University of Dhaka
Computer Science and Engineering
4th Year 2nd Semester B.Sc.: 2022
CSE4269– Parallel and Distributed Systems

Assignment Code: A4
Assignment Title: Floyd's all-pairs shortest path algorithm .

• Objectives

The objective of this assignment is to solve an embarrassingly parallel problem, with a certain class of MPI routines, called 'collective' .

• Programming Assignment

Given a weighed graph $G = (V, E)$ with n nodes, the cost of an edge from node i to j is $c_{i,j}$. Floyd's algorithm calculates the cost $d_{i,j}$ of the shortest path between each pair of nodes (i, j) in V. A recurrence formulation for calculating $d_{i,j}$ using k intermediate nodes is provided in formula 12.6 of the book [Introduction to Parallel Computing: Ananth Grama]. You are required to calculate $d_{i,j}$ using the formula for each pair of nodes in the given graph, using a parallel dynamic programming solution strategy. In your parallel solution, the cost matrix at level k which computes $d^k_{i,j}$, i.e., the $d_{i,j}$ values using k intermediate nodes, is calculated from the cost matrix at level $k-1$. Given a graph with n nodes, the final goal is to calculate the cost matrix at level n, which includes all n nodes. The possible solution strategies:

1. Strategy : Row- and column-wise one-to-all broadcasts are used while computing the level k matrix. The algorithm is discussed in the textbook (Chapter 12.4.1 and 10.4.2) .
2. Strategy 2: Pipelining replaces the one-to-all broadcasts.

Here is what you are required to do:

1. In this assignment, you are required to implement the parallel programs for each of the above two solution strategies and test on a graph of n nodes using P processors.
2. Write a sequential version of the algorithm (Chapter 12.4.1) and execute it on a single node of the cluster. Measure the execution time.
3. You are also required to measure the speed-up with different values of n and P for both the parallel versions.
4. Explain your experimental findings, in terms of which of the two versions is better (if any) and why

• Your Task

- Complete the programming part and run the experiment in your own machine. See the performance and plot that accordingly.
- Repeat the same in the cluster.

- **Submission**

  - Name of Submitted file:
    - Serial Version: serialFloyd.c/ serialFloyd.cpp
    - Parallel Version: parallelFloyd.c/ parallelFloyd.cpp
    - Pipeline Version: pipelineFloyd.c/ pipelineFloyd.cpp
  - Create "makefile" for assignment4.
  - Put proper comments on the program where you incorporate or modify the code. Please use standard comment style.
  - Submit all the source file (serial, parallel, pipelined) and the makefile.
  - Input File: input.txt contains the input matrix for the graph. It needs to be large enough to split, so that multiple processes/processors have sufficient number of elements to work with concurrently.
  - The generated Output File: output.txt, contains the shortest path among all the nodes of the graph
  - The report contain:
    1. Speed-up plot and explanation in two different configuration (i) own machine, (ii) cluster.

- **Submission Deadline**

  - Programming assignment: 05.11.2023
  - Marks distribution will be announced later

Thank You

12.4.1 Floyd's All-Pairs Shortest-Paths Algorithm

Consider a weighted graph G, which consists of a set of nodes V and a set of edges E. An edge from node i to node j in E has a weight $c_{i,j}$. Floyd's algorithm determines the cost $d_{i,j}$ of the shortest path between each pair of nodes (i, j) in V (Section 10.4.2). The cost of a path is the sum of the weights of the edges in the path.

Let $d_{i,j}^k$ be the minimum cost of a path from node i to node j, using only nodes $v_0, v_1, \ldots, v_{k-1}$. The functional equation of the DP formulation for this problem is

Equation 12.6

$$d_{i,j}^k = \begin{cases} c_{i,j} & k = 0 \\ \min\{d_{i,j}^{k-1}, (d_{i,k}^{k-1} + d_{k,j}^{k-1})\} & 0 \leq k \leq n-1 \end{cases} .$$

Since $d_{i,j}^n$ is the shortest path from node i to node j using all n nodes, it is also the cost of the overall shortest path between nodes i and j . The sequential formulation of this algorithm requires n iterations, and each iteration requires time $\Theta(n^2)$. Thus, the overall run time of the sequential algorithm is $\Theta(n^3)$.

