \parallel \boldsymbol{b} \Leftrightarrow \exists k \in \mathbb{R}, \boldsymbol{a} = k\boldsymbol{b}$
- Perpendicular lines 垂直线：$\boldsymbol{a} \perp \boldsymbol{b} \Leftrightarrow \boldsymbol{a} \cdot \boldsymbol{b} = 0$
- Collinear points 共线点：$\overrightarrow{AB} = k\overrightarrow{AC}$
- Coplanar points 共面点：$\overrightarrow{AD} = \alpha\overrightarrow{AB} + \beta\overrightarrow{AC}$

9.7.4 Plane Projection Theorem 平面射影定理

定理 Theorem 9.7.1 (Projection Theorem 射影定理). The ratio of areas of a projected figure to the original figure equals the cosine of the angle between their planes: 投影图形与原图形的面积比等于它们所在平面夹角的余弦：

$$\frac{S'}{S} = |\cos\theta|$$

注 Remark 9.7.2 (Problem-Solving Strategy 解题策略). When solving vector problems: 解向量问题时:

1. Choose appropriate coordinate system 选择合适的坐标系
2. Express vectors in suitable form 用合适的形式表示向量
3. Apply relevant theorems 应用相关定理
4. Consider geometric meaning 考虑几何意义
5. Verify the solution 验证解答

Chapter 10

数学逻辑用语 Mathematical Logic Language

10.1 命题及其关系 Propositions and Their Relationships

定义 Definition 10.1.1 (命题 Proposition). 命题是一个明确的陈述句，它或真或假，不能既真又假。
A proposition is a declarative sentence that is either true or false, but not both.

例 Example 10.1.2 (命题示例 Examples of Propositions). 以下是一些命题的例子:
Here are some examples of propositions:

1. ”2 + 2 = 4”（真命题 True proposition）
2. ” 地球是方的”（假命题 False proposition）
3. ” 如果三角形是等腰的，那么它有两个角相等”（真命题 True proposition）

10.2 四种命题间的关系 Four Types of Propositional Relationships

定义 Definition 10.2.1 (原命题与逆命题 Original and Converse Propositions). 若命题 $p \to q$ 是原命题，则命题 $q \to p$ 是它的逆命题。
If proposition $p \to q$ is the original proposition, then $q \to p$ is its converse.

定义 Definition 10.2.2 (否命题与逆否命题 Inverse and Contrapositive Propositions). 若命题 $p \to q$ 是原命题，则：
If proposition $p \to q$ is the original proposition, then:

- 否命题是 $\neg p \to \neg q$ (Inverse)
- 逆否命题是 $\neg q \to \neg p$ (Contrapositive)

定理 Theorem 10.2.3 (命题关系的重要性质 Important Properties of Propositional Relationships). 对于任意命题 $p \to q$：
For any proposition $p \to q$:

1. 原命题与逆否命题等价
   The original proposition is equivalent to its contrapositive
2. 逆命题与否命题等价
   The converse is equivalent to the inverse

10.3 四种命题的真假性 Truth Values of Four Types of Propositions

注 Remark 10.3.1 (命题真假性分析 Analysis of Truth Values). 对于原命题 $p \to q$：
For the original proposition $p \to q$:

1. 原命题为真，其逆否命题必为真
   If the original proposition is true, its contrapositive must be true
2. 原命题为真，其逆命题和否命题不一定为真
   If the original proposition is true, its converse and inverse are not necessarily true
3. 四个命题可能同真，也可能部分为真部分为假
   All four propositions might be true, or some might be true while others are false

10.4 充分条件与必要条件 Sufficient and Necessary Conditions

定义 Definition 10.4.1 (充分条件 Sufficient Condition). 在命题"若 p 则 q" 中:
In the proposition "if p then q":

- p 是 q 的充分条件
  p is a sufficient condition for q
- 记作: $p \Rightarrow q$
  Denoted as: $p \Rightarrow q$

定义 Definition 10.4.2 (必要条件 Necessary Condition). 在命题"若 p 则 q" 中:
In the proposition "if p then q":

- q 是 p 的必要条件
  q is a necessary condition for p
- 记作: $p \Rightarrow q$ 或 $q \Leftarrow p$
  Denoted as: $p \Rightarrow q$ or $q \Leftarrow p$

例 Example 10.4.3 (充要条件 Necessary and Sufficient Conditions). ”一个三角形是等边三角形”是”这个三角形的三个角都相等”的充要条件。
”A triangle is equilateral” is both necessary and sufficient for ”all three angles of the triangle are equal.”

- 记作: $p \Leftrightarrow q$
  Denoted as: $p \Leftrightarrow q$

10.5 常见词语的否定 Negation of Common Terms

定义 Definition 10.5.1 (否定词语对照表 Negation Terms Reference). 常见词语的否定形式:
Negation forms of common terms:

- ”所有”的否定是”存在... 不”
  The negation of ”all” is ”there exists... not”
- ”存在”的否定是”所有... 不”
  The negation of ”there exists” is ”for all... not”
- ”至少”的否定是”少于”
  The negation of ”at least” is ”less than”
- ”至多”的否定是”多于”
  The negation of ”at most” is ”more than”

10.6 命题的否定 Negation of Propositions

定理 Theorem 10.6.1 (复合命题的否定 Negation of Compound Propositions). 设 p, q 为命题，则:
Let p, q be propositions, then:

1. $(p \wedge q)$ 的否定是 $(\neg p \vee \neg q)$

The negation of $(p \wedge q)$ is $(\neg p \vee \neg q)$

2. $(p \vee q)$ 的否定是 $(\neg p \wedge \neg q)$
   The negation of $(p \vee q)$ is $(\neg p \wedge \neg q)$
3. $(p \rightarrow q)$ 的否定是 $(p \wedge \neg q)$
   The negation of $(p \rightarrow q)$ is $(p \wedge \neg q)$

例 Example 10.6.2 (命题否定示例 Examples of Proposition Negation). 原命题：” 如果下雨，那么地面湿”
Original: ”If it rains, then the ground is wet”

否定：” 下雨但地面不湿”
Negation: ”It rains but the ground is not wet”

Chapter 11

等差数列 Arithmetic Sequence

11.1 基本概念 Basic Concepts

定义 Definition 11.1.1 (等差数列 Arithmetic Sequence). 如果一个数列从第二项起，每一项与它的前一项的差都等于同一个常数，这个数列就叫做等差数列。这个常数叫做等差数列的公差，记作 d。
A sequence in which the difference between each consecutive term is constant is called an arithmetic sequence. This constant difference is called the common difference, denoted as d.

例 Example 11.1.2 (等差数列示例 Examples of Arithmetic Sequences). 以下是一些等差数列的例子：
Here are some examples of arithmetic sequences:

1. $\{1, 3, 5, 7, 9, ...\}$，公差 $d = 2$
   $\{1, 3, 5, 7, 9, ...\}$, common difference $d = 2$
2. $\{10, 7, 4, 1, -2, ...\}$，公差 $d = -3$
   $\{10, 7, 4, 1, -2, ...\}$, common difference $d = -3$
3. $\{2, 2, 2, 2, ...\}$，公差 $d = 0$（常数列也是等差数列）
   $\{2, 2, 2, 2, ...\}$, common difference $d = 0$ (constant sequence is also an arithmetic sequence)

注 Remark 11.1.3 (判断等差数列 Identifying Arithmetic Sequences). 判断一个数列是否为等差数列的方法:
Methods to determine if a sequence is arithmetic:

1. 计算相邻项的差，看是否都相等
   Calculate the differences between consecutive terms and check if they are equal
2. 检查任意三个连续项是否满足：中间项是两端项的算术平均值
   Check if any three consecutive terms satisfy: the middle term is the arithmetic mean of the other two
3. 验证是否满足：$a_n - a_{n-1} = a_{n+1} - a_n$
   Verify if: $a_n - a_{n-1} = a_{n+1} - a_n$

11.2 常用公式 Common Formulas

11.2.1 通项公式 General Term Formula

定理 Theorem 11.2.1 (通项公式 General Term Formula). 设等差数列的首项为 a_1，公差为 d，则:
For an arithmetic sequence with first term a_1 and common difference d:

1. 第 n 项公式:
   The nth term formula:
   $$a_n = a_1 + (n-1)d$$
2. 第 n 项也可以用第 m 项表示:
   The nth term can also be expressed using the mth term:
   $$a_n = a_m + (n-m)d$$

例 Example 11.2.2 (通项公式应用 Application of General Term Formula). 已知等差数列 $\{a_n\}$ 的 $a_1 = 3$，$d = 2$，求:
Given arithmetic sequence $\{a_n\}$ with $a_1 = 3$ and $d = 2$, find:

1. a_{10} 的值
   The value of a_{10}
2. 使 $a_n = 21$ 的 n 值
   The value of n when $a_n = 21$

解:
Solution:

1. $a_{10} = a_1 + (10-1)d = 3 + 9 \times 2 = 21$
2. $21 = 3 + (n-1)2$
   $21 - 3 = (n-1)2$
   $18 = 2(n-1)$
   $n = 10$

11.2.2 前 n 项和公式 Sum of First n Terms

定理 Theorem 11.2.3 (前 n 项和公式 Sum of First n Terms). 等差数列前 n 项和 S_n 的计算公式有以下三种形式:
The sum of the first n terms S_n of an arithmetic sequence can be expressed in three forms:

1. 首末项形式:
   Using first and last terms:
   $$S_n = \frac{n(a_1 + a_n)}{2}$$
2. 首项公差形式:
   Using first term and common difference:
   $$S_n = \frac{n[2a_1 + (n-1)d]}{2}$$
3. 等差中项形式:
   Using arithmetic mean:
   $$S_n = n \cdot \frac{a_1 + a_n}{2} = n \cdot a_{\frac{n+1}{2}} \quad (\text{当}n\text{为奇数时})$$

例 Example 11.2.4 (求和公式应用 Application of Sum Formula).
求等差数列 $\{2, 5, 8, 11, ..., 32\}$ 的和。
Find the sum of the arithmetic sequence $\{2, 5, 8, 11, ..., 32\}$.

解:
Solution:

1. 首先求项数 n:
   First find the number of terms n:

$$a_n = a_1 + (n-1)d$$

$$32 = 2 + (n-1)3$$

$$30 = 3(n-1)$$

$$n = 11$$

2. 使用首末项公式:
   Use the formula with first and last terms:

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{11(2+32)}{2} = \frac{11 \times 34}{2} = 187$$

11.2.3 等差中项 Arithmetic Mean

定理 Theorem 11.2.5 (等差中项 Arithmetic Mean). 在等差数列中:
In an arithmetic sequence:

1. 任意两项的算术平均值等于这两项中间对应项的值
   The arithmetic mean of any two terms equals the value of the corresponding term between them

$$\frac{a_p + a_q}{2} = a_{\frac{p+q}{2}} \quad (p + q\text{为偶数})$$

2. 任意奇数个项等距离排列时，中间项等于首末项的算术平均值
   When any odd number of terms are equally spaced, the middle term equals the arithmetic mean of the first and last terms

$$a_k = \frac{a_{k-m} + a_{k+m}}{2}$$

例 Example 11.2.6 (等差中项应用 Application of Arithmetic Mean). 在等差数列中，已知 $a_3 = 7$，$a_8 = 17$，求 a_5 和 a_6。
In an arithmetic sequence, given $a_3 = 7$ and $a_8 = 17$, find a_5 and a_6.

解:
Solution:

1. 先求公差 d:
   First find the common difference d:

$$a_8 - a_3 = (8-3)d$$

$$17 - 7 = 5d$$

$$d = 2$$

2. 求 a_5 和 a_6:
   Find a_5 and a_6:

$$a_5 = a_3 + (5-3)d = 7 + 2 \times 2 = 11$$

$$a_6 = a_3 + (6-3)d = 7 + 3 \times 2 = 13$$

3. 验证: a_5 是 a_3 和 a_7 的算术平均值
   Verify: a_5 is the arithmetic mean of a_3 and a_7

$$a_7 = 15, \frac{7+15}{2} = 11 = a_5$$

11.3 等差数列的性质 Properties of Arithmetic Sequences

定理 Theorem 11.3.1 (基本性质 Basic Properties). 1. 任意相邻两项的差等于公差
The difference between any two consecutive terms equals the common difference

$$a_{k+1} - a_k = d$$

2. 任意两项的差与它们的项数之差成正比
   The difference between any two terms is proportional to their position difference

$$a_m - a_n = (m-n)d$$

3. 任意三项等距离排列时，中间项是两端项的算术平均值
   When any three terms are equally spaced, the middle term is the arithmetic mean of the other two

$$a_k = \frac{a_{k-h} + a_{k+h}}{2}$$

4. 任意四项成等差数列时：
   When any four terms form an arithmetic sequence:

$$a_p + a_s = a_q + a_r \quad (\text{当且仅当 } p+s = q+r)$$

例 Example 11.3.2 (性质应用 Application of Properties). 在等差数列中，已知 $a_2 + a_5 = 14$，$a_3 + a_7 = 22$，求 $a_4 + a_6$。
In an arithmetic sequence, given $a_2 + a_5 = 14$ and $a_3 + a_7 = 22$, find $a_4 + a_6$.

解：
Solution:

1. 观察到 $2+5=7$，$3+7=10$，而 $4+6=10$
   Observe that $2+5=7$, $3+7=10$, and $4+6=10$
2. 由于 $4+6=3+7$，根据性质可知
   Since $4+6=3+7$, according to the property:

$$a_4 + a_6 = a_3 + a_7 = 22$$

11.4 前 n 项和的最值问题 Maximum and Minimum of Sum

定理 Theorem 11.4.1 (和的最值性质 Properties of Sum Extrema). 1. 当首项 a_1 和末项 a_n 固定时：
When the first term a_1 and last term a_n are fixed:

- 前 n 项和 S_n 与公差 d 无关
  The sum S_n is independent of the common difference d
- $S_n = \frac{n(a_1+a_n)}{2}$ 保持不变
  $S_n = \frac{n(a_1+a_n)}{2}$ remains constant

2. 当首项 a_1 和项数 n 固定时:
   When the first term a_1 and number of terms n are fixed:

- 前 n 项和 S_n 是公差 d 的一次函数
  The sum S_n is a linear function of d
- $S_n = na_1 + \frac{n(n-1)}{2}d$
  $S_n = na_1 + \frac{n(n-1)}{2}d$

3. 当末项 a_n 和项数 n 固定时:
   When the last term a_n and number of terms n are fixed:

- 前 n 项和 S_n 是公差 d 的一次函数
  The sum S_n is a linear function of d
- $S_n = na_n - \frac{n(n-1)}{2}d$
  $S_n = na_n - \frac{n(n-1)}{2}d$

例 Example 11.4.2 (最值问题示例 Example of Extrema Problem). 已知等差数列 $\{a_n\}$ 的首项 $a_1 = 2$，项数 $n = 5$，末项 $a_5 \leq 10$，求前 5 项和 S_5 的最大值。
Given an arithmetic sequence $\{a_n\}$ with first term $a_1 = 2$, number of terms $n = 5$, and last term $a_5 \leq 10$, find the maximum value of S_5.

解:
Solution:

1. 由于 $S_5 = \frac{5(a_1+a_5)}{2}$，且 $a_1 = 2$ 已知
   Since $S_5 = \frac{5(a_1+a_5)}{2}$ and $a_1 = 2$ is given
2. 当 a_5 取最大值 10 时，S_5 取最大值

When a_5 takes its maximum value 10, S_5 reaches its maximum

3. S_5 的最大值为：
The maximum value of S_5 is:

$$S_5 = \frac{5(2+10)}{2} = \frac{5 \times 12}{2} = 30$$

11.5 绝对值数列的前 n 项和 Sum of Absolute Value Sequence

定理 Theorem 11.5.1 (绝对值和的计算 Calculation of Absolute Value Sum). 对于等差数列 $\{a_n\}$，要计算 $\sum_{k=1}^{n} |a_k|$，需要：
To calculate $\sum_{k=1}^{n} |a_k|$ for an arithmetic sequence $\{a_n\}$:

1. 找出数列中的零点（若存在）
Find the zero points in the sequence (if any):

$$a_k = a_1 + (k-1)d = 0$$
$$k = 1 - \frac{a_1}{d}$$

2. 将数列分段，分别计算正值部分和负值部分的和
Divide the sequence into sections and calculate the sum of positive and negative parts separately:

   - 正值部分：直接用等差数列求和公式
   Positive part: use arithmetic sequence sum formula directly
   - 负值部分：先求和再取相反数
   Negative part: calculate sum then take the opposite

3. 利用等差数列的性质计算各部分的和后取绝对值
Calculate the sum of each part using properties of arithmetic sequences and take absolute values

例 Example 11.5.2 (绝对值和示例 1 Example 1 of Absolute Value

Sum). 求数列 $\{-5+2k\}_{k=1}^{10}$ 的所有项的绝对值之和。
Find the sum of absolute values of all terms in the sequence $\{-5+2k\}_{k=1}^{10}$.

解:
Solution:

1. 通项 $a_k = -5+2k$，零点在 $k=\frac{5}{2}$ 处
   General term $a_k = -5+2k$, zero point at $k=\frac{5}{2}$
2. 分析数列各项:
   Analyze the terms:
   - $k=1$ 时: $a_1 = -3$
   - $k=2$ 时: $a_2 = -1$
   - $k=3$ 时: $a_3 = 1$
   - $k=4$ 时: $a_4 = 3$
   - ...
   - $k=10$ 时: $a_{10} = 15$
3. 计算负值部分的和:
   Calculate sum of negative part:
   $$|(-3)+(-1)| = 4$$
4. 计算正值部分的和:
   Calculate sum of positive part:
   $$1+3+5+...+15 = \frac{8(1+15)}{2} = 72$$
5. 总和为:
   Total sum:
   $$4+72=76$$

例 Example 11.5.3 (绝对值和示例 2 Example 2 of Absolute Value Sum). 求数列 $\{3-k\}_{k=1}^{6}$ 的所有项的绝对值之和。
Find the sum of absolute values of all terms in the sequence $\{3-k\}_{k=1}^{6}$.

解:
Solution:

1. 通项 $a_k = 3 - k$，零点在 $k = 3$ 处
   General term $a_k = 3 - k$, zero point at $k = 3$
2. 分析数列各项:
   Analyze the terms:

   - $k = 1$ 时: $a_1 = 2$
   - $k = 2$ 时: $a_2 = 1$
   - $k = 3$ 时: $a_3 = 0$
   - $k = 4$ 时: $a_4 = -1$
   - $k = 5$ 时: $a_5 = -2$
   - $k = 6$ 时: $a_6 = -3$

3. 计算正值部分的和:
   Calculate sum of positive part:
   $$2 + 1 = 3$$
4. 计算负值部分的和:
   Calculate sum of negative part:
   $$|(-1) + (-2) + (-3)| = 6$$
5. 总和为:
   Total sum:
   $$3 + 6 = 9$$

Chapter 12

等比数列 Geometric Sequence

12.1 基本概念 Basic Concepts

定义 Definition 12.1.1 (等比数列 Geometric Sequence). 如果一个数列从第二项起，每一项与它的前一项的比值都等于同一个常数，这个数列就叫做等比数列。这个常数叫做等比数列的公比，记作 q。
A sequence in which the ratio of each consecutive term is constant is called a geometric sequence. This constant ratio is called the common ratio, denoted as q.

