# Class Title: Homework 1

*Professor's Name - Section 001*

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## Problem 1. (Question 1, Section 7.1)

(a.) $2x + y = (2x_1, 2x_2, 2x_3) + (y_1, y_2, y_3) = (1, 3, 5)$
(b.) $x \cdot y = -1 + 0 + 2 = 1$
(c.) $||x|| = \sqrt{x \cdot x} = \sqrt{1+0+4} = \sqrt{5}$, $||y|| = \sqrt{y \cdot y} = \sqrt{1+9+1} = \sqrt{11}$
(d.) The cosine of the angle is given by

$$\frac{u \cdot v}{||u||\ ||v||} = \cos\theta$$

Therefore the cosine of the angle between $x$ and $y$ is

$$\frac{x \cdot y}{||x||\ ||y||} = \frac{1}{\sqrt{5}\sqrt{11}} = \frac{1}{\sqrt{55}} \approx 0.13484$$

(e.) The distance is given by the metric $||x - y||$ giving us

$$\sqrt{(x-y)\cdot(x-y)} = \sqrt{(2,-3,1)\cdot(2,-3,1)} = \sqrt{14}$$

## Problem 2. (Question 6, Section 7.1)

Let $u$ and $v$ be vectors in $\mathbb{R}^d$ where $v = au$ and $a \in \mathbb{R}$ and $a \neq 0$ (trivial case). Then by the Cuachy Schwartz inequality we know that

$$|u \cdot v| \leq ||u||\ ||v||$$

Substituting values for $v$ we get

$$\begin{aligned} |u \cdot v| &\leq ||u||\ ||v|| \\ |u \cdot au| &\leq ||u||\ ||au|| \\ |a(u \cdot u)| &\leq ||u||\ ||au|| && \text{by 7.1.4(c)} \\ |a||u \cdot u| &\leq |a|\ ||u||\ ||u|| && \text{by 7.1.10(b)} \\ |u \cdot u| &\leq \sqrt{u \cdot u}\sqrt{u \cdot u} \\ |u \cdot u| &\leq (u \cdot u) \end{aligned}$$

By by 7.1.4(d) it the inner product must always be positive, which means that

$$|u \cdot u| \leq (u \cdot u) \implies |u \cdot u| = (u \cdot u)$$

Therefore if $v$ is a scalar multiple of $u$ it must be an equality.

## Problem 3. Question 8, Section 7.1

The prove that $|| \cdot ||_\infty$ is a norm of $\mathbb{R}^d$ we must show that the following three conditions hold:

(1) $||x + y|| \leq ||x|| + ||y||$

(2) $||ax|| = |a|||x||$

(3) $||x|| = 0 \implies x = 0$

**Condition 1:**
Let $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$. Also, let $x_m = \max\{|x_1|, \cdots, |x_d|\}$ and let $y_m = \max\{|y_1|, \cdots, |y_d|\}$. By the triangle inequality we know that

$$|x_m + y_m| \leq |x_m| + |y_m| \implies ||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$$

Therefore it satisfies condition 1.

**Condition 2:**
Distributing out the absolute value we see that

$$|ax_m| = |a||x_m| \implies |a|||x||_\infty$$

There it satisfies condition 2.

**Condition 3:**
If $||x||_\infty = 0$, then $\max\{|x_1, \cdots, x_d\}] = 0$. Since the maximum value of any $x_j \leq x_m$ and $x_m = 0$, then all $x_j = 0$.. Since this forces all $x_j = 0$, then $x = (0, \cdots, 0)$ for every entry in the vector so $x = 0$

As these three conditions are satisfied, we then know that $|| \cdot ||_\infty$ is a norm on $\mathbb{R}^d$.

