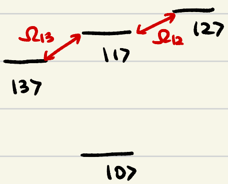
Feb. 16. Gyeonghun

Let's follow J. Randall's approach:



$$\hat{H}_0 = \sum_{i=1}^{3} \Delta_i |i\rangle\langle i|$$

$$\hat{H}_D = \sum_{i=2}^{3} \Omega_{1i} e^{i(\omega_i t+\varphi_i)} |1\rangle\langle i| + h.c. \quad , \quad \bar{\Omega}_{1i} = \Omega_{1i} e^{i(\omega_i t+\varphi_i)}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\Omega}_{12} & \bar{\Omega}_{13} \\ 0 & \bar{\Omega}_{12}^* & 0 & 0 \\ 0 & \bar{\Omega}_{13}^* & 0 & 0 \end{pmatrix}$$

$\hat{H}_D$ gives three dressed states and $|0\rangle$.

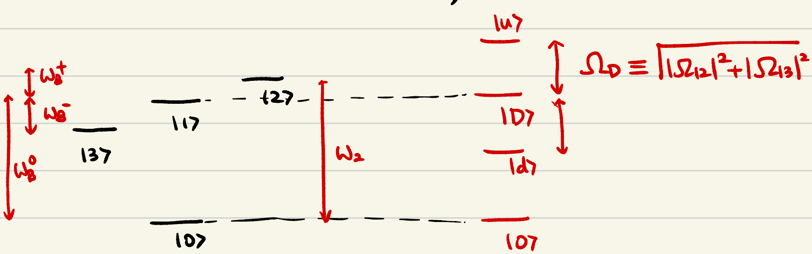
$$|D\rangle = \frac{1}{\sqrt{|\Omega_{12}|^2+|\Omega_{13}|^2}}\left(\bar{\Omega}_{13}|2\rangle - \bar{\Omega}_{12}|3\rangle\right) \quad \text{with eigenvalue } 0$$

$$|u\rangle = \frac{1}{\sqrt{2}\sqrt{|\Omega_{12}|^2+|\Omega_{13}|^2}}\cdot\left(\sqrt{|\Omega_{12}|^2+|\Omega_{13}|^2}\,|1\rangle + \bar{\Omega}_{12}^*|2\rangle + \bar{\Omega}_{13}^*|3\rangle\right) \quad \text{with eigenvalue } \sqrt{|\Omega_{12}|^2+|\Omega_{13}|^2}$$

$$|d\rangle = \frac{1}{\sqrt{2}\cdot\sqrt{|\Omega_{12}|^2+|\Omega_{13}|^2}}\left(-\sqrt{|\Omega_{12}|^2+|\Omega_{13}|^2}\,|1\rangle + \bar{\Omega}_{12}^*|2\rangle + \bar{\Omega}_{13}^*|3\rangle\right) \quad \text{with eigenvalue } -\sqrt{|\Omega_{12}|^2+|\Omega_{13}|^2}$$

Then,

$$\hat{H}_D = \sqrt{|\Omega_{12}|^2+|\Omega_{13}|^2}\cdot\left(|u\rangle\langle u| - |d\rangle\langle d|\right) = \Omega_D\left(|u\rangle\langle u| - |d\rangle\langle d|\right)$$



Now, let's calculate probing Hamiltonian.

With bare states,

$$\hat{H}_P = \sum_{i=1}^{3} \Omega_{0i} e^{i(\omega_s t+\varphi_s)} |0\rangle\langle i| + h.c. \quad , \quad \bar{\Omega}_{0i} = \Omega_{0i} e^{i(\omega_s t+\varphi_s)}$$

Let's calculate interaction picture Hamiltonian about H_0

interaction picture w.r.t. H_0

$$\hat{H}_P^{(I_0)} = \exp(iH_0 t) H_P \exp(-iH_0 t)$$

$$= \exp\left(i\sum_{i=1}^{3}\Delta_i|i\rangle\langle i|t\right)\left(\sum_{j=1}^{3}\Omega_{0j}e^{i(\omega_s t+\varphi_s)}|0\rangle\langle j|\right)\cdot\exp\left(-i\sum_{k=1}^{3}\Delta_k|k\rangle\langle k|t\right) + h.c.$$

