# Carnegie Mellon University
# MSCF Program
# 46-956 Introduction to Fixed Income

Fall 2018

Mini 1

Lecture Notes for Week 1


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# Course Information

Instructor: *Bill Hrusa*

Teaching Assistants: *David Itkin* and *Ben Weber*

Official Textbook: *Fixed Income Securities:Tools for Today's Markets* by Bruce Tuckman and Angel Serrat ($3^{rd}$ ed). Exercises from Tuckman & Serrat are posted on Canvas under "Assignments".

Additional Recommended Textbook: *Fixed Income Securities: Valuation, Risk, and Risk Management* by Pietro Veronesi

# Lectures and TA Sessions

Lectures: Monday & Wednesday 10:30 - 11:50 AM

TA Sessions: Fridays from 2:15 - 3:45 PM ($1^{st}$ session on August 31)

# Grading Components

Homework Assignments: 6 to hand in; Lowest Score will be Dropped, i.e. use the Best 5

Final Exam: Thursday October 18 from 3:00 - 6:00 PM

Course Grade:

- 25% Homework Average
- 75% Final Exam Score

Make-Up Final: Students with a course grade of C+ or below can take a make-up final. The maximum course grade for students taking the make-up is B-. The make-up exam will not be used to lower your course grade.

# Homework Write-Ups

- Each student must turn in his or her own write-up.
- You may discuss the problems with one another, but must work independently on your write-up.
- "New York" or "Pittsburgh" clearly marked on each paper.
- You must explain clearly how you arrive at your answer, showing a sufficient number of intermediate steps, with the possible exception of a few short-answer questions.
- It is expressly forbidden to access course materials from previous renditions of 46-956 or from any rendition of 21-378.
- Late HWs cannot be accepted for credit. (Late means any time after 10:30 AM on the due date.)
- You should be sure to keep a copy of your solutions.

# Homework Submission

Homework should be submitted by attaching a single PDF file to the assignment on Canvas. Spreadsheets can also be attached if relevant.

# Homework Grading

- Some of the homework problems might not be graded. (In fact, it will typically be the case that a few of the questions will not be graded.)
- Complete solutions to *all* problems will be posted on Canvas, so you can check your work on all problems. Solutions will be posted at the beginning of lecture on the due date. Homework that is submitted after the solutions are posted cannot be accepted for credit.
- A subset of the problems will be selected for careful grading. It will not be announced in advance which problems will be graded.

# Trips to New York

I will be in NY on Wednesday September 5, Wednesday September 19, and Wednesday, October 3.

# Securities

In this course, we shall use the term *security* to mean any tradable financial instrument. (In some contexts, the term "security" has a more precise legal meaning that might exclude some of the financial instruments we talk about from legally being considered securities.)

Some very important types of securities are:

- Equity Securities (e.g., common stocks)
- Debt Securities (e.g., bonds)
- Derivative Contracts

A financial derivative is a contract between two parties that specifies conditions on the values of underlying variables that determine how payments are to be exchanged between the parties. The underlying variables are often prices of assets. Some important examples of derivatives are put and call options on stocks, futures contracts, and interest rate swaps.

# A Bond is Really a Loan

An agent who purchases a bond is actually making a loan to the agency that issues the bond. The initial price paid for the bond is the *principal* of the loan. (Assuming that the issuing agency does not *default*) the bond holder receives *interest* payments as well as repayment of the principal. There are several common repayment schemes:

- The loan is repaid with a single lump-sum payment (principal plus interest) at maturity. (Zero Coupon Bond)
- There are periodic interest-only payments and the principal (together with a final interest payment) is paid at maturity. (Coupon Bond)
- There are periodic level payments with a portion of each payment reflecting interest on the outstanding principal and the rest of the payment is applied to reduce the outstanding principal balance. (Annuity or Self-Amortizing Loan)

# Fixed-Income Securities

**Literal Definition**: A security whose future payments (dates and amounts) are known (with certainty) at the time that the security is issued.

It is standard practice to use the term *fixed income security* to also include derivatives written on debt securities. For such derivatives, the payment dates and amounts generally will not be known with certainty in advance. A better term might be *interest-rate product*.

We shall refer to securities whose future payments are known with certainty as *securities with deterministic cash flows* or *securities with fixed payments*. Many US treasuries fall into this category.

# Default Risk

Some securities (e.g., corporate bonds) that promise fixed future payments might not actually make the promised payments because the issuer of the security might default. *Default Risk*, *Credit Risk*, or *Counterparty Risk*, although extremely important, will not be treated with any degree of depth in this course. **Unless stated otherwise, we assume there is no risk of default**.

# Some Important Examples of Fixed-Income Securities

- Zero-Coupon Bonds
- Coupon Bonds
- Annuities
- Inflation Protected Bonds
- Floaters and Inverse Floaters
- Callable and Putable Bonds
- Interest Rate Swaps, Caps, Floors, Swaptions
- Interest Rate Futures (especially Eurodollar Futures)
- Mortgage Backed Securities
- Bond Options
- Bond Futures
- Options on Bond Futures

Prices of fixed income securities are frequently described using *interest rates*. In practice, interest rates depend on many factors, including the initiation date of the loan, the length of the loan, and the schedule according to which payments are to be made. **Interest rates that will prevail at future dates are generally not known in advance.**

Much can be said about securities with deterministic cash flows without employing a model describing the manner in which interest rates will evolve in time. Roughly the first half of the course will be devoted to this topic. (Classical Material)

In most cases, in order to analyze fixed-income securities with uncertain payments, it is crucial to make a mathematical model that reflects the uncertainty. This is a very serious (and interesting) endeavor. The last half of the course will be devoted to this topic. We will consider only very simple models (discrete time and finite sample space), but the same ideas can be used with much more sophisticated models.

# Players in Fixed-Income Markets

- **Issuers** - include federal, state and local governments; agencies (GSE); corporations
- **Intermediaries** - facilitate the sale of securities - include dealers; investment banks; credit rating agencies
- **Investors** - include individuals; governments; insurance companies; pension funds; mutual funds; commercial banks

This course will focus on the mathematical principles used to analyze fixed-income securities rather than on "institutional practice". However, I will make a serious effort to respect actual market conventions as much as possible in examples and exercises.

# Risks for Fixed-Income Securities

In addition to default risk, there are many types of risks that need to be considered for fixed income securities. These include

- Inflation Risk
- Liquidity Risk
- Interest Rate Risk (Market Risk)
- Reinvestment Risk
- Currency Risk (FX Risk)
- Timing Risk (Call Risk)

Traders are most concerned with liquidity risk, interest rate risk, and timing risk. Default risk can sometimes be hedged using *credit default swaps* (CDS). It is important note that all of the types of risk listed above, except for "Timing Risk" are relevant for securities with deterministic cash flows.

# Securities with Deterministic Cash Flows

**Remark 1.1:** Unless stated otherwise: Time will be measured in years and the present time is taken to be $t = 0$. We assume that there is no risk of default. We ignore transaction costs (including bid-ask spread) and we assume that all securities can be purchased or sold short in any amounts we please (including fractional shares). Moreover, we assume that the size of an order does not have any impact on the price per share.

*Ideal Frictionless Liquid Market with no Default Risk*

# Zero-Coupon Bonds

Characterized by a *face value* $F$ and a *maturity* $T$. The holder receives a single payment of amount $F$ at time $T$.

Building blocks for all securities with deterministic cash flows:
Every security with deterministic cash flows can be expressed as a portfolio of zero-coupon bonds.

They are also called *pure discount bonds*, *zeros*, or *ZCBs*.

# Coupon Bonds

Characterized by a *face value* $F$, a *maturity* $T$, an *annual coupon rate* $q$, and a number $m$ of *coupon payments* per year.

The holder receives coupon payments of amount

$$F\frac{q}{m}$$

at each of the times $\frac{i}{m}$ for $i = 1, 2, \cdots, mT$ plus the face value at maturity. (Notice that that the holder receives $F(1 + \frac{q}{m})$ at time $T$.)

For the vast majority of bonds in the US $m = 2$, i.e. the coupon payments are made once every six months. Unless stated otherwise, we assume that $m = 2$.

**Remark 1.2**: The face value of a bond is often referred to as the *par value*. When a coupon bond is issued, the coupon rate is typically chosen so that the initial price of the bond is very close to the face value. A bond is said to trade *above par*, *at par*, or *below par*, depending on whether the current price of the bond is above, equal to, or below the face value. The terminology *premium to par* and *discount to par* is also commonly used.

# Price Quotations

Unless stated otherwise, bond prices will be given per \$100 of face value (or as a percentage of face value). I will use decimal amounts in lecture, homework exercises, and on the final exam. (Read about *ticks* on your own.)

Unless stated otherwise, when we encounter coupon bonds that were issued previously, we assume that a coupon payment has just been made and that the next coupon payment will be made in precisely $\frac{1}{m}$ years, where $m$ is the number of coupon payments per year. The current price of the bond *does not* include the coupon payment that has just been made. When a bond trades between coupon payments there is *accrued interest* that is separate from the quoted bond price. (Accrued interest will be discussed next week.)

# Annuities

Characterized by a *maturity* $T$, a payment amount $A$, and a number $m$ of payments per year. The holder receives payments of amount $A$ at each of the times $\frac{i}{m}$ for $i = 1, 2, \cdots, mT$.

- A *perpetuity* is an annuity that has maturity $T = \infty$.
- There are also *lifetime annuities* (having unknown maturity) and annuities making variable payments.

# Discount Factors & the Time-Value of Money

Under typical economic conditions, it is better to receive \$1 now than in the future. Similarly, it is better to pay \$1 in the future than to pay \$1 now.

Question: How much would you pay today in order to receive \$1 $t$-years from today? (No risk of default.)

Although different investors may have different feelings about this, there is a single market price for this privilege, namely the *discount factor* for time $t$; it is denoted by

$$d(t).$$

# Discount Factors (Cont.)

We must have $d(t) > 0$ in order to avoid arbitrage.

In the US, it is almost always the case that

$$d(t) < 1 \text{ for } t > 0$$

and that $d(t)$ decreases as $t$ increases.

Situations in which $d(t) > 1$ for some $t > 0$ or $d(t_2) > d(t_1)$ for some $t_2 > t_1$ correspond to some interest rate being negative. There are certain important economies (e.g., Japan and Germany) that currently have some negative interest rates. There have been negative interest rates in the US (e.g. during the 2008 financial crisis), but these situations have not lasted for very long. Negative interest rates have been present in Japan and Germany for several years.

**Remark 1.3:** The price today for receiving an amount $F$ $t$-years from today is $Fd(t)$. Consequently, the price today of a zero with face value $F$ and maturity $T$ should be $Fd(T)$.

**Law of One Price:** Two securities (or portfolios) with exactly the same future payments should have the same current price.

**Remark 1.4:** The price today for a security that will make payments of amounts $F_i$ at each of the times $t_i$ for $i = 1, 2, \cdots, N$ is given by

$$P = \sum_{i=1}^{N} F_i d(t_i) = F_1 d(t_1) + F_2 d(t_2) + \cdots + F_N d(t_N).$$

# Arbitrage

An extremely important concept in mathematical finance is the notion of **arbitrage**

By an *arbitrage strategy* we mean a trading strategy in which there is no input of capital, zero probability of a loss, and a strictly positive probability of a (strictly positive) profit.

By a *strong arbitrage* we mean an arbitrage strategy in which a strictly positive profit is certain (i.e. has probability 1). Clearly, every strong arbitrage strategy is an arbitrage strategy.

Although arbitrage opportunities do sometimes exist in the real world, they usually disappear shortly after they are discovered, because prices will adjust once there is heavy trading to attempt to make an arbitrage profit. Unless stated otherwise, we assume that arbitrage is not possible. The absence of arbitrage implies that the law of one price holds.

In general, the absence of arbitrage implies the absence of strong arbitrage, and the absence of strong arbitrage implies the law of one price. (It is possible to construct models in which the law of one price is satisfied, but there is strong arbitrage. It is also possible to construct models in which there is no strong arbitrage, but arbitrage is possible.) The concept of arbitrage will be treated in detail in MPAP (Multiperiod Asset Pricing).

Traders and other practitioners sometimes use the term arbitrage to indicate a strategy that requires no input of capital, the probability of a loss is small (but not necessarily zero), and the probability of a profit is high. If there is any doubt about what someone means by arbitrage, you should as whether the probability of a loss must really zero, or whether a loss is simply considered to be unlikely.

**Example 1.1**: Suppose that the following three bonds are trading:

- Bond #1: A zero with maturity 6 months and current price 97.53 (per $100 of face)
- Bond #2: A coupon bond with maturity 1 year, annual coupon rate 5% and current price 99.87
- Bond #3: A coupon bond with maturity 18 months and annual coupon rate 10% and current price 106.52

Bond #2 and Bond #3 pay coupons every 6 months.

Let us find the discount factors implicit in these bond prices.

- For Bond #1: $97.53 = 100d(.5)$,
  which gives $d(.5) = .9753$.
- For Bond #2: coupon payments (per 100 face) are $100(.05)/2 = 2.5$, so that

$$99.87 = 2.5d(.5) + 102.5d(1) = 2.43825 + 102.5d(1),$$

  which gives $d(1) = .9506$.
- For Bond #3: coupon payments (per 100 face) are $100(.1)/2 = 5$, so that

$$106.52 = 5d(.5) + 5d(1) + 105d(1.5),$$

  which gives $d(1.5) = .9228$.

