Strong spatial mixing for the matching polynomial: A simple proof via Heilmann–Lieb Theorem

Abstract

We study the connections between strong spatial mixing and zero-freeness, the two main notions used to devise deterministic approximation algorithms for counting problems. Following a recent framework introduced by [Reg23] for the vertex model, we show that the implication from zero-freeness of the partition function to strong spatial mixing also works for the matching polynomial, a typical counting problem in the edge model. Based on Heilmann and Lieb's results, a zero-free region $C_\Delta = \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$ of the partition function and a Christoffel–Darboux type identity, we give a very simple proof for strong spatial mixing of the matching polynomial on graphs of degree bounded by Δ with any complex edge parameter $z \in C_\Delta$. This proof does *not* rely on the tree recursion which is commonly used for proving strong spatial mixing in previous results.

1 Introduction

A *matching* of a graph G is a set of edges such that no two edges share a common vertex. We denote by $\mathcal{M}(G)$ the set of all matchings of G. In the paper, we study the matching (generating) polynomial defined as $M_G(z) = \sum_{M \in \mathcal{M}(G)} z^{|M|}$. The evaluation of $M_G(z)$ at $z = 1$ counts the number of matchings of G, which is a canonical problem in the #P class for counting problems and in particular, the framework of Holant problems [CLX08]. The matching polynomial is also referred to as the partition function of the monomer-dimer model, one of the most fundamental models in statistical physics. In the classical work of Heilmann and Lieb [HL72], the partition function $M_G(z)$ is defined as a complex function and by studying its complex zeros, a notion of (absence of) phase transition is proved in terms of smoothness properties of the free energy function $\log M_G(z)$. Furthermore, in quantum theory, it is shown that the quantum evolution of a system in thermodynamic equilibrium is equivalent to the partition function of the system with complex parameters [WCPL14]. Thus, it is meaningful to consider $M_G(z)$ with complex parameters z.

The exact computation of $M_G(z)$ is #P-hard for all $z \in \mathbb{C}$ except for the trivial point $z = 0$ [CGW13]. So the focus on $M_G(z)$ has been to find approximate solutions. The pioneering algorithm developed by Jerrum and Sinclair [JS89, JS96] gives the first fully polynomial randomized approximation scheme (FPRAS) for counting matchings. This algorithm is based on the Markov chain Monte Carlo (MCMC) approach which uses rapidly mixing Markov chains to obtain appropriate random samples for a sampling problem equivalent to the original counting problem. Since this is a probabilistic approach, it gives randomized algorithms and works only for non-negative real parameters z.

When it comes to deterministic approximation algorithms, there are two approaches, which are related to the two standard notions of phase transitions in statistical physics respectively. One notion is the location of zeros of the partition function mentioned above. The other notion is correlation decay which refers to that correlations between edges decay exponentially with the distance between them. In [vdB99], a form of correlation decay known as the complete analyticity condition was established using a probabilistic argument. The method associated with correlation decay, or more precisely *strong spatial mixing* (SSM) was originally developed by Weitz [Wei06] for the hard-core model. By establishing SSM on computation trees or equivalently self-avoiding walk trees using a recursion formula, [BGK+07] apply this approach to give a fully polynomial deterministic approximation scheme (FPTAS) for the matching polynomial with non-negative parameters on graphs of bounded degree. Later, the SSM result and FPTAS were extended to arbitrary complex parameters that are not negative and the requirement of bounded degree for graphs was relaxed to bounded connective constant in [BGGŠ21]. The approach turning complex zero-free regions of the partition function into FPTASes was developed by Barvinok [Bar16], and extended by Patel and Regts [PR17]. Based on Heilmann–Lieb theorem [HL72] which gives a zero-free region $\mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$ denoted by $\mathbb{C}_\Delta$ for the matching polynomial $M_G(z)$ on graphs of degree bounded by Δ, (i.e., $M_G(z) \neq 0$ for any G of degree bounded by Δ and $z \in \mathbb{C}_\Delta$), an FPTAS can be obtained for the matching polynomial on the zero-free region (via a reduction to the hard-core model using line graphs).

As a series of approximation algorithms have been devised using these three methods for various counting problems including the matching polynomial, it is becoming a very intriguing question that whether these different methods, in particular the two deterministic methods related to the notions of phase transitions show some inherent connections. This question attracts much research.

Recent results showed that one can extend real parameters in which correlation decay exists to their complex neighborhoods where the partition function is zero-free for the hard-core model [PR19], the Ising model [LSS19a,PR20,LSS19b], and the general 2-spin system [SS21] on graphs of bounded degree. On the other hand, Gamarnik [Gam23] showed that zero-freeness of the partition function directly implies a weak form of SSM for the hard-core model and other graph homomorphism models on graphs with certain properties. Later, Regts [Reg23] formally showed that zero-freeness in fact implies SSM for these models, and then the implication was extended to the general 2-spin system with complex parameters [SY24].

