From Zero-freeness to Strong Spatial Mixing of Matching Polynomial

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Abstract

We study the problem of approximating the matching polynomial on graphs with maximum degree Δ and show that the matching polynomial exhibits strong spatial mixing for all complex edge parameter $z \in \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$. Additionally, we provide a FPTAS for approximating the matching polynomial using correlation decay.

1 Introduction

Matching polynomial is a fundamental object in mathematics, computer science and statistical physics. In statistical physics, the matching polynomial is called the partition function of the dimer model. A *matching* in a graph G is a set of edges such that no two edges share a common vertex. We denote the set of all matchings in G as $\mathcal{M}(G)$. In this paper, we focus on matching generating polynomial, which is defined as

$$M_G(z) = \sum_{M \in \mathcal{M}(G)} z^{|M|},$$

When edge parameter z is a positive real number, Jerrum and Sinclair [JS89] gave a FPRAS for approximating $M_G(z)$. Based on the correlation decay technique, Bayati et al. [BGK$^+$07] gave the a FPTAS with graph is bouned degree Δ.

When z is a complex number, using the method of Barvinok [Bar16], Patel and Regts [PR17] gave a FPTAS for graphs with bounded degree Δ when $z \notin (-\infty, -\frac{1}{4(\Delta-1)}]$, fitting the hardness result in [BGGŠ21]. Bezáková et al. [BGGŠ21] relaxed the bounded degree condition to the bounded connective constant, gave a FPTAS when $z \notin (-\infty, 0)$ using the correlation decay technique.

A well known zero-freeness result of matching polynomial is the Heilmann–Lieb Theorem [HL72], which is Barvinok's algorithm based on. The relationship of zero-freeness and correlation decay or strong spatial mixing (SSM) for some graph polynomials or the partition function of some models is studied in [DS85, DS87, PR19, LSS19, PR20, LSS22, Gam23, Reg23, SY24].

In this paper, we use the from zero-freeness result to the SSM of matching polynomial of Δ bounded degree graph for all complex edge parameter $z \in \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$ and provide a FPTAS for approximating the matching polynomial using self avoiding walk tree.

2 Preliminary

2.1 Matching Polynomial

There are various definition of the matching polynomial. An equivalent type of matching generating polynomial is matching defect polynomial, which is defined as

$$\mu_G(z) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} z^{n-2|M|}.$$

The weighted matching defect polynomial is defined as

$$\mu_G(z) = \sum_{M \in \mathcal{M}(G)} \prod_{e \in M} w_e z^{n-2|M|}.$$

Matching generating polynomial and defect polynomial are equivalent by the following relationship

$$\mu_G(z) = z^n M_G(-z^{-2}), \quad M_G(z) = (-i)^n z^{n/2} \mu_G(iz^{-1/2}).$$

The next theorem is the Heilmann–Lieb Theorem [HL72], stating the zero-free region of $M_G(z)$ and $\mu_G(z)$.

Theorem 2.1. *Let Δ be the maximum degree of G. The roots of $M_G(z)$ lie in the interval $(-\infty, -\frac{1}{4(\Delta-1)}]$ and the roots of $\mu_G(z)$ lie in the interval $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$*

We define the edge type ratio and vertex type ratio as follows. Both of them have the probabilistic meaning, vertex type ratio is the probability that the vertex v is not matched and edge type ratio is the probability that the edge e is not matched.

Definition 2.2. *Let G be a graph, v is a vertex in G and e is an edge in G, then we define*

$$P_{G,v}(z) = \frac{M_{G-v}(z)}{M_G(z)}, \quad P_{G,e}(z) = \frac{M_{G-e}(z)}{M_G(z)}.$$

A recursive formula of $P_{G,v}(z)$ is given by

$$P_{G,v}(z) = \frac{M_{G-v}(z)}{M_{G-v}(z) + z\sum_{u\sim v} M_{G-\{u,v\}}(z)} = \frac{1}{1 + z\sum_{u\sim v} P_{G-\{v\},u}(z)}$$

where $u \sim v$ means u is adjacent to v.

Our algorithm is based on the following result of Godsil.

