INVESTIGATION OF RESOLVING SETS AND METRIC DIMENSION

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Abstract. For any graph G it is possible to describe each of its vertices uniquely with respect to an ordered subset of vertices of G called a resolving set. In this project, we investigate properties of associated with minimal resolving sets, known as bases and conditions under which changing G affects its bases.

1. introduction

For this investigation we will begin with several basic definitions that might not be part of a typical introduction to graph theory.

Definition 1.1 (Representation of vertex(with respect to W))**.** :
For an ordered subset $W = (w_1, w_2, ...)$ of vertices in G, the representation of a vertex $v \in G$ with respect to W denoted $rep(v/W)$ is:

$$rep(v/W) = (d(v, w_1), d(v, w_2), ...)$$

Definition 1.2 (Resolving Set)**.** :
An ordered subset $W \subset V(G)$ is called a resolving set of G if for all $v_1, v_2 \in V(G)$:

$$rep(v_1/W) \neq rep(v_2/W)$$

Definition 1.3 (Basis of a graph)**.** :
A resolving set W of G is called a $basis\ of\ G$ if for any other resolving set H of G,

$$|H| \geq |W|$$

Definition 1.4 (Metric Dimension)**.** :
The metric dimension of a graph G denoted $MD(G)$ is the order of any basis for G.

Finding the metric dimension or any basis for a graph is currently considered to be an NP-Hard problem. Metric dimension is known for some subsets of graphs which will be discussed later, but in general it is very difficult or simply time consuming to determine. The intent of this paper is to investigate whether there is a set of graphs for which having a known metric dimension allows us to determine metric dimension after adding or removing an edge. Furthermore we will attempt to investigate what properties of these graphs make it possible to determine metric dimension for them after adding or removing an edge.

Example 1.5. The path graph P_3 has a metric dimension of 1. Simply calling W the set containing one leaf node of P_3 is sufficient to construct a basis for P_3. By adding any edge to P_3, P_3 becomes K_3 which has a metric dimension of 2.

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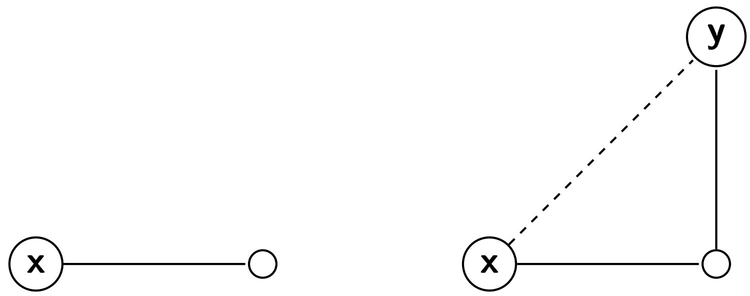


2. ENCODABILITY OF GRAPHS WITH RESPECT TO THEIR BASES

Definition 2.1 (Representation set of a graph)**.** :
The representation set of a graph with respect to a basis W is the set $H = \{r(v/W) \mid v \in G\}$.

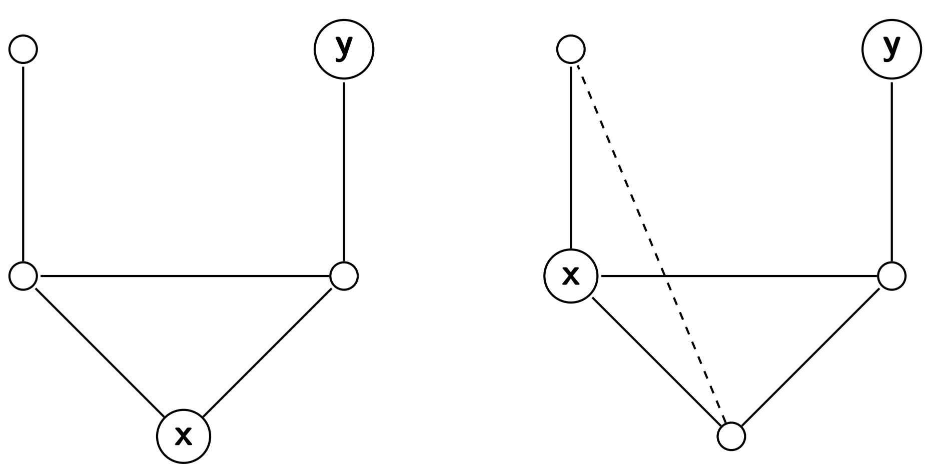
Definition 2.2 (Unique Encoding)**.** :
If we can construct a graph G from its representation set H by connecting vertices in G if their representations differ by no more than 1 in each place then we say that G is uniquely encoded by H. Furthermore we say that G is uniquely encodeable with respect to its basis W or G is enccodable under W.

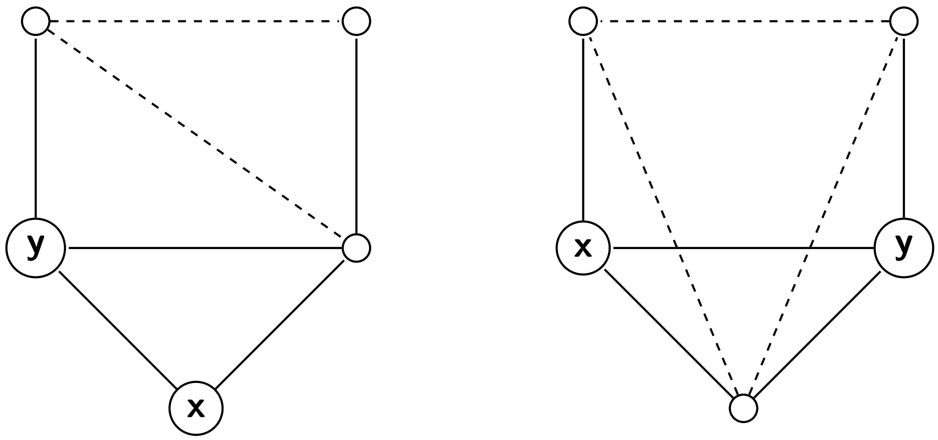
Lemma 2.3. *G is encodable under W if and only if for all nonadjacent $v_1,\ v_2 \in G$ there exists $w \in W$ such that:*