The models for which connections between correlation decay and zero-freeness have been established including the 2-spin system and the graph homomorphism model are all vertex models in which values are assigned to vertices and constraint functions are labelled on edges. Another framework for counting problems is the edge-model, also known as the Holant problem, in which values are assigned to edges and constraint functions are labelled on vertices. The matching polynomial is a problem in the edge model in which each edge takes a value 0 or 1 and each vertex of degree d is labelled by a d-ary function $f_d(x)$ where $f_d(x) = 1$ if its Hamming weight $\mathrm{wt}(x) \leq 1$ and $f_d(x) = 0$ otherwise. It is proved that the edge-model is more expressive than the vertex model. There are problems in the edge model, for example the problem of counting perfect matchings which are provably not expressible as a problem in the vertex model [FLS07, CG22]. In this paper, we show that the implication from zero-freeness to SSM also holds for the matching polynomial, an edge-model problem. Our proof is based on the framework introduced by [Reg23] and developed by [SY24], namely local dependence of coefficients (LDC) and a uniform bound implies SSM, and a Christoffel–Darboux type identity implies LDC. A Christoffel–Darboux type identity is proved for the matching polynomial also in the same paper [HL72]. Thus, based on Heilmann and Lieb's results on the zero-free region and a Christoffel–Darboux type identity, we give a very simple proof for SSM of the matching polynomial. In addition, by reducing the matching polynomial to the independence polynomial using line graphs, we prove an edge type SSM for the matching polynomial using a Christoffel–Darboux type identity for the independence polynomial. The proof is even slightly simpler than the proof of the standard vertex type SSM. The two types of SSM are indeed equivalent.

Below, we formally describe our results and we give some necessary notations. Let G be a graph. We denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G. For a set $S \subseteq V(G)$, we use $G \backslash S$ to denote the graph obtained from G by deleting all vertices in S and their incident edges. For a vertex $v \in V(G)$, let $G \backslash v = G \backslash \{v\}$. For a set $A \subseteq E(G)$, we use $G - A$ to denote the graph obtained from G by removing all edges in A. For an edge $e \in E(G)$, let $G - e = G - \{e\}$. We define the following vertex-type and edge-type ratio functions respectively, $P_{G,v}(z) = \frac{M_{G \backslash v}(z)}{M_G(z)}$, and $P_{G,e}(z) = \frac{M_{G-e}(z)}{M_G(z)}$. Note that both $P_{G,v}(z)$ and $P_{G,e}(z)$ are well-defined and analytic on the zero-free region of $M_G(z)$. In particular, when z is a positive number $P_{G,v}(z)$ refers to the marginal probability that v is unmatched and $P_{G,e}(z)$ refers to the marginal probability that e is a unmatched edge in the Gibbs distribution.

Let $d_G(u, v)$ denote the distance between two vertices u and v in a graph G. Also, Let $d_G(e, e')$ denote the distance between two edges e and e' in a graph G, i.e., the shortest distance between their endpoints. For a vertex $v \in V(G)$ and a vertex set $S \subseteq V(G)$, we define $d_G(v, S) = \min_{u \in S} d_G(v, u)$. Also, for an edge $e \in E(G)$ and an edge set $A \subseteq E(G)$, we define $d_G(e, A) = \min_{e' \in A} d_G(e, e')$. The symmetric difference of any two sets A and B is $(A \backslash B) \cup (B \backslash A)$, denoted by $A \neq B$.

Definition 1.1 (vertex-type SSM). *Let $\mathcal{G}$ be a family of graphs closed under taking subgraphs and $z \in \mathbb{C}$. The matching polynomials defined on $\mathcal{G}$ with the parameter z is said to satisfy vertex-type strong spatial mixing (SSM) if exist constants $C > 0$ and $r > 1$ such that for any graph $G = (V, E) \in \mathcal{G}$, any vertex $v \in V$, any two vertex sets $A, B \subseteq V \backslash v$, we have*

$$\left|P_{G\backslash A,v}(z) - P_{G\backslash B,v}(z)\right| \leq Cr^{-d_G(v,A\neq B)}.$$

This form of SSM is essentially the one proved in [BGK+07, BGGŠ21]. We roughly explain why it leads to an FPTAS. By Godsil's SAW tree construction, for a graph $G = (V, E)$ and a vertex $v \in V$, let T be the SAW tree rooted of G at v. Then $P_{G,v}(z) = P_{T,v}(z)$. If the above SSM property is satisfied, then one can truncate the SAW tree T at a depth of $O(\log n)$ where $n = |V|$ is the size of G, and compute $P_{T',v}(z)$ for the remaining tree T' in polynomial time given the graph G has a constant bounded degree. Here, the SSM property ensures that the error introduced by the truncation on the tree T is bounded by $O(1/n)$, which gives an FPTAS. Note that although our proof of SSM is based on zero-freeness, and does not rely on the tree recursion which is a commonly used argument for SSM as in [BGK+07, BGGŠ21], the tree recursion is still needed in order to get an FPTAS from SSM. The next definition of edge-type SSM is actually the original form of correlation decay proved in [vdB99]. It is indeed equivalent to the vertex-type SSM.