This illustrates the fact that discount factors should decrease as time increases.

Suppose that another bond is also trading:

- Bond #4: A zero coupon bond with maturity 1 year and current price 94.93.

We see that Bond #4 is trading below the price we calculate using the value for $d(1)$ computed above:

$$100d(1) = 95.06.$$

We say that Bond #4 is *trading cheap* relative to the other bonds. (A bond that is trading at a price higher than the price calculated using other bonds is said to *trade rich* relative to the other bonds.)

There may be an opportunity to make a riskless profit by purchasing Bond #4 and selling a portfolio that *replicates* Bond #4 in terms of Bond #1 and Bond #2.

If we purchase 100 face of Bond #2 and sell short 2.5 face of Bond #1, we create a zero with maturity 1 year and face value 102.5. Consequently, 100 face of Bond #4 can be replicated by purchasing 100/1.025 face of Bond #2 and shorting 2.5/1.025 face of Bond #1. The cost of the replicating portfolio (per 100 face) is

$$\frac{1}{1.025}99.87 - \frac{2.5}{102.5}97.53 = 95.06.$$

Ignoring transaction costs, one can make a riskless profit (immediately) of .13 per 100 face by purchasing Bond #4 and selling the replicating portfolio. This is an example of an *arbitrage strategy* (in fact, a strong arbitrage). Let us look at the associated cash flows:

Time $t = 0$: Purchase 100 face of Bond #4, sell short 100/1.025 face of Bond #2, and purchase 2.5/1.025 face of Bond #1. Net cash flow is $-94.93 + (1/1.025)99.87 - (2.5/102.5)97.53 = .1254$.

Time $t = .5$: Pay coupon associated with the short sale of Bond #2 and receive payment from purchase of Bond #1. Net cash flow is $-(2.5/1.025) + (2.5/1.025) = 0$.

Time $t = 1$: Receive payment from Bond #4 and close out short position on Bond #2. Net cash flow is $100 - (1/1.025)102.5 = 0$.

**Remark**: On $500 million face of Bond #4, the profit would be $626,829.27. There may be practical reasons that a transaction cannot be executed at the posted prices. If this kind of trade could actually be executed in practice, prices of the bonds would adjust until the arbitrage opportunity disappeared.

# US Treasury Securities

**T-Bills** - Zeros with maturities of 28 days (4 weeks), 91 days (.25 years), 182 (.5 years) days, and 364 days (1 year) when issued.

**T-Notes** - Coupon bonds with maturities between 1 and 10 years when issued.

**T-Bonds** - Coupon bonds with maturities greater than 10 years when issued (typically 30 years).

**STRIPS** - Bonds can be *stripped* into individual coupon payments and principal payments (C-strips and P-strips) thereby creating zeros of long maturities. (STRIPS stands for *Separate Trading of Registered Principal and Interest Securities*)

# US Treasury Securities (Cont.)

All Treasury bills with maturities less than one year are issued weekly. One-year bills are issued every 4 weeks.

Treasury notes are currently being issued with maturities of 2, 3, 5, 7, and 10 years. Bonds are being issued with maturity of 30 years. Notes and Bonds pay coupons every 6 months.

2, 3, 5, and 7-year notes are currently being issued monthly. 10-year notes and 30-year bonds are currently being issued 4 times per year.

The most recently issued treasury securities of each maturity are said to be *on the run*. Earlier issues are said to be *off the run*. On-the-run issues are typically much more liquid than off-the-run issues.

# US Treasury Securities (Cont.)

The treasury also issues floating rate notes (FRNs) with maturity 2 years and inflation-protected securities (TIPS) with maturities of 5, 10, and 30 years. FRNs pay coupons 4 times per year and TIPS pay coupons twice per year.

C-strips and P-strips can be used to re-assemble a bond. Coupon payments must come from a C-strip and the principal payment must come from a P-strip. Any C-strip paying the correct amount on the correct date can be used for a coupon payment. However, the P-strip must come from a bond having the exact same CUSIP number as the bond being assembled. (CUSIP stands for *Committee on Uniform Security Identification Procedures*.) For this reason, P-strips and C-strips paying the exact same amount on the exact same day might trade at different prices.

# Web Resources for US Treasury Info

www.treasurydirect.gov/indiv/indiv.htm

www.treasury.gov/resource-center/data-chart-center/Pages/index.aspx

www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield

www.bloomberg.com/markets/rates-bonds/government-bonds/us/

www.wsj.com/mdc/public/page/2_3020-tstrips.html

# Interest Rates

Discount factors are intrinsic (i.e., they do not depend on any kind of compounding convention). However, they are not very intuitive. Investors often find it useful to quantify the time-value of money using *interest rates*. There are numerous different conventions used to describe interest rates and it is essential to know what convention is being employed in each particular situation.

# Semiannual Compounding

Because most bonds in the US pay coupons every 6 months, bond investors in the US often focus on *semiannual compounding*: If an amount $x$ is invested at an annual rate $r$ compounded semiannually for $T$ years, the value of the investment will be

$$x\left(1+\frac{r}{2}\right)^{2T}$$

after $T$ years. If we know the initial investment, the final value of the investment, and the length of time the money was invested we can work backward and determine the interest rate. (When we use semiannual compounding, the maturity $T$ should be a multiple of 6 months.)

Indeed, if an initial investment of $x$ grows to the amount $w$ after $T$ years, we have the equation

$$w = x\left(1+\frac{r}{2}\right)^{2T},$$

which can be solved for $r$:

$$r = 2\left[\left(\frac{w}{x}\right)^{\frac{1}{2T}} - 1\right].$$

# Spot Rates

The *spot rate* for maturity $t$ is the interest rate on a spot loan in which the lender gives money to the borrower at the time of the agreement ($t = 0$) and the loan is settled with a single lump-sum payment at time $t$. Following Tuckman & Serrat, the $t$-year spot rate (semiannual compounding) will be denoted by

$$\hat{r}(t).$$

Although spot rates can be described using any compounding convention, we shall use semiannual compounding unless stated otherwise. Notice that

$$\hat{r}(t) = 2\left[\left(\frac{1}{d(t)}\right)^{\frac{1}{2t}} - 1\right].$$

# Effective Spot Rates

It is also convenient to introduce the *effective spot rate* $\hat{R}(t)$ for time $t$ defined by

$$\hat{R}(t) = \left(\frac{1}{d(t)}\right)^{\frac{1}{t}} - 1,$$

so that an amount $x$ invested at time 0 is worth

$$x(1 + \hat{R}(t))^t$$

at time $t$. (We do not require $t$ to be a multiple of 6 months.) This convention will prove especially useful in situations when cash flows arrive at dates that are not uniformly spaced.

**Example 1.2**: Compute the spot rates and effective spot rates corresponding to the discount factors

$$d(.5) = .9753, \quad d(1) = .9506, \quad d(1.5) = .9228$$

from Example 1.1. We have

$$\hat{r}(.5) = 2\left[\left(\frac{1}{.9753}\right)^{1} - 1\right] = .05065,$$

$$\hat{r}(1) = 2\left[\left(\frac{1}{.9506}\right)^{\frac{1}{2}} - 1\right] = .05131,$$

$$\hat{r}(1.5) = 2\left[\left(\frac{1}{.9228}\right)^{\frac{1}{3}} - 1\right] = .05429,$$

$$\hat{R}(.5) = \left(\frac{1}{.9753}\right)^2 - 1 = .05129,$$

$$\hat{R}(1) = \left(\frac{1}{.9506}\right) - 1 = .05197,$$

$$\hat{R}(1.5) = \left(\frac{1}{.9228}\right)^{\frac{2}{3}} - 1 = .055022.$$

**Remark 1.5**: Notice that in Example 1.2, the effective spot rates $\hat{R}(t)$ are greater than the corresponding spot rates $\hat{r}(t)$. This is always the case. Also in this example, the spot rates increase with $t$. This is not always the case.

# Discount Factors from Spot Rates

We can express the discount factor in terms of the spot rates via the formulas

$$d(t) = \frac{1}{\left(1 + \frac{\hat{r}(t)}{2}\right)^{2t}},$$

or

$$d(t) = \frac{1}{\left(1 + \hat{R}(t)\right)^{t}}.$$

Except under extraordinary circumstances, discount factors in the real world decrease as maturity increases. (There are important models which allow discount factors to increase over some time intervals, but with very small probability.)

# Spot Rate Curve

The graph of $\hat{r}(t)$ versus $t$ is called the *spot rate curve* or the *zero-coupon yield curve*. Spot rate curves can have different shapes depending on economic conditions. The graph of $\hat{R}(t)$ versus $t$ is called the *effective spot rate curve*. We will look at spot rate curves in detail a bit later on.

# Forward Interest Rates

A *forward loan* is an agreement to lend money at some future date. The interest rate (as well as the size of the loan) is set at the time of the agreement. (Nothing is paid by either party to enter into the agreement.) We really need 3 time variables and a compounding convention. We will use the notation

$$R^{for}_{\tau,\eta,T}$$

to indicate that the agreement is made at time $\tau$, the loan is initiated at time $\eta$ and settled at time $T$, and the compounding convention is "effective". We assume $\tau \leq \eta < T$. Notice that $T - \eta$ is the length of the loan. The amount to be repaid at time $T$, per \$1 borrowed at time $\eta$ is

$$\left(1 + R^{for}_{\tau,\eta,T}\right)^{T-\eta}.$$

The forward rates $R^{for}_{\tau,\eta,T}$ are known at time $\tau$, but not earlier.

Consider a forward loan in which it is agreed at time 0 to borrow \$1 at time $\eta$ and repay the loan at time $T$, at the effective forward rate $R^{for}_{0,\eta,T}$. Let us put

$$A_T = \left(1 + R^{for}_{0,\eta,T}\right)^{T-\eta},$$

the amount to be repaid at time $T$. The forward loan can be replicated by purchasing a ZCB with maturity $\eta$ and face value \$1 and shorting a ZCB with maturity $T$ and face value $A_T$ at time 0. By the no-arbitrage principle, the initial cost of the replicating portfolio must be zero, which implies that

$$d(\eta) - A_T d(T) = 0.$$

This tells us that

$$A_T = \frac{d(\eta)}{d(T)}.$$

Recalling the definition of $A_T$ we find that

$$\left(1 + R^{for}_{0,\eta,T}\right)^{T-\eta} = \frac{d(\eta)}{d(T)} = \frac{(1+\hat{R}(T))^T}{(1+\hat{R}(\eta))^\eta}.$$

The above equation can easily be solved to obtain formulas for $R^{for}_{0,\eta,T}$ in terms of the discount factors or spot rates for maturities $\eta$ and $T$.

It is much more important to understand how to replicate a forward loan using ZCBs than it is to "memorize" formulas for forward rates. Simply memorizing formulas could lead to an error because of a different convention being used to for compounding or for describing the time variables.

**Remark 1.6**: The third time variable in the specification of a forward interest rate is sometimes taken to be the length of the loan, rather than the maturity date of the loan. Be sure to ask what convention is being used if you have any doubt. Different practitioners use different conventions, and it is much better to ask a question than to make a pricing error.

Forward rates can also be quoted using other compounding conventions. For semiannual compounding, we use the notation

$$r^{for}_{\tau,\eta,T}$$

to denote the rate agreed upon at time $\tau$ for a loan to be initiated at time $\eta$ and settled at time $T$. Here we assume that $T-\eta$ is a multiple of 6 months.

Observe that

$$(1 + R^{for}_{\tau,\eta,T})^{T-\eta} = \left(1 + \frac{r^{for}_{\tau,\eta,T}}{2}\right)^{2(T-\eta)}.$$

Following Tuckman & Serrat we use the notation $f(t)$ to denote the (semiannually compounded) rate agreed upon at time 0 for a loan made at time $t - .5$ and settled with a single lump-sum payment at time $t$. Notice that

$$f(.5) = \hat{r}(.5).$$

The forward rates $f(t)$ can be deduced from the spot rates $\hat{r}(t)$. For example we must have

$$\left(1+\frac{\hat{r}(1)}{2}\right)^2 = \left(1+\frac{f(.5)}{2}\right)\left(1+\frac{f(1)}{2}\right),$$

which can be solved for $f(1)$. More generally, we must have

$$\left(1+\frac{f(.5)}{2}\right)\times\left(1+\frac{f(1)}{2}\right)\times\cdots\times\left(1+\frac{f(t)}{2}\right) = \left(1+\frac{\hat{r}(t)}{2}\right)^{2t}.$$

It is useful to observe that the forward rates can also be obtained from the equation

$$(1) \qquad \left(1+\frac{\hat{r}(t)}{2}\right)^{2t} = \left(1+\frac{\hat{r}(t-.5)}{2}\right)^{2t-1}\left(1+\frac{f(t)}{2}\right)$$

There are many other similar relationships, e.g.

$$\left(1+\frac{\hat{r}(t)}{2}\right)^{2t} = \left(1+\frac{\hat{r}(t-1)}{2}\right)^{2t-2}\left(1+\frac{f(t-.5)}{2}\right)\left(1+\frac{f(t)}{2}\right).$$

**Example 1.3**: Find the forward rates $f(.5)$, $f(1)$, and $f(1.5)$ implied by the spot rates

$$\hat{r}(.5) = .05065, \quad \hat{r}(1) = .05131, \quad \hat{r}(1.5) = .05429$$

of Example 1.2. We observe first that $f(.5) = \hat{r}(.5) = .05065$. We next observe that

$$\left(1 + \frac{.05065}{2}\right)\left(1 + \frac{f(1)}{2}\right) = \left(1 + \frac{.05131}{2}\right)^2,$$

which gives $f(1) = .05197$. Finally, we observe that

$$\left(1 + \frac{.05065}{2}\right)\left(1 + \frac{.05197}{2}\right) \times \left(1 + \frac{f(1.5)}{2}\right) = \left(1 + \frac{.05429}{2}\right)^3,$$

which gives $f(1.5) = .06026$.

