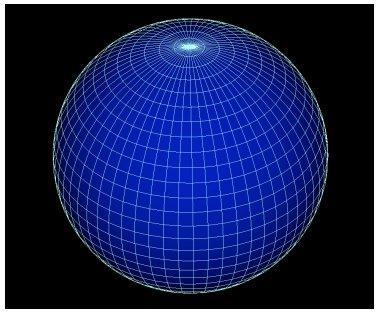
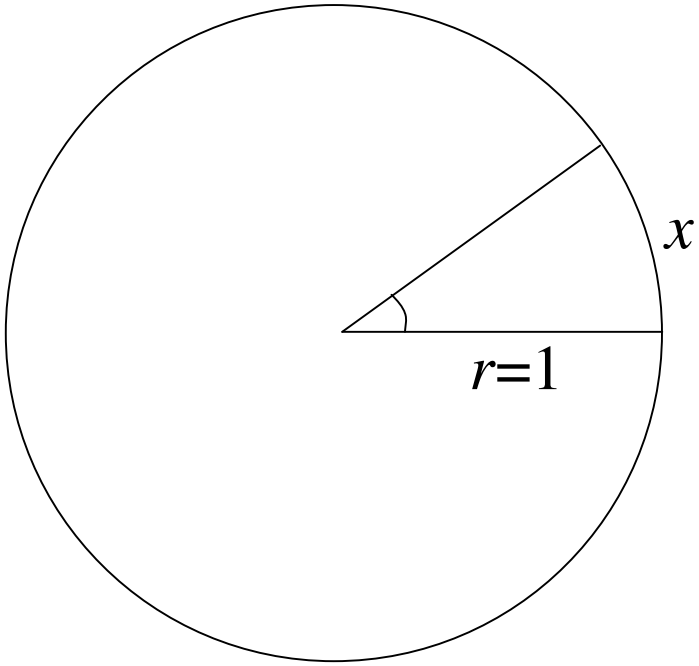
The diffusion equation on a circle



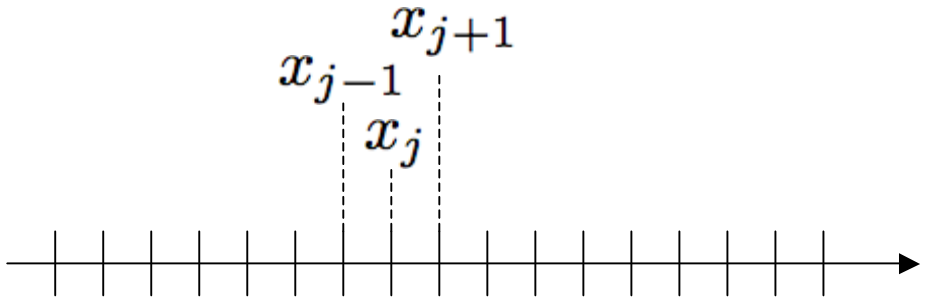


$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}$$

$$0 \leq x < 2\pi$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}$$

FDM: Finite Difference Method



$$\frac{d\psi}{dx} = \frac{\psi_{j+1} - \psi_{j-1}}{2\Delta x} + O(\Delta x)^2$$

$$\frac{d^2\psi}{dx^2} = \frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{(\Delta x)^2} + O(\Delta x)^2$$

$$\frac{d\psi_j}{dt} = \frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{(\Delta x)^2}$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}$$

FDM: Finite Difference Method

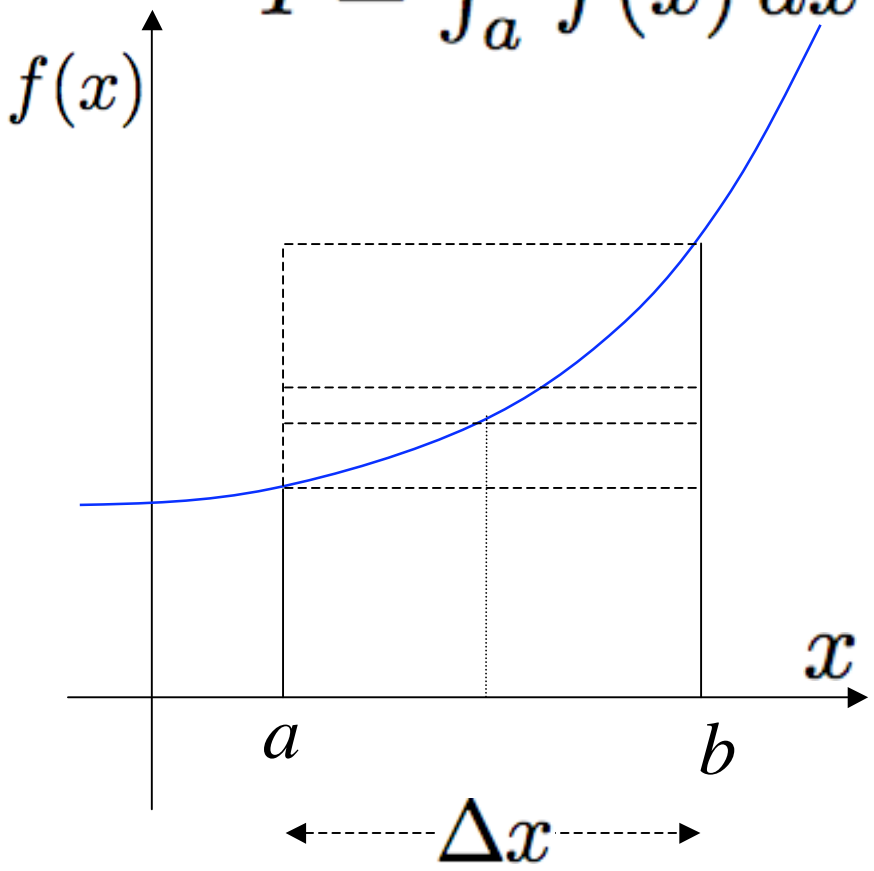
$$\frac{d\psi_j}{dt} = \frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{(\Delta x)^2}$$

$$\frac{d\psi_j}{dt} = f(\psi_1, \psi_2, \cdots, \psi_N)$$

==> Time integration.

