

Trapped Ion Notes - Part 1

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I recommend downloading the latest version from the following link: <https://github.com/Ke-Sun96/Notes.git>

SPAM

Error budget

a. $|0\rangle$ state { PMT dark count. ①

SPAM related { optical pumping ②
 $|0\rangle \rightarrow |1\rangle$ leakage. ③

b. $|1\rangle$ state { Gate error (μ -wave / Raman) ④

SPAM related { optical pumping ⑤
 $|1\rangle \rightarrow |0\rangle$ leakage. ⑥

a. Measure the overall error of detecting $|0\rangle$ state.

Method. - Calibrate pump / detection time

- Run the circuit repeatedly: pump \rightarrow detect, to get m points, each point contains n experiments. So the total sampling number $N = mn$.
- The average prob of these N samplings is the error

$$\bar{\epsilon}_{|0\rangle} = \frac{\sum P_{mn-|0\rangle}}{N}$$

b. Measure the overall error of detecting $|1\rangle$ state.

Method. - Calibrate pump / detection time (This should be the same as those when measuring $|0\rangle$ state error)

- Calibrate gate π -time $\xrightarrow{\text{prop } |1\rangle}$
- Run the circuit repeatedly: pump \rightarrow π -gate \rightarrow detect to get $N = mn$ samples.
- The error $\bar{\epsilon}_{|1\rangle} = 1 - \frac{\sum P_{mn-|1\rangle}}{N}$

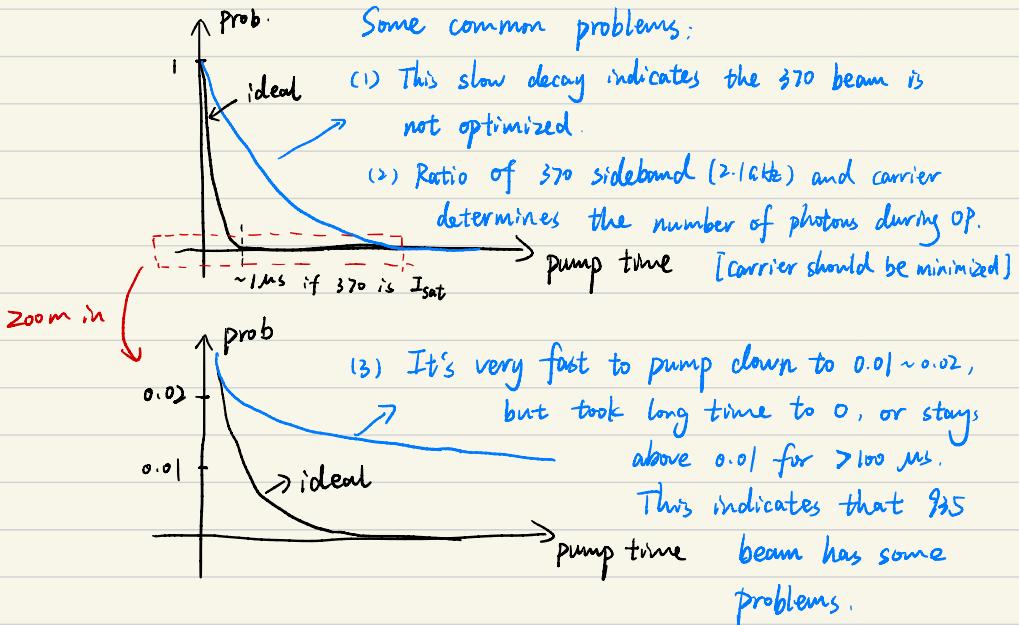
① PMT dark count:

Method: - Run the EXACT same scan as measuring $I_{>}$ error
 but only dumping the ion. If ϵ_{PMT} is large, we should
 - The error $\epsilon_{PMT} = \frac{\sum P_{\text{ion}} - P_{\text{PMT}}}{N}$ check it in detail (turn off 370/855
 dumping PMT
 block PMT
 etc.)

② Optical pumping:

This is influenced by $\begin{cases} 370 & \left\{ \text{polarization, frequency, power.} \right. \\ & \left. 2.1 \text{ GHz intensity, frequency.} \right. \\ 935 & \left\{ \text{Power, polarization, freq.} \right. \\ & \left. 3.1 \text{ GHz sideband intensity, freq.} \right. \end{cases}$

The typical way to examine: run pump time scan:



③ Leaksage error: $|1\rangle \rightarrow |0\rangle$

what can be measured

P_{det} : PMT count when detection on only

Method: Prep $|1\rangle$, measure with detection on for T_1 , get the average phonon number from histogram. Change detection time (T_2, T_3, \dots)

Repeat, get avg phonon number N , $P_{\text{det}} = \frac{N}{T}$.

τ : Decay rate when turning on detection beam after prep $|1\rangle$

T : Detection time.

P_{1S} — — — Note: The detection beam off-resonantly couples $|1\rangle \rightarrow |P_{1S}, F=1\rangle$ (-2.1 GHz)
 $S_{1/2}$  and $|0\rangle \rightarrow |P_{1S}, F=1\rangle$ (14.7 kHz), which results in leakage as well as a dark state $\propto |0\rangle + \beta|1\rangle$. where $|\alpha|^2 = a$, $|\beta|^2 = b$

Based on Rachel M. Noek's thesis (Duke), page 71.

The rate at which an ion initially in the $|1\rangle$ state will pump to $|0\rangle$ state when detection is $R_d \approx \left(\frac{2}{3}\right) \left(\frac{1}{3}\right) \left(\frac{1}{2}\right) \left(\frac{2\Omega^2}{T^2}\right) \left(\frac{T}{2\Delta_{HFP}}\right)^2$ $\Delta_{HFP} = 2.1 \text{ GHz}$.

The first factor of $\left(\frac{2}{3}\right)$ is due to the fact that one out of three states in the ${}^3S_{1/2} |F=1\rangle$ manifold is a coherent dark state at any given time.

The factor of $\left(\frac{1}{3}\right)$ is the branching ratio of the ${}^3P_{1/2} |F=1\rangle$ states decaying into $|0\rangle$ state.

The rate for $|0\rangle$ to pump into one of the bright ${}^3S_{1/2} |F=1\rangle$ states is

$$R_b \approx \left(\frac{2}{3}\right) \left(\frac{1}{2}\right) \left(\frac{2\Omega^2}{T^2}\right) \cdot \left(\frac{T}{2(\Delta_{HFP} + \Delta_{HFS})}\right)^2 \Delta_{HFS} = 12.6 \text{ kHz}$$

The factor of $\left(\frac{2}{3}\right)$ is the branching ratio of the ${}^3P_{1/2} |F=1\rangle$ states decaying into ${}^3S_{1/2} |F=1\rangle$ states.

$$\text{For } {}^{171}\text{Yb}^+, \quad R_d/R_b = \frac{1}{3} \left(\frac{2.1 + 12.6}{2.1} \right)^2 = 16.3.$$

Therefore, when applying DETCTION beam, the population of bright and dark state ($P_b(t)$ & $P_d(t)$) involves as follows.

$$\frac{dp_b}{dt} = -R_d \cdot P_b(t) + R_b \cdot P_d(t)$$

$$\frac{dp_d}{dt} = -R_b \cdot P_b(t) + R_d \cdot P_d(t)$$

$$\text{At equilibrium, } \frac{dp_b}{dt} = \frac{dp_d}{dt} = 0$$

$$\Rightarrow \begin{cases} \frac{P_b}{P_d} = \frac{R_b}{R_d} = \frac{1}{16.3} \\ P_b + P_d = 1 \end{cases} \Rightarrow P_b \approx 0.058.$$

Assume the solution is $P_b = c_1 + c_2 e^{\beta t}$, $P_d = (1 - c_1) + c_3 e^{\beta t}$

$$\Rightarrow \begin{cases} c_2 \beta e^{\beta t} = -R_d \cdot (c_1 + c_2 e^{\beta t}) + R_b \cdot [(1 - c_1) + c_3 e^{\beta t}] \\ c_3 \beta e^{\beta t} = -R_b [(1 - c_1) + c_3 e^{\beta t}] + R_d (c_1 + c_2 e^{\beta t}) \end{cases}$$

$$\Rightarrow \begin{cases} -R_d c_1 + R_b (1 - c_1) = 0 \\ c_2 \beta = -R_d c_2 + R_b c_3 \\ c_3 \beta = -R_b c_3 + R_d c_2 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{R_b}{R_b + R_d} \approx 0.058 \\ c_2 = -c_3 = C, \\ \beta = -(R_b + R_d) \end{cases}$$

$$\Rightarrow P_b(t) = 0.058 + C \cdot e^{-(R_b + R_d)t}, \text{ where } C \text{ is determined by the initial condition.}$$

$$\text{if } P_b(t=0) = 1, \quad P_b(t) = 0.058 + 0.942 e^{-(R_b + R_d)t}$$

Prep bright

$$\text{if } P_b(t=0) = 0, \quad P_b(t) = 0.058 - 0.058 e^{-(R_b + R_d)t}$$

Prep dark.

For simplicity. Let's assign $a = 0.942$, $b = 0.058$, $\tau = \frac{1}{R_b + R_d}$

$$\text{If prep bright, } P_b(t) = a e^{-\frac{t}{\tau}} + b$$

To get the leakage error, we need to calculate how much error it will cause if the leakage happens.

Firstly, if the ion is at $|1\rangle$ state, the prob of detecting more than 1 photon (determine it's $|1\rangle$ state) during detection time T is

$$P_0(T) = 1 - \exp(-P_{\text{det}} T)$$

- The part that doesn't decay to $|0\rangle$ during T : $P_b(T) = a \exp(-\frac{T}{\tau}) + b$

$$\text{Prob of measuring } |1\rangle: P_\alpha(T) = [a \exp(-\frac{T}{\tau}) + b] \cdot P_0(T)$$

- The part that decays to $|0\rangle$ during T :

$$\text{Prob that it still stays at } |1\rangle \text{ at } t: P_b(t) = a \exp(-\frac{t}{\tau}) + b$$

Within $(t, t+dt)$, the portion that decays is

$$dP_{|1\rangle} = \frac{dP_{|0\rangle}}{dt} = -\frac{a}{\tau} \exp(-\frac{t}{\tau}) dt$$

During detection time T , the prob we can still measure $|1\rangle$:

$$P_\beta(T) = \int_0^T \underbrace{|dP_{|1\rangle}(t)| \cdot P_0(t)}_{|dP_{|1\rangle}(t)|}$$

The detection success rate before it decays.

So, the prob that we can measure $|1\rangle$ is

$$P(T) = P_\alpha(T) + P_\beta(T)$$

For $|0\rangle \rightarrow |1\rangle$, Prob of getting $|1\rangle$ is $P_{|1\rangle} = -b \exp(-\frac{T}{\tau}) + b$

$$\frac{dP_{|1\rangle}}{dt} = \frac{b}{\tau} \exp(-\frac{t}{\tau})$$

Prob of measuring $|1\rangle$ state is

$$P(T) = \int_0^T |dP_{|1\rangle}(t)| P(T-t)$$

④ Gate error:

Two methods: (1) composed gates like SK1.

(2) Normal gates.

Method: Scan the gate number and fit the gate error

Assuming gate error is linear[↑], linear curve is enough.

(1) SK1 is robust to gate over-rotation error, but gate time is 5x longer.

(2) Normal gate is fast, but the over-rotation error becomes very significant when the gate number increases.

- To mitigate this error, it's important to use multiple gates to calibrate the gate time.

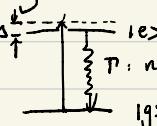
e.g. - Scan π -time with 1 gate first, get T_π , error σ_π

- Then scan between $[100.5 T_\pi, 101.5 T_\pi]$ to get $T_{101\pi}$,

The more precise π -time becomes $T_\pi' = \frac{T_{101\pi}}{101}$, error also decreases to $\sigma_\pi' = \frac{\sigma_\pi}{101}$

Doppler cooling

Doppler cooling uses a laser detuned to form an optical transition between two levels:



Γ : natural radiative linewidth of decay.

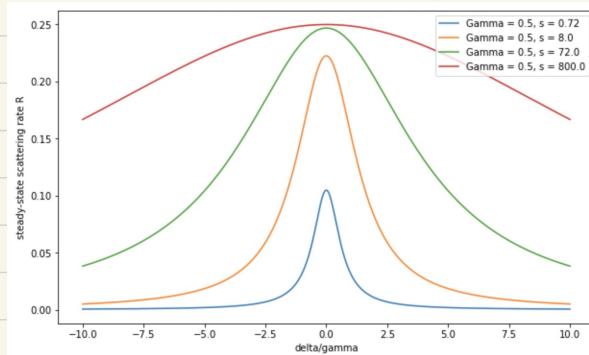
$$\text{The photon scattering rate } P_s = T \cdot \text{fee} = \frac{s \cdot \frac{\Gamma}{2}}{1 + s + 4 \frac{\Delta^2}{\Gamma^2}}$$

where $s = I / I_s$, $I_s = \pi h c / 3 \lambda T$ is the saturation intensity.

λ is the wavelength of light that resonates with the transition.

T is the lifetime of excited state. $T = 1/\Gamma$.

Γ is the natural linewidth.



Red arrows indicate the state transition paths during Doppler cooling.