**Remark 1.7**: An upward sloping spot-rate curve corresponds to the forward rates $f(t)$ being above the spot rates $\hat{r}(t)$. A downward sloping spot-rate curve corresponds to the forward rates $f(t)$ being below spot rates $\hat{r}(t)$. To see why this is the case, let us fix $t \in \{.5, 1, 1.5, 2, 2.5, \cdots\}$ and put

$$\lambda = \frac{1 + \frac{\hat{r}(t)}{2}}{1 + \frac{\hat{r}(t-.5)}{2}}. \tag{2}$$

Observe that

$$\begin{array}{ll} \lambda > 1 \text{ if and only if } & \hat{r}(t) > \hat{r}(t - .5), \\ \lambda < 1 \text{ if and only if } & \hat{r}(t) < \hat{r}(t - .5). \end{array}$$

Using (1) and (2) we find that

$$\left(1 + \frac{f(t)}{2}\right) = \lambda^{2t-1}\left(1 + \frac{\hat{r}(t)}{2}\right),$$

which yields the desired conclusion. $\square$

# An Investment Scenario

**Question**: Consider two investors, say $A$ and $B$, each with the same initial capital to invest. Investor $A$ uses all of the initial capital to purchase a 6-month zero and holds this bond until maturity; at $t = .5$ she uses all of the money she receives from the 6-month zero to purchase another 6-month zero and hold it until maturity. Investor $B$ uses uses all of the initial capital to purchase a one-year zero and holds the bond until maturity. Which investor will be better off in one year?

**Answer**: It depends. Let's look at their capitals at time 1 (per $1 of initial investment).

$$\text{Investor B}: \quad \left(1+\frac{\hat{r}(1)}{2}\right)^2 = \left(1+\frac{\hat{r}(.5)}{2}\right)\left(1+\frac{f(1)}{2}\right),$$

$$\text{Investor A}: \quad \left(1+\frac{\hat{r}(.5)}{2}\right)\left(1+\frac{r_{.5,1}}{2}\right),$$

where $r_{.5,1}$ is the spot rate that will prevail at time $.5$ for 6-month loans (i.e., initiated at $t=.5$ and settled at $t=1$.)

**Remark 1.8**: There is no way of knowing for sure at $t=0$ whether $r_{.5,1}$ will be greater than, less than, or equal to $f(1)$.

# Interpretations of Interest Rates for Zeros

Investing in a $T$-year zero can be interpreted as:

- **(i)**: a $T$-year investment in which the investor commits at $t = 0$ to leave the principal and all accumulated interested on deposit until $t = T$. The interest rate will be constant at $\hat{r}(T)$ over the entire period of investment and will be compounded semiannually; or
- **(ii)**: a $T$-year investment in which the investor commits at $t = 0$ to leave the principal and all accumulated interest on deposit until $t = T$. The interest rate will be $f(.5)$ during the first 6 months, $f(1)$ between $t = .5$ and $t = 1$, $f(1.5)$ between $t = 1$ and $t = 1.5$, $\cdots$, $f(T)$ between $t = T - .5$ and $t = T$.

**However**, the idea of *accumulated interest* on a zero is just a device to help us think about this kind of investment: If the investor wants to bail out at a time $t < T$, then he or she must accept the market price of the bond at time $t$ (which might actually be below the initial purchase price).

# Dependence of Price on Maturity for Coupon Bonds

Suppose that we have two coupon bonds with the same coupon rate $q > 0$ and the same face value $F$. Bond #1 has maturity $T$, whereas Bond #2 has maturity $T + .5$.

**Question**: How do we tell quickly which bond should cost more at $t = 0$?

**Answer**:

- An investor who buys Bond #2 and shorts Bond #1 will have to pay $F$ at time $T$ and will receive $F(1+\frac{q}{2})$ at time $T+.5$.
- An investor who agrees at time 0 to invest $F$ between time $T$ and time $T+.5$ (and pays nothing to enter the agreement) will receive $F(1+\frac{f(T+.5)}{2})$ at time $T+.5$.
- Consequently if $q > f(T+.5)$ then Bond #2 will have a higher price than Bond #1 at $t=0$. If $q < f(T+.5)$ then Bond #1 will have a higher price than Bond #2 at $t=0$.

# A Reinvestment Scenario

Investors A and B have the same initial capital to invest at time 0.

- Investor A invests all of the initial capital in a 6-month zero and keeps rolling over the investment into new 6-month zeros every 6 months until maturity $T$.
- Investor B invests all of the initial capital in a coupon bond with maturity $T$ and coupon rate $q > 0$, and all of the coupon payments will be invested in 6-month zeros and rolled over into new 6-month zeros every 6 months.

- If all forward rates $f(t)$ are higher than the corresponding 6-month spot rates $r_{t-.5,t}$ then investor B will be better off than investor A at time $T$.
- If all forward rates $f(t)$ are lower than the corresponding 6-month spot rates $r_{t-.5,t}$ then investor A will be better off than investor B at time $T$.

Here, $r_{t-.5,t}$ is the spot rate that will prevail at time $t-.5$ for loans initiated at time $t-.5$ and settled at time $t$.

You should try to write out a careful proof of this assertion. It is not difficult if you look at things the right way, but some thought may be required to find the right way.

# Additional Comment on Forward Rates

**Question**: "What kind of formula can be obtained for the forward rates $R^{for}_{\tau,\eta,T}$ when $\tau > 0$?"

Following the notation you will use in Stochastic Calculus: for $0 \leq t_1 \leq t_2$, let $B(t_1, t_2)$ denote the price at time $t_1$ for a ZCB that pays \$1 at time $t_2$. (Notice that $B(t_1, t_2)$ is not known until $t_1$.) If we use ZCBs to replicate a forward loan in which the rate is agreed upon at time $\tau$, \$1 is received at time $\eta$ and $(1 + R^{for}_{\tau,\eta,T})^{T-\eta}$ is repaid at time $T$, we find that

$$B(\tau, \eta) - (1 + R^{for}_{\tau,\eta,T})^{T-\eta} B(\tau, T) = 0.$$

This expression can be solved for the forward rate.

# Carnegie Mellon University
# MSCF Program
# 46-956 Introduction to Fixed Income

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Mini 1

Lecture Notes for Week 2


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# Discount Factors & Bond Prices

Discount Factor for time $t$:

$$d(t)$$

Price at time 0 in order to receive \$1 at time $t$ (no risk of default).

Zero Coupon Bond: Single payment of amount $F$ at maturity $T$. The price is given by

$$P = Fd(T).$$

Annuity: Makes equal payments of amount $A$, $m$ times per year until maturity $T$. Assuming that the annuity has just been issued (or that a payment has just been made), the price is given by

$$P = A \sum_{i=1}^{mT} d\left(\frac{i}{m}\right).$$

# Discount Factors & Bond Prices (Continued)

Coupon Bond: Coupon payments of amount $F\frac{q}{2}$ every 6 months prior to maturity plus a payment of $F(1+\frac{q}{2})$ at maturity $T$. $q$ is the annual coupon rate. The price is given by

$$P = Fd(T) + F\frac{q}{2}\sum_{i=1}^{2T} d\left(\frac{i}{2}\right),$$

assuming that the bond has just been issued (or that a coupon has just been paid).

# Discount Factors & Interest Rates

Discount factors incorporate a rate per unit time for growth of capital as well as a length of time. (Intrinsic, but not so intuitive.)

Interest rates just describe a rate of growth per unit time, but we must worry about various conventions:

$\hat{r}(t)$ – $t$-year spot rate (semiannual compounding)

$\hat{R}(t)$ – $t$-year effective spot rate

$$d(t) = \frac{1}{\left(1 + \frac{\hat{r}(t)}{2}\right)^{2t}} = \frac{1}{\left(1 + \hat{R}(t)\right)^{t}}.$$

The spot rates $\hat{r}(t)$ and $\hat{R}(t)$ apply to loans where the money is received at time 0 and repaid in a single lump sum at time $t$. These rates are known at time 0.

# Forward Rates

$f(t)$ – forward rate (agreed upon at time 0) for loans or investments between time $t - .5$ and time $t$

**Remark 2.1**: The forward rates $f(t)$ are known at time 0. The actual spot rates that will prevail at future dates are not known at time 0.

**Remark 2.2**: The forward rates $f(t)$ can be expressed in terms of discount factors through the formula

$$f(t) = 2\left[\frac{d(t-.5)}{d(t)} - 1\right].$$

We observed last time that

$$(1) \quad \left(1 + \frac{\hat{r}(t)}{2}\right)^{2t} = \left(1 + \frac{f(.5)}{2}\right)\left(1 + \frac{f(1)}{2}\right)\cdots\left(1 + \frac{f(t)}{2}\right).$$

**Remark 2.3**: The previous formula indicates that the spot rate $\hat{r}(t)$ is some kind of geometric average of the forward rates $f(.5), f(1), f(1.5), \cdots, f(t)$. In fact, the spot rate $\hat{r}(t)$ is usually very close numerically to the arithmetic average of the forward rates $f(.5), f(1), f(1.5), \cdots, f(t)$, i.e.

$$\hat{r}\left(\frac{N}{2}\right) \approx \frac{1}{N}\sum_{i=1}^{N} f\left(\frac{i}{2}\right).$$

Here $N$ is a positive integer.

**Example 2.2**: In Example 1.3, we had

$$f(.5) = .05065,\ \ f(1) = .05197,\ \ f(1.5) = .06026, \hat{r}(1.5) = .05429.$$

Observe that

$$\frac{1}{3}[f(.5) + f(1) + f(1.5)] = .05429.$$

# Justification of Remark 2.3

Let $N$ be a positive integer and put $t = \frac{N}{2}$. Let

$$x_i = f\left(\frac{i}{2}\right), \quad i = 1, 2, \cdots, N.$$

Solving (1) for $\hat{r}\left(\frac{N}{2}\right)$ we see that

$$\hat{r}\left(\frac{N}{2}\right) = g(x_1, x_2, \cdots, x_N),$$

where

$$g(\vec{x}) = 2\left[\left(1 + \frac{x_1}{2}\right)\left(1 + \frac{x_2}{2}\right)\cdots\left(1 + \frac{x_N}{2}\right)\right]^{\frac{1}{N}} - 2.$$

Observe that

$$\frac{\partial g}{\partial x_i} = \frac{1}{N}\left[\left(1 + \frac{x_1}{2}\right)\cdots\left(1 + \frac{x_N}{2}\right)\right]^{\frac{1}{N}-1} h_i(\vec{x}),$$

where

$$h_i(\vec{x}) = \left(1 + \frac{x_1}{2}\right) \cdots \left(1 + \frac{x_{i-1}}{2}\right) \left(1 + \frac{x_{i+1}}{2}\right) \cdots \left(1 + \frac{x_N}{2}\right).$$

In particular, we have

$$\vec{\nabla} g(\vec{0}) = \frac{1}{N}(1, 1, \cdots, 1).$$

Since $g(\vec{0}) = 0$, the linear approximation for $g(\vec{x})$ about $\vec{0}$ is

$$g(\vec{x}) \approx \vec{\nabla} g(\vec{0}) \cdot \vec{x} = \frac{1}{N} \sum_{i=1}^{N} x_i.$$

# Forward Loans and Discount Factors

**Remark 2.4**: To compute payments for forward loans it is often simpler to use discount factors than interest rates. If we agree at $t = 0$ to borrow $A_\eta$ at time $\eta$ and repay the loan with a single payment of amount $A_T$ at time $T$ (with $T > \eta$), then we have

$$A_\eta d(\eta) - A_T d(T) = 0,$$

which gives

$$\frac{A_T}{A_\eta} = \frac{d(\eta)}{d(T)}.$$

Recall that nothing is paid initially to enter into a forward loan agreement.

# Bond Yields

*Coupon Yield*: $q$ the annual coupon rate

*Current Yield*:

$$q\frac{F}{P} \text{ where } P \text{ is the current price of the bond}$$

The coupon yield and the current yield are, of course, identical if the bond is trading at par. In this case they provide a reasonable measurement of the investment yield over the period between the present time and the bond's maturity. For bonds that are not trading at par, these measures of return can be quite misleading, because the face value (which will be paid at maturity) does not match the initial investment.

# Yield to Maturity

It is very useful to consider a bond's *yield to maturity* which is defined to be the single interest rate that when used to discount all of the bond's future payments produces the bond's current market price.