Numerical integration

$$I = \int_a^b f(x)\,dx$$



1) $I = \Delta x\, f(a)$

Error $\propto O(\Delta x^2)$

2) Trapezoid rule

$$I = \frac{\Delta x}{2}\,[f(a) + f(b)]$$

Error $\propto O(\Delta x^3)$

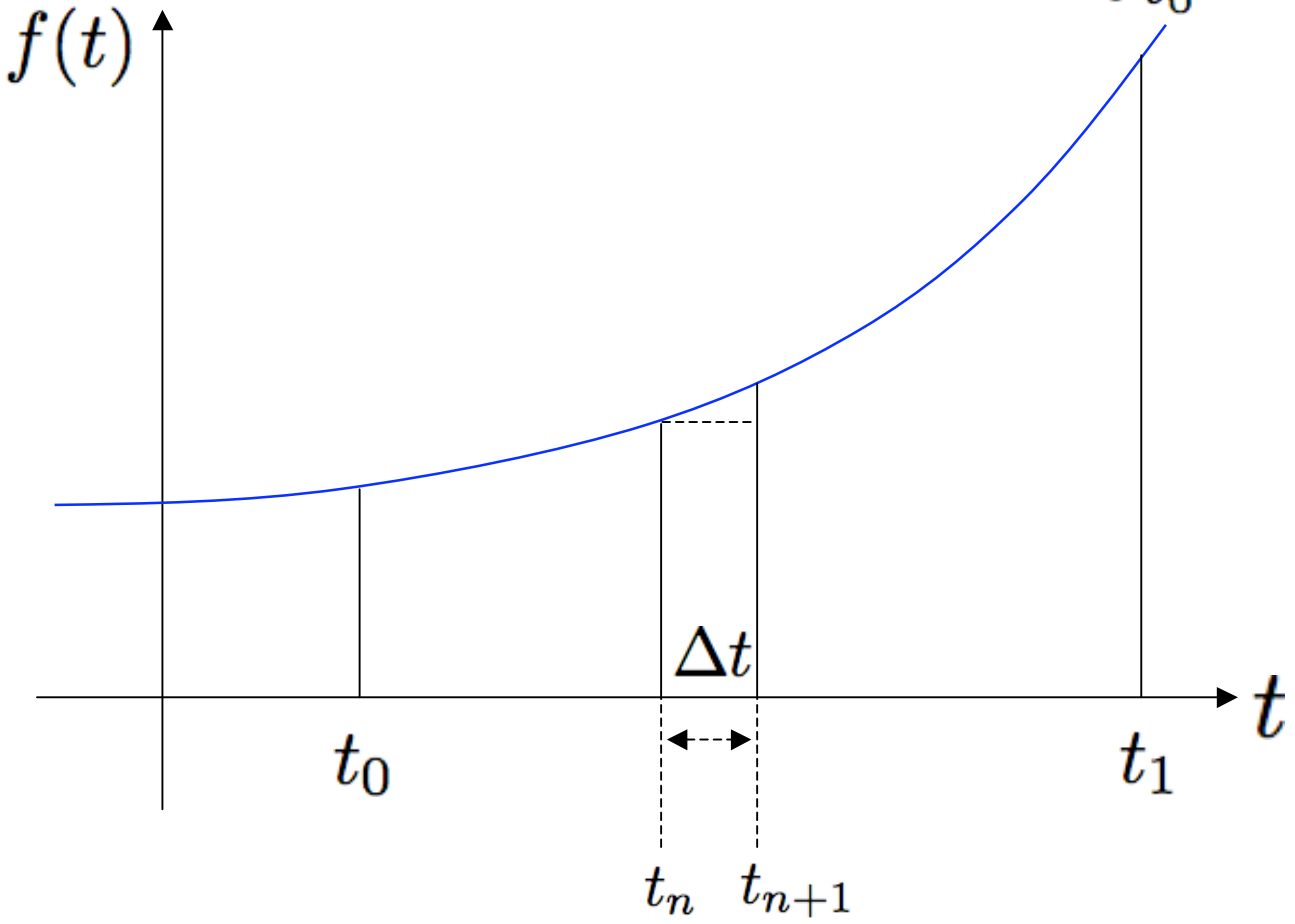
3) Simpson's rule

$$I = \frac{\Delta x}{6}\,\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Error $\propto O(\Delta x^5)$

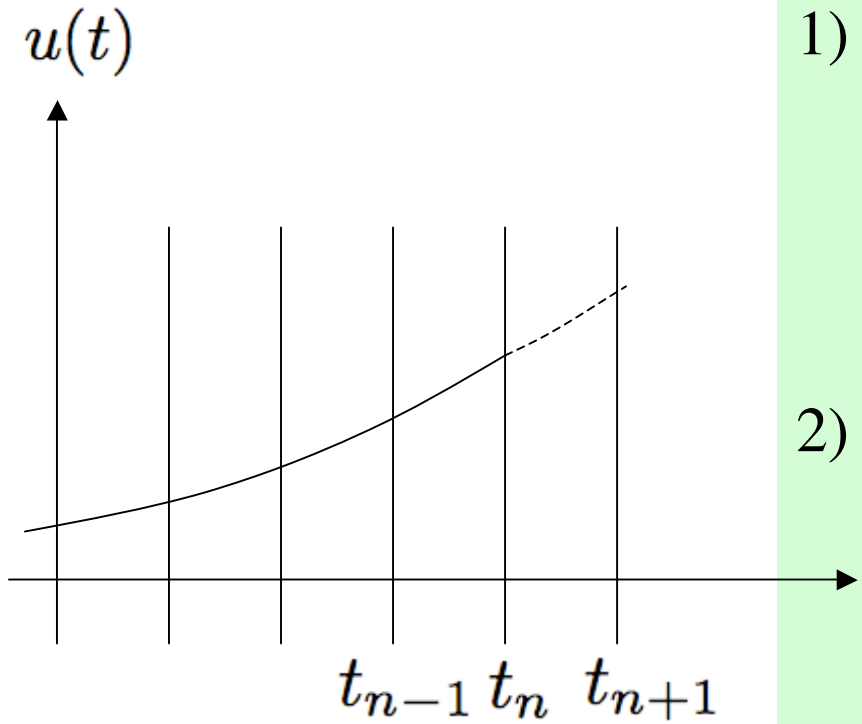
$$\frac{du(t)}{dt} = f(u(t), t)$$

$$u(t_1) = u(t_0) + \int_{t_0}^{t_1} f\, dt$$



$$\frac{du(t)}{dt} = f(u(t), t)$$

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} f\, dt$$



1)

