Quiz: question 1

What is the storage required to represent analysis solution to a matrix completion problem

$$\text{minimize} \sum_{(i,j)\in E} (x_{ij} - b_{ij})^2 \quad \text{s.t.} \|X\| \leq \alpha$$

with variable $X \in \mathbf{R}^{m\times n}$ if all solutions have rank $\leq r$?

- $O(mnr \log n)$
- $O(mnr)$
- $O(r(m+n)\log(m+n))$
- $O(r(m+n))$
- $O(r\log(m+n))$

Quiz: question 2

What is the main advantage of the Conditional Gradient method for designing an optimal storage algorithm?

- the rank of any iterate $\leq$ the rank of the solution
- the rank of the iterate grows by at most 1 at each iteration
- the iteration can be expressed in terms of a low dimensional "dual" variable
- the method converges linearly for underconstrained problems like matrix completion and phase retrieval

Sketchy Decisions: Convex Low-Rank Matrix Optimization with Optimal Storage

Madeleine Udell

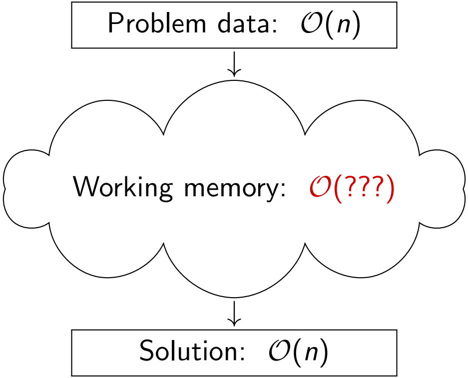
Operations Research and Information Engineering
Cornell University

Based on joint work with
Alp Yurtsever (MIT), Volkan Cevher (EPFL),
and Joel Tropp (Caltech)

ORIE 7191, February 2019

Goal

Can we develop algorithms that provably solve a problem using **storage** bounded by the size of the **problem data** and the size of the **solution**?



Model problem: low rank matrix optimization

consider a convex problem with decision variable $X \in \mathbb{R}^{m \times n}$

compact matrix optimization problem:

$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X) \\ \text{subject to} & \|X\|_{S_1} \leq \alpha \end{array} \qquad \text{(CMOP)}$$

- $\mathcal{A} : \mathbb{R}^{m \times n} \to \mathbb{R}^d$
- $f : \mathbb{R}^d \to \mathbb{R}$ convex and smooth
- $\|X\|_{S_1}$ is Schatten-1 norm: sum of singular values

assume

- **compact specification**: problem data use $\mathcal{O}(n)$ storage
- **compact solution**: $\operatorname{rank} X_\star = r$ constant

Note: Same ideas work for $X \succeq 0$

Are desiderata achievable?

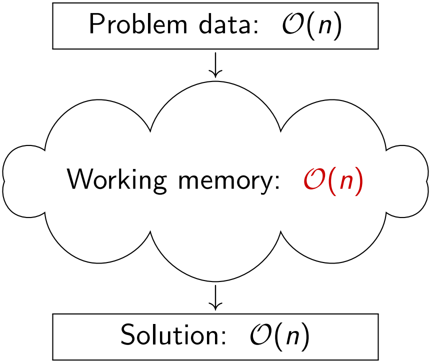
$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X) \\ \text{subject to} & \|X\|_{S_1} \leq \alpha \end{array}$$

CMOP, using any first order method:

$$\boxed{\text{Problem data: } \mathcal{O}(n)}$$
$$\downarrow$$
$$\boxed{\text{Working memory: } \mathcal{O}(n^2)}$$
$$\downarrow$$
$$\boxed{\text{Solution: } \mathcal{O}(n)}$$

Are desiderata achievable?