If all of the relevant spot rates had the same value $y$, then the discount factors would be given by

$$d(t) = \frac{1}{\left(1+\frac{y}{2}\right)^{2t}},$$

so that the price of the bond would be given by

$$P = F\frac{q}{2}\left[\frac{1}{\left(1+\frac{y}{2}\right)} + \frac{1}{\left(1+\frac{y}{2}\right)^2} + \cdots + \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right] + \frac{F}{\left(1+\frac{y}{2}\right)^{2T}}.$$

We can work this process in reverse and ask which value of $y$ makes the above equation correct. (Assuming that all payments are nonnegative, it can be shown that there is exactly one such $y$.) This value of $y$ is called the *yield to maturity* of the bond. In general it must be computed numerically by solving the equation

$$P = F\frac{q}{2}\left[\frac{1}{\left(1+\frac{y}{2}\right)} + \frac{1}{\left(1+\frac{y}{2}\right)^2} + \cdots + \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right] + \frac{F}{\left(1+\frac{y}{2}\right)^{2T}}$$

for $y$.

# Some Remarks on Yield to Maturity

Since the yield to maturity does not account for what an investor does with the coupon payments, several caveats are in order.

**Remark 2.5**: If two different bonds with the same maturity (but different coupon rates) have different yields to maturity, it is not necessarily the case that the security with the higher yield represents a "better investment".

**Remark 2.6**: Even if a bond is held until maturity, the yield to maturity does not necessarily represent "the return" on the investment. In fact, when cash flows arrive at multiple dates, it is not clear how to define a return on the investment unless some assumptions are made concerning what happens to the payments received prior to maturity.

# Remarks on Yield to Maturity (Continued)

**Remark 2.7**: However, it is true that if the yield to maturity remains unchanged between two successive coupon payments, then this common value represents the yield associated with holding the bond over that period of time.

**Remark 2.8**: It is theoretically possible to have a negative yield to maturity. This hardly ever occurs in practice in the US. Unless warned otherwise, you may assume that all bonds in this course have $y > 0$.

# Computing the Yield to Maturity

Let $\lambda = \frac{1}{1+\frac{y}{2}}$. Then we have

$$P = F\lambda^{2T} + F\frac{q}{2}\sum_{i=1}^{2T}\lambda^i. \tag{2}$$

Using the formula for summing a geometric series:

$$\sum_{i=1}^{N}\lambda^i = \frac{\lambda(1-\lambda^N)}{1-\lambda}, \quad \lambda \neq 1,$$

we find that

$$P = F\left(\lambda^{2T} + \frac{q}{2}\frac{\lambda(1-\lambda^{2T})}{1-\lambda}\right). \tag{3}$$

We solve equation (3) for $\lambda$ and recover $y$ via the formula

$$y = 2\left(\frac{1}{\lambda} - 1\right).$$

For reasonably small maturities, it is simpler to use (2) directly rather than (3). For $T \geq 1.5$, we will typically need to solve numerically for $\lambda$.

# An Important Result

The following proposition concerning prices, yields to maturity and coupon rates is of crucial importance.

**Proposition**: For coupon bonds, we have

(i) $P > F$ if and only if $q > y$.

(ii) $P = F$ if and only if $q = y$.

(iii) $P < F$ if and only if $q < y$.

**Remark 2.9**: In practice, prices of coupon bonds are often quoted by giving the yield to maturity.

**Proof**: To prove the proposition, we observe that

$$\frac{\lambda}{1-\lambda} = \frac{2}{y},$$

so that (3) becomes

$$P = F\left[\lambda^{2T} + \frac{q}{y}(1-\lambda^{2T})\right]. \tag{4}$$

The desired conclusions follow from (4). Here's how: Keeping $y > 0$ fixed, we can think of the right-hand side of (2) as a linear function in the variable $x = \frac{q}{y}$ with positive slope. When $x = 1$ this function takes the value $F$; for $x > 1$ the value is greater than $F$ and for $x < 1$ the value is less than $F$. $\square$

# Effective Yield to Maturity

It is sometimes useful to use the *effective yield to maturity* $Y$. If we have a security that will make payments $F_i > 0$ at each of the times $T_i > 0$ then the effective yield to maturity is the unique number $Y$ satisfying

$$P = \sum_{i=1}^{N} \frac{F_i}{(1+Y)^{T_i}},$$

where $P$ is the current price of the security.

**Remark 2.10**: If the payments $F_i$ can be both positive and negative then it can happen that there is no yield to maturity or there can be more than one. (Exactly one is still possible.) See Assignment 2. The term *internal rate of return (IRR)* is sometimes used in place of yield to maturity. I will use "yield to maturity" in situations with nonnegative payments, and use "IRR" in the rare situations when we allow the payments to change sign.

**Example 2.2**: Consider Bond #3 from Example 1.1. The maturity is 18 months, the coupon rate is $q = .1$ and the price (per 100 face) is 106.52. Since the bond is trading above par, we must have $y < .1$, by virtue of the proposition. To determine $y$ precisely, we consider first the equation

$$106.52 = 105\lambda^3 + 5\lambda^2 + 5\lambda.$$

Solving numerically (by bisection, for example) we find $\lambda = .973635$, which gives

$$y = 2\left(\frac{1}{.973635} - 1\right) = .054158.$$

We could also determine $y$ by solving

$$106.52 = 100\left[\lambda^3 + .05\left(\frac{\lambda - \lambda^4}{1 - \lambda}\right)\right].$$

# Some Important Facts

**Yield-To-Maturity and Spot Rates**: (i) The yield to maturity is alway between the lowest spot rate and the highest spot rate for times up to the maturity of the bond. (ii) If the term structure is flat (i.e. if the spot rates are constant), then the common spot rate is equal to the yield to maturity.

**Pull to Par**: As the time to maturity shortens, the prices of premium bonds fall to par and the prices of discount bonds rise to par. (To make this notion precise, we must use the *flat price*, which will be introduced shortly.)

# Annuities

Recall that an annuity is a security with fixed cash flows that pays the same amount, say $A$, at equally spaced times, until maturity $T$. Let us assume here that the payments will be made every 6 months, with the first payment to be made 6 months from today. (In practice, annuities frequently make monthly or quarterly payments.) If the current price of such an annuity is $P$, then the yield to maturity is defined to be the unique number $y$ satisfying the equation

$$P = \sum_{i=1}^{2T} \frac{A}{\left(1 + \frac{y}{2}\right)^{i}} = A\lambda \frac{(1 - \lambda^{2T})}{1 - \lambda},$$

Here, as before, we have put

$$\lambda = \frac{1}{1 + \frac{y}{2}}.$$

# Perpetuities

A *perpetuity* or *perpetual annuity* is an annuity that makes payments indefinitely. Although they are not actually used in practice, they are useful theoretical devices. If the yield to maturity $y$ of an annuity is constant and $T \to \infty$ we find

$$P \to A \frac{\lambda}{1-\lambda} = \frac{2A}{y},$$

so that

$$y = \frac{2A}{P}.$$

# Perpetuities (Cont.)

The formula for the price of a perpetuity in terms of the payment amount $A$ and the yield to maturity $y$ has a very natural financial interpretation. Since there is no repayment of principal, each payment $A$ represents 6 months interest on the amount $P$ at the annual rate $y$ (compounded semiannually). Therefore

$$A = P\frac{y}{2},$$

which easily gives

$$P = \frac{2A}{y}.$$

For perpetuities making payments $m$ times per year and having yield to maturity $y$ (expressed according to compounding $m$ times per year), the same logic gives

$$P = \frac{mA}{y}.$$

# Yield Curves

**Zero-Coupon Yield Curve**: Spot-rate curve

**Par-Coupon Yield Curve**: Yield to maturity of coupon bonds that are currently trading at par, plotted as a function of maturity

**Annuity Yield Curve**: Yield to maturity of annuities plotted as a function of maturity

# Yield to Maturity and Coupon Rate

Assuming that the bond has just been issued or that a coupon payment has just been made, the YTM of a coupon bond is the unique number $y > 0$ satisfying

$$P = F\lambda^{2T} + F\frac{q}{2}\sum_{i=1}^{2T}\lambda^{i},$$

where $\lambda = (1 + \frac{y}{2})^{-1}$. Using the formula for the price in terms of the discount factors, we have

$$Fd(T) + F\frac{q}{2}\sum_{i=1}^{2T} d\left(\frac{i}{2}\right) = F\lambda^{2T} + F\frac{q}{2}\sum_{i=1}^{2T}\lambda^{i}.$$

We can cancel out $F$ in the equation above, but not $q$. The yield to maturity of a coupon bond generally depends on the coupon rate $q$ and on the maturity $T$.

# The Coupon Effect

In general, there is no simple relationship between the coupon rate $q$ and the yield to maturity $y$ for coupon bonds. However, if the spot rates are monotonic (i.e., always increasing or always decreasing) we can order the yields of coupon bonds if we know the ordering of the coupon rates. More precisely, we have the following proposition.

# Coupon Effect (Cont.)

**Proposition**: Consider two coupon bonds having the same maturity $T$, but different coupon rates $q^{(2)} \geq q^{(1)}$. (Both bonds are assumed to pay coupons twice per year, with the next coupon payment at time $t = .5$.) Let $y^{(1)}$ and $y^{(2)}$ denote the yields to maturity of these bonds.

**(i)** If $\hat{r}(t) \geq \hat{r}(t - .5)$ for $t = 1, 1.5, 2, \cdots, T$, then $y^{(1)} \geq y^{(2)}$.

**(ii)** If $\hat{r}(t) \leq \hat{r}(t - .5)$ for $t = 1, 1.5, 2, \cdots, T$, then $y^{(2)} \geq y^{(1)}$.

(The proposition remains valid if all inequalities are replaced with strict inequalities.)

This result follows from a more general one that you will prove as part of Assignment 2.

# Par Coupon Bonds

**Remark 2.11**: For a given maturity $T$, the yield to maturity of a coupon bond depends on the coupon rate. Yields are often quoted for par coupon bonds. For such bonds, the coupon rate is given by

$$q = \frac{2(1 - d(T))}{\sum_{i=1}^{2T} d(\frac{i}{2})};$$

moreover, by the proposition, the coupon rate is the same as the yield to maturity $y$. (Be sure that you know how to derive this formula yourself.)

# Yield to Maturity for Annuities

For an annuity making payments (of amount $A$) twice per year, the yield to maturity is the unique number $y > 0$ satisfying

$$P = A \sum_{i=1}^{2T} \lambda^i,$$

where $\lambda = (1 + \frac{y}{2})^{-1}$. Using the formula for the price in terms of the discount factors, we find that

$$A \sum_{i=1}^{2T} d\left(\frac{i}{2}\right) = A \sum_{i=1}^{2T} \lambda^i.$$

We can, of course, cancel out $A$ and we see that the yield to maturity of an annuity is independent of the payment size. Of course, the yield maturity will generally depend on the maturity $T$.

# Yield to Maturity of a Zero Coupon Bond

For a ZCB with maturity $T$, the yield to maturity is simply the spot rate $\hat{r}(T)$:

$$P = Fd(T) = \frac{F}{\left(1 + \frac{\hat{r}(T)}{2}\right)^{2T}} = \frac{F}{\left(1 + \frac{y}{2}\right)^{2T}}.$$

This is, of course, consistent with simply putting $q = 0$ in the formulas for coupon bonds.

# Yield of a Perpetuity

Consider a perpetuity that pays the amount $A > 0$ at each of the times $t = .5, 1, 1.5, 2, \cdots$. Assuming that we have discount factors available for arbitrarily long maturities, the price of the perpetuity is given by

$$P = A \sum_{i=1}^{\infty} d\left(\frac{i}{2}\right).$$

We assume that the infinite series converges.

Recall that the yield to maturity of a perpetuity satisfies

$$P = \frac{2A}{y}.$$

It follows that

$$y = \frac{2}{\sum_{i=1}^{\infty} d\left(\frac{i}{2}\right)}.$$

# An Approximation for Annuity Yields

**Remark 2.12**: For annuities having very long (but finite) maturity $T$ and making payments twice per year, the approximation

$$y \approx \frac{2}{\sum_{i=1}^{2T} d\left(\frac{i}{2}\right)}.$$

is reasonable.

# Yield Curves

For US Treasury securities, there are three *yield curves* that we will look at:

- Zero-Coupon Yield Curve: Same as the spot-rate curve
- Annuity Yield Curve: Plot of the yield to maturity for annuities making payments every 6 months, versus maturity of the annuity (also called the *self-amortizing yield curve*)
- Par-Coupon Yield Curve: Plot of the yield to maturity for par coupon bonds versus maturity of the bond

Each one of these yield curves completely determines the other two.

The par-coupon yield is always between the zero-coupon yield and the annuity yield, but whether the zero-coupon yield is above or below the annuity yield depends on the shape of the zero-coupon yield curve. (The zero-coupon yield is always higher than the annuity yield if the entire zero-coupon yield curve is upward sloping.)

# Some Comments on Constructing Yield Curves

In practice, it is useful to have a "smooth" discount function $d(t)$ available for all $t$ between 0 and some maturity $T$. To construct such a function in practice, one obtains as many "reliable" data points as possible (from prices of liquidly traded instruments) and computes the implied spot rates. Some kind of interpolation is used to fill in spot rates for intermediate values of $t$. Then discount factors and forward rates are computed from the interpolated spot-rate curve. The discount factors are smoother than the spot rates, while the forward rates are more "jagged" than the spot rates. We will not carry out such an analysis in this course. It will be discussed in *Data Science Numerical Methods* and *Financial Computing IV*. (You can also interpolate the discount factors, and determine rates from the discount function.)

