Lecture 2

Other important gates

- Y gate

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- S gate

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix}$$

- T gate

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

The gates X, Y and Z are also called, together with the identity, the Pauli Gates. Another notation is: σ_x, σ_y and σ_z

The Bloch Sphere

A common way to represent a qubit is by a point in the surface of a Bloch Sphere

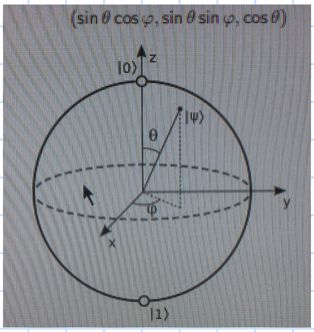
If $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2+|b|^2 = 1$, we can find angles λ, δ, θ such that:

$$\alpha = e^{i\lambda}\cos\frac{\theta}{2} \qquad B = e^{i\delta}\sin\frac{\theta}{2}$$

Since an overall phase is physically irrelevant, we can rewrite:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\delta}\sin\frac{\theta}{2}|1\rangle \quad \text{while } 0 \leqslant \theta \leqslant \pi \text{ and } 0 \leqslant \delta < 2\pi$$

From that equation we can obtain spherical coordinates for a point in R^3



Rotation Gates

$$R_X(\theta) = e^{-i\frac{\theta}{2}X} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_Y(\theta) = e^{-i\frac{\theta}{2}Y} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_Z(\theta) = e^{-i\frac{\theta}{2}Z} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Z = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

Notice that: $R_x(\pi) = X$ $\quad R_z(\frac{\pi}{2}) = S$

$R_y(\pi) = Y$ $\quad R_z(\frac{\pi}{4}) = T$

$R_z(\pi) = Z$

Using rotation gates to generate one-qubit gates

For any one-qubit gate U there exist a unit vector $r = (r_x, r_y, r_z)$ and an angle θ such that

$$U = e^{-i\frac{\theta}{2}r\sigma} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(r_xX + r_yY + r_zZ)$$

For instance, choosing $\theta = \pi$ and $r = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ we can see that:

$$H = e^{-i\frac{\theta}{2}r\sigma} = -i\frac{1}{\sqrt{2}}(X+Z)$$

Additionally, it can also be proved that there exist angles α, B and λ such that

$$U = R_x(\alpha)R_y(B)R_z(\lambda)$$

Inner Product, Dirac's Notation and Bloch Sphere

The inner product of two states $|\psi_1\rangle = \alpha_1|0\rangle + B_1|1\rangle$ and $|\psi_2\rangle = \alpha_2|0\rangle + B_2|1\rangle$ is given by:

$$\langle\psi_1|\psi_2\rangle = \alpha_1\alpha_2 + B_1B_2$$

Notice that $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$

Proof:

$$\begin{aligned} \langle\psi_1|\psi_2\rangle &= (\overline{\alpha_1}\langle 0| + \overline{\beta_1}\langle 1|)(\alpha_2|0\rangle + \beta_2|1\rangle) \\ &= \overline{\alpha_1}\alpha_2\langle 0|0\rangle + \overline{\alpha_1}\beta_2\langle 0|1\rangle + \overline{\beta_1}\alpha_2\langle 1|0\rangle + \overline{\beta_1}\beta_2\langle 1|1\rangle \\ &= \overline{\alpha_1}\alpha_2 + \overline{\beta_1}\beta_2 \end{aligned}$$

Two-qubit systems: more than the sum of their parts

Each of the qubits can be in state $|0\rangle$ or in state $|1\rangle$

So, for two qubits we have four possibilities:

$|00\rangle$, $|10\rangle$, $|01\rangle$ and $|11\rangle$

Of course, we can have superpositions so a generic state would be:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

where α_{xy} are complex numbers such that: $\sum_{x,y=0}^{1} |\alpha_{xy}|^2 = 1$

Measuring a two-qubit system

If we measure both qubits, we will obtain:

00 with probability $|\alpha_{00}|^2$ and the new state will be $|00\rangle$

$\vdots$

It is an analogous situation to what we had with one qubit, but now with four possibilities

Measuring just one qubit in a two-qubit system

If we have a state:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

we can also measure just one qubit

If we measure the first (for the second will be analogous):

- We will get 0 with prob: $|\alpha_{00}|^2 + |\alpha_{01}|^2$
- In that case, the new state of $|\psi\rangle$ will be:

$$\frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

Vector Representation

If we have a state:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

We can represent with the column vector

$$\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

A two-qubit quantum gate is a unitary matrix U of size 4×4

Tensor product of one-qubit gates

The simplest way of obtaining a two-qubit gate is by having a pair of one-qubit gates A and B acting on each of the qubits

In this case, the matrix for the two-qubit gate is the tensor product $A \otimes B$

It holds that: $(A \otimes B)(|\psi_1\rangle \otimes |\psi_2\rangle) = (A|\psi_1\rangle) \otimes (B|\psi_2\rangle)$

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{1,2}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \\ a_{2,1}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{2,2}\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}$$

The CNOT gate

The CNOT (or controlled-NOT or cX) gate is given by the unitary matrix

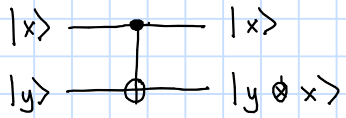
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

If the first qubit is $|0\rangle$, nothing changes. If it is $|1\rangle$, we flip the second bit (and the first stays the same)

That is: $|00\rangle \rightarrow |00\rangle \qquad |01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |11\rangle \qquad |11\rangle \rightarrow |10\rangle$

Schematics:



This is an extremely important gate for it allows to:

- Create entanglement
- Copy classical information
- Construct other controlled gates

The no-cloning theorem

There is no quantum gate that makes copies of an arbitrary qubit

Proof: Suppose that we have a gate U such that $U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$

Then $U|00\rangle = |00\rangle$ and $U|10\rangle = |11\rangle$ and by linearity

$$U\left[\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right] = \frac{1}{\sqrt{2}}\cdot[\underbrace{U|00\rangle}_{|00\rangle} + \underbrace{U|10\rangle}_{|11\rangle}] = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$$

But $\dfrac{|00\rangle + |10\rangle}{\sqrt{2}} = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}|0\rangle$

So we should have

$$U\left[\frac{|00\rangle + |10\rangle}{\sqrt{2}}\right] = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}\cdot\frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \neq \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Quantum Entanglement : the spooky action at a distance

We say that a state $|\psi\rangle$ is a product state if it can be written in the form :

$$|\psi\rangle = |\psi_1\rangle |\psi_2\rangle$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are two states (of at least one qubit)

An entangled state is a state that is not a product state

Examples (Bell gates) :

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$