CMOP, using **SketchyCGM**:



Application: matrix completion

find X matching M on observed entries

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in\Omega}(X_{ij} - M_{ij})^2 \\ \text{subject to} & \|X\|_{S_1} \le \alpha \end{array}$$

- $m =$ rows, $n =$ columns of matrix to complete
- $d = |\Omega|$ number of observations
- $\mathcal{A}$ selects observed entries X_{ij}, $(i, j) \in \Omega$
- $f(z) = \|z - \mathcal{A}M\|^2$

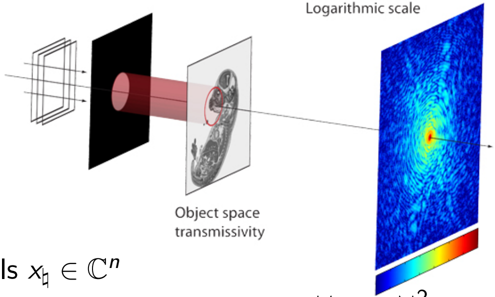
Matrix completion is a CMOP

find X matching M on observed entries

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in\Omega}(X_{ij} - M_{ij})^2 \\ \text{subject to} & \|X\|_{S_1} \leq \alpha \end{array}$$

- compact specification if $d = \mathcal{O}(m+n)$ observations
  e.g., constant $\#$ observations / person
- compact solution if $\text{rank}(X)$ constant
  i.e., constant $\#$ parameters / person
- in practice, usually find rank $\ll 200$ even with m and n in the millions...

Application: Phase retrieval



[1]

- image with n pixels $x_\natural \in \mathbb{C}^n$
- acquire noisy nonlinear measurements $b_i = |\langle a_i, x_\natural \rangle|^2 + \omega_i$
- relax: if $X = x_\natural x_\natural^*$, then

$$|\langle a_i, x_\natural \rangle|^2 = x_\natural a_i^* a_i x_\natural^* = \operatorname{tr}(a_i^* a_i x_\natural^* x_\natural) = \operatorname{tr}(a_i^* a_i X)$$

- recover image by solving

$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X; b) \\ \text{subject to} & \operatorname{tr} X = \alpha \\ & X \succeq 0. \end{array}$$

[1]image courtesy of Manuel Guizar-Sicairos

Phase retrieval is a CMOP

find X matching observations

$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X; b) \\ \text{subject to} & \operatorname{tr} X = \alpha \\ & X \succeq 0. \end{array}$$

- compact specification if $d = \mathcal{O}(n)$ observations
  e.g., constant $\#$ observations / pixels
- compact solution if $\operatorname{rank}(X)$ constant
  e.g., if correctly recover the rank-1 solution!

Why compact?

why a compact specification?

- data is expensive
- collect constant data per column (=user or sample)
- if solution is compact, compact specification should suffice

why a compact solution?

- the world is simple and structured
- given d observations, there is a solution with rank $\mathcal{O}(\sqrt{d})$
  (Barvinok 1995, Pataki 1998)
- nice latent variable models are of log rank
  (Udell & Townsend 2019)

Optimal Storage

What kind of storage bounds can we hope for?

- Assume black-box implementation of

$$\mathcal{A}(uv^*) \qquad u^*(\mathcal{A}^*z) \qquad (\mathcal{A}^*z)v$$

  where $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, and $z \in \mathbb{R}^d$
- Need $\Omega(m+n+d)$ storage to apply linear map
- Need $\Theta(r(m+n))$ storage for a rank-r approximate solution

Definition. An algorithm for the model problem has **optimal storage** if its working storage is

$$\Theta(d + r(m+n)).$$

Optimal Storage

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Definition. An algorithm for the model problem has **optimal storage** if its working storage is

$$\Theta(d + r(m + n)).$$

If we write down X, we've already failed.

A brief biased history of matrix optimization (I)

- 1990s: **Interior-point methods**
  - Storage cost $\Theta((m+n)^4)$ for Hessian
- 2000s: **Convex first-order methods (FOM)**
  - (Accelerated) proximal gradient and others
  - Store matrix variable $\Theta(mn)$

(**Interior-point:** Nemirovski & Nesterov 1994; . . . ; **First-order:** Rockafellar 1976; Auslender & Teboulle 2006; . . .)

A brief biased history of matrix optimization (I)

- 2008–Present: **Storage-efficient convex FOM**
  - Conditional gradient method (CGM) and extensions
  - Store matrix in low-rank form $\mathcal{O}(t(m+n))$ after t iterations
  - Requires storage $\Theta(mn)$ for $t \geq \min(m, n)$
  - Variants: prune factorization, or seek rank-reducing steps

- 2003–Present: **Nonconvex methods**
  - Burer–Monteiro factorization idea + various opt algorithms
  - Store low-rank matrix factors $\Theta(r(m+n))$
  - For guaranteed solution, need statistical assumptions or $\mathcal{O}(n^{3/2})$ storage

(**CGM:** Frank & Wolfe 1956; Levitin & Poljak 1967; Hazan 2008; Clarkson 2010; Jaggi 2013; . . . ; **CGM+pruning:** Rao Shah Wright 2015; Freund Grigas Mazumder 2017; . . . ; **Nonconvex:** Burer & Monteiro 2003; Keshavan et al. 2009; Jain et al. 2012; Bhojanapalli et al. 2015; Candès et al. 2014; Boumal et al. 2015; Bhojanapalli et al. 2018; Waldspurger & Waters 2018; . . .)

The dilemma

- convex methods: slow memory hogs with guarantees
- nonconvex methods: fast, lightweight, but brittle

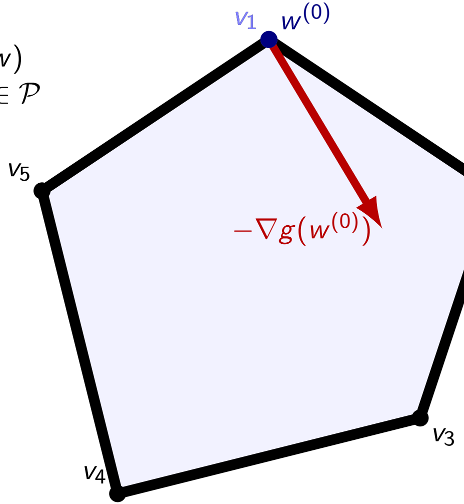
The dilemma

- convex methods: slow memory hogs with guarantees
- nonconvex methods: fast, lightweight, but brittle

goal: low memory and guaranteed convergence

Conditional gradient method (Frank-Wolfe)

$$\begin{array}{ll} \text{minimize} & g(w) \\ \text{subject to} & w \in \mathcal{P} \end{array}$$



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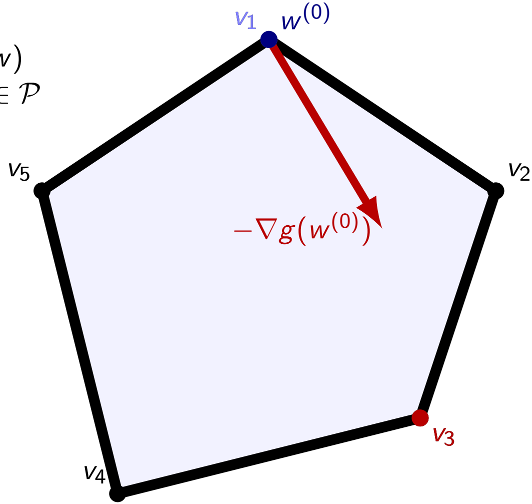

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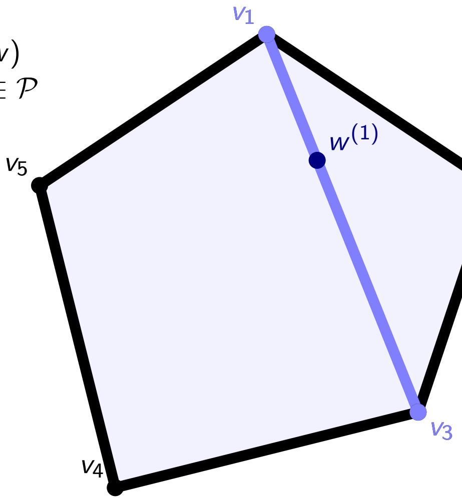

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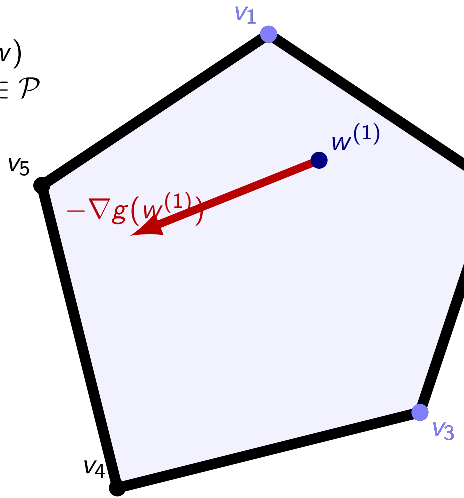

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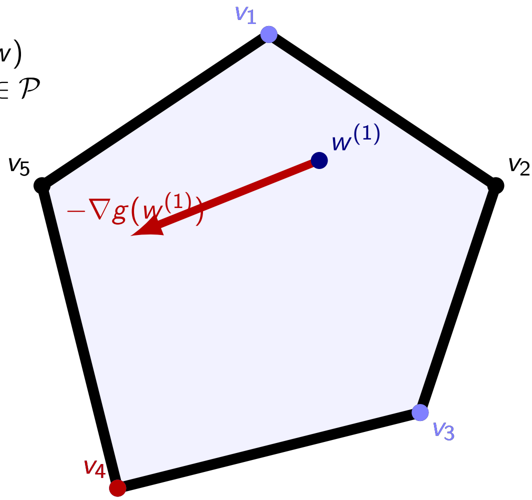

Conditional gradient method (Frank-Wolfe)

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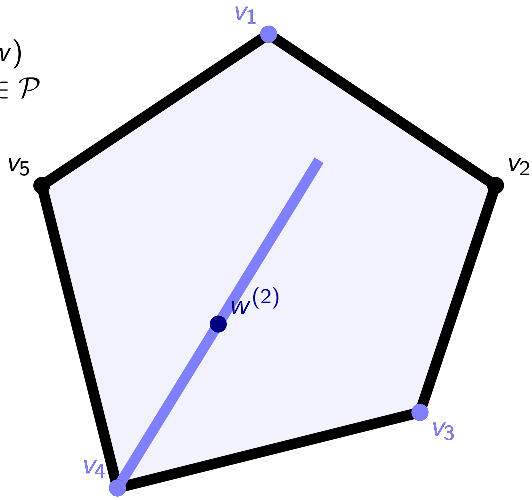

Conditional Gradient Method

$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X) \\ \text{subject to} & \|X\|_{S_1} \leq \alpha \end{array}$$

CGM. set $X^0 = 0$. for $t = 0, 1, \ldots$

- compute $G^t = \mathcal{A}^* \nabla f(\mathcal{A}X^t)$
- set search direction

$$H^t = \underset{\|X\|_{S_1} \leq \alpha}{\operatorname{argmax}} \langle X, -G^t \rangle$$

- set stepsize $\eta^t = 2/(t+2)$
- update $X^{t+1} = (1 - \eta^t) X^t + \eta^t H^t$

Conditional gradient method (CGM)

features:

- relies on efficient **linear optimization oracle** to compute

$$H^t = \operatorname*{argmax}_{\|X\|_{S_1} \le \alpha} \langle X, -G^t \rangle$$

- bound on suboptimality follows from subgradient inequality

$$\begin{aligned} f(\mathcal{A}X^t) - f(\mathcal{A}X^\star) &\le \langle X^t - X^\star, G^t \rangle \\ &\le \langle X^t - X^\star, \mathcal{A}^* \nabla f(\mathcal{A}X^t) \rangle \\ &\le \langle \mathcal{A}X^t - \mathcal{A}X^\star, \nabla f(\mathcal{A}X^t) \rangle \\ &\le \langle \mathcal{A}X^t - \mathcal{A}H^t, \nabla f(\mathcal{A}X^t) \rangle \end{aligned}$$

  to provide stopping condition
- faster variants: linesearch, away steps, ...

Linear optimization oracle for MOP

compute search direction

$$\underset{\|X\|_{S_1} \leq \alpha}{\operatorname{argmax}} \langle X, -G \rangle$$

- solution given by maximum singular vector of $-G$:

$$-G = \sum_{i=1}^{n} \sigma_i u_i v_i^* \quad \Longrightarrow \quad X = \alpha u_1 v_1^*$$

- use Lanczos method: only need to apply G and G^*

Conditional gradient descent

Algorithm 1 CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality ε
Output: Solution $X_\star$

```
function CGM
    X ← 0
    for t ← 0, 1, ... do
        (u, v) ← MaxSingVec(−A*(∇f(AX)))
        H ← −α uv*
        if ⟨AX − AH, ∇f(AX)⟩ ≤ ε then break for
        η ← 2/(t + 2)
        X ← (1 − η)X + ηH
    return X
```

$$\begin{aligned}
&\textbf{function } \mathrm{CGM}\\
&\quad X \leftarrow 0\\
&\quad \textbf{for } t \leftarrow 0, 1, \ldots \textbf{ do}\\
&\quad\quad (u, v) \leftarrow \texttt{MaxSingVec}(-\mathcal{A}^*(\nabla f(\mathcal{A}X)))\\
&\quad\quad H \leftarrow -\alpha\, uv^*\\
&\quad\quad \textbf{if } \langle \mathcal{A}X - \mathcal{A}H, \nabla f(\mathcal{A}X)\rangle \le \varepsilon \textbf{ then break for}\\
&\quad\quad \eta \leftarrow 2/(t+2)\\
&\quad\quad X \leftarrow (1-\eta)X + \eta H\\
&\quad \textbf{return } X
\end{aligned}$$

Two crucial ideas

To solve the problem using optimal storage:

- Use the low-dimensional "dual" variable

$$z_t = \mathcal{A}X_t \in \mathbb{R}^d$$

  to drive the iteration.
- Recover solution from small (randomized) sketch.

Never write down X until it has converged to low rank.

Conditional gradient descent

Algorithm 2 CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality ε
Output: Solution $X_\star$

function CGM
    $X \leftarrow 0$
    for $t \leftarrow 0, 1, \ldots$ **do**
        $(u, v) \leftarrow \texttt{MaxSingVec}(-\mathcal{A}^*(\nabla f(\mathcal{A}X)))$
        $H \leftarrow -\alpha\, uv^*$
        if $\langle \mathcal{A}X - \mathcal{A}H, \nabla f(\mathcal{A}X)\rangle \le \varepsilon$ **then break for**
        $\eta \leftarrow 2/(t+2)$
        $X \leftarrow (1-\eta)X + \eta H$
    return X

Conditional gradient descent

Introduce "dual variable" $z = \mathcal{A}X \in \mathbb{R}^d$; eliminate X.

Algorithm 3 Dual CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality ε
Output: Solution $X_\star$

function $\textsc{dualCGM}$
  $z \leftarrow 0$
  for $t \leftarrow 0, 1, \ldots$ **do**
    $(u, v) \leftarrow \texttt{MaxSingVec}(-\mathcal{A}^*(\nabla f(z)))$
    $h \leftarrow \mathcal{A}(-\alpha u v^*)$
    if $\langle z - h,\, \nabla f(z) \rangle \leq \varepsilon$ **then break for**
    $\eta \leftarrow 2/(t+2)$
    $z \leftarrow (1-\eta) z + \eta h$

Conditional gradient descent

Introduce "dual variable" $z = \mathcal{A}X \in \mathbb{R}^d$; eliminate X.

Algorithm 4 Dual CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality ε
Output: Solution $X_\star$

function DUALCGM
  $z \leftarrow 0$
  for $t \leftarrow 0, 1, \ldots$ **do**
    $(u, v) \leftarrow \texttt{MaxSingVec}(-\mathcal{A}^*(\nabla f(z)))$
    $h \leftarrow \mathcal{A}(-\alpha u v^*)$
    if $\langle z - h, \nabla f(z) \rangle \le \varepsilon$ **then break for**
    $\eta \leftarrow 2/(t+2)$
    $z \leftarrow (1-\eta)z + \eta h$

we've solved the problem... but where's the solution?

Two crucial ideas

1. Use the low-dimensional "dual" variable

$$z_t = \mathcal{A}X_t \in \mathbb{R}^d$$

   to drive the iteration.
2. Recover solution from small (randomized) sketch.

How to catch a low rank matrix

if $\hat{X}$ has the same rank as $X^\star$,
and $\hat{X}$ acts like $X^\star$ (on its range and co-range),
then $\hat{X}$ is $X^\star$

use single-pass randomized sketch (Tropp Yurtsever U Cevher 2017)

- see a series of additive updates
- remember how the matrix acts on random subspace
- reconstruct a low rank matrix that acts like $X^\star$
- storage cost for sketch and arithmetic cost of update are $\mathcal{O}(r(m+n))$; reconstruction is $\mathcal{O}(r^2(m+n))$

Single-pass randomized sketch

- Draw and fix two independent standard normal matrices

$$\Omega \in \mathbb{R}^{n\times k} \quad \text{and} \quad \Psi \in \mathbb{R}^{\ell\times m}$$

with $k = 2r + 1$, $\ell = 4r + 2$.

Single-pass randomized sketch

- Draw and fix two independent standard normal matrices

$$\Omega \in \mathbb{R}^{n\times k} \quad \text{and} \quad \Psi \in \mathbb{R}^{\ell\times m}$$

  with $k = 2r + 1$, $\ell = 4r + 2$.
- The sketch consists of two matrices that capture the range and co-range of X:

$$Y = X\Omega \in \mathbb{R}^{n\times k} \quad \text{and} \quad W = \Psi X \in \mathbb{R}^{\ell\times m}$$

Single-pass randomized sketch

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- Rank-1 updates to X can be performed on sketch:

$$X' = \beta_1 X + \beta_2 uv^*$$
$$\Downarrow$$
$$Y' = \beta_1 Y + \beta_2 uv^* \Omega \quad \text{and} \quad W' = \beta_1 W + \beta_2 \Psi uv^*$$

Single-pass randomized sketch

- Draw and fix two independent standard normal matrices
$$\Omega \in \mathbb{R}^{n\times k} \quad \text{and} \quad \Psi \in \mathbb{R}^{\ell\times m}$$
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$$X' = \beta_1 X + \beta_2 uv^*$$
$$\Downarrow$$
$$Y' = \beta_1 Y + \beta_2 uv^*\Omega \quad \text{and} \quad W' = \beta_1 W + \beta_2 \Psi uv^*$$
- Both the storage cost for the sketch and the arithmetic cost of an update are $\mathcal{O}(r(m + n))$.

Recovery from sketch

To recover rank-r approximation $\hat{X}$ from the sketch, compute

1. $Y = QR$ (tall-skinny QR)
2. $B = (\Psi Q)^\dagger W$ (small QR + backsub)
3. $\hat{X} = Q[B]_r$ (tall-skinny SVD)

Recovery from sketch

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1. $Y = QR$ (tall-skinny QR)
2. $B = (\Psi Q)^{\dagger} W$ (small QR + backsub)
3. $\hat{X} = Q[B]_r$ (tall-skinny SVD)

Theorem (Reconstruction (Tropp Yurtsever U Cevher, 2016))

Fix a target rank r. Let X be a matrix, and let (Y, W) be a sketch of X. The reconstruction procedure above yields a rank-r matrix $\hat{X}$ with

$$\mathbb{E}\,\|X - \hat{X}\|_{\mathrm{F}} \leq 2\,\|X - [X]_r\|_{\mathrm{F}}\,.$$

Similar bounds hold with high probability.

Previous work (Clarkson Woodruff 2009) algebraically but not numerically equivalent.

Recovery from sketch: intuition

let

$$Y = X\Omega \in \mathbb{R}^{n\times k} \quad \text{and} \quad W = \Psi X \in \mathbb{R}^{\ell\times m}$$

- if Q is an orthonormal basis for $\mathcal{R}(X)$, then

$$X = QQ^*X$$

- if $QR = X\Omega$, then Q is (approximately) a basis for $\mathcal{R}(X)$
- and if $W = \Psi X$, we can estimate

$$\begin{aligned} W &= \Psi X \\ &\approx \Psi QQ^*X \\ (\Psi Q)^\dagger W &\approx Q^*X \end{aligned}$$

- hence we may reconstruct X as

$$X \approx QQ^*X \approx Q(\Psi Q)^\dagger W$$

SketchyCGM

Algorithm 5 SketchyCGM for the model problem (CMOP)

Input: Problem data; suboptimality ε; target rank r
Output: Rank-r approximate solution $\hat{X} = U\Sigma V^*$

function $\textsc{SketchyCGM}$
    $\textsc{Sketch.Init}(m, n, r)$
    $z \leftarrow 0$
    for $t \leftarrow 0, 1, \dots$ **do**
        $(u, v) \leftarrow \texttt{MaxSingVec}(-\mathcal{A}^*(\nabla f(z)))$
        $h \leftarrow \mathcal{A}(-\alpha u v^*)$
        if $\langle z - h, \nabla f(z) \rangle \le \varepsilon$ **then break for**
        $\eta \leftarrow 2/(t+2)$
        $z \leftarrow (1-\eta)z + \eta h$
        $\textsc{Sketch.CGMUpdate}(-\alpha u, v, \eta)$
    $(U, \Sigma, V) \leftarrow \textsc{Sketch.Reconstruct}(\,)$
    return (U, Σ, V)

Guarantees

Suppose

- $X_{\mathrm{cgm}}^{(t)}$ is tth CGM iterate
- $\lfloor X_{\mathrm{cgm}}^{(t)} \rfloor_r$ is best rank r approximation to CGM solution
- $\hat{X}^{(t)}$ is SketchyCGM reconstruction after t iterations

Theorem (Convergence to CGM solution)

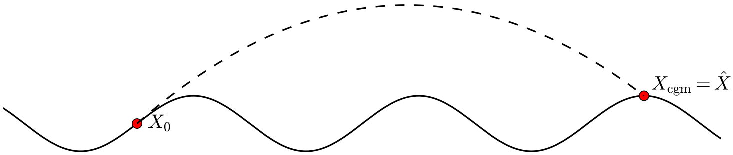
After t iterations, the SketchyCGM reconstruction satisfies

$$\mathbb{E}\,\|\hat{X}^{(t)} - X_{\mathrm{cgm}}^{(t)}\|_{\mathrm{F}} \leq 2\,\|\lfloor X_{\mathrm{cgm}}^{(t)} \rfloor_r - X_{\mathrm{cgm}}^{(t)}\|_{\mathrm{F}}\,.$$

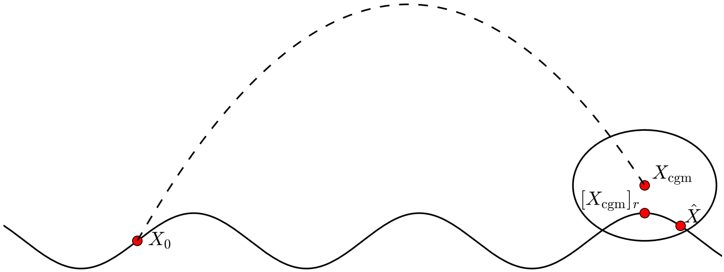
If in addition $X^\star = \lim_{t\to\infty} X_{\mathrm{cgm}}^{(t)}$ has rank r, then RHS $\to 0$!

(Tropp Yurtsever U Cevher, 2016)

Convergence when $\text{rank}(X_{\textbf{cgm}}) \leq r$



Convergence when $\mathrm{rank}(X_{\mathbf{cgm}}) > r$



Guarantees (II)

Theorem (Convergence rate)

Fix $\kappa > 0$ and $\nu \geq 1$. Suppose the (unique) solution $X_\star$ of (CMOP) *has* $\mathrm{rank}(X_\star) \leq r$ *and*

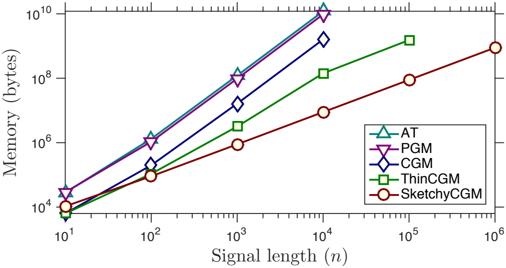
$$f(\mathcal{A}X) - f(\mathcal{A}X_\star) \geq \kappa \, \|X - X_\star\|_{\mathrm{F}}^{\nu} \quad \text{for all} \quad \|X\|_{S_1} \leq \alpha. \quad (1)$$

Then we have the error bound

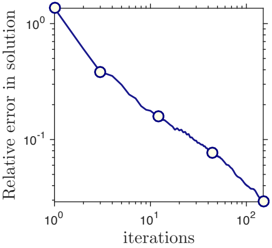
$$\mathbb{E}\,\|\hat{X}_t - X_\star\|_{\mathrm{F}} \leq 6 \left(\frac{2\kappa^{-1} C}{t+2} \right)^{1/\nu} \quad \text{for } t = 0, 1, 2, \ldots$$

where C is the curvature constant (Eqn. (3), Jaggi 2013) of the problem (CMOP).

SketchyCGM is scalable



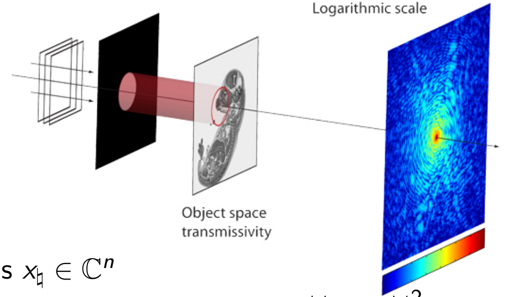
(A) Memory usage for five algorithms



(B) Convergence for $n = 8 \cdot 10^6$.

PGM	=	proximal gradient (via TFOCS (Becker Candès Grant, 2011))
AT	=	accelerated PGM (Auslander Teboulle, 2006) (via TFOCS),
CGM	=	conditional gradient method (Jaggi, 2013)
ThinCGM	=	CGM with thin SVD updates (Yurtsever Hsieh Cevher, 2015)
SketchyCGM	=	ours, using $r = 1$

Application: Phase retrieval



- image with n pixels $x_\natural \in \mathbb{C}^n$
- acquire noisy nonlinear measurements $b_i = |\langle a_i, x_\natural \rangle|^2 + \omega_i$
- relax: if $X = x_\natural x_\natural^*$, then

$$|\langle a_i, x_\natural \rangle|^2 = x_\natural a_i^* a_i x_\natural^* = \mathrm{tr}(a_i^* a_i x_\natural^* x_\natural) = \mathrm{tr}(a_i^* a_i X)$$

- recover image by solving

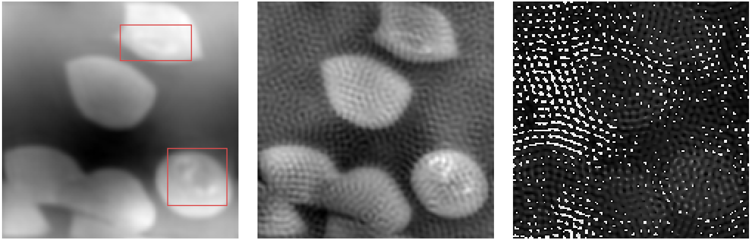
$$\begin{array}{ll} \text{minimize} & f(\mathcal{A}X; b) \\ \text{subject to} & \mathrm{tr}\, X \leq \alpha \\ & X \succeq 0. \end{array}$$

compact if $d = \mathcal{O}(n)$ observations and $\mathrm{rank}(X^\star)$ constant

SketchyCGM is reliable

Fourier ptychography:

- imaging blood cells with $\mathcal{A}$ = subsampled FFT
- $n = 25,600$, $d = 185,600$
- $\text{rank}(X_\star) \approx 5$ (empirically)



(A) SketchyCGM (B) Burer–Monteiro (C) Wirtinger Flow

- brightness indicates phase of pixel (thickness of sample)
- red boxes mark malaria parasites in blood cells

Conclusion

SketchyCGM offers a proof-of-concept **convex method** with **optimal storage** for low rank matrix optimization using two new ideas:

- Drive the algorithm using a smaller (dual) variable.
- Sketch and recover the decision variable.

References:

- J. A. Tropp, A. Yurtsever, M. Udell, and V. Cevher. Randomized single-view algorithms for low-rank matrix reconstruction. SIMAX 2017.
- A. Yurtsever, M. Udell, J. A. Tropp, and V. Cevher. Sketchy Decisions: Convex Optimization with Optimal Storage. AISTATS 2017.