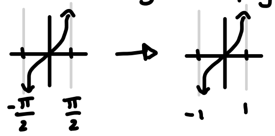
1) Find function $f:(-1,1) \to \mathbb{R}$ thats onto and one to one

$f(x) = \tan x$ has domain $(-\frac{\pi}{2}, \frac{\pi}{2})$ but question said to find domain from $(-1,1)$ so we increase it by multiplying x by $\frac{\pi}{2}$ so now, $f(x) = \tan(\frac{\pi}{2}x)$ where domain is $(-1,1)$



Now we have to show its onto and one to one to show its a bijection:

Onto

$\forall y \in \mathbb{R} \Rightarrow \exists x, f(x) = y, -1 \le x \le 1$ $\tan(\frac{\pi}{2}y) = x$

$\tan(\frac{\pi}{2}x) = y$ [switch x & y]

$\frac{\pi}{2}y = \tan^{-1}(x)$

$y = \frac{2}{\pi}\tan^{-1}(x)$

$x = \frac{2}{\pi}\tan^{-1}(y)$ [switch x & y]

$\frac{2}{\pi}\tan^{-1}$ domain is $(-1,1)$ so f is onto!

One to one

$f(x_1) = f(x_2) \to x_1 = x_2$

$\tan(\frac{\pi}{2}x_1) = \tan(\frac{\pi}{2}x_2)$

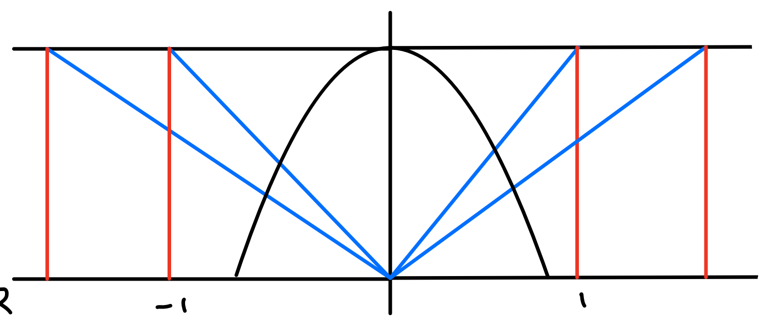
$\frac{\pi}{2}x_1 = \frac{\pi}{2}x_2$

$x_1 = x_2$

This means that f is one to one!

Since the domain of f is $(-1,1)$ and its onto and one to one, it's a bijection

y = 1



y = 0 ℝ

— = draw secont line from origin, projecting to every point of semicircle and hit points on the y=1 line, extending and maintaining some projection

— = draw line vertically down to hit $\mathbb{R}$, y=0 line

Since every point on $\mathbb{R}$ can map to a point on semicircle uniquely and each point on the semi circle can uniquely map back down to one unique $\mathbb{R}$, its one to one and onto, hence a bijection!

2) $P_F(\mathbb{N})$ is all finite subsets of $\mathbb{N}$

Every finite subset has a largest element.

# of subsets where 1 is the largest number = 1 $\underbrace{\{1\}}_{1}$

# of subsets where 2 is the largest number = 2 $\underbrace{\{2\}}_{1}, \underbrace{\{1,2\}}_{2}$ (disregard other sets as 2 is not largest # in there)

# of subsets where 4 is the largest number = 4 $\underbrace{\{3\}}_{1}, \underbrace{\{1,3\}}_{2}, \underbrace{\{2,3\}}_{3}, \underbrace{\{1,2,3\}}_{4}$

# of subsets where 8 is the largest number = 8 $\underbrace{\{4\}}_{1}, \underbrace{\{1,4\}}_{2}, \underbrace{\{2,4\}}_{3}, \underbrace{\{3,4\}}_{4}, \underbrace{\{1,2,4\}}_{5}, \underbrace{\{1,3,4\}}_{6}$
$\underbrace{\{2,3,4\}}_{7}, \underbrace{\{1,2,3,4\}}_{8}$

We can observe how the # of subsets where i is the largest number is always 2^{i-1}

$P_F(\mathbb{N}) = \{x \in P(\mathbb{N}) \mid x \text{ is finite}\}$

$P_F(\mathbb{N})$ is countable if it can be ordered.

Largest element in $P_F(\mathbb{N})$ is i

# of numbers less than i is $i-1$

Size of power set is defined as 2^x

Since we can order the elements in $P_F(\mathbb{N})$ like

$\{2^0, 2^1, 2^2, 2^3, \ldots 2^{i-1}\}$, $P_F(\mathbb{N})$ is countable!