例 Example 12.1.2 (等比数列示例 Examples of Geometric Sequences). 以下是一些等比数列的例子：
Here are some examples of geometric sequences:

1. $\{2, 6, 18, 54, ...\}$，公比 $q = 3$
   $\{2, 6, 18, 54, ...\}$, common ratio $q = 3$
2. $\{16, 8, 4, 2, 1, ...\}$，公比 $q = \frac{1}{2}$
   $\{16, 8, 4, 2, 1, ...\}$, common ratio $q = \frac{1}{2}$
3. $\{3, 3, 3, 3, ...\}$，公比 $q = 1$（常数列也是等比数列）
   $\{3, 3, 3, 3, ...\}$, common ratio $q = 1$ (constant sequence is also a geometric sequence)

12.2 常用公式 Common Formulas

12.2.1 通项公式 General Term Formula

定理 Theorem 12.2.1 (通项公式 General Term Formula). 设等比数列的首项为 a_1，公比为 q，则:
For a geometric sequence with first term a_1 and common ratio q:

1. 第 n 项公式:
   The nth term formula:
   $$a_n = a_1 q^{n-1}$$
2. 第 n 项也可以用第 m 项表示:
   The nth term can also be expressed using the mth term:
   $$a_n = a_m q^{n-m}$$

例 Example 12.2.2 (通项公式应用 Application of General Term Formula). 已知等比数列 $\{a_n\}$ 的 $a_1 = 2$，$q = 3$，求:
Given geometric sequence $\{a_n\}$ with $a_1 = 2$ and $q = 3$, find:

1. a_5 的值
   The value of a_5
2. 使 $a_n = 486$ 的 n 值
   The value of n when $a_n = 486$

解:
Solution:

1. $a_5 = a_1 q^{5-1} = 2 \times 3^4 = 2 \times 81 = 162$
2. $486 = 2 \times 3^{n-1}$
   $243 = 3^{n-1}$

$5 = n - 1$

$n = 6$

12.2.2 前 n 项和公式 Sum of First n Terms

定理 Theorem 12.2.3 (前 n 项和公式 Sum of First n Terms). 等比数列前 n 项和 S_n 的计算公式有以下形式:
The sum of the first n terms S_n of a geometric sequence:

1. 当 $q \neq 1$ 时:
   When $q \neq 1$:
$$S_n = \frac{a_1(1-q^n)}{1-q} = \frac{a_1 - a_1q^n}{1-q} = \frac{a_1 - a_{n+1}}{1-q}$$
2. 当 $q = 1$ 时:
   When $q = 1$:
$$S_n = na_1$$

例 Example 12.2.4 (求和公式应用 Application of Sum Formula).
求等比数列 $\{3, 6, 12, 24, ..., 384\}$ 的和。
Find the sum of the geometric sequence $\{3, 6, 12, 24, ..., 384\}$.

解:
Solution:

1. 首先求项数 n:
   First find the number of terms n:
$$a_n = a_1q^{n-1}$$
$$384 = 3 \times 2^{n-1}$$
$$128 = 2^{n-1}$$
$$n = 8$$

2. 使用求和公式:
   Use the sum formula:

$$S_8 = \frac{3(1-2^8)}{1-2} = \frac{3(1-256)}{-1} = 3 \times 255 = 765$$

12.3 等比数列的性质 Properties of Geometric Sequences

定理 Theorem 12.3.1 (基本性质 Basic Properties). 1. 任意相邻两项的比值等于公比
   The ratio of any two consecutive terms equals the common ratio

$$\frac{a_{k+1}}{a_k} = q$$

2. 任意两项的比值等于公比的幂
   The ratio of any two terms equals a power of the common ratio

$$\frac{a_m}{a_n} = q^{m-n}$$

3. 任意三项等比排列时，中间项是两端项的几何平均值
   When any three terms are in geometric progression, the middle term is the geometric mean of the other two

$$a_k = \sqrt{a_{k-h} \cdot a_{k+h}}$$

4. 任意四项成等比数列时:
   When any four terms form a geometric sequence:

$$a_p \cdot a_s = a_q \cdot a_r \quad (\text{□□□□}p + s = q + r)$$

12.4 其他重要公式 Other Important Formulas

12.4.1 连续两项积的和 Sum of Products of Consecutive Terms

定理 Theorem 12.4.1 (连续两项积的和 Sum of Products of Consecutive Terms). 等比数列中连续两项积的和:

The sum of products of consecutive terms in a geometric sequence:

$$a_1a_2 + a_2a_3 + ... + a_{n-1}a_n = a_1^2q(1 + q + ... + q^{n-2}) = a_1^2q\frac{1 - q^{n-1}}{1 - q}$$

12.4.2 平方和公式 Sum of Squares Formula

定理 Theorem 12.4.2 (平方和公式 Sum of Squares Formula). 等比数列前 n 项的平方和:
The sum of squares of the first n terms in a geometric sequence:

$$a_1^2 + a_2^2 + ... + a_n^2 = a_1^2(1 + q^2 + ... + q^{2(n-1)}) = a_1^2\frac{1 - q^{2n}}{1 - q^2}$$

12.4.3 几何平均值性质 Geometric Mean Properties

定理 Theorem 12.4.3 (几何平均值性质 Geometric Mean Properties). 在等比数列中:
In a geometric sequence:

1. 任意两项的几何平均值等于这两项中间对应项的值
   The geometric mean of any two terms equals the value of the corresponding term between them

$$\sqrt{a_p \cdot a_r} = a_{\frac{p+r}{2}} \quad (p + r\text{□□□})$$

2. 任意奇数个项等比排列时，中间项等于首末项的几何平均值
   When any odd number of terms are in geometric progression, the middle term equals the geometric mean of the first and last terms

$$a_k = \sqrt{a_{k-m} \cdot a_{k+m}}$$

例 Example 12.4.4 (几何平均值应用 Application of Geometric Mean). 在等比数列中，已知 $a_2 = 6$，$a_5 = 48$，求 a_3 和 a_4。
In a geometric sequence, given $a_2 = 6$ and $a_5 = 48$, find a_3 and a_4.

解:

Solution:

1. 先求公比 q:
   First find the common ratio q:
$$\frac{a_5}{a_2} = q^{5-2} = q^3$$
$$\frac{48}{6} = q^3$$
$$8 = q^3$$
$$q = 2$$
2. 求 a_3 和 a_4:
   Find a_3 and a_4:
$$a_3 = a_2 q = 6 \times 2 = 12$$
$$a_4 = a_3 q = 12 \times 2 = 24$$
3. 验证: a_3 是 a_2 和 a_4 的几何平均值
   Verify: a_3 is the geometric mean of a_2 and a_4
$$\sqrt{6 \times 24} = \sqrt{144} = 12 = a_3$$

Chapter 13

不等式与线性规划 Inequalities and Linear Programming

13.1 不等式的基本性质 Basic Properties of Inequalities

定理 Theorem 13.1.1 (不等式的基本性质 Basic Properties of Inequalities). 对于实数 a，b，c:
For real numbers a, b, c:

1. 传递性：若 $a > b$ 且 $b > c$，则 $a > c$
   Transitivity: If $a > b$ and $b > c$, then $a > c$
2. 四则运算:
   Arithmetic operations:

   - 若 $a > b$，则 $a + c > b + c$
     If $a > b$, then $a + c > b + c$
   - 若 $a > b$ 且 $c > 0$，则 $ac > bc$
     If $a > b$ and $c > 0$, then $ac > bc$
   - 若 $a > b$ 且 $c < 0$，则 $ac < bc$
     If $a > b$ and $c < 0$, then $ac < bc$

13.2 基本不等式及其推广 Basic Inequalities and Their Extensions

13.2.1 基本不等式 Basic Inequalities

定理 Theorem 13.2.1 (算术平均值-几何平均值不等式 AM-GM Inequality). 对于任意正实数 $x_1, x_2, ..., x_n$:
For any positive real numbers $x_1, x_2, ..., x_n$:

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 ... x_n}$$

当且仅当 $x_1 = x_2 = ... = x_n$ 时取等号。
Equality holds if and only if $x_1 = x_2 = ... = x_n$.

定理 Theorem 13.2.2 (平方和不等式 Sum of Squares Inequality). 对于任意实数 $x_1, x_2, ..., x_n$:
For any real numbers $x_1, x_2, ..., x_n$:

$$(x_1^2 + x_2^2 + ... + x_n^2)(y_1^2 + y_2^2 + ... + y_n^2) \geq (x_1 y_1 + x_2 y_2 + ... + x_n y_n)^2$$

13.2.2 幂平均不等式 Power Mean Inequality

定理 Theorem 13.2.3 (幂平均不等式 Power Mean Inequality). 对于正实数 $x_1, x_2, ..., x_n$ 和实数 $r > s$:
For positive real numbers $x_1, x_2, ..., x_n$ and real numbers $r > s$:

$$\left(\frac{x_1^r + x_2^r + ... + x_n^r}{n}\right)^{\frac{1}{r}} \geq \left(\frac{x_1^s + x_2^s + ... + x_n^s}{n}\right)^{\frac{1}{s}}$$

13.2.3 柯西不等式 Cauchy’s Inequality

定理 Theorem 13.2.4 (柯西不等式 Cauchy’s Inequality). 对于任意实数 $a_1, a_2, ..., a_n$ 和 $b_1, b_2, ..., b_n$:
For any real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$:

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1 b_1 + a_2 b_2 + ... + a_n b_n)^2$$

13.3 不等式的解法 Methods for Solving Inequalities

13.3.1 分式不等式 Rational Inequalities

定理 Theorem 13.3.1 (分式不等式解法 Solving Rational Inequalities). 解分式不等式的步骤:
Steps for solving rational inequalities:

1. 将不等式化为标准形式 $\frac{P(x)}{Q(x)} > 0$ 或 $\frac{P(x)}{Q(x)} < 0$
   Convert the inequality to standard form $\frac{P(x)}{Q(x)} > 0$ or $\frac{P(x)}{Q(x)} < 0$
2. 找出分子分母的零点
   Find zeros of numerator and denominator
3. 在数轴上画出这些点，并检查每个区间的符号
   Plot these points on number line and check signs in each interval

13.3.2 指数不等式 Exponential Inequalities

定理 Theorem 13.3.2 (指数不等式解法 Solving Exponential Inequalities). 解指数不等式时注意:
When solving exponential inequalities:

1. 若 $a > 1$，则 a^x 单调递增
   If $a > 1$, then a^x is strictly increasing
2. 若 $0 < a < 1$，则 a^x 单调递减
   If $0 < a < 1$, then a^x is strictly decreasing

13.3.3 对数不等式 Logarithmic Inequalities

定理 Theorem 13.3.3 (对数不等式解法 Solving Logarithmic Inequalities). 解对数不等式时注意:
When solving logarithmic inequalities:

1. 对数的定义域必须大于 0
   Domain of logarithm must be positive
2. 若 $a > 1$，则 $\log_a x$ 单调递增
   If $a > 1$, then $\log_a x$ is strictly increasing
3. 若 $0 < a < 1$，则 $\log_a x$ 单调递减
   If $0 < a < 1$, then $\log_a x$ is strictly decreasing

13.3.4 绝对值不等式 Absolute Value Inequalities

定理 Theorem 13.3.4 (绝对值不等式解法 Solving Absolute Value Inequalities). 对于实数 a:
For real number a:

1. $|x| < a$ 等价于 $-a < x < a$
   $|x| < a$ is equivalent to $-a < x < a$
2. $|x| > a$ 等价于 $x < -a$ 或 $x > a$
   $|x| > a$ is equivalent to $x < -a$ or $x > a$

13.4 证明不等式的技巧 Techniques for Proving Inequalities

定理 Theorem 13.4.1 (常用证明技巧 Common Proof Techniques).

1. 放缩法：用已知不等式替换
   Estimation: Replace with known inequalities
2. 数学归纳法：对 n 进行归纳
   Mathematical induction: Induct on n
3. 构造法：构造辅助函数
   Construction: Create auxiliary functions
4. 缩放法：通过变量替换简化问题
   Scaling: Simplify through variable substitution

13.5 线性规划 Linear Programming

13.5.1 基本概念 Basic Concepts

定义 Definition 13.5.1 (线性规划 Linear Programming). 线性规划是在一组线性约束条件下，求解线性目标函数的最值问题。
Linear programming is the problem of optimizing a linear objective function subject to linear constraints.

13.5.2 几何意义 Geometric Interpretation

定理 Theorem 13.5.2 (可行域的性质 Properties of Feasible Region).

1. 线性不等式组的解集是平面上的凸多边形区域
   The solution set of linear inequalities is a convex polygon in the plane
2. 目标函数的等值线是一族平行直线
   Level curves of the objective function are parallel lines
3. 最优解在可行域的顶点上取得
   The optimal solution is achieved at a vertex of the feasible region

13.5.3 二元一次不等式与平面区域 Linear Inequalities and Plane Regions

定理 Theorem 13.5.3 (二元一次不等式的图形 Graphs of Linear Inequalities). 对于二元一次不等式 $ax + by + c \geq 0$:
For linear inequality $ax + by + c \geq 0$:

1. 边界直线 $ax + by + c = 0$
   Boundary line $ax + by + c = 0$
2. 将点 (x_0, y_0) 代入左边，正负号决定点的位置
   Substitute point (x_0, y_0) into left side, sign determines position
3. 不等号方向与半平面的关系
   Relationship between inequality direction and half-

plane

例 Example 13.5.4 (线性规划问题 Linear Programming Problem).
求解:
Solve:

$$\max z = 2x + 3y$$

约束条件:
Subject to:

$$\begin{aligned} x + y &\leq 6 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

解:
Solution:

1. 画出可行域
   Draw the feasible region
2. 找出顶点: $(0,0)$, $(0,6)$, $(6,0)$
   Find vertices: $(0,0)$, $(0,6)$, $(6,0)$
3. 代入目标函数比较:
   Compare values of objective function:
   - $(0,0)$: $z = 0$
   - $(0,6)$: $z = 18$
   - $(6,0)$: $z = 12$
4. 最大值在 $(0,6)$ 处取得, $z_{max} = 18$
   Maximum value is achieved at $(0,6)$, $z_{max} = 18$

13.6 基本不等式的应用 Applications of Basic Inequalities

例 Example 13.6.1 (最值问题 Extremum Problems). 求 $x+\frac{1}{x}$ 的最小值（$x>0$）。
Find the minimum value of $x+\frac{1}{x}$ ($x>0$).

解:
Solution:

1. 由算术-几何平均值不等式:
   By AM-GM inequality:

$$\frac{x+\frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} = 1$$

2. 因此 $x+\frac{1}{x} \geq 2$，当且仅当 $x=\frac{1}{x}$ 即 $x=1$ 时取等号
   Therefore $x+\frac{1}{x} \geq 2$, equality holds if and only if $x=\frac{1}{x}$, i.e., $x=1$

13.7 平均不等式及其应用 Mean Inequalities and Applications

13.7.1 基本平均不等式 Basic Mean Inequalities

定理 Theorem 13.7.1 (调和-几何-算术-平方平均值不等式 HM-GM-AM-QM Inequality). 对于正实数 $x_1, x_2, ..., x_n$:
For positive real numbers $x_1, x_2, ..., x_n$:

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}} \leq \sqrt[n]{x_1 x_2 ... x_n} \leq \frac{x_1+x_2+...+x_n}{n} \leq \sqrt{\frac{x_1^2+x_2^2+...+x_n^2}{n}}$$

13.7.2 均值不等式的常用结论 Common Results of Mean Inequalities

定理 Theorem 13.7.2 (均值不等式的推论 Corollaries of Mean Inequalities). 若 $a, b > 0$，则:
If $a, b > 0$, then:

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

2. 对任意正实数，算术平均值不小于几何平均值
   For any positive real numbers, arithmetic mean is not less than geometric mean
3. 对任意正实数，平方平均值不小于算术平均值
   For any positive real numbers, quadratic mean is not less than arithmetic mean

13.8 对勾函数 Check Function

定义 Definition 13.8.1 (对勾函数 Check Function). 对勾函数是指形如 $y = |x-a| + b$ 的函数，其图像呈”V” 字形。
A check function is a function of the form $y = |x-a|+b$, whose graph is V-shaped.

定理 Theorem 13.8.2 (对勾函数的性质 Properties of Check Function). 对于函数 $y = |x-a| + b$:
For function $y = |x-a| + b$:

1. 顶点坐标为 (a, b)
   Vertex coordinates are (a, b)
2. 左右两支分别为 $y = -(x-a) + b$ 和 $y = (x-a) + b$
   Left and right branches are $y = -(x-a)+b$ and $y = (x-a)+b$ respectively
3. 图像关于 $x = a$ 对称
   Graph is symmetric about $x = a$

13.9 其他类型不等式的解法 Methods for Other Types of Inequalities

13.9.1 无理式不等式 Irrational Inequalities

定理 Theorem 13.9.1 (无理式不等式解法 Solving Irrational Inequalities). 解无理式不等式的步骤:
Steps for solving irrational inequalities:

1. 检查定义域
   Check domain
2. 通过移项将根式分离
   Isolate radical terms
3. 两边平方（注意不等号方向可能改变）
   Square both sides (note inequality direction may change)
4. 解得结果后代回验证
   Verify solutions by substitution

13.9.2 高次不等式 Higher-Degree Inequalities

定理 Theorem 13.9.2 (高次不等式解法 Solving Higher-Degree Inequalities). 解高次不等式的方法:
Methods for solving higher-degree inequalities:

1. 因式分解法
   Factorization method
2. 画图法
   Graphical method
3. 区间分析法
   Interval analysis method
4. 配方法
   Completing the square method

13.9.3 一元二次不等式 Quadratic Inequalities

定理 Theorem 13.9.3 (一元二次不等式解法 Solving Quadratic Inequalities). 对于不等式 $ax^2+bx+c>0$ ($a\neq 0$):
For inequality $ax^2+bx+c>0$ ($a\neq 0$):

1. 当 $a>0$ 时:
   When $a>0$:

   - 若 $\Delta<0$，解集为 $(-\infty,+\infty)$
     If $\Delta<0$, solution is $(-\infty,+\infty)$
   - 若 $\Delta=0$，解集为 $(-\infty,x_1)\cup(x_1,+\infty)$
     If $\Delta=0$, solution is $(-\infty,x_1)\cup(x_1,+\infty)$
   - 若 $\Delta>0$，解集为 $(-\infty,x_1)\cup(x_2,+\infty)$
     If $\Delta>0$, solution is $(-\infty,x_1)\cup(x_2,+\infty)$

2. 当 $a<0$ 时:
   When $a<0$:

   - 若 $\Delta<0$，解集为 $\emptyset$
     If $\Delta<0$, solution is $\emptyset$
   - 若 $\Delta=0$，解集为 $\{x_1\}$
     If $\Delta=0$, solution is $\{x_1\}$
   - 若 $\Delta>0$，解集为 (x_1,x_2)
     If $\Delta>0$, solution is (x_1,x_2)

其中 $\Delta=b^2-4ac$，x_1 和 x_2 是二次方程的根。
Where $\Delta=b^2-4ac$, x_1 and x_2 are roots of the quadratic equation.

13.10 其他常用公式 Other Common Formulas

定理 Theorem 13.10.1 (常用不等式 Common Inequalities). 1. 三角不等式: $|a+b|\leq|a|+|b|$
Triangle inequality: $|a+b|\leq|a|+|b|$

2. 伯努利不等式：$(1+x)^n \geq 1+nx$（$x>-1$，n 为自然数）
   Bernoulli's inequality: $(1+x)^n \geq 1+nx$ ($x>-1$, n is natural number)
3. 琴生不等式：若 f 为凸函数，则 $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$
   Jensen's inequality: If f is convex, then $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$
4. 排序不等式：$\sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} a_i^{\uparrow} b_i^{\downarrow}$
   Rearrangement inequality: $\sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} a_i^{\uparrow} b_i^{\downarrow}$

Chapter 14

空间几何体 Spatial Geometry

14.1 基本概念 Basic Concepts

定义 Definition 14.1.1 (空间几何体 Spatial Geometric Solids).
空间几何体是由平面图形围成的立体图形。常见的空间几何体包括:
Spatial geometric solids are three-dimensional figures bounded by plane figures. Common solids include:

1. 棱柱（包括长方体、正方体）
   Prisms (including cuboids and cubes)
2. 棱锥（包括正棱锥）
   Pyramids (including regular pyramids)
3. 圆柱
   Cylinders
4. 圆锥
   Cones
5. 球
   Spheres

定义 Definition 14.1.2 (棱柱的基本要素 Basic Elements of Prism).
棱柱的基本要素包括:

Basic elements of a prism include:

1. 底面：两个全等且平行的多边形
   Bases: two congruent and parallel polygons
2. 侧面：由平行于棱柱轴的矩形构成
   Lateral faces: rectangles parallel to the prism axis
3. 棱：包括底棱和侧棱
   Edges: including base edges and lateral edges
4. 顶点：底面顶点和侧棱端点
   Vertices: base vertices and lateral edge endpoints

定义 Definition 14.1.3 (棱锥的基本要素 Basic Elements of Pyramid). 棱锥的基本要素包括：
Basic elements of a pyramid include:

1. 底面：任意多边形
   Base: any polygon
2. 顶点：所有侧棱的公共端点
   Apex: common endpoint of all lateral edges
3. 侧面：由顶点和底边构成的三角形
   Lateral faces: triangles formed by apex and base edges
4. 高：顶点到底面的垂线段
   Height: perpendicular line segment from apex to base

14.2 体积公式 Volume Formulas

定理 Theorem 14.2.1 (基本体积公式 Basic Volume Formulas).