Theorem 2.3 ([God81])**.** *Let G be a graph with bounded degree Δ and $z \in \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$. Let $T_{saw}(G, v)$ be the self-avoiding walk tree rooted at v in G. Then*

$$P_{G,v}(z) = P_{T_{saw}(G,v),v}(z).$$

2.2 Tools from Complex Analysis

For some $\rho > 0$ and $z_0 \in \mathbb{C}$, we denote the open disk $\{z \in \mathbb{C} : |z - z_0| < \rho\}$ centered at z_0 of radius ρ by $\mathbb{D}_\rho(z_0)$, its boundary by $\partial\mathbb{D}_\rho(z_0)$, and its closure by $\overline{\mathbb{D}_\rho}(z_0)$. For simplicity, we write $\mathbb{D}_\rho(0)$ as $\mathbb{D}_\rho$ and $\mathbb{D}_1(0)$ as $\mathbb{D}$. For a holomorphic (or analytic) function f on a complex neighborhood $U \subseteq \mathbb{C}$ of z_0, f can be expanded as a convergent series $f(z) = \sum_{k\geq 0} a_k(z - z_0)^k$ near z_0, called the Taylor expansion series of f at z_0.

Lemma 2.4 (c.f. Lemma 16 in [SY24])**.** *Let $P(z)$ and $Q(z)$ be two analytic functions on some complex neighborhood U of z_0. Suppose that the Taylor series $\sum_{k\geq 0} a_k(z - z_0)^k$ of $P(z)$ and $\sum_{k\geq 0} b_k(z - z_0)^k$ of $Q(z)$ near z_0 satisfy $a_k = b_k$ for $k = 0, 1, \cdots, n$ for some $n \in \mathbb{N}$. Also, suppose that there exists an $M > 0$ such that both $|P(z)| \leq M$ and $|Q(z)| \leq M$ on some circle $\partial\mathbb{D}_\rho(z_0) \subseteq U$ $(\rho > 0)$. Then for every $z \in \mathbb{D}_\rho(z_0)$, we have*

$$|P(z) - Q(z)| \leq \frac{2M}{\rho(r-1)r^n}, \quad \textit{with} \quad r = \frac{\rho}{|z - z_0|}.$$

Definition 2.5. *Let U be an open subset of $\mathbb{C}$. A family $\mathcal{F}$ of holomorphic functions $f : U \to \mathbb{C}$ is said to be a normal family if each infinite sequence of functions in $\mathcal{F}$ has a subsequence that converges locally uniformly to a holomorphic function on U or converges locally uniformly to ∞ on U.*

The next tool is the Montel's theorem. Fot the detail of the proof, see [NN12].

Theorem 2.6. *(Montel) Let U be an open subset of $\mathbb{C}$ and $\mathcal{F}$ be a family of holomorphic functions $f : U \to \mathbb{C}$ such $f(U) \subset \mathbb{C}\backslash\{0, 1\}$ for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family.*

The next theorem is the Riemann Mapping Theorem, which allow us to map some open subset of $\mathbb{C}$ to the unit disk $\mathbb{D}$.

Theorem 2.7. *(Riemann Mapping Theorem) Let U be a non-empty, not the whole of $\mathbb{C}$ and simply connected open subset of $\mathbb{C}$, $z_0 \in U$. There exists a unique bijective holomorphic function $f : U \to \mathbb{D}$, such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

3 SSM of vertex type ratio

3.1 Uniform bound of vertex type ratio

Definition 3.1 (interlace)**.** *Let $p(z) = C_1 \prod_{i=1}^{n}(z - \lambda_i)$ and $q(z) = C_2 \prod_{i=1}^{m}(z - \gamma_i)$ be two real-roots polynomials where $C_1C_2 \neq 0$ and $m = n - 1$ or $m = n$. We say $q(z)$ interlaces $p(z)$ if $\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \gamma_2 \geq \cdots$. If all the inequalities are strict, we say $q(z)$ strictly interlaces $p(z)$.*

The next two lemma are stated in the proof of the Heilmann–Lieb Theorem [HL72]. The second lemma can be directly derived from the first one by a continuity argument.

Lemma 3.2. *Assume G is a weighted complete graph with $w_{uv} > 0$ for all pairs $u, v \in V(G)$. Then $\mu_G(z)$ is real rooted, with distinct roots and for all $v \in V(G)$, $\mu_{G-v}(z)$ strictly interlaces $\mu_G(z)$. Also, $\sum_{u\sim v} w_{ij}\mu_{G-\{u,v\}}(z)$ is real-rooted and strictly interlaces $\mu_{G-v}(z)$.*

Lemma 3.3. *Assume G is a non-negative weighted graph. Then $\mu_G(z)$ is real rooted and for all $v \in V(G)$, $\mu_{G-v}(z)$ interlaces $\mu_G(z)$. Also, if $\sum_{u\sim v} w_{uv}\mu_{G-\{u,v\}}(z)$ is not identically zero, then it is real-rooted and interlaces $\mu_{G-v}(z)$.*

We will use the above two lemmas to prove the ratio avoid 0 and 1.

Lemma 3.4. *Let G be a graph with n vertices and bounded degree Δ and v be an vertex in G. If $z \in \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$, $z \neq 0$ and v is not isolated in G , then $P_{G,v}(z)$ avoid 0 and 1.*

Proof. By theorem 2.1, avoid 0 is trivial. We prove the avoid 1 case. Note

$$M_G(z) - M_{G-v}(z) = z\sum_{u\sim v} M_{G-\{u,v\}}(z).$$

Prove all roots of $\sum_{u\sim v} M_{G-\{u,v\}}(z)$ is in the interval $(-\infty, -\frac{1}{4(\Delta-1)}]$ is enough to show $P_{G,v}(z)$ avoid 1. Since $\sum_{u\sim v}\mu_{G-\{u,v\}}(z) = z^{n-2}\sum_{u\sim v} M_{G-\{u,v\}}(-z^2)$, show all roots of $\sum_{u\sim v}\mu_{G-\{u,v\}}(z)$ in the interval $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$ is enough.

Still by theorem 2.1, all $n-1$ roots of $\mu_{G-v}(z)$ lie in the interval $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$. Since v is not isolated in G, $\sum_{u\sim v}\mu_{G-\{u,v\}}(z)$ is not identically zero, by lemma 3.3, $\sum_{u\sim v}\mu_{G-\{u,v\}}(z)$ is real-rooted and interlaces $\mu_{G-v}(z)$. Thus all $n-2$ roots of $\sum_{u\sim v}\mu_{G-\{u,v\}}(z)$ lie in the interval $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$. Then $P_{G,v}(z)$ avoid 1.

□

Lemma 3.5. *Let $\Delta \geq 2$, and S is a compact subset of $U = \mathbb{C}\backslash((-\infty, -\frac{1}{4(\Delta-1)}]\cup\{0\})$ which contain a positive real number. There exists two positive constants C_1, C_2 such that for any graph G with bounded degree Δ, $v \in V(G)$, $z \in S$,*

$$C_1 \leq |P_{G,v}(z)| \leq C_2.$$

Proof. We prove the upper bound first. When v is isolated in G, $P_{G,v}(z) = 1$. Thus we only need to consider the case when v is not isolated in G.

Suppose the contrary, then there exists a sequence of tuples $(G_n, v_n, z_n \in S)_{n\in\mathbb{N}}$ such that

$$|P_{G_n,v_n}(z_n)| \geq n.$$

We denote $\{f_n(z)\}_{n\in\mathbb{N}}$ as the subsequence of functions $\{P_{G_n,v_n}(z)\}_{n\in\mathbb{N}}$. By lemma 4.1, we know that $f_n(z)$ avoid 0 and 1 for all $z \in U$. Note U is a open set, by Montel's theorem, $\{f_n(z)\}_{n\in\mathbb{N}}$ is a normal family. Thus $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ converges locally uniformly to a holomorphic function $f(z)$ on U or converges locally uniformly ∞ on U.

There exists a positive real number z_0 in S, then $0 < f_{n_i}(z_0) < 1$, thus subsequence can't converge locally uniformly to ∞ on U. Then the subsequence converges to a holomorphic function $f(z)$ on U.

Since S is a compact subset of U, the convergence is uniform on S, the convergence is uniform on S and also the function f is bounded by some $C > 0$ on S. Then, there existed some $k \in \mathbb{N}$, such that for all $i \geq k$, we have $|f_{n_i}(z)| \leq 2C$ on S. This is a contradiction.

Since S is a compact subset of U, let $R = \sup_{z\in S}|z|$, then

$$\left|\frac{1}{P_{G,v}(z)}\right| = \left|1 + z\sum_{u\sim v} P_{G-v,u}(z)\right| \leq 1 + \sum_{u\sim v}|z|\,|P_{G-v,u}(z)| \leq 1 + \Delta RC,$$

the lower bound is got. □

3.2 LDC

We use the Christoffel–Darboux type identities of the matching polynomial to prove the LDC of the vertex type ratio.

Lemma 3.6 ([HL72]). *Let G be a graph and $u, v \in V(G)$, $u \neq v$. Let $\mathcal{P}_{u,v}$ be the set of paths from u to v. Then*

$$\mu_{G-u}(z)\mu_{G-v}(z) - \mu_G(z)\mu_{G-u-v}(z) = \sum_{P \in \mathcal{P}_{u,v}} \mu_{G-P}(z)^2$$

where $G - P$ is the graph obtained by deleting the vertices in P from G.

Lemma 3.7. *Let G be a graph with n vertices and $u, v \in V(G)$, $u \neq v$, then*

$$z^{d(u,v)} \mid M_{G-u}(z)M_{G-v}(z) - M_G(z)M_{G-u-v}(z).$$

Proof. By the relationship of matching generating polynomial and matching defect polynomial, we have

$$\begin{aligned}
&M_{G-u}(z)M_{G-v}(z) - M_G(z)M_{G-u-v}(z)\\
=&(-i)^{n-1}z^{\frac{n-1}{2}}\mu_{G-u}(iz^{-\frac{1}{2}})(-i)^{n-1}z^{\frac{n-1}{2}}\mu_{G-v}(iz^{-\frac{1}{2}})-\\
&(-i)^{n}z^{\frac{n}{2}}\mu_{G}(iz^{-\frac{1}{2}})(-i)^{n-2}z^{\frac{n-2}{2}}\mu_{G-u-v}(iz^{-\frac{1}{2}})\\
=&(-z)^{n-1}\left[\mu_{G-u}(iz^{-\frac{1}{2}})\mu_{G-v}(iz^{-\frac{1}{2}}) - \mu_G(iz^{-\frac{1}{2}})\mu_{G-u-v}(iz^{-\frac{1}{2}})\right]\\
=&\sum_{P \in \mathcal{P}_{u,v}} (-z)^{n-1}\mu_{G-P}(iz^{-\frac{1}{2}})^2\\
=&\sum_{P \in \mathcal{P}_{u,v}} (-z)^{|P|-1}\left[(-i)^{n-|P|}z^{\frac{n-|P|}{2}}\mu_{G-P}(iz^{-\frac{1}{2}})^2\right]^2\\
=&\sum_{P \in \mathcal{P}_{u,v}} (-z)^{|P|-1}M_{G-P}(z)^2.
\end{aligned}$$

Since P has at least $d(u, v) + 1$ vertices, $z^{d(u,v)} \mid M_{G-u}(z)M_{G-v}(z) - M_G(z)M_{G-u-v}(z)$. ∎

Using the above lemma, we can prove the LDC of the vertex type ratio.

Lemma 3.8. *Let G be a graph, $v \in V(G)$, vertices set $A \subseteq V(G) - v$, then the taylor series near 0 of $P_{G,v}(z)$ and $P_{G-A,v}(z)$ satisfy*

$$z^{d_G(v,A)} \mid P_{G,v}(z) - P_{G-A,v}(z).$$

Proof. We prove it by induction on $|A|$. For the base case $|A| = 1$, write $A = \{u\}$, then

$$P_{G,v}(z) - P_{G-A,u}(z) = \frac{M_{G-v}(z)}{M_G(z)} - \frac{M_{G-u-v}(z)}{M_{G-u}(z)} = \frac{M_{G-u}(z)M_{G-v}(z) - M_G(z)M_{G-u-v}(z)}{M_G(z)M_{G-u}(z)}.$$

Since $M_G(0)M_{G-u}(0) = 1$, $\frac{1}{M_G(z)M_{G-u}(z)}$ is an analytic function near 0. Combine lemma 3.7, we have $z^{d_G(v,A)} \mid P_{G,v}(z) - P_{G-A,u}(z)$.

Suppose $k \geq 2$, the statement is true for all $|A| < k$. Let A be a vertices set with $|A| = k$, $u \in A$, write $A' = A - u$, then

$$P_G(z) - P_{G-A,v}(z) = \left[P_G(z) - P_{G-A',v}(z)\right] + \left[P_{G-A',v}(z) - P_{G-A,v}(z)\right].$$

By the induction hypothesis, we have $z^{d_G(v,A')} \mid P_G(z) - P_{G-(A-u),v}(z)$ and $z^{d_{G-A'}(v,u)} \mid P_{G-A',v}(z) - P_{G-A',v}(z)$. Since $d_G(v,A) \leq d_G(v,A')$ and $d_G(v,A) \leq d_G(v,u) \leq d_{G-A'}(v,u)$, we have $z^{d_G(v,A)} \mid P_G(z) - P_{G-A,v}(z)$. $\square$

Lemma 3.9. *Let G be a graph, $v \in V(G)$, vertices sets $A, B \subseteq V(G) - v$, then the taylor series near 0 of $P_{G,v}(z)$ and $P_{G-A,v}(z)$ satisfy*

$$z^{d_G(v,A\neq B)} \mid P_{G-A,v}(z) - P_{G-B,v}(z).$$

Proof. Let $G' = G - (A \cap B)$, $A' = A - B$, $B' = B - A$, then

$$P_{G-A,v}(z) - P_{G-B,v}(z) = P_{G'-A',v}(z) - P_{G'-B',v}(z) = \left[P_{G'-A',v}(z) - P_{G',v}(z)\right] + \left[P_{G',v}(z) - P_{G'-B',v}(z)\right]$$

By the previous lemma, we have $z^{d_{G'}(v,A')} \mid P_{G',v}(z) - P_{G'-A',v}(z)$ and $z^{d_{G'}(v,B')} \mid P_{G',v}(z) - P_{G'-B',v}(z)$. Since $d_G(v, A \neq B) = \min\{d_G(v,A'), d_G(v,B')\} \leq \min\{d_{G'}(v,A'), d_{G'}(v,B')\}$, we are done. $\square$

3.3 Strong spatial mixing

Definition 3.10 (SSM)**.** *Fix z be a complex parameter, $\mathcal{G}$ be a family of graphs. The matching polynomial defined on $\mathcal{G}$ with parameter z is said to satisfy strong spatial mixing (SSM) if exists a positive constant C such that for any graph $G = (V,E) \in \mathcal{G}$ and any vertex $v \in V(G)$, any two vertices sets $A, B \subseteq V(G) - v$, we have*

$$|P_{G-A,v}(z) - P_{G-B,v}(z)| \leq C r^{-d_G(v,A\neq B)}$$

where $A \neq B$ means $(A - B) \cup (B - A)$.

Theorem 3.11. *Fix $\Delta \geq 2$, let $\mathcal{G}$ be the graph family bounded degree Δ and $z \in \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$. Then the matching polynomial with parameter z exhibits SSM.*

Proof. By Riemann mapping theorem (Theorem 2.7), there exists a bijective holomorphic function $f : \mathbb{D} \to U$, mapping the unit disk $\mathbb{D}$ to $U = \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$ with $f(0) = 0$.

By lemma 3.5, we know $P_G(f(z))$ is uniformly bounded on any compact subset of $\mathbb{D}$, especially on the circle $\partial\mathbb{D}_\rho$ for $0 < \rho < 1$. Since $f(0) = 0$, we can write $f(z) = zg(z)$, for some analytic function $g(z)$. By lemma 3.9, we have $(zg(z))^{d_G(v,A\neq B)} \mid P_{G-A,v}(f(z)) - P_{G-B,v}(f(z))$, thus $z^{d_G(v,A\neq B)} \mid P_{G-A,v}(z) - P_{G-B,v}(z)$.

Combine lemma 2.4 , for some $r > 1$, we have

$$|P_{G-A,v}(f(z)) - P_{G-B,v}(f(z))| \leq O\left(r^{-d_G(v,A\neq B)}\right).$$

For any $z \in U$, consider ρ such that $f^{-1}(z) \in \mathbb{D}_\rho$, we are done. $\square$

Theorem 3.12. *Fix $\Delta \geq 2$, $z \in \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$, for any $\varepsilon \in (0,1)$, we can compute $\hat{M}_G(z) = M_G(z)e^a$ for some complex a with $|a| < \varepsilon$ in $\mathrm{poly}(\frac{n}{\epsilon})$ time.*

Proof. We express the matching polynomial as the product of the vertex type ratio

$$M_G(z) = \frac{1}{\prod_{i=1,n} P_{G_i,v_i}(z)}$$

where $v_1, v_2, \cdot, v_n$ is an arbitrary ordering of the vertices in G and $G_i = G - \{v_1, v_2, \cdots, v_{i-1}\}$.

By theorem 2.3, we know $P_{G_i,v_i}(z) = P_{T_{saw}(G_i,v_i),v_i}(z)$, where $T_{saw}(G_i, v_i)$ is the self-avoiding walk tree rooted at v_i in G_i. By theorem 3.11, we know the matching polynomial with parameter z exhibits SSM. By lemma 3.5, we know the vertex type ratio has a lower bound $C_1 > 0$.

For a ε-approximating of matching polynomial, $\frac{\varepsilon}{n}$-approximating of the ratio is enough. Since the ratio has a lower bound $C_1 > 0$, since $|e^z - 1| \geq (1 - \frac{1}{e})|z| \geq \frac{|z|}{2}$ for $z \in \mathbb{D}$, $\frac{\varepsilon C_1}{2n}$ additive error for the ratio is enough. By the SSM, truncating the self avoiding walk tree in $O(\log(\frac{n}{\varepsilon}))$ depth is sufficient to get such an additive error. Thus we get a $n\Delta^{O(\log(\frac{n}{\varepsilon}))} = \mathrm{poly}(\frac{n}{\varepsilon})$ time algorithm. □

4 SSM of edge type ratio

Except the SSM of the vertex type ratio, we also define and prove the SSM of the edge type ratio (we call it ESSM). Moveover, the two type SSM are equivalent in matching polynomial.

The proof is similar and more simple than the vertex type ratio.

4.1 Uniform bound

Edge type ratio has the same avoiding 0 and 1 property as the vertex type ratio.

Lemma 4.1. *Let G be a graph with bouned degree Δ and e be an edge in G. If $z \in \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$ and $z \neq 0$, then $P_{G,e}(z)$ avoid 0 and 1.*

Proof. Avoid 0 is trivial. We prove the avoid 1 case, write $e = (u, v)$, a simple observation is $M_G(z) - M_{G-e}(z) = zM_{G\backslash\{u,v\}}(z)$. Since $z \neq 0$ and $M_{G\backslash\{u,v\}}(z) \neq 0$, we have $M_G(z) \neq M_{G-e}(z)$. Thus $P_{G,v}(z) \neq 1$. □

Lemma 4.2. *Let $\Delta \geq 2$, and S is a compact subset of $U = \mathbb{C}\backslash((-\infty, -\frac{1}{4(\Delta-1)}] \cup \{0\})$ which contain a positive real number. There exists two positive constants C_1, C_2 such that for any graph G with bounded degree Δ, $e \in E(G)$, $z \in S$,*

$$C_1 \leq |P_{G,e}(z)| \leq C_2.$$

Proof. Same as the proof of lemma 3.5, still by Montel's theorem, we have the upper bound of the edge type ratio.

For the lower bound, take the positive real number z_0 in S, then $\frac{1}{P_{G,e}(z_0)} > 1$ and

$$\frac{1}{P_{G,e}(z_0)} = \frac{M_G(z_0)}{M_{G-e}(z_0)} = 1 + z_0 \frac{M_{G-\{u,v\}}(z_0)}{M_{G-e}(z_0)} \leq 1 + z_0,$$

then same as the proof of the upper bound, we have the upper bound of $\frac{1}{P_{G,e}(z)}$, i.e. the lower bound of $P_{G,e}(z)$. □

4.2 LDC

A connection between the matching generating polynomial and the independence polynomial is $M_G(z) = I_{L(G)}(z)$ where $L(G)$ is the line graph of G, I is the independence polynomial. The lemma is directly from the Christoffel–Darboux type identities of the independence polynomial.

Lemma 4.3 ([Ben18]). *Let G be a graph, $u, v \in V(G)$. Let $\mathcal{B}_{u,v}$ be the set of induced connected, bipartite graphs containing the vertices u and v. Then*

$$z^{d_G(u,v)+1} | I_{G-u}(z)I_{G-v}(z) - I_G(z)I_{G-u-v}(z).$$

Lemma 4.4. *Let G and e_1, e_2 be two different edges in G. Then*

$$z^{d_G(e_1,e_2)+2} \mid M_{G-e_1}(z)M_{G-e_2}(z) - M_G(z)M_{G-e_1-e_2}(z).$$

Proof. Let u, v be two vertices in $L(G)$ corresponding to the edges e_1, e_2 respectively. By the relationship of matching generating polynomial and independence polynomial, we have

$$M_{G-e_1}(z)M_{G-e_2}(z) - M_G(z)M_{G-e_1-e_2}(z) = I_{L(G)-u}(z)I_{L(G)-v}(z) - I_{L(G)}(z)I_{L(G)-u-v}(z).$$

Nothing $d_{L(G)}(u, v) = 1 + d_G(e_1, e_2)$, by lemma 4.3, we have $z^{d_G(e_1,e_2)+2} \mid M_{G-e_1}(z)M_{G-e_2}(z) - M_G(z)M_{G-e_1-e_2}(z)$.

□

Having the above lemma, by the same technique in lemma 3.9, we can get the next lemma.

Lemma 4.5. *Let G be a graph, $e \in E(G)$, two edges sets $A, B \subseteq E(G) - e$, then the taylor series near 0 of $P_{G,v}(z)$ and $P_{G-A,v}(z)$ satisfy*

$$z^{d_G(e,A \neq B)+2} \mid P_{G-A,v}(z) - P_{G-B,v}(z).$$

4.3 SSM

Definition 4.6 (edge type SSM). *Fix z be a complex parameter, $\mathcal{G}$ be a family of graphs. The matching polynomial defined on $\mathcal{G}$ with parameter z is said to satisfy strong spatial mixing (SSM) if exists a positive constant C such that for any graph $G = (V, E) \in \mathcal{G}$ and any vertex $e \in E(G)$, any two edges sets $A, B \subseteq E(G) - e$, we have*

$$|P_{G-A,e}(z) - P_{G-B,e}(z)| \leq Cr^{-d_G(e,A \neq B)}$$

where $A \neq B$ means $(A - B) \cup (B - A)$.

Having the bound and LDC, immediately, we have the SSM result of the edge type ratio.

Theorem 4.7. *Fix $\Delta \geq 2$, let $\mathcal{G}$ be the graph family bounded degree Δ and $z \in \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$. Then the matching polynomial with parameter z exhibits edge type SSM.*

Theorem 4.8 (FPTAS). *Fix $z \in \mathbb{C}\backslash(-\infty, -\frac{1}{4(\Delta-1)}]$ and $\Delta \in \mathbb{N}^+$. Then there is an algorithm that given a graph $G(V, E)$ with maximum degree Δ and $\epsilon \in (0, 1)$, outputs a $\hat{M} = M_G(z)e^{\delta}$ with complex number $|\delta| < \epsilon$ in time $O(\text{poly}(n/\epsilon))$.*

4.4 Equivalent between the two SSM

In matching generating polynomial, delete a vertex is equivalent to delete all edges incident to the vertex (noting the polynomial of a single vertex is 1). In the bounded degree graph, vertex ratio can be write the product of at most Δ edge type ratio. Thus we can get the vertex ratio SSM from the edge ratio SSM. Also, write $e = (u, v)$, since $P_{G,e}(z) = 1 - z\frac{M_{G-u-v}(z)}{M_G(z)} = 1 - zP_{G,v}(z)P_{G-v,u}(z)$, we can get the edge ratio SSM from the vertex ratio SSM.

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