3)

infinity binary word	subset of $\mathbb{N}$
0000000000...	$\emptyset$
1000000000...	$\{1\}$
011 1000000...	$\{2,3,4\}$
0101 010000...	$\{2,4,6\}$
1010 101010...	$\{1,3,5,7,9\}$ or $\{2k-1 \mid k \in \mathbb{N}\}$
1001 001001...	$\{3k-2 \mid k \in \mathbb{N}\}$
1111 111111 ...	$\mathbb{N}$

$\rightarrow 3k-2$

$k=1 = 3(1)-2 = 1$
$k=2 = 3(2)-2 = 4$
$k=3 = 3(3)-2 = 7$

$\left.\begin{matrix} 1 \\ 4 \\ 7 \end{matrix}\right\} 3,\ 3 = \underbrace{1001001\ldots}_{3\ 3}$

every 3, there is a 1

- An infinite word is made of 0's and 1's
- Cantors diagonal theorem is only used when you want to show something is uncountable
- The diagonal is made by choosing the i th element of the i th row like bit 9 in row 9 etc
- Making an infnte word out of the numbers the diagonal hits:

$S_n = 0011101$

- A complement of something is its opposite thing so 1 turs to 0, 0 turns into 1 so:

$S_n^c = 11\ 00010$

We know an infinite binary word is made up of 0's and 1's so S_n^c is an infinite binary word. The table shows all the infinite binary words we can make. However, the complement of S_n, S_n^c will never show up there as atleast one digit will always not match from the ones given in the table. If something is a bijection, $|$subset of $\mathbb{N}| = |$infinite binary word$|$. Since S_n^c is an infinite word that is not in subset of $\mathbb{N}$, $|$subset of $\mathbb{N}| \neq |$infinite binary word$|$ so no bijection. So $P(\mathbb{N})$ is uncountable

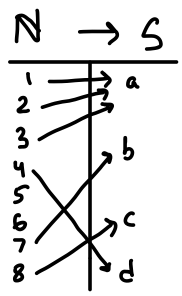
4) $f: \mathbb{N} \to S$ is onto so all y's have an x. So all elements of S have an element of $\mathbb{N}$ that is mapped onto S

$f: \mathbb{N} \to S$; f is onto

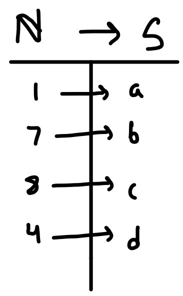
↓

S has to be countable

Something is countable if it is a bijection. A bijection is when something is onto and 1 to 1. We can make something by restricting the domain. Like if 2 x values have the same y, choose the smallest or largest one. So,



This is onto as all y values are used up but to make it one to one, we make each input tied to one output. So choose min or max from $\mathbb{N}$ when they have same y so:



1, 2, 3 all had output b, min is 1 so 1 has a. Repeat this for others and now the function is onto + one to one so a bijection

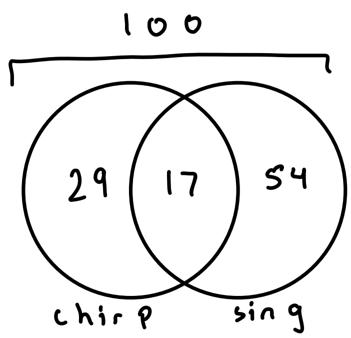
$$\boxed{\text{Now that we limited the domain, every element of } \mathbb{N} \text{ maps to one element of } S \text{ which we can call } \mathbb{N}'. \text{ So } f: \mathbb{N}' \to S \text{ is a bijection which means } S \text{ is countable}}$$

5) Total birds = 100

Chirping birds = 46

Chirping + singing birds = 17

Singing birds = ?



chirp + sing = 46

only chirp = 46 − 17 = 29

total sing = total − chirp

= 100 − 29

= 71

$n = 100$

chirp = c = 46

chirp $\wedge$ sing = $c \wedge s$ = 17

sing s = ?

without using a venn diagram we can use the inclusion-exclusion way

$$n = |c| + |s| - |c \wedge s|$$

$$100 = 46 + |s| - 17$$

$$100 = 29 + |s|$$

$$100 - 29 = |s|$$

$$71 = |s|$$

$\boxed{71}$

6) Total amount of numbers $= 546$

Negation = divisible by 2: $\frac{x}{2} = a \geq 1$ S_2 = divisible by 2

divisible by 3: $\frac{x}{3} = b \geq 1$ S_3 = divisible by 3

divisible by 7: $\frac{x}{7} = c \geq 1$ S_7 = divisible by 7

$$|S_2| = \left\lfloor \frac{546}{2} \right\rfloor = 273$$

$$|S_3| = \left\lfloor \frac{546}{3} \right\rfloor = 182$$

$$|S_7| = \left\lfloor \frac{546}{7} \right\rfloor = 78$$

$$|S_2 \wedge S_3| = \left\lfloor \frac{546}{2 \cdot 3} \right\rfloor = \left\lfloor \frac{546}{6} \right\rfloor = 91$$

$$|S_2 \wedge S_7| = \left\lfloor \frac{546}{2 \cdot 7} \right\rfloor = \left\lfloor \frac{546}{14} \right\rfloor = 39$$

$$|S_3 \wedge S_7| = \left\lfloor \frac{546}{3 \cdot 7} \right\rfloor = \left\lfloor \frac{546}{21} \right\rfloor = 26$$

$$|S_2 \wedge S_3 \wedge S_7| = \left\lfloor \frac{546}{2 \cdot 3 \cdot 7} \right\rfloor = 13$$