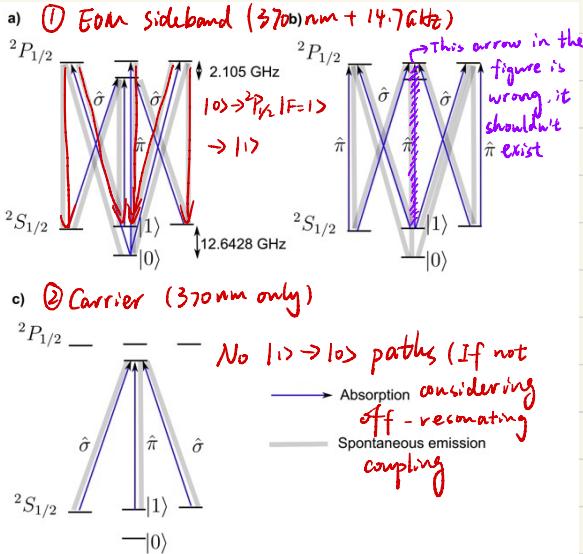


Figure 2.5: $^{171}\text{Yb}^+$ Doppler cooling, optical pumping and state detection schemes. The solid arrows indicate applied frequency and polarization components in the 369.5 nm beams and gray lines show the possible routes for spontaneous emission to the ground state manifold. a) Doppler cooling frequencies and polarization. b) Optical pumping frequencies and polarization required for qubit state preparation in $|0\rangle$, c) Frequency and polarization components for qubit state detection .

Doppler cooling:

Ensure every ion (both in $|0\rangle$ and $|1\rangle$ states) can be stimulated.

So we need:

- { ① 370 nm + 14.7 kHz
- { ② 370 nm

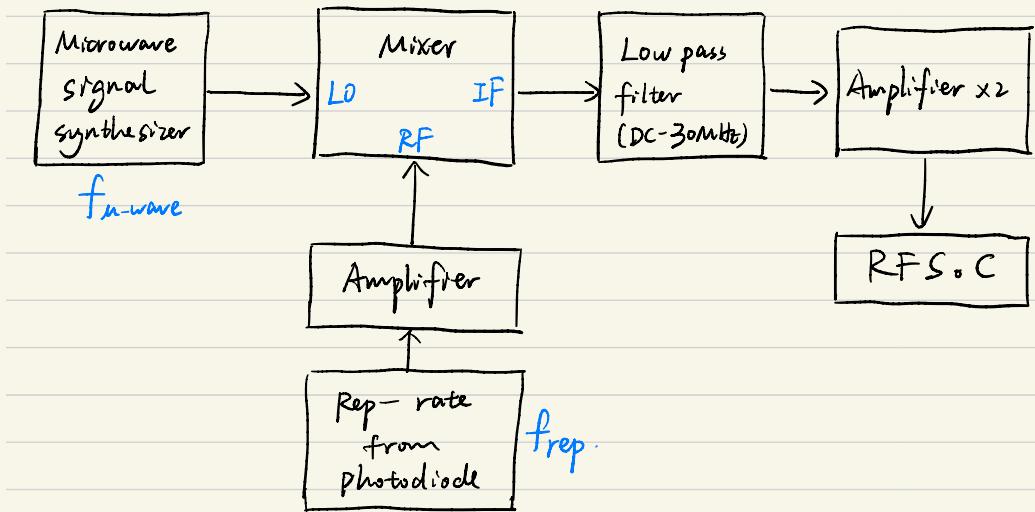
\Rightarrow After Doppler cooling, the ion should be at a mixed state $p = \{ p_0 |0\rangle\langle 0|, p_1 |1\rangle\langle 1| \}$, with $p_1 \gg p_0$, i.e. $p_1 \approx 1$, $p_0 \approx 0$. Because the speed of $|1\rangle \rightarrow |0\rangle$ is much slower than $|0\rangle \rightarrow |1\rangle$, As a result, most transitions happens btw $\{ ^2S_{1/2}, F=1 \} \leftrightarrow \{ ^2P_{1/2}, F=0 \}$ during Doppler cooling

Doppler cooling limit: $t_{\text{BT}} \sim \hbar T / k$ { $\begin{cases} ^2P_{3/2}: T_2 = 2\pi \times 25.9 \text{ MHz} \\ ^2P_{1/2}: T_1 = 2\pi \times 19.7 \text{ MHz} \end{cases}$

$$\text{Neh} = \frac{1}{e^{\frac{t_{\text{BT}}}{kT}} - 1} = \frac{1}{e^{2\pi \times 1.8 / 19.7} - 1} \approx 4.99.$$

Additionally, 0.5% $^2P_{1/2}$ is decayed to $^3D_{3/2}$. Therefore, a 935.2 nm & 932.5 nm + 3.07 GHz beam

Rep-rate lock



NCO mixer Frequency

$$f_{diff} = f_{u\text{-wave}} - n_h \cdot freq$$

Detection Freq Shift

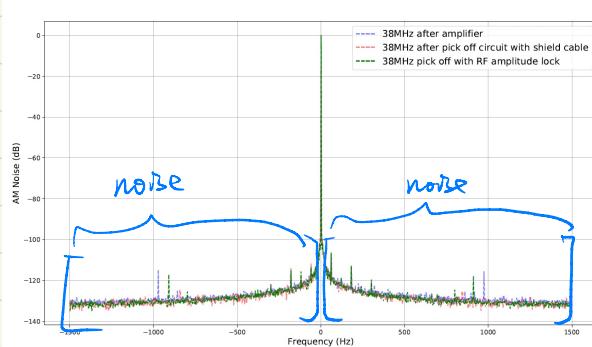
Harmonic divider (Red labels are the names in PSoC)

The selection of $f_{u\text{-wave}}$ & n_h should make sure

1. $|f_{diff}|$ is the lowest freq among all harmonics.
2. The nearest harmonic can be suppressed by the LPF.
3. n_h is better to be 2^k (eg, 8, 32...)

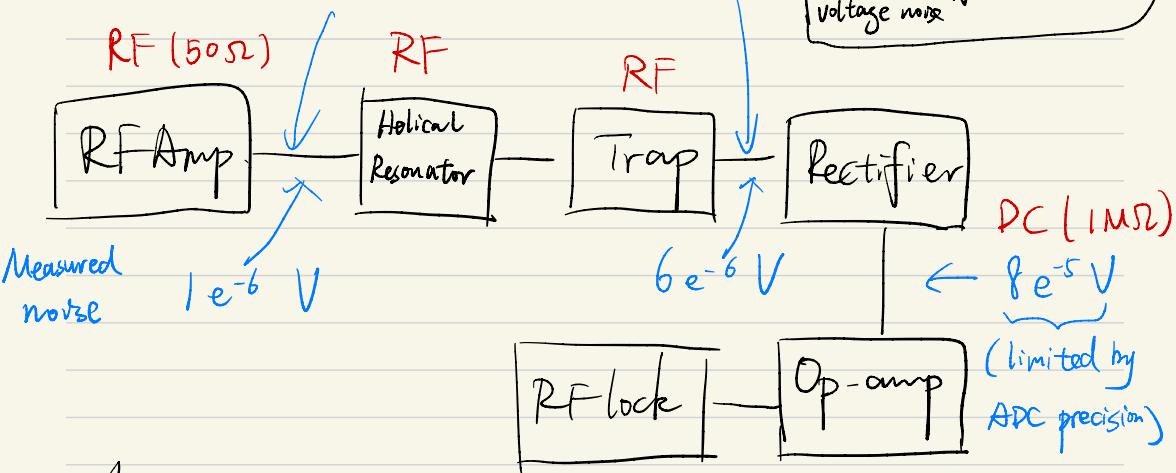
RF Lock circuit

Methods of measuring noise (V_{rms})

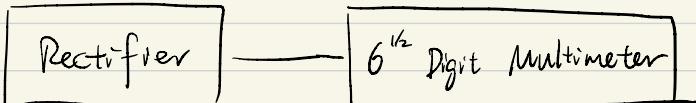


The bandwidth of this data is 1 Hz. So summing over all the spectrum other than the central 1 Hz will give us the noise level in dB. ($\propto \text{dB}$).

$$\text{The rms voltage noise} = \sqrt{10 \cdot \frac{(\text{X})}{10}} \cdot 50\Omega$$



When

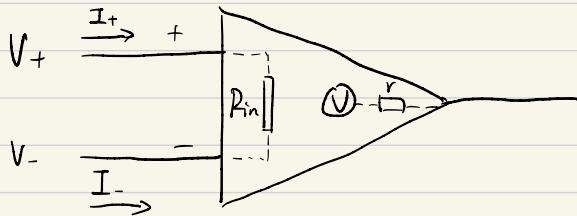


noise $V_{pp} \approx 20 \mu\text{V}$ @ 600Hz BW.

$$V_{rms} = \frac{V_{pp}}{\sqrt{6.6}} \cdot \sqrt{\frac{50\text{K}}{600}} \approx 28 \mu\text{V}$$

The only part that we can optimize is the op-amps after Rectifier.

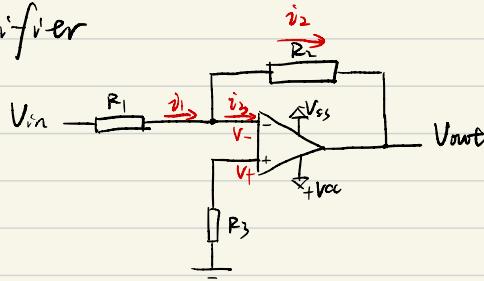
Op-amp:



Principles: Virtual short circuit: $V_+ \approx V_-$

Virtual open circuit: $I_+ = I_- = 0$

1. Inverting amplifier



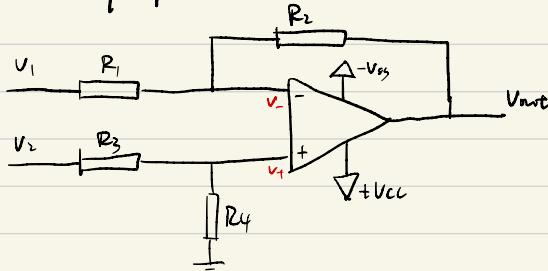
Virtual open: $i_3 = 0 \Rightarrow i_1 = i_2$.

$$V_- = \frac{R_2}{R_1 + R_2} \cdot V_{in} + \frac{R_1}{R_1 + R_2} \cdot V_{out}$$

Virtual short: $V_- = V_+ = 0 \text{ V}$

$$\Rightarrow \boxed{V_{out} = -\frac{R_2}{R_1} V_{in}}$$

2. Differential amplifier.



$$V_- = V_1 \cdot \frac{R_2}{R_1+R_2} + V_{\text{out}} \cdot \frac{R_1}{R_1+R_2}$$

$$V_+ = V_2 \cdot \frac{R_4}{R_3+R_4}$$

$$V_- = V_+$$

$$\Rightarrow V_{\text{out}} \cdot \frac{R_1}{R_1+R_2} + V_1 \frac{R_2}{R_1+R_2} = V_2 \frac{R_4}{R_3+R_4}$$

$$V_{\text{out}} \cdot \frac{R_1}{R_1+R_2} = V_2 \frac{R_4}{R_3+R_4} - V_1 \frac{R_2}{R_1+R_2}$$

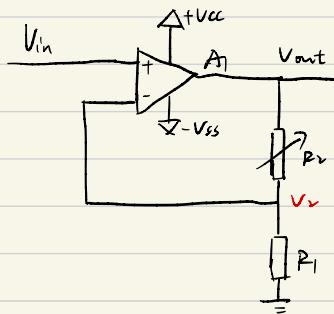
$$\boxed{V_{\text{out}} = \frac{R_1+R_2}{R_3+R_4} \cdot \frac{R_4}{R_1} \cdot V_2 - \frac{R_2}{R_1} V_1}$$

When $R_1 = R_3, R_2 = R_4$

$$\Rightarrow V_{\text{out}} = \frac{R_2}{R_1} (V_2 - V_1)$$

3. Non-inverting amplifier circuit.

(In general, it's only used to amplify the DC voltage).



$$\left\{ \begin{array}{l} V_2 = \frac{R_1}{R_1 + R_2} \cdot V_{out} \\ V_2 = V_{in} \end{array} \right.$$

$$\Rightarrow V_{out} = \left(1 + \frac{R_2}{R_1}\right) V_{in}$$

RF lock noise analysis.

Noise figure (NF)

$$NF (\text{dB}) = 10 \log (F)$$

$$F = \frac{SNR_I}{SNR_0} \rightarrow \text{input SNR}$$
$$\rightarrow \text{output SNR.}$$

Output signal $S_o = S_I \times G$

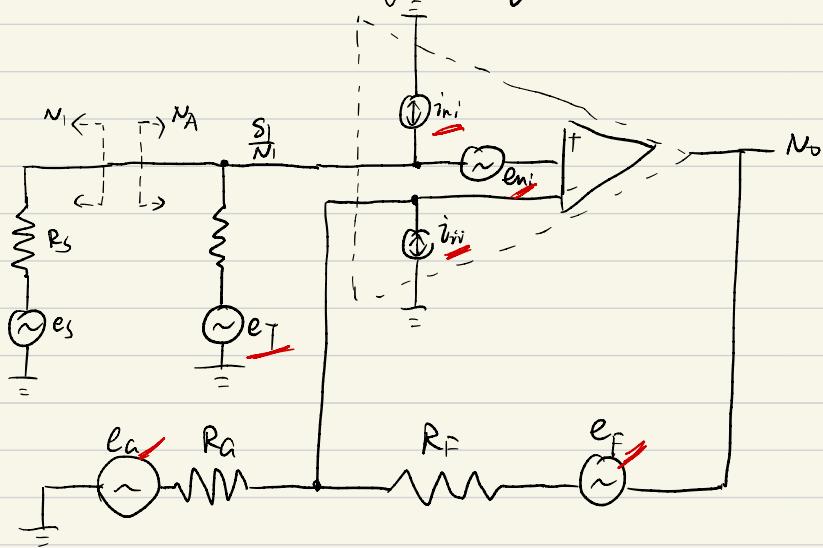
Input signal Gain

Output noise $N_o = (N_I + N_A) \times G$

noise from input noise of the device

$$\Rightarrow F = \frac{SNR_I}{SNR_0} = \frac{S_I / N_I}{S_o / N_o} = 1 + \frac{N_A}{N_I}$$

1. Non-inverting noise analysis diagram



Input noise: voltage noise: $\sqrt{4kTR_S}$

Adding terminating resistor: R_T ($= R_S$)

$$N_2 = 4kTR_S \left(\frac{R_T}{R_S + R_T} \right)^2 = kTR_S$$

Device noise: $N_A = C_1 e_{in_i}^2 + C_2 i_{in_i}^2 + C_3 i_{in_i}^2 + C_4 e_T^2 + C_5 e_A^2 + C_6 e_F^2$

where C_i : are scaling factors.