Definition 1.2 (edge-type SSM). *Let $\mathcal{G}$ be a family of graphs closed under taking subgraphs and $z \in \mathbb{C}$. The matching polynomials defined on $\mathcal{G}$ with the parameter z is said to satisfy edge-type strong spatial mixing (SSM) if exist constants $C > 0$ and $r > 1$ such that for any graph $G = (V, E) \in \mathcal{G}$, any edge $e \in E$, and any two edge sets $A, B \subseteq E \backslash e$, we have*

$$|P_{G-A,e}(z) - P_{G-B,e}(z)| \leq Cr^{-d_G(e,A\neq B)}.$$

Now, we give our main result.

Theorem 1.3. *For any $z \in \mathbb{C}_\Delta$, the matching polynomials on all graphs of degree bounded by Δ with the parameter z exhibit both vertex-type SSM and edge-type SSM.*

2 Preliminaries

2.1 Matching defect polynomial and independence polynomial

The matching defect polynomial is defined as $\mu_G(z) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} z^{n-2|M|}$. It is related to the matching generating polynomial $M_G(z)$ by the following identities

$$\mu_G(z) = z^n M_G(-z^{-2}), \quad M_G(z) = (-z)^{\frac{n}{2}} \mu_G((-z)^{-\frac{1}{2}}).$$

Heilmann-Lieb Theorem gives locations of zeros for both $M_G(z)$ and $\mu_G(z)$.

Theorem 2.1. *Let Δ be the maximum degree of G. The roots of $M_G(z)$ lie in the interval $(-\infty, -\frac{1}{4(\Delta-1)}]$ and the roots of $\mu_G(z)$ lie in the interval $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$.*

The matching polynomial $M_G(z)$ can also be expressed as the independence polynomial of some special graphs called line graphs. An independent set of graph G is a set of vertices such that no two vertices are adjacent by an edge. For a graph G, the independence polynomial of G is defined as $I_G(z) = \sum_{S\in\mathcal{I}} z^{|S|}$ where $\mathcal{I}$ is the set of all independent sets of G. For a graph G, the line graph $L(G)$ of G is a graph whose vertices are one-to-one mapped to edges of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a common vertex. Let $I_{L(G)}(z)$ be the independent polynomial of the line graph $L(G)$, then $M_G(z) = I_{L(G)}(z)$.

It is easy to check that $M_{G_1\cup G_2}(z) = M_{G_1}(z)M_{G_2}(z)$ for the disjoint union $G_1 \cup G_2$ of any two graphs G_1 and G_2, and $M_{\{v\}}(z) = 1$ for any isolated vertex v.

2.2 LDC and uniform bound implies SSM

The framework that LDC and uniform bound implies SSM was originally introduced in [Reg23]. Here, we adopt the definition of LDC from [SY24]. For two complex functions $f(z)$ and $g(z)$ analytic near 0, we denote by $z^k \mid f(z) - g(z)$ the property that their Taylor series $f(z) = \sum_{i=0}^{\infty} a_i z^i$ and $g(z) = \sum_{i=0}^{\infty} b_i z^i$ near $z = 0$ satisfy $a_i = b_i$ for $0 \le i \le k-1$.

Definition 2.2 (LDC)**.** *Let $\mathcal{G}$ be a family of graphs closed under taking subgraphs, and U be a complex region containing 0. The matching polynomials defined on $G \in \mathcal{G}$ and $z \in U$ is said to satisfy local dependence of coefficients (LDC) on vertices if for any graph $G \in \mathcal{G}$, any vertex $v \in V(G)$, and any two vertex sets $A, B \subseteq V(G)\backslash v$, the functions $P_{G\backslash A,v}(z)$ and $P_{G\backslash B,v}(z)$ are analytic on U, and $z^{d_G(v,A\neq B)} \mid P_{G\backslash A,v}(z) - P_{G\backslash B,v}(z)$.*

Similarly, one can define edge-type LDC for edge-type ratio functions $P_{G-A,e}(z)$ and $P_{G-B,e}(z)$.

Remark 2.3. *By Heilmann–Lieb Theorem, $M_G(z) \neq 0$ for G of degree bounded by Δ and $z \in \mathbb{C}_\Delta$. Then, for any graph G of degree bounded by Δ, and $v \in V(G)$ and $e \in E(G)$, the ratio functions $P_{G,v}(z)$ and $P_{G,e}(z)$ are always analytic on $z \in \mathbb{C}_\Delta$. Thus, in order to prove LDC, we only need to focus on the Taylor series of ratio functions near $z = 0$.*

Exactly following the proofs in [SY24], we have the following lemma.

Lemma 2.4. *Let $\mathcal{G}$ be a family of graphs closed under taking subgraphs, U and U' be complex regions containing 0 such that $\partial U' \subseteq U$. Suppose that the matching polynomials defined on $G \in \mathcal{G}$ satisfy vertex-type LDC for $z \in U$, and there exists a constant C such that for all $G \in \mathcal{G}$ and $v \in V(G)$, $|P_{G\backslash v}(z)| \le C$ holds on $z \in \partial U'$. Then, for any $z \in U'$, the matching polynomials defined on $\mathcal{G}$ with parameter z satisfy vertex-type SSM.*

Similarly, edge-type LDC and a uniform bound for $|P_{G,e}(z)|$ implies edge-type SSM.

