1) Counter example:

Assume $n^2 - 79n + 1601$ where $n \in \mathbb{N}$ will always produce a prime number is true

$n = 1 \rightarrow (1)^2 - 79(1) + 1601 = 1523$ ✓

$n = 2 \rightarrow (2)^2 - 79(2) + 1601 = 1447$ ✓

$\vdots \qquad \vdots \qquad \vdots$

$n = 1601 \rightarrow (1601)^2 - 79(1601) + 1601 = 1681$ ×

1681 is not prime as it can be divided by 41

As 1681 can be divided by another number other than 1681 and 1, the assumption can be concluded to be false so $n^2 - 79n + 1601$ does not always produce a prime number

2) $a_n = 1 + \prod_{k=1}^{n} p_k$

$a_1 (3)$, $a_2 (7)$, $a_3 (31)$, $a_4 (211)$, $a_5 (2311)$ all prime

$a_n = 1 + \prod_{k-1}^{n} p_k$

if $n = 6$: $a_6 = 1 + (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13)$

$= 1 + 30031$

$= 30031$

$$\boxed{\text{30031 is divisible by 59 and 509 so it is not prime. Hence } k^{th} \text{ prime is not always prime.}}$$

3) Fibonacci numbers: $a_n = a_{n-1} + a_{n-2}$ $\quad n > 1 \; n \in \mathbb{N}$

0 1 1 2 3 5 8 13 21 34 55 89 144

233 377

$$\boxed{\begin{array}{l} a_{14} = a_{13} + a_{12} = 233 + 144 = 377 \\ \text{so there is a fibonacci number that ends} \\ \text{in digit } 7 \end{array}}$$

4) if $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$ then $x \cdot y \in \mathbb{Q}$

Rational $= \frac{a}{b}$, a & $b \in \mathbb{Z}$, $\frac{a}{b}$ irreducible

$x = \sqrt{3}$ is not $\in \mathbb{Q}$

$y = \frac{1}{\sqrt{3}}$ is not $\in \mathbb{Q}$

$x \cdot y = \sqrt{3} \cdot \frac{1}{\sqrt{3}} = 1$ whis is $\in \mathbb{Q}$

As a result, there exists two irrational numbers that when multiplied produces rational

5) Prove : $\underbrace{x^4 - x - 1 = 0}$ has more than one solution

existential statement
can be proved by
constructive or non-constructive

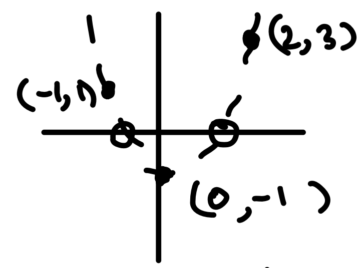
$x^4 - x - 1 = 0$ is continuous

A root is when $y = 0$. To show $x^4 - x - 1 = 0$ has more than one solution, we can prove that for certain x values, the y values would be positive, negative then positive again which would mean the graph is crossing $y=0$ line as known by intermediate value theorem.

$x = -1 \rightarrow (-1)^4 - (-1) - 1 = 1$

$x = 0 \rightarrow (0)^4 - (0) - 1 = -1$

$x = 2 \rightarrow (2)^4 - (2) - 1 = 13$



• = points

o = solutions

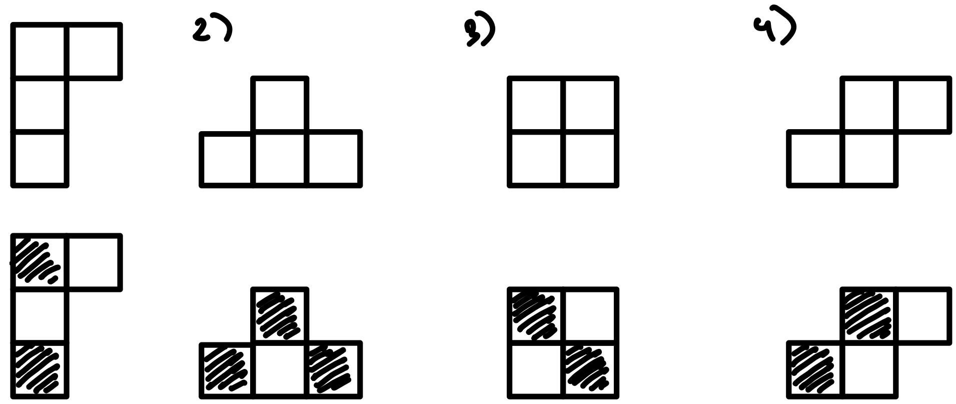
$$\boxed{\text{As a result, this implies there are more than one solution to } x^4 - x - 1 = 0}$$

6) $x = \sqrt{3}^{\sqrt{2}} \to y$

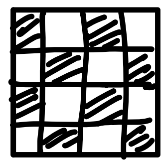
$$x^{y} = \left(\sqrt{3}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{3}\right)^{\sqrt{2}\cdot\sqrt{2}} = \left(\sqrt{3}\right)^{2} = 3$$