- $C_1 = 1$ b.c. voltage directly at the amplifier's input

$$- C_2 i_{in_i}^2 = i_{in_i}^2 \left(\frac{R_S R_T}{R_S + R_T} \right)^2$$

$$- C_3 i_{ii}^2 = \bar{V}_{ii}^2 \left(\frac{R_F R_G}{R_F + R_G} \right)^2$$

$$- C_4 e_T^2 = 4kT R_T \left(\frac{R_G}{R_S + R_T} \right)^2 \xrightarrow{R_S=R_T} kTR_T$$

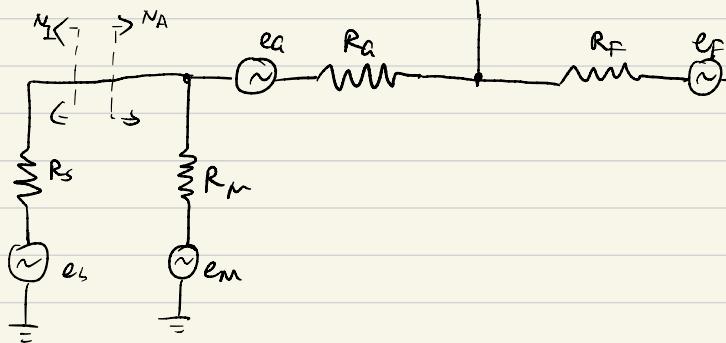
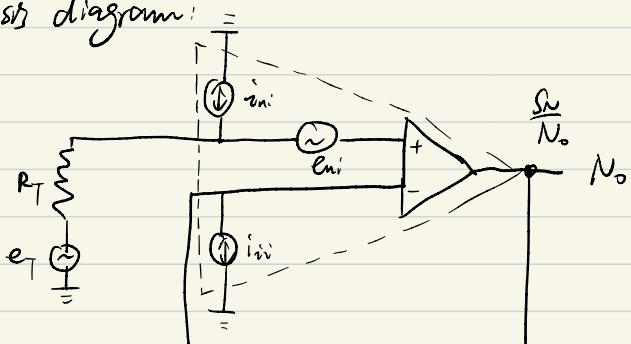
$$- C_5 e_a^2 = 4kTR_G \left(\frac{R_F}{R_F + R_G} \right)^2$$

$$- C_6 e_F^2 = 4kTR_F \left(\frac{R_G}{R_F + R_G} \right)^2$$

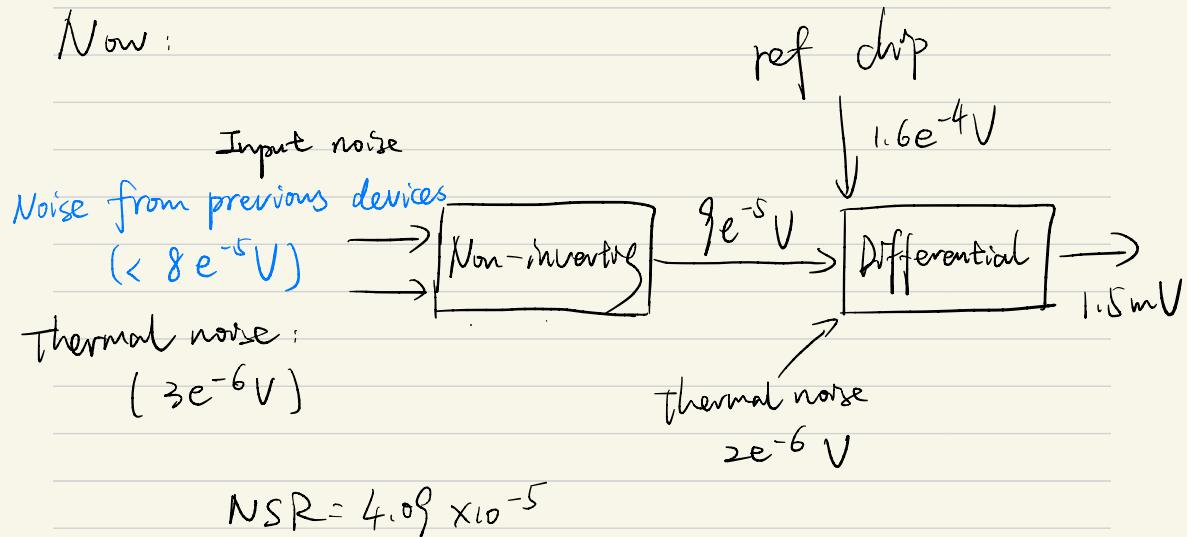
Unit: $kTR \cdot : \text{eV} \cdot \text{sL} = \text{C} \cdot \text{V} \cdot \frac{\text{V}}{\text{I}} = \text{C} \cdot \text{V} \cdot \frac{\text{V}}{\text{C/s}} = \text{V}^2 \cdot \text{s}$

$e^2 : V^2$

Inverting noise analysis diagram:

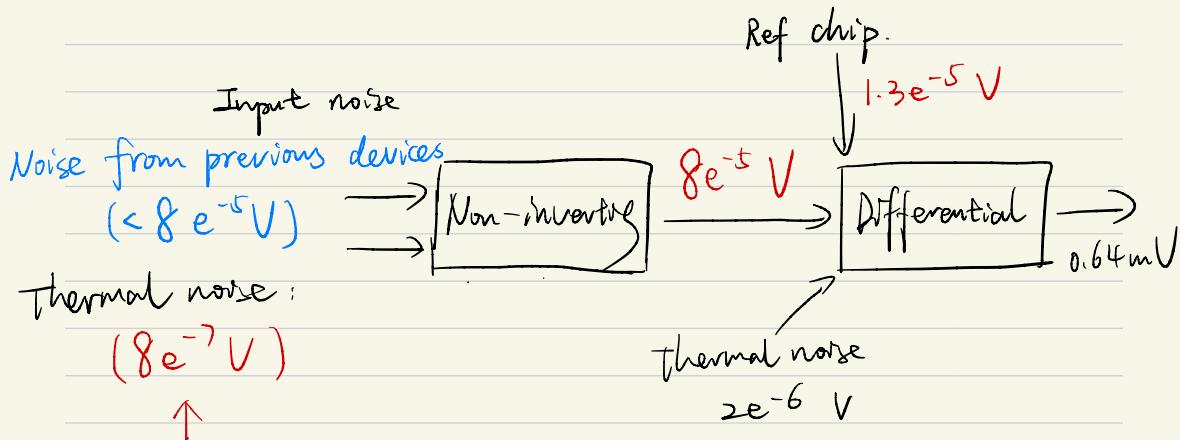


Now:



Optimize:

- ref chip: change to a low-noise one ($V_{rms}: 1.6e^{-4}V \rightarrow 1.3e^{-5}V$)
- Resistors: Reduce thermal noise



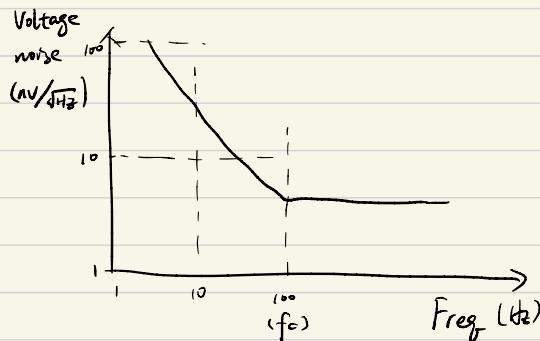
By changing the
resistance of R_1 ,
and R_2

$$NSR = 1.76 \times 10^{-5}$$

Tips :

1. Detailed optimizations and calculation codes can be found on github.com/Ke-Sun96/Ion-simulation, under the "RF-lock-noise" folder.
2. Calculate the input noise of a device based on datasheet.

Input noise calculation:



$$\text{eg: } V_n = \frac{100}{f} \quad f < 10\text{K}$$

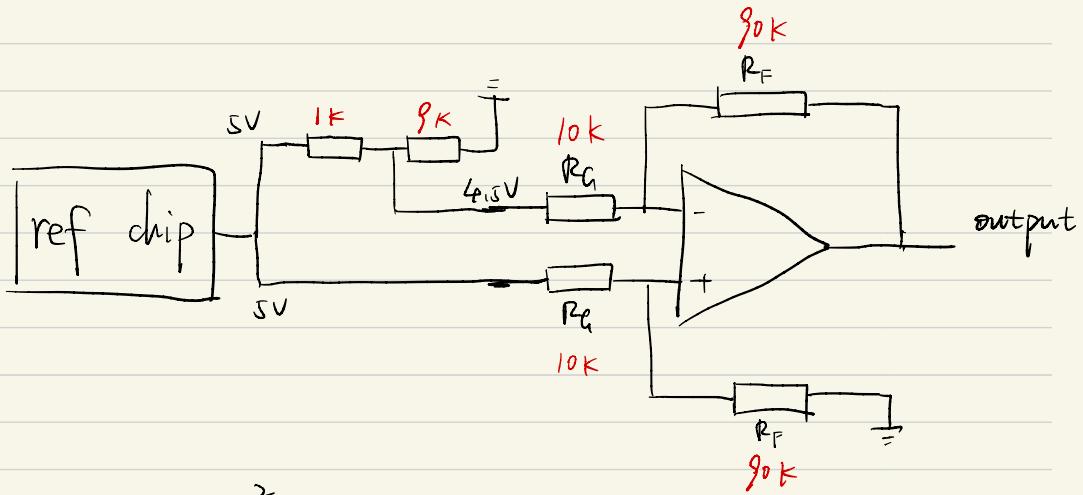
$$V_n = 2.7 \quad f > 10\text{K}$$

$$\text{Integral: } f < 10\text{K} \quad V_{n1}^2 = \int_1^{10000} \left(\frac{100}{f}\right)^2 df = 92103.4 \text{ nV}^2$$

$$f > 10\text{K} \quad V_{n2}^2 = \int_{10000}^{50000} (2.7)^2 df = 291600 \text{ nV}^2$$

$$V_{nI} = \sqrt{V_{n1}^2 + V_{n2}^2} = 619.438 \text{ nV}$$

3. Measure the noise of a Ref chip by yourself.



$$(V_{\text{noise-out}})^2 = \left(V_{\text{chip}}^2 + \left(\frac{4.5}{5} \cdot V_{\text{chip}} \right)^2 \right) \times \text{gain}$$

$$V_{\text{noise-out}} = 4.03 \cdot V_{\text{chip}} \quad \text{measured value: } 0.108 \text{ mV}$$

$$\Rightarrow V_{\text{chip}} = \frac{V_{\text{noise-out}}}{4.03} = 0.027 \text{ mV}$$

DC solutions

Mode frequency:

The potential $\phi = ax^2 + by^2 + cz^2$. (general expression)

the coefficients a, b, c contains both DC & RF terms.

$$\begin{cases} a = a_{DC} + a_{RF} \\ b = b_{DC} + b_{RF} \\ c = c_{DC} + c_{RF} \end{cases}$$

According to Maxwell equation: $\nabla^2 \phi = 0$

$$\nabla^2 (a_{DC}x^2 + b_{DC}y^2 + c_{DC}z^2) = 0$$

$$\Rightarrow a_{DC} + b_{DC} + c_{DC} = 0$$

$$\Rightarrow a + b + c = a_{RF} + b_{RF} + c_{RF}$$

In a particular case, where x -direction has no confinement. $a_{RF} = 0$, $b_{RF} = c_{RF} = \frac{1}{2}m\omega_{sec}^2$, where ω_{sec} is the secular frequency.

$$\Rightarrow a + b + c = 2 \cdot \frac{1}{2}m\omega_{sec}^2$$

$$\text{On the other hand, } a = \frac{1}{2}m\omega_x^2, \quad b = \frac{1}{2}m\omega_y^2, \quad c = \frac{1}{2}m\omega_z^2$$

$$\Rightarrow \omega_x^2 + \omega_y^2 + \omega_z^2 = 2 \cdot \omega_{sec}^2$$

Mathematical derivation (Radial direction)

Potential with Rotation (yz) and squeeze ($y^2 - z^2$)

$$\phi = a(y^2 + z^2) + \underbrace{2b yz}_{\text{Rotation}} + \underbrace{c(y^2 - z^2)}_{\text{Squeeze}}$$

$$= (y, z) \underbrace{\left(a\hat{I} + b\hat{\sigma}_x + c\hat{\sigma}_z \right)}_{\text{Matrix}} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} a+c & b \\ b & a-c \end{pmatrix}$$

↓ diagonalize.