In [Reg23], Regts introduce Montel's theorem to get a uniform bound for a family of analytic functions. In particular, the following lemma is proved.

Lemma 2.5. *Let U be a complex region and $\mathcal{F}$ be a family of holomorphic functions $f : U \to \mathbb{C}$ such that $f(U) \subset \mathbb{C}\backslash\{0, 1\}$ for all $f \in \mathcal{F}$. If there exists $z_0 \in U$ and $C > 0$ such that $|f(z_0)| \le C$ for all $f \in \mathcal{F}$. Then for any compact subset $S \subset U$, there exists a positive constant C_1 such that for all $f \in \mathcal{F}$ and $z \in S$, we have $|f(z)| \le C_1$.*

With these tools in hands, we prove vertex-type and edge-type SSM respectively in Section 3 and 4. In Section 3.1, we prove vertex-type LDC using a Christoffel–Darboux type identity for the matching polynomial, and show that the vertex ratio functions $P_{G,v}(z)$ avoid 0 and 1 using an interlace lemma which is a key part in the proof of Heilmann–Lieb theorem. In Section 4.1, we reduce the matching polynomial to the independence polynomial on corresponding line graphs and prove edge-type LDC using a Christoffel–Darboux type identity for the independence polynomial. In the case, the fact that the edge ratio functions $P_{G,e}(z)$ avoid 0 and 1 is a straightforward corollary of Heilmann–Lieb theorem.

3 Vertex-type SSM

3.1 Vertex-type LDC

The following is a Christoffel–Darboux type identity for the matching defect polynomial.

Lemma 3.1 ([HL72])**.** *Let G be a graph, $u, v \in V(G)$ $(u \neq v)$, and $\mathcal{P}_{u,v}$ be the set of paths from u to v in G. Then*

$$\mu_{G\setminus u}(z)\mu_{G\setminus v}(z) - \mu_G(z)\mu_{G\setminus\{u,v\}}(z) = \sum_{P\in\mathcal{P}_{u,v}} \mu_{G\setminus P}(z)^2.$$

Then, we have a Christoffel–Darboux type identity for the matching generating polynomial.

Lemma 3.2. *Let G be a graph, $u, v \in V(G)$ $(u \neq v)$, and $\mathcal{P}_{u,v}$ be the set of paths from u to v in G. Then*

$$M_{G\setminus u}(z)M_{G\setminus v}(z) - M_G(z)M_{G\setminus\{u,v\}}(z) = \sum_{P\in\mathcal{P}_{u,v}} (-z)^{|P|-1} M_{G\setminus P}(z)^2.$$

Proof. Recall that $\mu_G(z) = z^n M_G(-z^{-2})$, and $M_G(z) = (-z)^{\frac{n}{2}}\mu_G((-z)^{-\frac{1}{2}})$. Let $t = -z$. Then,

$$\begin{aligned}
&M_{G\setminus u}(z)M_{G\setminus v}(z) - M_G(z)M_{G\setminus\{u,v\}}(z)\\
=&t^{\frac{n-1}{2}}\mu_{G\setminus u}(t^{-\frac{1}{2}})t^{\frac{n-1}{2}}\mu_{G\setminus v}(t^{-\frac{1}{2}}) - t^{\frac{n}{2}}\mu_G(t^{-\frac{1}{2}})t^{\frac{n-2}{2}}\mu_{G\setminus\{u,v\}}(t^{-\frac{1}{2}})\\
=&t^{n-1}\left[\mu_{G-u}(t^{-\frac{1}{2}})\mu_{G-v}(t^{-\frac{1}{2}}) - \mu_G(t^{-\frac{1}{2}})\mu_{G-u-v}(t^{-\frac{1}{2}})\right]\\
=&\sum_{P\in\mathcal{P}_{u,v}} t^{n-1}\mu_{G\setminus P}(t^{-\frac{1}{2}})^2\\
=&\sum_{P\in\mathcal{P}_{u,v}} t^{|P|-1}\left[t^{\frac{n-|P|}{2}}\mu_{G\setminus P}(t^{-\frac{1}{2}})\right]^2\\
=&\sum_{P\in\mathcal{P}_{u,v}} (-z)^{|P|-1}M_{G\setminus P}(z)^2.
\end{aligned}$$

□

Now we can prove vertex-type LDC using the above Lemma.

