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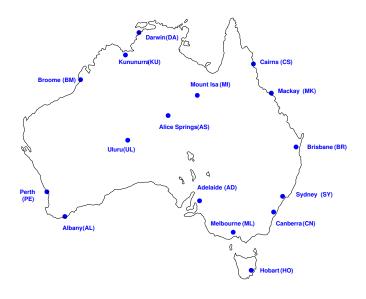
In other words, What if we just want to connect all the vertices together in a network?

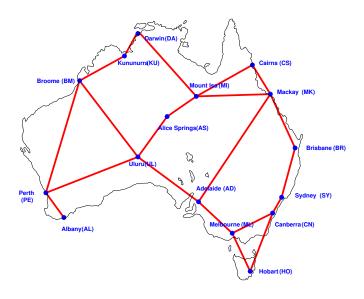
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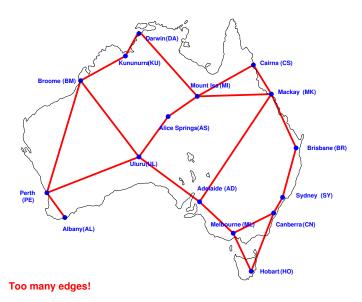
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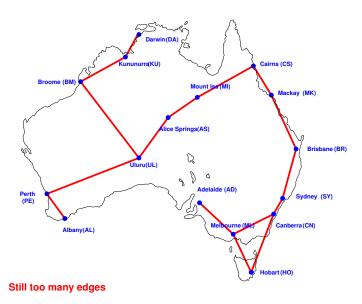
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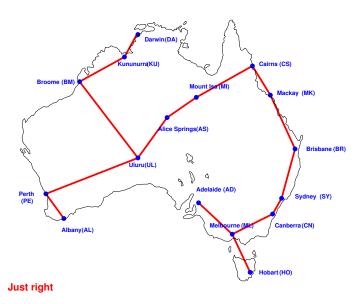
- Roads, railroads
- Telephone lines
- Fiber-optic cable

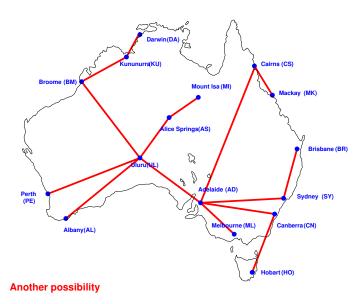


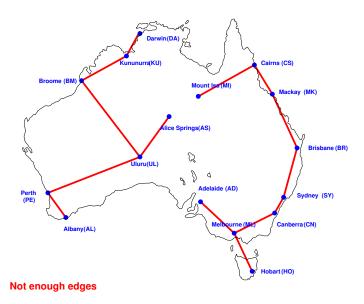












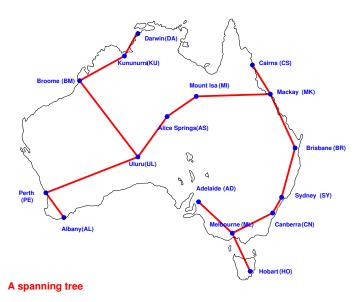
Networks and Spanning Trees

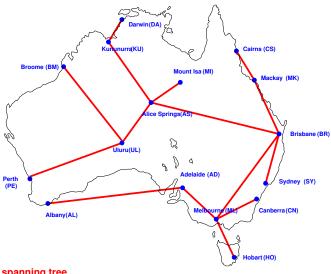
Definition: A **network** is a connected graph.

Definition: A **spanning tree** of a network is a subgraph that

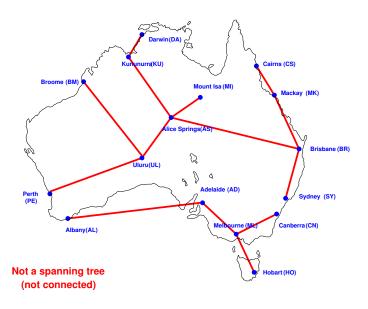
- 1. connects all the vertices together; and
- 2. contains no circuits.

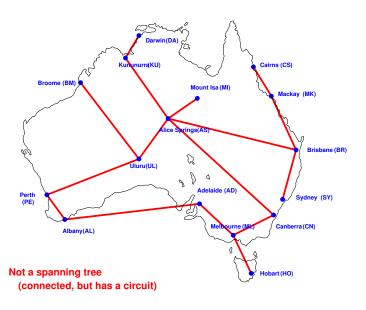
In graph theory terms, a spanning tree is a subgraph that is both **connected** and **acyclic**.

