HW5 Solution to practice problem.

The `Dense Subgraph Problem` takes as input an undirected graph $G = (V, E)$ and two positive integers a and b, and returns a subset of *exactly* a vertices such that there are *at least* b edges connecting them, if such set exists, or return NO otherwise.

Show that the `Dense Subgraph Problem` is `NP-complete`.

Solution:

Given an input $G = (V, E)$ and a candidate solution S we check that `Dense Subgraph is in NP` counting the vertices in S and checking if there are exactly a in time $O(n)$, and checking all pairs of vertices $x, y \in S$ to see if $(xy) \in E$ in time $O(n^2)$. The number of such pairs is compared to b to verify that S is indeed a solution. Since this procedure runs in polynomial time our problem is in the class `NP`.

We now reduce `Clique-search` to `Dense Subgraph`. Given an input G and $g > 0$ of the first, we consider the input of `Dense Subgraph` given by $G, a = g$ and $b = \frac{g(g-1)}{2}$. This transformation is polynomial in the input size since copying the graph G is bounded by $O(n + m)$ and arithmetic operations are considered $O(1)$.

We show that there is a solution for Clique-search for the given input if and only if there is a solution for `Dense Subgraph` and the input we built. Note that there are at most $\frac{g(g-1)}{2}$ many edges connecting a set of g vertices. Thus, if we find a set of $a = g$ vertices connected by at least $b = \frac{g(g-1)}{2}$ many edges, we must have exactly $\frac{g(g-1)}{2}$ edges, which means this set is a clique. This shows that both solutions are exactly the same, since we are passing the same graph to both problems. Furthermore, recovering the solution of Clique-search is simply taking the same solution returned by `Dense Subgraph`, which can be done in time $O(n + m)$.

These prove that `Dense Subgraph is NP-complete`, as desired.