23.8 $f(x,y)=\frac{xy}{1+x^2+y^2}$, $P_0(1,1)$, $P_1(2,2)$

$\overline{P_0P_1}=(1,1)=\bar{i}+\bar{j}$ $\quad |\overline{P_0P_1}|=\sqrt{1+1}=\sqrt{2}$

$l=\frac{\overline{P_0P_1}}{|\overline{P_0P_1}|}=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ $\quad \cos\alpha=\frac{1}{\sqrt{2}}$, $\cos\beta=\frac{1}{\sqrt{2}}$

част. похідні $f(x,y)$ в т. $(1,1)$

$f_x'=\frac{y-x^2y+y^3}{(1+x^2+y^2)^2}\Big|_{P_0}=\frac{1}{9}$, $\quad f_y'=\frac{x-xy^2+x^3}{(1+x^2+y^2)^2}\Big|_{P_0}=\frac{1}{9}$

$\frac{\partial f}{\partial l}(P_0)=\frac{\partial f}{\partial x}(P_0)\cdot\cos\alpha+\frac{\partial f}{\partial y}(P_0)\cos\beta=\frac{1}{9}\cdot\frac{1}{\sqrt{2}}+\frac{1}{9}\cdot\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{9}$ //

$\text{grad} f=\frac{\partial f}{\partial x}\bar{i}+\frac{\partial f}{\partial y}\bar{j}=\frac{1}{9}\bar{i}+\frac{1}{9}\bar{j}$ $\quad \text{grad} f=\left(\frac{1}{9},\frac{1}{9}\right)$ //

23.9 $f(x,y)=e^{x^2+y^2+z^2}$, $P_0(1,0,0)$, $\bar{a}=(0,1,1)$

$\frac{\partial f}{\partial x}\Big|_{P_0}=e^{x^2+y^2+z^2}\cdot 2x\Big|_{P_0}=2e$

$\frac{\partial f}{\partial y}\Big|_{P_0}=e^{x^2+y^2+z^2}\cdot 2y\Big|_{P_0}=0$

$\frac{\partial f}{\partial z}\Big|_{P_0}=e^{x^2+y^2+z^2}\cdot 2z\Big|_{P_0}=0$

$|a|=\sqrt{0^2+1^2+1^2}=\sqrt{2}$ $\quad \cos\alpha=0$, $\cos\beta=\frac{1}{\sqrt{2}}$, $\cos\gamma=\frac{1}{\sqrt{2}}$

$\frac{\partial f}{\partial a}\Big|_{P_0}=2e\cdot 0+0\cdot\frac{1}{\sqrt{2}}+0\cdot\frac{1}{\sqrt{2}}=0$ //

$\text{grad} f=\frac{\partial f}{\partial x}\bar{i}+\frac{\partial f}{\partial y}\bar{j}+\frac{\partial f}{\partial z}\bar{k}=2e\bar{i}$

$\text{grad} f=(2e,0,0)$ //

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23.12 $\sqrt{3,98^2+3,01^2} = f(x,y)$

$f(x,y) = f(P_0) + f'_x(P_0)\Delta x + f'_y(P_0)\Delta y$

$f(x,y) = \sqrt{x^2+y^2}$, $P_0 = (4,3)$

$\Delta x = x - x_0 = -0,02$ $\Delta y = y - y_0 = 0,01$

$f'_x = \frac{x}{\sqrt{x^2+y^2}}$, $f'_y = \frac{y}{\sqrt{x^2+y^2}}$

$f(x,y) \approx 5 - \frac{4}{5}\cdot 0,02 + \frac{3}{5}\cdot 0,01 = 4,99$ //

23.17 $f(x,y,z) = \frac{z}{x^2+y^2}$, $x^2+y^2 \neq 0$

$df = \frac{z}{x^2+y^2} d\left(\frac{z}{x^2+y^2}\right) = \frac{z}{x^2+y^2}\left(\frac{dz}{2x\,dx + 2y\,dy}\right)$

$f'_x = -\frac{2xz}{(x^2+y^2)^2}$, $f'_y = -\frac{2yz}{(x^2+y^2)^2}$, $f'_z = \frac{1}{x^2+y^2}$

$df = -\frac{2xz}{(x^2+y^2)^2}dx - \frac{2yz}{(x^2+y^2)^2}dy + \frac{1}{x^2+y^2}dz$ //

$f''_{xx} = \frac{6x^2z - 2y^2z}{(x^2+y^2)^3}$, $f''_{yy} = \frac{6y^2z - 2x^2z}{(x^2+y^2)^3}$, $f''_{zz} = 0$

$f''_{xy} = \frac{8xyz}{(x^2+y^2)^3}$, $f''_{xz} = -\frac{2x}{(x^2+y^2)^2}$, $f''_{yz} = -\frac{2y}{(x^2+y^2)^2}$

