INVESTIGATION OF RESOLVING SETS AND METRIC DIMENSION

KYLE RYAN

Abstract. Metric dimension is known for some subsets of graphs which will be discussed later, but in general it is very difficult or simply time consuming to determine. The intent of this paper is to investigate whether there is a set of graphs for which having a known metric dimension or basis allows us to determine metric dimension or basis after adding or removing a vertex or an edge. We will introduce a new construction which can be used to describe the encodability of a graph with respect to one or more of its bases. We will also introduce a new proposition governing where elements of a basis can lie in a graph and a new theorem describing conditions under which subdividing bridges has no effect on the basis of a graph.

1. introduction

For any graph G it is possible to describe each of its vertices uniquely with respect to an ordered subset of vertices of G called a resolving set. Describing a graph by its basis is often very difficult. Every known algorithm for finding a basis of an arbitary graph is greedy and even finding the metric dimension for a graph is currently considered to be of NP-Hard complexity. It is assumed for the remainder of this paper that we are dealing with simple connected graphs.

For this investigation we will begin with several basic definitions that might not be part of a typical introduction to graph theory.

Definition 1.1 (Representation of vertex(with respect to W)).
For an ordered subset $W = (w_1, w_2, ...)$ of vertices in G, the representation of a vertex $v \in G$ with respect to W denoted $r(v/W)$ where $r : G \mapsto \mathbb{N}^{|W|}$ is defined by:

$$r(v/W) = (d(v, w_1), d(v, w_2), ...)$$

Then $r(G/W)$ is the image of r with respect to W when applied to G.

Definition 1.2 (Resolving Set).
An ordered subset $W \subset V(G)$ is called a resolving set of G if r for all $v_1, v_2 \in V(G)$:

$$r(v_1/W) \neq r(v_2/W)$$

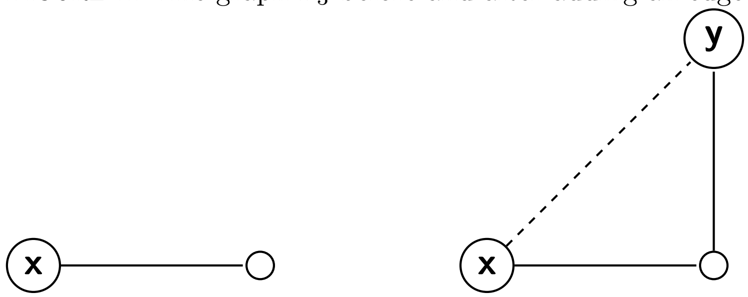
Equivalently, whenever. $r(G/W)$ is injective.

Definition 1.3 (Basis of a graph (Also called reference set)).
A resolving set W of G is called a *basis of G* if for any other resolving set H of G,

$$|H| \geq |W|$$

Definition 1.4 (Metric Dimension).
The metric dimension of a graph G denoted $MD(G)$ is the order of any basis for G.

FIGURE 1. The graph P_3 before and after adding an edge



Example 1.5. The path graph P_3 as shown in figure 1 has a metric dimension of 1. Simply calling W the set containing one leaf node of P_3 is sufficient to construct a basis for P_3. By adding any edge to P_3(shown dotted), P_3 becomes K_3 which has a metric dimension of 2.

2. ENCODABILITY OF GRAPHS WITH RESPECT TO THEIR BASES

Nearly any collection of information can be described as a graph. As such, any method for reducing the space required to represent a graph can be an incredibly useful tool. In this section we will focus on properties that allow a graph to be encoded by its image $r(G/W)$ for some basis W.

Definition 2.1 (Representation set of a graph)**.**
The representation set of a graph with respect to a basis W is the set $H = \{r(v/W) \mid v \in G\}$.

Construction 2.2. *Let G be a graph with a basis W. Let G' be the graph constructed from $r(G/W)$ by connecting elements of $r(G/W)$ if and only if $r(v_i/W) - r(v_j/W) \in \{0, \pm 1\}$. We will denote this construction $\mathbb{N}^{|W|} \mapsto G'$ as $\xi(r(G/W))$.*

Definition 2.3 (Unique Encoding)**.**
We call G uniquely encodeable with respect to a basis W or W-encodable if

$$\xi(r(G/W)) = G.$$

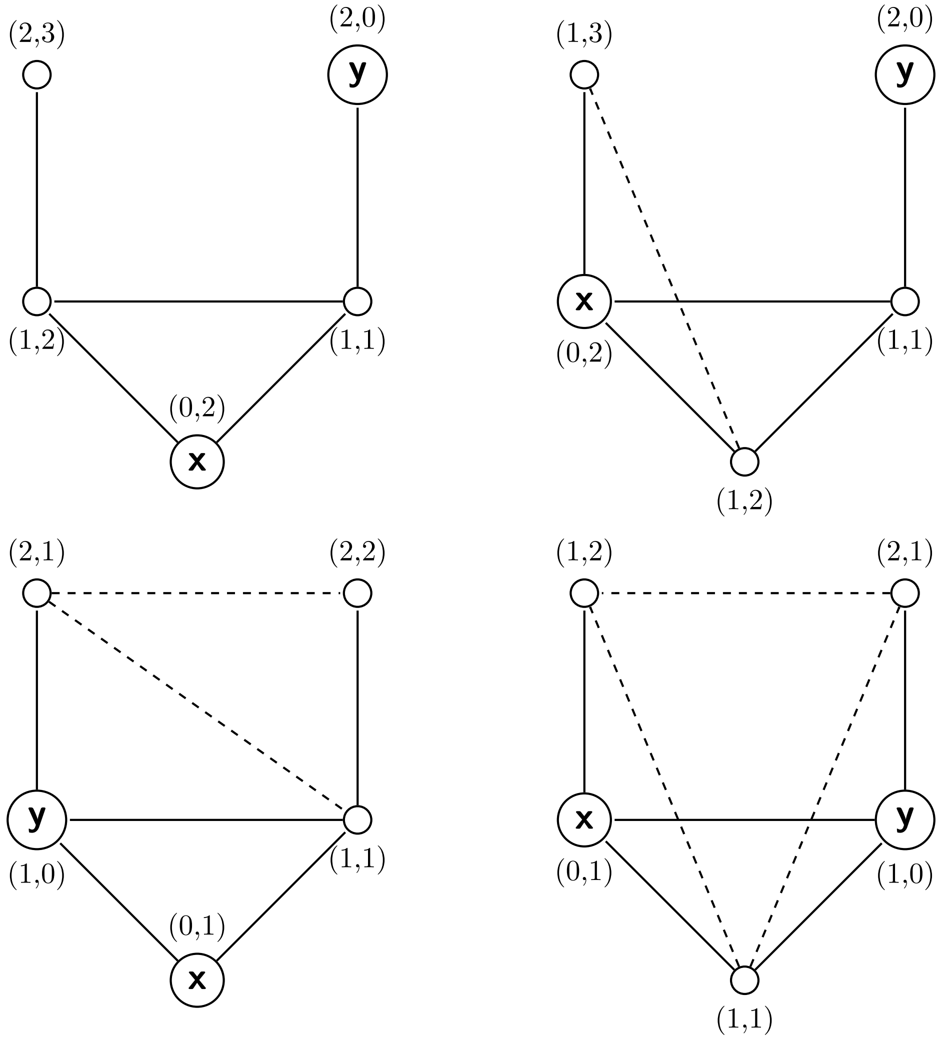
Proposition 2.4. *A graph G is uniquely encodable under W if and only if for all nonadjacent $v_1,\ v_2 \in G$ there exists $w \in W$ such that:*

$$|d(v_1, w) - d(v_2, w)| > 1$$

Proof. Let G be a graph with basis W. Let R be the representation set of G with respect to W. Assume that for some pair of non adjacent vertices $v_1,\ v_2\ \in G,\ \exists\, w \in W$ such that $|d(v_1, w) - d(v_2, w)| \leq 1$. If we construct a graph H from R following the encoding scheme described above, $v_1,\ v_2$ will be adjacent in H. Then $H \neq G$ and so G is not encodable under W.
If for all nonadjacent $v_1,\ v_2 \in G$ there exists $w \in W$ such that $|d(v_1, w) - d(v_2, w)| > 1$, no pair of nonadjacent vertices will be adjacent in H. For every pair of adjacent vertices $v_1,\ v_2$ in G and $w \in W$, $|d(v_1, w) - d(v_2, w)| \leq 1$ and so they will be adjacent in H. Thus G is encodable under W if and only if for all nonadjacent $v_1,\ v_2 \in G$ there exists $w \in W$ such that $|d(v_1, w) - d(v_2, w)| > 1$. $\square$

Advisors: Trevor McGuire, Warren Shreve.

FIGURE 2. The four nonisomorphic bases of the bull graph



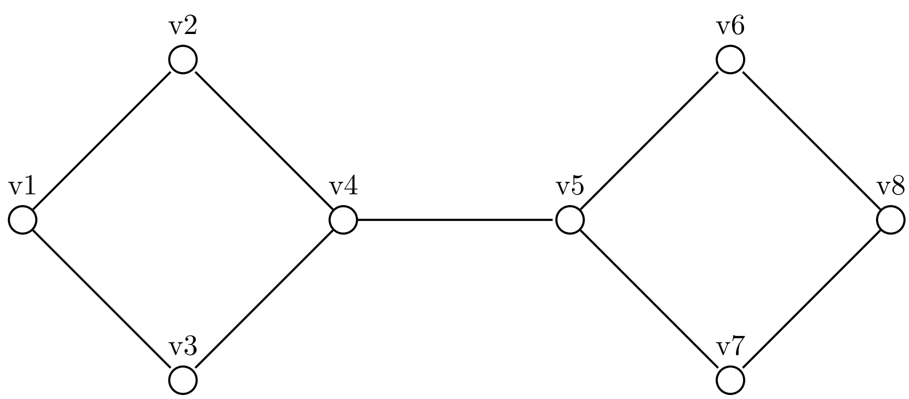
Example 2.5. The bull graph pictured above has basis size 2. There are 4 possible bases for it up to isomorphism but it is W-encodable only under one of them. For the other 3 bases it is possible to add edges (shown dotted) without affecting $r(v/W)$ for any $v \in G$.

3. Properties of bases in graphs with cut vertices

In this section we focus on properties of graphs with cut vertices. Generally the literature for resolving sets focuses on a very specific class of graphs, like wheel graphs or trees. Cut vertices and bridges are common in a very broad range of graphs and so this section focuses on properties of graphs with cut vertices in the hope that these simple tools may lead to the development of more complex tools in the future.

Proposition 3.1. *Let G be a graph with a cut vertex v and resolving set W. Let H be a component of $G - v$. If $G - v$ has more than two components or if for any component L of $G - v$, L is not a path, then $W \not\subseteq H$.*

Proof. Suppose that $W \subset H$. Let v be a cut vertex of G such that $G - v$ has more than 2 components. Since $W \subset H$, the path from any vertex not in H to any element of W must pass through v. Then there exist vertices u_1, u_2 adjacent to v such that $d(u_1, w_i) = d(u_2, w_i) = d(v, w_i) + 1$ for all $w_i \in W$. In this case, $r(u_1/W) = r(u_2/W)$. Then W cannot be not a resolving set for G.
We can show without loss of generality that the same is true for any cut vertex v whenever some component of $G - v$ other than H is not path.
Observe that if G is not a path it contains at least one $deg > 2$ vertex. Let l be a $deg > 2$ vertex in G such that for allu_i such that $deg(u_i) > 2, d(u_i, v) \geq d(l, v)$. Then by the same argument as above we see there are 2 adjacent vertices with the same representation and again W cannot be a resolving set of G. ☐



Example 3.2. k

Theorem 3.3 (Bridge Subdivision Lemma)**.**
If G is a graph with a bridge L separating two non path subraphs which are only connected through L then subdivision of L leaves the bases of G unchanged.

Proof.
Let G be a graph with a bridge (u, v).
Let H, S be the largest connected induced subgraphs of G such that
$u \in H,\ v \notin H, v \in S,\ u \notin S$.
Let L be the uv path separating H and S.
Let W be a basis of G.
Let G', H', S', L', W' be the same objects described above after subdivision of (u, v) with vertex e and the additional constraint that $e \notin H' \cup S'$.

i)If H and S both contain basis elements in G, then W resolves G'.
Suppose that for some basis W of G, W does not resolve G', this can only be true if for some $u_h \in H$, $v_s \in S$

$$r(u_h/W) + (\underbrace{0, \dots}_{n}, 1, \dots) = r(v_s/W) + (\underbrace{1, \dots}_{n}, 0, \dots),$$

where the first n elements of each representation correspond to the distances from elements of W_h up to rearrangement.
Suppose that such a pair of vertices does exist. Then for all $w_h \in W_H \subset H$, $d(v_s, w_h) < d(u_h, w_h)$.
Every $v_s w_h$ path for all w_h passes through L. Then $d(v_s, w_h) = d(v_s, u) + d(u, w_h)$. By the triangle inequality, $d(u_h, w_h) \leq d(u_h, u) + d(u, w_h)$ for all u_h in H. Then $d(v_s, u) < d(u_h, u)$ and consequently $d(v_s, e) < d(u_h, e)$.
Without loss of generality we can show that because for all $w_s \in W_S \subset S$, $d(u_h, w_s) < d(v_s, w_s)$ we must have $d(u_h, e) < d(v_s, e)$. This is a contradiction. Then such a pair of vertices cannot possibly exist and so we have W resolves G'.

ii)If H' and S' both contain basis elements in G', then W' resolves G'.
In this case we see that we must have a pair of vertices $u_h \in H'$, $v_s \in S'$ such that

$$r(u_h/W') + (\underbrace{1, \dots}_{n}, 0, \dots) = r(v_s/W') + (\underbrace{0, \dots}_{n}, 1, \dots),$$

where the first n elements of each representation correspond to the distances from elements of W'_h up to rearrangement. Again we will suppose that such a pair of vertices exists.
Then we can see that for basis elements in H':

$$\begin{aligned}
d(u_h, w_h) + 1 &= d(v_s, w_h)\\
d(u_h, u) + d(u, w_h) + 1 &= d(v_s, u) + d(u, w_h)\\
d(u_h, u) + 1 &= d(v_s, u)\\
d(u_h, u) + 1 &> d(v_s, u) - 1\\
d(u_h, u) + d(u, e) &> d(v_s, u) - d(u, e)\\
d(u_h, e) &> d(v_s, e).
\end{aligned}$$

Without loss of generality, we consider basis elements in S' and notice:

$$d(v_s, e) > d(u_h, e).$$

Then $d(v_s, e) >< d(u_h, e)$, a clear contradiction.
Thus W' resolves G, and so $W = W'$.

□

Example 3.4. j

4. Conclusion

In this paper we introduced several new tools for finding bases of graphs and for describing graphs once a basis has already been found. [3]

References

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Department of Mathematics 2750, North Dakota State University, PO BOX 6050, Fargo, ND 58108-6050, USA

E-mail address: kyleryanmn@gmail.com, trevor.mcguire@ndsu.edu, warren.shreve@ndsu.edu