Equation 12.6 is a serial polyadic formulation. Nodes $d_{i,j}^k$ can be partitioned into n levels, one for each value of k. Elements at level $k + 1$ depend only on elements at level k. Hence, the formulation is serial. The formulation is polyadic since one of the solutions to $d_{i,j}^k$ requires a composition of solutions to two subproblems $d_{i,k}^{k-1}$ and $d_{k,j}^{k-1}$ from the previous level. Furthermore, the dependencies between levels are sparse because the computation of each element in $d_{i,j}^{k+1}$ requires only three results from the preceding level (out of n^2).

A simple CREW PRAM formulation of this algorithm uses n^2 processing elements. Processing elements are organized into a logical two-dimensional array in which processing element $P_{i,j}$ computes the value of $d_{i,j}^k$ for $k = 1, 2, \ldots, n$. In each iteration k, processing element $P_{i,j}$ requires the values $d_{i,j}^{k-1}$, $d_{i,k}^{k-1}$, and $d_{k,j}^{k-1}$. Given these values, it computes the value of $d_{i,j}^k$ in constant time. Therefore, the PRAM formulation has a parallel run time of $\Theta(n)$. This

Figure 1:

formulation is cost-optimal because its processor-time product is the same as the sequential run time of $\Theta(n^3)$. This algorithm can be adapted to various practical architectures to yield efficient parallel formulations (Section 10.4.2).

As with serial monadic formulations, data locality is of prime importance in serial polyadic formulations since many such formulations have sparse connectivity between levels.

Figure 2:

10.4.2 Floyd's Algorithm

Floyd's algorithm for solving the all-pairs shortest paths problem is based on the following observation. Let $G = (V, E, w)$ be the weighted graph, and let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertices of G. Consider a subset $\{v_1, v_2, \ldots, v_k\}$ of vertices for some k where $k \leq n$. For any pair of vertices v_i , $v_j \in V$, consider all paths from v_i to v_j whose intermediate vertices belong to the set $\{v_1, v_2, \ldots, vk\}$. Let $p_{i,j}^{(k)}$ be the minimum-weight path among them, and let $d_{i,j}^{(k)}$ be the weight of $p_{i,j}^{(k)}$. If vertex v_k is not in the shortest path from v_i to v_j, then $p_{i,j}^{(k)}$ is the same as $p_{i,j}^{(k-1)}$. However, if v_k is in $p_{i,j}^{(k)}$, then we can break $p_{i,j}^{(k)}$ into two paths – one from v_i to v_k and one from v_k to v_j. Each of these paths uses vertices from $\{v_1, v_2, \ldots, v_{k-1}\}$. Thus, $d_{i,j}^{(k)} = d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)}$. These observations are expressed in the following recurrence equation:

Equation 10.5

$$d_{i,j}^{(k)} = \begin{cases} w(v_i, v_j) & \text{if} \quad k = 0 \\ \min\left\{d_{i,j}^{(k-1)}, d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)}\right\} & \text{if} \quad k \geq 1 \end{cases}$$

Figure 3:

The length of the shortest path from v_i to v_j is given by $d_{i,j}^{(n)}$. In general, the solution is a matrix $D^{(n)} = (d_{i,j}^{(n)})$.

Floyd's algorithm solves Equation 10.5 bottom-up in the order of increasing values of k. Algorithm 10.3 shows Floyd's all-pairs algorithm. The run time of Floyd's algorithm is determined by the triple-nested for loops in lines 4–7. Each execution of line 7 takes time $\Theta(1)$; thus, the complexity of the algorithm is $\Theta(n^3)$. Algorithm 10.3 seems to imply that we must store n matrices of size n xn. However, when computing matrix $D^{(k)}$, only matrix $D^{(k-1)}$ is needed. Consequently, at most two n x n matrices must be stored. Therefore, the overall space complexity is $\Theta(n^2)$. Furthermore, the algorithm works correctly even when only one copy of D is used (Problem 10.6).