Eigenvalues:

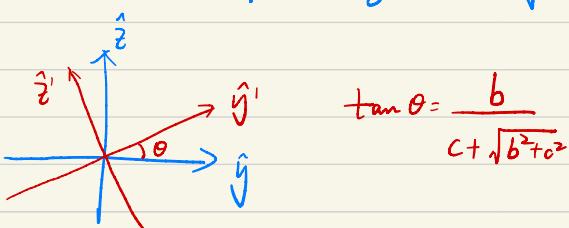
$$\omega_1^2 = a - \sqrt{b^2 + c^2}$$

Eigenvectors

$$v_1 = \left\{ \frac{c - \sqrt{b^2 + c^2}}{b}, 1 \right\}$$

$$\omega_2^2 = a + \sqrt{b^2 + c^2}$$

$$v_2 = \left\{ \frac{c + \sqrt{b^2 + c^2}}{b}, 1 \right\}.$$

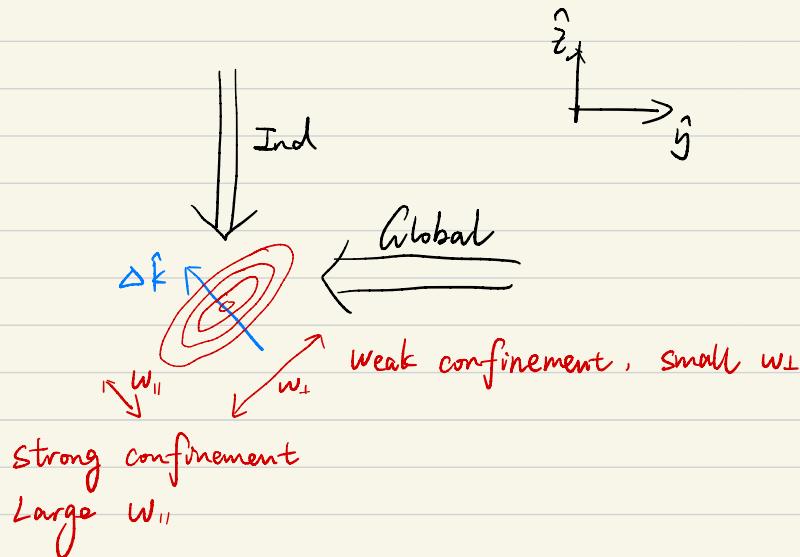


$$\tan \theta = \frac{b}{c + \sqrt{b^2 + c^2}}$$

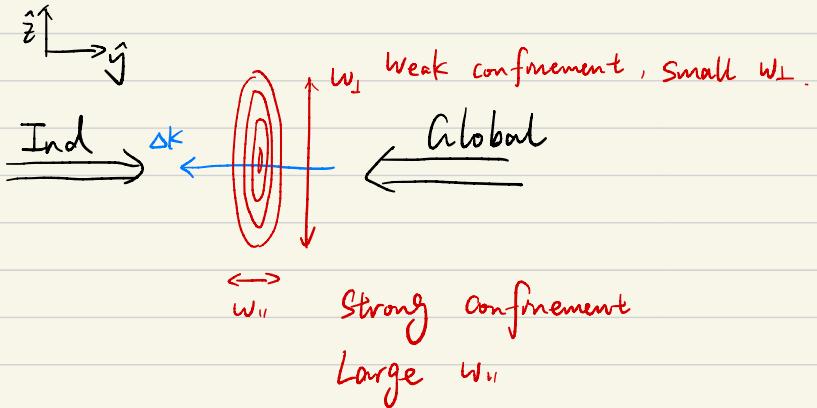
As long as $c=0$, $\tan \theta = 1$, $\theta = 45^\circ$

$$\text{Or } b \gg c, \quad \tan \theta \approx \frac{1}{1 + \frac{c}{b} + \frac{c^2}{2b^2}} \approx 1 - \frac{c}{b} - \frac{c^2}{2b^2}$$

In red chamber:



In Cryo:



We want $\tan \theta = 0 \Rightarrow b = 0$.

Only squeeze is required, no rotation is required.

Calibration steps:

1. Adjust X_1 (x comp) to move the DC null point, s.t. when
A little bit $Y_1 \& Z_1$.

adjusting Trap solution the ion doesn't move.

— By the end of this step, "Trap solution" will be independent to the X_1 ($Y_1 \& Z_1$ are hard b.c. you need to adjust $Y_1 \& Z_1$ for in-motion)

For Red 2. Adjust Rotation (YZ):

chamber:

① When tweaking the "SimpleRot", the ion will move. This is because at DC null point, it's indep. to X_1 , but not necessarily to YZ . So we need to either

a. Quantify how much "SimpleRot" is related to "Y comp" and "Z comp", e.g. α "SimpleRot" + β "Y comp" + γ "Z comp" makes sure the ion goes back to the original position (PMT counts stay the same). Then define a new rotation operation "Rot_new" = α "SimpleRot" + β "Y comp" + γ "Z comp"

or b. Adjust "SimpleRot", the ion will move away from DC null. Adjust "Y comp" and "Z comp" to move the ion back

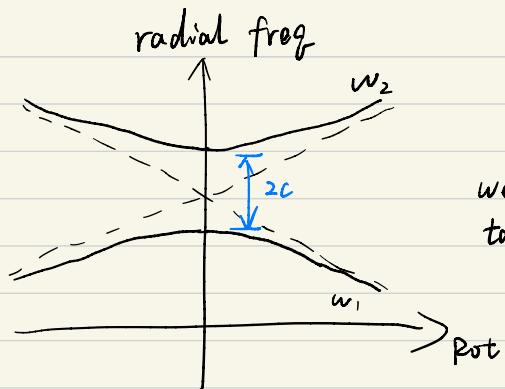
② Scan the sideband frequency.

③ Repeat ① & ② to get desired radial mode frequency.

Normally, the desired SB freq is to make sure the Raman only couples to one of the two directions, whichever has higher frequency.

3. Adjust Squeeze ($y^2 - z^2$)

When scanning Rotation, one will find the following relation:



By adjusting squeezing

we want to make sure $c = 0$, s.t.
 $\tan \theta = 1$, $\theta = 45^\circ$.

Recall: Eigenvalues:

$$\omega_1^2 = a - \sqrt{b^2 + c^2}$$

Eigenvectors

$$v_1 = \left\{ \frac{c - \sqrt{b^2 + c^2}}{b}, 1 \right\}$$

$$\omega_2^2 = a + \sqrt{b^2 + c^2}$$

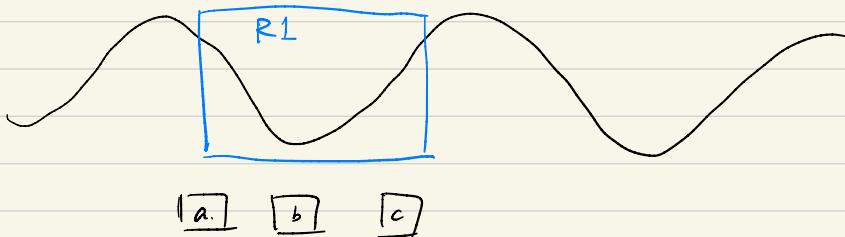
$$v_2 = \left\{ \frac{c + \sqrt{b^2 + c^2}}{b}, 1 \right\}.$$

$$\tan \theta = \frac{b}{c + \sqrt{b^2 + c^2}}$$

* Important: Step 2 & 3 only applies for Red chamber since
 $\angle(\vec{\alpha F}, \vec{B}) = 45^\circ$

For Gyro and STAQ, $\angle(\vec{\alpha F}, \vec{B}) = 0^\circ$, the influence of Rotation and Squeeze are flipped. But it's easy to analyze.

Ideas of generating DC solutions.



1. (a) Calculate the potential generated by electrode α ($\alpha = a, b, c$) in $R1$, and get V_α .

(b) Fit $V_\alpha = \sum_{ijk} C_{ijk}^{(\alpha)} x^i y^j z^k$. Store $C_{ijk}^{(\alpha)}$

2. Fix: Desired potential V .

$$V = \sum_{\alpha} b_{\alpha} V_{\alpha}$$

$$= \sum_{\alpha} b_{\alpha} \sum_{ijk} C_{ijk}^{(\alpha)} x^i y^j z^k = \sum_{ijk} \left(\sum_{\alpha} b_{\alpha} C_{ijk}^{(\alpha)} \right) x^i y^j z^k$$

$$= \sum_{ijk} \beta_{ijk} x^i y^j z^k$$

$$\text{Def: } \#_i \cdot \#_j \cdot \#_k = m \\ \#_{\alpha} = n$$

$$\Rightarrow \sum_{\alpha} b_{\alpha} C_{ijk}^{(\alpha)} = \beta_{ijk}$$

$$\Rightarrow \begin{pmatrix} C \\ \vdots \\ C \end{pmatrix}_{m \times n} \begin{bmatrix} b \\ \vdots \\ b \end{bmatrix}_{n \times 1} \approx \begin{bmatrix} \beta \\ \vdots \\ \beta \end{bmatrix}_{m \times 1}$$

Normally $m < n$. under-damped

Step 1. Find b_0 s.t. $C \cdot b_0 \approx \beta$

2. Find b_i 's s.t. $C \cdot b_i = 0$

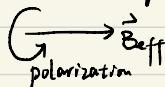
3. $b = b_0 + \sum_i t_i \cdot b_i$ is also a solution for
 $\hat{C} \cdot b \approx \hat{\beta}$.

4. Optimize t_i st. the maximum absolute value of b
is the smallest maximum voltage on the
electrode.

Co-propagation

An easy way to determine the polarization that is needed to drive co-prop transition is as follows:
 necessary condition

The laser with circular polarization will generate an effective magnetic field \vec{B}_{eff} of which the direction follows the right hand rule:



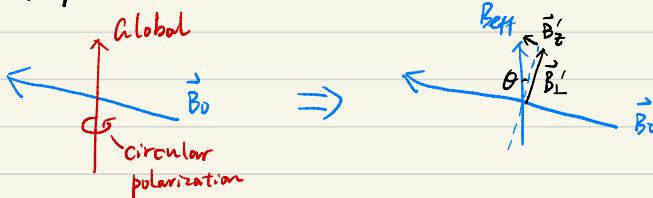
Actually, the direction of \vec{B}_{eff} is $\vec{\epsilon}_{B_{\text{eff}}} = \vec{\epsilon}_E^y \times \vec{\epsilon}_E$.

* Ref: Madjarov, Ivaylo Sashkov
 Thesis, Caltech, Sec. 2.3.2.

$$\begin{aligned} \vec{\epsilon}_E^y \times \vec{\epsilon}_E &= \begin{pmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ a(t) & b(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} = (a^* \beta - b^* \beta^*) \hat{e}_z \\ &= i a a_2 \sin[2(kz - \omega t)] \hat{e}_z \end{aligned}$$

Red chamber

Global co-prop



\vec{B}_L' has the same effect of the static \vec{B}_0 , driving $|0,0\rangle \rightarrow |1,0\rangle$

\vec{B}_L can drive the transition $|0,0\rangle \rightarrow |1,-1\rangle \Delta |1,+1\rangle$

$$\vec{B}_L = \vec{B}_{0+}' + \vec{B}_{0-}' . \quad |\vec{B}_L|^2 = |\vec{B}_{0+}'|^2 + |\vec{B}_{0-}'|^2$$

$$|\tan \theta| = \frac{B_z'}{B_L} = \frac{\mu B_z'}{\mu B_L} = \frac{\sqrt{2}|0,0\rangle \rightarrow |1,0\rangle}{\sqrt{2}|0,0\rangle \rightarrow |1,-1\rangle}$$

For Individual beam (also apply to Global beam in Purple system)



The $|0,0\rangle \rightarrow |1,0\rangle$ transition should be faster than $|0,0\rangle \rightarrow |1,-1\rangle$

At non-linear polarization, we can get

$$|\tan \theta| = \frac{\sqrt{2} \sum |0,0\rangle \rightarrow |1,-1\rangle}{\sqrt{2} |0,0\rangle \rightarrow |1,0\rangle}$$

Frequencies for co-prop: Atom freq settings.

Global: Tone 0: f_0 , Tone 1: f_1

$$f_1 - f_0 + n_{rep} \cdot freq = f_{target}$$

$$\Rightarrow f_1 = f_0 + f_{target} - n_{rep} \cdot freq$$

Scan this freq.

Set as a constant

rep-rate { measured - no fb. set : with feedback }

multiple of rep-rate, need to be correctly set in the RFsoC, normally it should be a negative value

Individual: double pass in upstream

$$2f_1 - 2f_0 + n_{rep} \cdot freq = f_{target}$$

$$\Rightarrow f_1 = f_0 + \frac{1}{2} f_{target} - \left(\frac{n_{rep}}{2} \right) freq$$

Note that when setting the # of Harmonic for fb in RFsoC, the value should be $-\frac{n_{rep}}{2}$.