$$u(t_{n+1}) = u(t_n) + \Delta t\, f(t_n)$$

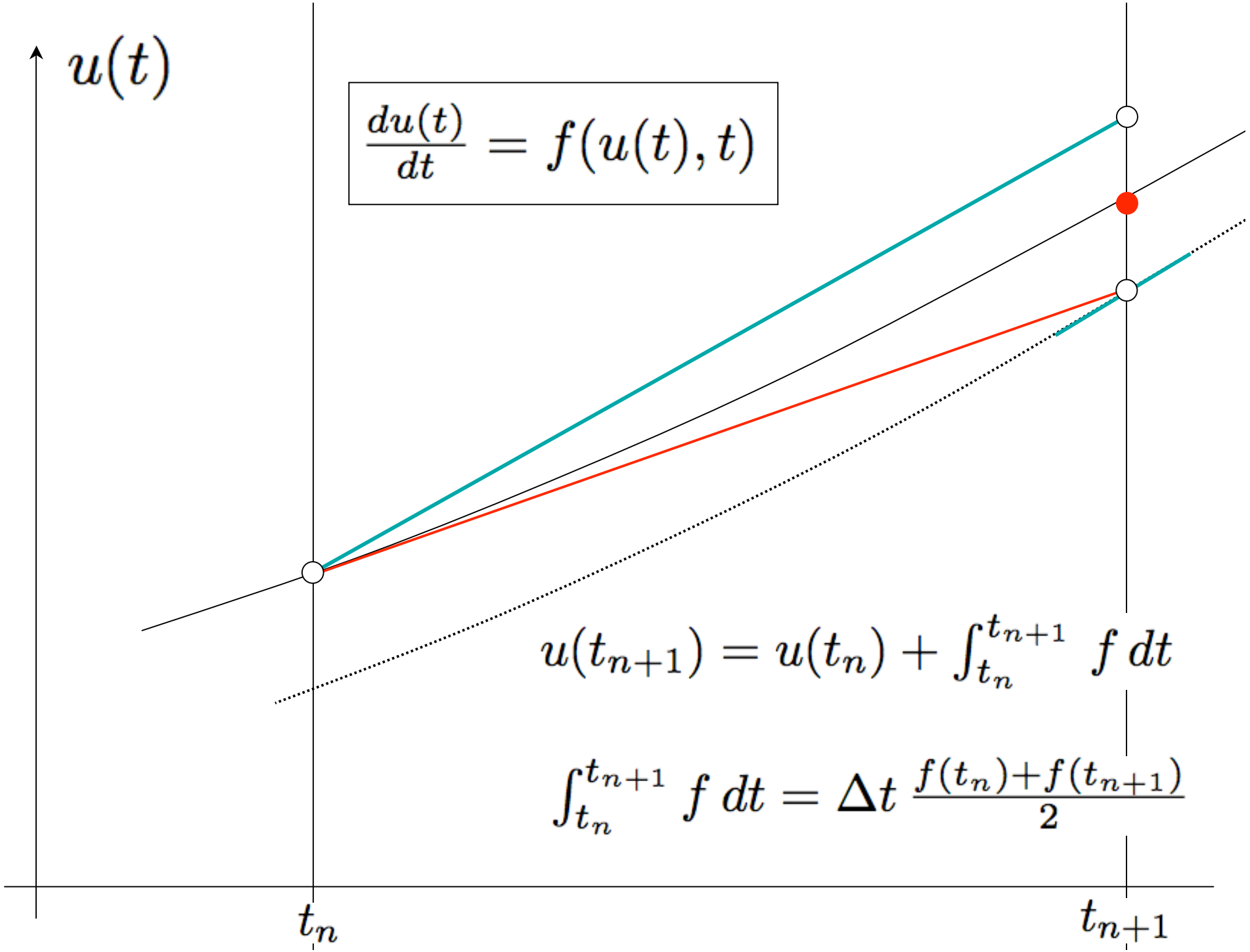
⇨ 1st order Euler method

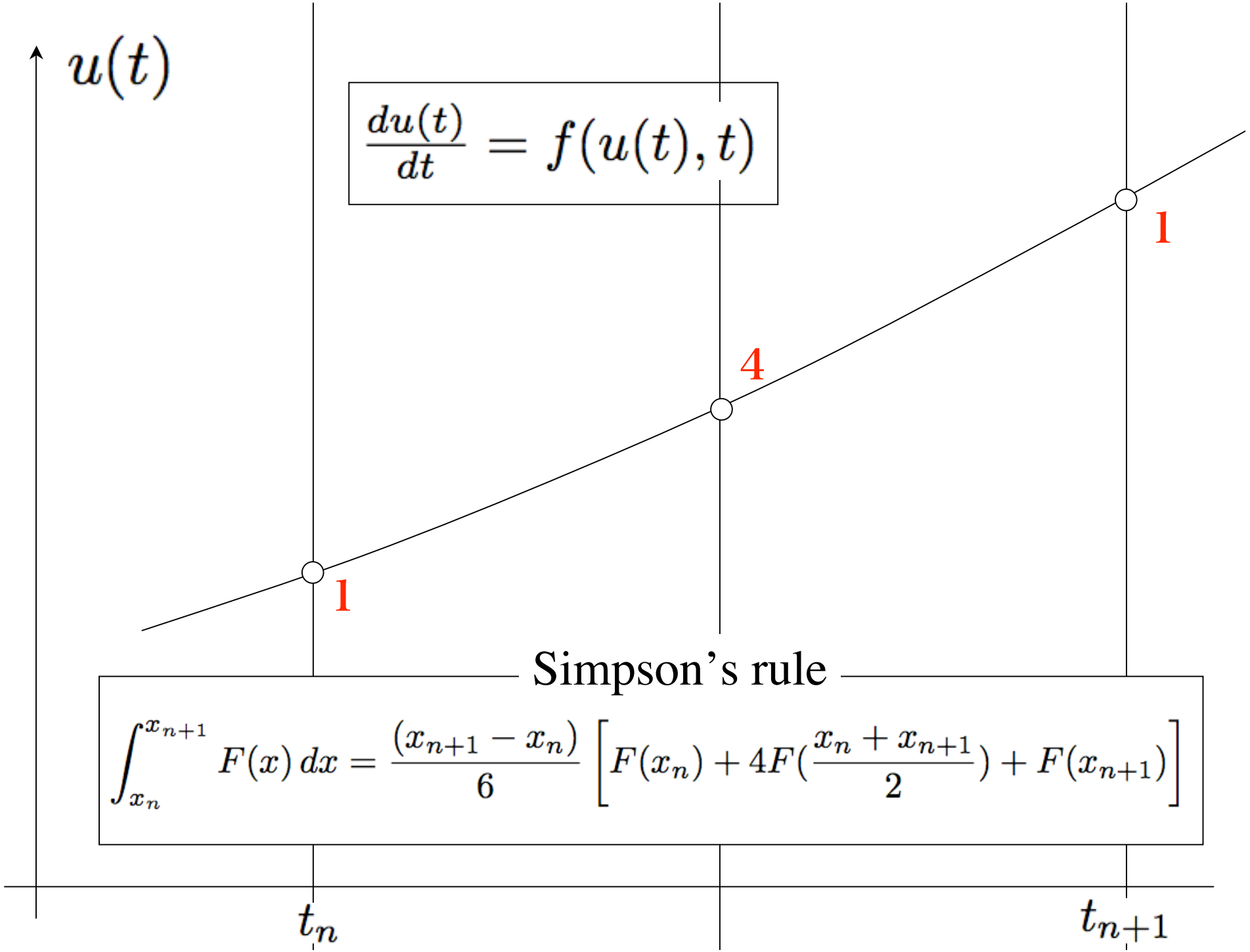
2) Trapezoid rule

⇨ 2nd order Runge-Kutta method

3) Simpson's rule