With

$$\left(\sum_{k=1}^{3}\Delta_k|k\rangle\langle k|\right)^{\ell}=\begin{cases}\sum_{k=1}^{3}\Delta_k^{\ell}\cdot|k\rangle\langle k| & (\ell\neq 0)\\ \mathbb{1} & (\ell=0)\end{cases}$$

$$\mathbb{1}\rightarrow\left(\sum_k(\Delta_k)^0\cdot|k\rangle\langle k|\right)+\overbrace{\left(\mathbb{1}-\sum_k|k\rangle\langle k|\right)}^{=|0\rangle\langle 0|}$$

$$\therefore\ \sum_{\ell}\left(\sum_{k=1}^{3}\Delta_k|k\rangle\langle k|\right)^{\ell}=\sum_{\ell}\left(\sum_{k=1}^{3}\Delta_k^{\ell}|k\rangle\langle k|\right)+|0\rangle\langle 0|$$

$$=\sum_{\ell,m}\sum_{ijk}\frac{(it)^{\ell}(-it)^{m}}{\ell!\,m!}\Omega_j\Delta_i^{\ell}\Delta_k^{m}e^{i(\omega_2 t+\varphi_2)}\ |i\rangle\underbrace{\cancel{\langle i|0\rangle}}_{0}\langle j|k\rangle\langle k|$$

$$+\sum_{\ell}\sum_{ij}\frac{(it)^{\ell}}{\ell!}\Omega_{0j}\Delta_i^{\ell}e^{i(\omega_2 t+\varphi_2)}\ |i\rangle\langle i|0\rangle\underbrace{\cancel{\langle j|0\rangle}}_{0}\langle 0|$$

$$+\sum_{mjk}\frac{(-it)^{m}}{m!}\Omega_{0j}\Delta_k^{m}e^{i(\omega_2 t+\varphi_2)}\ |0\rangle\underbrace{\cancel{\langle 0|0\rangle}}_{1}\underbrace{\langle j|k\rangle}_{\delta_{jk}}\langle k|$$

$$+\sum_{j}\Omega_{0j}e^{i(\omega_2 t+\varphi_2)}\ |0\rangle\langle 0|0\rangle\underbrace{\cancel{\langle j|0\rangle}}_{0}\langle 0|\ +h.c.$$

$$\therefore\ \hat{H}_p^{(I)}=\sum_{j=1}^{3}\Omega_{0j}\exp\left(i\left((-\Delta_j+\omega_2)t+\varphi_2\right)\right)|0\rangle\langle j|\ +h.c.$$

$$=\sum_{i=1}^{3}\Omega_{0i}e^{i\varphi_2}\cdot\exp(i\delta_i t)|0\rangle\langle i|\ +h.c.$$

Let $\delta_i=\omega_2-\Delta_i$

With dressed states

$$\begin{cases}|D\rangle=\frac{1}{\Omega_D}e^{i(\omega_1 t+\varphi_1)}\left(\Omega_{13}|2\rangle-\Omega_{12}|3\rangle\right)\\ |u\rangle=\frac{1}{\sqrt{2}\Omega_D}\left(\Omega_D|1\rangle+\Omega_{12}^{*}e^{-i(\omega_1 t+\varphi_1)}|2\rangle+\Omega_{13}^{*}e^{-i(\omega_1 t+\varphi_1)}|3\rangle\right)\\ |d\rangle=\frac{1}{\sqrt{2}\Omega_D}\left(-\Omega_D|1\rangle+\Omega_{12}^{*}e^{-i(\omega_1 t+\varphi_1)}|2\rangle+\Omega_{13}^{*}e^{-i(\omega_1 t+\varphi_1)}|3\rangle\right)\end{cases}$$

$$\begin{cases}|u\rangle-|d\rangle=\sqrt{2}|1\rangle\ \Rightarrow\ |1\rangle=\frac{1}{\sqrt{2}}(|u\rangle-|d\rangle)\\ \frac{1}{\sqrt{2}}(|u\rangle+|d\rangle)=\frac{1}{\Omega_D}\cdot e^{-i(\omega_1 t+\varphi_1)}\cdot\left(\Omega_{12}^{*}|2\rangle+\Omega_{13}^{*}|3\rangle\right)\\ |D\rangle=\frac{1}{\Omega_D}e^{i(\omega_1 t+\varphi_1)}\left(\Omega_{13}|2\rangle-\Omega_{12}|3\rangle\right)\end{cases}$$