Yield curve data for US treasuries is available at

www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield

# Remarks on Yield Curves (Continued)

**Remark 2.13**: Rather than interpolating to fit the yield curve (or discount function) exactly at all observed points, it is sometimes better to use a "least-squares" type of approximation based on a family of curves whose shape is adapted to the term structures produced by certain types of models. A very important example of such a family of curves is the Nelson-Siegel family of curves with four parameters $a, b, c, \lambda$:

$$a + b\frac{1 - e^{-\lambda t}}{\lambda t} + c\left(\frac{1 - e^{-\lambda t}}{\lambda t} - e^{-\lambda t}\right),$$

for fitting continuously compounded yield curves. This topic will be discussed in *Data Science* and *Financial Computing IV*.

# Accrued Interest

**Question**: What happens when a bond trades in between coupon dates?

The "value" of a coupon bond must drop by the coupon payment amount $F\frac{q}{2}$ immediately after a coupon is paid. In order to quote prices that are "continuous" the price of a coupon bond is decomposed into

$$\text{“flat price”} + \text{“accrued interest”}.$$

It is the accrued interest that "jumps" at coupon payments.

**Remark 2.14**: The accrued interest should equal 0 just after a coupon is paid and should be equal to the full coupon amount $F\frac{q}{2}$ just before a coupon is paid.

# Accrued Interest (Continued)

The flat price plus the accrued interest is called the *full price* or *invoice price* of the bond. It is the full price that is determined by the market. The choice of a formula for the accrued interest is simply a market convention.

$$P^{full} = P^{flat} + AI \ = \ PV(\text{future payments}).$$

The full price is determined by the market, the accrued interest is defined by a market convention, and the *flat price* is defined to be the full price minus the accrued interest. The convention that has been adopted for accrued interest is based on the notion of *simple interest*.

**Remark 2.15**: The terms *clean price* and *dirty price* are sometimes used for the flat price and full price, respectively. It is the flat price that is quoted.

# Simple Interest

The notion of compounding does not suit every situation. In the *money market* in which investors borrow or lend for short terms (usually less than one year), an investor commits to a fixed term and interest is paid at the end of the term. The simple interest convention

$$Interest \;=\; (Principal) \times (Rate) \times (Time)$$

is typically used.

# Day Count Conventions

There is more than one convention for converting the time interval between two calendar dates into a fraction of a year. Such a convention is called a *day count convention*. For example, it is common practice with mortgages (or car loans) to treat each month as being $\frac{1}{12}$ of a year.

With simple interest, sometimes the convention *actual/360* is used and sometimes the convention *actual/actual* is used. Under actual/360, the interest earned in $n$ days is $rn/360$ times the principal, whereas the interest for $n$ days is $rn/365$ under the actual/actual convention (in a non-leap year). Here $r$ is the annual simple interest rate.

**Example 2.3**: Money markets $A$ and $B$ are both quoting an annual simple interest rate of $r = .05$ for investments of term one year. Market $A$ uses the actual/360 convention, whereas market $B$ uses the actual/actual convention.

(i) If \$50,000 is deposited for one year the interest is

$$\frac{50,000(.05)365}{360} = 2,534.72 \text{ in } A, \quad 50,000(.05) = 2,500 \text{ in } B,$$

a difference of \$34.72.

(ii) If \$500 million is deposited for one year, the difference between the two accounts is \$347,222.22.

# Accrued Interest

For a coupon bond the accrued interest is computed using a simple interest convention by the formula

$$AI = \eta F \frac{q}{2},$$

where $\eta$ is the fraction of a coupon period that elapses between the last coupon payment and settlement. For US treasury notes and bonds the actual/actual convention is used; more precisely accrued interest is computed according to the formula

$$AI = \frac{n}{N} F \frac{q}{2},$$

where $N$ is the actual number of days between the last coupon payment and the next coupon payment and $n$ is the number of days between the last coupon payment and settlement.

**Example 2.4**: Suppose that there are 182 days between the last coupon date and the next coupon date. Assume that the last coupon date was 77 days ago and that settlement occurs today. On a bond with $F = \$10,000$ and $q = 5.5\%$, the accrued interest is

$$AI = \frac{10,000(77)(.055)}{182(2)} = 116.35.$$

# Yield to Maturity Between Coupon Payments

Let $\tau$ denote the fraction of a coupon period between the present time and the next coupon payment (so that $\tau = 1 - \eta$) and let $k$ denote the number of additional coupon payments remaining (after the next coupon). The yield to maturity $y$ is determined by solving the equation

$$P^{full} = F\lambda^{k+\tau} + F\frac{q}{2}\sum_{i=0}^{k}\lambda^{\tau+i},$$

where $\lambda = (1 + \frac{y}{2})^{-1}$.

# YTM Between Coupon Payments (Continued)

**Remark 2.16**: There are $k+1$ future coupon payments. Each future payment is discounted by

$$d(t) = \frac{1}{(1+\frac{y}{2})^{2t}},$$

where $t$ is the time until the payment is received. (Here $t$ need not be a multiple of 6 months.)

# Current Yield Between Coupon Payments

The current yield is defined to be $q\frac{F}{P^{flat}}$ for bonds in between coupon payments.

**Remark 2.17**: Coupon payments are not always made on the "official coupon date" because of holidays. This may lead to complications that can be significant in interpreting the yield to maturity when coupon payments do not arrive exactly 6 months apart. (See pages 60-61 in the second edition of Tuckman for an interesting example.)

# Conventions for T-Bills

Prices of T-Bills are often quoted in terms of the so-called *discount yield* $y_d$. The formula is

$$P = 100\left[1 - \frac{ny_d}{360}\right],$$

where $P$ is the price per 100 face and $n$ is the number of days between settlement and maturity. Notice that

$$y_d = \frac{(100 - P)360}{100n}.$$

**Remark 2.18**: The definition of discount yield has two peculiar features:

(i) It divides the dollar gain by 100 rather than $P$; and

(ii) it assumes a 360-day year.

The *bond-equivalent yield* $y_{be}$ corrects these two defects. For $n \leq 182$, we have

$$y_{be} = \frac{(100 - P)365}{Pn}.$$

For $n > 182$, we have

$$P\left(1 + \frac{y_{be}}{2}\right) + \frac{y_{be}}{365}\left(n - \frac{365}{2}\right)\left(1 + \frac{y_{be}}{2}\right)P = 100.$$

The bond-equivalent yield is also called the *coupon-equivalent yield*.

**Remark 2.19**: There is a veritable "snake pit" of conventions concerning day counts, compounding, and yields. Software accounts properly for these in most cases, but you must be aware that subtle differences in conventions can lead to big dollar differences on large transactions. Be careful in practice!

**Remark 2.20**: Even those fixed income securities having no risk of default and whose payment dates and amounts are known with certainty at the time of issue face *interest rate risk*. For example, it can happen that the price of a zero coupon bond will drop below its original purchase price. Roughly speaking, as interest rates rise, the market prices of previously issued bonds will drop.

Financial news and changes in investors' views of the market leads to changes in bond prices and spot rates. The changes that can occur in the spot rate curve can be quite complex. In particular, it can happen that long-term rates and short-term rates move in opposite directions.

For certain applications, it may be possible to capture the essential features of changes in the spot rate curve by keeping track of one, or several, key *interest rate factors*. In other words, we think of changes in the spot rate curve as being driven by some (small) list of interest rate variables such as a long-term rate, a medium-term rate, and a short-term rate; or in very special situations by a parallel shift.

Clearly, we cannot retain all of the information about the spot-rate curve by monitoring just a few variables, but sometimes we can do extremely well.

## ONE-FACTOR MODELS

If we try to measure changes in the spot-rate curve by means of a single interest rate factor (variable) $y$.

**Trade-Off**: Very simple, but not always appropriate.

*The challenge of measuring price sensitivity lies in quantifying what is meant by changes in interest rates.*

# One Factor Models: Basic Approximations

Suppose that the price of a security is given by

$$P = f(y),$$

where $y$ is some interest rate factor (or variable).

**First-Order Approximation**

$$\Delta P \approx f'(y)\Delta y$$

**Second-Order Approximation**

$$\Delta P \approx f'(y)\Delta y + \frac{1}{2}f''(y)(\Delta y)^2$$

# A Quick look at ZCBs

The price of a zero coupon bond with face value $F$ and maturity $T$ is given by

$$P = f(y) = F \cdot (1 + \frac{y}{2})^{-2T},$$

where $y$ is the yield to maturity of the bond. (Recall that $y = \hat{r}(T)$.) Taking the derivative of $f$, we find that

$$\begin{aligned} f'(y) &= -2TF \cdot (1 + \tfrac{y}{2})^{-2T-1}(\tfrac{1}{2}) \\ &= -\left(\frac{T}{1+\frac{y}{2}}\right) F \cdot (1 + \tfrac{y}{2})^{-2T} \\ &= -\left(\frac{T}{1+\frac{y}{2}}\right) f(y). \end{aligned}$$

# ZCBs: First-Order Approximation

The first-order approximation gives

$$\Delta P = -\left(\frac{T}{1+\frac{y}{2}}\right) P \Delta y.$$

- The price decreases when "rates go up".
- Change in price is proportional to current price.
- Change in price is proportional to time to maturity.

**Dollar Change in Price**

$$\Delta P = -\left[\left(\frac{T}{1+\frac{y}{2}}\right) P\right] \Delta y$$

**Relative Change in Price**

$$\frac{\Delta P}{P} = -\left(\frac{T}{1+\frac{y}{2}}\right) \Delta y$$

**Back to the General Situation**

- Bond is not necessarily a zero.
- $y$ can be any single interest rate factor.

# Basis Points

Changes in interest rates are frequently measured in *basis points*. One basis point is equal to one one hundredth of a percentage points. In terms of decimals, one basis point equals .0001.

**Example 2.5**: Suppose that the two-year spot rate is 4.853%, i.e. $\hat{r}(2) = .04853$. If the rate increases by 2 basis points, then

$$\hat{r}(2) = 4.873\% = .04873.$$

# DV01

The abbreviation *DV01* stands for *dollar value of an '01*, i.e. the change in dollar value in price resulting from a change in the interest rate factor of one basis point. Sometimes other names are used for the same concept – however DV01 is by far the most common term.

The abbreviation *PVBP* stands for *price value of a basis point*. This means the same thing as DV01..

Since an increase in interest rates leads to a decrease in price for most of the basic fixed-income securities, a minus sign is included in the definition:

$$DV01 = -\frac{\Delta P}{10,000 \Delta y}.$$

**Remark 2.14**: DV01 is often quoted for some specified amount of face. Be careful to give units.

**Warning**: Occasionally the opposite sign convention is used. Be sure to check the sign convention if in doubt!

# First-Order Approximation and DV01

Recall that $P = f(y)$. The quantity $\Delta P/\Delta y$ represents the slope of a secant line to the graph of the price function $f$. If we have an analytical formula for $f(y)$ we can use techniques from calculus to compute the derivative $f'(y)$ which gives the slope of the tangent line to the graph of the price function. For small changes in $y$, the slope of the secant line is for all practical purposes the same as the slope of the tangent line. In such a case we can write

$$DV01 = -\frac{f'(y)}{10,000}.$$

# Yield-Based DV01

**Remark 2.15**: Most market participants use the term DV01 to mean yield-based DV01, i.e. DV01 computed when $y$ is the yield to maturity of the security. Of course, other interest rate variables could be used. If you have any doubt about what "interest rate variable" is being used, be sure to insist on clarification.

**Example 2.6**: Consider a European call option (with exercise date $T_E = 1$ year) struck at par on a coupon bond having maturity $T > 1$. Let $P$ denote the current price of the option (per \$100 face of the bond) and let $y$ denote the yield to maturity of the bond. Suppose that the yield of the bond is currently at $y = 4\%$ and we know that

$$P = 8.0866 \text{ when } y = 4.01\% \qquad P = 8.2148 \text{ when } y = 3.99\%.$$

We are interested in what happens as the yield of the bond moves away from 4%. We calculate

$$DV01 = -\frac{8.0866 - 8.2148}{10,000(.0401 - .0399)} = .0641.$$

This means that for every *increase* in rate of one basis point, we can expect the price of the option to *decrease* by approximately 6.41 cents for every 100 face of the bond.

This observation is certainly **not** rocket science – we should expect the change in price to be approximately proportional to the change in rate (for small changes in the interest rate factor). Given two prices for nearby values of $y$ we should be able to predict prices for other nearby values of $y$ by a linear interpolation of the two given values.

However, we can use DV01 to draw some very important conclusions. In particular, we gain some insight into how to hedge a short position on the option from Example 2.6.

**Example 2.7**: Consider the call option of Example 2.6. Suppose that we know the DV01 of the underlying bond (per \$100 face) as well:

| Rate Level | Option DV01 | Bond DV01 |
|---|---|---|
| 4.00% | .0641 | .0857 |

Suppose also that an trader is short \$100 million face of the call. How might she hedge the interest rate exposure by trading in the underlying bond?

The trader stands to lose money if rates fall, so bonds should be purchased. Let $F$ be the face amount of bond to be purchased. She should choose $F$ so that price change in the bond portfolio associated with a one basis point change in bond yield equals the price change in the option position. This leads to

$$F \times \frac{.0857}{100} = 100,000,000 \times \frac{.0641}{100},$$

or

$$F = 100,000,000 \times \frac{.0641}{.0857} = 74,795,799.30.$$

In other words, the trader should purchase \$74,795,799.30 face of the underlying bond.