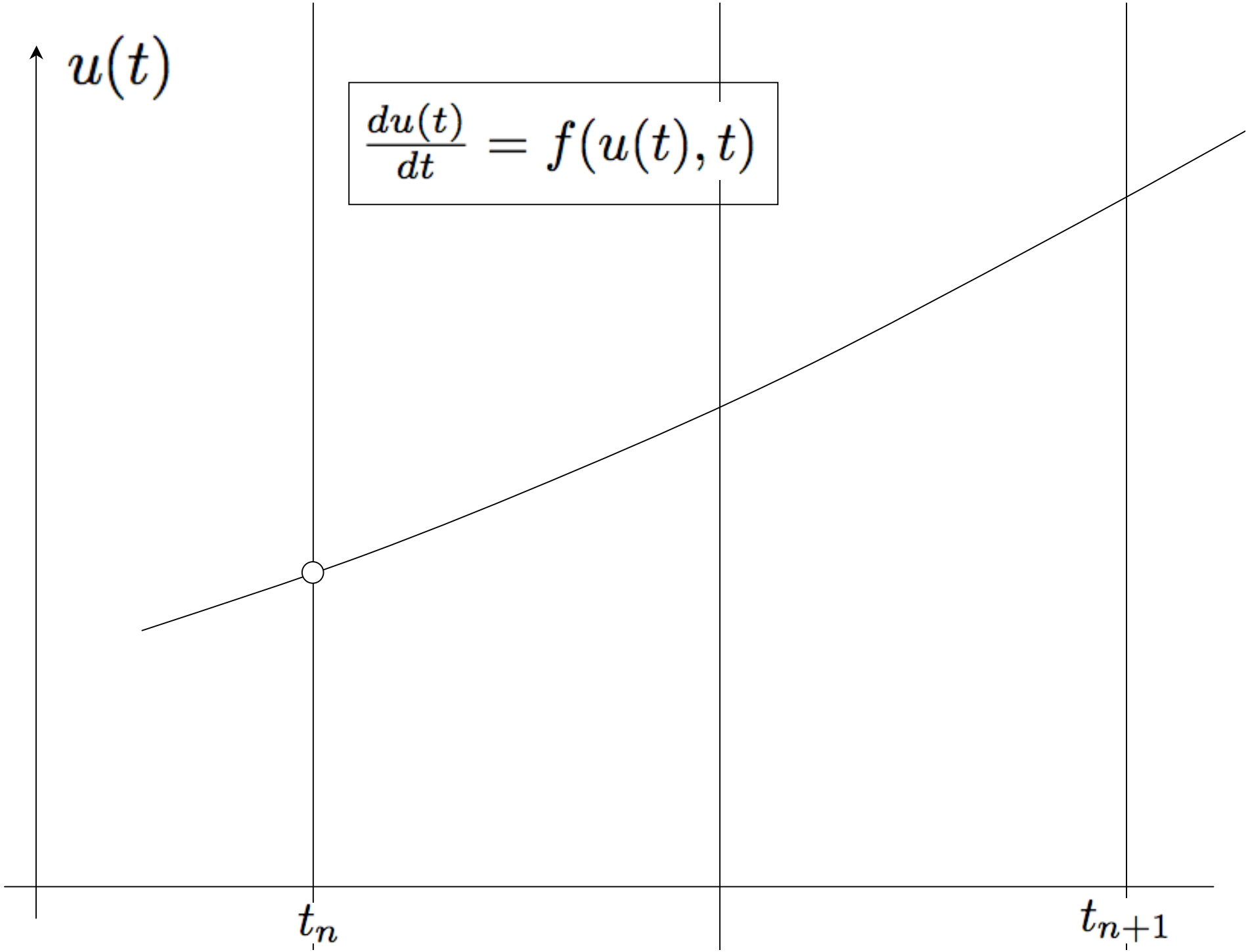
⇨ 4th order Runge-Kutta method

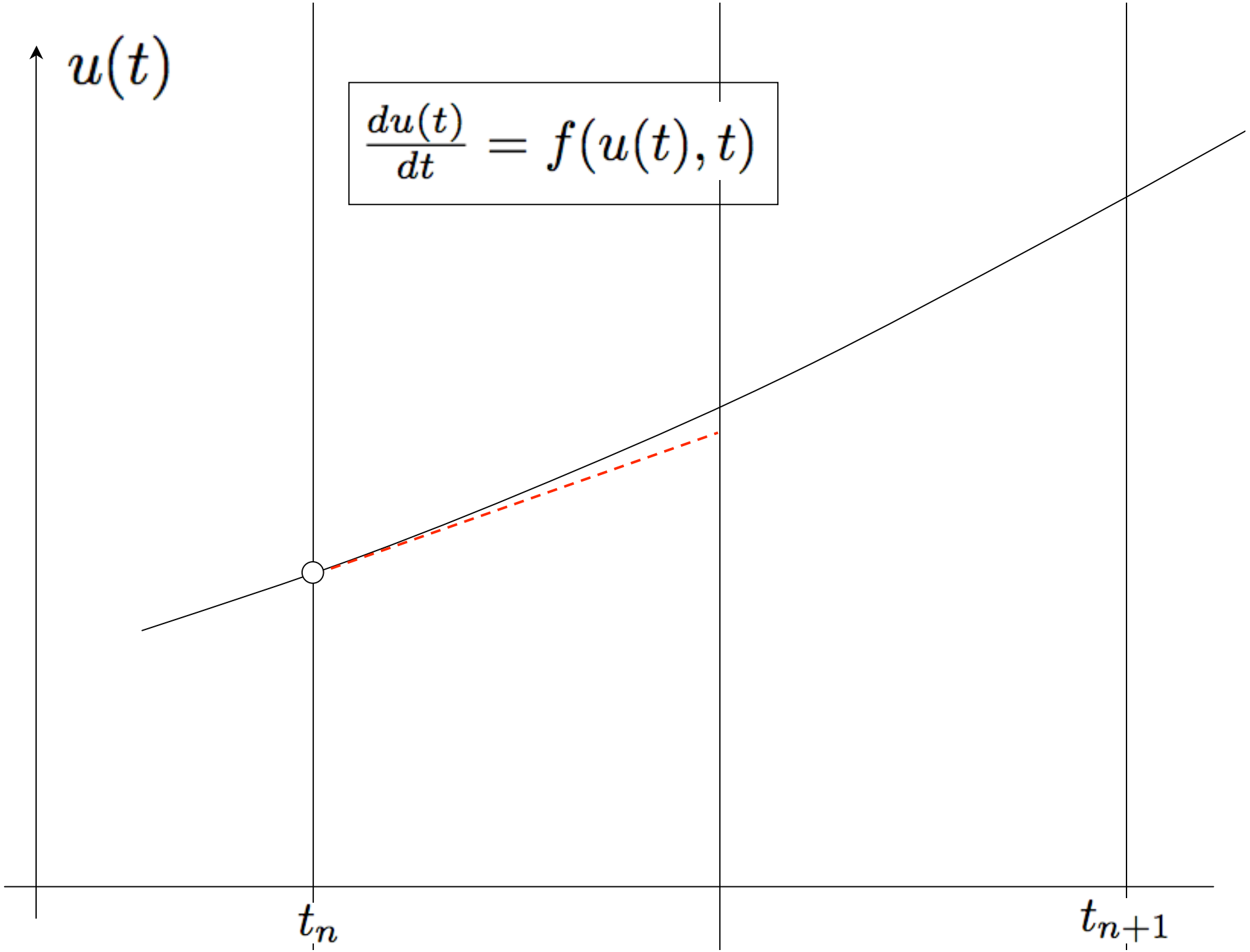

$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} f\, dt$
$\int_{t_n}^{t_{n+1}} f\, dt = \Delta t\, \frac{f(t_n)+f(t_{n+1})}{2}$
t_n
t_{n+1}

