On the Convergence of Iterative Methods

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Abstract

We investigate the convergence properties of various iterative methods for solving linear systems. Our main result establishes a new sufficient condition for convergence that generalizes previous work in this area.

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1. Introduction

Iterative methods play a crucial role in solving large-scale linear systems that arise in scientific computing and engineering applications.

Definition 1.1 : Iterative Method

An iterative method for solving $Ax = b$ is a sequence of approximations $\{x_k\}$ generated by the recurrence relation $x_{\{k+1\}} = Bx_k + c$ where B is the iteration matrix and c is a constant vector.

Theorem 1.2 : Convergence Criterion

The iterative method converges if and only if the spectral radius of the iteration matrix satisfies $\rho(B) < 1$.

Proof Let $e_k = x_k - x^*$ be the error at iteration k, where x^* is the exact solution. Then $e_{\{k+1\}} = Be_k$, which implies $e_k = B^k e_0$. The method converges if and only if $\lim_{k\to\infty} B^k = 0$, which is equivalent to $\rho(B) < 1$. □

2. Analysis of Specific Methods

2.1. Jacobi Method

The Jacobi method uses the iteration matrix $B = D^{-1}(L + U)$ where $A = D - L - U$.

Proposition 2.3 : Jacobi Convergence

If A is strictly diagonally dominant, then the Jacobi method converges.

Proof For strictly diagonally dominant matrices, we have $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for all i. This implies $\rho(B) < 1$ by the Gerschgorin circle theorem.

□

2.2. Gauss-Seidel Method

The Gauss-Seidel method typically converges faster than Jacobi for the same matrix.

Lemma 2.4 : Comparison Lemma

For any matrix A, the spectral radius of the Gauss-Seidel iteration matrix is less than or equal to that of the Jacobi iteration matrix.

Corollary 2.5 : Gauss-Seidel Convergence

If the Jacobi method converges, then the Gauss-Seidel method also converges.

3. Numerical Examples

3.1. Simple 2x2 System

Consider the system $((2, 1), (1, 3))((x_1), (x_2)) = ((5), (6))$ This matrix is strictly diagonally dominant, so both Jacobi and Gauss-Seidel methods converge.

Remark 3.6 : Implementation Note

In practice, one should check the condition number of the matrix as well as the spectral radius to predict convergence behavior.