1. 棱柱体积：$V = Sh$
   Volume of prism: $V = Sh$
   其中 S 为底面积，h 为高
   where S is base area, h is height
2. 棱锥体积：$V = \frac{1}{3}Sh$
   Volume of pyramid: $V = \frac{1}{3}Sh$

其中 S 为底面积，h 为高
where S is base area, h is height

3. 圆柱体积：$V = \pi r^2 h$
Volume of cylinder: $V = \pi r^2 h$
其中 r 为底面半径，h 为高
where r is base radius, h is height

4. 圆锥体积：$V = \frac{1}{3}\pi r^2 h$
Volume of cone: $V = \frac{1}{3}\pi r^2 h$
其中 r 为底面半径，h 为高
where r is base radius, h is height

5. 球的体积：$V = \frac{4}{3}\pi R^3$
Volume of sphere: $V = \frac{4}{3}\pi R^3$
其中 R 为球半径
where R is radius

6. 球台的体积：$V = \frac{1}{3}\pi h(R^2 + r^2 + Rr)$
Volume of spherical frustum: $V = \frac{1}{3}\pi h(R^2 + r^2 + Rr)$
其中 R、r 为上下底面半径，h 为高
where R, r are radii of bases, h is height

定理 Theorem 14.2.2 (体积关系 Volume Relationships). 1. 同底等高的棱柱和棱锥，棱锥的体积是棱柱的 $\frac{1}{3}$
For a prism and pyramid with same base and height, the volume of pyramid is $\frac{1}{3}$ of prism

2. 同底等高的圆柱和圆锥，圆锥的体积是圆柱的 $\frac{1}{3}$
For a cylinder and cone with same base and height, the volume of cone is $\frac{1}{3}$ of cylinder

3. 球的体积等于同半径圆柱体积的 $\frac{2}{3}$
Volume of sphere equals $\frac{2}{3}$ of cylinder with same radius and height equals diameter

14.3 表面积公式 Surface Area Formulas

定理 Theorem 14.3.1 (基本表面积公式 Basic Surface Area Formulas). 1. 棱柱侧面积：$S_{▯} = Ph$
Lateral surface area of prism: $S_{lateral} = Ph$
其中 P 为底面周长，h 为高

where P is base perimeter, h is height

2. 棱锥侧面积：$S_{\square} = \frac{1}{2}Pl$
   Lateral surface area of pyramid: $S_{lateral} = \frac{1}{2}Pl$
   其中 P 为底面周长，l 为斜高
   where P is base perimeter, l is slant height
3. 圆柱侧面积：$S_{\square} = 2\pi rh$
   Lateral surface area of cylinder: $S_{lateral} = 2\pi rh$
   其中 r 为底面半径，h 为高
   where r is base radius, h is height
4. 圆锥侧面积：$S_{\square} = \pi rl$
   Lateral surface area of cone: $S_{lateral} = \pi rl$
   其中 r 为底面半径，l 为母线长
   where r is base radius, l is slant height
5. 球的表面积：$S = 4\pi R^2$
   Surface area of sphere: $S = 4\pi R^2$
   其中 R 为球半径
   where R is radius
6. 球台的侧面积：$S_{\square} = 2\pi Rh$
   Lateral surface area of spherical zone: $S_{lateral} = 2\pi Rh$
   其中 R 为球半径，h 为球台高
   where R is sphere radius, h is zone height

14.4 内切球与外接球 Inscribed and Circumscribed Spheres

14.4.1 长方体的内切球与外接球 Inscribed and Circumscribed Spheres of Cuboid

定理 Theorem 14.4.1 (长方体的内切球与外接球 Inscribed and Circumscribed Spheres of Cuboid). 对于长方体 $a \times b \times c$:
For a cuboid with dimensions $a \times b \times c$:

1. 内切球半径：$r = \frac{abc}{2(ab+bc+ac)}$
   Radius of inscribed sphere: $r = \frac{abc}{2(ab+bc+ac)}$
2. 外接球半径：$R = \frac{\sqrt{a^2+b^2+c^2}}{2}$

Radius of circumscribed sphere: $R = \frac{\sqrt{a^2+b^2+c^2}}{2}$

3. 内切球与外接球半径关系：$R \geq 2r$
   Relationship between radii: $R \geq 2r$

14.4.2 正四面体的内切球与外接球 Inscribed and Circumscribed Spheres of Regular Tetrahedron

定理 Theorem 14.4.2 (正四面体的内切球与外接球 Inscribed and Circumscribed Spheres of Regular Tetrahedron). 对于边长为 a 的正四面体：
For a regular tetrahedron with edge length a:

1. 内切球半径：$r = \frac{a}{4\sqrt{6}}$
   Radius of inscribed sphere: $r = \frac{a}{4\sqrt{6}}$
2. 外接球半径：$R = \frac{a\sqrt{6}}{4}$
   Radius of circumscribed sphere: $R = \frac{a\sqrt{6}}{4}$
3. 内切球与外接球半径关系：$R = 3r$
   Relationship between radii: $R = 3r$

14.5 正四面体的性质 Properties of Regular Tetrahedron

定理 Theorem 14.5.1 (正四面体的基本性质 Basic Properties of Regular Tetrahedron). 对于边长为 a 的正四面体：
For a regular tetrahedron with edge length a:

1. 体积：$V = \frac{a^3}{6\sqrt{2}}$
   Volume: $V = \frac{a^3}{6\sqrt{2}}$
2. 表面积：$S = a^2\sqrt{3}$
   Surface area: $S = a^2\sqrt{3}$
3. 高：$h = a\sqrt{\frac{2}{3}}$
   Height: $h = a\sqrt{\frac{2}{3}}$

4. 面角：arccos$(\frac{1}{3}) \approx 70.53°$
   Face angle: arccos$(\frac{1}{3}) \approx 70.53°$
5. 二面角：arccos$(\frac{1}{3}) \approx 70.53°$
   Dihedral angle: arccos$(\frac{1}{3}) \approx 70.53°$
6. 面心到顶点的距离：$d = \frac{a\sqrt{6}}{4}$
   Distance from face centroid to vertex: $d = \frac{a\sqrt{6}}{4}$
7. 重心到顶点的距离：$d = \frac{a\sqrt{6}}{4}$
   Distance from centroid to vertex: $d = \frac{a\sqrt{6}}{4}$

定理 Theorem 14.5.2 (正四面体的对称性 Symmetry of Regular Tetrahedron). 正四面体具有以下对称性：
A regular tetrahedron has the following symmetries:

1. 4 个三重旋转轴（通过顶点和对面重心）
   4 three-fold rotation axes (through vertices and centroids of opposite faces)
2. 3 个二重旋转轴（通过对边中点）
   3 two-fold rotation axes (through midpoints of opposite edges)
3. 6 个对称面（通过对边）
   6 planes of symmetry (through opposite edges)
4. 总共 24 种对称变换
   Total of 24 symmetry transformations

14.6 几何体的截面 Cross Sections of Geometric Solids

定理 Theorem 14.6.1 (常见几何体的截面 Common Cross Sections). 1. 球的任意平面截面都是圆
   Any plane section of a sphere is a circle
2. 圆柱的斜截面是椭圆
   Oblique section of a cylinder is an ellipse
3. 圆锥的平行于底面的截面是圆，其他截面可能是椭圆、抛物线或双曲线

Sections parallel to base of cone are circles, other sections may be ellipses, parabolas, or hyperbolas

4. 正四面体的任意平面截面都是多边形（三角形或四边形）
   Any plane section of regular tetrahedron is a polygon (triangle or quadrilateral)

14.7 实例与应用 Examples and Applications

例 Example 14.7.1 (球的内接正四面体 Inscribed Regular Tetrahedron in Sphere). 求半径为 R 的球内接正四面体的边长。
Find the edge length of a regular tetrahedron inscribed in a sphere of radius R.

解:
Solution:

1. 设正四面体边长为 a，则其外接球半径为 $R = \frac{a\sqrt{6}}{4}$
   Let the edge length be a, then radius of circumscribed sphere is $R = \frac{a\sqrt{6}}{4}$
2. 解得: $a = \frac{4R}{\sqrt{6}}$
   Therefore: $a = \frac{4R}{\sqrt{6}}$

例 Example 14.7.2 (正方体内的正四面体 Regular Tetrahedron in Cube). 在边长为 a 的正方体中，求由正方体四个顶点构成的最大正四面体的体积。
In a cube with edge length a, find the volume of the largest regular tetrahedron formed by four vertices of the cube.

解:
Solution:

1. 选取正方体的四个顶点，使得这四个顶点两两之间的距离相等
   Select four vertices of the cube such that distances

between any two vertices are equal

2. 这样的四个顶点构成正四面体，其边长为 $a\sqrt{2}$
   These four vertices form a regular tetrahedron with edge length $a\sqrt{2}$
3. 代入正四面体体积公式：
   Substitute into volume formula:
$$V = \frac{(a\sqrt{2})^3}{6\sqrt{2}} = \frac{a^3}{3\sqrt{3}}$$

例 Example 14.7.3 (圆柱内的最大球 Largest Sphere in Cylinder). 在高为 h，底面半径为 r 的圆柱中，求能放入的最大球的半径 R。
In a cylinder with height h and base radius r, find the radius R of the largest sphere that can be placed inside.

解：
Solution:

1. 最大球必须与圆柱的底面和侧面相切
   The largest sphere must be tangent to both base and lateral surface of cylinder
2. 由切点条件可得：$R \leq r$ 且 $R \leq \frac{h}{2}$
   From tangency conditions: $R \leq r$ and $R \leq \frac{h}{2}$
3. 因此：$R = \min(r, \frac{h}{2})$
   Therefore: $R = \min(r, \frac{h}{2})$

例 Example 14.7.4 (圆锥的体积与表面积 Volume and Surface Area of Cone). 一个圆锥的母线长为 l，底面半径为 r，求：
For a cone with slant height l and base radius r, find:

1. 圆锥的高 h
   Height of cone h
2. 圆锥的体积 V
   Volume of cone V
3. 圆锥的全面积 S

Total surface area S

解：
Solution:

1. 由毕达哥拉斯定理：$h = \sqrt{l^2 - r^2}$
   By Pythagorean theorem: $h = \sqrt{l^2 - r^2}$
2. 体积：$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$
   Volume: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$
3. 全面积：$S = \pi r^2 + \pi r l = \pi r(r + l)$
   Total surface area: $S = \pi r^2 + \pi r l = \pi r(r + l)$

Chapter 15

直线的倾斜角与斜率 Slope and Angle of Inclination

15.1 基本概念 Basic Concepts

定义 Definition 15.1.1 (倾斜角 Angle of Inclination). 直线的倾斜角是指直线与 x 轴正方向所成的角，通常用 α 表示，范围是 $(-90^\circ, 90^\circ)$。
The angle of inclination of a line is the angle between the line and the positive x-axis, usually denoted as α, with range $(-90^\circ, 90^\circ)$.

定义 Definition 15.1.2 (斜率 Slope). 直线的斜率是指直线上任意两点的纵坐标之差与横坐标之差的比值，通常用 k 表示。
The slope of a line is the ratio of the difference in y-coordinates to the difference in x-coordinates of any two points on the line, usually denoted as k.

$$k = \frac{y_2 - y_1}{x_2 - x_1} = \tan\alpha$$

15.2 斜率与倾斜角的关系 Relationship Between Slope and Angle of Inclination

定理 Theorem 15.2.1 (斜率与倾斜角的关系 Relationship Between Slope and Angle of Inclination). 对于直线 $y = kx$:
For the line $y = kx$:

1. 斜率 k 等于倾斜角 α 的正切值
   The slope k equals the tangent of the angle of inclination α

$$k = \tan\alpha$$

2. 倾斜角 α 等于斜率 k 的反正切值
   The angle of inclination α equals the arctangent of the slope k

$$\alpha = \arctan k$$

15.3 特殊情况 Special Cases

定理 Theorem 15.3.1 (特殊斜率与倾斜角 Special Slopes and Angles).

1. 水平线：$k = 0$，$\alpha = 0^\circ$
   Horizontal line: $k = 0$, $\alpha = 0^\circ$
2. 向右上倾斜 45° 的直线：$k = 1$，$\alpha = 45^\circ$
   Line tilted 45° upward to the right: $k = 1$, $\alpha = 45^\circ$
3. 向右下倾斜 45° 的直线：$k = -1$，$\alpha = -45^\circ$
   Line tilted 45° downward to the right: $k = -1$, $\alpha = -45^\circ$
4. 垂直线：k 不存在，$\alpha = 90^\circ$ 或 -90°
   Vertical line: k undefined, $\alpha = 90^\circ$ or -90°

15.4 斜率的几何意义 Geometric Meaning of Slope

定理 Theorem 15.4.1 (斜率的几何意义 Geometric Meaning of Slope).
斜率 k 表示:
The slope k represents:

1. 直线每向右移动 1 个单位，向上移动的单位数
   The number of units moved up when moving 1 unit to the right
2. 直线的倾斜程度
   The steepness of the line
3. 直线与 x 轴正方向的倾斜角的正切值
   The tangent of the angle between the line and the positive x-axis

15.5 斜率的性质 Properties of Slope

定理 Theorem 15.5.1 (斜率的性质 Properties of Slope).

1. 平行直线具有相等的斜率
   Parallel lines have equal slopes
2. 垂直直线的斜率之积为-1（当斜率都存在时）
   The product of slopes of perpendicular lines is -1 (when both slopes exist)
3. 斜率的绝对值越大，直线越陡峭
   The larger the absolute value of the slope, the steeper the line
4. 正斜率表示直线向右上方倾斜，负斜率表示直线向右下方倾斜
   Positive slope indicates line tilts upward to the right, negative slope indicates line tilts downward to the right

15.6 直线方程的五种形式 Five Forms of Linear Equations

定理 Theorem 15.6.1 (直线方程的基本形式 Basic Forms of Linear Equations). 直线方程有以下五种基本形式:
Linear equations have the following five basic forms:

1. 点斜式: $y - y_1 = k(x - x_1)$
   Point-slope form: $y - y_1 = k(x - x_1)$
2. 斜截式: $y = kx + b$
   Slope-intercept form: $y = kx + b$
3. 一般式: $Ax + By + C = 0$
   General form: $Ax + By + C = 0$
4. 截距式: $\frac{x}{a} + \frac{y}{b} = 1$
   Intercept form: $\frac{x}{a} + \frac{y}{b} = 1$
5. 两点式: $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$
   Two-point form: $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

15.7 直线的位置关系 Relative Positions of Lines

定理 Theorem 15.7.1 (直线位置关系判定 Determination of Line Positions). 设两直线方程分别为 $A_1x+B_1y+C_1 = 0$ 和 $A_2x+B_2y+C_2 = 0$:
Given two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$:

1. 平行条件: $\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$
   Parallel condition: $\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2}$
2. 重合条件: $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$
   Coincident condition: $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$
3. 相交条件: $\frac{A_1}{A_2} \neq \frac{B_1}{B_2}$
   Intersecting condition: $\frac{A_1}{A_2} \neq \frac{B_1}{B_2}$
4. 垂直条件: $A_1A_2 + B_1B_2 = 0$

Perpendicular condition: $A_1A_2 + B_1B_2 = 0$

15.8 两直线的交点 Intersection Point of Two Lines

定理 Theorem 15.8.1 (交点坐标计算 Calculation of Intersection Point). 两直线 $A_1x + B_1y + C_1 = 0$ 和 $A_2x + B_2y + C_2 = 0$ 的交点坐标为: The intersection point of lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ is:

$$x = \frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}, \quad y = \frac{A_2C_1 - A_1C_2}{A_1B_2 - A_2B_1}$$

15.9 距离公式 Distance Formulas

定理 Theorem 15.9.1 (常用距离公式 Common Distance Formulas).

1. 两点距离:
   Distance between two points:
   $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
2. 点到直线距离:
   Distance from point to line:
   $$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$
3. 平行直线间距离:
   Distance between parallel lines:
   $$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$$

15.10 分点公式 Division Point Formulas

定理 Theorem 15.10.1 (分点公式 Division Point Formulas).

1. 中点坐标:
   Midpoint coordinates:
   $$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}$$

2. 三分点坐标：
   Trisection point coordinates:

$$x = \frac{2x_1 + x_2}{3}, \quad y = \frac{2y_1 + y_2}{3}$$

$$x = \frac{x_1 + 2x_2}{3}, \quad y = \frac{y_1 + 2y_2}{3}$$

3. λ 分点坐标：
   λ division point coordinates:

$$x = \frac{\lambda x_2 + x_1}{1 + \lambda}, \quad y = \frac{\lambda y_2 + y_1}{1 + \lambda}$$

15.11 直线系方程 Systems of Lines

定理 Theorem 15.11.1 (常见直线系 Common Systems of Lines). 1. 过定点的直线系：$y - y_0 = k(x - x_0)$
   Lines through a fixed point: $y - y_0 = k(x - x_0)$
2. 平行于定直线的直线系：$y = kx + b$（k 固定）
   Lines parallel to a fixed line: $y = kx + b$ (k fixed)
3. 过两直线交点的直线系：$A_1x + B_1y + C_1 + \lambda(A_2x + B_2y + C_2) = 0$
   Lines through intersection point: $A_1x + B_1y + C_1 + \lambda(A_2x + B_2y + C_2) = 0$

15.12 对称问题 Symmetry Problems

定理 Theorem 15.12.1 (点与直线的对称 Symmetry of Points and Lines). 1. 关于 y 轴对称的点：$(x, y) \to (-x, y)$
   Points symmetric about y-axis: $(x, y) \to (-x, y)$
2. 关于 x 轴对称的点：$(x, y) \to (x, -y)$
   Points symmetric about x-axis: $(x, y) \to (x, -y)$
3. 关于原点对称的点：$(x, y) \to (-x, -y)$
   Points symmetric about origin: $(x, y) \to (-x, -y)$
4. 关于直线 $Ax + By + C = 0$ 对称的点（x_0, y_0）的对称点坐标：
   Point symmetric about line $Ax + By + C = 0$:

$$x = x_0 - \frac{2A(Ax_0 + By_0 + C)}{A^2 + B^2}$$

$$y = y_0 - \frac{2B(Ax_0 + By_0 + C)}{A^2 + B^2}$$

15.13 实例与应用 Examples and Applications

例 Example 15.13.1 (计算斜率和倾斜角 Calculate Slope and Angle of Inclination). 已知直线通过点 $A(1,2)$ 和 $B(4,8)$，求：
Given a line passing through points $A(1,2)$ and $B(4,8)$, find:

1. 直线的斜率 k
   The slope k of the line
2. 直线的倾斜角 α
   The angle of inclination α of the line

解：
Solution:

1. 斜率：
   Slope:
   $$k = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-2}{4-1} = 2$$
2. 倾斜角：
   Angle of inclination:
   $$\alpha = \arctan k = \arctan 2 \approx 63.43^\circ$$

例 Example 15.13.2 (判断直线的位置关系 Determine the Relative Position of Lines). 判断以下直线对的位置关系：
Determine the relative position of the following pairs of lines:

1. $y = 2x + 1$ 和 $y = 2x - 3$
   $y = 2x + 1$ and $y = 2x - 3$
2. $y = 2x + 1$ 和 $y = -\frac{1}{2}x + 4$
   $y = 2x + 1$ and $y = -\frac{1}{2}x + 4$

解:
Solution:

1. 两直线斜率相等（$k_1 = k_2 = 2$），所以两直线平行
   The two lines have equal slopes ($k_1 = k_2 = 2$), so they are parallel
2. 两直线斜率之积为-1（$k_1 \cdot k_2 = 2 \cdot (-\frac{1}{2}) = -1$），所以两直线垂直
   The product of slopes is -1 ($k_1 \cdot k_2 = 2 \cdot (-\frac{1}{2}) = -1$), so they are perpendicular

例 Example 15.13.3 (实际应用：坡度计算 Practical Application: Slope Calculation). 一条山路长 100 米，上升高度为 20 米，求：
A mountain road is 100 meters long with a vertical rise of 20 meters, find:

1. 这条山路的斜率
   The slope of this road
2. 山路与水平面的夹角
   The angle between the road and horizontal plane

解:
Solution:

1. 斜率：
   Slope:

$$k = \frac{\text{上升高度}}{\text{水平距离}} = \frac{20}{100} = 0.2$$

2. 倾斜角：
   Angle of inclination:

$$\alpha = \arctan 0.2 \approx 11.31^\circ$$

例 Example 15.13.4 (对称点问题 Symmetry Point Problem). 求点 $(3,4)$ 关于直线 $x+y=5$ 的对称点坐标。
Find the coordinates of the point symmetric to $(3,4)$ about the line $x+y=5$.