Lemma 3.3. *Let G be a graph, $v \in V(G)$, and $A \subseteq V(G)\backslash\{v\}$, then the Taylor series of $P_{G,v}(z)$ and $P_{G\setminus A,v}(z)$ near 0 satisfy $z^{d_G(v,A)} \mid P_{G,v}(z) - P_{G\setminus A,v}(z)$.*

Proof. We prove this by induction on $|A|$. For the base case $|A| = 1$, for instance $A = \{u\}$, we have

$$P_{G,v}(z) - P_{G\setminus A,v}(z) = \frac{M_{G\setminus v}(z)}{M_G(z)} - \frac{M_{G\setminus\{u,v\}}(z)}{M_{G\setminus u}(z)} = \frac{M_{G\setminus u}(z)M_{G\setminus v}(z) - M_G(z)M_{G\setminus\{u,v\}}(z)}{M_G(z)M_{G\setminus u}(z)}.$$

Clearly $\frac{1}{M_G(z)M_{G-u}(z)}$ is an analytic function on $\mathbb{C}_\Delta$ containing 0. Combining Lemma 3.2, we have $z^{d_G(v,u)} \mid P_{G,v}(z) - P_{G\setminus A,v}(z)$.

Now consider the case that $k \geq 2$. Suppose that the statement is true for all $|A| < k$. Consider a vertex set A with $|A| = k$. Let $u \in A$, and $A' = A\backslash\{u\}$, then

$$P_G(z) - P_{G\setminus A,v}(z) = \left[P_G(z) - P_{G\setminus A',v}(z)\right] + \left[P_{G\setminus A',v}(z) - P_{G\setminus A,v}(z)\right].$$

By the induction hypothesis, we have $z^{d_G(v,A')} \mid P_G(z) - P_{G\backslash A',v}(z)$ and $z^{d_{G\backslash A'}(v,u)} \mid P_{G\backslash A',v}(z) - P_{G-A,v}(z)$. Since $d_G(v,A) \leq d_G(v,A')$ and $d_G(v,A) \leq d_G(v,u) \leq d_{G\backslash A'}(v,u)$, we have $z^{d_G(v,A)} \mid P_G(z) - P_{G\backslash A,v}(z)$. $\square$

Lemma 3.4. *Let G be a graph, $v \in V(G)$, and $A, B \subseteq V(G)\backslash\{v\}$, then the Taylor series of $P_{G\backslash A,v}(z)$ and $P_{G\backslash B,v}(z)$ near 0 satisfy $z^{d_G(v,A\neq B)} \mid P_{G\backslash A,v}(z) - P_{G\backslash B,v}(z)$.*

Proof. Let $G' = G\backslash(A \cap B)$, $A' = A\backslash B$, $B' = B\backslash A$, then

$$P_{G\backslash A,v}(z) - P_{G\backslash B,v}(z) = P_{G'\backslash A',v}(z) - P_{G'\backslash B',v}(z) = \left[P_{G'\backslash A',v}(z) - P_{G',v}(z)\right] + \left[P_{G',v}(z) - P_{G'\backslash B',v}(z)\right]$$

By the previous lemma, we have $z^{d_{G'}(v,A')} \mid P_{G',v}(z) - P_{G'\backslash A',v}(z)$ and $z^{d_{G'}(v,B')} \mid P_{G',v}(z) - P_{G'\backslash B',v}(z)$. Since $d_G(v, A \neq B) = \min\{d_G(v,A'), d_G(v,B')\} \leq \min\{d_{G'}(v,A'), d_{G'}(v,B')\}$, we are done. $\square$

3.2 Uniform bound of vertex type ratio

Definition 3.5 (interlace)**.** *Let $p(z) = C_1 \prod_{i=1}^{n}(z - \lambda_i)$ and $q(z) = C_2 \prod_{i=1}^{m}(z - \gamma_i)$ be two real-rooted polynomials (i.e., all $\lambda_i, \gamma_i \in \mathbb{R}$) where $C_1 C_2 \neq 0$ and $m = n-1$ or $m = n$. We say $q(z)$ interlaces $p(z)$ if $\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \gamma_2 \geq \cdots$.*

The next interlace lemma is a key step in the proof of Heilmann–Lieb Theorem [HL72].

Lemma 3.6. *Let $G = (V,E)$ be a graph, for all $v \in V$, then $\mu_{G\backslash v}(z)$ interlaces $\mu_G(z)$. Also, if $f(z)$ is a convex combination of $\{\mu_{G\backslash v}(z) | v \in V\}$, then $f(z)$ interlaces $\mu_G(z)$.*

In particular, for a non-isolated vertex $v \in V$, $\sum_{u\sim v} \mu_{G\backslash\{u,v\}}(z)$ interlaces $\mu_{G\backslash v}(z)$ where $u \sim v$ denotes that vertices u and v are adjacent.

We are ready to prove the vertex-type ratio functions avoid 0 and 1.

Lemma 3.7. *Let G be a graph with n vertices and bounded degree Δ and v be an vertex in G. If $z \in \mathbb{C}_\Delta\backslash\{0\}$ and v is not isolated in G , then $P_{G,v}(z)$ avoid 0 and 1.*

Proof. By Heilmann–Lieb Theorem, it is trivial that $P_{G,v}(z) \neq 0$. We prove $P_{G,v}(z) \neq 1$. Note

$$M_G(z) - M_{G\backslash v}(z) = z \sum_{u\sim v} M_{G\backslash\{u,v\}}(z).$$

Prove all roots of $\sum_{u\sim v} M_{G\backslash\{u,v\}}(z)$ is in the interval $(-\infty, -\frac{1}{4(\Delta-1)}]$ is enough to show $P_{G,v}(z)$ avoid 1. Since $\sum_{u\sim v} \mu_{G\backslash\{u,v\}}(z) = z^{n-2} \sum_{u\sim v} M_{G\backslash\{u,v\}}(-z^2)$, show all roots of $\sum_{u\sim v} \mu_{G\backslash\{u,v\}}(z)$ in the interval $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$ is enough.

