12.7 $f(x) = \dfrac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$

$$\begin{array}{l|l} ax+b & cx+d \\ -\left(ax+\frac{ad}{c}\right) & \frac{a}{c} \\ \hline b-\frac{ad}{c} & \end{array}$$

$$\frac{ax+b}{cx+d} = \frac{a}{c} + \frac{b-\frac{ad}{c}}{cx+d} = \frac{a}{c} + \frac{bc-ad}{c(cx+d)} = \frac{a}{c} + \frac{bc-ad}{c}\left(\frac{1}{cx+d}\right)$$

$$f^{(n)}(x) = \frac{bc-ad}{c} \cdot c^n \cdot (-1)^n \frac{n!}{(cx+d)^{n+1}}$$ //

12.8 $f(x) = \dfrac{x^2}{1-x}$, $x \neq 1$

$$f(x) = \frac{x^2}{1-x} = -x-1+\frac{1}{1-x}$$

$$f^{(n)}(x) = \cancel{(-1)^n} \cdot \frac{n! \cdot \cancel{(-1)^n}}{(\cancel{1+}\,x)^{n+1}} = n! \cdot (1-x)^{-(n+1)}$$ //

12.10 $f(x)=\frac{1}{\sqrt{1-2x}}=(1-2x)^{-\frac{1}{2}}$

$f'(x)=\frac{1}{(1-2x)^{\frac{3}{2}}}$, $f''(x)=\frac{3}{(1-2x)^{\frac{5}{2}}}$

$f^{(n)}=\frac{(2n-1)!!}{(1-2x)^{n+\frac{1}{2}}}$ при $n=1$ – істина; припустимо при n – істина

при $n+1$: $f^{(n+1)}=\left(\frac{(2n-1)!!}{(1-2x)^{n+\frac{1}{2}}}\right)'=(2n-1)!!\cdot\left(\frac{1}{(1-2x)^{n+\frac{1}{2}}}\right)'=\frac{(2n-1)!!\cdot(2n+1)}{(1-2x)^{(n+1)+\frac{1}{2}}}$

12.11 $f(x)=\sin^2 x=\frac{1}{2}(1-\cos 2x)$

$f^{(n)}(x)=-\frac{1}{2}(\cos 2x)^{(n)}=-2^{n-1}\cos\left(2x+\frac{\pi n}{2}\right)$

12.13 $f(x)=\operatorname{arctg} x$

$f'(x)=\frac{1}{1+x^2}$, $f''=-\frac{2x}{(1+x^2)^2}$

$(1+x)\cdot f''=\frac{-2x}{(1+x^2)}=-2x\cdot f'$

$(1+x^2)\cdot f^{(n)}(x)+C^1_{n-2}\cdot 2x\cdot f^{(n-1)}(x)+2C^2_{n-2}\cdot f^{(n-2)}(x)=$

$=-2x\cdot f^{(n-1)}(x)-2C^1_{n-2}\cdot f^{(n-2)}(x)$

$f^{(n)}(x)=(n^2-3n+10)f^{(n-2)}(x)$

Оскільки $f(0)=0$, $f'(0)=1$, $f''(0)=0 \Rightarrow$

$\Rightarrow f^{(2k)}(0)=0$, $f^{(2k+1)}(0)=(-1)^k\cdot(2k)!$, $k\in\mathbb{Z}_+$

12.14 $f(x)=(\arcsin x)^2$

$f'(x)=2\arcsin x\,\frac{1}{\sqrt{1-x^2}}\Rightarrow\sqrt{1-x^2}\cdot f'(x)=2\arcsin x$

$\frac{-2x}{2\sqrt{1-x^2}}f'(x)+\sqrt{1-x^2}f''(x)=\frac{2}{\sqrt{1-x^2}}$

$(1-x^2)f''(x)-xf'(x)=2$

$f^{(n)}(x)(1-x^2)-(n-2)\cdot f^{(n-1)}(x)\,2x-\frac{(n-2)!}{2!(n-4)!}f^{(n-2)}(x)\,2-f^{(n-1)}(x)\cdot x-(n-2)f^{(n-2)}(x)=0$

$f^{(n)}(0)-(n-3)(n-2)f^{(n-2)}(0)-(n-2)f^{(n-2)}(0)=0$

$f^{(n)}(0)=(n-2)^2f^{(n-2)}(0)$

оскільки $f(0)=0$, $f'(0)=0$, $f''(0)=2\Rightarrow$

$\Rightarrow f^{(2k-1)}(0)=0$, $f^{(2k)}(0)=2^{2k-1}((k-1)!)^2$, $k\in N$

12.18 $f = \frac{u}{v}$

$df = \frac{v\,du - u\,dv}{v^2}$

$d^2 f = \frac{v(v\,d^2u - u\,d^2v) - 2v\,dv(v\,du - u\,dv)}{v^3}$, $v \neq 0$

13.24 $\lim\limits_{x\to+\infty} \frac{\pi - 2\operatorname{arctg} x}{e^{\frac{1}{x}} - 1} = \lim\limits_{x\to+\infty} \frac{0}{0}$

$f(x) = \pi - 2\operatorname{arctg} x \underset{x\to+\infty}{=} 0$

$g(x) = e^{\frac{1}{x}} - 1 \underset{x\to+\infty}{=} 0$

$f'(x) = -\frac{2}{x^2+1}$

$g'(x) = -\frac{e^{\frac{1}{x}}}{x^2}$

$\lim\limits_{x\to+\infty} \frac{\pi - 2\operatorname{arctg} x}{e^{\frac{1}{x}} - 1} = \lim\limits_{x\to+\infty} \frac{2x^2}{(x^2+1)\cdot e^{\frac{1}{x}}} = \underbrace{\lim\limits_{x\to+\infty} \frac{1}{e^{\frac{1}{x}}}}_{=1} \cdot \lim\limits_{x\to+\infty} \frac{x^2}{x^2+1}$

$= 2\lim\limits_{x\to+\infty} \frac{x^2}{x^2+1}$

$f(x) = x^2 \underset{x\to+\infty}{=} +\infty$

$g(x) = x^2 + 1 \underset{x\to+\infty}{=} +\infty$

$f'(x) = 2x$

$g'(x) = 2x$

$2\lim\limits_{x\to+\infty} 1 = 2$ //

13.26 $\lim\limits_{x\to1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \infty - \infty =$

$= \lim\limits_{x\to1} \left(\frac{x-1-\ln x}{\ln x \cdot (x-1)}\right) = \lim\limits_{x\to1} \left(\frac{1-\frac{1}{x}}{1-\frac{1}{x}+\ln x}\right) = \lim\limits_{x\to1} \left(\frac{\frac{x-1}{x}}{\frac{x-1+x\cdot\ln x}{x}}\right) = \lim\limits_{x\to1} \left(\frac{x-1}{x-1+x\cdot\ln x}\right) =$

$= \lim\limits_{x\to1} \left(\frac{1}{2+\ln(x)}\right) = \frac{1}{2+\ln 1} = \frac{1}{2}$ //

13.27 $\lim\limits_{x \to 0+0} x^x = 0^0 = \lim\limits_{x \to 0+0} e^{x \ln(x)}$; $\lim\limits_{x \to 0+0} x \ln x$

підстановка $x = \frac{1}{t} \Rightarrow t = \frac{1}{x}$; $x \to +0 \Rightarrow t \to +\infty$

$\lim\limits_{t \to +\infty} -\frac{\ln(t)}{t} = \lim\limits_{t \to +\infty} -\frac{1}{t} = 0.$

$\lim\limits_{x \to 0+0} e^{x \ln x} = e^0 = 1$ //

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