解:
Solution:

1. 将直线方程化为标准形式: $x+y-5=0$
   Convert line equation to standard form: $x+y-5=0$
2. 代入公式:
   Apply formula:
$$x = 3 - \frac{2(1)(3+4-5)}{1^2+1^2} = 3-2 = 1$$
$$y = 4 - \frac{2(1)(3+4-5)}{1^2+1^2} = 4-2 = 2$$
3. 对称点坐标为 $(1,2)$
   The symmetric point is $(1,2)$

例 Example 15.13.5 (直线系问题 System of Lines Problem). 求过点 $(1,2)$ 且平行于直线 $2x-y+3=0$ 的直线方程。
Find the equation of the line passing through point $(1,2)$ and parallel to line $2x-y+3=0$.

解:
Solution:

1. 原直线斜率: $k=2$
   Slope of original line: $k=2$
2. 使用点斜式:
   Use point-slope form:
$$y-2 = 2(x-1)$$
3. 化简得: $y=2x$
   Simplify to: $y=2x$

Chapter 16

圆与方程 Circle and Its Equations

16.1 圆的方程 Equations of Circle

定义 Definition 16.1.1 (圆的标准方程 Standard Form of Circle Equation). 圆心为 (a,b)，半径为 R 的圆的标准方程为:
The standard form of a circle equation with center (a,b) and radius R is:

$$(x-a)^2+(y-b)^2=R^2$$

定理 Theorem 16.1.2 (圆的一般方程 General Form of Circle Equation). 圆的一般方程为:
The general form of a circle equation is:

$$x^2+y^2+Dx+Ey+F=0$$

其中:
where:

1. 圆心坐标: $(-\frac{D}{2},-\frac{E}{2})$
   Center coordinates: $(-\frac{D}{2},-\frac{E}{2})$
2. 半径: $R=\sqrt{\frac{D^2+E^2}{4}-F}$

Radius: $R = \sqrt{\frac{D^2+E^2}{4} - F}$

定理 Theorem 16.1.3 (圆的一般方程转化 Conversion of Circle Equations). 一般方程 $x^2+y^2+Dx+Ey+F=0$ 可以配方转化为标准方程: General equation $x^2+y^2+Dx+Ey+F=0$ can be converted to standard form by completing the square:

1. 对 x 项配方: $(x^2+Dx) = (x+\frac{D}{2})^2 - \frac{D^2}{4}$
   Complete square for x terms: $(x^2+Dx) = (x+\frac{D}{2})^2 - \frac{D^2}{4}$
2. 对 y 项配方: $(y^2+Ey) = (y+\frac{E}{2})^2 - \frac{E^2}{4}$
   Complete square for y terms: $(y^2+Ey) = (y+\frac{E}{2})^2 - \frac{E^2}{4}$
3. 最终形式: $(x+\frac{D}{2})^2 + (y+\frac{E}{2})^2 = \frac{D^2+E^2}{4} - F$
   Final form: $(x+\frac{D}{2})^2 + (y+\frac{E}{2})^2 = \frac{D^2+E^2}{4} - F$

16.2 点与圆的位置关系 Position Relationship between Point and Circle

定理 Theorem 16.2.1 (点与圆的位置关系 Position Relationship between Point and Circle). 设点 $P(x_0, y_0)$ 到圆心 (a, b) 的距离为 d, 圆的半径为 R:
Let d be the distance from point $P(x_0, y_0)$ to center (a, b), and R be the radius:

1. 点在圆外: $d > R$ 或 $(x_0-a)^2 + (y_0-b)^2 > R^2$
   Point outside circle: $d > R$ or $(x_0-a)^2 + (y_0-b)^2 > R^2$
2. 点在圆上: $d = R$ 或 $(x_0-a)^2 + (y_0-b)^2 = R^2$
   Point on circle: $d = R$ or $(x_0-a)^2 + (y_0-b)^2 = R^2$
3. 点在圆内: $d < R$ 或 $(x_0-a)^2 + (y_0-b)^2 < R^2$
   Point inside circle: $d < R$ or $(x_0-a)^2 + (y_0-b)^2 < R^2$

16.3 直线与圆的位置关系 Position Relationship between Line and Circle

定理 Theorem 16.3.1 (直线与圆的位置关系判定 Determination of Line-Circle Position). 设直线 $Ax+By+C=0$ 与圆 $(x-a)^2+(y-b)^2=R^2$:
For line $Ax+By+C=0$ and circle $(x-a)^2+(y-b)^2=R^2$:

1. 点到直线距离：$d=\frac{|Aa+Bb+C|}{\sqrt{A^2+B^2}}$
   Distance from center to line: $d=\frac{|Aa+Bb+C|}{\sqrt{A^2+B^2}}$
2. 相离：$d>R$
   No intersection: $d>R$
3. 相切：$d=R$
   Tangent: $d=R$
4. 相交：$d<R$
   Intersect: $d<R$

定理 Theorem 16.3.2 (直线与圆的交点 Intersection Points of Line and Circle). 设直线 $Ax+By+C=0$ 与圆 $(x-a)^2+(y-b)^2=R^2$ 相交:
For line $Ax+By+C=0$ and circle $(x-a)^2+(y-b)^2=R^2$:

1. 将直线方程代入圆的方程可得一个关于 x 的二次方程
   Substituting line equation into circle equation gives a quadratic equation in x
2. 判别式 $\Delta>0$ 时有两个交点
   When discriminant $\Delta>0$, there are two intersection points
3. 判别式 $\Delta=0$ 时有一个交点（切点）
   When discriminant $\Delta=0$, there is one intersection point (tangent point)
4. 判别式 $\Delta<0$ 时无交点
   When discriminant $\Delta<0$, there are no intersection

points

16.4 圆与圆的位置关系 Position Relationship between Circles

定理 Theorem 16.4.1 (圆与圆的位置关系判定 Determination of Circle-Circle Position). 设两圆圆心距为 d，半径分别为 R_1 和 R_2：
For two circles with center distance d and radii R_1, R_2:

1. 外离：$d > R_1 + R_2$
   External separation: $d > R_1 + R_2$
2. 外切：$d = R_1 + R_2$
   External tangency: $d = R_1 + R_2$
3. 相交：$|R_1 - R_2| < d < R_1 + R_2$
   Intersection: $|R_1 - R_2| < d < R_1 + R_2$
4. 内切：$d = |R_1 - R_2|$
   Internal tangency: $d = |R_1 - R_2|$
5. 内含：$d < |R_1 - R_2|$
   Internal containment: $d < |R_1 - R_2|$

16.5 圆系方程 System of Circle Equations

定理 Theorem 16.5.1 (圆系方程 System of Circle Equations).
1. 过定点的圆系：$(x-a)^2 + (y-b)^2 = \lambda$
   Circles through fixed point: $(x-a)^2 + (y-b)^2 = \lambda$
2. 与定直线相切的圆系：$[(x-a)^2+(y-b)^2-R^2]^2 = 4R^2[(x-x_0)^2+(y-y_0)^2]$
   Circles tangent to fixed line: $[(x-a)^2 + (y-b)^2 - R^2]^2 = 4R^2[(x-x_0)^2 + (y-y_0)^2]$
3. 与定圆相切的圆系：$[(x-a)^2 + (y-b)^2 - R^2]^2 = 4r^2[(x-h)^2 + (y-k)^2]$
   Circles tangent to fixed circle: $[(x-a)^2 + (y-b)^2 - R^2]^2 = 4r^2[(x-h)^2 + (y-k)^2]$

定理 Theorem 16.5.2 (圆系的参数方程 Parametric Equations of Circle Systems
过两点的圆系：$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=\lambda$
Circles through two points: $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=\lambda$

2. 与两直线相切的圆系：$(\frac{A_1x+B_1y+C_1}{\sqrt{A_1^2+B_1^2}})^2=(\frac{A_2x+B_2y+C_2}{\sqrt{A_2^2+B_2^2}})^2$
Circles tangent to two lines: $(\frac{A_1x+B_1y+C_1}{\sqrt{A_1^2+B_1^2}})^2=(\frac{A_2x+B_2y+C_2}{\sqrt{A_2^2+B_2^2}})^2$
3. 与两圆相切的圆系：复杂的方程，需要具体情况具体分析
Circles tangent to two circles: Complex equations, need case-by-case analysis

16.6 圆的切线方程 Tangent Line Equations

定理 Theorem 16.6.1 (圆的切线方程 Tangent Line Equations). 1.
过圆上一点 (x_0,y_0) 的切线方程：
Tangent line through point (x_0,y_0) on circle:

$$(x-a)(x_0-a)+(y-b)(y_0-b)=R^2$$

2. 过圆外一点 (x_1,y_1) 的切线方程：
Tangent line through external point (x_1,y_1):

$$(x-a)(x_1-a)+(y-b)(y_1-b)=\pm R\sqrt{(x_1-a)^2+(y_1-b)^2-R^2}$$

16.7 切线长定理 Tangent Length Theorem

定理 Theorem 16.7.1 (切线长定理 Tangent Length Theorem). 从圆外一点到圆的两条切线长度相等。
The lengths of two tangent lines from an external point to a circle are equal.

设点 $P(x_0,y_0)$ 到圆心的距离为 d，圆的半径为 R，则：
If point $P(x_0,y_0)$ is at distance d from center and circle radius is R:

$$\text{切线长}=\sqrt{d^2-R^2}$$

$$\text{Tangent length}=\sqrt{d^2-R^2}$$

定理 Theorem 16.7.2 (切线长的应用 Applications of Tangent Length). 1.

两圆公切线长：$l = \sqrt{d^2 - (R_1 \pm R_2)^2}$
Length of common tangent lines: $l = \sqrt{d^2 - (R_1 \pm R_2)^2}$

2. 内公切线长：$l = \sqrt{d^2 - (R_1 - R_2)^2}$
   Length of internal common tangent: $l = \sqrt{d^2 - (R_1 - R_2)^2}$
3. 外公切线长：$l = \sqrt{d^2 - (R_1 + R_2)^2}$
   Length of external common tangent: $l = \sqrt{d^2 - (R_1 + R_2)^2}$

16.8 实例与应用 Examples and Applications

例 Example 16.8.1 (求圆的标准方程 Find Standard Form of Circle Equation). 已知圆过点 $A(1,2)$，$B(3,4)$，$C(5,2)$，求圆的标准方程。
Given a circle passing through points $A(1,2)$, $B(3,4)$, $C(5,2)$, find its standard equation.

解：
Solution:

1. 设圆心为 (h,k)，则：
   Let the center be (h,k), then:
   $$(1-h)^2 + (2-k)^2 = (3-h)^2 + (4-k)^2 = (5-h)^2 + (2-k)^2$$
2. 解方程组得：
   Solve equations:
   $$h = 3, k = 2$$
3. 代入任一点求半径：
   Find radius using any point:
   $$R^2 = (1-3)^2 + (2-2)^2 = 4$$
4. 圆的标准方程为：
   Standard equation is:
   $$(x-3)^2 + (y-2)^2 = 4$$

例 Example 16.8.2 (圆与直线的位置关系 Position of Line and Circle). 判断直线 $2x - y + 1 = 0$ 与圆 $x^2 + y^2 = 4$ 的位置关系。

Determine the position relationship between line $2x-y+1=0$ and circle $x^2+y^2=4$.

解:
Solution:

1. 圆心为原点 $(0,0)$，半径 $R=2$
   Circle center is origin $(0,0)$, radius $R=2$
2. 点到直线距离:
   Distance from center to line:
   $$d=\frac{|2(0)-0+1|}{\sqrt{4+1}}=\frac{1}{\sqrt{5}}$$
3. 因为 $\frac{1}{\sqrt{5}}<2$，所以直线与圆相交
   Since $\frac{1}{\sqrt{5}}<2$, the line intersects the circle

例 Example 16.8.3 (圆的切线问题 Circle Tangent Problem). 求过点 $P(3,4)$ 到圆 $x^2+y^2=1$ 的切线方程。
Find the equations of tangent lines from point $P(3,4)$ to circle $x^2+y^2=1$.

解:
Solution:

1. 圆心为原点，半径 $R=1$
   Circle center is origin, radius $R=1$
2. 代入切线方程公式:
   Apply tangent line formula:
   $$3x+4y=\pm\sqrt{25-1}$$
3. 整理得:
   Simplify:
   $$3x+4y=\pm\sqrt{24}$$

例 Example 16.8.4 (圆的一般方程转化 Converting General Circle

Equation). 将圆的一般方程 $x^2 + y^2 - 6x + 4y + 9 = 0$ 化为标准形式。
Convert the general circle equation $x^2 + y^2 - 6x + 4y + 9 = 0$ to standard form.

解:
Solution:

1. 对 x 项配方:
   Complete square for x terms:
   $$x^2 - 6x = (x-3)^2 - 9$$
2. 对 y 项配方:
   Complete square for y terms:
   $$y^2 + 4y = (y+2)^2 - 4$$
3. 代入原方程:
   Substitute back:
   $$(x-3)^2 + (y+2)^2 = 4$$
4. 因此圆心为 $(3, -2)$，半径为 2
   Therefore, center is $(3, -2)$ and radius is 2

例 Example 16.8.5 (圆与圆的位置关系 Position Relationship between Circles). 判断圆 $x^2 + y^2 = 4$ 与圆 $(x-3)^2 + y^2 = 1$ 的位置关系。
Determine the position relationship between circles $x^2+y^2 = 4$ and $(x-3)^2 + y^2 = 1$.

解:
Solution:

1. 第一个圆：圆心 $O_1(0, 0)$，半径 $R_1 = 2$
   First circle: center $O_1(0, 0)$, radius $R_1 = 2$
2. 第二个圆：圆心 $O_2(3, 0)$，半径 $R_2 = 1$
   Second circle: center $O_2(3, 0)$, radius $R_2 = 1$

3. 圆心距：$d = 3$
   Center distance: $d = 3$
4. 因为 $R_1 + R_2 = 3 = d$，所以两圆外切
   Since $R_1 + R_2 = 3 = d$, the circles are externally tangent

例 Example 16.8.6 (圆的公切线 Common Tangent Lines of Circles). 求两个圆 $x^2 + y^2 = 4$ 和 $(x - 5)^2 + y^2 = 1$ 的外公切线长。
Find the length of external common tangent lines of circles $x^2 + y^2 = 4$ and $(x - 5)^2 + y^2 = 1$.

解：
Solution:

1. 第一个圆：$R_1 = 2$
   First circle: $R_1 = 2$
2. 第二个圆：$R_2 = 1$
   Second circle: $R_2 = 1$
3. 圆心距：$d = 5$
   Center distance: $d = 5$
4. 代入外公切线长公式：
   Apply external common tangent formula:

$$l = \sqrt{5^2 - (2 + 1)^2} = \sqrt{25 - 9} = 4$$

例 Example 16.8.7 (圆系方程应用 Application of Circle System). 求过点 $(1, 1)$ 且与直线 $x + y = 0$ 相切的圆系方程。
Find the equation of circle system passing through point $(1, 1)$ and tangent to line $x + y = 0$.

解：
Solution:

1. 设圆的方程为 $(x - h)^2 + (y - k)^2 = r^2$
   Let the circle equation be $(x - h)^2 + (y - k)^2 = r^2$

2. 过点 $(1,1)$ 条件:
   Condition for passing through $(1,1)$:

$$(1-h)^2+(1-k)^2=r^2$$

3. 与直线相切条件:
   Condition for tangency to line:

$$\frac{|h+k|}{\sqrt{2}}=r$$

4. 联立得圆系方程:
   Combine to get circle system:

$$(x-h)^2+(y-k)^2=(\frac{h+k}{\sqrt{2}})^2$$

   其中 h 和 k 满足 $(1-h)^2+(1-k)^2=(\frac{h+k}{\sqrt{2}})^2$
   where h and k satisfy $(1-h)^2+(1-k)^2=(\frac{h+k}{\sqrt{2}})^2$

Chapter 17

空间距离 Spatial Distance

17.1 基本概念 Basic Concepts

定义 Definition 17.1.1 (空间向量 Spatial Vector). 空间向量是有大小和方向的量，可以用有序数组 (x, y, z) 表示。
A spatial vector is a quantity with magnitude and direction, represented by ordered triple (x, y, z).

1. 向量的模：$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$
   Magnitude of vector: $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$
2. 向量的方向：由起点指向终点
   Direction of vector: from start point to end point
3. 向量的坐标：终点坐标减去起点坐标
   Coordinates of vector: end point coordinates minus start point coordinates

定义 Definition 17.1.2 (向量运算 Vector Operations).
1. 向量加减：对应坐标相加减
   Vector addition/subtraction: add/subtract corresponding coordinates
2. 向量数乘：每个坐标乘以该数

Scalar multiplication: multiply each coordinate by the scalar

3. 向量点乘：$\vec{a}\cdot\vec{b} = x_1x_2 + y_1y_2 + z_1z_2$
   Dot product: $\vec{a}\cdot\vec{b} = x_1x_2 + y_1y_2 + z_1z_2$
4. 向量叉乘：$\vec{a}\times\vec{b} = (y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2)$
   Cross product: $\vec{a}\times\vec{b} = (y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2)$

17.2 点到平面的距离 Distance from Point to Plane

定理 Theorem 17.2.1 (点到平面的距离公式 Distance Formula from Point to Plane). 设平面 π 的方程为 $Ax+By+Cz+D=0$, 点 $P(x_0, y_0, z_0)$ 到平面 π 的距离为:
Given plane $\pi : Ax + By + Cz + D = 0$ and point $P(x_0, y_0, z_0)$, the distance is:

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

公式解释：
Formula explanation:

1. 分子是点坐标代入平面方程的结果的绝对值
   Numerator is absolute value of plane equation evaluated at point
2. 分母是平面法向量的模
   Denominator is magnitude of plane normal vector
3. 平面的法向量为 (A, B, C)
   Normal vector of plane is (A, B, C)

例 Example 17.2.2 (点到平面距离计算 Calculate Distance from Point to Plane). 求点 $P(1, 2, 3)$ 到平面 $2x - y + 2z - 3 = 0$ 的距离。
Find the distance from point $P(1, 2, 3)$ to plane $2x - y + 2z - 3 = 0$.

解：
Solution:

1. 确定平面方程的系数：
   Identify coefficients of plane equation:
   $$A = 2, B = -1, C = 2, D = -3$$
2. 代入点的坐标计算分子：
   Calculate numerator using point coordinates:
   $$|2(1) - (2) + 2(3) - 3| = |2 - 2 + 6 - 3| = |3| = 3$$
3. 计算分母：
   Calculate denominator:
   $$\sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$$
4. 计算距离：
   Calculate distance:
   $$d = \frac{3}{3} = 1$$
5. 检验：距离应为非负数
   Verify: distance should be non-negative

例 Example 17.2.3 (点到平面距离的应用 Application of Point--Plane Distance). 已知四面体 $ABCD$ 的四个顶点坐标为 $A(0,0,0)$, $B(2,0,0)$, $C(0,2,0)$, $D(0,0,2)$，求点 D 到平面 ABC 的距离。
Given tetrahedron $ABCD$ with vertices $A(0,0,0)$, $B(2,0,0)$, $C(0,2,0)$, $D(0,0,2)$, find the distance from point D to plane ABC.

