1) n = different type of coupons

c = identical coupons

You want to guarantee c identical coupons

If there were 4 different coupons (pigeonholes) and you want to collect atleast 3 identical ones (pigeons) you'd assume the worst case scenario where your first two passes are each of different type. As a result, the third pass you make will guarantee 3 identical ones no matter what you choose.

	A	B	C	D	first pass, 4 diff coupons, you chose all diff ones
	A A	B B	C C	D D	second pass, 4 diff coupons, you chose all diff ones again
$m =$	A A	B B B	C C	D D	third pass, no matter what coupon you choose, it'll guarantee you have 3 identical ones

m is # of things you chose in that last pass so counting all the letters gives you 9.

$$c = \left\lfloor \frac{m-1}{n} \right\rfloor + 1$$

$$c - 1 = \left\lfloor \frac{m-1}{n} \right\rfloor$$

$$n(c-1) = m - 1$$

$$n(c-1) + 1 = m$$

$$\boxed{f(m) = n(c-1)+1}$$

2) - Matching pair of socks like c lor perhaps, white + white = good pair

Assume worst case scenario where you picked all differnt colored socks first. After placing all these n socks into their boxes, the very next pair you pick will be a good pair thus n + 1. If you exhaust all colored socks → ...
but then you pick another one and as there are no other colors left so
→ ... Since the very next sock you choose will be a colored sock you already chose before, no matter the color it will be the same colored pair thus guaranteeing atleast 1 correct pair so $\underbrace{n}_{\#\text{ of passes}}+\underbrace{1}_{\#\text{ you guarantee}}$

- Just any pair that has one right, one left sock regardless of color so left + right = good pair

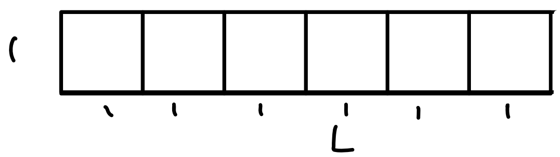
Assume worst case scenario where you picked all left or all right socks first. After placing n left or n right socks in their boxes, the very next pair you pick will be a good pair thus n + 1. If you exhaust all left socks → ...
but then you pick a right one as there are no left ones left so
→ ... Since the very next sock you choose will be a right sock, no matter what leftsock you choose, it will be 1 left and 1 right thus guaranteeing atleast 1 good pair so $\underbrace{n}_{\#\text{ of passes}}+\underbrace{1}_{\#\text{ you guarantee}}$

$\boxed{\text{In both cases, blindy grab } n+1 \text{ socks to guarantee a good pair}}$

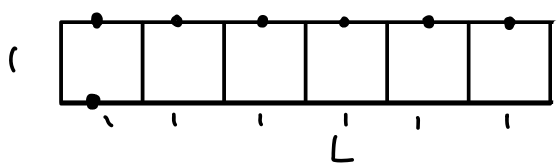
3) $1 \times L$ rectangle and we place $L+1$ points on the perimeter.
2 points must be within a distance of $\sqrt{2}$

rectangle

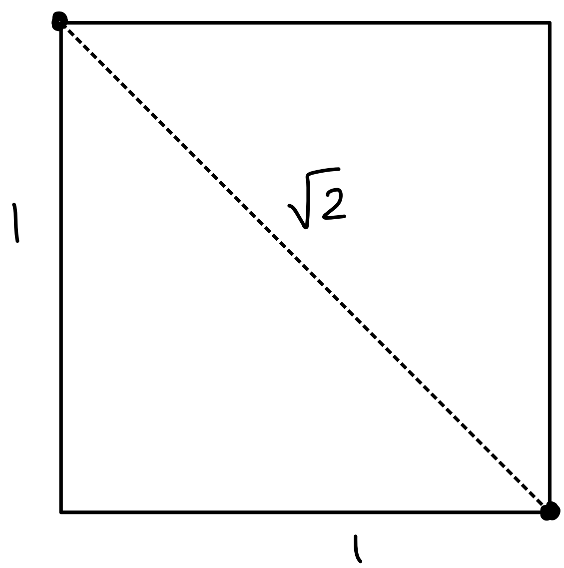
We can divide this into 1×1 squares



so in this example $L = 6$ and prompt says to place $L+1$ points on the perimeter



if L points are placed on the perimeter, each square will house one point. But $L+1$ points means atleast one square will house 2 points. If a 1×1 square contains two points, the greatest distance between them would be when they are on opposing corners which is equal to the length of the squares diagonal and according to the pythagorean theorem, $a^2 + b^2 = c^2$ where a and b are the sides and c is the hypotenuse.