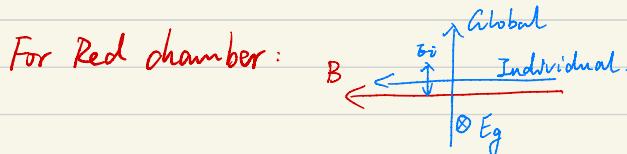
Counter-prop Raman transition

$$1. \text{ Freq: } f_{\text{ind}} - f_{\text{global}} + n_{\text{rep}} \cdot f_{\text{rep}} = f_{\text{RF}}$$

\downarrow $\downarrow f_g$ \downarrow \downarrow

For Red chamber ($2f_{\text{ind}} + f_{\text{shutter}}$) $n_{\text{rep}} = 104$ $f_{\text{rep}} \sim 117.801 \text{ MHz}$.
 $f_{\text{ind}} = 195.5 \text{ MHz}$. $f_{\text{shutter}} = 210 \text{ MHz}$. $f_g \sim 209.516 \text{ MHz}$

2. Adjust the individual & global waveplates to make sure they follow bin-perp-bin configuration.



① Use PBS to ensure E_g is linear and vertical.

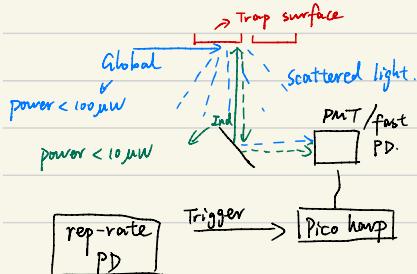
② Use individual co-prop to ensure that ind beam is linear (minimize co-prop Rabi frequency).

Optional: Use global co-prop to optimize QWP of global further more (minimize global co-prop Rabi freq.)

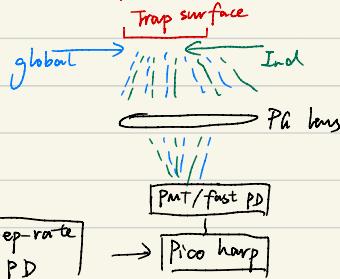
③ Drive counter-prop. Rotate HWP of individual to maximize counter-prop Rabi freq.

3. Beam path match.

For Red chamber:



For Purple chamber:



Tips of using Pico harp:

1. PMT / PD should be fast enough (faster than f-replate)
2. Important: Add sufficient proper filters before the PMT / PD.
3. PMT has readout delay (ns) if the optical power is too large. \Rightarrow Need to adjust the power settings and make sure it's small enough

Raman transition theory

1. Pulsed Laser

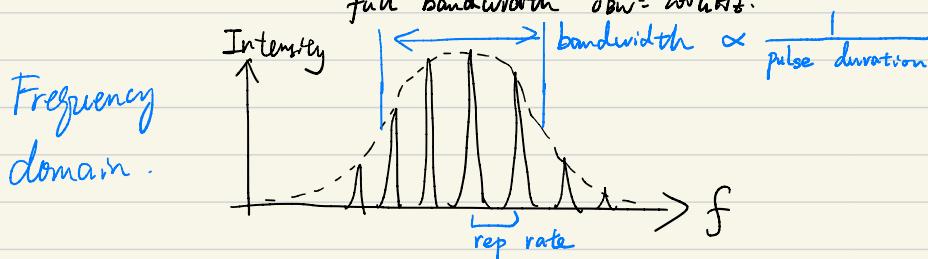
- Why not use cw laser?

A: ① We need two beams separated by 12.6 GHz, one could modulate that by AOM or BOM, but both of which is difficult to implement at the required freq due to the limited bandwidth of these devices.

② No high power CW 355nm laser.

- Pulsed laser: pulse duration ~ 10 ps.

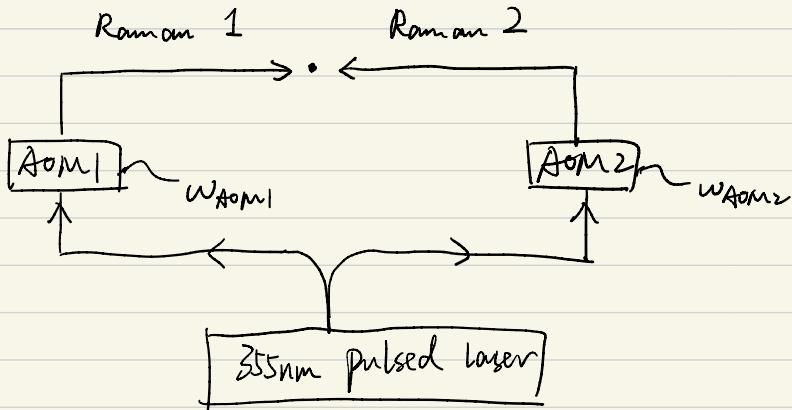
full bandwidth $\Delta_{\text{FW}} = 200$ GHz.



- It's not straightforward to implement Raman transitions:

- the central wavelength of the laser (355 nm) couples the ground state to $^2\text{P}_{1/2}$ & $^2\text{B}_{1/2}$ state with a detuning of $\Delta = 33$ THz and $\Delta = -66$ THz, respectively.
- In the freq comb regime there are multiple frequency components in the opticle spectrum that can potentially drive resonant and off-resonant Raman transitions between the qubit levels.

2. Freq - comb picture :



E - field at ion:

$$\vec{E}_i(t) = \vec{E}_0 \sum_{n=0}^{\infty} f(t-nT) e^{-i(w_c + w_{\text{atom}})_i t + \phi} \quad (i=1, 2)$$

where $f(t)$ is the pulse shape in time domain.

T is the time separation between pulses.

w_c is the center freq of 355 laser

ϕ is the spatially dependent phase of the field.

\Rightarrow Freq domain: freq combs separated by $W_r = \frac{2\pi}{T}$
 (Assuming large N)

$$E = E_0 \sum_{j=-\infty}^{\infty} \bar{f}(jw_r) e^{-i(jw_r + w_c + w_{\text{atom}})_i t} e^{i(k_j^i x + \phi)}$$

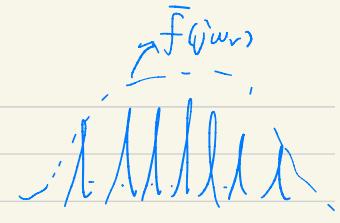
where $\bar{f}(w)$ is the Fourier trans of pulse shape $f(t)$
 k_j^i is the wave vector for j -th freq component.

The E field at ion will be:

$$E_T = \bar{E}_0 \sum_{j=-\infty}^{+\infty} a_j e^{-i(jw_r + w_c + w_{\text{atom}})_1 t} + \bar{E}_1 \sum_{j=-\infty}^{\infty} b_j e^{-i(jw_r + w_c + w_{\text{atom}})_2 t}$$

where $a_j = \bar{f}(j\omega_r) e^{i(k_j^0 \cdot x + \phi_0)}$

$$b_j = \bar{f}(j\omega_r) e^{i(k_j^0 x + \phi_1)}$$



Freq Comb:

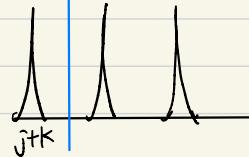
$$\omega_c + j\omega_r + \omega_{\text{atom}}$$

$$\omega_{\text{HF}} + \mu$$

$$\omega_c + (j+k)\omega_r + \omega_{\text{atom}}$$

Raman 1

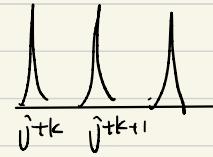
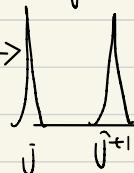
$$\omega_{\text{atom}_1}$$



Raman 2.

$$\omega_c$$

$$\omega_{\text{atom}_2}$$



$$\omega_r \approx 119 \text{ MHz}$$

$$\omega_{\text{HF}} = 12.6 \text{ GHz}$$

f_{reg}

3.1 State evolution: (Not considering pulse laser)

Details see S. Debnath's paper.

$$i \begin{pmatrix} \dot{c}_{1(t)} \\ \dot{c}_{2(t)} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} c_{1(t)} \\ c_{2(t)} \end{pmatrix}$$

Hermitian requires $\beta = \beta^*$

Assumptions: $\omega_r + \omega_{\text{atom}_2} - \omega_{\text{atom}_1} = \omega_{\text{HF}} + \mu$.

$$\frac{\mu}{\text{kHz}} \ll \frac{\omega_r}{\text{MHz}}, \frac{\omega_{\text{atom}_2}}{\text{MHz}} \ll \frac{\omega_{\text{HF}}}{\text{GHz}} \ll \frac{\Delta}{\text{THz}}$$

$$\Rightarrow \alpha = \frac{1}{4} \left(|\bar{d}_{\text{die}} \cdot \bar{E}_0|^2 \sum_j \frac{|a_j|^2}{\Delta + jwr + w_{\text{atom}}} + |\bar{d}_{\text{die}} \cdot \bar{E}_1|^2 \sum_j \frac{|b_j|^2}{\Delta + jwr + w_{\text{atom2}}} \right)$$

$$\delta = \frac{1}{4} \left(|\bar{d}_{\text{die}} \cdot \bar{E}_0|^2 \sum_j \frac{|a_j|^2}{\Delta - w_{\text{HF}} + jwr + w_{\text{atom}}} + |\bar{d}_{\text{die}} \cdot \bar{E}_1|^2 \sum_j \frac{|b_j|^2}{\Delta - w_{\text{HF}} + jwr + w_{\text{atom2}}} \right)$$

Taylor expansion in terms of $\frac{jwr}{\Delta}$, $\frac{w_{\text{HF}} + jwr}{\Delta}$

$$\begin{aligned} \alpha &= \frac{1}{4} \left[|\bar{d}_{\text{die}} \cdot \bar{E}_0|^2 \frac{1}{\Delta} \sum_j |a_j|^2 \left(1 + \frac{jwr}{\Delta} + \frac{w_{\text{atom}}}{\Delta} \right)^{-1} \\ &\quad + |\bar{d}_{\text{die}} \cdot \bar{E}_1|^2 \frac{1}{\Delta} \sum_j |b_j|^2 \left(1 + \frac{jwr}{\Delta} + \frac{w_{\text{atom2}}}{\Delta} \right)^{-1} \right] \\ &= \frac{1}{4} \left[|\bar{d}_{\text{die}} \cdot \bar{E}_0|^2 \frac{1}{\Delta} \left(1 - \frac{w_{\text{atom}}}{\Delta} - \sum_{j=\infty}^{\infty} |a_j|^2 \frac{jwr}{\Delta} \right) \right. \\ &\quad \left. + |\bar{d}_{\text{die}} \cdot \bar{E}_1|^2 \frac{1}{\Delta} \left(1 - \frac{w_{\text{atom2}}}{\Delta} - \sum_{j=\infty}^{\infty} |b_j|^2 \frac{jwr}{\Delta} \right) \right] \end{aligned}$$

$\sum_{j=\infty}^{\infty} jwr = \infty$
 $|a_j| \approx |a_j+| \approx \dots \approx |a_{j+k}|$

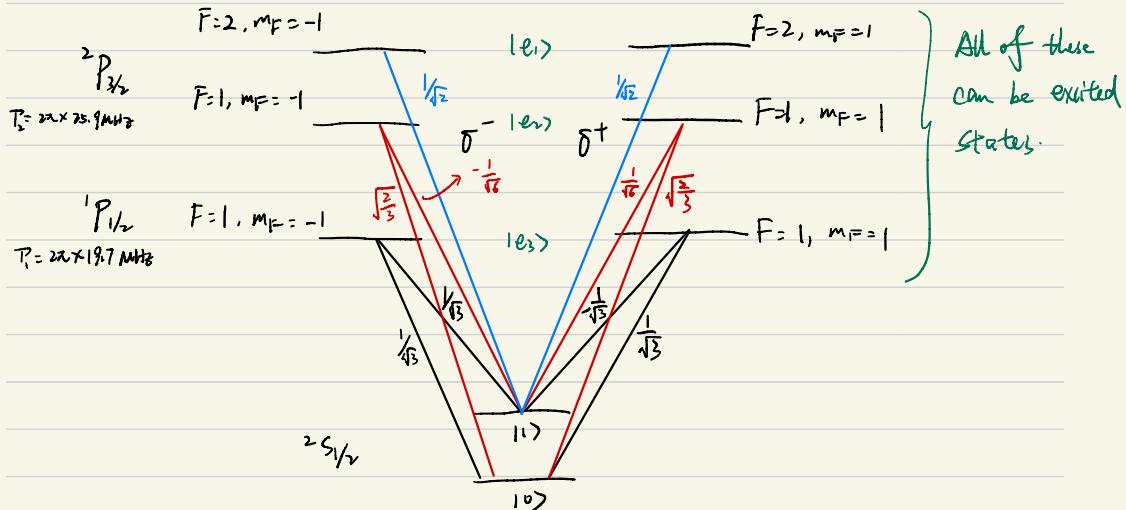
$$\Rightarrow \alpha = \frac{1}{4} (|\bar{d}_{\text{die}} \cdot \bar{E}_0|^2 + |\bar{d}_{\text{die}} \cdot \bar{E}_1|^2) \frac{1}{\Delta}$$

Similarly, $\delta = \frac{1}{4} (|\bar{d}_{\text{die}} \cdot \bar{E}_0|^2 + |\bar{d}_{\text{die}} \cdot \bar{E}_1|^2) \frac{1}{\Delta} \left(1 + \frac{w_{\text{HF}}}{\Delta} \right)$

$$\beta = \frac{e^{i\omega t}}{4} (\bar{d}_{\text{die}}^* \cdot \bar{E}_1^*) (\bar{d}_{\text{die}} \cdot \bar{E}_0) \sum_j a_j^* b_{j+k}^* \frac{1}{\Delta + jwr + w_{\text{atom}} + \mu}$$

$$\beta = \beta^*$$

3.2. Raman transitions via multiple excited states.



$$\frac{dI_{oe_1} \cdot \bar{E}_0}{\downarrow I_0} = \sqrt{\frac{I_0}{Ch \text{ coeff}}} \frac{Ch(a, e_1)}{\downarrow Ch \text{ coeff}} \frac{(\hat{a}_0 \cdot \hat{b}_{\pm})}{\downarrow \text{Unit vector of } E\text{-field}} \frac{\frac{T}{\sqrt{2}I_{sat}}}{\downarrow \frac{T}{I_{sat}}} \xrightarrow{\text{natural radiative linewidth}} \text{natural radiative linewidth}$$

$$I_0 = |E_0|^2 \quad \text{intensity} \quad I_{sat} = \pi hc / 3kT \quad T = 1/f$$

$$\xrightarrow{\text{saturation intensity.}}$$

$$\Rightarrow \alpha = \frac{1}{4\Delta_1} \left(\frac{1}{3} \right) (I_0 + I_1) \frac{T_1^2}{2I_{sat1}} + \frac{1}{4\Delta_2} \left(\frac{2}{3} \right) (I_0 + I_1) \frac{T_2^2}{2I_{sat2}}$$

$$\delta = \frac{1}{4\Delta_1} \left(1 + \frac{w_{HF}}{\Delta_1} \right) \left(\frac{1}{3} \right) (I_0 + I_1) \frac{T_1^2}{2I_{sat1}} + \frac{1}{4\Delta_2} \left(\frac{2}{3} \right) \left(1 + \frac{w_{HF}}{\Delta_2} \right) (I_0 + I_1) \frac{T_2^2}{2I_{sat2}}$$

$$\beta = \frac{\sqrt{I_0 I_1}}{12} e^{i\omega t} \left(\frac{T_1^2 D_{k,\Delta_1}}{I_{sat1}} - \frac{T_2^2 D_{k,\Delta_2}}{I_{sat2}} \right) (-\sigma_o + \sigma_{o+}^* + \sigma_{o-}^*)$$

where $\Delta_1 = 33 \text{ THz}$, $\Delta_2 = -67 \text{ THz}$ are detuning of 365 nm from $^2P_{3/2}$
and $^2P_{3/2}$ states. $D_{k,\Delta_i} = \sum_j \frac{\alpha_j \beta_j^*}{\Delta_i + j\omega_r + \omega_{RPM_i}}$, $E_i = \sqrt{I_i} (\hat{\sigma}_{o+} \hat{\sigma}_{o+} + \hat{\sigma}_{o-} \hat{\sigma}_{o-} + \chi_i \hat{\chi}_i)$

So, Stark shift is

$$\begin{aligned}\Delta_s &= \delta - \alpha \\ &= \frac{1}{4\sigma_1} \frac{W_{HF}}{\sigma_1} \cdot \frac{1}{3} (I_0 + I_1) \frac{T_1^2}{2I_{sat1}} + \frac{1}{4\sigma_2} \frac{W_{HF}}{\sigma_2} \frac{2}{3} (I_0 + I_1) \frac{T_2^2}{2I_{sat2}} \\ &= \frac{W_{HF}}{24\sigma_1^2} \frac{(I_0 + I_1) T_1^2}{I_{sat1}} + \frac{W_{HF}}{12\sigma_2^2} \frac{(I_0 + I_1) T_2^2}{I_{sat2}}\end{aligned}$$

≈ 24 . Hz.

↙ This value is only the scalar term of Stark shift. (part of two-photon SS)

Not considering { vector term
Four-photon SS
pulsed laser .

See Stark shift part for details.

Stark shift

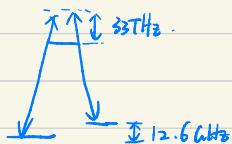
$$\gamma = \alpha_3 Ig + \alpha_v |\vec{B}_0 + \vec{B}' Ig|^2 + \alpha_4 Ig^2$$

Two-photon Stark shift:

- This comes from diagonalizing the matrix of a single beam:

$$\begin{pmatrix} 10s & 10s \\ 0 & \Omega \end{pmatrix} \xrightarrow[S]{P}$$

① Scalar term of the laser. This term comes from the detuning difference between $10s$ and $11s$ state



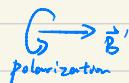
$$S_{\text{laser}} = \frac{\Omega^2}{\Delta} \gtrsim 25 \text{ kHz}.$$

(pi time < 20 μs)

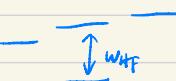
$$\begin{aligned} \text{Scalar term } \delta_s &= \frac{\Omega^2}{\Delta} - \frac{\Omega^2}{\Delta + \text{WRF}} \\ &= \frac{\Omega^2}{\Delta} \left(1 - \frac{1}{1 + \frac{\text{WRF}}{\Delta}} \right) \\ &\approx \frac{\Omega^2}{\Delta} \cdot \frac{\text{WRF}}{\Delta} \\ &= 25 \text{ kHz} \cdot \frac{12.6 \text{ GHz}}{33 \text{ THz}}. \end{aligned}$$

A more detailed calculation of this term can be referred to "Raman transition Theory" section.

② Vector term: The laser with circular polarization will generate an effective magnetic field \vec{B}' , which

follows the right-hand rule: 

Thus \vec{B}' will generate frequency shift the same as a real magnetic field:

e.g. 

$$\begin{aligned} S_{\text{laser}} F = 1 : \Delta_{\text{Zeman}} &= \pm 1.4 (\text{MHz}/\text{a}) \cdot B \\ \text{or } \text{WRF} &= 12.6 \text{ GHz} + 310 (\text{Hz}/\text{a}) \cdot B \end{aligned}$$

Red chamber:



\vec{B}'_z has the same effect of the static \vec{B}_0 , driving $|0,0\rangle \rightarrow |1,0\rangle$

\vec{B}'_L can drive the transition $|0,0\rangle \rightarrow |1,-1\rangle$ & $|1,+1\rangle$

$$\vec{B}'_L = \vec{B}'_{0+} + \vec{B}'_{0-}, \quad |\vec{B}'_L|^2 = |\vec{B}'_{0+}|^2 + |\vec{B}'_{0-}|^2$$

$$|\tan \theta| = \frac{B'_z}{B'_L} = \frac{\mu B'_z}{\mu B'_L} = \frac{\sqrt{S_{10,0} \rightarrow 11,0}}{\sqrt{S_{10,0} \rightarrow 11,-1}}$$

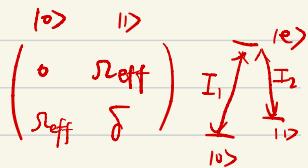
- $\Delta\nu$ in $\Delta\nu |\vec{B}_0 + \vec{B}'_L| g$ is $\Delta\nu = 310 B^2 \text{ Hz}$ (B in Gauss)

③ Four photon Stark shift:

$$\delta_4 = \propto_4 I g^2$$

$$S_{\text{eff}} \propto I_1 I_2$$

This comes from diagonalizing the Raman transition (Two beams)



• \propto_4 is a function of

- polarization: P

- Rep-rate : freq

- Detuning : δ

$$\left. \right\} \propto_4 \propto P \cdot \beta(\text{freq}, \delta)$$

function calculated

In M. Goldman's Note (not published)

For global beam only } we want the beam
 Ind beam only }

to be lin-perp-lin, meaning that both the beams are linearly polarized.

$$\Rightarrow P = 0 \quad \Rightarrow \quad \delta_4 = 0.$$

Summary:

- For global only (Consider polarization and intensity only)

$$\delta = \alpha_s I_g + \alpha_v |\vec{B}_0 + \vec{B}'(P)|^2 I_g + \alpha_4(P) I_g^2$$

$$= \alpha_s I_g + \alpha_v |\vec{B}_0|^2 + \alpha_v |\vec{B}'|^2 I_g^2 + \alpha_4(P) I_g^2$$

$$+ \alpha_v |\vec{B}_0| |\vec{B}'| \cos(\frac{\pi}{2} - \theta) I_g$$

$$= \underbrace{\alpha_v |\vec{B}_0|^2}_{\text{This term exists}} + \left(\alpha_s + \alpha_v \vec{B}_0 \cdot \vec{B}'(P) \right) I_g + \left(\alpha_v |\vec{B}'(P)|^2 + \alpha_v(P) \right) I_g^2$$

even no light, so

it's not Stark shift

for linear polarization.

$$\delta_{\text{stark_shift}} = \alpha_s I_g$$

For circular polarization:

$$\delta_{\text{stark-shift}} = [\alpha_s + \omega_r \vec{B}_0 \cdot \vec{B}_z(p)] I_g + [\omega_r |\vec{B}(p)|^2 + \omega_4(p)] I_g$$

What we can get:

① At linear polarization, get $\delta_{\text{stark shift}}$ vs. I_g .

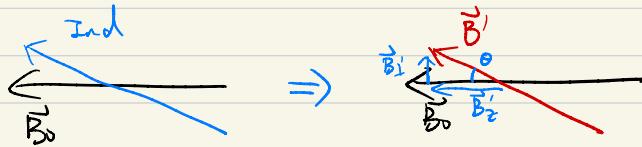
$$\delta_S = k_1 \cdot I_g, \quad k_1 = \alpha_s$$

② At non-linear polarization:

- Calibrate the global beam intensity for $|0,0\rangle \rightarrow \begin{cases} |1,-1\rangle \\ |1,0\rangle \\ |1,-1\rangle \end{cases}$
- . we can get the

$$|\tan \theta| = \frac{\mathcal{R}_{|0,0\rangle \rightarrow |1,0\rangle}}{\sqrt{2} \mathcal{R}_{|0,0\rangle \rightarrow |1,-1\rangle}}$$

For Individual beam:



The $|0,0\rangle \rightarrow |1,0\rangle$ transition should be faster than $|0,0\rangle \rightarrow \{ |1,-1\rangle, |1,+1\rangle \}$

Similar to the global only case.

① At linear polarization, we can get

$$\delta_{ss} = k_i \cdot I_i \quad , \quad k_i = \alpha_s \quad (\text{Scalar term})$$

② At non-linear polarization, we can get

$$|\tan \theta| = \frac{\sqrt{\sum S^z_{|0,0\rangle \rightarrow |1,-1\rangle}}}{\sqrt{\sum S^z_{|0,0\rangle \rightarrow |1,0\rangle}}}$$

For counter-prop:

$$\delta_{ss} = \delta_{\text{global only}} + \delta_{\text{ind only}} + \delta_{g\&i}.$$

Conditions: lin-perp-lin

$$\Rightarrow \delta_{\text{global only}} = \alpha_{sg} \cdot I_g$$

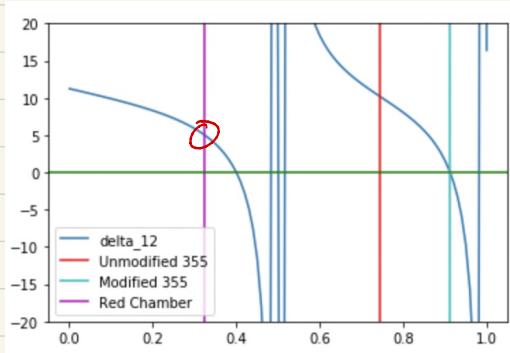
$$\delta_{\text{ind only}} = \alpha_{si} \cdot I_i.$$

$\delta_{g\&i}$ is 4-photon Stark shift, assuming the detuning δ_{z0}

$$\delta_{g\&i} = P \cdot \delta_0(\text{f.rep}) \cdot I_g I_i$$

Polarization term, For lin-perp-lin, P is a non-zero constant.

→ $\delta_0(\text{f.rep}) :$



Total Stark shift for Raman transition is:

$$\delta_{ss} = \underbrace{\alpha_{sg} I_g}_{\alpha_4} + \underbrace{\alpha_{si} I_i}_{\alpha_4} + \underbrace{P \delta_0(\text{f.rep}) I_g \cdot I_i}_{\alpha_4}$$

By measurement, $\alpha_{sg} = \alpha_{si} < 0$.
 $\alpha_4 > 0$.

• M. Goldman's Note

(Not published, but people in EURIQA project should has a copy)

- For the specific case of driving Raman transitions between two clock states ($Cmp=0$) via ${}^2P_{1/2}$ and ${}^2P_{3/2}$, the Rabi frequency Ω

$$\Omega = P_{1/2} \frac{g_1 g_2}{3} \frac{\omega_p}{\Delta(\omega - \omega_p)} \quad (1)$$

where ω_{wp} is the energy difference between ${}^2P_{1/2}$ & ${}^2P_{3/2}$

Δ is the detuning from ${}^2P_{1/2}$ ($\Delta \gg \omega_0$)

$g_i = \frac{e_i |\vec{d}_0 \cdot \hat{\vec{e}}_i|}{\hbar}$, \vec{d}_0 is the dipole operator between

$$P_{1/2} = i (\vec{e}_i^* \times \hat{\vec{e}}_i) \cdot \hat{\vec{B}}$$

where $E_i(t) = E_i e^{i\omega_i t} \hat{\vec{e}}_i$. $\hat{\vec{B}}$ is the direction of ion's quantization axis.