$$|d(v_1, w) - d(v_2, w)| > 1$$

Proof. Let G be a graph with basis W. Let R be the representation set of G with respect to W. Assume that for some pair of vertices $v_1,\ v_2\ \in G,\ \exists\ w \in W$ such that $|d(v_1, w) - d(v_2, w)| > 1$. If we construct a graph H from R following the encoding scheme described above, $v_1,\ v_2$ will be adjacent in H. Then $H \neq G$ and so G is not encodable under W. $\square$

Example 2.4. Observe the following





The bull graph pictured above has basis size 2. There are 4 possible bases for it up to isomorphism but is W-encodable only under one of them. For the other 3 bases it is possible to add edges (shown dotted) without affecting $r(v/W)$ for any $v \in G$.

3. Upper and Lower Bounds on Metric Dimension

Lemma 3.1. *Let G be a graph with a cut vertex v and resolving set W. Let H be a component of $G - v$. If $G - v$ has more than two components or if for any component L of $G - v$, L is not a path, then $W \nsubseteq H$.*

Proof. Suppose that $W \subset H$. Let v be a cut vertex of G such that $G - v$ has more than 2 components. Since $W \subset H$, the path from any vertex not in H to any element of W must pass through v. Then there exist vertices u_1, u_2 adjacent to v such that $d(u_1, w_i) = d(u_2, w_i) = d(v, w_i) + 1$ for all $w_i \in W$. In this case, $r(u_1/W) = r(u_2/W)$. Then W cannot be not a resolving set for G.
We can show without loss of generality that the same is true for any cut vertex v whenever some component of $G - v$ other than H is not path.
Observe that if G is not a path it contains at least one $deg > 2$ vertex. Let l be a $deg > 2$ vertex in G such that for allu_i such that $deg(u_i) > 2, d(u_i, v) \geq d(l, v)$. Then by the same argument as above we see there are 2 adjacent vertices with the same representation and again W cannot be a resolving set of G. $\square$

Lemma 3.2 (Bridge Subdivision Lemma)**.** *Let G be a graph with a bridge (u, v).*
Let H, S be the largest connected induced subgraphs of G such that $v \notin H, u \notin S$.
Let L be the uv path separating H and S.
Let W be a basis of G.
Let G', H', S', L', W' be the same objects described above after subdivision of (u, v) with vertex e and the additional constraint that $e \notin H' \cup S'$.

If H and S both contain basis elements then $MD(G') = MD(G)$

Proof.
Suppose that $MD(G') > MD(G)$, this can only be true if for some $u_h \in H$, $v_s \in S$

$$r(u_h/W) + (\underbrace{0, \ldots}_{n}, 1, \ldots) = r(v_s/W) + (\underbrace{1, \ldots}_{n}, 0, \ldots)$$

Where the first n elements of each representation correspond to the distances from elements of W_h up to rearrangement.
Suppose that such a pair of vertices does exist. Then for all $w_h \in W_H \subset H$,
$d(v_s, w_h) < d(u_h, w_h)$.
Every $v_s w_h$ path for all w_h passes through L. Then $d(v_s, w_h) = d(v_s, u) + d(u, w_h)$. By the triangle inequality, $d(u_h, w_h) \leq d(u_h, u) + d(u, w_h)$ for all u_h in H. Then $d(v_s, u) < d(u_h, u)$ and consequently $d(v_s, e) < d(u_h, e)$.
Without loss of generality we can show that because for all $w_s \in W_S \subset S$, $d(u_h, w_s) < d(v_s, w_s)$ we must have $d(u_h, e) < d(v_s, e)$. This is a contradiction. Then such a pair of vertices cannot possibly exist and so we have $MD(G') \leq MD(G)$.
Now suppose that $MD(G) > MD(G')$
In this case we see that we must have a pair of vertices $u_h \in H$, $v_s \in S$ such that

$$r(u_h/W') + (\underbrace{1, \dots,}_{n} 0, \dots) = r(v_s/W') + (\underbrace{0, \dots,}_{n} 1, \dots)$$

Where the first n elements of each representation correspond to the distances from elements of W'_h up to rearrangement. Again we will suppose that such a pair of vertices exists.
Then we can see that for basis elements in H:

$$\begin{aligned} d(u_h, w_h) + 1 &= d(v_s, w_h) \\ d(u_h, u) + d(u, w_h) + 1 &= d(v_s, u) + d(u, w_h) \\ d(u_h, u) + 1 &= d(v_s, u) \\ d(u_h, u) + 1 &> d(v_s, u) - 1 \\ d(u_h, u) + d(u, e) &> d(v_s, u) - d(u, e) \\ d(u_h, e) &> d(v_s, e) \end{aligned}$$

And without loss of generality by considering basis elements in V:

$$d(v_s, e) > d(u_h, e)$$

Then $d(v_s, e) >< d(u_h, e)$, a clear contradiction.
Thus $MD(G) = MD(G')$. $\square$

[3]

References

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