$$\begin{cases}\begin{cases}\frac{\Omega_D}{\sqrt{2}}e^{i(\omega_1 t+\varphi_1)}\cdot\Omega_{13}(|u\rangle+|d\rangle)=\Omega_{12}^{*}\Omega_{13}|2\rangle+|\Omega_{13}|^{2}|3\rangle\\ \Omega_D e^{-i(\omega_1 t+\varphi_1)}\Omega_{12}^{*}|D\rangle=\Omega_{12}^{*}\Omega_{13}|2\rangle-|\Omega_{12}|^{2}|3\rangle\end{cases}\\ \begin{cases}\frac{\Omega_D}{\sqrt{2}}\bar{\Omega}_{12}(|u\rangle+|d\rangle)=|\Omega_{12}|^{2}|2\rangle+\Omega_{13}^{*}\Omega_{12}|3\rangle\\ \Omega_D\bar{\Omega}_{13}^{*}|D\rangle=|\Omega_{13}|^{2}|2\rangle-\Omega_{13}^{*}\Omega_{12}|3\rangle\end{cases}\end{cases}$$

$$\Omega_D^2 |3\rangle = \Omega_D \left[\frac{\Omega_{13}}{\sqrt{2}} e^{i(\omega_1 t + \varphi_1)} (|u\rangle + |d\rangle) - \Omega_D^* e^{-i(\omega_1 t + \varphi_1)} \cdot |D\rangle \right]$$

$$\therefore |3\rangle = \frac{1}{\Omega_D} \left[\frac{\bar{\Omega}_{13}}{\sqrt{2}} |u\rangle + \frac{\bar{\Omega}_{13}}{\sqrt{2}} |d\rangle - \bar{\Omega}_{12}^* |D\rangle \right]$$

$$\Omega_D^2 |2\rangle = \Omega_D \left[\frac{\bar{\Omega}_{12}}{\sqrt{2}} (|u\rangle + |d\rangle) + \bar{\Omega}_{13}^* |D\rangle \right]$$

$$\therefore |2\rangle = \frac{1}{\Omega_D} \left[\frac{\bar{\Omega}_{12}}{\sqrt{2}} |u\rangle + \frac{\bar{\Omega}_{12}}{\sqrt{2}} |d\rangle + \bar{\Omega}_{13}^* |D\rangle \right]$$

$$\therefore \begin{cases} |1\rangle = \frac{1}{\sqrt{2}} (|u\rangle - |d\rangle) \\ |2\rangle = \frac{1}{\Omega_D} \left[\frac{\bar{\Omega}_{12}}{\sqrt{2}} |u\rangle + \frac{\bar{\Omega}_{12}}{\sqrt{2}} |d\rangle + \bar{\Omega}_{13}^* |D\rangle \right] \\ |3\rangle = \frac{1}{\Omega_D} \left[\frac{\bar{\Omega}_{13}}{\sqrt{2}} |u\rangle + \frac{\bar{\Omega}_{13}}{\sqrt{2}} |d\rangle - \bar{\Omega}_{12}^* |D\rangle \right] \end{cases}$$

$$\hat{H}_p^{(I_0)} = \sum_{j=1}^{3} \Omega_{0j} \exp\left(i\left((-\Delta_j + \omega_2)t + \varphi_2\right)\right) |0\rangle\langle j| + h.c.$$

$$= \bar{\Omega}_{01} e^{-i\Delta_1 t} |0\rangle \cdot \frac{1}{\sqrt{2}} (\langle u| - \langle d|)$$

$$+ \bar{\Omega}_{02} e^{-i\Delta_2 t} |0\rangle \cdot \frac{1}{\Omega_D} \left[\frac{\bar{\Omega}_{12}^*}{\sqrt{2}} \langle u| + \frac{\bar{\Omega}_{12}^*}{\sqrt{2}} \langle d| + \bar{\Omega}_{13} \langle D| \right]$$

$$+ \bar{\Omega}_{03} e^{-i\Delta_3 t} |0\rangle \cdot \frac{1}{\Omega_D} \left[\frac{\bar{\Omega}_{13}^*}{\sqrt{2}} \langle u| + \frac{\bar{\Omega}_{13}^*}{\sqrt{2}} \langle d| - \bar{\Omega}_{12} \langle D| \right] + h.c.$$

$$= \frac{1}{\sqrt{2}} \left(\bar{\Omega}_{01} e^{-i\Delta_1 t} + \frac{\bar{\Omega}_{02} \bar{\Omega}_{12}^*}{\Omega_D} e^{-i\Delta_2 t} + \frac{\bar{\Omega}_{03} \bar{\Omega}_{13}^*}{\Omega_D} e^{-i\Delta_3 t} \right) |0\rangle\langle u|$$