$\boxed{\text{So if you place } L+1 \text{ points on a } 1 \times L \text{ rectangle, 2 points must be within a distance of } \sqrt{2}}$

4) - 40 Pokemons

- 18 sky
- 23 ground
- ? water
- 9 sky and ground
- 7 sky and water
- 12 ground and water
- 4 ground and sky and water

Total

$|S|$

$|G|$

$|W|$

$|S \wedge G|$

$|S \wedge W|$

$|G \wedge W|$

$|S \wedge G \wedge W|$

$$\text{Total} = |S| + |G| + |W| - |S \wedge G| - |S \wedge W| - |G \wedge W| + |S \wedge G \wedge W|$$

$$40 = 18 + 23 + |W| - 9 - 7 - 12 + 4$$

$$40 = 17 + |W|$$

$$40 - 17 = |W|$$

$$23 = |W|$$

$\boxed{\text{23 pokemons like to fight in the water!}}$

5) Binary word length = 70 consisting of 0s and 1s
Atleast 2 occurences of a sequence of 6 bits.

There are 6 digits in a bit
Each digit has 2 choices it can either be a 0 or 1

$$2^6 = 64$$

Worst case scenario = The first 64 6-bit sequences are all different so first = $\underbrace{\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot}$ but the second bit starts from first bit = $\underbrace{\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot}$ and so on

If the first 64 6 bit sequences are all diffeent then the 65^{th} sequence is bound to be a repeat of the already creted 64 possible sequences.

6) - Total = 20

- Boys = 12
- Glasses = 13
- Girls = Total - Boys = 20 - 12 = 8
- Boys with glasses = 2 (Girls with no glasses)

= Boys with glasses = 2 (total girls - girls with glasses)

= Boys with glasses = 2 (8 - girls with glasses)

Glasses = boys with glasses + girls with glasses

13 = 2(8 - girls with glasses) + girls with glasses

13 = 16 - 2 girls with glasses + girls with glasses

13 - 16 = -1 girls with glasses

-3 = -1 girls with glasses

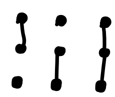
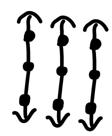
$\frac{-3}{-1}$ = girls with glasses

$\boxed{3 = \text{girls with glasses}}$

7) $n \in \mathbb{N}$, grid contains $3n$ points

Choose $2n+1$ points and when you connect them the total amount of lines will be less than $n(2n+1)$

We are choosing 2 points from $2n+1$ so $\binom{2n+1}{2} = \frac{(2n+1)(2n)}{2}$
$= n(2n+1)$

If $n = 3$ and the graph contains $3n$ points, theres $3(1)$ points.
Out of the 3 points you are making $2n+1$ lines so $2(1)+1=3$
These three lines would be  but these lines extend infinitely. so it'd be  This means that the points are colinear. Choosing $2n+1$ points means you are going around 2 times placing points in n rows and $+1$ and pigeonhole says the very next point you place in any rows will mean you have 3 colinear points in one row thus a repeating line.

$$\boxed{\text{As colinear points result in repeat lines, connecting } 2n+1 \text{ points will always be less than } 2(2n+1)}$$

8a) $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

Base case = $P(0)$ is true

Inductive step = for all $k \geq 0$, $P(k) \Rightarrow P(k+2)$

This would not constitute a valid proof that $P(n)$ is true for all $n \in \mathbb{N}$ as it stops defining true at $P(1)$. If $\mathbb{N}$ starts from 0 and its $k+2$ as $P(0) \Rightarrow P(0+2) = P(2)$ is all good but then we move on to $P(1) \Rightarrow$??. We cannot do anything as the base case only defines $P(0)$ as true and we don't know what $P(1)$ implies. So to fix this issue we should change the base case to $P(0) \wedge P(1)$ is true as this would allow us to constitute that for all $n \in \mathbb{N}$, $P(n)$ is true.