- Pulsed Laser:

$$E_i(t) = \sum_{n=1}^{\infty} p_i(t - n/f_{\text{rep}}) \quad (2)$$

If the pulse width $\tau \ll 1/f_{\text{rep}}$, Fourier trans of (2) is

$$\tilde{E}_i(w) = \sum_k \tilde{E}_{i,k}(w) = \sum_k \delta(w - w_k) \tilde{p}_i(w - w_i) \hat{\vec{e}}_i. \quad (3)$$

where $\delta(w)$ is a sharply peaked function of width $\sim \frac{f_{\text{rep}}}{N}$

$\tilde{p}_i(w)$ is the envelope function.

$$\text{if } p_i(t) = e_i \sqrt{\pi} \operatorname{sech}(at/\tau)$$

$$\tilde{p}_i(w) = e_i \sqrt{\pi} \operatorname{sech}(aw/\tau)$$

\Rightarrow Time-averaged E-field amplitude

$$\begin{aligned} \text{ith comb} \quad \text{kth tooth.} \\ \vec{E}_{i,k} &= \text{freq} \cdot \int_{-\infty}^{\infty} |\vec{E}_{i,k}(w)| / dw \\ &= \text{freq} \cdot \tilde{p}_i(w_k - w_i) \end{aligned} \quad (4)$$

* Condition: pulse width τ is short enough s.t. Δw of freq comb is larger than the qubit splitting w_0 .

$$|f\tau + n\text{freq}| = w_0/2\pi \quad (5)$$

$$(4) \Rightarrow g_{i,k} = \frac{\vec{E}_{i,k} |\vec{d}_0 \cdot \hat{\sigma}_+|}{\hbar} = \frac{|\vec{d}_0 \cdot \hat{\sigma}_+|}{\hbar} \cdot \text{freq} \tilde{p}_i(w_k - w_i) \quad (6)$$

(6) insert into (1) \Rightarrow

$$\begin{aligned} S_{12} &= \sum_k P_{1,2} \frac{g_{1,k} g_{2,k+n}}{3} \frac{w_p}{\Delta(\Delta - w_p)} \\ &= \sum_k P_{1,2} \frac{|\vec{d}_0 \cdot \hat{\sigma}_+|^2}{3\hbar^2} \text{freq}^2 \cdot \tilde{p}_1(w_k - w_i) \cdot \tilde{p}_2(w_{k+n} - w_i) \cdot \frac{w_p}{\Delta(\Delta - w_p)} \\ &= P_{1,2} \cdot \underbrace{g_1 g_2 \frac{|\vec{d}_0 \cdot \hat{\sigma}_+|^2}{3\hbar^2}}_{\equiv S_{1,2}} \frac{w_p}{\Delta(\Delta - w_p)} \cdot \sum_k \text{freq}^2 \cdot \left(I \cdot \frac{\sqrt{2}}{2}\right)^2 \operatorname{sech}\left(\frac{(w_k - w_i)\tau}{2}\right) \\ &\quad \cdot \operatorname{sech}\left(\frac{(w_{k+n} - w_i)\tau}{2}\right) \end{aligned}$$

So if $P = \frac{\sqrt{2}}{2} g_0 \operatorname{sech}\left(\frac{w\tau}{2}\right)$ $\equiv h_n = \frac{\sqrt{2}}{2} h_{k,n} \quad (7)$

$\begin{aligned} y = wx &= \int_{-\infty}^{\infty} \operatorname{sech}(ax) e^{iwx} dx \\ &= \int_{-\infty}^{\infty} \operatorname{sech}(y) e^{iwy/a} dy \\ &= \frac{1}{a} \int_{-\infty}^{\infty} \operatorname{sech}(y) e^{iwy} dy \\ &= \frac{1}{a} \pi \operatorname{sech}\left(\frac{\pi w}{2a}\right) \end{aligned}$	$\left \begin{array}{l} \text{if } a = \frac{\pi}{2}, \\ \tilde{p}_i(w) = \sqrt{\frac{\pi}{2}} g_0 \frac{\pi}{2} \cdot \pi \operatorname{sech}\left(\frac{\pi w\tau}{2\pi}\right) = \sqrt{\frac{\pi}{2}} g_0 \tau \operatorname{sech}\left(\frac{w\tau}{2}\right) \checkmark \\ \text{Let } (w_k - w_i) = 2\pi \text{freq} \cdot k. \xrightarrow{\text{continuous to discrete}} \tilde{p}_k = \sqrt{\frac{\pi}{2}} \tau g_0 \operatorname{sech}\left(k \pi \text{freq} \tau\right) \\ \text{So } h_k = \text{freq}^2 \cdot \left(I \cdot \frac{\sqrt{2}}{2}\right)^2 \operatorname{sech}\left(k \pi \text{freq} \tau\right) \operatorname{sech}\left[\left(k \pi \text{freq} \tau\right) \pi \text{freq} \tau\right] \end{array} \right.$
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$$So \quad h_m = \sum_k \text{freq} \cdot (\pi \sqrt{\frac{k}{2}})^2 \operatorname{sech}(kx\text{freq}\tau) \operatorname{sech}[(ktn)x\text{freq}\tau]$$

$$= \text{freq} \cdot (\pi \sqrt{\frac{k}{2}})^2 \sum_k \frac{1}{\cosh(kx\text{freq}\tau)} \cdot \frac{1}{\cosh[(ktn)x\text{freq}\tau]}$$

$$\frac{\theta = x\text{freq}\tau}{k} \sum_k \frac{1}{\frac{e^{k\theta} + e^{-k\theta}}{2}} \cdot \frac{1}{\frac{e^{(ktn)\theta} + e^{-(ktn)\theta}}{2}}$$

$$= \sum_k \frac{4}{e^{(ktn)\theta} + e^{-(ktn)\theta} + e^{n\theta} + e^{-n\theta}}$$

$$= \sum_k \frac{2}{\cosh[(2ktn)\theta] + \cosh(n\theta)}$$

$$= \sum_k \frac{2}{\cosh(2k\theta) \cosh(n\theta) + \sinh(2k\theta) \sinh(n\theta) + \cosh(n\theta)}$$

$$= \sum_k \frac{2}{[\cosh(2k\theta) + 1] \cosh(n\theta) + \sinh(2k\theta) \sinh(n\theta)}$$

$$\text{when } n\theta \ll 1, \quad = \sum_k \frac{2}{[\cosh(2k\theta) + 1] \cosh(n\theta)}$$

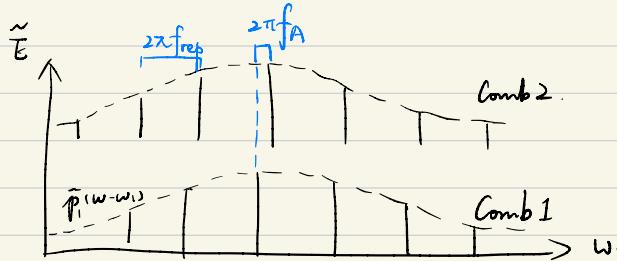
$$\approx \frac{1}{\cosh(n\theta)} \cdot \frac{2 \cdot 0.0408}{\theta}$$

$$\text{when } n\theta \geq 1, \quad = \sum_k \frac{2}{[\cosh(2k\theta) + 1 + \sinh(2k\theta)] \sinh(n\theta)}$$

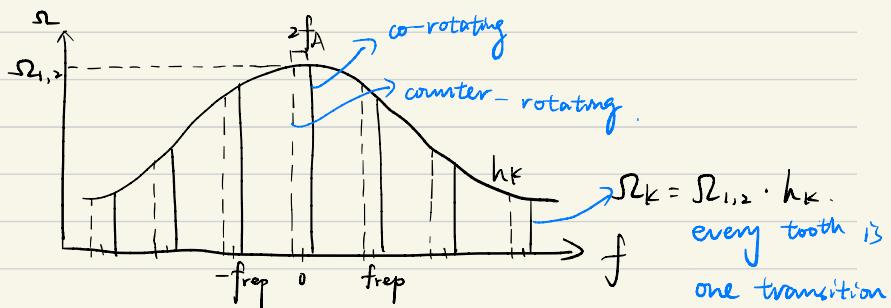
$$= \frac{N}{\sum_k} \frac{2}{(e^{2k\theta} + 1) \sinh(n\theta)}$$

$$= 2N \frac{1}{\sinh(n\theta)}$$

$$= \begin{cases} \text{freq} \tau / \cosh(nz\text{freq}\tau), & n\theta \ll 1 \\ k \text{freq}^2 \tau^2 \pi / \sinh(n\theta), & n\theta \geq 1 \end{cases}$$



Beat note:



Co-rotating:

$$n_frep + f_A$$

Comb 2

Comb 1

Counter-rotating

$$n_frep - f_A$$

Comb 2

Aom2

Aom1

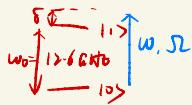
Comb 1

$$\text{Define } x = \frac{w_0/2\pi \bmod f_{rep}}{f_{rep}} \in [0, 1].$$

The resonance condition is $|f_A + n_frep| = w_0/2\pi$
is $f_A = (k \pm x) f_{rep}$.

→ This Stark shift is the effective single-photon Stark shift:

$$\delta_{\text{sc}} = -\frac{|\Omega_k|^2}{\delta}$$

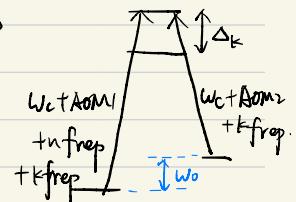


The Stark shift due to the k th comb tooth is

$$\delta_k = -\frac{|\Omega_k|^2}{\Delta_k} \quad (\text{Actually it's } \delta_k)$$

Summing over all the δ_k :

$$\begin{aligned} \delta &= \sum_k \delta_k = -\sum_k \frac{|\Omega_k|^2}{\Delta_k} \rightarrow \delta_k = \begin{cases} (k \cdot \text{freq} + f_A) - (n+x) \cdot \text{freq} & \text{co-rotating} \\ (k \cdot \text{freq} - f_A) - (n+x) \cdot \text{freq} & \text{counter-rotating} \end{cases} \\ &= -\underbrace{\frac{|\Omega_{1,2}|^2}{2 \cdot \text{freq}}}_{\delta_{1,2}} \sum_k \frac{\hbar \omega}{\Delta_k} \cdot \left(\frac{1}{k-n-x + f_A/\text{freq}} + \frac{1}{k-n-x - f_A/\text{freq}} \right) \end{aligned} \quad (8)$$



Before calculating specific Stark shift, we define some independent parameters:

1. Raman Rabi frequency:

$$\Omega_0 = \frac{\Omega_{ij}}{P_{ij} \sqrt{I_i I_j}} \quad \text{independent of } \begin{cases} \text{intensity } I_{ij} \\ \text{polarization } P_{ij} \end{cases}$$

2. Stark shift:

$$\delta_0 = \frac{\delta_{ij}}{P_{ij}^2 I_i I_j} = -\frac{|\Omega_0|^2}{2 \cdot \text{freq}}$$

3. Polarization. Beam 1 before QWP:

$$\vec{E} = \vec{z} \cdot (\hat{x} + \hat{y}) e^{i \omega t}$$

After QWP which has an angle ϕ relevant to \hat{y} :

$$\hat{\epsilon} = \hat{x}x e^{i\omega t} + \hat{y}y e^{i(\omega t+2\phi)}$$

$$\begin{aligned}\hat{\epsilon}^* \times \hat{\epsilon} &= \hat{x}^* \hat{x} \hat{y} \hat{z} \\ &= \hat{z} \cdot \hat{x} \hat{y} e^{i2\phi} - \hat{x} \hat{y} e^{-i2\phi} \\ &= \hat{z} \cdot \hat{x} \hat{y} z i \sin 2\phi\end{aligned}$$

$$\propto \sin 2\phi$$

$$\begin{aligned}\text{Define } S_1 &= \hat{\epsilon}_1^* \times \hat{\epsilon}_1 \\ &= -\sin(2\phi)\end{aligned}$$

$$|P_{1,2}| = \sqrt{1 - S_1^2/2} \quad |P_{1,1}| = |S_1|$$

\Rightarrow ① Dual-beam transitions are most strongly driven when both beams are linearly and orthogonally polarized. ($S_1=0$) ($S_1=0$) ($\hat{\epsilon}_1^* \times \hat{\epsilon}_2$ is max when ortho)

② We can drive single-beam transition only when the beam has some degree of circular polarization. ($S_1 \neq 0$)

4. Intensity imbalance: $\alpha = \sqrt{I_{1b}/I_{1r}}$

If one beam goes into one AOM with two RF signals: $I_{1b} + I_{1r} = I_1$, when $\alpha=1$, $\sqrt{I_{1b} I_{1r}} = I_1/2$.