$$+ \frac{1}{\sqrt{2}} \left(-\bar{\Omega}_{01} e^{-i\Delta_1 t} + \frac{\bar{\Omega}_{02} \bar{\Omega}_{12}^*}{\Omega_D} e^{-i\Delta_2 t} + \frac{\bar{\Omega}_{03} \bar{\Omega}_{13}^*}{\Omega_D} e^{-i\Delta_3 t} \right) |0\rangle\langle d|$$

$$+ \frac{1}{\sqrt{2}} \left(\frac{\bar{\Omega}_{02} \bar{\Omega}_{13}}{\Omega_D} e^{-i\Delta_2 t} - \frac{\bar{\Omega}_{03} \bar{\Omega}_{12}}{\Omega_D} e^{-i\Delta_3 t} \right) |0\rangle\langle D| + h.c.$$

$$= C_{0u} |0\rangle\langle u| + C_{0d} |0\rangle\langle d| + C_{0D} |0\rangle\langle D| + h.c.$$

$$C_{ou} = \frac{1}{\sqrt{2}}\left(\Omega_{01} e^{i\varphi_1} e^{i(\omega_2-\Delta_1)t} + \frac{\Omega_{02}\Omega_{12}^*}{\Omega_D} e^{i(\varphi_1-\varphi_2)} e^{i(\omega_2-\omega_1-\Delta_2)t} + \frac{\Omega_{03}\Omega_{13}^*}{\Omega_D} e^{i(\varphi_1-\varphi_2)} \cdot e^{i(\omega_2-\omega_1-\Delta_3)t}\right)$$

$$C_{od} = \frac{1}{\sqrt{2}}\left(-\Omega_{01} e^{i\varphi_1} e^{i(\omega_2-\Delta_1)t} + \frac{\Omega_{02}\Omega_{12}^*}{\Omega_D} e^{i(\varphi_1-\varphi_2)} e^{i(\omega_2-\omega_1-\Delta_2)t} + \frac{\Omega_{03}\Omega_{13}^*}{\Omega_D} e^{i(\varphi_1-\varphi_2)} \cdot e^{i(\omega_2-\omega_1-\Delta_3)t}\right)$$

$$C_{oD} = \frac{1}{\sqrt{2}}\left(\frac{\Omega_{02}\Omega_{13}}{\Omega_D} e^{i(\varphi_1+\varphi_2)} e^{i(\omega_1+\omega_2-\Delta_2)t} - \frac{\Omega_{03}\Omega_{12}}{\Omega_D} e^{i(\varphi_1+\varphi_2)} e^{i(\omega_1+\omega_2-\Delta_3)t}\right)$$

Let's transform $H_P^{(I_0)}$ to $H_P^{(I_D)}$

$$H_P^{(I_D)} = e^{iH_0 t} H_P^{(I_0)} e^{-iH_0 t}$$

from 1st order, they will be cancelled $|0\rangle$ since $\langle u|0\rangle = \langle d|0\rangle = 0$.

$$= \exp\left(i\Omega_D t\left(|u\rangle\langle u| - |d\rangle\langle d|\right)\right)\cdot\left(C_{ou}|0\rangle\langle u| + C_{od}|0\rangle\langle d| + C_{oD}|0\rangle\langle D|\right)\exp\left(-i\Omega_D t\left(|u\rangle\langle u| - |d\rangle\langle d|\right)\right)$$

$$\exp\left(-i\Omega_D t\left(|u\rangle\langle u| - |d\rangle\langle d|\right)\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-i\Omega_D t)^k}{k!}\cdot\left(|u\rangle\langle u| - |d\rangle\langle d|\right)^k$$

$$= \begin{cases} |u\rangle\langle u| + (-1)^k |d\rangle\langle d| & (k\neq 0) \\ \mathbb{1} = |u\rangle\langle u| + |d\rangle\langle d| + (\mathbb{1} - |u\rangle\langle u| - |d\rangle\langle d|) & (k=0) \\ \quad = |u\rangle\langle u| + (-1)^0 |d\rangle\langle d| + |D\rangle\langle D| + |0\rangle\langle 0| & \end{cases}$$

$$= \sum_{k=0}^{\infty}\frac{(-i\Omega_D t)^k}{k!}|u\rangle\langle u| + \sum_{k=0}^{\infty}\frac{(i\Omega_D t)^k}{k!}|d\rangle\langle d| + |D\rangle\langle D| + |0\rangle\langle 0|$$

