19.24 $a_n = n\sum\limits_{k=1}^{n} \ln\left(1+\frac{k}{n^2}\right)\cdot \operatorname{arctg}\frac{k^2}{n^3} =$

$\operatorname{arctg}\frac{k^2}{n^3} = \frac{k^2}{n^3} + O\left(\frac{k^6}{n^9}\right)$ $\quad \ln\left(1+\frac{k}{n^2}\right) = \frac{k}{n^2} + O\left(\frac{k^2}{n^4}\right)$

$= n\sum\limits_{k=1}^{n}\left(\frac{k}{n^2}+O\left(\frac{k^2}{n^4}\right)\right)\left(\frac{k^2}{n^3}+O\left(\frac{k^6}{n^9}\right)\right) =$

$= n\sum\limits_{k=1}^{n}\left(\frac{k^3}{n^5}+O\left(\frac{k^7}{n^{11}}\right)+O\left(\frac{k^4}{n^7}\right)+O\left(\frac{k^8}{n^{13}}\right)\right) =$

$= n\sum\limits_{k=1}^{n}\left(\frac{k^3}{n^5}+O\left(\frac{k^4}{n^7}\right)\right) = \frac{1}{n}\sum\limits_{k=1}^{n}\left(\frac{k}{n}\right)^3 + \frac{1}{n}\sum\limits_{k=1}^{n} O\left(\frac{k^4}{n^7}\right)$

$\Rightarrow \int\limits_0^1 x^3\,dx = \frac{x^4}{4} = \frac{1}{4}$ //

20.2 $\int\limits_0^a x^3 f(x^2)\,dx = \frac{1}{2}\int\limits_0^{a^2} t f(t)\,dt,\ a>0,\ f\in C([0,a^2])$

$\int\limits_0^a x^3 f(x^2)\,dx = \left| \begin{array}{l} x^2=t,\ x=\sqrt{t} \\ dx = \frac{dt}{2\sqrt{t}} \end{array}\right| = \int\limits_0^{a^2} t\cancel{\sqrt{t}} f(t)\frac{dt}{2\cancel{\sqrt{t}}} = \frac{1}{2}\int\limits_0^{a^2} t f(t)\,dt$ //

20.12 $\int\limits_1^2 \frac{dx}{x+x^3} = \int\limits_1^2 \frac{dx}{x(1+x^2)} = \int\limits_1^2 \frac{A}{x}dx + \int\limits_1^2 \frac{Bx+C}{1+x^2} = \ln x\Big|_1^2 - \frac{1}{2}\ln(1+x^2)\Big|_1^2 =$

$A(1+x^2)+Bx^2+Cx=1 \quad A=1 \quad A+B=0 \quad B=-1$

$= \frac{3}{2}\ln 2 - \ln\sqrt{5} = \frac{1}{2}\ln\frac{8}{5}$ //

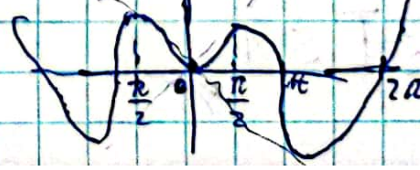
20.18 $\int\limits_4^7 (x^2-6x+13)\,dx,\quad t=x^2-6x+13$

$\int\limits_4^7 (x^2-6x+13)\,dx = \left| \begin{array}{l} t=x^2-6x+13 \\ dt=(2x-6)dx \\ dt=(2\sqrt{t-4})dx \end{array}\right| = \int\limits_5^{20} \frac{t\,dt}{2\sqrt{t-4}}$ //

20.24 $\lim\limits_{x\to+\infty} \frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt} = \lim\limits_{x\to+\infty} \frac{\frac{d}{dx}\left(\int_0^x e^{t^2}dt\right)^2}{\frac{d}{dx}\left(\int_0^x e^{2t^2}dt\right)} =$

$= \lim\limits_{x\to+\infty} \frac{2\int_0^x e^{t^2}dt\cdot \cancel{e^{x^2}}}{e^{\cancel{2}x^2}} = \lim\limits_{x\to+\infty} \frac{\cancel{2e^{x^2}}}{\cancel{e^{x^2}}\cdot 2x} = \lim\limits_{x\to+\infty}\frac{1}{x} = 0$ //