Algorithm 10.3 Floyd's all-pairs shortest paths algorithm. This program computes the all-pairs shortest paths of the graph $G = (V, E)$ with adjacency matrix A.

```
1.  procedure FLOYD_ALL_PAIRS_SP( A)
2.  begin
3.     D(0) = A;
4.     for k := 1 to n do
5.        for i := 1 to n do
6.           for j := 1 to n do
                 d(k)i,j := min(d(k-1)i,j, d(k-1)i,k + d(k-1)k,j);
7.
8.  end FLOYD_ALL_PAIRS_SP
```

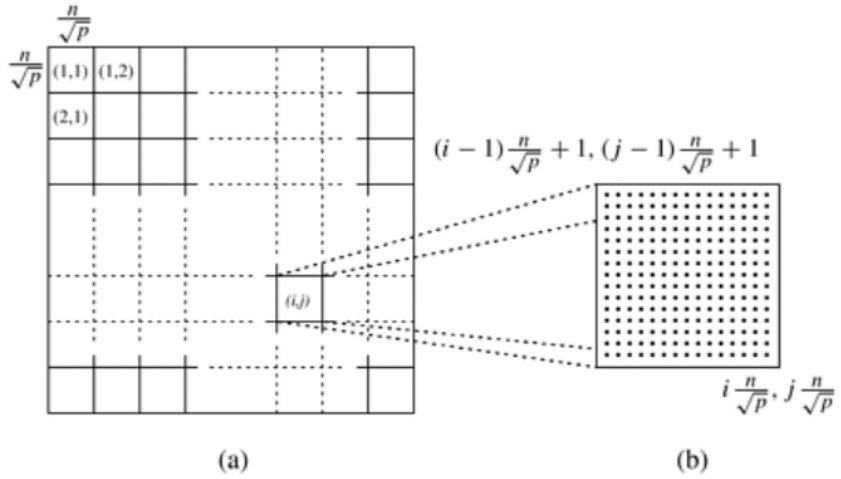
Parallel Formulation

A generic parallel formulation of Floyd's algorithm assigns the task of computing matrix $D^{(k)}$ for each value of k to a set of processes. Let p be the number of processes available. Matrix $D^{(k)}$ is partitioned into p parts, and each part is assigned to a process. Each process computes the $D^{(k)}$ values of its partition. To accomplish this, a process must access the corresponding segments of the k^{th} row and column of matrix $D^{(k-1)}$. The following section describes one technique for partitioning matrix $D^{(k)}$. Another technique is considered in Problem 10.8.

2-D Block Mapping One way to partition matrix $D^{(k)}$ is to use the 2-D block mapping (Section 3.4.1). Specifically, matrix $D^{(k)}$ is divided into p blocks of size $(n/\sqrt{p}) \times (n/\sqrt{p})$, and each block is assigned to one of the p processes. It is helpful to think of the p processes as arranged in a logical grid of size $\sqrt{p} \times \sqrt{p}$. Note that this is only a conceptual layout and does not necessarily reflect the actual interconnection network. We refer to the process on the i^{th} row and j^{th} column as $P_{i,j}$. Process $P_{i,j}$ is assigned a subblock of $D^{(k)}$ whose upper-left corner is $((i-1)n/\sqrt{p}+1, (j-1)n/\sqrt{p}+1)$ and whose lower-right corner is $(in/\sqrt{p}, jn/\sqrt{p})$. Each process updates its part of the matrix during each iteration. Figure 10.7(a) illustrates the 2-D block mapping technique.

Figure 4:

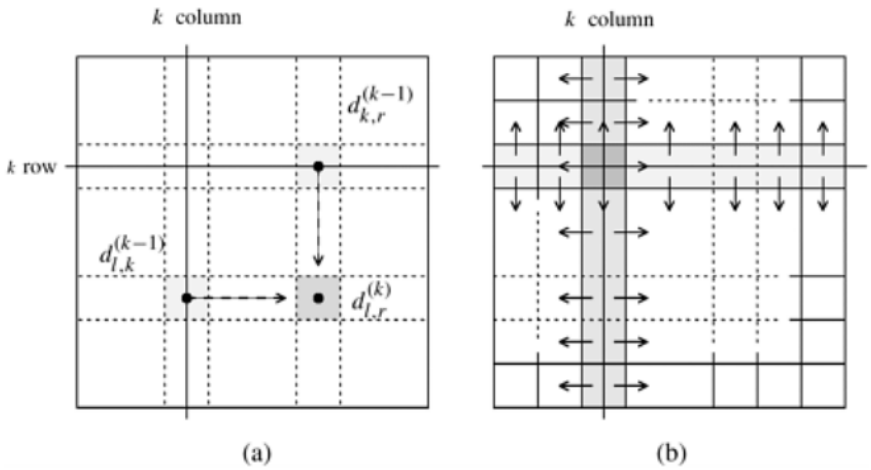
Figure 10.7. (a) Matrix $D^{(k)}$ distributed by 2-D block mapping into $\sqrt{p} \times \sqrt{p}$ subblocks, and (b) the subblock of $D^{(k)}$ assigned to process $P_{i,j}$.



During the k^{th} iteration of the algorithm, each process $P_{i,j}$ needs certain segments of the k^{th} row and k^{th} column of the $D^{(k-1)}$ matrix. For example, to compute $d_{l,r}^{(k)}$ it must get $d_{l,k}^{(k-1)}$ and $d_{k,r}^{(k-1)}$. As Figure 10.8 illustrates, $d_{l,k}^{(k-1)}$ resides on a process along the same row, and element $d_{k,r}^{(k-1)}$ resides on a process along the same column as $P_{i,j}$. Segments are transferred as follows. During the k^{th} iteration of the algorithm, each of the $\sqrt{p}$ processes containing part of the k^{th} row sends it to the $\sqrt{p}-1$ processes in the same column. Similarly, each of the $\sqrt{p}$ processes containing part of the k^{th} column sends it to the $\sqrt{p}-1$ processes in the same row.

Figure 10.8. (a) Communication patterns used in the 2-D block mapping. When computing $d_{i,j}^{(k)}$, information must be sent to the highlighted process from two other processes along the same row and column. (b) The row and column of $\sqrt{p}$ processes that contain the k^{th} row and column send them along process columns and rows.

Figure 5:



Algorithm 10.4 shows the parallel formulation of Floyd's algorithm using the 2-D block mapping. We analyze the performance of this algorithm on a p-process message-passing computer with a cross-bisection bandwidth of $\Theta(p)$. During each iteration of the algorithm, the k^{th} row and k^{th} column of processes perform a one-to-all broadcast along a row or a column of $\sqrt{p}$ processes. Each such process has $n/\sqrt{p}$ elements of the k^{th} row or column, so it sends $n/\sqrt{p}$ elements. This broadcast requires time $\Theta^{\Theta((n \log p)/\sqrt{p})}$. The synchronization step on line 7 requires time $\Theta(\log p)$. Since each process is assigned n^2/p elements of the $D^{(k)}$ matrix, the time to compute corresponding $D^{(k)}$ values is $\Theta(n^2/p)$. Therefore, the parallel run time of the 2-D block mapping formulation of Floyd's algorithm is

$$T_P = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta\left(\frac{n^2}{\sqrt{p}} \log p\right)}^{\text{communication}}.$$

Since the sequential run time is $W = \Theta(n^3)$, the speedup and efficiency are as follows:

Equation 10.6

$$S = \frac{\Theta(n^3)}{\Theta(n^3/p) + \Theta((n^2 \log p)/\sqrt{p})}$$

$$E = \frac{1}{1 + \Theta((\sqrt{p} \log p)/n)}$$

Figure 6:

From Equation 10.6 we see that for a cost-optimal formulation $(\sqrt{p}\log p)/n = O(1)$; thus, 2-D block mapping can efficiently use up to $O(n^2/\log^2 n)$ processes. Equation 10.6 can also be used to derive the isoefficiency function due to communication, which is $\Theta(p^{1.5}\log^3 p)$. The isoefficiency function due to concurrency is $\Theta(p^{1.5})$. Thus, the overall isoefficiency function is $\Theta(p^{1.5}\log^3 p)$.

