1) Base case: $n=0$; $\sum_{i=0}^{0} f_i = f_{0+2} - 1 = f_2 - 1 = 1 - 1 = 0$ ✓

$\hookrightarrow f_2 = f_1 + f_0 = 1 + 0 = 1$

Inductive step: $\forall k \geq n_0$, $P(k) \Rightarrow P(k+1)$

$P(k) = \sum_{i=0}^{k} f_i = f_{k+2} - 1$

$P(k+1) = \sum_{i=0}^{k+1} f_i = f_{(k+1)+2} - 1 = f_{k+3} - 1$

$P(k+1) = P(k) + f_{k+1} = f_{k+2} - 1 + f_{k+1} = f_{k+3} - 1$

Note:

Property does not apply for $n=-1$ becase fibonocci sequence always starts off with $n=0$ so its impossible to find $n=-1$ as it's not included in the sequence

Base case: $P(0) = \sum_{i=0}^{0} f_i = F_2 - 1$

$n = -1$: $P(-1) = \sum_{i=0}^{-1} f_i = f_1 - 1$

empty sum

$0 = f_1 - 1$

$0 = 1 - 1$ ✓

Inductive step: $P(k) \Rightarrow P(k+1)$

$P(k) = \sum_{i=0}^{k} F_i = f_{k+2} - 1$

$\sum_{i=0}^{k+1} F_i = \sum_{i=0}^{k} f_i + F_{k+1}$

$= F_{k+2} - 1 + F_{k+1}$

$= f_{k+3} - 1$ ✓

2) <u>Base case:</u>

$n=0$ $\quad a_0 = \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^0 + \left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right)(2-p)^0$

$= \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right) + \left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right)$

$= \frac{1}{2}+\frac{1}{2} = \frac{2}{2} = 1$ ✓

$n=1$ $\quad a_1 = \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^1 + \left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right)(2-p)^1$

$= \frac{p}{2} + \frac{p\sqrt{3}}{3} + \frac{(2-p)}{2} - \frac{(2-p)\sqrt{3}}{3}$

$= \frac{p+2-p}{2} + \frac{p\sqrt{3}-\sqrt{3}(2-p)}{p}$

$= 1 + \frac{p\sqrt{3}-2\sqrt{3}+p\sqrt{3}}{3}$

$= 1 + \frac{\sqrt{3}+3-2\sqrt{3}+\sqrt{3}+3}{3}$

$= 1 + \frac{2\sqrt{3}-2\sqrt{3}+6}{3}$

$= 1+2 = 3$ ✓

<u>Inductive step:</u>

Prove: $a_n = \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^n + \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)(2-p)^n$

$a_n = 2a_{n-1} + 2a_{n-2}$

$a_n = 2\left[\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^{n-1} + \left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right)(2-p)^{n-1}\right] + 2\left[\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^{n-2} + \left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right)(2-p)^{n-2}\right]$

$a_n = 2\left[\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^n p^{-1} + \left(\frac{1}{2}-\frac{\sqrt{3}}{3}\right)(2-p)^n(2-p)^{-1}\right] + 2\left[\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^n p^{-2} + \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)(2-p)^n(2-p)^{-2}\right]$

$a_n = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}p^n p^{-1} + 2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)(2-p)^n(2-p)^{-1} + 2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^n p^{-2} + 2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)(2-p)^n(2-p)^{-2}$

$a_n = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^n p^{-1} + 2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^n p^{-2}) + 2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)(2-p)^n(2-p)^{-1} + 2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)(2-p)^n(2-p)^{-2}$

$a_n = \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^n\left(2p^{-1}+2p^{-2}\right) + 2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)(2-p)^n(2-p)^{-1} + 2\left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)(2-p)^n(2-p)^{-2}$

$a_n = \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^n + \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)(2-p)^n\left(2(2-p)^{-1}+2(2-p)^{-2}\right)$

$a_n = \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)p^n + \left(\frac{1}{2}+\frac{\sqrt{3}}{3}\right)(2-p)^n$

Hint: $2p^{-1}+2p^{-2} = 2(2-p)^{-1}+2(2-p)^{-2} = 1$

Note: The base case must cover $n=0$ & $n=1$ because to find the next term, you need the two previous cases
So to find $n=2$, you need $n=1$ and $n=0$ etc