**Remark 2.23**: (i) If the DV01s of securities $A$ and $B$ have the same sign, then hedging a short position in security $A$ requires a long position in security $B$.
(ii) If the DV01s of security $A$ and $B$ have opposite signs then hedging a short position in security $A$ requires a short position in security $B$.

# Duration

Another popular measure of interest-rate risk is the so-called *duration* which is defined by

$$D = -\frac{1}{P}\frac{\Delta P}{\Delta y}.$$

Observe that duration is simply $10,000 \times DV01/P$. Duration measures the *relative change* in price resulting from a change in $y$. In fact, duration gives the percentage decrease in price due to an increase in the interest rate factor of 100 percentage points.

When we apply sensitivity analysis in practice, the $\Delta y$ values should be very small.

Once again, if we have an analytical expression for $P = f(y)$ we use calculus to compute $f'(y)$ and write

$$D = -\frac{f'(y)}{P} = -\frac{1}{P}\frac{dP}{dy}.$$

Notice that

$$DV01 = \frac{PD}{10,000}.$$

**Remark 2.24**: We don't really need both DV01 and duration. However, practitioners use both. Each is more convenient for certain purposes.

# Convexity

The *convexity* of a bond measures the how the interest rate sensitivity changes when the interest rate factor changes. The mathematical definition is

$$C = \frac{f''(y)}{P} = \frac{1}{P}\frac{d^2P}{dy^2}.$$

This assumes that we have an explicit expression for the price as a function of $y$. Without such a formula, convexity must be approximated numerically.

# Carnegie Mellon University
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Mini 1

Lecture Notes for Week 3


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# Comment on Yield to Maturity

If the spot-rate curve is not flat, then the yield to maturity **cannot** be used to discount individual cash flows of a bond (except in the case of a ZCB where there is only one payment).

The yield to maturity **can always** be used to discount all of the bond's cash flows, add the results together, and obtain a price for the bond. In other words, the YTM applies to the bond *as a whole* rather than to *individual payments*.

# DV01, Duration, & Convexity (Continued)

Single interest rate factor (variable): $y$

Security with current price a function of the interest rate factor: $P = f(y)$

DV01: $DV01 = -\frac{f'(y)}{10{,}000}$

Duration: $D = -\frac{f'(y)}{P}$

Convexity: $C = \frac{f''(y)}{P}$

If we don't have an analytical expression giving the price in terms of $y$, we can use difference quotients instead of derivatives. In particular, we can use

$$DV01 = -\frac{\Delta P}{10,000 \Delta y},$$

$$D = -\frac{\Delta P}{P \Delta y}.$$

The reason for the minus sign in the definitions of DV01 and duration is that for traditional bonds with fixed cash flows, an increase in the interest rate factor leads to a decrease in price. With the sign convention as above, DV01 and duration will be positive for ZCBs, Coupon Bonds, and Annuities. For these securities, the interest rate factor is usually taken to be the yield to maturity.

# First- and Second-Order Approximations

**First-Order Approximation**

$$\frac{\Delta P}{P} = -D\Delta y$$

**Second-Order Approximation**

$$\frac{\Delta P}{P} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

**Remark 3.1**: Some authors define convexity as

$$\frac{f''(y)}{2P}.$$

In this case, there is no factor of $\frac{1}{2}$ in the second-order approximation.

**Remark 3.2**: It is very important to notice that

(i) Positive duration leads to price increases when rates fall and price decreases when rates rise.

(ii) Negative duration leads to price decreases when rates fall and price increases when rates rise.

(iii) Positive convexity makes a positive contribution to the price both when rates rise and when they fall.

(iv) Negative convexity makes a negative contribution to the price both when rates rise and when they fall.

- Positive Convexity – Long on Volatility
- Negative Convexity – Short on Volatility

# Portfolios of Fixed-Income Securities

Although DV01 is frequently quoted per some amount of face, here we shall regard prices as *dollar prices* and the DV01 of a portfolio will be based on the total price of the portfolio (rather than the price per some amount of face). Since DV01 reflects total price change, it scales with the amount invested. Since duration represents a relative price change, it does not scale with the amount invested.

In other words, suppose that two investors $A$ and $B$ each invest all of their capital in one and the same bond and that investor $A$ invests twice as much as investor $B$. Investor $A$'s DV01 will be twice as much as investors $B$'s DV01. However, their durations will be the same (as will their convexities).

Suppose that a portfolio is being built using fixed-income securities $S^{(1)}, S^{(2)}), \cdots, S^{(N)}$. Here we think of each of the securities as having dollar prices $P^{(1)}, P^{(2)}, \cdots, P^{(N)}$ (not expressed as a percentage of face). An investor builds a portfolio with total initial capital $X > 0$. The investor buys $\alpha^{(i)}$ shares of $S^{(i)}$, so that

$$X = \alpha^{(1)} P^{(1)} + \alpha^{(2)} P^{(2)} + \cdots + \alpha^{(N)} P^{(N)}.$$

(Some of the $\alpha^{(i)}$ can be negative here.)

For the entire portfolio we have

$$DV01 = \sum_{i=1}^{N} \alpha^{(i)} DV01^{(i)},$$

$$D = \sum_{i=1}^{N} \left( \frac{\alpha^{(i)} P^{(i)}}{X} \right) D^{(i)},$$

$$C = \sum_{i=1}^{N} \left( \frac{\alpha^{(i)} P^{(i)}}{X} \right) C^{(i)},$$

where $DV01^{(i)}$, $D^{(i)}$, and $C^{(i)}$ are the DV01, duration, and convexity of the $i^{th}$ security.

**Remark 3.3**: The duration and convexity of each piece is weighted by the percentage of capital associated with that piece..

**Remark 3.4**: In this class I will use the terminology *duration* and *convexity* for portfolios *only* when the net position is *long*, i.e. for portfolios with a positive price. The meanings of duration and convexity can become confusing for a portfolio with negative price. Some of the statements in the book make sense only for portfolios with a positive price. If we want to measure the interest-rate exposure of a portfolio with a negative (or zero) net price, we can use duration and convexity to analyze components of the portfolio and combine the results. Talking about the DV01 of a portfolio with zero or negative net price is perfectly ok.

# Hedging Based on Duration and Convexity

**Remark 3.5**: If two portfolios have the same price and the same duration, then for small changes in $y$, the price changes in the two portfolios will be approximately the same (both in magnitude and sign), i.e. they have the same DV01s. This very simple idea is frequently used to construct hedges. A better hedge might be obtained by matching prices, durations, and convexities.

**Remark 3.6**: Given any three numbers $P, \delta, \gamma$ with $P > 0$ and any three maturities $T_3 > T_2 > T_1 > 0$, it is possible to construct a portfolio consisting only of ZCBs with maturities $T_1$, $T_2$, $T_3$ such that the portfolio has price $P$, duration $\delta$ and convexity $\gamma$. You will be asked to prove this result for homework.

# Yield Based DV01 for Coupon Bonds

$$P = f(y) = F\left[(1+\frac{y}{2})^{-2T} + \frac{q}{2}\sum_{i=1}^{2T}(1+\frac{y}{2})^{-i}\right]$$

$$DV01 = -\frac{f'(y)}{10,000}$$

$$DV01 = \frac{F}{10,000} \times \frac{1}{1+\frac{y}{2}}\left[\frac{T}{(1+\frac{y}{2})^{2T}} + \frac{q}{2}\sum_{i=1}^{2T}\frac{\frac{i}{2}}{(1+\frac{y}{2})^{i}}\right]$$

**Remark 3.7**: A more compact expression for the DV01 can be obtained by summing the geometric series in the formula for $P$ to obtain

$$P = f(y) = F\left[(1+\frac{y}{2})^{-2T} + \frac{q}{y}(1-(1+\frac{y}{2})^{-2T})\right],$$

and then differentiating to obtain

$$DV01 = \frac{F}{10,000}\left[\frac{q}{y^2}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right) + \left(1-\frac{q}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right].$$

# Macaulay Duration

For securities with deterministic cash flows, a notion of duration based on yield to maturity was introduced by F.R. Macaulay in 1938. It is closely related to (but not identical to) the duration obtained by differentiating the price with respect to the yield.

Let $T_i = i/2$, for $i = 1, 2, \cdots, 2T$ and consider a security (with maturity $T$) that makes payments $F_i \geq 0$ at the times $T_i$ and has yield to maturity $y$, so that the price is given by

$$P = \sum_{i=1}^{2T} \frac{F_i}{(1 + \frac{y}{2})^i}.$$

Observe that

$$D = -\frac{f'(y)}{P} = \frac{1}{P\left(1 + \frac{y}{2}\right)} \sum_{i=1}^{2T} \frac{\frac{i}{2} F_i}{(1 + \frac{y}{2})^i}.$$

# Macaulay Duration (Continued)

The Macaulay duration is defined by

$$D_{Mac} = \frac{1}{P}\sum_{i=1}^{2T}\frac{\frac{i}{2}F_i}{(1+\frac{y}{2})^i} = \frac{1}{P}\sum_{i=1}^{2T}\frac{T_i F_i}{(1+\frac{y}{2})^i}.$$

The Macaulay duration is simply a weighted average of the payment times with weights being the present values of the payment amounts with discount factors computed using the yield to maturity.

Observe that $D_{Mac}$, can be expressed in terms of duration $D$ via the formula

$$D_{Mac} = \left(1+\frac{y}{2}\right)D.$$

# Macaulay Duration (Continued)

The numerical value of the Macaulay duration is fairly close to the numerical value of the duration. The Macaulay duration has the very nice property that it is exactly equal to the maturity for a zero coupon bond.

Notice that in terms of $D_{Mac}$, the first-order approximation becomes

$$\frac{\Delta P}{P} = -\frac{D_{Mac}}{(1+\frac{y}{2})}\Delta y.$$

# Duration for Coupon Bonds

Unless stated otherwise, when we talk about the duration of a coupon bond, it is understood that the interest rate factor is the yield to maturity. It follows from the expressions for DV01 that

$$D = \frac{F}{P} \times \frac{1}{1+\frac{y}{2}} \left[ \frac{T}{(1+\frac{y}{2})^{2T}} + \frac{q}{2} \sum_{i=1}^{2T} \frac{\frac{i}{2}}{(1+\frac{y}{2})^{i}} \right],$$

and

$$D = \frac{F}{P} \left[ \frac{q}{y^2} \left( 1 - \frac{1}{(1+\frac{y}{2})^{2T}} \right) + \left( 1 - \frac{q}{y} \right) \frac{T}{(1+\frac{y}{2})^{2T+1}} \right].$$

# Duration for Coupon Bonds (Continued)

For coupon bonds, the Macaulay duration is given by

$$D_{Mac} = \frac{F}{P}\left[\frac{T}{(1+\frac{y}{2})^{2T}} + \frac{q}{2}\sum_{i=1}^{2T}\frac{\frac{i}{2}}{(1+\frac{y}{2})^{i}}\right],$$

and

$$D_{Mac} = \frac{F}{P}\left(1+\frac{y}{2}\right)\left[\frac{q}{y^2}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right) + \left(1-\frac{q}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right].$$

**Example 3.1**: Consider a coupon bond with $q = .04$ and $T = 20$, and yield to maturity $y = .04231$. Let us find the duration and Macaulay duration. Using the relation between price and yield, we find that with $\lambda = (1 + .04231/2)^{-1}$:

$$P = 100\lambda^{40} + 2\frac{\lambda}{1-\lambda}(1 - \lambda^{40}) = 96.90351$$

per 100 face. Substituting into the formulas, we find that

$$D = 13.555$$

$$D_{Mac} = \left(1 + \frac{.04231}{2}\right) 13.555 = 13.842$$

**Example 3.2**: Consider a coupon bond with $F = 100,000$, $q = .04$, $T = 20$, and yield to maturity $y = .04231$. Find the DV01 of the bond and use the DV01 to estimate the price of the bond if $y$ is decreased by 17 basis points.

From Example 3.1, we know that $P = 96,903.51$ and $D = 13.555$. It follows that

$$DV01 = \frac{PD}{10,000} = \frac{96,903.51(13.555)}{10,000} = 131.353.$$

If the yield decreases by 17 basis points the price of the bond will increase by approximately

$$131.353(17) = 2,233.00.$$

If we use the exact relation between price and yield, we find that when $y = .04061$ (a decrease of 17 bp), the price of the bond is 99,170.10. The exact change in bond price is an increase of 2266.59.

# Duration and DV01 for Par Coupon Bonds

For par coupon bonds, we have

$$\frac{F}{P} = 1, \quad \frac{q}{y^2} = \frac{1}{y}, \quad \left(1 - \frac{q}{y}\right) = 0,$$

and the formulas for duration and DV01 simplify considerably:

$$D = \frac{1}{y}\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right],$$

$$D_{Mac} = \left(\frac{1}{y} + \frac{1}{2}\right)\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right],$$

$$DV01 = \frac{F}{10,000y}\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right].$$

**Example 3.3**: The current value of the 10-year par-coupon yield is approximately $y = 2.98\%$. The duration of the 10-year par bond is therefore

$$D = \frac{1}{.0298}\left[1 - \frac{1}{(1+\frac{.0298}{2})^{20}}\right] \approx 8.593.$$

This means that the DV01 of the 10-year par bond is approximately $8.593 per $10,000 face (i.e. 8.593 cents per $100 face)

**Remark**: With a value of $2.17\%$ (September 12, 2017) for the 10-year par-coupon yield, we would get $D = 8.946$. With a yield of $3.59\%$ (April 4, 2011), we would get $D = 8.33979$. With a yield of $6.01\%$ (July 17, 2000) we would get $D = 7.43530$. With a yield of $1\%$, we would get $D = 9.49371$.