Speeding Things Up In the 2-D block mapping formulation of Floyd's algorithm, a synchronization step ensures that all processes have the appropriate segments of matrix $D^{(k-1)}$ before computing elements of matrix $D^{(k)}$ (line 7 in Algorithm 10.4). In other words, the k^{th} iteration starts only when the $(k-1)^{th}$ iteration has completed and the relevant parts of matrix $D^{(k-1)}$ have been transmitted to all processes. The synchronization step can be removed without affecting the correctness of the algorithm. To accomplish this, a process starts working on the k^{th} iteration as soon as it has computed the $(k-1)^{th}$ iteration and has the relevant parts of the $D^{(k-1)}$ matrix. This formulation is called *pipelined 2-D block mapping*. A similar technique is used in Section 8.3 to improve the performance of Gaussian elimination.

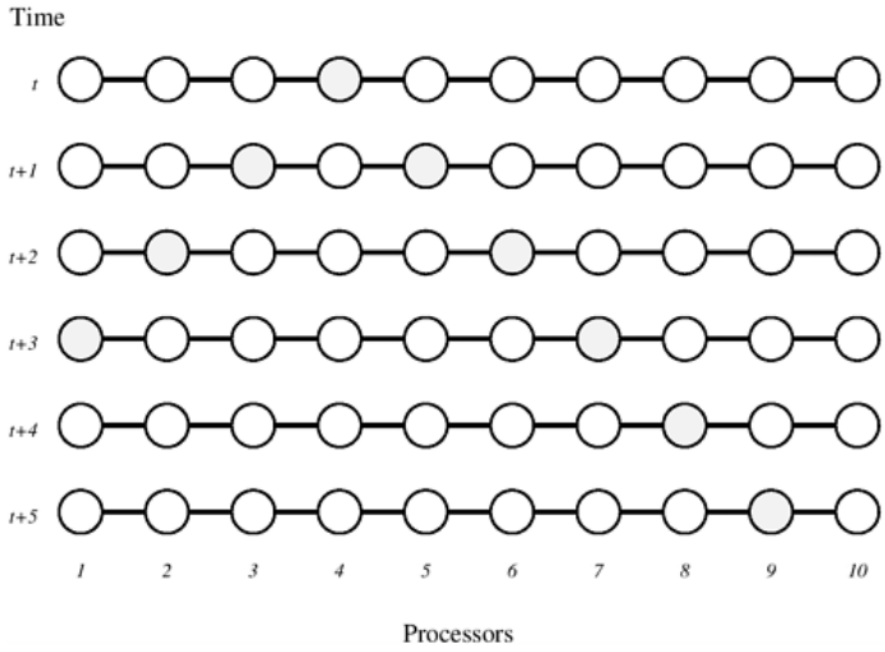
Algorithm 10.4 Floyd's parallel formulation using the 2-D block mapping. $P_{*,j}$ denotes all the processes in the j^{th} column, and $P_{i,*}$ denotes all the processes in the i^{th} row. The matrix $D^{(0)}$ is the adjacency matrix.

```
1.   procedure FLOYD_2DBLOCK(D^(0))
2.   begin
3.      for k := 1 to n do
4.      begin
5.         each process P_{i,j} that has a segment of the k^th row of D^(k-1);
              broadcasts it to the P_{*,j} processes;
6.         each process P_{i,j} that has a segment of the k^th column of D^(k-1);
              broadcasts it to the P_{i,*} processes;
7.         each process waits to receive the needed segments;
8.         each process P_{i,j} computes its part of the D^(k) matrix;
9.      end
10.  end FLOYD_2DBLOCK
```