Still by Heilmann–Lieb Theorem, all $n-1$ roots of $\mu_{G\backslash v}(z)$ lie in the interval $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$. Since v is not isolated in G, $\sum_{u\sim v} \mu_{G-\{u,v\}}(z)$ is not identically zero, by Lemma 3.6, $\sum_{u\sim v} \mu_{G\backslash\{u,v\}}(z)$ is real-rooted and interlaces $\mu_{G\backslash v}(z)$. Thus all $n-2$ roots of $\sum_{u\sim v} \mu_{G\backslash\{u,v\}}(z)$ lie in the interval $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$. Then $P_{G,v}(z)$ avoid 1. $\square$

Lemma 3.8. *Let $\Delta \geq 2$, and S is a compact subset of $\mathbb{C}_\Delta\backslash\{0\}$. There exists a constant $C > 0$ such that for any graph G with bounded degree Δ, any $v \in V(G)$, and any $z \in S$, we have $|P_{G,v}(z)| \leq C$.*

Proof. When v is isolated in G, $P_{G,v}(z) = 1$. Thus we only need to consider the case when v is not isolated in G. By Lemma 3.7, we know that $P_{G,v}(z)$ avoid 0 and 1 for all $z \in \mathbb{C}_\Delta \backslash \{0\}$. Since $0 < P_{G,v}(1) < 1$ for all $G, v \in V(G)$. Then by Lemma 2.5, the upper bound is got. $\square$

Remark 3.9. *By the same approach, we can also prove a lower bound for the ratio functions $P_{G,v}(z)$. Although such a lower bound is not used for the proof of SSM, it is necessary for the FPTAS. Such an FPTAS computes the matching polynomial by telescoping, i.e., $M_G(z) = \prod_{i=1,n} \frac{1}{P_{G_i,v_i}(z)}$ where $v_1, v_2, \cdots, v_n$ are all vertices in G and $G_i = G\backslash\{v_1, v_2, \cdots, v_{i-1}\}$. A lower bound of $P_{G,v}(z)$ together with an upper bound ensures that the telescoping computation eventually gives an $(1 \pm \epsilon)$ multiplicative approximation.*

Combining Lemmas 3.4, 3.8 and 2.4, we can establish vertex-type SSM.

Theorem 3.10. *For any $z \in \mathbb{C}_\Delta$, the matching polynomials on all graphs of degree bounded by Δ with the parameter z exhibit vertex-type SSM.*

4 Edge-type SSM

We prove edge-type SSM in a similar way to vertex-type SSM. Technically, the proof is even simpler.

4.1 Edge-type LDC

We prove edge-type LDC using the following result from the Christoffel–Darboux type identities for the independence polynomial [Ben18].

Lemma 4.1. *Let G be a graph, $u, v \in V(G)$ where $u \neq v$. Then*

$$z^{d_G(u,v)+1} | I_{G\backslash u}(z) I_{G\backslash v}(z) - I_G(z) I_{G\backslash\{u,v\}}(z).$$

Lemma 4.2. *Let G and e_1, e_2 be two different edges in G. Then*

$$z^{d_G(e_1,e_2)+2} \mid M_{G-e_1}(z) M_{G-e_2}(z) - M_G(z) M_{G-\{e_1,e_2\}}(z).$$

Proof. Let u, v be the two vertices in the line graph $L(G)$ of G corresponding to the edges e_1, e_2 in G respectively. Recall that $M_G(z) = I_{L(G)}(z)$. Thus, we have

$$M_{G-e_1}(z) M_{G-e_2}(z) - M_G(z) M_{G-\{e_1,e_2\}}(z) = I_{L(G)\backslash u}(z) I_{L(G)\backslash v}(z) - I_{L(G)}(z) I_{L(G)\backslash\{u,v\}}(z).$$

Note $d_{L(G)}(u, v) = 1 + d_G(e_1, e_2)$. By Lemma 4.1, we have $z^{d_G(e_1,e_2)+2} \mid M_{G-e_1}(z) M_{G-e_2}(z) - M_G(z) M_{G-\{e_1,e_2\}}(z)$. $\square$

Then, same as the proofs Lemmas 3.3 and 3.4, we get the next lemma.

