1. e) $a_n = \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}$

$S_n = \sum_{k=1}^{n} \sqrt{k+2} - 2\sqrt{k+1} + \sqrt{k} = \cancel{\sqrt{3}} - \cancel{2\sqrt{2}} + 1 + \cancel{\sqrt{4}} - \cancel{2\sqrt{3}} + \cancel{\sqrt{2}} +$

$+ \sqrt{5} - \cancel{2\sqrt{4}} + \cancel{\sqrt{3}} + \ldots + \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} =$

$= 1 - \sqrt{2} + \sqrt{3} - 2 + \ldots + \sqrt{n-2} - \cancel{2\sqrt{n-1}} + \cancel{\sqrt{n}} + \cancel{\sqrt{n+1}} - \cancel{2\sqrt{n-1+1}} + \cancel{\sqrt{n-1}} + \sqrt{n+2} - \cancel{2\sqrt{n+1}} + \cancel{\sqrt{n}} =$

$= 1 - \sqrt{2} + \sqrt{n-2} - \sqrt{n-1} + \sqrt{n+2} - \sqrt{n+1} = 1 - \sqrt{2} + \frac{1}{\sqrt{n+2} + \sqrt{n+1}} - \frac{1}{\sqrt{n-2} + \sqrt{n+1}}$

$\lim_{n\to\infty} = 1 - \sqrt{2}$ //

ж) $a_n = q^n \cos n\alpha, \quad |q| < 1$

$S_n = \sum_{k=1}^{n} q^k \cdot \cos k \cdot \alpha$

$q^n \cdot \cos n\alpha + i q^n \sin n\alpha = q^n e^{in\alpha} = (q e^{i\alpha})^n$

$r \overset{def}{=} q e^{i\alpha}, \ |r| = |q| < 1$

$\sum_{n=1}^{\infty} q^n \cdot \cos n\alpha + i \sum_{n=1}^{\infty} q^n \cdot \sin n\alpha = \sum_{n=1}^{\infty} r^n = \frac{r}{1-r} = \frac{q e^{i\alpha}}{1 - q e^{i\alpha}}$

$\sum_{n=1}^{\infty} q^n \cos n\alpha = Re\left(\frac{qe^{i\alpha}}{1-qe^{i\alpha}}\right)$ – дійсна частина //

к) $a_n = \frac{1}{(n+1)(n+2)(n+3)}$

$S_n = \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)(k+3)} = \sum_{k=1}^{n} \left(\frac{1}{2} \cdot \frac{1}{k+1} - \frac{1}{k+2} + \frac{1}{2} \cdot \frac{1}{k+3}\right) =$

$= \sum_{k=1}^{n} \left(\frac{1}{2}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) + \frac{1}{2}\left(\frac{1}{k+3} - \frac{1}{k+2}\right)\right) = \frac{1}{2} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} - \frac{1}{4}$

$\lim_{n\to\infty} S_n = \frac{1}{4}$ //

л) $a_n = \text{arctg} \frac{1}{2n^2}$

$S_n = \sum_{k=1}^{n} \text{arctg} \frac{1}{2k^2} = \sum_{k=1}^{n} (\text{arctg}(2k+1) - \text{arctg}(2k-1)) =$

$= \text{arctg}\, 3 - \text{arctg}\, 1 + \text{arctg}\, 5 - \text{arctg}\, 3 + \ldots + \text{arctg}(2n+1) - \text{arctg}(2n-1) =$

$= -\operatorname{arctg} 1 + \operatorname{arctg}(2n+1)$

$\lim\limits_{n\to\infty} S_n = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$ //

2. к) $a_n = \frac{1}{\sqrt{(2n-1)(2n+3)}}$

$\lim\limits_{n\to\infty} a_n = 0$

$\frac{1}{\sqrt{(2n-1)(2n+3)}} = \frac{1}{\sqrt{4n^2+4n-3}} \cdot a_n = O\left(\frac{1}{n}\right)$ – розбіжний //

л) $a_n = \operatorname{tg}\frac{\pi}{4n}$.