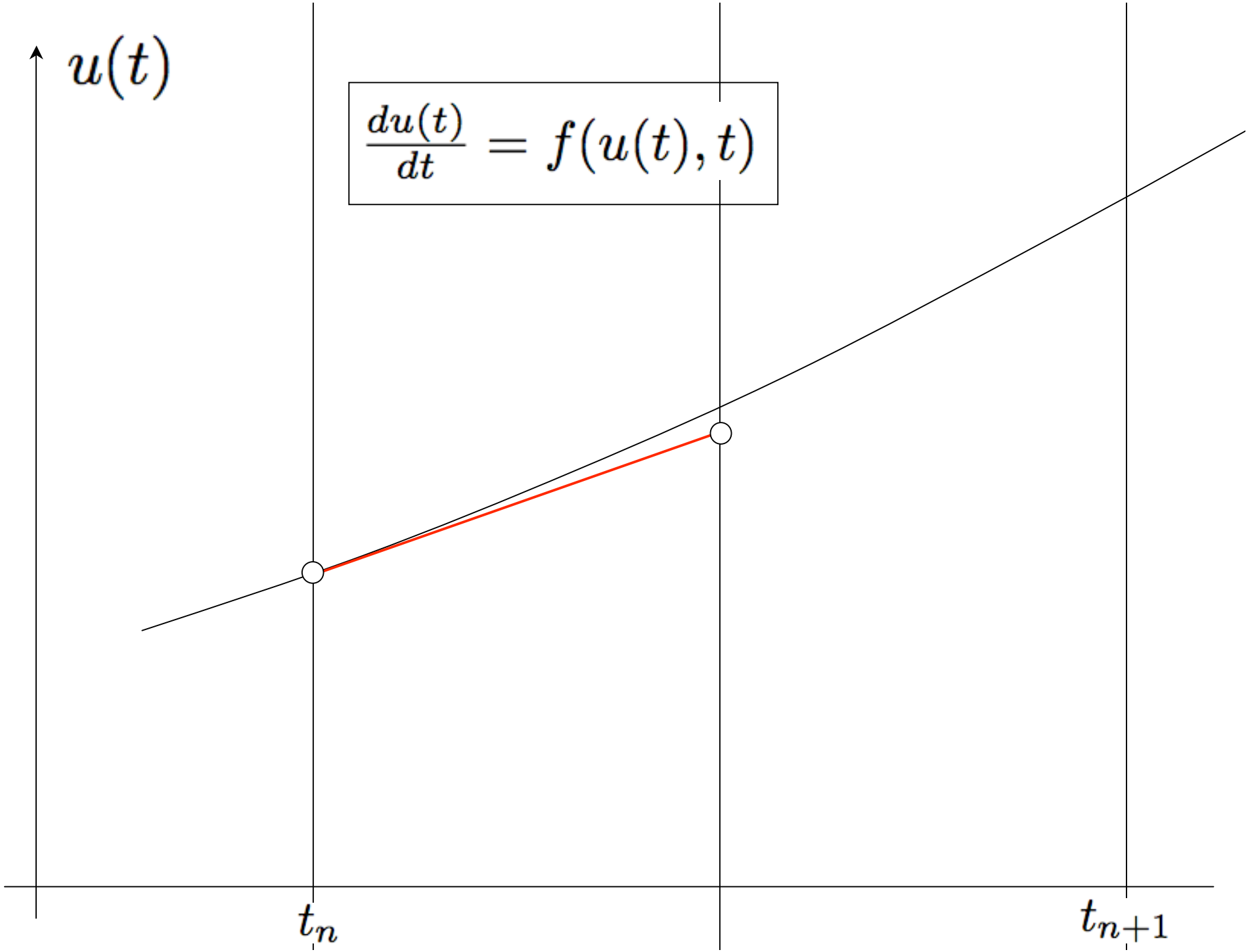


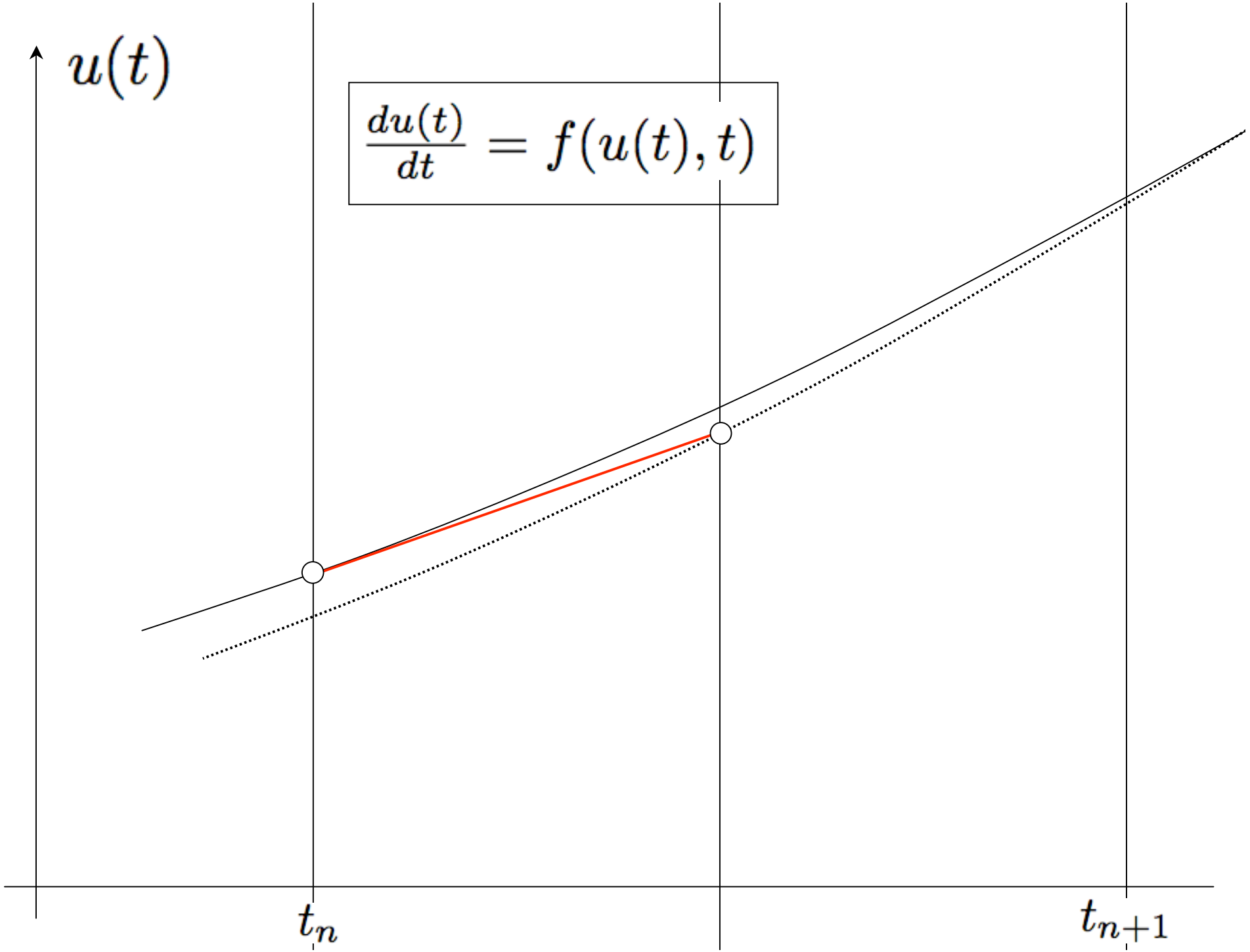
Simpson's rule

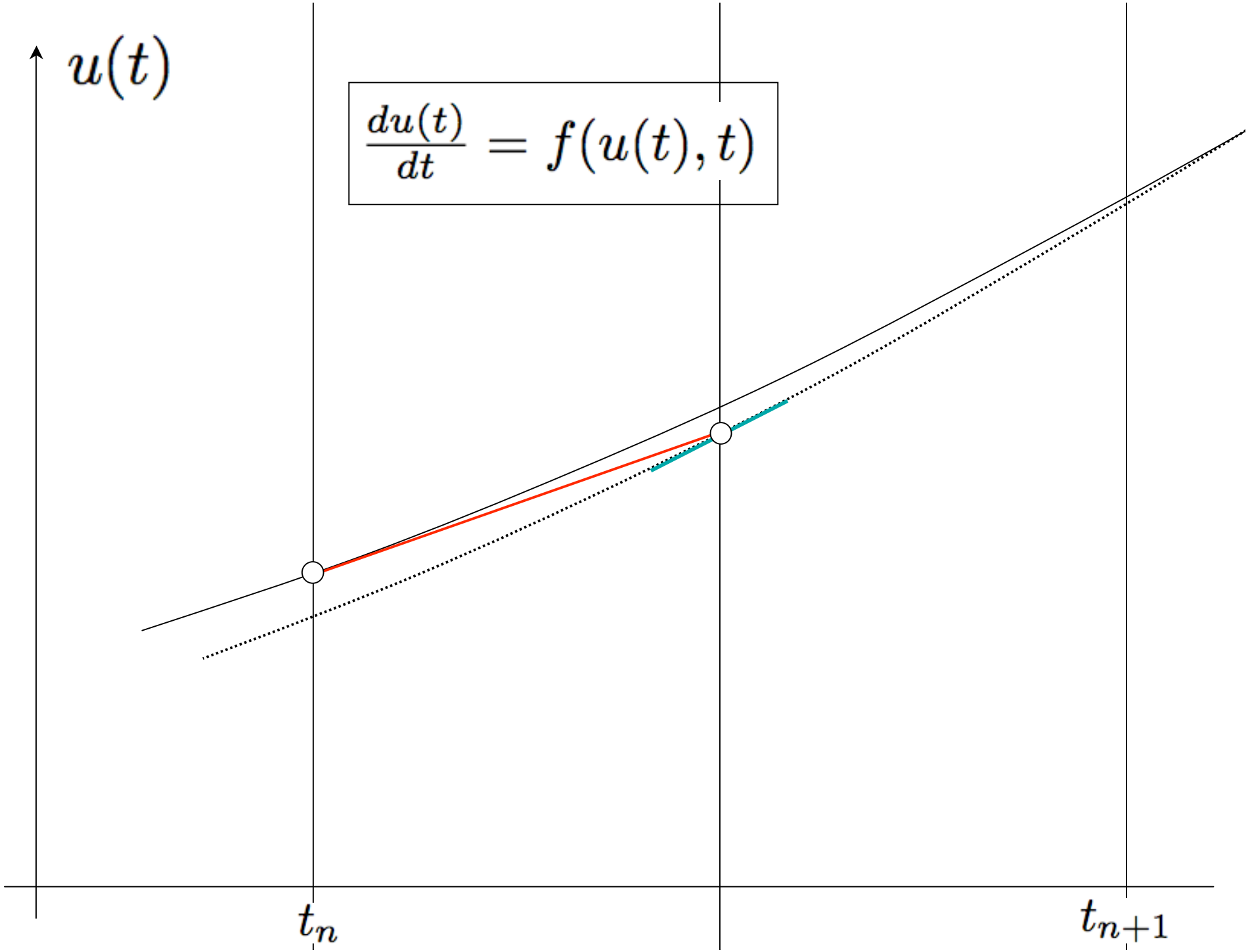
$$\int_{x_n}^{x_{n+1}} F(x)\,dx = \frac{(x_{n+1} - x_n)}{6}\left[F(x_n) + 4F(\frac{x_n + x_{n+1}}{2}) + F(x_{n+1})\right]$$

