- ANGLE MAP

  Cos (θ) obtains an interval $[-1, 1]$ with 1 being the desired quantity;

  We must obtain some mapping $f : [-1, 1] \to (-\infty, 0]$ such that $f'[1] = 0$;

  Preferably with a high resolution (slope) near

  1 so that steepness is expounded and convergence is simpler.

- Some candidate wrapper functions,

  where $x = \cos(\theta)$ and k = steepness general and q = steepness specific (close to best)

  Quadratic :

  $$k\,(x-1)^2$$

  Asymptotic (Dangerous pole at $x == 1$) :

  $$\frac{k}{(x-1)^2 - 4}$$

  Sinh :

  $$k\mathrm{Sinh}\left[(x-1)^2\right] / \mathrm{Sinh}[4]$$

  Sharp :

  $$1 - \mathrm{Exp}[-q\,(x-1)\verb|^|2]$$

  To counteract the flatness at the worst condition $(x = -1)$ in the dip function,

  We can combine it with the quadratic function.

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(* Q such that the second derivative of the angle map has only one zero between -
 1 and 1. K is a scaling factor and so does not matter. *)
k = 5;
q = 0.5 E^1.5; (* Smoothest,
with positive concavity and singular critical point.*)
AngMap[x_, k_, q_] := k ((x - 1) ^2 + 1 - Exp[-q (x - 1) ^2]) / 5;
AngMapx[x_] = AngMap[x, k, q];
Plot[{AngMapx[x], -AngMapx'[x]}, {x, -1, 1}, PlotRange → All]

deq =
 DeleteDuplicates@(Simplify[Solve[D[AngMap[x, k, qq], {x, 3}] == 0 && x < 1, x, Reals],
    Assumptions → {qq > 0, x < 1}])
Solve[0 == D[AngMap[x, k, qq], {x, 2}] /. deq, qq] (*Obtain the value of q
  which gives only a single critical point before the minimum at x=1*)
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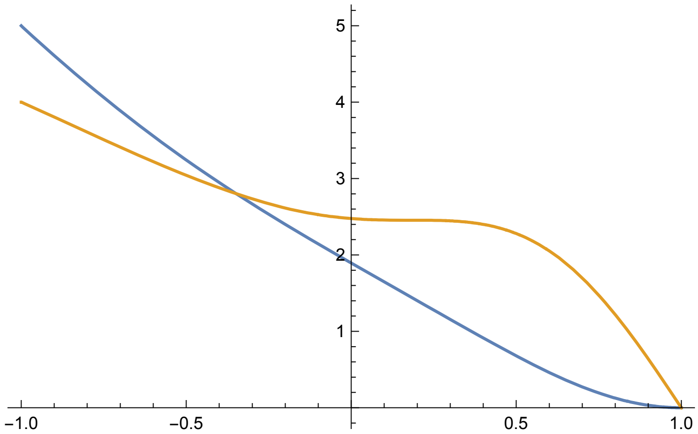

$$\left\{\left\{x \to 1 - \frac{\sqrt{\frac{3}{2}}}{\sqrt{qq}}\right\}\right\}$$

$$\left\{\left\{qq \to \frac{e^{3/2}}{2}\right\}\right\}$$