$\lim\limits_{n\to\infty} a_n = 0$

$a_n \simeq \frac{\pi}{4n} + \frac{1}{3}\left(\frac{\pi}{4n}\right)^3 + O\left(\frac{1}{n^3}\right) \sim O\left(\frac{1}{n}\right)$ – розбіжний //

м) $a_n = \frac{1}{\ln(n+e)}$

$\lim\limits_{n\to\infty} a_n = 0$

$a_n = \frac{1}{\ln(n+e)} = \frac{1}{\ln(n(1+\frac{e}{n}))} = \frac{1}{\ln n + \ln(1+\frac{e}{n})} = \log_{n+e} e$ //

3. б) $a_n = \frac{\cos x^n}{n^2}$

$|S_{n+p} - S_n| = \left|\frac{\cos x^{n+1}}{(n+1)^2} + \frac{\cos x^{n+2}}{(n+2)^2} + \ldots + \frac{\cos x^{n+p}}{(n+p)^2}\right| \leq$

$\leq \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+p)^2} < \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(n+p-1)(n+p)} =$

$= \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}$

$\Rightarrow \forall p \in N, \forall n > \frac{1}{\varepsilon}$ – ряд збіжний //

4. д) $a_n = \frac{3^n \cdot n!}{n^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}(n+1)!\, n^n}{3^n n!\, (n+1)^{n+1}} = \lim_{n\to\infty} \frac{3 \cdot n^n}{(n+1)^n} =$$

$$= \lim_{n\to\infty} 3\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{3}{e} > 1 - \text{розбіжний} //$$

е) $a_n = \frac{(n!)^2}{2^{n^2}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^2 \cdot 2^{n^2}}{(n!)^2 \cdot 2^{(n+1)^2}} = \lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^{n^2}}{2^{n^2+2n+1}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{2^{2n+1}} \to \infty - \text{розбіжний} //$$

к) $a_n = n^2\left(2+\frac{1}{n}\right)^{-n-2}$

$$\lim_{n\to\infty} \sqrt[n]{n^2\left(2+\frac{1}{n}\right)^{-(n+2)}} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} \cdot \sqrt[n]{\frac{n^2}{(2+\frac{1}{n})^2}} =$$

$$= \frac{1}{2} < 1 - \text{збіжний} //$$

5. а) $a_n = \frac{n!\, e^n}{n^{n+p}}$

$$\frac{a_n}{a_{n+1}} = \frac{n! \cdot e^n (n+1)^{n+p+1}}{n^{n+p}(n+1)!\, e^{n+1}} = \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+p} = \frac{1}{e} \cdot e^{(n+p)\ln(1+\frac{1}{n})} =$$

$$= e^{-1+(n+p)\left(\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right)} = e^{\frac{p-0,5}{n} + o\left(\frac{1}{n}\right)} = 1 + \frac{p-0,5}{n} + o\left(\frac{1}{n}\right),\ n\to\infty$$

при $p > \frac{3}{2}$ ряд є збіжним за ознакою Раабе //

6. в) $\ln^p\left(\sec\frac{\pi}{n}\right) = \left(\ln\frac{1}{\cos\frac{\pi}{n}}\right)^p = \left(-\ln\cos\frac{\pi}{n}\right)^p =$

$$= \left(-\ln\left(1 - \frac{\pi^2}{2n^2} + o\left(\frac{1}{n^2}\right)\right)\right)^p = \left(\frac{\pi^2}{2n^2} + o\left(\frac{1}{n^2}\right)\right)^p = O\left(\frac{1}{n^{2p}}\right)$$

$2p > 1,\ p > \frac{1}{2}$ //

12. a) $a_n = \frac{\ln^{100} n}{n} \sin \frac{\pi n}{4}$

$$\left| \sum_{k=1}^{n} \sin \frac{k\pi}{4} \right| = \left(\sin \frac{\pi}{8}\right)^{-1} \left| \sin \frac{\pi n}{8} \sin \frac{n+1}{8} \pi \right| \le \frac{1}{\sin \frac{\pi}{8}}$$

$$\frac{\ln^{100} n}{n} \to 0, \quad n \to \infty$$

за озн. Діріхле ряд є збіжним //

г) $a_n = (-1)^n \sin \frac{1}{n}$

$$\sin\left(\frac{1}{n+1}\right) < \sin\left(\frac{1}{n}\right), \quad \sin\left(\frac{1}{n}\right) \sim \frac{1}{n} \to 0, \; n \to \infty$$

За озн. Лейбніца ряд є збіжним

13. a) $a_n = \frac{(-1)^n}{n^p}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ – збіжний за озн. Лейбніца при $p > 0$

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^p} \right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ – збіжний при $p > 1$

При $p \in (0; 1]$ ряд умовно збіжний, а при $p \in (1; +\infty)$ абсолютно збіжний. //