Lemma 4.3. *Let G be a graph, $e \in E(G)$, and $A, B \subseteq E(G) - e$. Then, the Taylor series near 0 of $P_{G-A,e}(z)$ and $P_{G-B,e}(z)$ satisfy $z^{d_G(e, A \neq B)+2} \mid P_{G-A,e}(z) - P_{G-B,e}(z)$.*

4.2 Uniform bound of edge-type ratio

Lemma 4.4. *Let G be a graph with bounded degree Δ and e be an edge in G. If $z \in \mathbb{C}_\Delta \backslash \{0\}$, then $P_{G,e}(z)$ avoids 0 and 1.*

Proof. By Heilmann–Lieb Theorem, $P_{G,e}(z) \neq 0$ is trivial. We prove $P_{G,e}(z) \neq 1$. Assume $e = (u, v)$. A simple observation is $M_G(z) - M_{G-e}(z) = zM_{G\backslash\{u,v\}}(z)$. Again by Heilmann–Lieb Theorem, $M_{G\backslash\{u,v\}}(z) \neq 0$. Also since $z \neq 0$, we have $M_G(z) - M_{G-e}(z) \neq 0$, and hence $P_{G,v}(z) \neq 1$. $\square$

Same as the proof of Lemma 3.8, we can get a uniform bound.

Lemma 4.5. *Let $\Delta \geq 2$, and S is a compact subset of $\mathbb{C}_\Delta \backslash \{0\}$. There exists a constant $C > 0$ such that for any graph G with bounded degree Δ, any $e \in E(G)$, and any $z \in S$, $|P_{G,e}(z)| \leq C$.*

Thus, we have the following edge-type SSM.

Theorem 4.6. *For any $z \in \mathbb{C}_\Delta$, the matching polynomials on all graphs of degree bounded by Δ with the parameter z exhibit edge-type SSM.*

5 Equivalence of vertex-type and edge-type SSM

We first give the following alternative definitions of vertex-type and edge-type SSM respectively. For a graph G, a vertex $v \in V(G)$, and an integer $k \geq 0$, define $N_G(v, k)$ to be the subgraph of G induced by the vertices within distance k from v. Similarly, we can define $N_G(e, k)$ for an edge $e \in E(G)$ as the subgraph of G induced by the vertices within distance k from the endpoints of e. $G \cong H$ indicates the standard graph isomorphism.

Definition 5.1 (vertex-type SSM)**.** *Let $\mathcal{G}$ be a family of graphs closed under taking subgraphs and $z \in \mathbb{C}$. The matching polynomials defined on $\mathcal{G}$ with the parameter z is said to satisfy vertex-type SSM if exists constants $C > 0, r > 1$ such that for two graph $G_1, G_2 \in \mathcal{G}$, $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, if $N_{G_1}(v_1, k) \cong N_{G_2}(v_2, k)$ where v_1 and v_2 are the corresponding vertices, then*

$$|P_{G_1,v_1}(z) - P_{G_2,v_2}(z)| \leq Cr^{-k}.$$

Definition 5.2 (edge type SSM)**.** *Let $\mathcal{G}$ be a family of graphs closed under taking subgraphs and $z \in \mathbb{C}$. The matching polynomials defined on $\mathcal{G}$ with the parameter z is said to satisfy edge-type SSM if exists a positive constants $C > 0, r > 1$ such that for any two graph $G_1, G_2 \in \mathcal{G}$, $e_1 \in E(G_1)$, $e_2 \in E(G_2)$, if $N_{G_1}(e_1, k) \cong N_{G_2}(e_2, k)$ where e_1 and e_2 are the corresponding edges, then*

$$|P_{G_1,e_1}(z) - P_{G_2,e_2}(z)| \leq Cr^{-k}.$$

One can easily check that Definitions 1.1 and 1.2 are equivalent to Definitions 5.1 and 5.2 respectively. Below, we prove the equivalence of vertex-type and edge-type SSM using Definitions 5.1 and 5.2. We need the following technical result.

Lemma 5.3. *Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be two sequences of complex numbers, ε and C be two positive constants, where $|x_i| \leq C, |y_i| \leq C$ and $|x_i - y_i| \leq \varepsilon$ for all i. Then $|\prod_{i=1}^n x_i - \prod_{i=1}^n y_i| \leq nC^{n-1}\varepsilon$.*

Proof. We prove it by induction on n. For the base case $n = 1$, the statement is trivial. Suppose the statement is true for $n = k$, then for $n = k + 1$, we have

$$\begin{aligned} & \left| \prod_{i=1}^{k+1} x_i - \prod_{i=1}^{k+1} y_i \right| \\ = & \left| x_{k+1} \left(\prod_{i=1}^{k} x_i - \prod_{i=1}^{k} y_i \right) + \prod_{i=1}^{k} y_i (x_{k+1} - y_{k+1}) \right| \\ \le & C \cdot k C^{k-1} \varepsilon + C^k \varepsilon = (k+1) C^k \varepsilon. \end{aligned}$$

Thus we are done. □

Theorem 5.4. *The matching polynomials defined on graphs of bounded degree Δ exhibit vertex-type SSM if and only if they exhibit edge-type SSM,*

Proof. In matching generating polynomial, delete a vertex is equivalent to delete all edges incident to the vertex (noting the polynomial of a single vertex is 1). In the bounded degree graph, vertex ratio can be write the product of at most Δ edge type ratio. Assuming the edges attached to v is $e_1, e_2, \cdots, e_k$ and $G_i = G - \{e_1, e_2, \cdots, e_{i-1}\}$, then $P_{G,v}(z) = \prod_{i=1}^{k} P_{G_i, e_i}(v)$. Thus we can get the vertex ratio SSM from the edge ratio SSM.

Also, write $e = (u, v)$, since $P_{G,e}(z) = 1 - z \frac{M_{G \setminus \{u,v\}}(z)}{M_G(z)} = 1 - z P_{G,v}(z) P_{G \setminus v, u}(z)$, we can get the edge ratio SSM from the vertex ratio SSM. □

References

[Bar16] Alexander Barvinok. *Combinatorics and complexity of partition functions*, volume 30. Springer, 2016.

[Ben18] Ferenc Bencs. Christoffel–darboux type identities for the independence polynomial. *Combinatorics, Probability and Computing*, 27(5):716–724, 2018.

[BGGŠ21] Ivona Bezáková, Andreas Galanis, Leslie Ann Goldberg, and Daniel Štefankovič. The complexity of approximating the matching polynomial in the complex plane. *ACM Transactions on Computation Theory (TOCT)*, 13(2):1–37, 2021.

[BGK+07] Mohsen Bayati, David Gamarnik, Dimitriy Katz, Chandra Nair, and Prasad Tetali. Simple deterministic approximation algorithms for counting matchings. In *Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, pages 122–127, 2007.

[CG22] Jin-Yi Cai and Artem Govorov. Perfect matchings, rank of connection tensors and graph homomorphisms. *Combinatorics, Probability and Computing*, 31(2):268–303, 2022.

[CGW13] Jin-Yi Cai, Heng Guo, and Tyson Williams. A complete dichotomy rises from the capture of vanishing signatures. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*, pages 635–644, 2013.

[CLX08] Jin-Yi Cai, Pinyan Lu, and Mingji Xia. Holographic algorithms by fibonacci gates and holographic reductions for hardness. In *2008 49th Annual IEEE Symposium on Foundations of Computer Science*, pages 644–653. IEEE, 2008.

[FLS07] Michael Freedman, László Lovász, and Alexander Schrijver. Reflection positivity, rank connectivity, and homomorphism of graphs. *Journal of the American Mathematical Society*, 20(1):37–51, 2007.

[Gam23] David Gamarnik. Correlation decay and the absence of zeros property of partition functions. *Random Structures & Algorithms*, 62(1):155–180, 2023.

[HL72] Ole J Heilmann and Elliott H Lieb. Theory of monomer-dimer systems. *Communications in mathematical Physics*, 25(3):190–232, 1972.

[JS89] Mark Jerrum and Alistair Sinclair. Approximating the permanent. *SIAM journal on computing*, 18(6):1149–1178, 1989.

[JS96] Mark Jerrum and Alistair Sinclair. The markov chain monte carlo method: an approach to approximate counting and integration. *Approximation algorithms for NP-hard problems*, pages 482–520, 1996.

[LSS19a] Jingcheng Liu, Alistair Sinclair, and Piyush Srivastava. Fisher zeros and correlation decay in the ising model. *Journal of Mathematical Physics*, 60(10), 2019.

[LSS19b] Jingcheng Liu, Alistair Sinclair, and Piyush Srivastava. The ising partition function: Zeros and deterministic approximation. *Journal of Statistical Physics*, 174(2):287–315, 2019.

[PR17] Viresh Patel and Guus Regts. Deterministic polynomial-time approximation algorithms for partition functions and graph polynomials. *SIAM Journal on Computing*, 46(6):1893–1919, 2017.

[PR19] Han Peters and Guus Regts. On a conjecture of sokal concerning roots of the independence polynomial. *Michigan Mathematical Journal*, 68(1):33–55, 2019.

[PR20] Han Peters and Guus Regts. Location of zeros for the partition function of the ising model on bounded degree graphs. *Journal of the London Mathematical Society*, 101(2):765–785, 2020.

[Reg23] Guus Regts. Absence of zeros implies strong spatial mixing. *Probability Theory and Related Fields*, 186(1):621–641, 2023.

[SS21] Shuai Shao and Yuxin Sun. Contraction: A unified perspective of correlation decay and zero-freeness of 2-spin systems. *Journal of Statistical Physics*, 185:1–25, 2021.

[SY24] Shuai Shao and Xiaowei Ye. From zero-freeness to strong spatial mixing via a christoffel-darboux type identity. *arXiv preprint arXiv:2401.09317*, 2024.

[vdB99] Jacob van den Berg. On the absence of phase transition in the monomer-dimer model. *Perplexing Problems in Probability: Festschrift in Honor of Harry Kesten*, pages 185–195, 1999.

[WCPL14] Bo-Bo Wei, Shao-Wen Chen, Hoi-Chun Po, and Ren-Bao Liu. Phase transitions in the complex plane of physical parameters. *Scientific reports*, 4(1):5202, 2014.

[Wei06] Dror Weitz. Counting independent sets up to the tree threshold. In *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*, pages 140–149, 2006.