1) Is there a graph that has the sum of degrees of its vertices equal to 13?

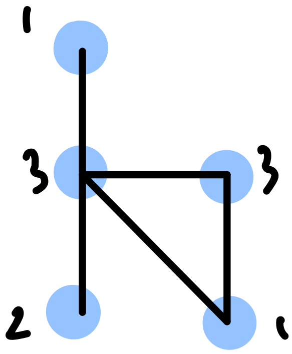
No because degree of vertex : # of edges touching it

$\sum d_v = 2|E| = 2 \cdot \#$ of edges

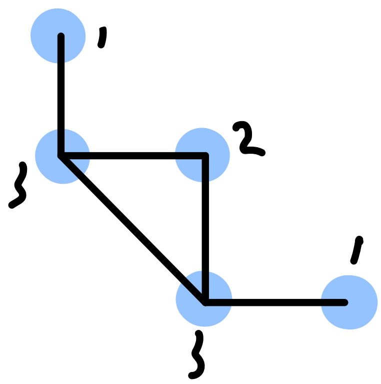
if $\forall E \in \mathbb{N}$ and $2E$ is even (2 times a positive integer is even) the sum of degrees of its vertices will never be 13

According to handshake lemma, sum of degree of vertices is always even because in any graph each edge is counted twice (from each side) and 13 is not even

2)

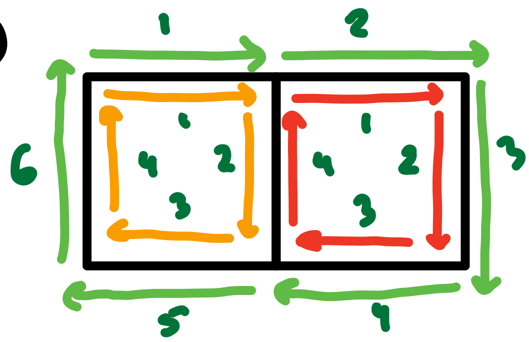


1, 1, 2, 3, 3



1, 1, 2, 3, 3

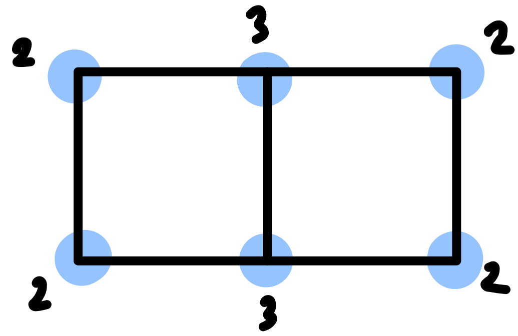
3)



degree of faces:

\# of edges of closed walk of boundary:

$6 + 4 + 4 = 14$



degree of vertices:

\# of edges touching it:

$2 + 2 + 3 + 2 + 2 + 3 = 14$

Both equal 14 so $\sum_f df = \sum dv$

4) s faces = 4 degrees, t faces = 3 degrees

all vertices = 3 degrees

$3v = 2e$ ($dv = 2|e|$, $e = \frac{3v}{2}$)

$\underbrace{4s + 3t}_{\text{\# of vertices for 1 face}} = 2e$

$4s + 3t = 3v$

$4s + 3t - 3v = 0$

$v - e + f = 2$ (Eulers formula)

$v - \frac{3v}{2} + s + t = 2$

$6v - \frac{18v}{2} + 6s + 6t = 12$

$6v - 9v + 6s + 6t = 12$

$3v + 6s + 6t = 12$

$e = \frac{3v}{2}$

f = # of faces = $s + t$

$4s + 3t - \cancel{3v} = 0$

$-(\cancel{-3v} + 6s + 6t = 12)$

$-2s - 3t = -12$

$2s + 3t = 12$

As s & $t > 0$, for $2s + 3t = 12$ to be true $2(3) + 3(2) = 12$ meaning $s = 3$ and $t = 2$. Therefore, $s + t = 3 + 2 = 5$ faces

5) $|E| \le \binom{n}{2}$

Graph with k vertices $\le \binom{k}{2}$ $\quad\quad 1 \le k \le n-1$

Graph with n-k vertices $\le \binom{n-k}{2}$

$|E| \le \binom{k}{2} + \binom{n-k}{2}$

$$\frac{(k)(k+1)}{2} + \frac{(n-k)(n-k-1)}{2} \le \frac{(n-1)(n-2)}{2}$$

$(k)(k-1) + (n-k)(n-k-1) \le (n-1)(n-2)$

$k^2 - k + n^2 - nk - n - kn + k^2 + k \le n^2 - 2n - n + 2$

$k^2 + k^2 - k + k + n^2 - nk - nk - n \le n^2 - 2n - n + 2$

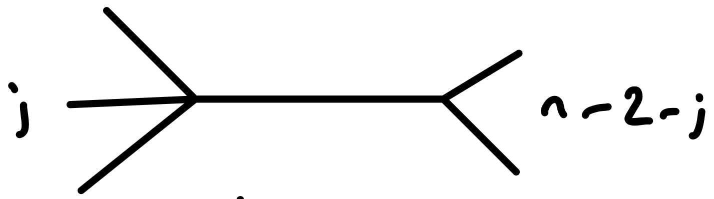
$2k^2 - 2kn \le -2n + 2$

$k^2 - kn \le -n + 1$

$k^2 - 1 \le kn - n$

Case 1	Case 2
$(k+1)(k-1) \le n(k-1)$	$1 \le k \le n-1$
$(1+1)(1-1) \le n(1-1)$	$+ \quad +1 \quad +1$
$0 \le 0$	$2 \le k+1 \le n$ ✓
True ✓	True

6)



n total vertices

if one node has j vertices, other has n-2-j vertices. j can have $\lfloor \frac{n-2}{2} \rfloor$ vertices as more would case overcount. So there are $\lfloor \frac{n-2}{2} \rfloor$ double-star trees

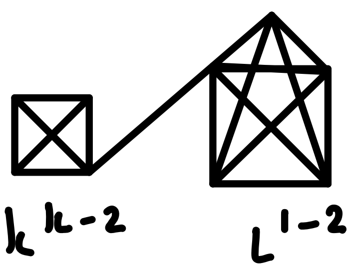
7) Cayley's Formula:

# of labeled trees on n vertices: n^{n-2}

complete graph with k labeled vertices: k^{k-2} labeled trees

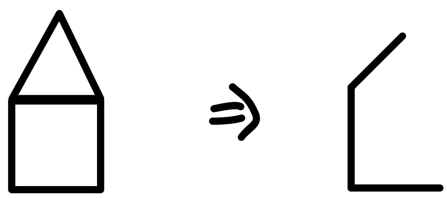
complete graph with l labeled vertices: l^{l-2} labeled trees

Product rule: $k^{k-2} \cdot l^{l-2}$



connected by a single edge tho only way to cross from $k \to l$

8)



Base : P(0)

$n - m = n - 0 = n$ connected components

Inductive step : $P(k-1) \wedge P(k) \Rightarrow P(k+1)$

1) graph with k+1 edges
2) remove vertex v with degree $d \geq 1$
3) We get graph with $k+1-d$ edges
4) from inductive hypothesis, this graph has $n-(k+1-d)$ connected components.
5) add back vertex v and this will add d edge that can collapse at most $(d+1)$ connected components
6)

$$\begin{array}{r} n - [(k+1-d)] \\ -(d+1)+1 \\ \hline n-(k+1) \end{array}$$

✓