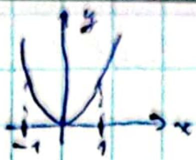
19.1 $f(x) = x^2;\ [a, b] = [-1, 1],\ m = 2$



$\Delta x_k = x_{k+1} - x_k = \frac{b-a}{2n} = \frac{1-(-1)}{2n} = \frac{1}{n}\ \forall k = \overline{0, 2n-1}$

$x_0 = -1,\ x_k = x_0 + \frac{k}{n} = -1 + \frac{k}{n},\ x_{k+1} = -1 + \frac{k+1}{n},\ [x_k, x_{k+1}]$

$x_{2n} = -1 + \frac{2n}{n} = 1$

$\overline{S_p}(f) = \sum_{k=0}^{2n-1} M_k \Delta x_k = \sum_{k=0}^{n-1} \left(-1 + \frac{k}{n}\right)^2 \cdot \frac{1}{n} + \sum_{k=n}^{2n-1} \left(-1 + \frac{k+1}{n}\right)^2 \cdot \frac{1}{n}$

$M_k = \sup\limits_{x \in [x_k, x_{k+1}]} f(x) = f(x_k) = \left(-1 + \frac{k}{n}\right)^2,\ [-1, 0] \supset x_k,\ k = \overline{0, n-1}$

$M_k = \sup\limits_{x \in [x_k, x_{k+1}]} f(x) = f(x_{k+1}) = \left(-1 + \frac{k+1}{n}\right)^2,\ [0, 1] \supset x_k,\ k = \overline{n, 2n-1}$

$\underline{S_p}(f) = \sum_{k=0}^{2n-1} m_k \Delta x_k = \sum_{k=0}^{n-1} \left(-1 + \frac{k+1}{n}\right)^2 \cdot \frac{1}{n} + \sum_{k=n}^{2n-1} \left(-1 + \frac{k}{n}\right) \cdot \frac{1}{n}$

19.2 $f(x) = x^3,\ [a, b] = [-2, 3],\ m = 1$

$f(x)$ – монотонно зростаюча і неперервна на $[-2, 3]$

$\Rightarrow \forall$ розбиття досягає найбільшого і найменшого значення відповідно в правому та лівому кінці

$[x_k, x_{k-1}],\ k = \overline{0, n-1} \quad \Delta x_k = \frac{b-a}{n} = \frac{3-(-2)}{n} = \frac{5}{n}$

$x_k = -2 + \frac{5k}{n},\ x_{k+1} = -2 + \frac{5(k+1)}{n}$

$\underline{S_p} = \frac{5}{n} \sum_{k=0}^{n-1} \left(-2 + \frac{5k}{n}\right)^3,\ \overline{S_p} = \frac{5}{n} \sum_{k=0}^{n-1} \left(-2 + \frac{5(k+1)}{n}\right)^3$

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},\ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},\ \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

$\int_{-2}^{3} x^3 = \frac{x^4}{4}\Big|_{-2}^{3} = \frac{81}{4} - 4 = 16\frac{1}{4}$

$\underline{S_p} = 16\frac{1}{4} - \frac{175}{2n} + \frac{125}{4n^2},\ \overline{S_p} = 16\frac{1}{4} + \frac{175}{2n} + \frac{125}{4n^2}$

$\lim\limits_{x \to \infty} \underline{S_p} = \lim\limits_{x \to \infty} \overline{S_p} = 16\frac{1}{4}$

19.3 $f(x) = \operatorname{sgn} x,\ [a, b] = [-2, 1],\ m = 3$

$\Delta x_k = x_{k+1} - x_k = \frac{b-a}{3n} = \frac{1-(-2)}{3n} = \frac{1}{n}\ \forall k = \overline{0, 3n-1}$

$\overline{S_p}(f) = \sum_{k=0}^{3n-1} M_k \Delta x_n = \sum_{k=0}^{2n-1} \operatorname{sgn}\left(-2+\frac{k}{n}\right)\frac{1}{n} + \sum_{k=2n+1}^{3n-1} \operatorname{sgn}\left(-2+\frac{k+1}{n}\right)\frac{1}{n}$