$d^2f = \frac{2}{(x^2+y^2)^2}\cdot\left(z\frac{(3x^2-y^2)}{x^2+y^2}dx^2 + z\frac{3y^2-x^2}{x^2+y^2}dy^2 + \frac{8xyz}{x^2+y^2}dx\,dy -\right.$

$\left. - x\,dx\,dz - y\,dy\,dz\right)$ //

23.19 $F(x,y) = h(x+y, x-y)$

Нехай $u = x+y$, $v = x-y$, u і v – лінійні ф-ції

$\Rightarrow d^2u = 0$ та $d^2v = 0$, тому форма диференціала

$\forall$ порядку зберігається

$$d^2h = \left(\frac{\partial}{\partial u}du + \frac{\partial}{\partial v}dv\right)^2 h$$

$dF = h_1'du + h_2'dv$, $d^2F = h_{11}''du^2 + 2h_{12}''dudv + h_{22}''dv^2$

$du = dx + dy$, $dv = dx - dy$

$\Rightarrow dF = h_1'(dx+dy) + h_2'(dx-dy)$ //

$d^2F = h_{11}''(dx+dy)^2 + 2h_{12}''(dx^2-dy^2) + h_{22}''(dx-dy)^2$ //

23.22 $F(x,y,z) = u(x^2+y^2, x^2-y^2, 2xz)$

$h = x^2+y^2$, $v = x^2-y^2$, $k = 2xz$

$dF = u_1'(2xdx + 2ydy) + u_2'(2xdx - 2ydy) + u_3'(2zdx + 2xdz) =$

$= (u_1'dh + u_2'dv + u_3'dk) = 2(xu_1' + xu_2' + zu_3')dx +$

$+ 2y(u_1' - u_2')dy + 2xu_3'dz$ //

$d^2F = u_{11}''dx^2 + u_{22}''dy^2 + u_{33}''dz^2 + 2u_{12}''dxdy + 2u_{13}''dxdz + u_{23}''dydz$

$d^2h = d(2xdx + 2ydy) = 2dx^2 + 2dy^2$

$d^2v = d(2xdx - 2ydy) = 2dx^2 - 2dy^2$

$d^2k = d(2zdx + 2xdz) = 2dxdz + 2dzdx = 4dxdz$

$d^2F = 2(u_1' + u_2' + 2x^2(u_{11}'' + u_{22}'') + 2z^2u_{33}'')dx^2 + 2(u_1' - u_2' +$

$+ 2y^2(u_{11}'' + u_{22}''))dy^2 + 4x^2u_{33}''dz^2 - 8y(xu_{11}'' - xu_{22}'' + zu_{13}'' - zu_{23}'')dxdy +$

$+ 4(u_3' + 2x(xu_{13}'' + xu_{23}'' + zu_{33}''))dxdz + 8xy(u_{13}'' + u_{23}'')dydz$ //

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23.23 $\dfrac{\partial^3 f}{\partial x^2 \partial y}$, $f(x,y) = (2x-5)\cdot\cos\dfrac{1}{y^2}$, $y \neq 0$

$$\frac{\partial f}{\partial x} = 2\cos\frac{1}{y^2}, \quad \frac{\partial^2 f}{\partial x^2} = 0 \Rightarrow \frac{\partial^3 f}{\partial x^2 \partial y} = 0 \,//$$

23.26 $\dfrac{\partial^{m+n} f}{\partial x^m \partial y^n}$, $m,n \in \mathbb{Z}^+$, $f(x,y) = \sin(xy)$, $(x,y) \in \mathbb{R}^2$

$$\frac{\partial^m f}{\partial x^m} = y^m \cdot \sin\left(xy + \frac{m\pi}{2}\right)$$

$$\frac{\partial^{m+n} f}{\partial x^m \partial y^n} = x^n \cdot y^m \cdot \sin\left(xy + \frac{(m+n)\pi}{2}\right) //$$

23.30 $x + y + z = e^z$ $\quad z = z(x,y)$

$$dx + dy + dz = e^z dz$$

$$(e^z - 1)\, dz = dx + dy$$

$$z'_x = \frac{1}{e^z - 1}, \quad z'_y = \frac{1}{e^z - 1} //$$

$$z''_{xx} = -\frac{e^z}{(e^z-1)^2} \cdot z'_x = -\frac{e^z}{(e^z-1)^3} //$$

$$z''_{xy} = -\frac{e^z}{(e^z-1)^2} \cdot z'_y = -\frac{e^z}{(e^z-1)^3} = z''_{yy} //$$

23.34 $f(x,y) = (1+x)^y$, $P_0(0,0)$

$$f(x,y) = 1 + yx + o(\|h\|^2)$$