$P(0) \Rightarrow P(0+2) = P(2)$ ✓

$P(1) \Rightarrow P(1+2) = P(3)$ ×

$P(0)$ is proven so we can deduce $P(2)$ and then $P(2) = P(4)$ and so on. However we don't know if $P(1)$ is true or not so we can't prove $P(3)$ which means we can't prove $P(3+2) = P(5)$ and so on. As all the P(odd number) values remain unproven, to rectify the situation we would simply change the base case to being that

$\boxed{P(0) \text{ and } P(1) \text{ are both true}}$

8b) Prove for all $n \in \mathbb{N}$, $\underbrace{\sum_{i=0}^{n}(4i+1) = 2n^2+3n+1}_{P(k) \text{ claim (usually after } n \in \mathbb{N})}$

Base Case: $(n=0)$

$$\sum_{i=0}^{n}(4i+1) = 2n^2+3n+1$$

$$\sum_{i=0}^{0}(4\cdot 0+1) \mid 2(0)^2+3(0)+1$$

$$1 = 1 \checkmark$$

Inductive step:

$P(k) \Rightarrow P(k+1)$, $k \in \mathbb{Z} \geq 0$ → starting point is $\geq$ the thing you subbed in as n in base

Inductive hypothesis:

$P(k): \sum_{i=0}^{k}(4i+1) = 2k^2+3k+1$ → replace n's with k's

$P(k+1): \sum_{i=0}^{k+1}(4i+1) = 2(k+1)^2+3(k+1)+1$

1) $P(k+1): \sum_{i=0}^{k+1}(4i+1) = 2(k+1)^2+3(k+1)+1$

2) $P(k+1): \sum_{i=0}^{k+1}(4i+1)$

3) $P(k+1): \sum_{i=0}^{k}(4i+1) + 4(k+1)+1$

4) $P(k+1): 2k^2+3k+1 + 4k+4+1$

5) $P(k+1): 2k^2+7k+6$

6) $P(k+1): 2k^2+4k+2+3k+3+1$

7) $P(k+1): 2(k^2+2k+1)+3k+3+1$

8) $P(k+1): 2k^2+7k+6$

* We know what $\sum_{i=0}^{k}(4i+1)$ equals from our inductive hypothesis so we sub it in

* We wanna show $2k^2+7k+6$ equals the thing given in the prompt

$$\boxed{\text{So, for all } n \in \mathbb{N}, \sum_{i=0}^{n}(4i+1) = 2n^2+3n+1}$$

8c) For all $n \in \mathbb{N}$, $\underbrace{4^n - 1 = \text{a multiple of } 3}_{P(k)\ \text{claim (usually after } n \in \mathbb{N})}$

Proof by induction general steps: State $P(n)$ [claim]

Prove Base case

↳ $P(0)$

Prove Inductive step

↳ $P(k) \Rightarrow P(k+1)$ with k domain starting from base case end

↳ state $P(k)$ from claim and $P(k+1)$

↳ come up with inductive hypothesis

↳ Prove something about $P(k+1)$

$P(n) = 4^n - 1$ is a multiple of 3

Base case: $(n = 0)$

$P(0) = 4^0 - 1$ is a multiple of 3

$= 1 - 1$ is a multiple of 3

$= 0$ is a multiple of 3 which is true as $3 \times 0 = 0$ and any integer multiplied by 3 is a multiple of 3 itself ✓

Inductive step:

$P(k) \Rightarrow P(k+1)$, $k \in \mathbb{Z} \geq 0$ → starting point is $\geq$ the thing you subbed in as n in base

Inductive hypothesis:

$P(k)$: $4^k - 1$ is a multiple of 3 → replace n's with k's

$P(k+1)$: $4^{k+1} - 1$

1) $P(k+1)$: $4^{k+1} - 1$

2) $P(k+1)$: $(4^k \cdot 4) - 1$

3) $P(k+1)$: $(4^k)(3+1) - 1$

4) $P(k+1)$: $3 \cdot 4^k + \underline{4^k - 1}$

5) $P(k+1)$: $3 \cdot 4^k + 3b$

6) $P(k+1)$: $3(4^k + b)$

7) $P(k+1)$: $3(m)$

* $4^k - 1$ is true as shown in inductive hypothesis $P(k)$: $4^k - 1$. So $4^k - 1$ is a multiple of 3 so we can call $4^k - 1$ as b and since its a multiple of 3, $3b$ where $b \in \mathbb{Z}$

* $4^k + b$ can just be represented as $m \in \mathbb{Z}$ as k and b are $\mathbb{Z}$ and so is 4.

If $P(k+1)$ is equal to $4^{k+1} - 1$ and is also equal to $3m$, $4^{k+1} - 1 = 3m$ which means its a multiple of 3 as anything multiplied by 3 is also a multiple of 3. As a result, for all $n \in \mathbb{N}$, $4^n - 1$ is a multiple of 3.