Consider a p-process system arranged in a two-dimensional topology. Assume that process $P_{i,j}$ starts working on the k^{th} iteration as soon as it has finished the $(k-1)^{th}$ iteration and has received the relevant parts of the $D^{(k-1)}$ matrix. When process $P_{i,j}$ has elements of the k^{th} row and has finished the $(k-1)^{th}$ iteration, it sends the part of matrix $D^{(k-1)}$ stored locally to processes $P_{i,j-1}$ and $P_{i,j+1}$. It does this because that part of the $D^{(k-1)}$ matrix is used to compute the $D^{(k)}$ matrix. Similarly, when process $P_{i,j}$ has elements of the k^{th} column and has finished the $(k-1)^{th}$ iteration, it sends the part of matrix $D^{(k-1)}$ stored locally to processes $P_{i-1,j}$ and $P_{i+1,j}$. When process $P_{i,j}$ receives elements of matrix $D^{(k)}$ from a process along its row in the logical mesh, it stores them locally and forwards them to the process on the side opposite from where it received them. The columns follow a similar communication protocol. Elements of matrix $D^{(k)}$ are not forwarded when they reach a mesh boundary. Figure 10.9 illustrates this communication and termination protocol for processes within a row (or a column).

Figure 7:

Figure 10.9. Communication protocol followed in the pipelined 2-D block mapping formulation of Floyd's algorithm. Assume that process 4 at time t has just computed a segment of the k^{th} column of the $D^{(k-1)}$ matrix. It sends the segment to processes 3 and 5. These processes receive the segment at time $t + 1$ (where the time unit is the time it takes for a matrix segment to travel over the communication link between adjacent processes). Similarly, processes farther away from process 4 receive the segment later. Process 1 (at the boundary) does not forward the segment after receiving it.



Consider the movement of values in the first iteration. In each step, $n/\sqrt{p}$ elements of the first row are sent from process $P_{i,j}$ to $P_{i+1,j}$. Similarly, elements of the first column are sent from process $P_{i,j}$ to process $P_{i,j+1}$. Each such step takes time $\Theta(n/\sqrt{p})$. After $\Theta(\sqrt{p})$ steps, process $P_{\sqrt{p},\sqrt{p}}$ gets the relevant elements of the first row and first column in time $\Theta(n)$. The values of successive rows and columns follow after time $\Theta(n^2/p)$ in a pipelined mode. Hence, process $P_{\sqrt{p},\sqrt{p}}$ finishes its share of the shortest path computation in time $\Theta(n^3/p) + \Theta(n)$. When process $P_{\sqrt{p},\sqrt{p}}$ has finished the $(n - 1)^{th}$ iteration, it sends the relevant values of the n^{th} row and column to the other processes. These values reach process $P_{1,1}$ in time $\Theta(n)$. The overall parallel run time of this formulation is

Figure 8:

$$T_P = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta(n).}^{\text{communication}}$$

Since the sequential run time is $W = \Theta(n^3)$, the speedup and efficiency are as follows:

Equation 10.7

$$S = \frac{\Theta(n^3)}{\Theta(n^3/p) + \Theta(n)}$$

$$E = \frac{1}{1 + \Theta(p/n^2)}$$

Table 10.1. The performance and scalability of the all-pairs shortest paths algorithms on various architectures with $O(p)$ bisection bandwidth. Similar run times apply to all k - d cube architectures, provided that processes are properly mapped to the underlying processors.

	Maximum Number of Processes for $E = \Theta(1)$	Corresponding Parallel Run Time	Isoefficiency Function
Dijkstra source-partitioned	$\Theta(n)$	$\Theta(n^2)$	$\Theta(p^3)$
Dijkstra source-parallel	$\Theta(n^2/\log n)$	$\Theta(n \log n)$	$\Theta((p \log p)^{1.5})$
Floyd 1-D block	$\Theta(n/\log n)$	$\Theta(n^2 \log n)$	$\Theta((p \log p)^3)$
Floyd 2-D block	$\Theta(n^2/\log^2 n)$	$\Theta(n \log^2 n)$	$\Theta(p^{1.5} \log^3 p)$
Floyd pipelined 2-D block	$\Theta(n^2)$	$\Theta(n)$	$\Theta(p^{1.5})$

From Equation 10.7 we see that for a cost-optimal formulation $p/n^2 = O(1)$. Thus, the pipelined formulation of Floyd's algorithm uses up to $O(n^2)$ processes efficiently. Also from Equation 10.7, we can derive the isoefficiency function due to communication, which is $\Theta(p^{1.5})$. This is the overall isoefficiency function as well. Comparing the pipelined formulation to the

Figure 9: