Bounding the rank of SDP

Lijun Ding

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Overview

1. SDP and Why Low Rank Solutions
   - What is SDP?
   - Why Low Rank Solutions?
2. Bounding the rank via number of constraints (m)
   - Bound of an exact solution
   - Bound of an approximate solution
3. Questions

Semidefinite Programming (SDP)

A semidefinite program is an optimization problem of the form:

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\begin{array}{ll}
\underset{X \in \mathbb{S}^n}{\text{minimize}} & \textbf{trace}(A_0 X) \\
\text{subject to} & \textbf{trace}(A_i X) = b_i, \; i = 1, \dots, m, \\
& X \succeq 0,
\end{array} \tag{P}
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where $A_i \in \mathbb{S}^n \subset \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^m$ are problem data.
Here $X \succeq 0$ means X is positive semidefinite and symmetric, i.e., exists some $k \in \mathbb{N}$ and $F \in \mathbb{R}^{n \times k}$ such that

$$X = FF^\top.$$

Applications of SDP

SDP is applicable to a variety of problems: meaning

- analyzing problem,
- providing useful solutions computationally.

An incomplete list of application includes

- Combinatorical optimization: Max-Cut, TSP
- Optics: Phase Retrieval
- Recommendation System: Matrix Completion
- Power Systems: Power flow
- Machine Learning: Community Detection
- Statistics: Experiment Design

Also see [Vandenberghe and Boyd 96] for more applications.

A detailed example: EDMC Set-up

Euclidean Distance Matrix Completion (EDMC):

1. n many vectors $\bar{v}_i \in \mathbb{R}^r$
2. An index set $\Omega \subset \{1, \ldots, n\} \times \{1, \ldots, n\}$
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Q: Can we recover the matrix $D = [d_{ij}^2]$ from d_{ij}, $(i, j) \in \Omega$ with $|\Omega| \ll \frac{n(n+1)}{2}$, e.g., $|\Omega| = \mathcal{O}(n)$?

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- Important in sensor network
- Infer all pairwise distance from a few pairwise distance
- $r = 3$ usually

A detail example : Convexification to SDP

Consider the variable matrix:

$$V = [v_1, \dots, v_n] \in \mathbb{R}^{r \times n},$$

and

$$X = V^\top V \in \mathbb{S}^n, X \succeq 0, \quad \text{and} \quad \mathbf{rank}(\bar{X}) \leq r$$

In particular, when $V = \bar{V} = [\bar{v}_1, \dots, \bar{v}_n]$, $\bar{X} = \bar{V}^\top \bar{V}$.

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The condition based on d_{ij}^2 is

$$d_{ij}^2 = \|v_i - v_j\|_2^2 = \|v_i\|_2^2 + \|v_j\|_2^2 - 2v_i^\top v_j = x_{ii} + x_{jj} - 2x_{ij}.$$

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Convexification to SDP: expect $\bar{X}$ to be a solution (up to a shift in $\bar{V}$)

Find	X, V	minimize	$\mathbf{trace}(X)$
s.t.	$d_{ij}^2 = x_{ii} + x_{jj} - 2x_{ij}$ $(i,j) \in \Omega$ $X = V^\top V$	subject to	$d_{ij}^2 = x_{ii} + x_{jj} - 2x_{ij}$ $(i,j) \in \Omega$ $X \succeq 0$

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Why Low Rank? Interpretability

Low Rank solutions $X^\star$ to SDP usually are more interpretable.

1. Euclidean distance matrix completion: $\mathbf{rank}(X^\star) > \mathbf{rank}(\bar{X})$, hard to interpret. Expect $\mathbf{rank}(X^\star) = \mathbf{rank}(\bar{X})$.

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2. Combinatorical optimization: $X^\star = xx^\top$ and $x = \{\pm 1\}^n$, binary assignment.
3. Phase Retrieval: $X^\star = xx^\top$, $x \in \mathbb{R}^n$ where x stands for an approximation of the underlying phase/object.

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Space	$\mathcal{O}(n^2)$	$\mathcal{O}(nr^\star)$

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- Time:

	$X \in \mathbb{S}^n$	$X = FF^\top$
Matrix-vector product	$\mathcal{O}(n^2)$	$\mathcal{O}(nr^\star)$

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Theorem 1 [Barvinok 95, Pataki 98]

If there is a solution to (P), then there is a solution $X^\star$ with rank $r^\star$ satisfying

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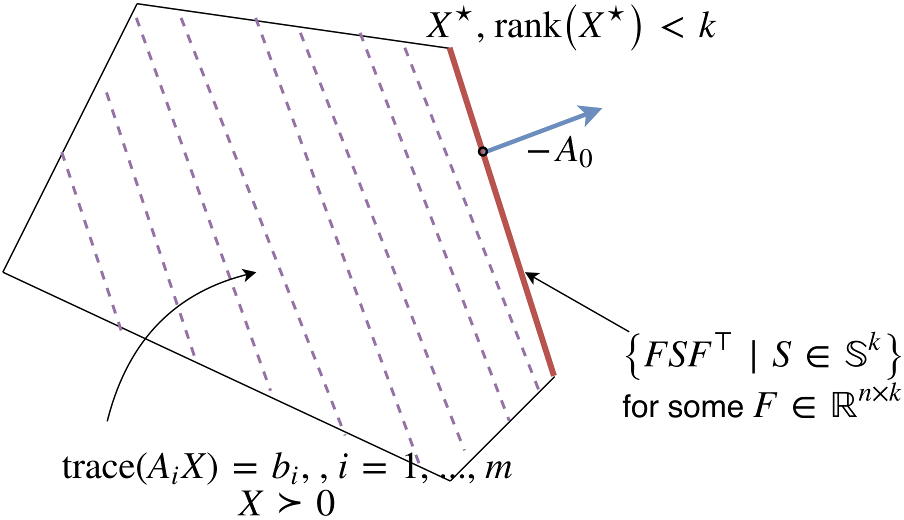

Exact Bound: Historical Remarks

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2. Proof strategy: relies on convex duality theory
3. Technique: Need results from algebraic geometry and semialgebraic set.

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[Pataki 98]

1. Motivation: eigenvalue optimization
2. Proof strategy: direct bound the rank of $X^\star$.
3. Technique: Linear algebra (only)
4. Generality: More general, deals with all solutions of (P)

Rank-Nullity Theorem

Given a linear map $\Psi : \mathbb{R}^l \to \mathbb{R}^h$, define the null space and image space as

$$\textbf{nullspace}(\Psi) := \{x \in \mathbb{R}^l \mid \Psi(x) = 0\}, \quad \textbf{im}(\Psi) := \{\Psi(x) \in \mathbb{R}^h \mid x \in \mathbb{R}^l\}.$$

Recall the rank-nullity theorem says that

$$\dim(\textbf{nullspace}(\Psi)) + \dim(\textbf{im}(\Psi)) = l.$$

In particular, if $h < l$, then $\textbf{nullspace}(\Psi) \neq \emptyset$.

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2. If $\frac{k(k+1)}{2} \leq m+1$, we are done. Otherwise, consider the linear map

$$\Psi_F : S \in \mathbb{S}^k \to \begin{bmatrix} \textbf{trace}(A_0 FSF^\top) \\ \textbf{trace}(A_1 FSF^\top) \\ \vdots \\ \textbf{trace}(A_m FSF^\top) \end{bmatrix} \in \mathbb{R}^{m+1}.$$

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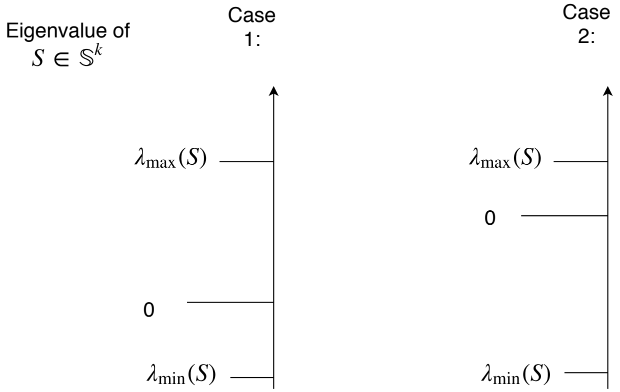
5. Set $-\frac{1}{\alpha}$ be a certain eigenvalue of S to reduce the rank and repeat step 1-4 until $\frac{k(k+1)}{2} \leq m+1$.

Exact bound: Illustration of Step 5

- Set $\frac{1}{\alpha}$ be a certain eigenvalue of S to reduce the rank



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Approximation bound: $\mathcal{O}(\frac{\ln(m)}{\epsilon^2})$

What if we look for an approximate solution?

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What if we look for an approximate solution?

Theorem 2 [Barvinok 2002]

Fix an approximation level $\epsilon \in (0,1)$. If A_is are positive semidefinite and (P) admits a solution $X^\star$ with optimal value b_0, then there is a matrix $X_0 \succeq 0$ such that for all $0 \leq i \leq m$

$$\mathbf{trace}(A_i X_0) \in b_i[1-\epsilon, 1+\epsilon]$$

and

$$\mathbf{rank}(X_0) \leq \frac{8}{\epsilon^2} \ln(4(m+1))$$

Fact: for any $A, B \succeq 0$, $\mathbf{trace}(AB) \geq 0$.

Approximation: Remarks

1. Dependence comparison:

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4. Proof technique in [Barvinok 2002]: Semidefinite quadratic form and Gaussian measures.

Approximation: Key Lemma

(Generalized) Johnson Lindernstrauss Lemma

Fix an approximation level $\epsilon \in (0,1)$, given m matrices $F_i \in \mathbb{R}^{n \times d_i}, i = 1, \ldots, m, d_i \leq n$, there is a matrix $U \in \mathbb{R}^{r \times n}$ with $r = \frac{8 \ln(4m)}{\epsilon^2}$ such that for all i,

$$\|UF_i\|_F^2 \in \|F_i\|_F^2 [1 - \epsilon, 1 + \epsilon].$$

- Take $d = 1$ reduces to normal J-L Lemma.
- Norma J-L will make the $r = \mathcal{O}(\frac{\ln(m + \sum d_i)}{\epsilon^2})$.

Approximation: Proof based on (generalized) J-L Lemma

1. First suppose the solution $X^\star = I$. Then

$$b_i = \textbf{trace}(A_i X^\star) = \textbf{trace}(A_i), \quad i = 0, 1, \ldots, m.$$

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4. Using generalized J-L to conclude there exists $U \in \mathbb{R}^{r \times n}$ with $r = \frac{8 \ln(4(m+1))}{\epsilon^2}$ such that
$$\|UF_i\|_F^2 \in \|F_i\|_F^2 [1 - \epsilon, 1 + \epsilon] = b_i [1 - \epsilon, 1 + \epsilon].$$

Approximation: Proof based on J-L Lemma conti.

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2. Now set $X_0 = U^\top U$ with rank no larger than $\frac{8 \ln(4(m+1))}{\epsilon^2}$, and for all i

$$\textbf{trace}(A_i X_0) = \textbf{trace}(F_i F_i^\top U^\top U) = \textbf{trace}((UF_i)^\top UF_i) = \|UF_i\|_F^2.$$

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4. Repeat previous step for A_i' and get X_0'. Set $X_0 = FX_0'F^\top$.

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- If a solution to (P) exists, there is a solution with rank $r^\star$ must satisfy

$$(a) r^\star > \frac{1}{2}m \quad (b) r^\star = \mathcal{O}(\sqrt{m}) \quad (c) r^\star > \frac{1}{2}n \quad (d) r^\star = \mathcal{O}(m^{\frac{1}{3}}).$$

- If a solution to (P) exists and all $A_i \succeq 0$, there is a matrix X_0 with ϵ-approximation level, i.e., $\mathbf{trace}(A_i X_0) \in b_i[1-\epsilon, 1+\epsilon]$, and has rank r no more than

$$(a) \mathcal{O}(\frac{\ln(m)}{\epsilon^2}) \quad (b) \mathcal{O}(\frac{\ln(\ln(m))}{\epsilon^2}) \quad (c) \mathcal{O}(\frac{\ln(m)}{\epsilon}) \quad (d) \mathcal{O}(\frac{\ln(n)}{\epsilon})$$

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