# Annuities

For an annuity that pays the same fixed amount every 6 months for $T$ years, it can be shown that

$$D = \frac{1}{y} - \left(\frac{1}{1+\frac{y}{2}}\right)\left[\frac{T}{(1+\frac{y}{2})^{2T}-1}\right],$$

$$D_{Mac} = \frac{1}{y} + \frac{1}{2} - \frac{T}{(1+\frac{y}{2})^{2T}-1}.$$

(These formulas do not seem to be in Tuckman & Serrat.)

# Perpetuities

Recall that a perpetuity pays the same amount $A$ every 6 months indefinitely. Recall also that for a perpetuity, we have

$$P = \frac{2A}{y},$$

from which we compute

$$\frac{dP}{dy} = -\frac{2A}{y^2}.$$

It follows that

$$D = \frac{1}{y}, \quad D_{Mac} = \frac{1}{y} + \frac{1}{2}.$$

# Example 3.4

Assume that the spot-rate curve is flat at .08. Portfolio A holds a par-coupon bond with price $P^{(1)} = \$500,000$ and maturity $T^{(1)} = 10$ and a ZCB with price $P^{(2)} = \$1,000,000$ and maturity $T^{(2)} = 20$. Let us find the Macaulay duration and DV01 for this portfolio. Observe that

$$D_{Mac}^{(1)} = 1.04 \left[ \frac{1}{.08} \left( 1 - \frac{1}{(1.04)^{20}} \right) \right] = 7.06697,$$

and that

$$D_{Mac}^{(2)} = 20.$$

# Example 3.4 (Continued)

It follows that

$$D_{Mac}^{(A)} = \frac{500,000(7.06697) + 1,000,000(20)}{1,500,000} = 15.689,$$

and

$$\begin{aligned} DV01^{(A)} &= \tfrac{1,500,000}{10,000} D^{(A)} \\ &= \tfrac{1,500,000}{10,000} \left(\tfrac{15.689}{1.04}\right) = 2,262.84. \end{aligned}$$

# Example 3.4 (Continued)

Portfolio B is to consist of a single par-coupon bond having maturity $T^{(3)} = 15$ and face value $F^{(3)}$. Find a value for $F^{(3)}$ so that for small parallel shifts in the spot rate curve, the price change in Portfolio A will be the same as the price change in Portfolio B (to a first-order approximation).
Observe that

$$DV01^{(B)} = \frac{F^{(3)}}{10,000}\left[\frac{1}{.08}\left(1 - \frac{1}{(1.04)^{30}}\right)\right].$$

Setting $DV01^{(A)} = DV01^{(B)}$ and solving for $F^{(3)}$, we find that

$$F^{(3)} = 2,617,200.$$

# Example 3.4 (Continued)

By approximately how much will the price of Portfolio A change if the there is a parallel shift downward of 24 basis points in the spot rate curve?

$$\Delta P \approx (2,262.84)24 = 54,308.$$

Notice that price should increase.

# Convexity for Coupon Bonds

Of course, it would be possible to introduce several different types of convexity. We shall not do so here. We shall make the convention that when we talk about convexity for securities with deterministic cash flows, that we mean convexity computed by differentiating the price twice with respect to the yield and dividing the result by the price.

# Convexity for Coupon Bonds (Continued)

Let $y$ denote the yield to maturity. Differentiating earlier expressions for $f'(y)$, we get

$$C = \frac{F}{P(1+\frac{y}{2})^2}\left[\frac{T(T+.5)}{(1+\frac{y}{2})^{2T}} + \frac{q}{2}\sum_{i=1}^{2T}\frac{i(i+1)}{4(1+\frac{y}{2})^i}\right].$$

$$\begin{aligned} C &= \frac{F}{P}\left[\frac{2q}{y^3}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right) - \frac{2q}{y^2}\frac{T}{(1+\frac{y}{2})^{2T+1}}\right. \\ &\qquad \left. + \left(1-\frac{q}{y}\right)\frac{T^2+.5T}{(1+\frac{y}{2})^{2T+2}}\right] \end{aligned}$$

# Dependence of DV01, Duration, and Convexity on Maturity and Yield for ZCBs

$$P = f(y) = F(1 + \frac{y}{2})^{-2T}$$

$$f'(y) = -TF \cdot \left(1 + \frac{y}{2}\right)^{-2T-1} = -\left(\frac{T}{1 + \frac{y}{2}}\right) f(y)$$

# ZCBs and Yield-Based Sensitivity (Continued)

$$
\begin{aligned}
f''(y) &= -\left(\frac{T}{1+\frac{y}{2}}\right) f'(y) + \frac{T}{2}\left(1+\frac{y}{2}\right)^{-2} f(y) \\
&= \left(\frac{T}{1+\frac{y}{2}}\right)^2 f(y) + \frac{1}{2}\frac{T}{(1+\frac{y}{2})^2} f(y) \\
&= \left[\frac{T^2+.5T}{(1+\frac{y}{2})^2}\right] f(y).
\end{aligned}
$$

# ZCBs and Yield-Based Sensitivity (Continued)

$$DV01 = \frac{1}{10,000}\left[\frac{FT}{(1+\frac{y}{2})^{2T+1}}\right]$$

$$D = \frac{T}{1+\frac{y}{2}}$$

$$D_{Mac} = T$$

$$C = \frac{T(T+\frac{1}{2})}{(1+\frac{y}{2})^2}$$

# Zero Coupon Bonds: DV01

Using Calculus, it is straightforward to show that (when the yield is held fixed at some value $y > 0$) the DV01 of a ZCB increases with increasing maturity until $T$ reaches the critical value

$$T = \frac{1}{2\ln\left(1 + \frac{y}{2}\right)} \approx \frac{1}{y},$$

and decreases for $T$ larger than this value, tending to zero as $T \to \infty$.

The phenomenon of DV01 decreasing as maturity increases for a ZCB can certainly occur in the actual markets.

# Zero Coupon Bonds: Duration and Convexity

- The Macaulay duration of a ZCB always equals the maturity.
- For fixed $y > 0$, the convexity of a ZCB increases (quadratically) with increasing $T$.
- For fixed $T > 0$, the convexity of a ZCB decreases with increasing $y$.

# Coupon Bonds

$$DV01 = \frac{F}{10,000}\left[\frac{q}{y^2}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right)+\left(1-\frac{q}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right]$$

$$D_{Mac} = \frac{F}{P}\left[\frac{T}{(1+\frac{y}{2})^{2T}}+\frac{q}{2}\sum_{i=1}^{2T}\frac{\frac{i}{2}}{(1+\frac{y}{2})^{i}}\right]$$

$$D_{Mac} = \frac{F}{P}\times\left(1+\frac{y}{2}\right)\left[\frac{q}{y^2}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right)+\left(1-\frac{q}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right]$$

# Limiting Duration for Coupon Bonds as Maturity Increases

Using the explicit formula for $D_{Mac}$ in terms of $y$ for a coupon bond, it can be shown that for fixed $q, y > 0$, we have

$$D_{Mac} \to \frac{1}{2} + \frac{1}{y} \text{ as } T \to \infty.$$

# Par Coupon Bonds

$$DV01 = \frac{F}{10,000y}\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right]$$

$$D = \frac{1}{y}\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right]$$

$$D_{Mac} = \frac{1+\frac{y}{2}}{y}\left[1 - \frac{1}{(1+\frac{y}{2})^{2T}}\right]$$

# Perpetuities

Consider a perpetuity that pays $A$ twice per year.

$$DV01 = \frac{2A}{10,000y^2}$$

$$D = \frac{1}{y}$$

$$D_{Mac} = \frac{1}{2} + \frac{1}{y}$$

$$C = \frac{2}{y^2}$$

# Rules of Thumb for Duration, DV01, and Convexity for Coupon Bonds

**Remark 3.8**: There are a number of useful "rules of thumb" for duration, DV01, and convexity for coupon bonds. However, some caution must be exercised in applying these rules in practice because, unless the yield curve is flat, changing the coupon rate or maturity of a bond will generally change the yield, so it may not be realistic to think of changing bond parameters "one at a time". Moreover, some of these rules (such as *duration increases with maturity*) are valid under typical market conditions, but could be violated under *extreme conditions*.

# Duration: Rules of Thumb for Coupon Bonds

Roughly speaking:

- Duration increases with increasing maturity. (Could be violated under extreme conditions.)
- Duration decreases with increasing coupon rate.
- Duration decreases with increasing yield to maturity.

To state these rules more precisely: For $q, y, T > 0$, let $D_{Mac}(q, y, T)$ denote the Macaulay duration of a bond with coupon rate $q$, yield to maturity $y$, and maturity $T$. Then we have the following result.

# Duration: Rules of Thumb (Continued)

**Proposition**:

(i) For fixed $q, y > 0$ $\lim_{T \to \infty} D_{Mac}(q, y, T) = \frac{1}{2} + \frac{1}{y}$.

(ii) For fixed $q, y > 0$ with $q \geq y$, $D_{Mac}(q, y, T)$ increases with increasing $T$.

(iii) For fixed $q, y > 0$ with $q < y$, there is a critical maturity $T^*$ (depending on $q$ and $y$) such that $D_{Mac}(q, y, T)$ increases with $T$ until $T$ reaches $T^*$ and then decrease as $T$ is increased beyond $T^*$. (The value of $T^*$ is typically much larger than the maturities of traded bonds.)

(iv) For fixed $y, T > 0$, $D_{Mac}(q, y, T)$ decreases with increasing $q$.

(v) For fixed $q, T > 0$, $D_{Mac}(q, y, T)$ decreases with increasing $y$.

# Duration:Rules of Thumb (Continued)

**Remark 3.9**: The phenomenon in item (iii) of the proposition is mostly a "mathematical curiosity" because for typical $q$ and $y$ the corresponding values of $T^*$ are very large. There is no simple closed-form analytical expression for the "critical maturity" $T^*$. Pianca (2006) gives a formula for $T^*$ in terms of the so-called Lambert function. I do not know of any actual bonds that have had their value of $T^*$ be less than or equal to their maturity. However, it could conceivably happen. With $y = .05$ and $q = .01$, $T^* \approx 53$ years. However, with $y = .10$ and $q = .02$, $T^* \approx 27$.

# DV01: Rules of Thumb for Coupon Bonds

Roughly speaking:

- DV01 increases with increasing maturity (exceptions can occur for discount bonds with large maturities).
- DV01 increases with increasing coupon.
- DV01 decreases with increasing yield.

To state these rules a bit more carefully: For $F, q, y, T > 0$, let $DV01(F, q, y, T)$ denote the DV01 of a coupon bond with face value $F$, coupon rate $q$, yield to maturity $y$ and maturity $T$. Then we have the following result.

# DV01: Rules of Thumb (Continued)

**Proposition**:

(i) For fixed $F, q, y > 0$, $\lim_{T \to \infty} DV01(F, q, y, T) = \frac{Fq}{10{,}000y^2}$.

(ii) For fixed $F, q, y > 0$ with $q \geq y$, $DV01(F, q, y, T)$ increases with increasing $T$.

(iii) For fixed $F, q, y > 0$ with $q < y$, there is a critical maturity $T^{**}$ such $DV01(F, q, y, T)$ increases with $T$ until $T$ reaches $T^{**}$ and then decreases for $T$ larger than $T^{**}$.

(iv) For fixed $F, y, T > 0$, $DV01(F, q, y, T)$ increases with increasing $q$.

(v) For fixed $F, q, T > 0$, $DV01(F, q, y, T)$ decreases with increasing $y$.

# Convexity for Coupon Bonds: Rules of Thumb

Roughly speaking:

- Convexity increases with increasing maturity. (This can be violated for deeply discounted bonds.)
- Convexity decreases with increasing coupon.
- Convexity decreases with increasing yield to maturity.

# Barbells Versus Bullets

In the asset-liability context, *barbelling* refers to the use of a portfolio of short-term and long-term bonds rather than intermediate term bonds.

Suppose that an asset manager has a portfolio of liabilities with Macaulay duration 9 years. The proceeds gained from those liabilities could be used to purchase several assets with duration 9 years, or alternatively, to purchase, say 2-year and 30-year securities that as a portfolio have a duration of 9 years. Let's look at an example.

**Example 3.5**: Let us assume that the spot-rate curve is flat at 6%. For a 9-year ZCB (a *bullet*) we have

$$D_{Mac} = 9, \quad C = \frac{9(9.5)}{(1+\frac{.06}{2})^2} = 80.59195.$$

For a portfolio with 75% of its value in 2-year zeros and 25% of its value in 30-year zeros we have

$$D_{Mac}^{Barbell} = 9, \quad C^{Barbell} = \frac{.75(2)(2.5)}{(1+\frac{.06}{2})^2} + .\frac{.25(30)(30.5)}{(1+\frac{.06}{2})^2} = 219.15355.$$

Notice that the convexity of the barbell is substantially greater than the convexity of the bullet.