3) $P(n): n = 4x + 5y \qquad x, y \in \mathbb{Z}^+$

Base case: $P(12) = 4(3) + 5(0)$

$P(13) = 4(2) + 5(1)$

$P(14) = 4(1) + 5(2) \qquad x, y \in \mathbb{Z}^+ \checkmark$

$P(15) = 4(0) + 5(3)$

Inductive Step: $p(i) \Rightarrow p(k+1) \qquad k \in \mathbb{Z} \geq 15$

$p(i): i = 4x + 5y \qquad i \in \mathbb{Z}, 12 \leq i \leq i$

$p(k+1): k+1 = 4x' + 5y'$

$p(k): k = 4x + 5y$

multiple of 4 $(x \geq 1)$

$k+1 = 4x + 5y + 1$

$k+1 = 4(x-1) + 5y + 4 + 1$

$k+1 = 4(x-1) + 5y + 5$

$k+1 = 4(x-1) + 5(y+1)$

let $x - 1 = x'$, $x \geq 1 \rightarrow x' \geq 0$

$k+1 = 4x' + 5y' \checkmark$

no multiples of 4 $(x = 0, y \geq 3)$

smallest n where $x = 0$ is $n = 15$ $\rightarrow y \geq 3$

$k+1 = 4x + 5y + 1$

$k+1 = 4x + 5(y-3) + 15 + 1$

$k+1 = 4x + 5(y-3) + 16$

$k+1 = 4(x+4) + 5(y-3)$

$x + 4 = x' \qquad y - 3 = y'$

$x = 0 \rightarrow x' > 0 \qquad y \geq 3 \rightarrow y' \geq 0$

$k+1 = 4(x') + 5(y') \checkmark$

We need multiple base cases to account for numbes that are not multiples of 4.

4)

length	possible choices	# of words
$n = 1$	x y z	3
$n = 2$	xyz xyz xy	8
$n = 3$	xyz xyz xy xyz xyz xy xyz xyz	$22 = 2(8) + 2(3)$

$a_n = 2a_{n-1} + 2a_{n-2}$, $n \geq 2$

# of words in length 1 : $|S_1| = 3$ (x, y, z)

$|S_2| = 8$ (xx, xy, xz
yz, yy, zx
zy)

a word of length n can be created by adding letter to word of length n-1. If word begins with x or y, we can add x, y, z to beginning. But if word begins with z, we can only add x or y to the beginning.

$$a_n = a_{n-1} + a_{n-1} + 2a_{n-2}$$

$$a_n = 2a_{n-1} + 2a_{n-2}$$

$$a_n = 2(a_{n-1} + a_{n-2})$$

5) $2n$ total disks

a_n = total # of moves for $2n$ disks

a_{n-1} = total # of ways to move $2n-2$ disks to second pole. We then use 2 moves to move the 2 largest disks to the third place. Then we need a_{n-1} moves to move all the poles from the second pole to the third pole.

$a_n = a_{n-1} + 2 + a_{n-1}$

a) $a_n = 2a_{n-1} + 2$

b) $a_1 = 2,\ a_2 = 6,\ a_3 = 14,\ a_4 = 30$

b) guess $a_n = 2^{n+1} - 2$

Base case: $a_1 = 2^2 - 2 = 2$ ✓

Inductive step: $P(k) \Rightarrow P(k+1)$

$$a_k = 2^{k+1} - 2$$
$$a_{k+1} = 2^{k+2} - 2$$
$$a_{k+1} = 2a_k + 2$$
$$a_{k+1} = 2(2^{k+1} - 2) + 2$$
$$a_{k+1} = 2^{k+2} - 4 + 2$$
$$a_{k+1} = 2^{k+2} - 2$$

6) $a_0 = 1 \quad a_1 = -2 \quad a_2 = -2(-2) - 1 = 4 - 1 = 3$

$a_3 = -2(3) - (-2) = -6 + 2 = -4$

$a_4 = -2(-4) - 3 = 8 - 3 = 5$

pattern = increases by 1, changes from + to −

possible recurrence = $(n+1)(-1)^n$

Base case: $n = 0;\ (0+1)(-1)^0 = (1)(1) = 1$

$n = 1;\ (1+1)(-1)^1 = (2)(-1) = -2$

$n = 2;\ (2+1)(-1)^2 = (3)(1) = 3$

Inductive step: $\forall k \geq 2,\ a_k \Rightarrow a_{k+1}$

$a_k = (k+1)(-1)^k$

$a_{k+1} = (k+1+1)(-1)^{k+1}$

$a_{k+1} = (k+2)(-1)^{k+1}$

if $a_n = -2a_{n-1} - a_{n-2}$ then $a_{k+1} = -2a_{k+1-1} - a_{k+1-2}$

$= -2a_k - a_{k-1}$

$(-1)^{k-1} = (-1)^{k+1}$

$= 2[(k+1)(-1)^k] - (k-1+1)(-1)^{k-1}$

$= (-1)(2)(k+1)(-1)^k - k(-1)^{k-1}$

$= (2k+2)(-1)^{k+1} - k(-1)^{k+1}$

$= (2k+2-k)(-1)^{k+1}$

$= (k+2)(-1)^{k+1}$

Problem

a) $a_1 = 1 = 1/1$

$a_2 = \frac{1}{2}(1)+1 = 3/2$

$a_3 = \frac{1}{2}(\frac{3}{2})+1 = 7/4$

$a_4 = \frac{1}{2}(\frac{7}{4})+1 = 15/8$

$a_5 = \frac{1}{2}(\frac{15}{8})+1 = 31/16$

$p(n) = a_n = \dfrac{2^n - 1}{2^{n-1}}$

Base case:

$a_1 = \dfrac{2^1-1}{2^{1-1}} = \dfrac{2-1}{2^0} = 1 \checkmark$

$a_k = \dfrac{2^k - 1}{2^{k-1}}$ $\quad k \in \mathbb{Z} \geq 1$

$a_{k+1} = \dfrac{2^{k+1}-1}{2^k}$

Inductive step:

$a_{k+1} = \frac{1}{2} a_k + 1$

$a_{k+1} = \frac{1}{2}\left(\dfrac{2^k - 1}{2^{k-1}}\right) + 1$

$a_{k+1} = \dfrac{2^k - 1}{2^k} + 1$

$a_{k+1} = \dfrac{2^k - 1}{2^k} + \dfrac{2^k}{2^k}$

$a_{k+1} = \dfrac{2(2^k) - 1}{2^k}$

$a_{k+1} = \dfrac{2^{k+1} - 1}{2^k}$ $\checkmark$

b) $a_n = \frac{1}{2}a_{n-1} + 1$

$- \quad a_{n-1} = \frac{1}{2}a_{n-2} + 1$

$a_n - a_{n-1} = \frac{1}{2}a_{n-1} - \frac{1}{2}a_{n-2}$

$a_n = \frac{3}{2}a_{n-1} - \frac{1}{2}a_{n-2}$

Characteristic Equation:

$a_n = \frac{3}{2}a_{n-1} - \frac{1}{2}a_{n-2}$

$a_n - \frac{3}{2}a_{n-1} + \frac{1}{2}a_{n-2} = 0$

$1x^2 - \frac{3}{2}x^1 + \frac{1}{2}x^0 = 0$

$x^2 - \frac{3}{2}x + \frac{1}{2}x^0 = 0$

$x^2 - \frac{3}{2}x + \frac{1}{2} = 0$

$2x^2 - 3x + 1 = 0(2)$

$2x^2 - 2x - x + 1 = 0$

$2x(x-1) - 1(x-1)$

$(x-1)(2x-1) = 0$

$p = 1,\ q = \frac{1}{2}$

$a_n = \alpha_1 p^n + \alpha_2 q^n$

$a_n = \alpha_2(1)^n + \alpha_2(\frac{1}{2})^n$

$a_n = \alpha_1(1)^n + \alpha_2(\frac{1}{2})^n$

$a_n = \alpha_1 + \alpha_2(\frac{1}{2})^n$

$a_1 = 1 = \alpha_1 + \alpha_2(\frac{1}{2})^1$

$1 = \alpha_1 + \alpha_2(\frac{1}{2})$

$1 - \alpha_1 = \alpha_2(\frac{1}{2})$

$2 - 2\alpha_1 = \alpha_2$

$2 - 3 + \frac{1}{2}\alpha_2 = \alpha_2$

$-1 = \alpha_2 - \frac{1}{2}\alpha_2$

$-1 = \frac{1}{2}\alpha_2$

$-2 = \alpha_2$

$a_2 = \frac{3}{2} = \alpha_1 + \alpha_2(\frac{1}{2})^2$

$\frac{3}{2} = \alpha_1 + \alpha_2\frac{1}{4}$

$\frac{3}{2} - \alpha_2\frac{1}{4} = \alpha_1$

$\frac{3}{2} - (-2)\frac{1}{4} = \alpha_1$

$\frac{3}{2} + \frac{1}{2} = \alpha_1$

$2 = \alpha_1$

$a_n = 2 - 2(\frac{1}{2})^n$

c) $a_0, a_1, a_2, a_3, \dots$

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 = \sum_{i=0}^{\infty} a_i x^i$$

$$f(x) = 1 \cdot x^0 + \frac{3}{2} x + \frac{7}{4} x^2 + \dots$$

n^{th} derivative of $f(x)$ at $x=0$ divided by $n!$ is a_n

$$a_n = \frac{f^n(0)}{n!}$$

$$a = \frac{f'(0)}{n!}$$

$\vdots$