① Stark shift in single-qubit rotation with one comb.

$$\text{Eqn (8): } \delta = \sum_k h_k^2 \left(\frac{1}{k-n-x + f_A/\text{freq}} + \frac{1}{k-n-x - f_A/\text{freq}} \right)$$

For one comb situation: $f_A = 0$

$f_A = 0$. one term within the sum should be dropped.

$$\delta_{11} = -\frac{|\Omega_{1,1}|^2}{2\pi\text{freq}} \rightarrow \Omega_{1,1} = |S_1| \varepsilon_1^2 \frac{|I_{10} \cdot \Omega_1|^2}{3h^2} \cdot \underbrace{\frac{w_p}{\Delta(\Delta-w_p)}}_{|\Omega_{2,0}|}$$

$$\text{Define } \xi_1(x) = \sum_k h_k^2 \frac{1}{k-n-x}$$

$$\Rightarrow \boxed{\delta^{(1)} = \delta_{1,1} \xi_1(x)}$$

$$\delta_0 = -\frac{|\Omega_{2,0}|^2}{2\pi\text{freq}}$$

$$\text{Because } |\Omega^{(1,\text{res})}| = |\Omega_{1,1}| \hbar n$$

$$= |S_1| \cdot I_1 \cdot |\Omega_2| \hbar n.$$

$$\boxed{\delta^{(1,\text{res})} = \delta_{1,1} \cdot \varepsilon_1(0)} \\ = S_1^2 I_1^2 \delta_0 \cdot \varepsilon_1(0)$$

$$\delta^{(1,\text{res})} = \underbrace{\xi_1^2 I_1^2}_{P_{ij}^2} \cdot \frac{\Omega_{2,j}^2}{P_{ij}^2 I_1^2} \cdot \frac{1}{2\pi\text{freq}}$$

$$= \frac{I_1^2}{I_1 \cdot 2j} \frac{\Omega_{2,j}^2}{2\pi\text{freq}}$$

② Stark-shift in single-qubit rotation with two combs.

$$f_A = \begin{cases} \pm x \text{ freq} & \text{if } 0 \leq x < 0.5 \\ (1 \mp x) \text{ freq} & \text{if } 0.5 \leq x < 1 \end{cases}$$

Two tones to a single AOM, one beam: I_{1r} & I_{1b} .

$$|\Omega^{(2)}| = |S_1| \cdot \sqrt{I_{lb} I_{lr}} \cdot |\Omega_0| h\nu$$

$$= \frac{1}{2} |S_1| I_1 |\Omega_0| h\nu.$$

$$\delta^{(2)} = \underbrace{\delta_{lb,lr} \cdot 2 \cdot \xi_2(x, x_{\text{freq}})}_{\text{inter-comb}} + \underbrace{(\delta_{lb,lb} + \delta_{lr,lr}) \xi_1(x)}_{\text{intra-comb}}$$

$$= |P_{1,1}|^2 \delta_0 \left[I_{lb} I_{lr} \cdot 2 \xi_2(x, x_{\text{freq}}) + (I_{lb}^2 + I_{lr}^2) \xi_1(x) \right]$$

$$= |S_1|^2 \delta_0 \cdot \underbrace{\frac{I_1^2}{4}}_{\text{Assume } \alpha \approx 1} [2 \xi_2(x, x_{\text{freq}}) + (\alpha^2 + \alpha^{-2}) \xi_1(x)]$$

$$\alpha \approx 1 \approx \pm |S_1|^2 \delta_0 I_1^2 [\xi_2(x, x_{\text{freq}}) + \xi_1(x)]$$

where $\xi_2(x, f_A) = \frac{1}{2} \sum_k h\nu \left(\frac{1}{k-n-x+f_A/\text{freq}} + \frac{1}{k-n-x-f_A/\text{freq}} \right)$

$$\xi_1(x) = \sum_k h\nu \frac{1}{k-n-x}$$

③ Stark shift for dual-qubit operations.

$$H = \frac{\hbar}{2} \sum_{i \in \{a,b\}} \eta |1\rangle \langle 1|_i (\Omega_{ra} a^- e^{-i\delta t} + \Omega_{rb} a^+ e^{i\delta t}) + \text{h.c.}$$

Effective Rabi frequency:

$$\Omega_{\text{MS}} = \frac{\eta^2 \Omega_{rb}^2 \Omega_{ra}^2 h\nu}{\delta}$$

$$|\Omega_0| = \frac{\Omega_{ij}}{|P_{ij}| \cdot \sqrt{I_i I_j}}$$

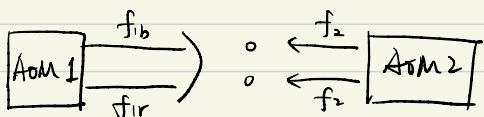
$$\text{Define : } \Omega_{\text{MS},0} = \frac{\Omega_{\text{MS}}}{|P_{1,1}|^2 \sqrt{I_{lb} I_{lr}}} = \frac{\eta^2 \Omega_0^2 h\nu}{\delta}$$

On the contrary

$$|S_{MS}| = |P_{1,2}|^2 \sqrt{I_{1b} I_{1r}} I_2 |P_{MS,0}| h\nu$$

$$= \frac{1}{2} \sqrt{-S^2/2} I_1 I_2 \underbrace{|S_{MS,0}| h\nu}_{\text{independent of experimental paras}}$$

- Resonance condition:



$$\text{Define } f_{Ab} = f_{1b} - f_2, \quad f_{Ar} = fir - f_2$$

phase-insensitive: Drive blue by causing the ion to absorb one ion from beam 1 and emit a photon into beam 2. Red is the opposite.

$$\begin{aligned} k\text{freq} \pm f_{Ab} &= \omega_0/2\pi + fm \\ k\text{freq} \mp f_{Ar} &= \omega_0/2\pi - fm \end{aligned} \quad \left. \right\}$$

Subtract
the two eqns $\Rightarrow f_{Ab} + f_{Ar} = k\text{freq} \pm 2fm \quad (\text{phase-insensitive})$

$$(f_{Ab} - f_{Ar} = k\text{freq} \pm 2fm \quad \text{for phase-sensitive})$$

Add
the two eqns $\Rightarrow (k_1 + k_2)\text{freq} + f_{Ab} - f_{Ar} = 2\omega_0/2\pi$

$$f_{Ab} - f_{Ar} = 2\omega_0/2\pi - (k_1 + k_2)\text{freq}$$

$$= 2x\text{freq}$$

To minimize f_{Ab} & f_{Ar}
Let $k_1 = k_2 = n$.

$$|f_{Ab} - f_{Ar}| = \begin{cases} 2x\text{freq} & 0 \leq x < 0.5 \\ 2(1-x)\text{freq} & 0.5 \leq x < 1 \end{cases}$$

- Stark shift: intra-beam shift $\delta^{(\text{MS}, 11)}$
 inter-beam shift $\delta^{(\text{MS}, 12)}$

$$\begin{aligned}\delta^{(\text{MS}, 11)} &= \underbrace{\delta_{\text{ib}, \text{ir}} \cdot 2 \cdot \frac{g_2(x, 2x\text{freq})}{f_{\text{Ab}-\text{far}}} + (\delta_{\text{ib}, \text{ib}} + \delta_{\text{ir}, \text{ir}}) g_1(x)}_{\text{inter-comb}} \\ &= \frac{1}{4} |S_1|^2 I_1^2 \delta_0 [2 g_2(x, 2x\text{freq}) + (\alpha^2 + \alpha^{-2}) g_1(x)] \\ &= \frac{1}{2} |S_1|^2 I_1^2 \delta_0 [g_2(x, 2x\text{freq}) + g_1(x)]\end{aligned}$$

* $\delta^{(\text{MS}, 11)}$ is very sensitive to x .
 but not sensitive to α

$$\begin{aligned}\delta^{(\text{MS}, 12)} &= \delta_{\text{ib}, 2} 2 \cdot \frac{g_2(x, f_{\text{Ab}})}{f_{\text{Ar}}} + \delta_{\text{ir}, 2} \cdot 2 \frac{g_2(x, f_{\text{Ar}})}{f_{\text{Ab}}} \\ &= 2|P_{1,2}|^2 I_2 \delta_0 [I_{\text{ib}} g_2(x, f_{\text{Ab}}) + I_1 g_2(x, f_{\text{Ar}})] \\ &= (1 - S_1^2/2) I_1 I_2 \delta_0 [\alpha g_2(x, f_{\text{Ab}}) + \alpha^{-1} g_2(x, f_{\text{Ar}})]\end{aligned}$$

Re-express this equation by assuming:

$$\begin{cases} f_{\text{Ab}} = \pm (x\text{freq} + f_m) \\ f_{\text{Ar}} = \mp (x\text{freq} - f_m) \end{cases}$$

$$f_m \ll x\text{freq}$$

$$\Rightarrow \delta^{(\text{MS}, 12)} = (1 - S_1^2/2) I_1 I_2 \delta_0 [(\alpha + \alpha^{-1}) g_2(x, x\text{freq})]$$

$$+ (\alpha - \alpha^{-1}) h_n^2 \frac{\text{freq}}{f_m}$$

drop high order

of $\frac{f_m}{\text{freq}} \ll 1$, Assume $\alpha \approx 1$

$$= (1 - S_1^2/2) I_2 \delta_0 [2 I_1 g_2(x, x\text{freq})]$$

$$+ (I_{\text{ib}} - I_{\text{ir}}) h_n^2 \frac{\text{freq}}{f_m}$$

For generality:

① Stark shift $\delta^{(1)}$ due to a single freq comb:

$$\begin{aligned} \frac{\delta^{(1)}}{|\Omega^{(1)}|^2} \cdot w_{\text{rep}} &= \frac{S_1^2 I_1^2 \delta_0 g_1(x)}{|S_1|^2 I_1^2 |\Omega^{(1)}|^2 \hbar^2} \cdot 2\pi f_{\text{rep}} \\ &= \frac{\delta_0}{|\Omega^{(1)}|^2} \cdot \frac{g_1(x)}{\hbar^2} \cdot 2\pi f_{\text{rep}} \\ &= - \frac{|\Omega^{(1)}|^2}{2\pi f_{\text{rep}}} / \frac{1}{|\Omega^{(1)}|^2} \cdot \frac{g_1(x)}{\hbar^2} \\ &= - \frac{g_1(x)}{\hbar^2} \end{aligned}$$

② Stark shift $\delta^{(2)}$ due to two freq combs.

$$\begin{aligned} \frac{\delta^{(2)}}{|\Omega^{(2)}|^2} \cdot w_{\text{rep}} &= \frac{\frac{1}{4} |S_1|^2 I_1^2 \delta_0 [2g_2(x, x_{\text{freq}}) + (\omega^2 + \omega^{-2}) g_1(x)]}{\frac{1}{4} |S_1|^2 I_1^2 |\Omega^{(2)}|^2 \hbar^2} \times 2\pi f_{\text{rep}} \\ &= \frac{\delta_0}{|\Omega^{(2)}|^2} \cdot 2\pi f_{\text{rep}} \cdot \frac{[2g_2(x, x_{\text{freq}}) + (\omega^2 + \omega^{-2}) g_1(x)]}{\hbar^2} \\ &= - \frac{[2g_2(x, x_{\text{freq}}) + (\omega^2 + \omega^{-2}) g_1(x)]}{\hbar^2} \end{aligned}$$

MS gate Stark shift:

$$\frac{\delta^{(MS,11)}}{|\Omega^{(2)}|^2} \times w_{rep} \underset{\alpha \approx 1}{\approx} \frac{\frac{1}{2} |S_1|^2 I_1 \int_0^{\infty} [\beta_2(x, 2x\text{freq}) + \beta_1(x)]}{(\frac{1}{2} |S_1| I_1 |\Omega^{(2)}| \hbar n)^2} \cdot w_{rep}$$

global beam ↙

$$= - \frac{2 [\beta_2(x, 2x\text{freq}) + \beta_1(x)]}{\hbar^2}$$

But the absolute value of $\delta^{(MS,11)}$ should be very small because we use lin-perp-lin. so $|S_1| \approx 0$.

$$\frac{\delta^{(MS,12)}}{|\Omega^{(MS)}|} \cdot \frac{w_{rep}}{\delta} = \frac{(-S_1^2) I_2 I_1 \int_0^{\infty} [(\alpha + \alpha^{-1}) \beta_2(x, x\text{freq}) + (\alpha - \alpha^{-1}) \hbar^2 \frac{\text{freq}}{f_m}]}{\frac{1}{2} (1 - S_1^2) I_1 I_2 \hbar^2 \gamma^2 |\Omega^{(2)}|^2 / \delta} \cdot \frac{w_{rep}}{\delta}^2$$

$$= - \frac{2 [(\alpha + \alpha^{-1}) \beta_2(x, x\text{freq}) + (\alpha - \alpha^{-1}) \hbar^2 \frac{\text{freq}}{f_m}]}{\hbar^2}$$

where $\alpha = \sqrt{I_{lb}/I_{ir}}$

$$\frac{\delta^{(MS,11)}}{\delta^{(MS,12)}} \underset{\alpha=1}{\approx} \frac{-2 [\beta_2(x, 2x\text{freq}) + \beta_1(x)]}{-2 [2 \beta_2(x, x\text{freq})]} \cdot \frac{|\Omega^{(2)}|^2 / w_{rep}}{|\Omega^{(MS)}| \cdot \delta / w_{rep}}^2$$

$$\downarrow$$

$$= \frac{[\beta_2(x, 2x\text{freq}) + \beta_1(x)]}{2 \beta_2(x, x\text{freq})} \cdot \frac{1}{2} \frac{S_1^2}{1 - S_1^2} \frac{I_1}{I_2} = \frac{1}{2} \frac{S_1^2}{1 - S_1^2} \frac{I_1}{I_2} \frac{\delta}{\delta^{(2)}} \xrightarrow{\text{global}} \frac{\delta}{\delta^{(2)}} \xrightarrow{\text{Ind}}$$

If no approximation:

$$\frac{\delta^{(\text{MS}, 12)}}{|\delta^{(\text{MS})}|} \cdot \frac{w_{\text{rep}} l^2}{\delta} = \frac{-2 [\alpha g_2(x, f_{Ab}) + \alpha^{-1} g_2(x, f_{Ar})]}{h^2}$$

$$\text{If } f_{Ab} = [(1-x) f_{\text{rep}} + f_m] , \quad f_{Ar} = [-(1-x) f_{\text{rep}} - f_m] \\ = -(1-x) f_{\text{rep}} + f_m$$