20.31 $I_1 = \int\limits_0^{\pi} x\sin x\,dx \quad ?\ I_2 = \int\limits_{\pi}^{2\pi} x\sin x\,dx$



$I_2:\ x-\pi=t,\ x=\pi+t \quad \int_0^{\pi}(\pi+t)\sin(\pi+t)\,dt =$

$= -\int_0^{\pi}(\pi+t)\sin t\,dt = -\int_0^{\pi}\pi\sin t\,dt - \int_0^{\pi} t\sin t\,dt = \cancel{\int_0^{\pi}\pi\sin t\,dt}$

$I_2 = -I_1 - \pi\int_0^{\pi}\sin t\,dt = -I_1 - 2\pi$

$\Rightarrow I_1 > I_2$ //

20.33 $I_1 = \int_0^{\pi} e^{-x^2}\cos^2 x\,dx \ ? \ \int_{\pi}^{2\pi} e^{-x^2}\cos^2 x\,dx$

$I_2:\ x-\pi=t,\ x=\pi+t \quad \int_0^{\pi} e^{-(\pi+t)^2}\cos^2(\pi+t)\,dt = \int_0^{\pi} e^{-(\pi+t)^2}\cos^2 t\,dt =$

$= \frac{1}{e^{\pi^2}}\int_0^{\pi} e^{-2\pi t}\cdot e^{-t^2}\cos^2 t\,dt$

$\Rightarrow I_1 > I_2$ //

20.34 $\int_0^{2} e^{x^2-x}\,dx = I$, За I теоремою про середнє; $e^{x^2-x}\in C[0;2]$

$I = e^{\xi^2-\xi}\int_0^{2} dx = e^{\xi^2-\xi}\cdot 2 \qquad \xi\in[0,2]$

$M = \sup_{\xi\in[0,2]} e^{\xi^2-\xi} = e^2, \quad m = \inf_{\xi\in[0,2]} e^{\xi^2-\xi} = e^{-\frac{1}{4}}$

$\frac{2}{\sqrt[4]{e}} \le 2e^{\xi^2-\xi} \le 2\cdot e^2 \qquad I\in\left[\frac{2}{\sqrt[4]{e}};\ 2e^2\right]$ //

20.37 $\int\limits_0^1 \frac{x^{19}dx}{\sqrt[3]{1+x^6}}$ $\quad f_1 = \frac{x^5}{\sqrt[3]{1+x^6}}$ – монотонна неспадна

$f_2 = x^{14}$ – монотонна неспадна

За II теоремою про середнє:

1) $I = f_1(1)\int\limits_{\xi_2}^1 f_2(x)\,dx = \frac{1}{\sqrt[3]{2}}\int\limits_{\xi_1}^1 x^{14}dx = \frac{1}{\sqrt[3]{2}}\left(\frac{1}{15} - \frac{\xi^{15}}{15}\right)$

Оскільки $\xi_2 \in [0, 1] \Rightarrow 0 \le I \le \frac{1}{15\sqrt[3]{2}}$

2) $I = f_2(1)\int\limits_{\xi_1}^1 f_1(x)\,dx = 1\cdot\int\limits_{\xi_1}^1 \frac{x^5}{\sqrt[3]{1+x^6}}dx = \left| \begin{array}{l} u = x^6+1 \\ du = 6x^5dx \\ dx = \frac{du}{6x^5} \end{array} \right| =$

$= \int\limits_{\xi_1}^1 \frac{\cancel{x^5}}{\sqrt[3]{u}}\cdot\frac{du}{6\cancel{x^5}} = \frac{1}{6}\int\limits_{\xi_1}^1 u^{-\frac{1}{3}}du = \frac{1}{6}\cdot\frac{3\sqrt[3]{u^2}}{2}\Big|_{\xi_1}^1 = \frac{\sqrt[3]{u^2}}{4}\Big|_{\xi_1}^1 =$

$= \frac{\sqrt[3]{(x^6+1)^2}}{4}\Big|_{\xi_1}^1 = \frac{\sqrt[3]{4}}{4} - \frac{\sqrt[3]{(\xi_1^6+1)^2}}{4}$

Оск. $\xi_1 \in [0, 1] \Rightarrow 0 \le I \le \frac{\sqrt[3]{4}-1}{4}$

Перша оцінка точніша, тому $I \in \left[0, \frac{1}{15\sqrt[3]{2}}\right]$ //