5.15) $x_n = \sum_{k=1}^{n} \frac{1}{2n+2k+1} = \sum_{k=1}^{2n} \frac{1}{2n+k} - \sum_{k=1}^{n} \frac{1}{2n+2k} = \sum_{k=1}^{2n} \frac{1}{2n+k} - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{n+k} =$

$= \sum_{k=1}^{4n} \frac{1}{k} - \sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{2} \left(\sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \right) = \ln \frac{4n}{2n} - \frac{1}{2} \ln 2 = \ln \sqrt{2}$ //

5.16 $x_n = \frac{1}{n^3} \sum_{k=1}^{n} k^2$

За теоремою Штольца:

$x_n = \frac{\sum_{k=1}^{n} k^2}{n^3} = \frac{z_n}{y_n}$, де $z_n = \sum_{k=1}^{n} k^2$ та $y_n = n^3$

$\lim_{n\to\infty} \frac{z_{n+1} - z_n}{y_{n+1} - y_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)^3 - n^3} = \lim_{n\to\infty} \frac{n^2+2n+1}{\cancel{n^3}+3n^2+3n-\cancel{n^3}+1} =$

$= \lim_{n\to\infty} \frac{n^2+2n+1}{3n^2+3n+1} = \frac{1 + \underbrace{\frac{2}{n}}_{\to 0} + \underbrace{\frac{1}{n^2}}_{\to 0}}{3 + \underbrace{\frac{3}{n}}_{\to 0} + \underbrace{\frac{1}{n^2}}_{\to 0}} = \frac{1}{3}$ //

6.1 $x_n = 1 + \frac{n}{n+1} \sin \frac{n\pi}{2}$

$4k-3 = 1 + \frac{4k-3}{4k-2}$ $\quad$ $4k-1 = 1 - \frac{4k-3}{4k}$

$4k-2 = 1$ $\quad$ $4k = 1$

	$\lim\limits_{n\to\infty} x_m$	$\inf\limits_{m\in N} x_m$	$\sup\limits_{m\in N} x_m$
$m = 4k$	1	1	1
$m = 4k-1$	0	0	$\frac{3}{4}$
$m = 4k-2$	1	1	1
$m = 4k-3$	2	$\frac{1}{2}$	2

$\inf\limits_{n\in N} x_n = \underline{\lim\limits_{n\to\infty}} x_n = 0, \sup\limits_{n\in N} x_n = \overline{\lim\limits_{n\to\infty}} x_n = 2$

6.7 $x_{n+1} = x_n^2 - 2x_n + 2, n \geq 1, x_1 \in (1,2)$

Припустимо $\exists \lim\limits_{n\to\infty} x_n = \alpha$

$\alpha = \alpha^2 - 2\alpha + 2 \Leftrightarrow \alpha^2 - 3\alpha + 2 = 0$

$\alpha_1 = 1, \alpha_2 = 2$

При $n = 1$: $x_1 > 1$

$n \in N$: $x_n > 1$

$x_{n+1} = x_n^2 - 2x_n + 2 > 1^2 - 2 \cdot 1 + 2 = 1$

Послідовність x_n є обмеженою

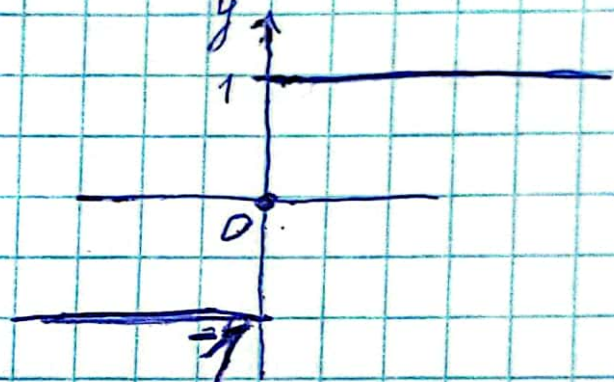
$x_{n+1} - x_n = x_n^2 - 2x_n + 2 - x_n = (x_n - 1)(x_n - 2) < 0$ при $x \in (1;2)$

x_n - незростаюча послідовність, за теоремою Вейєрштраса є обмеженою. Отже, $\lim\limits_{n\to\infty} x_n = 1$.

6.10 $f(x) = \lim\limits_{n\to\infty} \dfrac{n^{x} - n^{-x}}{n^{x} + n^{-x}} =$

$= \lim\limits_{n\to\infty} \left(\dfrac{n^{x} - \frac{1}{n^{x}}}{n^{x} + \frac{1}{n^{x}}} \right) = \lim\limits_{n\to\infty} \left(\dfrac{n^{x}\left(1 - \frac{1}{n^{2x}}\right)}{n^{x}\left(1 + \frac{1}{n^{2x}}\right)} \right) = \lim\limits_{n\to\infty} \dfrac{1 - \frac{1}{n^{2x}}}{1 + \frac{1}{n^{2x}}} =$

$= \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$



6.13 $f(x) = \lim\limits_{n\to\infty} \sqrt[n]{1 + x^{n} + \left(\frac{x^{2}}{2}\right)^{n}}, \quad x \geq 0$

$\lim\limits_{n\to\infty} \sqrt[n]{1 + x^{n} + \left(\frac{x^{2}}{2}\right)^{n}} = \lim\limits_{n\to\infty} \sqrt[n]{1 + x^{n} + \frac{x^{2n}}{2^{n}}} =$

$= \lim\limits_{n\to\infty} \sqrt[n]{1 + x^{2n}\left(\frac{1}{x^{n}} + \frac{1}{2^{n}}\right)} = 1$