14.7 $py^{p-1}(x-y) \le x^p - y^p \le px^{p-1}(x-y)$, $0<y<x$, $p>1$;

$x^p - y^p = p\xi^{p-1}(x-y)$, $0<y<\xi<x \Rightarrow y^{p-1} < \xi^{p-1} < x^{p-1}$

$\Rightarrow py^{p-1}(x-y) \le x^p - y^p \le px^{p-1}(x-y)$

14.18 $x^3+3x+6x\ln x+2 > 6x^2$, $x>1$;

$x_0 = 1$, $\varphi(x) = x^3+3x+6x\ln x+2$, $\psi(x) = 6x^2$

$\varphi(1) = 6$, $\psi(1) = 6$;

$\varphi'(x) = 3x^2+3+6\ln x+6$, $\psi'(x) = 12x$ $\varphi'(1) = \psi'(1) = 12$

Scanned with CamScanner

$\varphi''(x) = 6x + \frac{6}{x}$, $\psi''(x) = 12$, $\varphi''(1) = \psi''(1) = 12$

$\varphi'''(x) = 6 - \frac{6}{x^2}$, $\psi'''(x) = 0$

$\varphi'''(x) > \psi'''(x)$ при $x > 1 \Rightarrow \varphi(x) > \psi(x)$ //

14.18 $x\ln x + y\ln y \geq (x+y)\ln\frac{x+y}{2}$, $x>0, y>0$;

$f(t) = t\ln t$, $f'(t) = 1 + \ln t$, $f''(t) = \frac{1}{t} > 0$, при $t > 0$

$f(t)$ — опукла вниз $\cup$

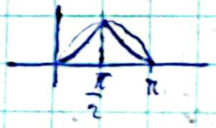
$f(q_1x_1 + q_2x_2) \leq q_1 f(x_1) + q_2 f(x_2)$

$\frac{1}{2}x\ln x + \frac{1}{2}y\ln y \geq \frac{x+y}{2}\ln\frac{x+y}{2} \Rightarrow$

$\Rightarrow x\ln x + y\ln y \geq (x+y)\ln\frac{x+y}{2}$ //

14.20 $\sqrt{\sin\frac{x+y}{2}} \geq \frac{1}{2}(\sqrt{\sin x} + \sqrt{\sin y})$, $\{x, y\} \subset [0, \pi]$

$f(t) = \sqrt{\sin t}$



$f'(t) = \frac{\cos t}{2\sqrt{\sin t}}$

$f''(t) = \frac{-2\sin t\sqrt{\sin t} - \frac{\cos t \cdot \cos t}{\sqrt{\sin t}}}{4\sin t} = \frac{-2\sin^2 t - \cos^2 t}{4\sin t\sqrt{\sin t}} =$

$= \frac{-2\sin^2 t - 1 + \sin^2 t}{4(\sin t)^{3/2}} = \frac{-(1+\sin^2 t)}{4(\sin t)^{3/2}} < 0 \quad \cap$

$f(q_1x_1 + q_2x_2) \geq q_1 f(x_1) + q_2 f(x_2)$

$\sqrt{\sin\frac{x+y}{2}} \geq \frac{1}{2}\sqrt{\sin x} + \frac{1}{2}\sqrt{\sin y} = \frac{1}{2}(\sqrt{\sin x} + \sqrt{\sin y})$ //

14.21 $\cos\left(\frac{x+y}{2}\right)^2 \geq \frac{1}{2}(\cos x^2 + \cos y^2)$, $\{x, y\} \subset [0, \sqrt{\frac{\pi}{2}}]$

$f(t) = \cos t^2$

$f'(t) = -2t \cdot \sin t^2$

$f''(t) = -2\sin t^2 - 4t^2 \cdot \cos t^2 = -(2\sin t^2 + 4t^2\cos t^2) < 0$ $\cap$

$f(g_1 x_1 + g_2 x_2) \geq g_1 f(x_1) + g_2 f(x_2)$

$\cos\left(\frac{x+y}{2}\right)^2 \geq \frac{1}{2}\cos x^2 + \frac{1}{2}\cos y^2 = \frac{1}{2}(\cos x^2 + \cos y^2)$ //

14.22 $\frac{x^n + y^n + z^n}{3} > \left(\frac{x+y+z}{3}\right)^n$, $x > 0, y > 0, z > 0, n > 1$,
$x \neq y, y \neq z, z \neq x$

$f(t) = t^n$

$f'(t) = n x^{n-1}$

$f''(t) = n^2 x^{n-2} - n x^{n-2} > 0$ при $n > 1$ $\Rightarrow f(t)$ - опукла вниз

$f(g_1 x_1 + g_2 x_2 + g_3 x_3) \leq g_1 f(x_1) + g_2 f(x_2) + g_3 f(x_3)$

$\frac{1}{3} x^n + \frac{1}{3} y^n + \frac{1}{3} z^n \geq \left(\frac{x+y+z}{3}\right)^n \Rightarrow$

$\Rightarrow \frac{x^n + y^n + z^n}{3} > \left(\frac{x+y+z}{3}\right)^n$ при $x > 0, y > 0, z > 0, n > 1$ //

13.16 $f(x) = \ln(\ln(4-x))$, $n = 3$

$\ln(\ln(4(1-\frac{x}{4}))) = \ln(\ln 4 + (-\frac{x}{4} - \frac{x^2}{32} - \frac{x^3}{192})) =$

$= \ln 2 + \ln\ln 2 - \frac{1}{8\ln 2}x - \frac{1+2\ln 2}{128\ln^2 2}x^2 - \frac{1+3\ln 2+4\ln^2 2}{1536\ln^3 2} + o(x^3)$ //

13.22 $\lim\limits_{x\to 0} \dfrac{\frac{1}{\operatorname{th} x} - \frac{1}{\operatorname{tg} x}}{x} = \lim\limits_{x\to 0} \dfrac{\frac{\operatorname{ch} x}{\operatorname{sh} x} - \frac{\cos x}{\sin x}}{x} = \lim\limits_{x\to 0} \dfrac{\frac{1+\frac{x^2}{2}}{x+\frac{x^3}{6}} - \frac{1-\frac{x^2}{2}}{x-\frac{x^3}{6}+o(x^3)}}{x} =$

$= \lim\limits_{x\to 0} \dfrac{\cancel{x} - \frac{x^3}{6} + \cancel{\frac{x^3}{2}} - \cancel{x} - \frac{x^3}{6} + \frac{x^3}{2}}{x^3} = \lim\limits_{x\to 0} \dfrac{2x^3}{3x^3} = \dfrac{2}{3}$ //

13.23 $\lim\limits_{x\to 0} \dfrac{\cos(\sin x) - \cos x}{2x^4} = \lim\limits_{x\to 0} \dfrac{1 - \frac{\left(x-\frac{x^3}{6}\right)^2}{2} + \frac{\left(x-\frac{x^3}{6}\right)^4}{4!} - 1 + \frac{x^2}{2} - \frac{x^4}{4!}}{2x^4}$

$= \lim\limits_{x\to 0} \dfrac{1 + \frac{\cancel{x^2}x^4}{3} - \cancel{\frac{x^2}{2}} + \frac{x^4}{4!} - 1 + \cancel{\frac{x^2}{2}} - \frac{x^4}{4!}}{2x^4} = \lim\limits_{x\to 0} \dfrac{\frac{x^4}{3}}{2x^4} = \dfrac{1}{6}$ //