19.18 $a_n = \sum_{k=1}^{n} \frac{(4k-3)^3}{k^4} = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{64}{\frac{k}{n}} - \frac{144}{\frac{k^2}{n}} + \frac{108}{\frac{k^3}{n}} - \frac{27}{\frac{k^4}{n}}\right)$

$\lim\limits_{n\to\infty} a_n = \int_0^1 \frac{64}{x}\,dx - \int_0^1 \frac{144}{x^2}\,dx + \int_0^1 \frac{108}{x^3}\,dx - \int_0^1 \frac{27}{x^4}\,dx =$

$= 144 - 54 + 9 = 99$ //

19.20 $a_n = \sum_{k=1}^{n-1} \sin\frac{k}{n^2}\cdot \operatorname{arctg}\frac{k}{n^3} = \sum_{k=1}^{n} \sin\frac{k}{n^2}\cdot \operatorname{arctg}\frac{k}{n^3} - \underbrace{\sin\frac{1}{n}\operatorname{arctg}\frac{1}{n^2}}_{\to 0}$

$\sin\frac{k}{n^2} = \frac{k}{n^2} + O\left(\frac{k^3}{n^6}\right)$ $\quad \operatorname{arctg}\frac{k}{n^3} = -k + \frac{k^3}{3n^9} + O\left(\frac{k^3}{n^9}\right)$

$a_n = \sum_{k=1}^{n}\left(\frac{k}{n^2} + O\left(\frac{k^3}{n^6}\right)\right)\left(-k + \frac{k^3}{3n^9} + O\left(\frac{k^3}{n^9}\right)\right) =$

$= \sum_{k=1}^{n}\left(-\frac{k^2}{n^2} + \frac{k^4}{3n^{11}} + \underbrace{O\left(\frac{k^3}{n^6}\right)}_{\to 0}\right)$

$\int_0^1 \left(-x^2 + \frac{x^4}{3}\right)dx = \left(-\frac{x^3}{3} + \frac{x^5}{15}\right)\Big|_0^1 = -\frac{1}{3} + \frac{1}{15} = -\frac{4}{15}$ //

19.23 $a_n = \left(\prod_{k=1}^{2^n-1}\left(1+\frac{k}{2^n}\right)\right)^{\frac{1}{2^n}} = e^{\frac{1}{2^n}\ln\left(\prod_{k=1}^{2^n-1}\left(1+\frac{k}{2^n}\right)\right)} =$

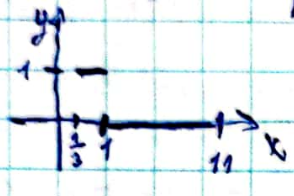
$= e^{\frac{1}{2^n}\sum_{k=1}^{2^n-1}\ln\left(1+\frac{k}{2^n}\right)} \Rightarrow e^{\int_0^1 \ln(1+x)\,dx}$

$\int_0^1 \ln(1+x)\,dx = \ln(1+x)\cdot x - \int_0^1 \frac{x\,dx}{1+x} = \ln 2 - x\Big|_0^1 + \ln|x+1|\Big|_0^1 =$

$= 2\ln 2 - 1 = \ln\frac{4}{e}$

$e^{\ln\frac{4}{e}} = \frac{4}{e}$ //

19.27 $f(x) = \left[\frac{1}{\sqrt{x}}\right],\ [a,b] = \left[\frac{1}{3}, 11\right]$



$\Delta x = \frac{11-\frac{1}{3}}{n} = \frac{32}{3n}$

$x_k = \frac{1}{3} + \frac{32k}{3n}$

$M_k = m_k = f(x_k) = \left[\frac{1}{\sqrt{\frac{1}{3}+\frac{32k}{3n}}}\right] \Rightarrow \overline{S}_p = \underline{S}_p \Rightarrow f$ - інтегровна за Ріманом.

20.1 $\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\cos x)\,dx,\ f \in C[0, 1]$

$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx = \left|\begin{matrix}\frac{\pi}{2}-x=t,\ dx=-dt\\ t=\left(\frac{\pi}{2}, 0\right)\end{matrix}\right| =$

$= -\int_{\pi/2}^{0} f(\sin t)\,dt = \int_0^{\pi/2} f(\sin t)\,dt$ //

20.3 $\int_0^1 \frac{\operatorname{arctg} x}{x}\,dx = \frac{1}{2}\int_0^{\pi/2}\frac{t\,dt}{\sin t}$

$\int_0^1 \frac{\operatorname{arctg} x}{x}\,dx = \left|\begin{matrix}\operatorname{arctg} x = y\\ x = \operatorname{tg} y\\ dx = \frac{1}{\cos^2 y}\,dy\\ y=0,\ y=\frac{\pi}{4}\end{matrix}\right| = \int_0^{\pi/4}\frac{y\,dy}{\operatorname{tg} y\cos^2 y} = \int_0^{\pi/4}\frac{2y}{\sin 2y}\,dy =$

$= \left|\begin{matrix}2y=t\\ 2dy=dt\\ y=t=0\\ y=\frac{\pi}{4},\ t=\frac{\pi}{2}\end{matrix}\right| = \frac{1}{2}\int_0^{\pi/2}\frac{t\,dt}{\sin t}$ //

20.7 $\int_0^{100\pi}\sqrt{1-\cos 2x}\,dx = \int_0^{100\pi}\sqrt{2}\,|\sin x|\,dx = \sqrt{2}\cdot 100\int_0^{\pi}\sin x\,dx = 200\sqrt{2}$ //

20.9 $\int_0^4 [x^2]\,dx = \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}} 3\,dx =$

$= \sum_{k=0}^{3} k\left(\sqrt{k+1}-\sqrt{k}\right) = (\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3}) = 5-\sqrt{2}-\sqrt{3}$ //

20.14 $\int_0^1 \sqrt{x^2+1}\,dx,\ x = \frac{1}{\cos t}\ \begin{cases}x=0\\ \frac{1}{\cos t}\neq 0\end{cases} \Rightarrow$ заміна неможлива