解：
Solution:

1. 求平面 ABC 的方程：
   Find equation of plane ABC:
   - 取三点坐标代入平面一般方程
     Substitute three points into general plane equation
   - 解得平面方程：$\frac{x}{2} + \frac{y}{2} + \frac{z}{2} = 1$
     Get plane equation: $\frac{x}{2} + \frac{y}{2} + \frac{z}{2} = 1$

- 化为标准形式：$x+y+z=2$
  Convert to standard form: $x+y+z=2$

2. 代入点 $D(0,0,2)$ 到距离公式：
   Apply distance formula with point $D(0,0,2)$:

$$d=\frac{|0+0+2-2|}{\sqrt{1^2+1^2+1^2}}=\frac{0}{\sqrt{3}}=0$$

3. 结论：点 D 在平面 ABC 上
   Conclusion: point D lies on plane ABC

17.3 直线到其平行平面的距离 Distance from Line to Its Parallel Plane

定理 Theorem 17.3.1 (直线到平行平面的距离 Distance from Line to Parallel Plane). 设直线 L 的方向向量为 $\vec{v}(a,b,c)$，平面 π 的法向量为 $\vec{n}(A,B,C)$，且 $\vec{v}\parallel\pi$（即 $\vec{v}\perp\vec{n}$），
则直线 L 上任意一点 $P(x_0,y_0,z_0)$ 到平面 $\pi: Ax+By+Cz+D=0$ 的距离为：
Given line L with direction vector $\vec{v}(a,b,c)$, plane π with normal vector $\vec{n}(A,B,C)$, and $\vec{v}\parallel\pi$ (i.e., $\vec{v}\perp\vec{n}$),
the distance from any point $P(x_0,y_0,z_0)$ on line L to plane $\pi: Ax+By+Cz+D=0$ is:

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

说明：
Notes:

1. 直线平行于平面时，直线上所有点到平面的距离相等
   When line is parallel to plane, all points on line have same distance to plane
2. 直线与平面平行的条件：$Aa+Bb+Cc=0$
   Condition for line parallel to plane: $Aa+Bb+Cc=0$

例 Example 17.3.2 (直线到平行平面距离计算 Calculate Distance from Line to Parallel Plane). 求直线 $L: \frac{x-1}{2} = \frac{y}{1} = \frac{z-1}{2}$ 到平面 $2x - y + 2z = 0$ 的距离。
Find the distance from line $L: \frac{x-1}{2} = \frac{y}{1} = \frac{z-1}{2}$ to plane $2x-y+2z = 0$.

解:
Solution:

1. 验证直线与平面平行:
   Verify line is parallel to plane:

   - 直线方向向量: $\vec{v}(2,1,2)$
     Line direction vector: $\vec{v}(2,1,2)$
   - 平面法向量: $\vec{n}(2,-1,2)$
     Plane normal vector: $\vec{n}(2,-1,2)$
   - 验证垂直: $2(2)+(-1)(1)+2(2) = 4-1+4 = 7 \neq 0$
     Verify perpendicular: $2(2)+(-1)(1)+2(2) = 4-1+4 = 7 \neq 0$
   - 结论: 直线与平面不平行
     Conclusion: line is not parallel to plane

2. 因为直线与平面不平行，所以没有固定距离
   Since line is not parallel to plane, there is no fixed distance

17.4 异面直线间的距离 Distance between Skew Lines

定理 Theorem 17.4.1 (异面直线间的距离 Distance between Skew Lines). 设两条异面直线 L_1 和 L_2 的方向向量分别为 $\vec{v_1}(a_1, b_1, c_1)$ 和 $\vec{v_2}(a_2, b_2, c_2)$,
L_1 上一点 $P_1(x_1, y_1, z_1)$, L_2 上一点 $P_2(x_2, y_2, z_2)$, 则两直线间的距离为:
Given two skew lines L_1 and L_2 with direction vectors $\vec{v_1}(a_1, b_1, c_1)$ and $\vec{v_2}(a_2, b_2, c_2)$,

points $P_1(x_1, y_1, z_1)$ on L_1 and $P_2(x_2, y_2, z_2)$ on L_2, the distance is:

$$d = \frac{|\vec{P_1P_2} \cdot (\vec{v_1} \times \vec{v_2})|}{|\vec{v_1} \times \vec{v_2}|}$$

例 Example 17.4.2 (异面直线间距离计算 Calculate Distance between Skew Lines). 求异面直线 $L_1 : \frac{x-1}{2} = \frac{y}{1} = \frac{z}{2}$ 和 $L_2 : \frac{x}{1} = \frac{y-2}{2} = \frac{z-1}{1}$ 间的距离。
Find the distance between skew lines $L_1 : \frac{x-1}{2} = \frac{y}{1} = \frac{z}{2}$ and $L_2 : \frac{x}{1} = \frac{y-2}{2} = \frac{z-1}{1}$.

解:
Solution:

1. L_1 的方向向量: $\vec{v_1}(2, 1, 2)$
   Direction vector of L_1: $\vec{v_1}(2, 1, 2)$
2. L_2 的方向向量: $\vec{v_2}(1, 2, 1)$
   Direction vector of L_2: $\vec{v_2}(1, 2, 1)$
3. L_1 上一点: $P_1(1, 0, 0)$
   Point on L_1: $P_1(1, 0, 0)$
4. L_2 上一点: $P_2(0, 2, 1)$
   Point on L_2: $P_2(0, 2, 1)$
5. 计算 $\vec{P_1P_2} = (-1, 2, 1)$
   Calculate $\vec{P_1P_2} = (-1, 2, 1)$
6. 计算 $\vec{v_1} \times \vec{v_2}$
   Calculate $\vec{v_1} \times \vec{v_2}$
7. 代入公式得: $d = \frac{1}{\sqrt{3}}$
   Apply formula: $d = \frac{1}{\sqrt{3}}$

17.5 平行平面间的距离 Distance between Parallel Planes

定理 Theorem 17.5.1 (平行平面间的距离 Distance between Parallel Planes). 设两个平行平面 $\pi_1 : Ax + By + Cz + D_1 = 0$ 和 $\pi_2 : Ax + By + Cz + D_2 = 0$，则它们之间的距离为:

Given two parallel planes $\pi_1 : Ax + By + Cz + D_1 = 0$ and $\pi_2 : Ax + By + Cz + D_2 = 0$, their distance is:

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

例 Example 17.5.2 (平行平面间距离计算 Calculate Distance between Parallel Planes). 求平行平面 $2x-y+2z+1=0$ 和 $2x-y+2z-5=0$ 间的距离。
Find the distance between parallel planes $2x - y + 2z + 1 = 0$ and $2x - y + 2z - 5 = 0$.

解:
Solution:

1. 代入公式:
   Apply formula:
   $$d = \frac{|1 - (-5)|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{6}{3} = 2$$

17.6 异面直线上两点间的最短距离 Shortest Distance between Points on Skew Lines

定理 Theorem 17.6.1 (异面直线上两点间的最短距离 Shortest Distance between Points on Skew Lines). 设两条异面直线的参数方程分别为:
Given parametric equations of two skew lines:

$$L_1 : \begin{cases} x = x_1 + a_1 t \\ y = y_1 + b_1 t \\ z = z_1 + c_1 t \end{cases} \qquad L_2 : \begin{cases} x = x_2 + a_2 s \\ y = y_2 + b_2 s \\ z = z_2 + c_2 s \end{cases}$$

则两直线上对应最短距离的两点的参数值为:
The parameter values for points with shortest distance are:

1. $t = \frac{(a_2^2+b_2^2+c_2^2)[(x_2-x_1)a_1+(y_2-y_1)b_1+(z_2-z_1)c_1]-(a_1a_2+b_1b_2+c_1c_2)[(x_2-x_1)a_2+(y_2-y_1)b_2+(z_2-z_1)c_2]}{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-(a_1a_2+b_1b_2+c_1c_2)^2}$

2. $s = \frac{(a_1a_2+b_1b_2+c_1c_2)[(x_2-x_1)a_1+(y_2-y_1)b_1+(z_2-z_1)c_1]-(a_1^2+b_1^2+c_1^2)[(x_2-x_1)a_2+(y_2-y_1)b_2+(z_2-z_1)c_2]}{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-(a_1a_2+b_1b_2+c_1c_2)^2}$

例 Example 17.6.2 (异面直线上最短距离点的确定 Determine Points with Shortest Distance on Skew Lines). 求异面直线 $L_1 : \frac{x}{1} = \frac{y-1}{1} = \frac{z}{1}$ 和 $L_2 : \frac{x-2}{1} = \frac{y}{1} = \frac{z-2}{1}$ 上的最短距离点。
Find the points with shortest distance on skew lines $L_1 : \frac{x}{1} = \frac{y-1}{1} = \frac{z}{1}$ and $L_2 : \frac{x-2}{1} = \frac{y}{1} = \frac{z-2}{1}$.

解:
Solution:

1. 将直线化为参数方程:
   Convert to parametric equations:

$$L_1 : \begin{cases} x = t \\ y = t+1 \\ z = t \end{cases} \qquad L_2 : \begin{cases} x = s+2 \\ y = s \\ z = s+2 \end{cases}$$

2. 代入公式求 t 和 s 值
   Apply formula to find t and s values
3. 代回参数方程得到两点坐标
   Substitute back to get coordinates of points

17.7 空间距离的应用 Applications of Spatial Distance

例 Example 17.7.1 (综合应用 Comprehensive Application). 一个正四面体的四个顶点分别为 $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$, $D(0,0,1)$。求:
A regular tetrahedron has vertices $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$, $D(0,0,1)$. Find:

1. 点 D 到面 ABC 的距离
   Distance from point D to face ABC

2. 边 AD 到边 BC 的距离
   Distance between edges AD and BC

解:

Solution:

1. 求面 ABC 的平面方程:
   Find equation of plane ABC:

$$x + y + z = 1$$

   点 D 到此平面的距离:
   Distance from D to this plane:

$$d = \frac{|0 + 0 + 1 - 1|}{\sqrt{1^2 + 1^2 + 1^2}} = 0$$

2. 边 AD 和 BC 为异面直线，使用异面直线距离公式:
   AD and BC are skew lines, use skew lines distance formula:

$$d = \frac{1}{\sqrt{6}}$$

Chapter 18

空间角 Spatial Angles

18.1 基本概念 Basic Concepts

定义 Definition 18.1.1 (空间角 Spatial Angle). 空间角是指在三维空间中由两条射线、一条直线与平面、或两个平面所形成的角。
A spatial angle is an angle formed by two rays, a line and a plane, or two planes in three-dimensional space.

1. 角的范围: $[0°, 180°]$ 或 $[0, \pi]$
   Range of angles: $[0°, 180°]$ or $[0, \pi]$
2. 角的度量: 通常用弧度或角度表示
   Measurement of angles: usually expressed in radians or degrees

18.2 异面直线所成的角 Angle between Skew Lines

定理 Theorem 18.2.1 (异面直线夹角 Angle between Skew Lines). 设两条异面直线 L_1 和 L_2 的方向向量分别为 $\vec{v_1}(x_1, y_1, z_1)$ 和 $\vec{v_2}(x_2, y_2, z_2)$, 则它们所成的角 θ 满足:

Given two skew lines L_1 and L_2 with direction vectors $\vec{v_1}(x_1, y_1, z_1)$ and $\vec{v_2}(x_2, y_2, z_2)$,
the angle θ between them satisfies:

$$\cos\theta = \frac{|x_1x_2 + y_1y_2 + z_1z_2|}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}}$$

说明:
Notes:

1. 异面直线夹角是指两个方向向量的夹角
   The angle between skew lines is the angle between their direction vectors
2. 夹角范围为 $[0°, 90°]$ 或 $[0, \frac{\pi}{2}]$
   The range of angle is $[0°, 90°]$ or $[0, \frac{\pi}{2}]$

例 Example 18.2.2 (计算异面直线夹角 Calculate Angle between Skew Lines). 求异面直线 $L_1 : \frac{x}{2} = \frac{y-1}{2} = \frac{z}{1}$ 和 $L_2 : \frac{x-1}{1} = \frac{y}{2} = \frac{z-1}{2}$ 所成的角。
Find the angle between skew lines $L_1 : \frac{x}{2} = \frac{y-1}{2} = \frac{z}{1}$ and $L_2 : \frac{x-1}{1} = \frac{y}{2} = \frac{z-1}{2}$.

解:
Solution:

1. 确定方向向量:
   Determine direction vectors:

   - L_1 的方向向量: $\vec{v_1}(2, 2, 1)$
     Direction vector of L_1: $\vec{v_1}(2, 2, 1)$
   - L_2 的方向向量: $\vec{v_2}(1, 2, 2)$
     Direction vector of L_2: $\vec{v_2}(1, 2, 2)$

2. 计算点积:

Calculate dot product:

$$\vec{v_1} \cdot \vec{v_2} = 2(1) + 2(2) + 1(2) = 2 + 4 + 2 = 8$$

3. 计算向量模:
Calculate vector magnitudes:
   - $|\vec{v_1}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$
   - $|\vec{v_2}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$
4. 代入公式:
Apply formula:
$$\cos\theta = \frac{8}{3 \cdot 3} = \frac{8}{9}$$
5. 求角度:
Find angle:
$$\theta = \arccos(\frac{8}{9}) \approx 27.27^\circ$$

18.3 直线与平面所成的角 Angle between Line and Plane

定理 Theorem 18.3.1 (直线与平面夹角 Angle between Line and Plane). 设直线 L 的方向向量为 $\vec{v}(a,b,c)$，平面 π 的法向量为 $\vec{n}(A,B,C)$，则直线与平面所成的角 θ 满足:
Given line L with direction vector $\vec{v}(a,b,c)$ and plane π with normal vector $\vec{n}(A,B,C)$,
the angle θ between them satisfies:

$$\sin\theta = \frac{|Aa + Bb + Cc|}{\sqrt{a^2 + b^2 + c^2}\sqrt{A^2 + B^2 + C^2}}$$

说明:
Notes:

1. 直线与平面的夹角是指直线与其在平面上的射影所成的角
The angle between a line and a plane is the angle between the line and its projection on the plane

2. 夹角范围为 $[0°, 90°]$ 或 $[0, \frac{\pi}{2}]$
   The range of angle is $[0°, 90°]$ or $[0, \frac{\pi}{2}]$

例 Example 18.3.2 (计算直线与平面夹角 Calculate Angle between Line and Plane). 求直线 $L : \frac{x-1}{1} = \frac{y}{2} = \frac{z+1}{2}$ 与平面 $\pi : x + 2y - 2z = 0$ 所成的角。
Find the angle between line $L : \frac{x-1}{1} = \frac{y}{2} = \frac{z+1}{2}$ and plane $\pi : x + 2y - 2z = 0$.

解:
Solution:

1. 确定向量:
   Determine vectors:
   - 直线方向向量: $\vec{v}(1, 2, 2)$
     Line direction vector: $\vec{v}(1, 2, 2)$
   - 平面法向量: $\vec{n}(1, 2, -2)$
     Plane normal vector: $\vec{n}(1, 2, -2)$
2. 计算点积:
   Calculate dot product:
   $$|\vec{v} \cdot \vec{n}| = |1(1) + 2(2) + 2(-2)| = |1 + 4 - 4| = 1$$
3. 计算向量模:
   Calculate vector magnitudes:
   - $|\vec{v}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$
   - $|\vec{n}| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$
4. 代入公式:
   Apply formula:
   $$\sin\theta = \frac{1}{9}$$
5. 求角度:

Find angle:

$$\theta = \text{arcsin}(\frac{1}{9}) \approx 6.37°$$

18.4 平面与平面所成的角 Angle between Two Planes

定理 Theorem 18.4.1 (平面夹角 Angle between Planes). 设两个平面 $\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$ 和 $\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$,
则它们所成的二面角 θ 满足:
Given two planes $\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$ and $\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$,
the dihedral angle θ between them satisfies:

$$\cos\theta = \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

说明:
Notes:

1. 二面角是指两个平面法向量的夹角
   The dihedral angle is the angle between the normal vectors of the two planes
2. 夹角范围为 $[0°, 90°]$ 或 $[0, \frac{\pi}{2}]$
   The range of angle is $[0°, 90°]$ or $[0, \frac{\pi}{2}]$

例 Example 18.4.2 (计算平面夹角 Calculate Angle between Planes).
求平面 $\pi_1 : 2x - y + z = 0$ 和 $\pi_2 : x + y + z = 0$ 所成的角。
Find the angle between planes $\pi_1 : 2x - y + z = 0$ and $\pi_2 : x + y + z = 0$.

解:
Solution:

1. 确定法向量:

Determine normal vectors:

- π_1 的法向量：$\vec{n_1}(2,-1,1)$
  Normal vector of π_1: $\vec{n_1}(2,-1,1)$
- π_2 的法向量：$\vec{n_2}(1,1,1)$
  Normal vector of π_2: $\vec{n_2}(1,1,1)$

2. 计算点积：
   Calculate dot product:

$$|\vec{n_1}\cdot\vec{n_2}| = |2(1)+(-1)(1)+1(1)| = |2-1+1| = 2$$

3. 计算向量模：
   Calculate vector magnitudes:

- $|\vec{n_1}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$
- $|\vec{n_2}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

4. 代入公式：
   Apply formula:

$$\cos\theta = \frac{2}{\sqrt{6}\sqrt{3}} = \frac{2}{\sqrt{18}}$$

5. 求角度：
   Find angle:

$$\theta = \arccos(\frac{2}{\sqrt{18}}) \approx 62.07^\circ$$

Chapter 19

圆锥曲线 Conic Sections

19.1 圆锥曲线的定义 Definitions of Conic Sections

19.1.1 第一定义 First Definition

定义 Definition 19.1.1 (圆锥曲线的第一定义 First Definition of Conic Sections). 圆锥曲线是圆锥面与平面相交所得的曲线。根据截面与母线的位置关系，可以得到：
A conic section is a curve obtained by intersecting a cone with a plane. Based on the position relationship between the cutting plane and the generatrix:

1. 当截面与母线平行时，得到抛物线
   When the cutting plane is parallel to the generatrix, we get a parabola
2. 当截面与母线的交角大于锥面母线与轴线的夹角时，得到椭圆
   When the angle between the cutting plane and generatrix is greater than the angle between the cone's generatrix and axis, we get an ellipse
3. 当截面与母线的交角小于锥面母线与轴线的夹角时，得到双曲线
   When the angle between the cutting plane and generatrix

is less than the angle between the cone's generatrix and axis, we get a hyperbola

19.1.2 统一定义 Unified Definition

定义 Definition 19.1.2 (圆锥曲线的统一定义 Unified Definition of Conic Sections). 平面上一点到一个定点（焦点）的距离与到一条定直线（准线）的距离之比为一常数（离心率 e）的点的轨迹。
The locus of points in a plane such that the ratio of the distance from a fixed point (focus) to the distance from a fixed line (directrix) is a constant (eccentricity e).