$\sqrt{3}$ is irrational (x), $\sqrt{2}$ is irrational (y)
but x^{y} $\left(\sqrt{3}^{\sqrt{2}}\right)^{\sqrt{2}}$ is rational as $3 \in \mathbb{Q}$.
As a result, there exists two irrational numbers x any y such that x^{y} is rational

7) 1) 2) 3) 4)



	1)	2)	3)	4)
# of white	2	1	2	2
# of black	2	3	2	2



A perfect square should be 4x4 with 16 squares and # of black squares = # of white squares. The second shape will always have either 3 black squares and one white square or 1 black square and three white squares. The other shapes have equal amounts of black, white squares but shape 2 does not. As the # of black squares are not equal to the # of white squares, these 4 shapes cannot be made into a perfect square

8) $\forall r \in \mathbb{R} - \{1\}, \frac{r}{r-1} \notin \mathbb{Q} \Rightarrow r \notin \mathbb{Q}$

contrapositive: double flip $\quad r \in \mathbb{Q} \Rightarrow \frac{r}{r-1} \in \mathbb{Q}$

$r = \frac{x}{y}$; irreducible; [if r is rational] x, y integers; y cannot be 0

$r \in \mathbb{Q} \Rightarrow r = \frac{a}{b}$ (sub into $\frac{r}{r-1}$)

$$\frac{\frac{a}{b}}{\frac{a}{b} - 1} = \frac{a}{a-b}$$

$$\boxed{\text{We know } a \neq b \text{ as } r \neq 1 \text{ so } \frac{a}{a-b} \in \mathbb{Q}}$$

$$\frac{r}{r-1} = \frac{\frac{x}{y}}{\frac{x}{y} - 1} = \frac{x}{y}\left(\frac{1}{\frac{x}{y} - 1}\right) = \frac{x \cdot 1}{y \cdot \left(\frac{x}{y} - 1\right)} = \underbrace{\frac{x}{x-y}}$$

for this to be $\mathbb{Q}$, (by definition) $x, x-y \in \mathbb{Z}$ ✓

$\underbrace{x - y \neq 0}$, prove by contradiction

$x - y = 0$

$x = y$

Then r is $\frac{x}{y}$ so $\frac{\text{some thing}}{\text{some thing}} = 1$

But the prompt said $\forall r \in \mathbb{R} - \{1\}$ so we just proved the statement

Reverse implication

$r \notin \mathbb{Q} \Rightarrow \frac{r}{r-1} \notin \mathbb{Q}$

$\frac{r}{r-1} = \frac{x}{y}$; $x, y \in \mathbb{Z}$; $y \neq 0$

$ry = x(r-1)$

$ry = rx - x$

$ry - rx = -x$

$r(y-x) = -x$

$r = \frac{-x}{y-x}$

$y - x$ cant be 0 but lets see if it $\quad y - x = 0 \quad y = x$

$\frac{r}{r-1} = \frac{x}{y}$

$\frac{r}{r-1} = \frac{x}{x}$ $(y = x)$

$\frac{r}{r-1} = 1 \rightarrow$ contradiction

$$\boxed{\text{So since } R \in \mathbb{Q} \text{ is true and true} \Rightarrow \text{true is true, statment is true even when reversed}}$$

9) prop: $\forall n \in \mathbb{N}$, n is even $\Rightarrow \binom{n}{3}$ is even

Hint: if $\frac{2x}{3}$ is an integer then $\frac{x}{3}$ is an integer because 2 and 3 have no common factors

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Proof:

1) n is even (given)
2) $n = 2k,\ k \in \mathbb{Z}$ (definition of being even)
3) $\binom{n}{3}$ (expression made from given information)
4) $\binom{n}{3} = \dfrac{n!}{3!(n-3)!}$ (definition of n choose k)
5) $\binom{n}{3} = \dfrac{n \cdot (n-1) \cdot (n-2)\cancel{(n-3)!}}{3!\ \cancel{(n-3)!}}$ (definition of factorial)
6) $\binom{n}{3} = \dfrac{n \cdot n-1 \cdot n-2}{3 \cdot 2 \cdot 1}$ (factorial expansion of 3!)
7) $\binom{n}{3} = \dfrac{2k \cdot 2k-1 \cdot 2k-2}{3 \cdot 2}$ (substitution of 2k as n)
8) $\binom{n}{3} = \dfrac{2 \cdot k \cdot 2k-1 \cdot \cancel{2}(k-1)}{3 \cdot \cancel{2}}$
9) $\binom{n}{3} = \dfrac{2 \cdot k \cdot 2k-1 \cdot k-1}{3}$
10) $\binom{n}{3} = 2\underbrace{\left(\dfrac{(k)(2k-1)(k-1)}{3}\right)}_{\text{this is } \in \mathbb{N}}$

2 times something from $\mathbb{N}$ is even and this follows 2k form so $\binom{n}{3}$ is indeed even