Inclusion - exclusion $= |S_2 \vee S_3 \vee S_7| = |S_2| + |S_3| + |S_7|$

$$- |S_2 \wedge S_3| - |S_2 \wedge S_7| - |S_3 \wedge S_7|$$

$$+ |S_2 \wedge S_3 \wedge S_7|$$

$$= 273 + 182 + 78$$

$$- 91 - 39 - 26$$

$$+ 13$$

$$= 390$$

390 is # of numbers divisible by 2,3 and 7 but we want the ones not divisible by 2,3 and 7 so we subtract that from the total which is 546. So 546 - 390 is 156

7a) _ _ _

8 choices because it cant be 0 (if its 0, itd be a 2 digit #)
it cant be 1 (prompt said no 1 first digit)

9 choices becase it cant be 2 (prompt said no 2 second digit)

9 choices becase it cant be 3 (prompt said no 3 third digit)

$\boxed{\text{Product rule} = 8 \cdot 9 \cdot 9 = 648}$

b) Total if prompt had no restrictions (1 can be 1^{st}, 2 can be 2^{nd}, $3 = 3^{rd}$)

$9 \cdot 10 \cdot 10 = 900$

		choice 1	choice 2	choice 3
Negation: $\lvert B_1 \rvert$	= numbers with 1 first digit =	($1 \cdot$	$10 \cdot$	10)
$\lvert B_2 \rvert$	= numbers with 2 second digit =	($9 \cdot$	$1 \cdot$	10)
$\lvert B_3 \rvert$	= numbers with 3 third digit =	($9 \cdot$	$10 \cdot$	1)
$\lvert B_1 \cap B_2 \rvert$	= numbers with 1 first digit and 2 second digit =	($1 \cdot$	$1 \cdot$	10)
$\lvert B_2 \cap B_3 \rvert$	= numbers with 2 second digit and 3 third digit =	($9 \cdot$	$1 \cdot$	1)
$\lvert B_3 \cap B_1 \rvert$	= numbers with 3 third digit and 1 first digit =	($1 \cdot$	$10 \cdot$	1)
$\lvert B_1 \cap B_2 \cap B_3 \rvert$	= numbers with 1 first digit 2 second digit and 3 third digit =	($1 \cdot$	$1 \cdot$	1)

Inclusion-exclusion: $|B_1 \cup B_2 \cup B_3| = |B_1| + |B_2| + |B_3|$

$$- |B_1 \cap B_2| - |B_2 \cap B_3| - |B_1 \cap B_3|$$

$$+ |B_1 \cap B_2 \cap B_3|$$

$$= 100 + 90 + 90$$

$$- 10 - 9 - 10$$

$$+ 1$$

$$= 252$$

252 is # without the restrictions but we want the # with restrictions so we subtract from the total which is 900 giving us $900 - 252 = 648$

c) If all 3 digits have to be different then

_ _ _

9 choices because it cant be 0 (if its 0, itd be a 2 digit #)

9 choices becase it cant be same as first digit

8 choices becase it cant be same as first or second digit

$9 \cdot 9 \cdot 8 = 648$

	choice 1	choice 2	choice 3
Negation: $\lvert B_1 \rvert$ = numbers with 1 first digit =	(1 ·	9 ·	8)
$\lvert B_2 \rvert$ = numbers with 2 second digit =	(9 ·	1 ·	8)
$\lvert B_3 \rvert$ = numbers with 3 third digit =	(9 ·	9 ·	1)
$\lvert B_1 \cap B_2 \rvert$ = numbers with a first digit and a second digit =	(1 ·	1 ·	8)
$\lvert B_2 \cap B_3 \rvert$ = numbers with a second digit and a third digit =	(9 ·	1 ·	1)
$\lvert B_3 \cap B_1 \rvert$ = numbers with a third digit and a first digit =	(1 ·	9 ·	1)
$\lvert B_1 \cap B_2 \cap B_3 \rvert$ = numbers with a first digit a second digit and a third digit =	(1 ·	1 ·	1)

Inclusion-exclusion: $|B_1 \cup B_2 \cup B_3| = |B_1| + |B_2| + |B_3|$

$- |B_1 \cap B_2| - |B_2 \cap B_3| - |B_1 \cap B_3|$

$+ |B_1 \cap B_2 \cap B_3|$

$= 72 + 72 + 81$

$- 8 - 9 - 9$

$+ 1$

$= 203$

$\boxed{\text{203 is \# of "bad digits" but we want the \# of "good digits" so we subtract from the totel which is 648 giving us } 648 - 203 = 445}$