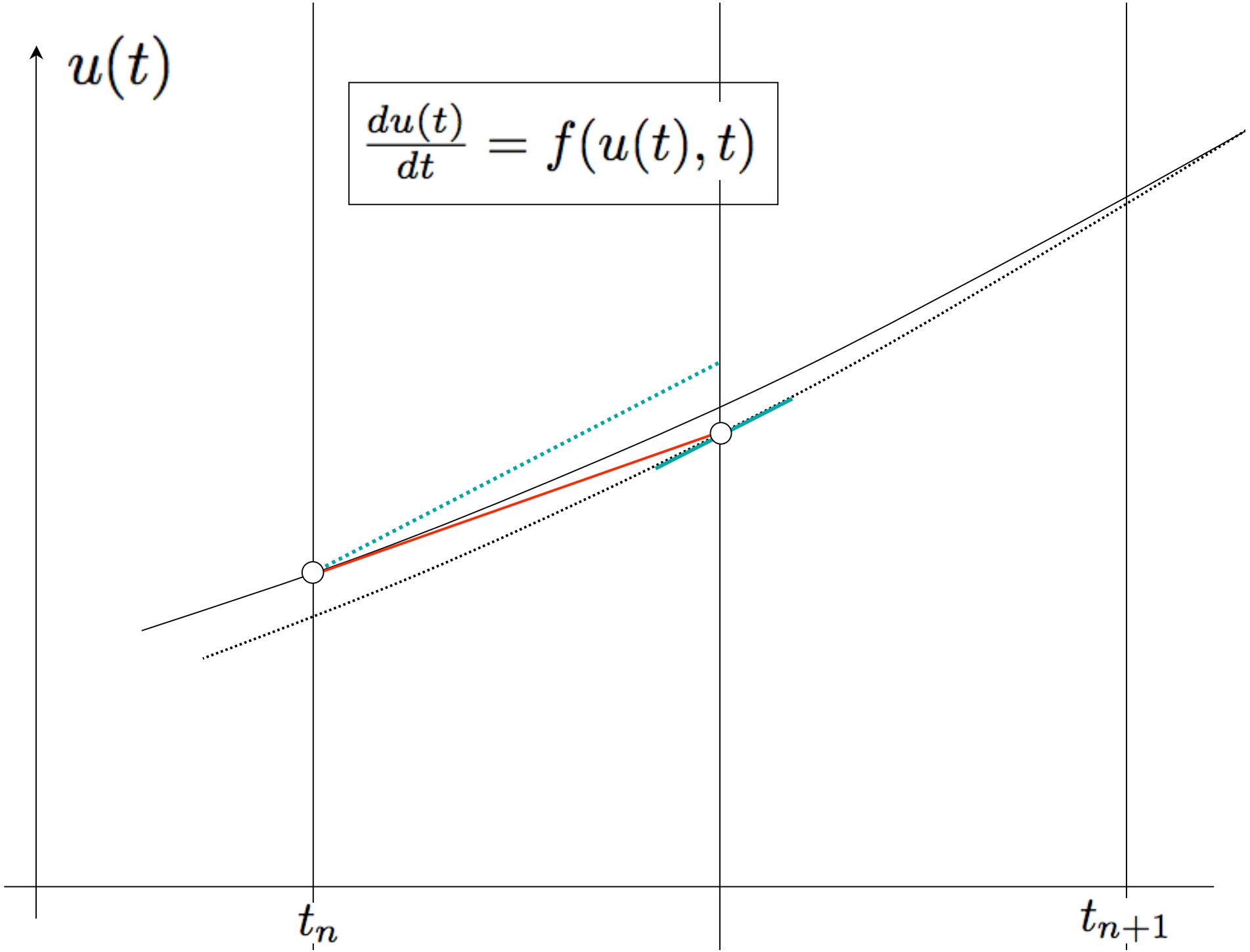

$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
t_n
t_{n+1}

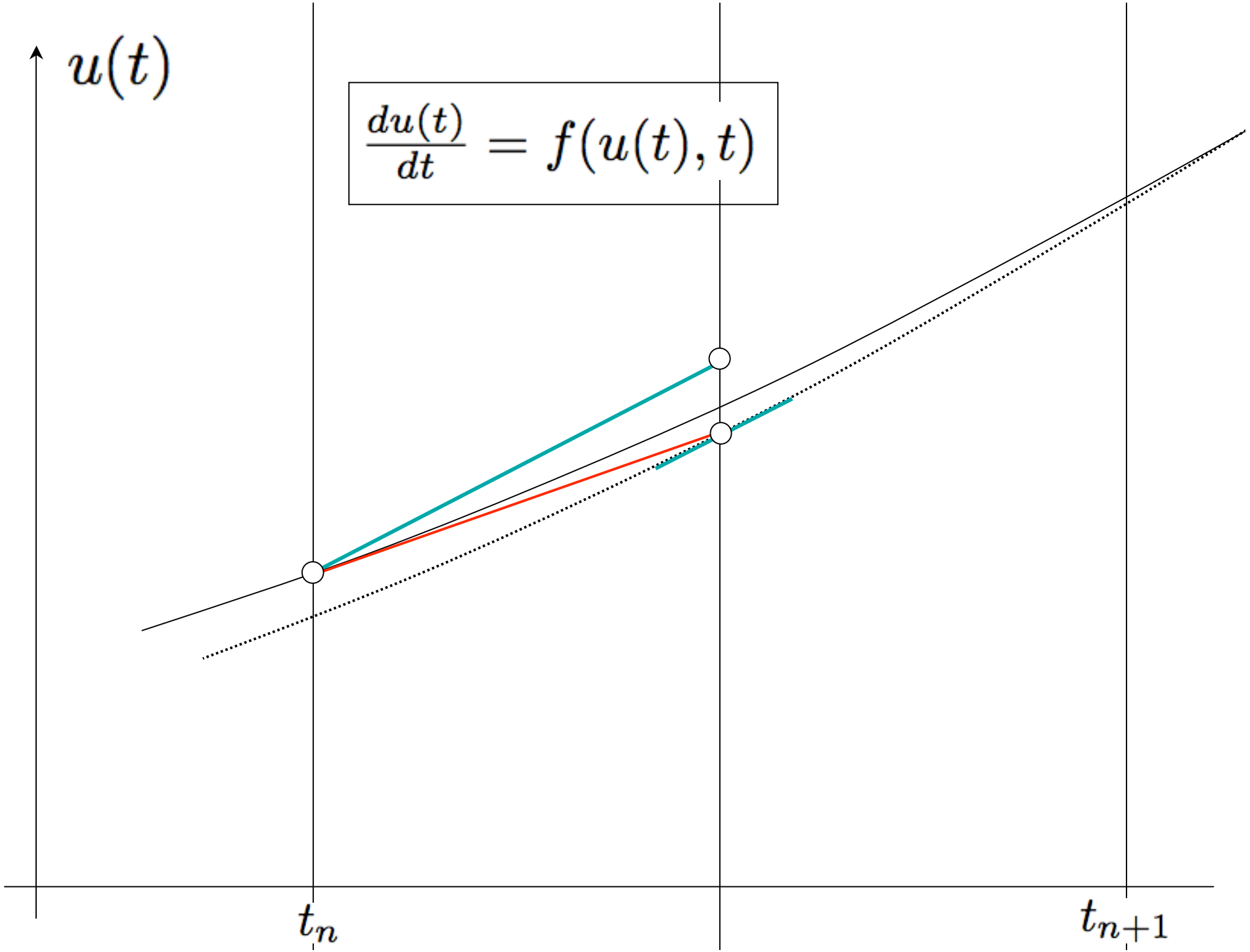

$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
t_n
t_{n+1}

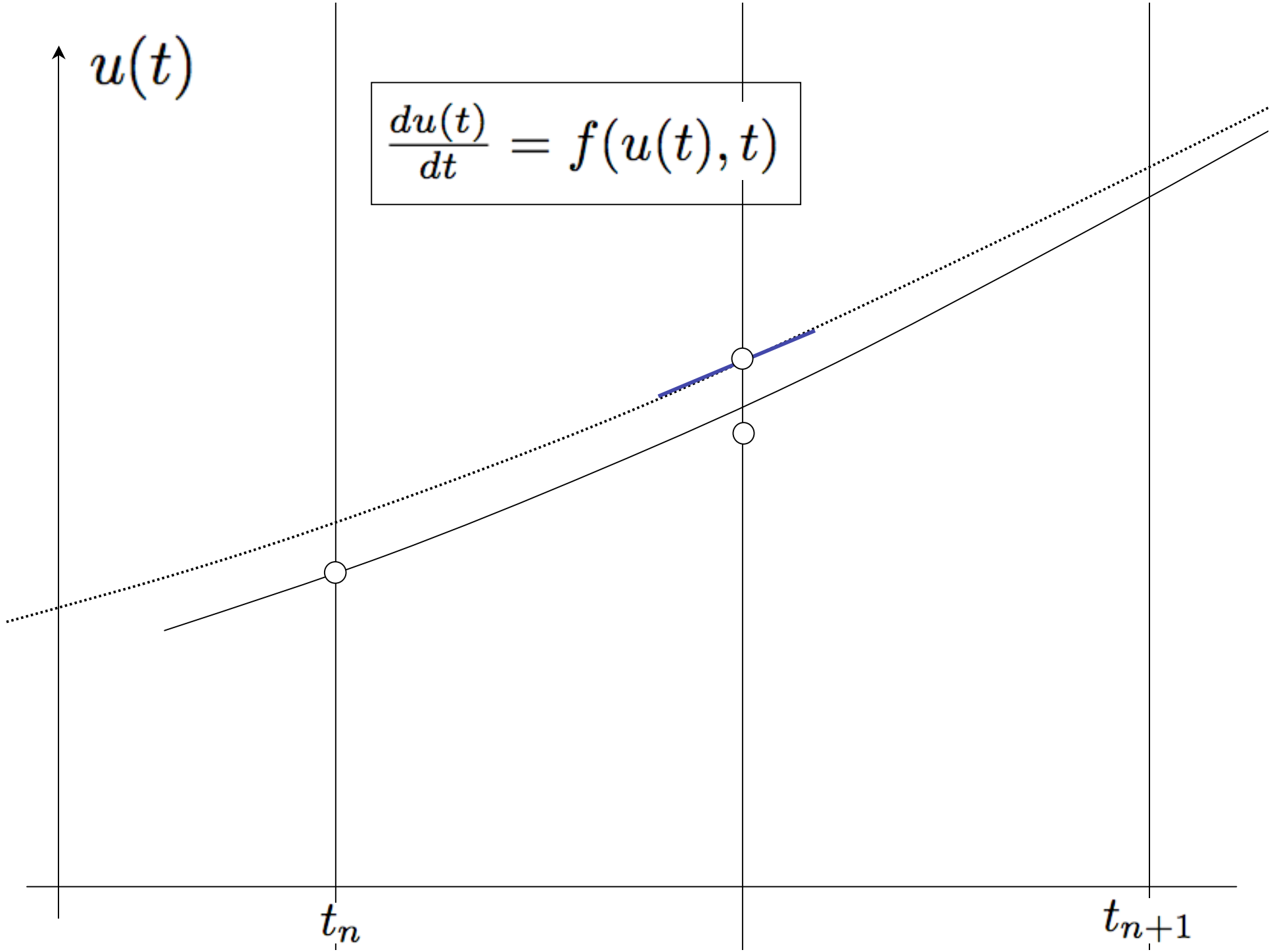

$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
t_n
t_{n+1}

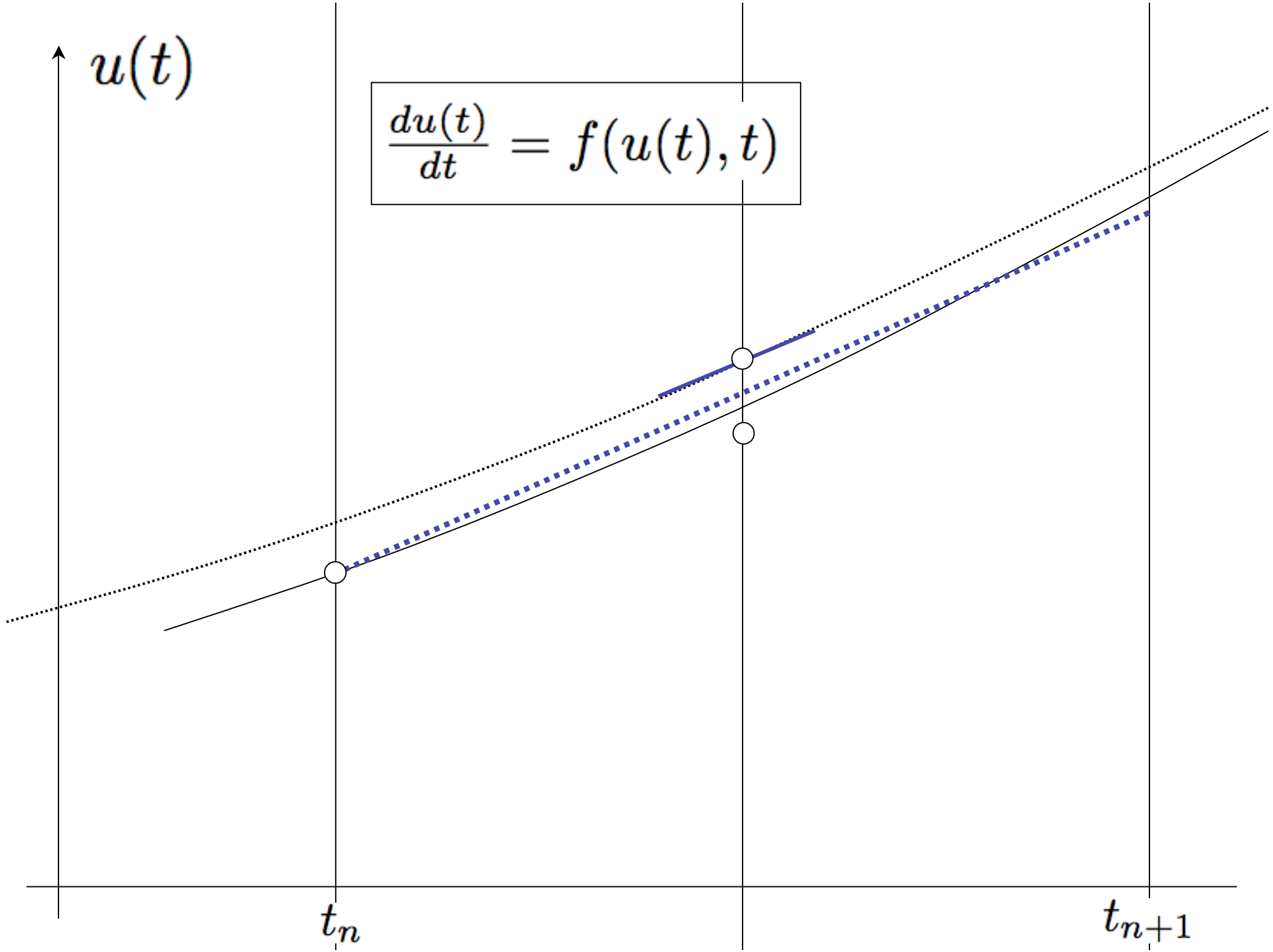

$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
t_n
t_{n+1}

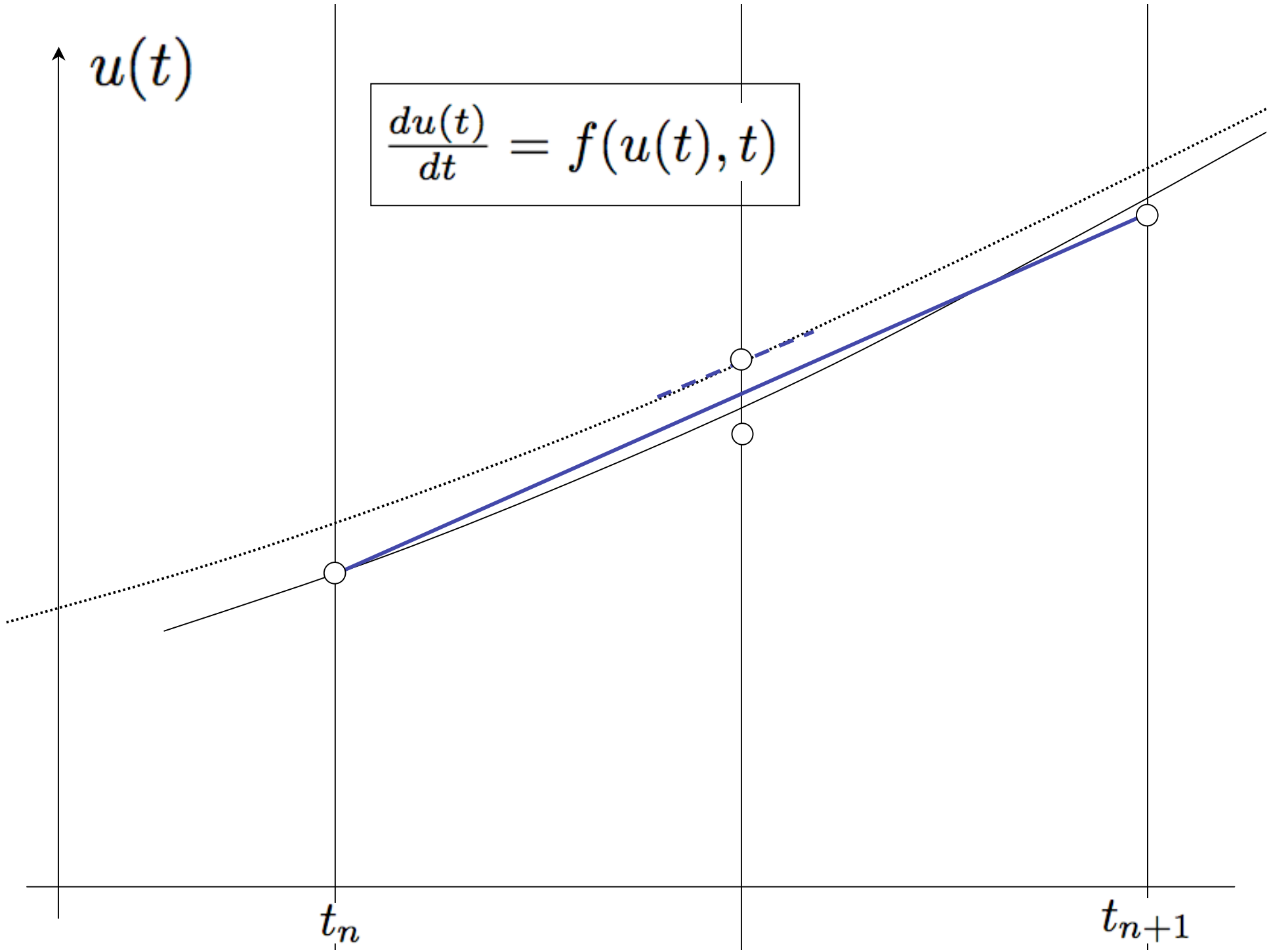

$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