## Problem 4 (Question 12, Section 7.1)

Note that

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \sum_{k=1}^{\infty} |x_k y_k|^2$$

Then summing up to $n$, we can see by Cauchy Schwartz that

$$\sum_{k=1}^{n} x_k^2 y_k^2 \leq \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2$$

Taking the limit gives us $n \to \infty$

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \sum_{k=1}^{\infty} x_k^2 y_k^2 \leq \sum_{k=1}^{\infty} x_k^2 \sum_{k=1}^{\infty} y_k^2 < \infty \implies \sum_{k=1}^{\infty} |x_k y_k| < \infty$$

## Problem 5 (Question 1, Section 7.2)

We have that $x_n - x = \left(\frac{n}{1+n} - 1, \frac{1-n}{n} + 1\right)$. Then

$$||x_n - x|| = \left|\left|\left(\frac{n}{1+n} - 1, \frac{1-n}{n} + 1\right)\right|\right| = \sqrt{\left(\frac{n}{1+n} - 1\right)^2 + \left(\frac{1-n}{n} + 1\right)^2} = \sqrt{\frac{1}{(n+1)^2} + \frac{1}{n^2}}$$

Then by approximation

$$\sqrt{\frac{1}{(n+1)^2} + \frac{1}{n^2}} \leq \sqrt{\frac{2}{n^2}} = \frac{\sqrt{2}}{n}$$

Set $N = \frac{\sqrt{2}}{\epsilon}$, then for all $\epsilon > 0$, if $n > N$

$$||x_n - x|| \leq \frac{\sqrt{2}}{n} \leq \frac{\sqrt{2}}{N} = \epsilon$$

Showing that $(1, 1)$ is indeed the limit.

## Problem 6. (Question 2, Section 7.2)

The sequence $\{x_n\}$ should converge. By Theorem 7.2.13 the sequence $\{x_n\}$ converges if and only if each component of $\{x_n\}$ converges to it's corresponding component of $x$. Let $x = (\frac{1}{3}, 1)$. If the limit for each component can be found then by 7.2.13 the vector should also converge.
From 3210, we know that

$$\frac{n^2 + n - 1}{3n^2 +} \to \frac{1}{3} \qquad \text{and} \qquad \frac{n-1}{n+1} \to 1$$

Therefore $x_n \to x$ by Theorem 7.2.13.

## Problem 7. (Question 12, Section 7.2)

To prove something is a metric we must show that the following conditions are satisfied:

(1) $d(x, y) = d(y, x)$

(2) $d(x, y) = 0$ iff $x = y$

(3) $d(x, z) \leq d(x, y) + d(y, z)$

From the problem description, condition (2) is given. Therefore we only need to show condition (1) and 3 are satisfied.

(1) If $x \neq y$ then $y \neq x \implies d(y, x) = 1 = d(x, y) \implies d(x, y) = d(y, x)$

(3) By cases, we can see that

- if $x = y$ and $y = z \implies d(x, z) \leq d(x, y) + d(y, z) \implies 0 \leq 0$
- if $x \neq y$ and $y = z \implies d(x, z) \leq d(x, y) + d(y, z) \implies 1 \leq 1$
- if $x = y$ and $y \neq z \implies d(x, z) \leq d(x, y) + d(y, z) \implies 1 \leq 1$
- if $x \neq y$ and $y \neq z$ and $x \neq z \implies d(x, z) \leq d(x, y) + d(y, z) \implies 1 \leq 2$
- if $x \neq y$ and $y \neq z$ and $x = z \implies d(x, z) \leq d(x, y) + d(y, z) \implies 0 \leq 2$

Since these are all true, then condition (3) is satisfied. Thus, since al three conditions are satisfied, we know that this is a metric on $\mathbb{R}$.

## Problem 8. (Question 17, Section 7.2)

Let $X$ be the set of rooms, and $d(x, y)$ be the shortest possible path from room $x$ to $y$. If we take the shortest possible path from room $x$ to $y$, then we can simply retrace our steps from room $y$ to $x$ to follow the exact same path. Therefore $d(x, y) = d(y, x)$

If we are in room $x$ which also happens to be the same room as $y$, then the shortest path from $x$ to $y$ is to not move, since we are already in the room. Therefore $d(x, y) = 0$ if $x = y$.

Finally, If we start at room $x$ and plan to go to room $z$, but stop at at least one other room $y$ before arriving at $z$, then the distance it takes it go from room $x$ to $z$ if $y$ is along the way is the same as just going to room $z$ directly. (Assuming that "going" to a room doesn't take any extra time). However, if the room $y$ is not along the way, then we will have to spend some more time going to $y$, and then going back to room $z$, so overall the distance we walked was more than if had just went to room $z$ directly. Therefore, $d(x, z) \leq d(x, y) + d(y, z)$

Since these three conditions are true in this situation, then our shortest possible path is a metric for the rooms in the building.