1. 当 $e = 1$ 时，为抛物线
   When $e = 1$, it's a parabola
2. 当 $0 < e < 1$ 时，为椭圆
   When $0 < e < 1$, it's an ellipse
3. 当 $e > 1$ 时，为双曲线
   When $e > 1$, it's a hyperbola
4. 当 $e = 0$ 时，为圆（椭圆的特例）
   When $e = 0$, it's a circle (special case of ellipse)

19.2 椭圆 Ellipse

19.2.1 标准方程 Standard Form

定理 Theorem 19.2.1 (椭圆标准方程 Standard Form of Ellipse). 设椭圆的中心在原点，长轴在 x 轴上，短轴在 y 轴上，则其标准方程为：
For an ellipse with center at origin, major axis on x-axis, and minor axis on y-axis:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b > 0)$$

其中：
Where:

- $2a$ 为长轴长
  $2a$ is the length of major axis
- $2b$ 为短轴长
  $2b$ is the length of minor axis
- $c^2 = a^2 - b^2$，其中 c 为半焦距
  $c^2 = a^2 - b^2$, where c is the semi-focal distance
- 离心率 $e = \frac{c}{a}$
  Eccentricity $e = \frac{c}{a}$

例 Example 19.2.2 (椭圆的基本参数计算 Calculate Basic Parameters of Ellipse). 已知椭圆 $\frac{x^2}{16} + \frac{y^2}{9} = 1$，求：
Given ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, find:

1. 长轴长和短轴长
   Length of major and minor axes
2. 焦点坐标
   Coordinates of foci
3. 离心率
   Eccentricity

解：
Solution:

1. 由标准方程得：
   From standard form:

   - $a^2 = 16$，故 $a = 4$
   - $b^2 = 9$，故 $b = 3$
   - 长轴长 $2a = 8$，短轴长 $2b = 6$

2. 计算半焦距：
   Calculate semi-focal distance:

- $c^2 = a^2 - b^2 = 16 - 9 = 7$
- $c = \sqrt{7}$
- 焦点坐标为 $F_1(-\sqrt{7}, 0)$，$F_2(\sqrt{7}, 0)$

3. 计算离心率：
   Calculate eccentricity:

$$e = \frac{c}{a} = \frac{\sqrt{7}}{4} \approx 0.661$$

19.2.2 椭圆的性质 Properties of Ellipse

定理 Theorem 19.2.3 (椭圆的基本性质 Basic Properties of Ellipse). 1. 对称性：椭圆关于坐标轴和原点对称
   Symmetry: Ellipse is symmetric about coordinate axes and origin
2. 顶点：$(\pm a, 0)$ 和 $(0, \pm b)$
   Vertices: $(\pm a, 0)$ and $(0, \pm b)$
3. 焦点弦：过焦点的弦
   Focal chord: Chord passing through focus
4. 准线：$x = \pm\frac{a}{e}$
   Directrix: $x = \pm\frac{a}{e}$

19.3 双曲线 Hyperbola

19.3.1 标准方程 Standard Form

定理 Theorem 19.3.1 (双曲线标准方程 Standard Form of Hyperbola). 设双曲线的中心在原点，实轴在 x 轴上，虚轴在 y 轴上，则其标准方程为：
For a hyperbola with center at origin, transverse axis on x-axis, and conjugate axis on y-axis:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (a, b > 0)$$

其中：
Where:

- $2a$ 为实轴长
  $2a$ is the length of transverse axis
- $2b$ 为虚轴长
  $2b$ is the length of conjugate axis
- $c^2 = a^2 + b^2$，其中 c 为半焦距
  $c^2 = a^2 + b^2$, where c is the semi-focal distance
- 离心率 $e = \frac{c}{a}$
  Eccentricity $e = \frac{c}{a}$

定理 Theorem 19.3.2 (双曲线的渐近线 Asymptotes of Hyperbola). 双曲线 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 的渐近线方程为:
The asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are:

$$y = \pm\frac{b}{a}x$$

性质:
Properties:

1. 渐近线是双曲线的极限位置
   Asymptotes are the limiting positions of hyperbola
2. 双曲线无限趋近于但永不与渐近线相交
   Hyperbola approaches but never intersects its asymptotes
3. 渐近线斜率为 $\pm\frac{b}{a}$
   Slopes of asymptotes are $\pm\frac{b}{a}$

例 Example 19.3.3 (双曲线的渐近线计算 Calculate Asymptotes of Hyperbola). 求双曲线 $\frac{x^2}{16} - \frac{y^2}{9} = 1$ 的渐近线方程。
Find the equations of asymptotes for hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

解:
Solution:

1. 由标准方程得:
   From standard form:

   - $a^2 = 16$，故 $a = 4$
   - $b^2 = 9$，故 $b = 3$

2. 渐近线斜率:
   Slopes of asymptotes:
   $$k = \pm\frac{b}{a} = \pm\frac{3}{4}$$

3. 渐近线方程:
   Equations of asymptotes:
   $$y = \pm\frac{3}{4}x$$

19.4 抛物线 Parabola

19.4.1 标准方程 Standard Form

定理 Theorem 19.4.1 (抛物线标准方程 Standard Form of Parabola).
设抛物线的顶点在原点，对称轴在 x 轴上，则其标准方程为:
For a parabola with vertex at origin and axis of symmetry on x-axis:
$$y^2 = 2px \quad (p \neq 0)$$

其中:
Where:

- p 为焦参数，表示焦点到准线的距离
  p is the focal parameter, representing the distance from focus to directrix
- 焦点坐标为 $(\frac{p}{2}, 0)$
  Focus coordinates are $(\frac{p}{2}, 0)$
- 准线方程为 $x = -\frac{p}{2}$
  Directrix equation is $x = -\frac{p}{2}$
- 离心率 $e = 1$
  Eccentricity $e = 1$

19.5 圆锥曲线的焦点位置判断 Focus Position of Conic Sections

定理 Theorem 19.5.1 (焦点位置判断 Determining Focus Position). 对于标准形式的圆锥曲线方程:
For standard form of conic section equations:

1. 椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:
   焦点在长轴上，到中心的距离为 $c = \sqrt{a^2 - b^2}$
   Foci are on major axis, distance to center is $c = \sqrt{a^2 - b^2}$
2. 双曲线 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:
   焦点在实轴上，到中心的距离为 $c = \sqrt{a^2 + b^2}$
   Foci are on transverse axis, distance to center is $c = \sqrt{a^2 + b^2}$
3. 抛物线 $y^2 = 2px$:
   焦点在对称轴上，到顶点的距离为 $\frac{p}{2}$
   Focus is on axis of symmetry, distance to vertex is $\frac{p}{2}$

19.6 圆锥曲线的任意弦长 Length of Arbitrary Chord

定理 Theorem 19.6.1 (任意弦长公式 Formula for Arbitrary Chord Length). 设点 $P_1(x_1, y_1)$ 和 $P_2(x_2, y_2)$ 是圆锥曲线上的两点，则弦 P_1P_2 的长度为:
For points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on a conic section, the length of chord P_1P_2 is:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

例 Example 19.6.2 (计算任意弦长 Calculate Arbitrary Chord Length). 求椭圆 $\frac{x^2}{16} + \frac{y^2}{9} = 1$ 上两点 $P_1(2, \sqrt{5})$ 和 $P_2(-2, \sqrt{5})$ 间的弦长。
Find the length of chord between points $P_1(2, \sqrt{5})$ and $P_2(-2, \sqrt{5})$ on ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

解:
Solution:

1. 代入公式:
   Apply formula:

$$|P_1P_2| = \sqrt{(-2-2)^2 + (\sqrt{5}-\sqrt{5})^2} = \sqrt{16+0} = 4$$

19.7 圆锥曲线的焦点弦长 Length of Focal Chord

定理 Theorem 19.7.1 (焦点弦长公式 Formula for Focal Chord Length).

1. 椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 的焦点弦长:
   Length of focal chord for ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

$$l = \frac{2b^2}{a}$$

2. 双曲线 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 的焦点弦长:
   Length of focal chord for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

$$l = \frac{2b^2}{a}$$

3. 抛物线 $y^2 = 2px$ 的焦点弦长:
   Length of focal chord for parabola $y^2 = 2px$:

$$l = 4p$$

例 Example 19.7.2 (计算焦点弦长 Calculate Focal Chord Length).
求椭圆 $\frac{x^2}{16} + \frac{y^2}{9} = 1$ 的焦点弦长。
Find the length of focal chord for ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

解:
Solution:

1. 由标准方程得:
   From standard form:

- $a^2 = 16$，故 $a = 4$
- $b^2 = 9$，故 $b = 3$

2. 代入焦点弦长公式:
   Apply focal chord formula:

$$l = \frac{2b^2}{a} = \frac{2 \cdot 9}{4} = \frac{9}{2}$$

19.8 圆锥曲线的通径 Diameter of Conic Sections

定理 Theorem 19.8.1 (通径定义与性质 Definition and Properties of Diameter). 通径是平行于某一方向的一组弦的中点的轨迹。
A diameter is the locus of midpoints of a set of parallel chords.

1. 椭圆的通径:
   Diameter of ellipse:

$$y = kx \quad 和 \quad y = -\frac{a^2}{b^2 k}x$$

   其中 k 为平行弦的斜率
   where k is the slope of parallel chords
2. 双曲线的通径:
   Diameter of hyperbola:

$$y = kx \quad 和 \quad y = \frac{a^2}{b^2 k}x$$

3. 抛物线的通径:
   Diameter of parabola:

$$x = \frac{p}{2k^2} + kt \quad (t为参数)$$

19.9 圆锥曲线的焦半径 Focal Radius of Conic Sections

定理 Theorem 19.9.1 (焦半径定义与公式 Definition and Formulas of Focal Radius). 焦半径是圆锥曲线上任意点到焦点的距离。
A focal radius is the distance from any point on the conic section to a focus.

1. 椭圆的焦半径:
   Focal radius of ellipse:
   $$r_1 + r_2 = 2a$$
   其中 r_1，r_2 分别是到两焦点的距离
   where r_1, r_2 are distances to the two foci
2. 双曲线的焦半径:
   Focal radius of hyperbola:
   $$|r_1 - r_2| = 2a$$
3. 抛物线的焦半径:
   Focal radius of parabola:
   $$r = \frac{p}{1 - \cos\theta}$$
   其中 θ 是与对称轴的夹角
   where θ is the angle with the axis of symmetry

例 Example 19.9.2 (计算焦半径 Calculate Focal Radius). 求椭圆 $\frac{x^2}{16} + \frac{y^2}{9} = 1$ 上点 $P(2, \sqrt{5})$ 的焦半径。
Find the focal radii of point $P(2, \sqrt{5})$ on ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

解:
Solution:

1. 焦点坐标: $F_1(-\sqrt{7}, 0)$，$F_2(\sqrt{7}, 0)$
   Focus coordinates: $F_1(-\sqrt{7}, 0)$, $F_2(\sqrt{7}, 0)$

2. 计算 r_1:
   Calculate r_1:

$$r_1 = \sqrt{(2+\sqrt{7})^2+5}$$

3. 计算 r_2:
   Calculate r_2:

$$r_2 = \sqrt{(2-\sqrt{7})^2+5}$$

4. 验证:
   Verify:

$$r_1 + r_2 = 2a = 8$$

19.10 焦点弦的性质 Properties of Focal Chords

定理 Theorem 19.10.1 (焦点弦的重要性质 Important Properties of Focal Chord
椭圆的任意焦点弦都被焦点二等分
Any focal chord of an ellipse is bisected by the focus

2. 双曲线的任意焦点弦都被焦点二等分
   Any focal chord of a hyperbola is bisected by the focus
3. 抛物线的任意焦点弦都被焦点二等分
   Any focal chord of a parabola is bisected by the focus
4. 焦点弦的中点轨迹是圆
   The locus of midpoints of focal chords is a circle
5. 焦点弦的端点到另一焦点的距离相等
   The distances from the endpoints of a focal chord to the other focus are equal

19.11 椭圆的焦点三角形 Focal Triangle of Ellipse

定理 Theorem 19.11.1 (椭圆焦点三角形的性质 Properties of Ellipse Focal Triangle). 设 P 为椭圆上一点，F_1，F_2 为两焦点，则三角形 PF_1F_2 满足:
Let P be a point on ellipse, F_1, F_2 be foci, then triangle PF_1F_2 satisfies:

1. 面积:
   Area:

$$S_{PF_1F_2} = \frac{1}{2}bc$$

2. 周长:
   Perimeter:

$$PF_1 + PF_2 + F_1F_2 = 2a + 2c$$

3. 内切圆半径:
   Inradius:

$$r = \frac{b}{2}$$

19.12 双曲线的焦点三角形 Focal Triangle of Hyperbola

定理 Theorem 19.12.1 (双曲线焦点三角形的性质 Properties of Hyperbola Focal Triangle). 设 P 为双曲线上一点，F_1，F_2 为两焦点，则三角形 PF_1F_2 满足:
Let P be a point on hyperbola, F_1, F_2 be foci, then triangle PF_1F_2 satisfies:

1. 面积:
   Area:

$$S_{PF_1F_2} = \frac{1}{2}bc$$

2. 周长:
   Perimeter:

$$|PF_1 - PF_2| + F_1F_2 = 2a + 2c$$

3. 外切圆半径:
   Circumradius:

$$R = \frac{c}{2}$$

19.13 椭圆和双曲线的公共焦点三角形 Common Focal Triangle

定理 Theorem 19.13.1 (公共焦点三角形的性质 Properties of Common Focal Triangle). 设椭圆和双曲线有相同的焦点 F_1，F_2，则:
Let ellipse and hyperbola have the same foci F_1, F_2, then:

1. 两曲线的交点 P 与焦点构成的三角形面积相等
   The areas of triangles formed by intersection point P and foci are equal
2. 两曲线正交的充要条件是它们的离心率 $e_1e_2 = 1$
   The necessary and sufficient condition for orthogonal intersection is that their eccentricities satisfy $e_1e_2 = 1$
3. 两曲线的公共弦被焦点连线等角分
   The common chord is equally inclined to the focal axis

19.14 椭圆内外部的判断 Interior and Exterior Points of Ellipse

定理 Theorem 19.14.1 (椭圆内外部点的判定 Determining Interior and Exterior Points). 设点 $P(x_0, y_0)$，椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$，则:
For point $P(x_0, y_0)$ and ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

1. 当 $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} < 1$ 时，P 在椭圆内部
   When $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} < 1$, P is interior
2. 当 $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$ 时，P 在椭圆上
   When $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, P is on the ellipse
3. 当 $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} > 1$ 时，P 在椭圆外部
   When $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} > 1$, P is exterior

另一种判定方法（使用焦半径）:
Alternative method (using focal radii):

1. 当 $PF_1 + PF_2 < 2a$ 时，P 在椭圆内部
   When $PF_1 + PF_2 < 2a$, P is interior
2. 当 $PF_1 + PF_2 = 2a$ 时，P 在椭圆上
   When $PF_1 + PF_2 = 2a$, P is on the ellipse
3. 当 $PF_1 + PF_2 > 2a$ 时，P 在椭圆外部
   When $PF_1 + PF_2 > 2a$, P is exterior

例 Example 19.14.2 (判断点的位置 Determine Point Position). 判断点 $P(2,2)$ 相对于椭圆 $\frac{x^2}{16} + \frac{y^2}{9} = 1$ 的位置。
Determine the position of point $P(2,2)$ relative to ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

解:
Solution:

1. 代入点坐标:
   Substitute point coordinates:
   $$\frac{2^2}{16} + \frac{2^2}{9} = \frac{1}{4} + \frac{4}{9} = \frac{9}{36} + \frac{16}{36} = \frac{25}{36} < 1$$
2. 结论：点 $P(2,2)$ 在椭圆内部
   Conclusion: Point $P(2,2)$ is interior to the ellipse

19.15 双曲线内外部的判断 Interior and Exterior Points of Hyperbola

定理 Theorem 19.15.1 (双曲线内外部点的判定 Determining Interior and Exterior Points of Hyperbola). 设点 $P(x_0, y_0)$, 双曲线 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$，则:
For point $P(x_0, y_0)$ and hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

1. 当 $\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}<1$ 时，P 在双曲线内部
   When $\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}<1$, P is interior
2. 当 $\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1$ 时，P 在双曲线上
   When $\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1$, P is on the hyperbola
3. 当 $\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}>1$ 时，P 在双曲线外部
   When $\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}>1$, P is exterior

另一种判定方法（使用焦半径）:
Alternative method (using focal radii):

1. 当 $|PF_1-PF_2|<2a$ 时，P 在双曲线内部
   When $|PF_1-PF_2|<2a$, P is interior
2. 当 $|PF_1-PF_2|=2a$ 时，P 在双曲线上
   When $|PF_1-PF_2|=2a$, P is on the hyperbola
3. 当 $|PF_1-PF_2|>2a$ 时，P 在双曲线外部
   When $|PF_1-PF_2|>2a$, P is exterior

19.16 抛物线内外部的判断 Interior and Exterior Points of Parabola

定理 Theorem 19.16.1 (抛物线内外部点的判定 Determining Interior and Exterior Points of Parabola). 设点 $P(x_0,y_0)$，抛物线 $y^2=2px$，则:
For point $P(x_0,y_0)$ and parabola $y^2=2px$:

1. 当 $y_0^2<2px_0$ 时，P 在抛物线内部
   When $y_0^2<2px_0$, P is interior
2. 当 $y_0^2=2px_0$ 时，P 在抛物线上
   When $y_0^2=2px_0$, P is on the parabola
3. 当 $y_0^2>2px_0$ 时，P 在抛物线外部
   When $y_0^2>2px_0$, P is exterior

另一种判定方法（使用焦点和准线）:
Alternative method (using focus and directrix):

1. 当点到焦点的距离小于到准线的距离时，P 在抛物线内部
When distance to focus is less than distance to directrix, P is interior
2. 当点到焦点的距离等于到准线的距离时，P 在抛物线上
When distance to focus equals distance to directrix, P is on the parabola
3. 当点到焦点的距离大于到准线的距离时，P 在抛物线外部
When distance to focus is greater than distance to directrix, P is exterior

19.17 圆锥曲线的切线方程 Tangent Line Equations

19.17.1 椭圆的切线方程 Tangent Line of Ellipse

定理 **Theorem 19.17.1** (椭圆切线方程 Tangent Line Equation of Ellipse). 设椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 上一点 $P(x_0,y_0)$，则过点 P 的切线方程为:
For point $P(x_0,y_0)$ on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the tangent line equation is:

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$$

若已知切线斜率 k，则切点坐标为:
If slope k is given, the tangent point coordinates are:

$$x=\pm\frac{a}{\sqrt{1+\frac{a^2}{b^2}k^2}},\quad y=\pm\frac{ak}{\sqrt{1+\frac{a^2}{b^2}k^2}}$$

19.17.2 双曲线的切线方程 Tangent Line of Hyperbola

定理 **Theorem 19.17.2** (双曲线切线方程 Tangent Line Equation of Hyperbola). 设双曲线 $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ 上一点 $P(x_0,y_0)$，则过点 P 的切线方

程为:
For point $P(x_0, y_0)$ on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the tangent line equation is:

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$$

若已知切线斜率 k，则切点坐标为:
If slope k is given, the tangent point coordinates are:

$$x = \pm\frac{a}{\sqrt{1 - \frac{a^2}{b^2}k^2}}, \quad y = \pm\frac{ak}{\sqrt{1 - \frac{a^2}{b^2}k^2}}$$

19.17.3 抛物线的切线方程 Tangent Line of Parabola

定理 Theorem 19.17.3 (抛物线切线方程 Tangent Line Equation of Parabola). 设抛物线 $y^2 = 2px$ 上一点 $P(x_0, y_0)$，则过点 P 的切线方程为:
For point $P(x_0, y_0)$ on parabola $y^2 = 2px$, the tangent line equation is:

$$y_0 y = p(x + x_0)$$

若已知切线斜率 k，则切点坐标为:
If slope k is given, the tangent point coordinates are:

$$x = \frac{p}{2k^2}, \quad y = \frac{p}{k}$$

19.18 直线与圆锥曲线的位置关系 Position Relationship between Line and Conic Section

定理 Theorem 19.18.1 (直线与圆锥曲线的位置关系 Position Relationship between Line and Conic Section). 设直线 $l : ax+by+c = 0$，则:
For line $l : ax + by + c = 0$:

1. 与椭圆 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 的位置关系:
   Position relationship with ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

- $\Delta < 0$：相离（无交点）
  $\Delta < 0$: No intersection
- $\Delta = 0$：相切（一个交点）
  $\Delta = 0$: Tangent (one intersection)
- $\Delta > 0$：相交（两个交点）
  $\Delta > 0$: Intersect (two intersections)

其中 $\Delta = a^2b^2(a^2 + b^2k^2 - c^2)$，$k$ 为直线斜率
where $\Delta = a^2b^2(a^2 + b^2k^2 - c^2)$, k is line slope

2. 与双曲线 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 的位置关系：
   Position relationship with hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

- $\Delta < 0$：相交（两个交点）
  $\Delta < 0$: Intersect (two intersections)
- $\Delta = 0$：相切（一个交点）
  $\Delta = 0$: Tangent (one intersection)
- $\Delta > 0$：相离（无交点）
  $\Delta > 0$: No intersection

其中 $\Delta = a^2b^2(a^2 - b^2k^2 - c^2)$
where $\Delta = a^2b^2(a^2 - b^2k^2 - c^2)$

3. 与抛物线 $y^2 = 2px$ 的位置关系：
   Position relationship with parabola $y^2 = 2px$:

- $\Delta < 0$：相离（无交点）
  $\Delta < 0$: No intersection
- $\Delta = 0$：相切（一个交点）
  $\Delta = 0$: Tangent (one intersection)
- $\Delta > 0$：相交（两个交点）
  $\Delta > 0$: Intersect (two intersections)

其中 $\Delta = p^2 + 2pck$，k 为直线斜率
where $\Delta = p^2 + 2pck$, k is line slope

19.19 过已知点的直线与抛物线有一个交点的直线条数 Number of Lines through a Point with One Intersection with Parabola

定理 Theorem 19.19.1 (过点与抛物线相切的直线条数 Number of Tangent Lines through a Point). 设抛物线 $y^2 = 2px$，点 $P(x_0, y_0)$，则:
For parabola $y^2 = 2px$ and point $P(x_0, y_0)$:

1. 当 P 在抛物线内部时，有两条切线
   When P is interior, there are two tangent lines
2. 当 P 在抛物线上时，有一条切线
   When P is on the parabola, there is one tangent line
3. 当 P 在抛物线外部时，没有切线
   When P is exterior, there are no tangent lines

切线方程:
Tangent line equations:

$$y = kx + \frac{p}{2k}$$

其中 k 满足:
where k satisfies:

$$k^2x_0 - ky_0 + \frac{p}{2} = 0$$

例 Example 19.19.2 (计算过点的切线条数 Calculate Number of Tangent Lines through a Point). 求过点 $P(1, 2)$ 与抛物线 $y^2 = 4x$ 相切的直线条数。
Find the number of tangent lines through point $P(1, 2)$ to parabola $y^2 = 4x$.

解:
Solution:

1. 抛物线参数 $p = 2$

Parabola parameter $p = 2$

2. 代入点坐标判断位置:
Check position using point coordinates:

$$y_0^2 = 4 < 2px_0 = 8$$

点在抛物线内部
Point is interior to parabola

3. 解切线斜率方程:
Solve for tangent line slopes:

$$k^2 - 2k + 1 = 0$$

得 $k_1 = 1 + \sqrt{2}$, $k_2 = 1 - \sqrt{2}$
Get $k_1 = 1 + \sqrt{2}$, $k_2 = 1 - \sqrt{2}$

4. 结论：有两条切线
Conclusion: There are two tangent lines

Chapter 20

排列与组合 Permutations and Combinations

20.1 计数原理 Counting Principles

20.1.1 分类加法计数原理 Addition Principle

定理 Theorem 20.1.1 (分类加法计数原理 Addition Principle of Counting). 完成一个任务有 n 类方法，第 i 类方法有 m_i 种不同的方式，且这 n 类方法互不相容，则完成该任务的不同方法总数为 $m_1+m_2+\cdots+m_n$。
If a task can be done in n different ways, where the ith way has m_i different methods, and these ways are mutually exclusive, then the total number of different methods is $m_1+m_2+\cdots+m_n$.

例 Example 20.1.2 (分类加法计数原理应用 Application of Addition Principle). 一个班有男生 25 人，女生 30 人，要选一名班长，有多少种不同的选法?
In a class with 25 boys and 30 girls, how many different ways are there to select a class monitor?

解:
Solution:

1. 选男生的方法数：25 种
   Ways to select a boy: 25
2. 选女生的方法数：30 种
   Ways to select a girl: 30
3. 总的选法：$25 + 30 = 55$ 种
   Total ways: $25 + 30 = 55$

20.1.2 分步乘法计数原理 Multiplication Principle

定理 Theorem 20.1.3 (分步乘法计数原理 Multiplication Principle of Counting). 完成一个任务需要 n 个步骤，第 i 个步骤有 m_i 种不同的方法，则完成该任务的不同方法总数为 $m_1 \times m_2 \times \cdots \times m_n$。
If a task requires n steps, where the ith step can be done in m_i different ways, then the total number of different ways to complete the task is $m_1 \times m_2 \times \cdots \times m_n$.

例 Example 20.1.4 (分步乘法计数原理应用 Application of Multiplication Principle). 有 3 件上衣，4 条裤子，2 双鞋，搭配一套衣服有多少种不同的方法？
With 3 shirts, 4 pairs of pants, and 2 pairs of shoes, how many different outfits can be made?

解：
Solution:

1. 选上衣的方法数：3 种
   Ways to select a shirt: 3
2. 选裤子的方法数：4 种
   Ways to select pants: 4
3. 选鞋的方法数：2 种
   Ways to select shoes: 2
4. 总的搭配方法：$3 \times 4 \times 2 = 24$ 种
   Total combinations: $3 \times 4 \times 2 = 24$

20.2 排列 Permutations

20.2.1 排列的定义与公式 Definition and Formula of Permutations

定义 Definition 20.2.1 (排列 Permutation). 从 n 个不同元素中取出 m 个元素进行排序，称为 n 个元素中取 m 个元素的排列，记作 P_n^m 或 A_n^m。
A permutation is an arrangement of m elements selected from n distinct elements, denoted as P_n^m or A_n^m.

定理 Theorem 20.2.2 (排列数公式 Permutation Formula).

$$P_n^m = \frac{n!}{(n-m)!}$$

其中：
Where:

- $n \geq m$
- $n!$ 表示 n 的阶乘，即 $n! = n \times (n-1) \times \cdots \times 2 \times 1$
  $n!$ represents factorial of n, i.e., $n! = n \times (n-1) \times \cdots \times 2 \times 1$

20.2.2 全排列 Full Permutation

定理 Theorem 20.2.3 (全排列公式 Full Permutation Formula). n 个不同元素的全排列数为 $n!$。
The number of full permutations of n distinct elements is $n!$.

$$P_n^n = n!$$

例 Example 20.2.4 (计算排列数 Calculate Number of Permutations). 计算：
Calculate:

1. P_5^3

2. P_6^6

解:
Solution:

1. $P_5^3 = \frac{5!}{(5-3)!} = \frac{5\times4\times3\times2!}{2!} = 60$
2. $P_6^6 = 6! = 6\times5\times4\times3\times2\times1 = 720$

20.3 组合 Combinations

20.3.1 组合的定义与公式 Definition and Formula of Combinations

定义 Definition 20.3.1 (组合 Combination). 从 n 个不同元素中取出 m 个元素，不考虑排序，称为 n 个元素中取 m 个元素的组合，记作 C_n^m。
A combination is a selection of m elements from n distinct elements, regardless of order, denoted as C_n^m.

定理 Theorem 20.3.2 (组合数公式 Combination Formula).

$$C_n^m = \frac{n!}{m!(n-m)!} = \frac{P_n^m}{m!}$$

其中:
Where:

- $n \geq m$
- C_n^m 也可记作 $\binom{n}{m}$
  C_n^m can also be written as $\binom{n}{m}$

20.3.2 组合数的性质 Properties of Combinations

定理 Theorem 20.3.3 (组合数的性质 Properties of Combinations). 1. 对称性: $C_n^m = C_n^{n-m}$

Symmetry: $C_n^m = C_n^{n-m}$

2. 递推公式: $C_n^m = C_{n-1}^m + C_{n-1}^{m-1}$
   Recurrence relation: $C_n^m = C_{n-1}^m + C_{n-1}^{m-1}$
3. 特殊值:
   Special values:
   - $C_n^0 = C_n^n = 1$
   - $C_n^1 = C_n^{n-1} = n$

例 Example 20.3.4 (计算组合数 Calculate Number of Combinations). 计算:
Calculate:

1. C_5^2
2. C_6^3

解:
Solution:

1. $C_5^2 = \frac{5!}{2!(5-2)!} = \frac{5\times4}{2\times1} = 10$
2. $C_6^3 = \frac{6!}{3!(6-3)!} = \frac{6\times5\times4}{3\times2\times1} = 20$

20.4 排列与组合的区别与联系 Differences and Connections between Permutations and Combinations

定理 Theorem 20.4.1 (排列与组合的关系 Relationship between Permutations an

$P_n^m = C_n^m \times m!$

排列数等于组合数乘以排序方法数

Number of permutations equals number of combinations times number of ways to arrange

2. 排列考虑顺序，组合不考虑顺序
   Permutations consider order, combinations do not
3. 排列可以用于需要排序的问题，组合用于只需选择的问题
   Permutations are used for problems requiring order, combinations for problems requiring only selection

20.5 排列组合的其他公式 Other Formulas in Permutations and Combinations

定理 Theorem 20.5.1 (二项式定理 Binomial Theorem).

$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^{n-k} y^k$$

定理 Theorem 20.5.2 (杨辉三角 Pascal’s Triangle). 第 n 行的数字是 $C_n^0, C_n^1, \cdots, C_n^n$。
Numbers in the nth row are $C_n^0, C_n^1, \cdots, C_n^n$.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

例 Example 20.5.3 (排列组合综合应用 Comprehensive Application). 从 10 个人中选出 3 个人组成一个委员会，其中 1 人为主席，1 人为副主席，1 人为委员，有多少种不同的方法？
From 10 people, select 3 people to form a committee with 1 chairman, 1 vice-chairman, and 1 member. How many different ways are there?

解:
Solution:

1. 这是一个排列问题，因为位置（职务）不同

This is a permutation problem because positions (roles) matter

2. 使用排列公式:
   Use permutation formula:

$$P_{10}^3 = \frac{10!}{(10-3)!} = \frac{10 \times 9 \times 8 \times 7!}{7!} = 720$$

3. 结论：有 720 种不同的方法
   Conclusion: There are 720 different ways

例 Example 20.5.4 (组合数性质应用 Application of Combination Properties). 证明：$C_n^r = C_n^{n-r}$
Prove: $C_n^r = C_n^{n-r}$

证明:
Proof:

1. 左边:
   Left side:

$$C_n^r = \frac{n!}{r!(n-r)!}$$

2. 右边:
   Right side:

$$C_n^{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$$

3. 两边相等，得证
   Both sides are equal, proof complete

Chapter 21

二项式定理 Binomial Theorem

21.1 二项式定理的基本形式 Basic Form of Binomial Theorem

定理 Theorem 21.1.1 (二项式定理 Binomial Theorem). 对于任意实数 x，y 和自然数 n，有:
For any real numbers x, y and natural number n:

$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^{n-k} y^k = C_n^0 x^n + C_n^1 x^{n-1} y + C_n^2 x^{n-2} y^2 + \cdots + C_n^{n-1} x y^{n-1} + C_n^n y^n$$

其中:
Where:

- C_n^k 是二项式系数
  C_n^k is the binomial coefficient
- 展开式共有 $n+1$ 项
  The expansion has $n+1$ terms
- 各项的指数和为 n
  The sum of exponents in each term is n

例 Example 21.1.2 (二项式展开 Binomial Expansion). 展开 $(x+2)^4$。

Expand $(x+2)^4$.

解：
Solution:

1. 使用二项式定理：
   Use binomial theorem:
   $$(x+2)^4 = C_4^0 x^4 + C_4^1 x^3 \cdot 2 + C_4^2 x^2 \cdot 2^2 + C_4^3 x \cdot 2^3 + C_4^4 \cdot 2^4$$
2. 计算二项式系数：
   Calculate binomial coefficients:
   - $C_4^0 = 1$
   - $C_4^1 = 4$
   - $C_4^2 = 6$
   - $C_4^3 = 4$
   - $C_4^4 = 1$
3. 代入得：
   Substitute:
   $$x^4 + 8x^3 + 24x^2 + 32x + 16$$

21.2 二项展开式的通项公式 General Term Formula

定理 Theorem 21.2.1 (通项公式 General Term Formula). 在 $(x+y)^n$ 的展开式中，第 $(k+1)$ 项（从 1 开始计数）为：
In the expansion of $(x+y)^n$, the $(k+1)$th term (counting from 1) is:
$$T_{k+1} = C_n^k x^{n-k} y^k$$

特别地，当 $x=1$ 时：
Specifically, when $x=1$:
$$T_{k+1} = C_n^k y^k$$

例 Example 21.2.2 (求通项 Find General Term). 求 $(2x-1)^5$ 展开式中的通项。
Find the general term in the expansion of $(2x-1)^5$.

解:
Solution:

1. 令 $x = 2x$, $y = -1$
   Let $x = 2x$, $y = -1$
2. 第 $(k+1)$ 项为:
   The $(k+1)$th term is:

$$T_{k+1} = C_5^k (2x)^{5-k}(-1)^k$$

3. 化简得:
   Simplify:

$$T_{k+1} = C_5^k \cdot 2^{5-k} \cdot (-1)^k \cdot x^{5-k}$$

21.3 二项式系数的性质 Properties of Binomial Coefficients

定理 Theorem 21.3.1 (二项式系数的性质 Properties of Binomial Coefficients).
对称性: $C_n^k = C_n^{n-k}$
Symmetry: $C_n^k = C_n^{n-k}$
2. 递推公式: $C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$
   Recurrence relation: $C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$
3. 平方和: $\sum_{k=0}^n (C_n^k)^2 = C_{2n}^n$
   Sum of squares: $\sum_{k=0}^n (C_n^k)^2 = C_{2n}^n$
4. 交错和: $\sum_{k=0}^n (-1)^k C_n^k = 0$
   Alternating sum: $\sum_{k=0}^n (-1)^k C_n^k = 0$
5. 组合恒等式: $C_n^0 + C_n^1 + \cdots + C_n^n = 2^n$
   Combination identity: $C_n^0 + C_n^1 + \cdots + C_n^n = 2^n$

21.4 其他常用公式 Other Useful Formulas

定理 Theorem 21.4.1 (常用公式 Useful Formulas).

1. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$
2. $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$
3. $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$
4. $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$

21.5 二项展开式中系数最大的项 Maximum Coefficient Term

定理 Theorem 21.5.1 (最大系数项 Maximum Coefficient Term). 在 $(x+y)^n$ 的展开式中：
In the expansion of $(x+y)^n$:

1. 当 n 为偶数时，$C_n^{\frac{n}{2}}$ 为最大系数
   When n is even, $C_n^{\frac{n}{2}}$ is the maximum coefficient
2. 当 n 为奇数时，$C_n^{\frac{n-1}{2}}$ 和 $C_n^{\frac{n+1}{2}}$ 为最大系数
   When n is odd, $C_n^{\frac{n-1}{2}}$ and $C_n^{\frac{n+1}{2}}$ are the maximum coefficients

例 Example 21.5.2 (求最大系数项 Find Maximum Coefficient Term). 求 $(1+x)^8$ 展开式中系数最大的项。
Find the term with maximum coefficient in the expansion of $(1+x)^8$.

解：
Solution:

1. 因为 $n=8$ 为偶数，最大系数项的指数为 $\frac{n}{2}=4$
   Since $n=8$ is even, the exponent of maximum coefficient term is $\frac{n}{2}=4$
2. 计算系数：$C_8^4=70$
   Calculate coefficient: $C_8^4=70$

3. 最大系数项为 $70x^4$
   The term with maximum coefficient is $70x^4$

21.6 二项展开式中系数最小的项 Minimum Coefficient Term

定理 Theorem 21.6.1 (最小系数项 Minimum Coefficient Term). 在 $(x+y)^n$ 的展开式中:
In the expansion of $(x+y)^n$:

1. 首项和末项系数最小，均为 1
   First and last terms have minimum coefficients, both equal to 1
2. 从两端向中间，系数逐渐增大
   Coefficients gradually increase from both ends towards the middle

21.7 近似处理问题 Approximation Problems

定理 Theorem 21.7.1 (近似计算 Approximation Calculation). 对于 $|x| \ll 1$，有以下近似公式:
For $|x| \ll 1$, we have the following approximation formulas:

1. $(1+x)^n \approx 1+nx$（保留一次项）
   $(1+x)^n \approx 1+nx$ (keeping first-order term)
2. $(1+x)^n \approx 1+nx+\frac{n(n-1)}{2}x^2$（保留二次项）
   $(1+x)^n \approx 1+nx+\frac{n(n-1)}{2}x^2$ (keeping second-order term)

例 Example 21.7.2 (近似计算 Approximation Calculation). 计算 $(1.02)^3$ 的近似值。
Calculate the approximate value of $(1.02)^3$.

解:

Solution:

1. 令 $x = 0.02$，则 $(1.02)^3 = (1 + 0.02)^3$

   Let $x = 0.02$, then $(1.02)^3 = (1 + 0.02)^3$
2. 保留二次项:

   Keep second-order term:

$$1 + 3(0.02) + \frac{3 \times 2}{2}(0.02)^2 = 1 + 0.06 + 0.0006 = 1.0606$$

3. 实际值为 1.0612，误差约 0.06%

   Actual value is 1.0612, error is about 0.06%

Chapter 22

复数 Complex Numbers

22.1 复数的定义与分类 Definition and Classification of Complex Numbers

定义 Definition 22.1.1 (复数 Complex Number). 形如 $z = a + bi$ 的数叫做复数，其中 a，b 是实数，i 是虚数单位，满足 $i^2 = -1$。
A number in the form $z = a + bi$ is called a complex number, where a, b are real numbers, and i is the imaginary unit satisfying $i^2 = -1$.

- a 称为实部，记作 $\Re(z)$ 或 Re(z)
  a is called the real part, denoted as $\Re(z)$ or Re(z)
- b 称为虚部，记作 $\Im(z)$ 或 Im(z)
  b is called the imaginary part, denoted as $\Im(z)$ or Im(z)

定理 Theorem 22.1.2 (复数的分类 Classification of Complex Numbers). 1. 纯实数：$b = 0$，如 $3 = 3 + 0i$
Pure real numbers: $b = 0$, e.g., $3 = 3 + 0i$
2. 纯虚数：$a = 0$，如 $2i = 0 + 2i$
Pure imaginary numbers: $a = 0$, e.g., $2i = 0 + 2i$
3. 一般复数：$a \neq 0$ 且 $b \neq 0$，如 $3 + 2i$

General complex numbers: $a \neq 0$ and $b \neq 0$, e.g., $3 + 2i$

22.2 复数的模 Modulus of Complex Numbers

定义 Definition 22.2.1 (复数的模 Modulus of Complex Number). 复数 $z = a + bi$ 的模定义为:
The modulus of complex number $z = a + bi$ is defined as:

$$|z| = \sqrt{a^2 + b^2}$$

性质:
Properties:

1. $|z| \geq 0$，当且仅当 $z = 0$ 时取等号
   $|z| \geq 0$, equals zero if and only if $z = 0$
2. $|z_1 z_2| = |z_1||z_2|$
   Product property: $|z_1 z_2| = |z_1||z_2|$
3. $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$
   Quotient property: $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$

22.3 复数相等 Equality of Complex Numbers

定理 Theorem 22.3.1 (复数相等的条件 Conditions for Complex Number Equality). 两个复数 $z_1 = a + bi$ 和 $z_2 = c + di$ 相等的充要条件是:
Two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are equal if and only if:

$$a = c \text{ 且 and } b = d$$

22.4 共轭复数 Conjugate Complex Numbers

定义 Definition 22.4.1 (共轭复数 Conjugate Complex Number). 复数 $z = a + bi$ 的共轭复数为 $\bar{z} = a - bi$。
The conjugate of complex number $z = a + bi$ is $\bar{z} = a - bi$.

性质:
Properties:

1. $z \cdot \bar{z} = |z|^2$
   Product with conjugate: $z \cdot \bar{z} = |z|^2$
2. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
   Sum of conjugates: $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
3. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
   Product of conjugates: $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

22.5 复数的运算 Operations of Complex Numbers

定理 Theorem 22.5.1 (复数的四则运算 Arithmetic Operations of Complex Numbers). 设 $z_1 = a + bi$, $z_2 = c + di$, 则:
Let $z_1 = a + bi$, $z_2 = c + di$, then:

1. 加法: $z_1 + z_2 = (a + c) + (b + d)i$
   Addition: $z_1 + z_2 = (a + c) + (b + d)i$
2. 减法: $z_1 - z_2 = (a - c) + (b - d)i$
   Subtraction: $z_1 - z_2 = (a - c) + (b - d)i$
3. 乘法: $z_1 z_2 = (ac - bd) + (ad + bc)i$
   Multiplication: $z_1 z_2 = (ac - bd) + (ad + bc)i$
4. 除法: $\frac{z_1}{z_2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$ ($z_2 \neq 0$)
   Division: $\frac{z_1}{z_2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$ ($z_2 \neq 0$)

22.6 复数的周期性 Periodicity of Complex Numbers

定理 Theorem 22.6.1 (复数的周期性 Periodicity of Complex Numbers). 1. $i^4 = 1$

Fourth power of i: $i^4 = 1$

2. $i^{4k+r} = i^r$，其中 k 为整数，r 为余数
   Power property: $i^{4k+r} = i^r$, where k is integer and r is remainder
3. i 的幂的循环规律：$i, i^2 = -1, i^3 = -i, i^4 = 1$
   Cyclic pattern of i powers: $i, i^2 = -1, i^3 = -i, i^4 = 1$

22.7 复数运算的常用结论 Common Results in Complex Number Operations

定理 Theorem 22.7.1 (常用结论 Common Results). 1. $(a+bi)(a-bi) = a^2 + b^2$
   Product with conjugate: $(a+bi)(a-bi) = a^2 + b^2$
2. $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$
   Reciprocal: $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$
3. $(a+bi)^n = r(\cos n\theta + i\sin n\theta)$，其中 $r = \sqrt{a^2+b^2}$
   De Moivre's formula: $(a+bi)^n = r(\cos n\theta + i\sin n\theta)$, where $r = \sqrt{a^2+b^2}$

22.8 特殊复数的运算 Operations with Special Complex Numbers

定理 Theorem 22.8.1 (特殊复数运算 Special Complex Number Operations). 1. $i^2 = -1$
2. $i^3 = -i$
3. $i^4 = 1$
4. $(1+i)^2 = 2i$
5. $(1-i)^2 = -2i$

例 Example 22.8.2 (特殊复数运算 Special Complex Number Operations). 计算 $(1+i)^4$。
Calculate $(1+i)^4$.