t_n
t_{n+1}

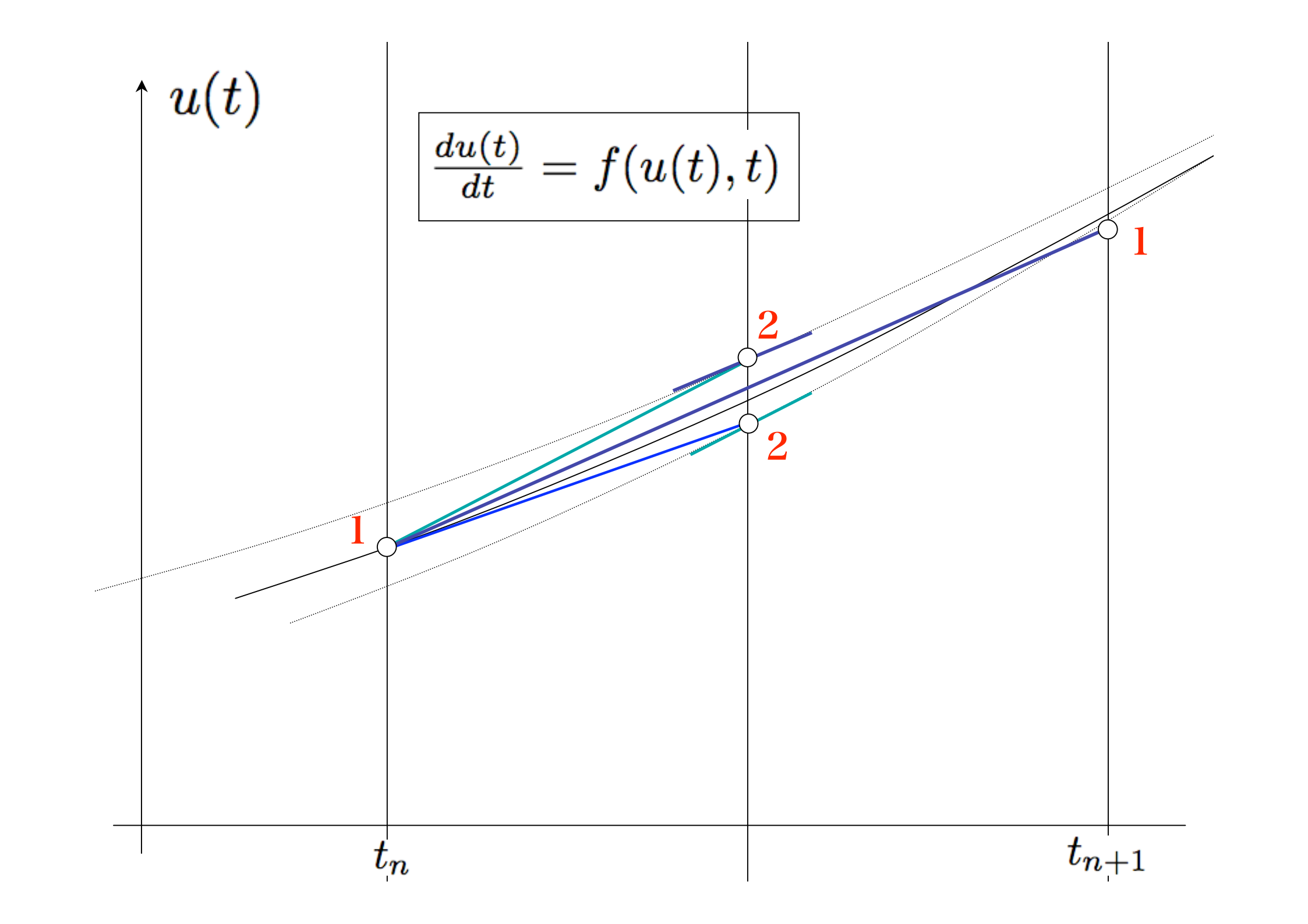

$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
t_n
t_{n+1}


$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
t_n
t_{n+1}


$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
t_n
t_{n+1}


$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
t_n
t_{n+1}


$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
t_n
t_{n+1}


$u(t)$
$\frac{du(t)}{dt} = f(u(t), t)$
1
2
2
1
t_n
t_{n+1}

4-step 4th-order Runge-Kutta Integration Method

$$\frac{du(t)}{dt} = f(t, u(t))$$

$$t_0 = t_n$$
$$u_0 = u(t_0)$$
$$df_1 = \Delta t\, f(t_0, u_0)$$

$$t_1 = t_0 + 0.5\,\Delta t$$
$$u_1 = u_0 + 0.5\, df_1$$
$$df_2 = \Delta t\, f(t_1, u_1)$$

$$t_2 = t_0 + 0.5\,\Delta t$$
$$u_2 = u_0 + 0.5\, df_2$$
$$df_3 = \Delta t\, f(t_2, u_2)$$

$$t_3 = t_0 + \Delta t$$
$$u_3 = u_0 + df_3$$
$$df_4 = f(t_3, u_3)$$

$$u_{n+1} = u_n + \tfrac{1}{6}(df_1 + 2\, df_2 + 2\, df_3 + df_4)$$

Error $O(\Delta t^5)$ for one step, $O(\Delta t^4)$ in total.