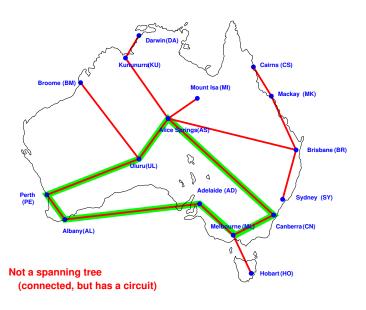




A spanning tree







In a network with N vertices, how many edges does a spanning tree have?



▶ Imagine starting with *N* isolated vertices and adding edges one at a time.

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In a network with ${\cal N}$ vertices, every spanning tree has exactly ${\cal N}-1$ edges.

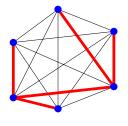
In a network with ${\cal N}$ vertices, every spanning tree has exactly ${\cal N}-1$ edges.

Must every set of N-1 edges form a spanning tree?

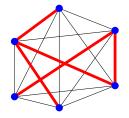


Answer: No.

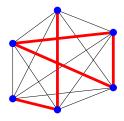
For example, suppose the network is K_4 .



Spanning tree

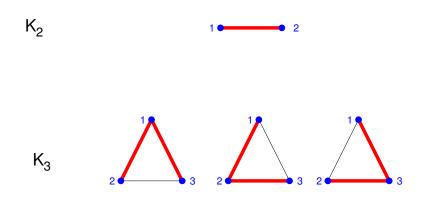


Spanning tree

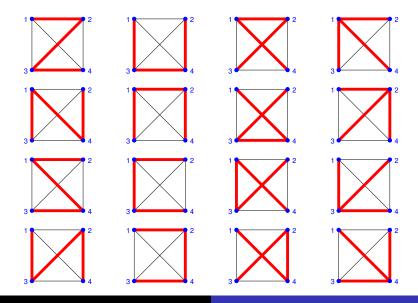


Not a spanning tree

Spanning Trees in K_2 and K_3



Spanning Trees in K_4



Suppose we have a network with N vertices.

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4. In a network with N vertices and M edges,

$$M \ge N - 1$$

(otherwise it couldn't possibly be connected!) That is,

$$M-N+1\geq 0$$
.

The number M-N+1 is called the **redundancy** of the network, denoted by R.

4. In a network with N vertices and M edges,

$$M > N - 1$$

(otherwise it couldn't possibly be connected!) That is,

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.

The number M-N+1 is called the **redundancy** of the network, denoted by R.

5. If R = 0, then the network is itself a tree. If R > 0, then there are usually several spanning trees.

Counting Spanning Trees

We now know that every spanning tree of an N-vertex network has exactly N-1 edges.

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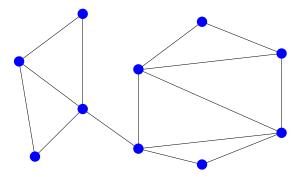
How many different spanning trees are there?

Of course, this answer depends on the network itself.

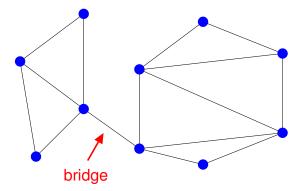
► If an edge of a network is a **loop**, then it is **not in any** spanning tree.

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- If an edge of a network is a bridge, then it must belong to every spanning tree.

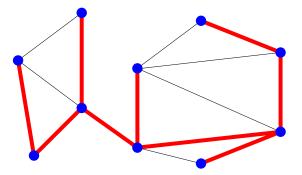
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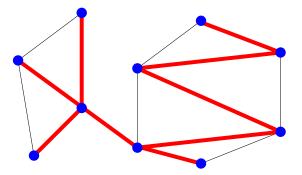
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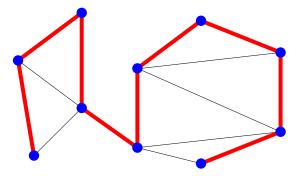
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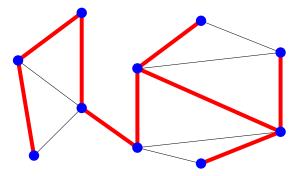
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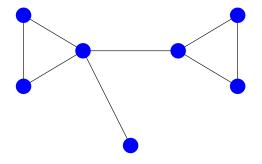
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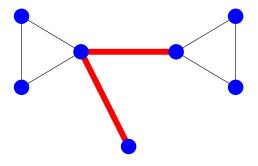
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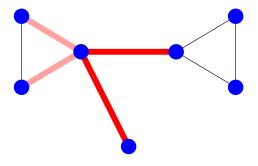
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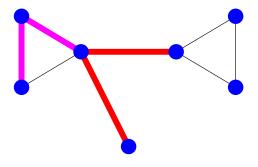
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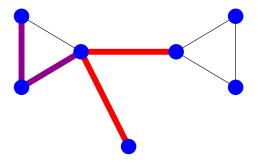
Of course, this answer depends on the network itself.

