Computing the Eigenvalues of a Matrix

To find the eigenvalues of a matrix, we need to find the roots of its characteristic polynomial. This can be done using the following algorithms:

Bisection Method: This algorithm uses the intermediate value theorem to find the root of a continuous function. The bounds [*a*,*b*] for the initial two points can be obtained using Laguerre-Samuelson's Inequality. If *f(a)f(b)<0*, a root lies in [*a*,*b*]. Then, the algorithm divides the interval [*a*,*b*] into two equal parts at [*c*]. If *f(a)f(c)<0*, the computation is repeated with b=c. If *f(b)f(c)<0*, the computation is repeated with a=c.

Consider a polynomial with all roots real:

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

Without loss of generality let $a_0 = 1$ and let

$$t_1 = \sum x_i \text{ and } t_2 = \sum x_i^2$$

Then

$$a_1 = -\sum x_i = -t_1$$

and

$$a_2 = \sum x_i x_j = \frac{t_1^2 - t_2}{2} \qquad \text{where } i < j$$

In terms of the coefficients

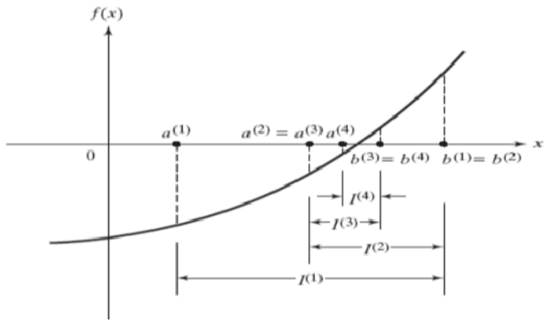
$$t_2 = a_1^2 - 2a_2$$

Laguerre showed that the roots of this polynomial were bounded by

$$-a_1/n \pm b\sqrt{n-1}$$

where

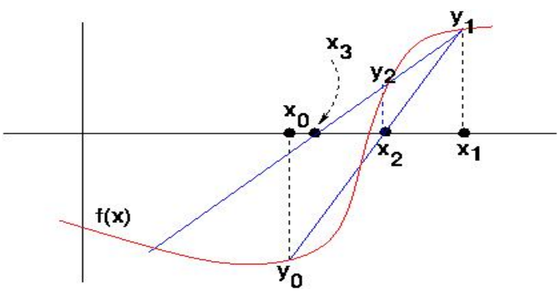
$$b = \frac{\sqrt{nt_2 - t_1^2}}{n} = \frac{\sqrt{na_1^2 - a_1^2 - 2na_2}}{n}$$



However, the bisection method does not always work for even-ordered polynomials. This is because *f(a)f(b)>0* is always greater than zero for even-powered polynomials.

Derivative Method: The derivative method is used to find the bounds of an even-ordered polynomial. One of the bounds from the Laguerre-Samuelson's Inequality is stored as [*a*]. The root of the differentiated function is stored as [*x*]. If *f(a)f(x)<0*, [*a*,*x*] is taken as the bound for the even-ordered polynomial. Using these bounds, the roots can now be calculated.

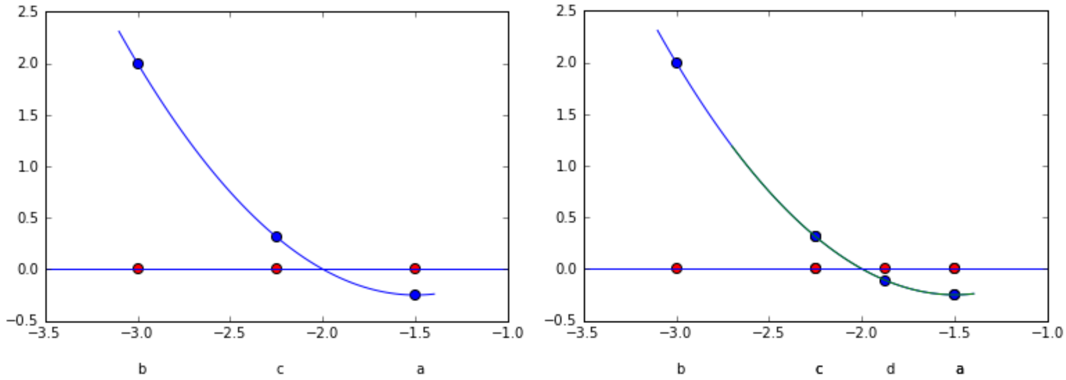
Secant Method:



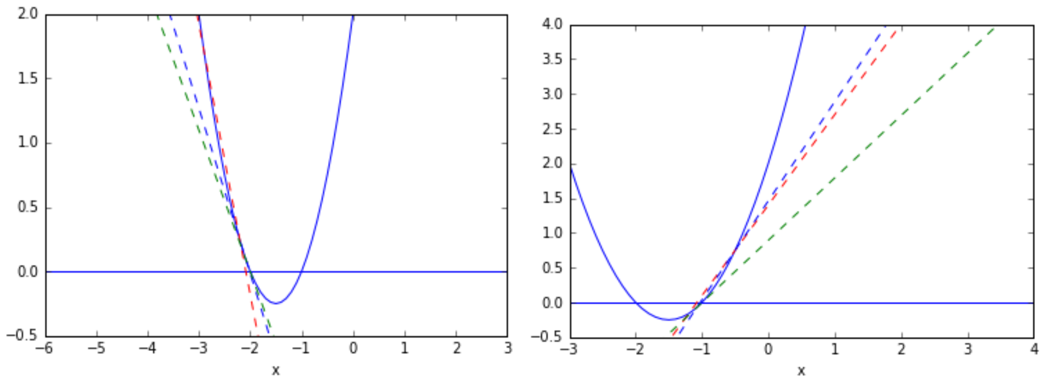
In the secant method, the root of the line from the initial two points of the function is found. This root is then set as the upper bound, and the lower bound is changed to the previous upper bound.

Working Of Algorithm:

$x^2 + 3x + 2 = 0$



Bisection Method



Secant Method

Generating the Polynomial as Co-effecients:

Let

$$\begin{aligned} p(s) &= a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \qquad (a_0 = 1) \\ &= (s - \lambda_1)\cdots(s-\lambda_n) \qquad (1) \end{aligned}$$

be the characteristic polynomial of an $n \times n$ matrix A, and define matrices $B_1, \cdots, B_n$ by

$$\operatorname{adj}(sI - A) = B_1 s^{n\ 1} + \cdots + B_{n-1}s + B_n,$$

so that

$$(sI - A)^{-1} = \frac{B_1 s^{n-1} + \cdots + B_{n-1}s + B_n}{p(s)}. \qquad (2)$$

A well-known algorithm, often attributed to Leverrier, Faddeev and others [1], indicates that the matrices B_k and the coefficients a_k can be obtained in a successive manner by means of the formulas

$$\begin{aligned} B_1 &= I, & a_1 &= -\tfrac{1}{1}\operatorname{tr}(AB_1), \\ B_2 &= AB_1 + a_1 I, & a_2 &= -\tfrac{1}{2}\operatorname{tr}(AB_2), \qquad (3) \\ &\vdots & &\vdots \\ B_n &= AB_{n-1} + a_{n-1}I, & a_n &= -\tfrac{1}{n}\operatorname{tr}(AB_n), \end{aligned}$$

To obtain the characteristic polynomial from the matrix, the Leverrier-Faddeev algorithm was used.
This algorithm can be derived using the Laplace transform. The link to the derivation is attached in the bibliography.

Synthetic Division Method: The previous methods give only one root of the polynomial. To find the other remaining roots, synthetic division is used. The polynomial is divided by [*x-(root)*]. The quotient is the new polynomial for which the roots are now found. This process is repeated until all the real roots are found.

Bibliography:

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