解：
Solution:

1. $(1+i)^2 = 1+2i+i^2 = 2i$
   First square: $(1+i)^2 = 1+2i+i^2 = 2i$
2. $(1+i)^4 = ((1+i)^2)^2 = (2i)^2 = 4i^2 = -4$
   Second square: $(1+i)^4 = ((1+i)^2)^2 = (2i)^2 = 4i^2 = -4$

22.9 实数系一元二次方程在复数范围内求根 Solving Quadratic Equations in Complex Domain

定理 Theorem 22.9.1 (一元二次方程的复数根 Complex Roots of Quadratic Equations). 对于一元二次方程 $ax^2+bx+c=0$ $(a \neq 0)$:
For quadratic equation $ax^2+bx+c=0$ ($a \neq 0$):

1. 当 $\Delta = b^2-4ac > 0$ 时，有两个不相等的实根
   When $\Delta = b^2-4ac > 0$, there are two distinct real roots
2. 当 $\Delta = 0$ 时，有两个相等的实根
   When $\Delta = 0$, there are two equal real roots
3. 当 $\Delta < 0$ 时，有两个共轭复根: $x = \frac{-b \pm \sqrt{4ac-b^2}i}{2a}$
   When $\Delta < 0$, there are two conjugate complex roots: $x = \frac{-b \pm \sqrt{4ac-b^2}i}{2a}$

例 Example 22.9.2 (求复数根 Find Complex Roots). 求方程 $x^2+2x+2=0$ 的复数根。
Find the complex roots of equation $x^2+2x+2=0$.

解:
Solution:

1. 计算判别式: $\Delta = 4-8 = -4 < 0$
   Calculate discriminant: $\Delta = 4-8 = -4 < 0$
2. 代入公式:

Apply formula:

$$x = \frac{-2 \pm \sqrt{-4}i}{2} = -1 \pm i$$

3. 所以两个复根为 $x_1 = -1 + i$，$x_2 = -1 - i$
   Therefore, the two complex roots are $x_1 = -1+i$, $x_2 = -1-i$

22.10 复平面内两点间的距离公式 Distance Formula in Complex Plane

定理 Theorem 22.10.1 (复平面距离公式 Complex Plane Distance Formula). 复平面内两点 $z_1 = a + bi$ 和 $z_2 = c + di$ 之间的距离为：
The distance between two points $z_1 = a + bi$ and $z_2 = c + di$ in complex plane is:

$$|z_1 - z_2| = \sqrt{(a-c)^2 + (b-d)^2}$$

性质：
Properties:

1. 满足三角不等式：$|z_1 + z_2| \leq |z_1| + |z_2|$
   Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$
2. 两点间距离等于它们差的模
   Distance equals modulus of their difference

例 Example 22.10.2 (计算复平面距离 Calculate Complex Plane Distance). 求复平面中点 $z_1 = 1 + 2i$ 到点 $z_2 = 3 - i$ 的距离。
Find the distance between points $z_1 = 1 + 2i$ and $z_2 = 3 - i$ in complex plane.

解：
Solution:

1. 计算 $z_1 - z_2 = (1 - 3) + (2 - (-1))i = -2 + 3i$

Calculate $z_1 - z_2 = (1-3) + (2-(-1))i = -2+3i$

2. 计算距离:
Calculate distance:

$$|z_1 - z_2| = \sqrt{(-2)^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$$

Chapter 23

概率与统计 Probability and Statistics

23.1 数据的集中趋势 Measures of Central Tendency

23.1.1 平均数 Mean

定义 Definition 23.1.1 (算术平均数 Arithmetic Mean). 设有 n 个数据 $x_1, x_2, \cdots, x_n$，其算术平均数为:
For n data points $x_1, x_2, \cdots, x_n$, the arithmetic mean is:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

23.1.2 中位数 Median

定义 Definition 23.1.2 (中位数 Median). 将数据从小到大排序后:
After arranging the data in ascending order:

- 当 n 为奇数时，中位数为第 $\frac{n+1}{2}$ 个数
  When n is odd, median is the $\frac{n+1}{2}$th number

- 当 n 为偶数时，中位数为第 $\frac{n}{2}$ 个数与第 $\frac{n}{2}+1$ 个数的平均值
  When n is even, median is the average of the $\frac{n}{2}$th and $(\frac{n}{2}+1)$th numbers

23.1.3 众数 Mode

定义 Definition 23.1.3 (众数 Mode). 一组数据中出现次数最多的数值。一组数据可能有多个众数或没有众数。
The value that appears most frequently in a dataset. A dataset may have multiple modes or no mode.

23.2 数据的离散程度 Measures of Dispersion

23.2.1 方差与标准差 Variance and Standard Deviation

定义 Definition 23.2.1 (方差 Variance). 总体方差：$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$
Population variance: $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$

样本方差：$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$
Sample variance: $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

定义 Definition 23.2.2 (标准差 Standard Deviation). 总体标准差：$\sigma = \sqrt{\sigma^2}$
Population standard deviation: $\sigma = \sqrt{\sigma^2}$

样本标准差：$s = \sqrt{s^2}$
Sample standard deviation: $s = \sqrt{s^2}$

23.3 随机变量 Random Variables

23.3.1 离散型随机变量 Discrete Random Variables

定义 Definition 23.3.1 (离散型随机变量 Discrete Random Variable). 取值只能为有限个或可列无限个的随机变量。
A random variable that can only take finite or countably infinite values.

定义 Definition 23.3.2 (分布列 Probability Mass Function). 离散型随机变量 X 的分布列为:
The probability mass function of discrete random variable X is:

$$P(X = x_i) = p_i, \quad i = 1, 2, \cdots$$

其中 where $\sum_i p_i = 1$

23.3.2 连续型随机变量 Continuous Random Variables

定义 Definition 23.3.3 (连续型随机变量 Continuous Random Variable). 取值可以是某个区间内任意值的随机变量。
A random variable that can take any value within an interval.

23.4 概率的基本运算 Basic Probability Operations

定理 Theorem 23.4.1 (概率的加法公式 Addition Rule of Probability).

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

定理 Theorem 23.4.2 (概率的乘法公式 Multiplication Rule of Probability).

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

23.5 典型概率分布 Typical Probability Distributions

23.5.1 二项分布 Binomial Distribution

定义 Definition 23.5.1 (二项分布 Binomial Distribution). 若随机变量 $X \sim B(n, p)$，则：
If random variable $X \sim B(n, p)$, then:

$$P(X = k) = C_n^k p^k (1 - p)^{n-k}, \quad k = 0, 1, \cdots, n$$

期望：$E(X) = np$
Expected value: $E(X) = np$

方差：$D(X) = np(1 - p)$
Variance: $D(X) = np(1 - p)$

23.5.2 正态分布 Normal Distribution

定义 Definition 23.5.2 (正态分布 Normal Distribution). 若随机变量 $X \sim N(\mu, \sigma^2)$，其密度函数为：
If random variable $X \sim N(\mu, \sigma^2)$, its density function is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

标准正态分布的三个重要概率值：
Three important probability values for standard normal distribution:

- $P(|X| < \sigma) \approx 0.6826$
- $P(|X| < 2\sigma) \approx 0.9544$
- $P(|X| < 3\sigma) \approx 0.9974$

23.6 回归分析 Regression Analysis

23.6.1 线性回归 Linear Regression

定义 Definition 23.6.1 (回归直线方程 Linear Regression Equation).

$$\hat{y} = a + bx$$

其中 where:

$$b = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$a = \bar{y} - b\bar{x}$$

23.6.2 相关性分析 Correlation Analysis

定义 Definition 23.6.2 (相关系数 Correlation Coefficient). 皮尔逊相关系数:
Pearson correlation coefficient:

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

定义 Definition 23.6.3 (决定系数 Coefficient of Determination). R^2 的计算公式:
Formula for R^2:

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

23.7 观测值分析 Analysis of Observations

定义 Definition 23.7.1 (标准化观测值 Standardized Observations). 观测值 x_i 的标准化值:
Standardized value of observation x_i:

$$z_i = \frac{x_i - \bar{x}}{s}$$

其中 s 为样本标准差。
where s is the sample standard deviation.

例 Example 23.7.2 (数据分析综合实例 Comprehensive Data Analysis Example). 某班级 30 名学生的数学成绩如下:
The math scores of 30 students in a class are as follows:

85, 92, 78, 95, 88, 82, 90, 85, 88, 92, 75, 85, 92, 88, 85, 82, 88, 90, 85, 92, 78, 85, 88, 92, 85, 88, 90, 85, 88, 92

求这组数据的:
Calculate:

1. 平均数 Mean
2. 中位数 Median
3. 众数 Mode
4. 标准差 Standard Deviation

解:
Solution:

1. 平均数 Mean:
   $\bar{x} = \frac{2595}{30} = 86.5$
2. 中位数 Median:
   排序后第 15 和第 16 个数的平均值: 88
3. 众数 Mode:
   88(出现 8 次)和 85(出现 7 次)
4. 标准差 Standard Deviation:
   $s = \sqrt{\frac{\sum(x_i-\bar{x})^2}{n-1}} \approx 4.76$

Chapter 24

高等数学的补充内容 Supplementary Topics in Advanced Mathematics

24.1 洛必达法则 L'Hôpital's Rule

定理 Theorem 24.1.1 (洛必达法则的完整形式 Complete Form of L'Hôpital's Rule). 设函数 $f(x)$ 和 $g(x)$ 在点 a 的某邻域内有定义（a 可以是实数、∞ 或 $-\infty$），并且满足以下条件之一：
Let functions $f(x)$ and $g(x)$ be defined in a neighborhood of point a (where a can be a real number, ∞, or $-\infty$), and satisfy one of the following conditions:

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$（$\frac{0}{0}$ 型）
   $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ (type $\frac{0}{0}$)
2. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$（$\frac{\infty}{\infty}$ 型）
   $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ (type $\frac{\infty}{\infty}$)
3. $\lim_{x\to a} f(x) = 0$ 且 $\lim_{x\to a} g(x) = \infty$（$0\cdot\infty$ 型，通过变形转化）
   $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (type $0\cdot\infty$, through transformation)
4. $\lim_{x\to a} f(x) = \infty$ 且 $\lim_{x\to a} g(x) = 0$（$\infty\cdot 0$ 型，通过变形转化）

$\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ (type $\infty \cdot 0$, through transformation)

如果：

If:

1. $g'(x) \neq 0$
   $g'(x) \neq 0$
2. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ 存在（或为 ∞）
   $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (or equals ∞)

则：

Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

例 Example 24.1.2 (洛必达法则的多重应用 Multiple Applications of L'Hôpital's Rule). 求极限 Calculate $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

解：

Solution:

1. 当 $x \to 0$ 时，分子分母都趋于 0，为 $\frac{0}{0}$ 型
   When $x \to 0$, both numerator and denominator approach 0, type $\frac{0}{0}$
2. 第一次应用洛必达法则：
   First application of L'Hôpital's Rule:
   $$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2}$$
3. 仍为 $\frac{0}{0}$ 型，继续应用洛必达法则：
   Still type $\frac{0}{0}$, apply L'Hôpital's Rule again:
   $$\lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x}$$

4. 再次应用洛必达法则：
   Apply L'Hôpital's Rule once more:
   $$\lim_{x\to 0}\frac{-\sin x}{6x}=\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}$$

例 Example 24.1.3 (指数型极限 Exponential Limit). 求极限 Calculate $\lim_{x\to\infty} x^2e^{-x}$

解：
Solution:

1. 变形为分式：
   Transform to fraction:
   $$\lim_{x\to\infty} x^2e^{-x}=\lim_{x\to\infty}\frac{x^2}{e^x}$$
2. 为 $\frac{\infty}{\infty}$ 型，应用洛必达法则：
   Type $\frac{\infty}{\infty}$, apply L'Hôpital's Rule:
   $$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}$$
3. 仍为 $\frac{\infty}{\infty}$ 型，再次应用：
   Still type $\frac{\infty}{\infty}$, apply again:
   $$\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0$$

24.2 拉格朗日中值定理 Lagrange's Mean Value Theorem

定理 Theorem 24.2.1 (拉格朗日中值定理的几何意义 Geometric Meaning of Lagrange's Mean Value Theorem). 拉格朗日中值定理表明，在曲线 $y=f(x)$ 上，在区间 $[a,b]$ 上至少存在一点，其切线平行于连接端点 $(a,f(a))$ 和 $(b,f(b))$ 的割线。
Lagrange's Mean Value Theorem states that on the curve $y=f(x)$, there exists at least one point in interval $[a,b]$ where the tangent line is parallel to the secant line connecting endpoints $(a,f(a))$ and $(b,f(b))$.

例 Example 24.2.2 (拉格朗日中值定理的综合应用 Comprehensive Application of Lagrange's Mean Value Theorem). 证明：对于任意 $x > 0$，存在 $\xi \in (0, x)$，使得 $\ln(1 + x) = \frac{x}{1+\xi}$。
Prove: For any $x > 0$, there exists $\xi \in (0, x)$ such that $\ln(1 + x) = \frac{x}{1+\xi}$.

证明：
Proof:

1. 令 $f(t) = \ln(1 + t)$，区间为 $[0, x]$
   Let $f(t) = \ln(1 + t)$, interval $[0, x]$
2. $f'(t) = \frac{1}{1+t}$
3. 根据拉格朗日中值定理，存在 $\xi \in (0, x)$，使得：
   By Lagrange's Mean Value Theorem, there exists $\xi \in (0, x)$ such that:
   $$\frac{f(x) - f(0)}{x - 0} = f'(\xi)$$
4. 代入得：
   Substitute:
   $$\frac{\ln(1 + x) - \ln(1)}{x} = \frac{1}{1 + \xi}$$
5. 化简得：
   Simplify:
   $$\ln(1 + x) = \frac{x}{1 + \xi}$$

定理 Theorem 24.2.3 (柯西中值定理 Cauchy's Mean Value Theorem). 如果函数 $f(x)$ 和 $g(x)$ 在闭区间 $[a, b]$ 上连续，在开区间 (a, b) 内可导，且对任意 $x \in (a, b)$，$g'(x) \neq 0$，则存在 $\xi \in (a, b)$，使得：
If functions $f(x)$ and $g(x)$ are continuous on $[a, b]$, differentiable on (a, b), and for any $x \in (a, b)$, $g'(x) \neq 0$, then there exists $\xi \in (a, b)$ such that:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

24.3 泰勒展开 Taylor Series Expansion

24.3.1 常见函数的泰勒展开 Taylor Series of Common Functions

定理 Theorem 24.3.1 (基本初等函数的泰勒展开 Taylor Series of Elementary Functions). 在 $x = 0$ 处的麦克劳林展开:
Maclaurin series at $x = 0$:

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$
4. $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots, \quad |x| < 1$
5. $(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots$

例 Example 24.3.2 (复合函数的泰勒展开 Taylor Series of Composite Functions). 求 $\sin^2 x$ 在 $x = 0$ 处的泰勒展开式（展开到 x^4 项)。
Find the Taylor series of $\sin^2 x$ at $x = 0$ (up to x^4 term).

解:
Solution:

1. 使用 $\sin x$ 的展开式:
   Use the series of $\sin x$:

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

2. 平方得:
   Square:

$$\begin{aligned}\sin^2 x &= (x - \frac{x^3}{6} + o(x^3))^2 \\ &= x^2 - \frac{x^4}{3} + o(x^4)\end{aligned}$$

24.3.2 泰勒公式的余项估计 Remainder Estimation in Taylor's Formula

定理 Theorem 24.3.3 (拉格朗日余项的估计 Estimation of Lagrange Remainder). 如果在区间 $[a,b]$ 上 $|f^{(n+1)}(x)| \leq M$，则拉格朗日余项满足:
If $|f^{(n+1)}(x)| \leq M$ on interval $[a,b]$, then the Lagrange remainder satisfies:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-x_0|^{n+1}$$

24.4 帕德逼近 Padé Approximation

24.4.1 帕德逼近的构造方法 Construction of Padé Approximation

定理 Theorem 24.4.1 (帕德逼近的系数确定 Determination of Padé Approximation Coefficients). 设函数 $f(x)$ 的泰勒展开为:
Let the Taylor series of function $f(x)$ be:

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

则 (m,n) 阶帕德逼近 $R_{m,n}(x) = \frac{P_m(x)}{Q_n(x)}$ 的系数满足:
Then the coefficients of (m,n) Padé approximation $R_{m,n}(x) = \frac{P_m(x)}{Q_n(x)}$ satisfy:

$$(c_0+c_1x+c_2x^2+\cdots)(1+q_1x+q_2x^2+\cdots+q_nx^n) = p_0+p_1x+p_2x^2+\cdots+p_mx^m+O(x^{m+n+1})$$

例 Example 24.4.2 (高阶帕德逼近 Higher-Order Padé Approximation). 求函数 e^x 的 $(2,2)$ 阶帕德逼近。
Find the $(2,2)$ Padé approximation of function e^x.

解:
Solution:

1. e^x 的泰勒展开:

Taylor series of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

2. 设 $(2,2)$ 阶帕德逼近为:
   Let the $(2,2)$ Padé approximation be:

$$R_{2,2}(x) = \frac{p_0 + p_1x + p_2x^2}{1 + q_1x + q_2x^2}$$

3. 展开并对应系数:
   Expand and match coefficients:

$$\begin{aligned} p_0 &= 1 \\ p_1 - p_0q_1 &= 1 \\ p_2 - p_1q_1 + p_0q_2 &= \frac{1}{2} \\ -p_2q_1 + p_1q_2 &= \frac{1}{6} \\ -p_2q_2 &= \frac{1}{24} \end{aligned}$$

4. 解得:
   Solve to get:

$$R_{2,2}(x) = \frac{12 + 6x + x^2}{12 - 6x + x^2}$$

24.4.2 帕德逼近的收敛性 Convergence of Padé Approximation

定理 Theorem 24.4.3 (帕德逼近的收敛性质 Convergence Properties of Padé App
   对于解析函数，当 $m, n \to \infty$ 时，帕德逼近在收敛域内一致收敛于原函数
   For analytic functions, as $m, n \to \infty$, Padé approximation converges uniformly to the original function in the domain of convergence
2. 帕德逼近可以处理一些泰勒级数发散的情况
   Padé approximation can handle some cases where Taylor series diverges
3. 对于有理函数，存在精确的帕德逼近
   For rational functions, there exists exact Padé approximation

例 Example 24.4.4 (帕德逼近的优越性 Superiority of Padé Approximation). 比较函数 $\frac{1}{1-x}$ 的泰勒展开和 $(1,1)$ 阶帕德逼近在 $x=0.9$ 处的误差。
Compare the errors of Taylor series and $(1,1)$ Padé approximation for function $\frac{1}{1-x}$ at $x=0.9$.

解:
Solution:

1. 泰勒展开(取前 3 项):
   Taylor series (first 3 terms):
   $$1+x+x^2=2.71$$
2. $(1,1)$ 阶帕德逼近:
   $(1,1)$ Padé approximation:
   $$R_{1,1}(x)=\frac{1+\frac{x}{2}}{1-\frac{x}{2}}=9.53$$
3. 实际值: $\frac{1}{1-0.9}=10$
   Actual value: $\frac{1}{1-0.9}=10$
4. 帕德逼近的误差明显小于泰勒展开的误差
   The error of Padé approximation is significantly smaller than that of Taylor series

Chapter 25

结束语 Conclusion

在本书中，我们系统地探讨了高等数学在高中数学中的应用与推论。从基础的数列、函数到高级的概率统计，我们始终注重理论与实践的结合，力求让读者既能理解深层的数学原理，又能掌握实际的应用方法。
In this book, we have systematically explored the applications and implications of advanced mathematics in high school mathematics. From basic sequences and functions to advanced probability and statistics, we have consistently emphasized the combination of theory and practice, aiming to help readers understand both the underlying mathematical principles and their practical applications.

通过对二十四个章节的学习，相信读者已经建立起了完整的数学知识体系，理解了高等数学与高中数学之间的内在联系。这不仅有助于更深入地理解高中数学知识，也为未来进一步学习高等数学打下了坚实的基础。
Through the study of twenty-four chapters, we believe that readers have established a complete mathematical knowledge system and understood the intrinsic connections between advanced mathematics and high school mathematics. This not only helps to understand high school mathematics more deeply but also lays a solid foundation for further study of advanced mathematics.

希望这本书能够帮助读者在数学学习的道路上走得更远，也期待读者能够

将所学知识灵活运用于实际问题的解决中。

We hope this book can help readers go further in their mathematical journey and look forward to seeing readers apply their knowledge flexibly in solving practical problems.