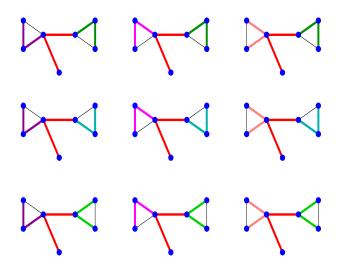


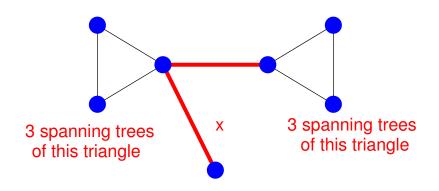




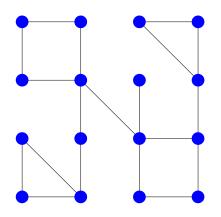




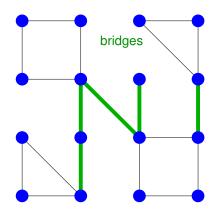


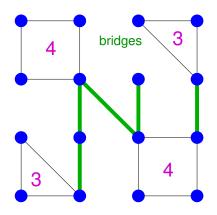


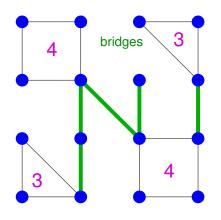
= 9 total spanning trees





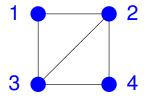




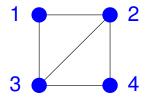


Answer:
$$4 \times 3 \times 3 \times 4 = 144$$
.

If the graph has circuits that overlap, it is trickier to count spanning trees. For example:

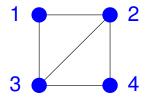


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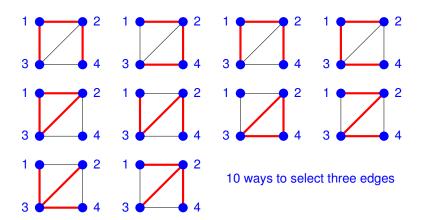


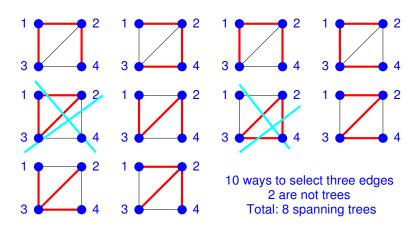
► There are N = 4 vertices \implies every spanning tree has N - 1 = 3 edges.

If the graph has circuits that overlap, it is trickier to count spanning trees. For example:



- ► There are N = 4 vertices \implies every spanning tree has N 1 = 3 edges.
- ▶ List all the sets of three edges and cross out the ones that are not spanning trees.





The Number of Spanning Trees of K_N (Not in Tannenbaum!)

Since K_N has N vertices, we know that every spanning tree of K_N has N-1 edges.

How many different spanning trees are there?



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We have already seen the answers for K_2 , K_3 , and K_4 .

Number of vertices (N)	Number of spanning trees in K_N
2	1
3	3
4	16
•	10

Number of spanning trees in K_N
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1
3
16

Number of spanning trees in K_N
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1
3
16
125

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What's the pattern?



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Number of spanning trees in K_N
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1
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$16 = 4^2$
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Cayley's Formula:

The number of spanning trees in K_N is N^{N-2} .

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The number of spanning trees in K_N is N^{N-2} .

For example, K_{16} (the Australia graph!) has

$$16^{14} = 72,057,594,037,927,936$$

spanning trees.

(By comparison, the number of Hamilton circuits is "only"

$$15! = 1,307,674,368,000.$$