$$= \exp(-i\Omega_D t)|u\rangle\langle u| + \exp(i\Omega_D t)|d\rangle\langle d| + |D\rangle\langle D| + |0\rangle\langle 0|$$

$$\therefore H_P^{(I_D)} = C_{ou}\exp(-i\Omega_D t)|0\rangle\langle u| + C_{od}\exp(i\Omega_D t)|0\rangle\langle d| + C_{oD}|0\rangle\langle D|$$

$$= \overline{C_{ou}}|0\rangle\langle u| + \overline{C_{od}}|0\rangle\langle d| + \overline{C_{oD}}|0\rangle\langle D|$$

$$\overline{C_{ou}} = \frac{1}{\sqrt{2}}\left(\Omega_{01} e^{i\varphi_1} e^{i(\omega_2-\Delta_1-\Omega_D)t} + \frac{\Omega_{02}\Omega_{12}^*}{\Omega_D} e^{i(\varphi_1-\varphi_2)} e^{i(\omega_2-\omega_1-\Delta_2-\Omega_D)t} + \frac{\Omega_{03}\Omega_{13}^*}{\Omega_D} e^{i(\varphi_1-\varphi_2)} \cdot e^{i(\omega_2-\omega_1-\Delta_3-\Omega_D)t}\right)$$

$$\overline{C_{ou}} = \frac{1}{\sqrt{2}}\left(\Omega_{01} e^{i\varphi_1} e^{i(\omega_2-\Delta_1+\Omega_D)t} + \frac{\Omega_{02}\Omega_{12}^*}{\Omega_D} e^{i(\varphi_1-\varphi_2)} e^{i(\omega_2-\omega_1-\Delta_2+\Omega_D)t} + \frac{\Omega_{03}\Omega_{13}^*}{\Omega_D} e^{i(\varphi_1-\varphi_2)} \cdot e^{i(\omega_2-\omega_1-\Delta_3+\Omega_D)t}\right)$$

$$\overline{C_{oD}} = \frac{1}{\sqrt{2}}\left(\frac{\Omega_{02}\Omega_{13}}{\Omega_D} e^{i(\varphi_1+\varphi_2)} e^{i(\omega_1+\omega_2-\Delta_2)t} - \frac{\Omega_{03}\Omega_{12}}{\Omega_D} e^{i(\varphi_1+\varphi_2)} e^{i(\omega_1+\omega_2-\Delta_3)t}\right)$$

$\Rightarrow$ peaks at $\omega_2 = \Delta_1 + \Omega_D$, $\omega_1 + \Delta_2 + \Omega_D$, $\omega_1 + \Delta_3 + \Omega_D$ $\Rightarrow$ $|0\rangle \leftrightarrow |u\rangle$ transition

$\omega_2 = \Delta_1 - \Omega_D$, $\omega_1 + \Delta_2 - \Omega_D$, $\omega_1 + \Delta_3 - \Omega_D$ $\Rightarrow$ $|0\rangle \leftrightarrow |d\rangle$ transition

$\omega_2 = \Delta_2 - \omega_1$, $\Delta_3 - \omega_1$ $\Rightarrow$ $|0\rangle \leftrightarrow |D\rangle$ transition

ω_0 : zero field F=0, F=1 transition, $x = \frac{g_J \mu_B B}{\hbar \omega_0} \ll 1$

$$\Delta_1 = \omega_0 \sqrt{1+x^2} = \omega_0 \left[1 + \tfrac{1}{2}x^2 + O(x^4) \right]$$

$$\Delta_2 = \frac{\omega_0}{2}\sqrt{1+x^2} + \frac{\omega_0}{2}(1+x) = \omega_0 \left[1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 + O(x^4) \right]$$

$$\Delta_3 = \frac{\omega_0}{2}\sqrt{1+x^2} + \frac{\omega_0}{2}(1-x) = \omega_0 \left[1 - \tfrac{1}{2}x + \tfrac{1}{4}x^2 + O(x^4) \right]$$

$$\Delta_2 - \Delta_1 = \omega_0 \left[\tfrac{1}{2}x - \tfrac{1}{4}x^2 + O(x^4) \right]$$

$$\Delta_1 - \Delta_3 = \omega_0 \left[\tfrac{1}{2}x + \tfrac{1}{4}x^2 + O(x^4) \right]$$