$\underline{S_p}(f) = \sum_{k=0}^{3n-1} m_k \Delta x_n = \sum_{k=0}^{2n-1} \operatorname{sgn}\left(-2+\frac{k}{n}\right)\frac{1}{n} + \sum_{k=2n+1}^{3n-1} \operatorname{sgn}\left(-2+\frac{k+1}{n}\right)\frac{1}{n}$

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19.4 $f(x)=|x|$, $[a,b]=[-3,2]$, $m=5$

$\Delta x_k = x_{k+1}-x_k = \frac{b-a}{5n} = \frac{2-(-3)}{5n} = \frac{1}{n}$ $\forall k=\overline{0,5n-1}$

$x_0=-3$, $x_k = x_0 + \frac{k}{n} = -3+\frac{k}{n}$, ..., $x_{5n} = -3+\frac{5n}{n} = 2$

$\overline{S_p}(f) = \sum_{k=0}^{5n-1} M_k \Delta x_k = \sum_{k=0}^{3n-1} \left|-3+\frac{k}{n}\right| \cdot \frac{1}{n} + \sum_{k=3n}^{5n-1} \left|-3+\frac{k+1}{n}\right| \cdot \frac{1}{n}$

$M_k = \sup_{x\in[x_k,x_{k+1}]} f(x) = f(x_k) = \left|-3+\frac{k}{n}\right|$, $[-3,0]\supset x_k$, $k=\overline{0,3n-1}$

$M_k = \sup_{x\in[x_k,x_{k+1}]} f(x) = f(x_{k+1}) = \left|-3+\frac{k+1}{n}\right|$, $[0,2]\supset x_k$, $k=\overline{3n,5n-1}$

$\underline{S_p}(f) = \sum_{k=0}^{5n-1} m_k \Delta x_k = \sum_{k=0}^{3n-1} \left|-3+\frac{k+1}{n}\right| \cdot \frac{1}{n} + \sum_{k=3n}^{5n-1} \left|-3+\frac{k}{n}\right| \cdot \frac{1}{n}$

19.8 $f(x)=x$, $[a,b]=[-1,1]$

$\Delta x_k = \frac{1-(-1)}{n} = \frac{2}{n}$

$x_k = -1+\frac{2k}{n}$, $x_{k+1} = -1+\frac{2(k+1)}{n}$

$M_k = \sup_{x\in[x_k,x_{k+1}]} f(x) = f(x_{k+1})$, $m_k = \inf_{x\in[x_k,x_{k+1}]} f(x) = f(x_k)$

$\overline{S_p} = \frac{2}{n} \sum_{k=0}^{n-1} -1+\frac{2(k+1)}{n} = \frac{2}{n}\left(-n+\frac{2}{n}\frac{n(n-1)}{2}+\frac{2}{n}\right) = -2+2-\frac{2}{n}+\frac{4}{n^2}$

$\lim_{n\to\infty} \overline{S_p} = 0$

$\underline{S_p} = \frac{2}{n} \sum_{k=0}^{n-1} \left(-1+\frac{2k}{n}\right) = -2+\frac{4}{n^2}\frac{n(n-1)}{2} = -2+2-\frac{2}{n}$

$\lim_{n\to\infty} \underline{S_p} = 0$

$\int_{-1}^{1} x\,dx = 0 \Leftrightarrow 0 \le \overline{S_p}(f) - \underline{S_p}(f) < \varepsilon$, $\lim_{\lambda_p\to 0} S_p(f,\xi_p) = I$

19.11 $\int\limits_{-3}^{2} \operatorname{sgn} x \, dx$

$\Delta x_k = \frac{5}{5n} = \frac{1}{n}, \quad m = 5$

$x_k = -2 + \frac{k}{n}, \quad \xi_k = x_k, \quad S_k = \frac{1}{n} \sum_{k=0}^{2n-1} \operatorname{sgn}\left(-2 + \frac{k}{n}\right) + \sum_{k=2n+1}^{5n} \operatorname{sgn}\left(-2 + \frac{k}{n}\right) =$

$= \frac{1}{n}(-1) \cdot 2n + \frac{1}{n}(1 \cdot 3n) = -2 + 3 = 1$