**Question**: What does this mean? **Answer**: Assuming parallel

shifts, if the yield moves away from 6%, the price of the barbell portfolio will be above the price of the bullet portfolio, whether rates rise or fall.

Let's look at some numbers (assuming 100 is invested in each portfolio) and that there is a parallel shift in the yield curve.

## Exact Price Changes

| **Shift** | **Bullet $\Delta P$** | **Barbell $\Delta P$** |
|---|---|---|
| 10 bp | -.8697698 | -.8629362 |
| 50 bp | -4.269802 | -4.108030 |
| -10 bp | .8778291 | .8848532 |
| -50 bp | 4.471323 | 4.656949 |

## $1^{st}$ Order Approximations

| **Shift** | **Bullet $\Delta P$** | **Barbell $\Delta P$** |
|---|---|---|
| 10 bp | -.8737864 | -.8737864 |
| 50 bp | -4.368932 | -4.368932 |
| -10 bp | .8737864 | .8737864 |
| -50 bp | 4.368932 | 4.368932 |

## Exact Price Changes

| Shift | Bullet $\Delta P$ | Barbell $\Delta P$ |
|---|---|---|
| 10 bp | -.8697698 | -.8629362 |
| 50 bp | -4.269802 | -4.108030 |
| -10 bp | .8778291 | .8848532 |
| -50 bp | 4.471323 | 4.656949 |

## $2^{nd}$ Order Approximations

| Shift | Bullet $\Delta P$ | Barbell $\Delta P$ |
|---|---|---|
| 10 bp | -.869756 | -.8628288 |
| 50 bp | -4.268192 | -4.09499 |
| -10 bp | .877816 | .8847441 |
| -50 bp | 4.46967 | 4.6428740 |

**Question**: Does this mean that there are real arbitrage opportunities trading with bullets and barbells in practice?

**Answer**: No. In the real world, the term structure will not be flat, so the barbell and bullet will have different yields. Also, we cannot be sure that rates will move in parallel. As we shall see on Assignment 4, an assumption of parallel shifts in the yield curve leads to "difficulties" when the term structure is flat.

# A Nonparallel Shift

Let's look at a nonparallel shift. Suppose that $\hat{r}(2) = .062$, $\hat{r}(9) = .064$, and $\hat{r}(30) = .066$. Then we have

$$P^{bullet} = 96.5685 \qquad P^{barbell} = 95.7063$$

In this case the bullet outperforms the barbell.

# General Barbell vs Bullet (flat yield curve)

For fixed $y > 0$ define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = \frac{t(t + .5)}{(1 + \frac{y}{2})^2}, \quad t \in \mathbb{R},$$

and assume that the term structure is flat at $y$. Notice that $g$ is a convex function because $g''(t) \geq 0$. Let $T_1, T_2$ with $0 < T_1 < T_2$ be given. Let $\alpha$ with $0 < \alpha < 1$ be given. Suppose that we build a barbell portfolio by investing $\alpha$ in a zero with maturity $T_1$ and $1 - \alpha$ in a zero with maturity $T_2$.

# General Barbell Versus Bullet (Cont.)

The duration and convexity of the barbell are given by

$$D^{barbell} = \frac{\alpha T_1 + (1-\alpha) T_2}{1 + \frac{y}{2}},$$

$$D^{barbell}_{Mac} = \alpha T_1 + (1-\alpha) T_2,$$

$$C^{barbell} = \alpha g(T_1) + (1-\alpha) g(T_2).$$

The bullet portfolio having the same price (namely 1) and the same duration is obtained by investing \$1 in a zero with maturity

$$T = \alpha T_1 + (1-\alpha) T_2.$$

The convexity of the bullet portfolio is

$$C^{bullet} = g(T) = g(\alpha T_1 + (1-\alpha) T_2).$$

By the definition of a convex function, we have

$$g(\alpha T_1 + (1-\alpha)T_2) \leq \alpha g(T_1) + (1-\alpha)g(T_2),$$

which says that

$$C^{bullet} \leq C^{barbell}.$$

In fact, the inequality is strict because $g$ is *strictly convex*.

**Remark 3.10**: Trades in which an intermediate-maturity security is purchased (or sold) and two securities whose maturities straddle the intermediate maturity are sold (or purchased) are referred to as *butterfly trades*.

# Duration and Convexity for a Callable Bond

**Example 3.6** A callable bond is a bond that the issuer has the right to buy back at some fixed set of prices on some fixed dates before maturity. Suppose that there is a bond with $q = .05$ and maturity $T = 10$ callable in one year at par. (Here, for simplicity, we assume that there is only one possible call date.) The value of the callable bond should be the difference in value between the underlying bond and a European call option on the underlying bond struck at par with exercise date $T_E = 1$ year. (Here the underlying is a non-callable coupon bond with $q = .05$ and $T = 10$; we shall refer to the underlying here as the "bond".)

Suppose we know that the prices, durations, and convexities of the (underlying) bond and the call option are as is in the table below. Here $y$ represents the yield to maturity of the (underlying) bond. In all tables below, the prices are for $100 face of the (underlying) bond.

| Rate Level | Bond Price | Bond D | Bond C |
| --- | --- | --- | --- |
| 4.00% | 108.1757 | 7.9263 | 75.4725 |
| 5.00% | 100.0000 | 7.7983 | 73.6287 |
| 6.00% | 92.5613 | 7.6686 | 71.7854 |

| Rate Level | Option Price | Option D | Option C |
| --- | --- | --- | --- |
| 4.00% | 8.1506 | 78.7438 | 2,800.9970 |
| 5.00% | 3.0501 | 121.2927 | 9,503.3302 |
| 6.00% | .6879 | 180.9833 | 25,627.6335 |

$$P^{Callable} = P^{Bond} - P^{Option}$$

Let us compute the duration of the callable bond when $y = .04$. At this level for $y$ the price of the long position on the bond is 108.1757/100.0251 times the price of the callable and the price of the short position on the option is -8.1506/100.0251 times the price of the callable. This leads to

$$D^{Callable} = \frac{108.1757(7.9263)}{100.0251} - \frac{8.1506(78.7438)}{100.0251} = 2.155692.$$

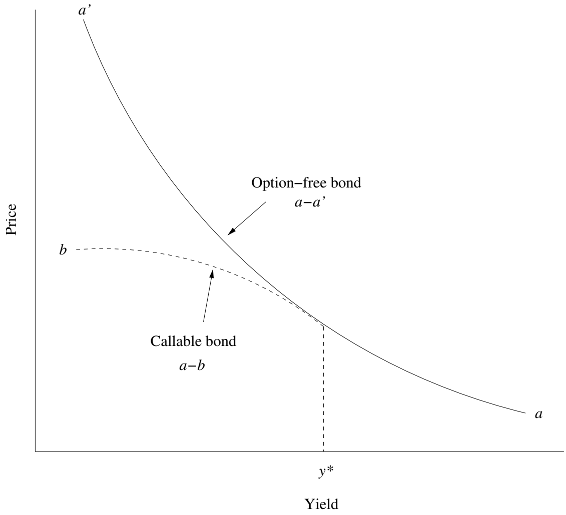
The calculation of convexity for the callable bond is similar. The results are summarized in a table.

| **Rate Level** | **Callable Price** | **Callable D** | **Callable C** |
|---|---|---|---|
| 4.00% | 100.0251 | 2.1557 | -146.618 |
| 5.00% | 96.9499 | 4.2277 | -223.035 |
| 6.00% | 91.8734 | 6.3709 | -119.563 |

The callable bond exhibits *negative convexity*.

**Remark 3.11**: In general callable bonds exhibit positive convexity when market rates are high (relative to the coupon rate) because there is little likelihood of the bond being called. However, when current market rates are low, the bond is subject to "price compression" and the price-yield curve lies below its tangent lines. Also for callable bonds a *yield to call* is frequently quoted (especially when current conditions suggest that the bond may be called). The yield to call is simply the yield to maturity computed under the assumption that the bond will be called at the earliest future call date. We shall have several homework exercises concerning callable (and putable) bonds after we introduce some term structure models.

The graph below is taken from *Fixed Income Mathematics* by Frank J. Fabozzi.



# Parallel Shifts of a Nonflat Yield Curve

One consider parallel shifts of a nonflat spot-rate curve. However, a couple of caveats are in order.

- Different securities will have different yields, so we must be careful.
- For a parallel shift of amount $\alpha$, the yields of typical securities will change by an amount close to (but not exactly equal to) $\alpha$.

# Swap Payments & Forward Rates

Let $r_{t-.5,t}$ denote the spot rate that will prevail at time $t-.5$ for loans to be settled with a single payment at time $t$. The floating payment at time $t$ in a standard interest rate swap (with semiannual payments) is given by

$$\frac{F}{2} r_{t-.5,t}.$$

It is very useful to observe that the time-0 price of a single payment of this amount at time $t$ is

$$\frac{F}{2} f(t) d(t),$$

where $f(t)$ is the forward (agreed upon at time 0) for a loan initiated at time $t-.5$ and settled at time $t$.

To verify the claim, we use replication. At time 0, we purchase a ZCB with face value $F$ and maturity $t - .5$ and also short a ZCB with face value $F$ and maturity $t$. At time $t - .5$, we invest $F$ until time $t$ at the rate $r_{t-.5,t}$ and then at time $t$, we pay $F$ to close out the short bond position. The time-0 price of this portfolio is

$$F[d(t-.5) - d(t)] = Fd(t)\left[\frac{d(t-.5)}{d(t)} - 1\right] = Fd(t)\frac{f(t)}{2}.$$

**Warning**: This does not mean that $r_{t-.5,t}$ is the same as $f(t)$ when $t > .5$. Remember that $f(t)$ is know at time 0, but $r_{t-.5,t}$ is not known until time $t - .5$.

# Carnegie Mellon University
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# 46-956 Introduction to Fixed Income

Fall 2018

Mini 1

## Short Rate Models are Free of Arbitrage

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Consider an $N$-period short-rate model with interest rate process $(R_n)_{0\le n\le N-1}$, risk-neutral measure $\tilde{\mathbb{P}}$, and actual probability measure $\mathbb{P}$. We assume that each sample path has strictly positive probability under both measures. The corresponding discount process $(D_n)_{0\le n\le N}$ satisfies

$$D_n(\omega) > 0 \ \text{ for all } \omega\in\Omega,\ \ n = 0, 1, 2, \cdots, N, \tag{1}$$

because of the assumption that $R_n > -1$.

Let $(X_n)_{0\le n\le N}$ be the capitals of a self-financing strategy satisfying

$$X_N(\omega) \ge 0 \ \text{ for all } \omega\in\Omega, \tag{2}$$

$$\mathbb{P}[X_N > 0] > 0. \tag{3}$$

Using our formula for computing prices, we see that the initial capital $X_0$ of the strategy is given by

$$X_0 = \tilde{\mathbb{E}}[D_N X_N]. \tag{4}$$

By (3), there is at least one sample path $\hat{\omega} \in \Omega$ with $X_N(\hat{\omega}) > 0$ and therefore also

(5).
$$D_N(\hat{\omega}) X_N(\hat{\omega}) > 0.$$

We conclude that

$$X_0 = \tilde{\mathbb{E}}[D_N X_N] \geq \tilde{\mathbb{P}}(\hat{\omega}) D_N(\hat{\omega}) X_N(\hat{\omega}) > 0.$$

It follows that there are no arbitrage strategies.

# Carnegie Mellon University
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Fall 2018

Mini 1

Lecture Notes for Week 4

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# Multiple Interest Rate Factors

Although a single-factor approach provides a valuable quick approximation, it is based on a premise that rates of all maturities are perfectly correlated, i.e. that movements of the entire term structure can be adequately described by a single random variable. There is little reason to believe that the change in the 6-month rate will perfectly predict the change in the 30-year rate, for example.

At the other extreme is complete *immunization*, which uses one ZCB for each individual cash flow being hedged, requires no assumptions about how rates move, but is almost certainly very costly and usually impractical.

Very significant improvements over a one-factor hedge can be obtained by using 3 or 4 interest rate factors.

# Key Rates

We shall concentrate here on an approach known as *key rate shifts*. The same kind of reasoning is applicable to other choices of interest-rate factors. When we implement key rate shifts, we can use spot rates or par-coupon yields (or other types of rates). If we will be hedging using zero coupon bonds, then spot rates are more natural. If we plan to hedge with par-coupon bonds, then it is more natural to use par-coupon yields.

We must first choose a set of key rates. For purposes of illustration, let us use par-coupon yields. Suppose that we decide to use 2, 5, 10, and 30 year bonds. (The details are quite a bit simpler with spot rates. You will use spot rates for homework.) The approach described here is appropriate for securities with deterministic cash flows.

Once we have decided what type of rates, how many key rates, and the terms of the key rates, there is still a very important decision to be made. We need to decide on how a change in each key rate will influence the rates of other maturities. Of course, we want set things up so that a change in any individual key rate does not force a change in any of the other chosen key rates. In other words, a change in the 10-year rate, should not force changes in the 2-year, 5-year, or 30-year rates, but it should force a change in the rates for terms near 10 years. We also want a change in a given key rate to have more of an influence on nearby rates. There are many ways to accomplish this. We focus on a particularly simple one here. Before describing the precise nature of the perturbations that we shall use, let's take a look at the form of the first- and second-order approximations when there are multiple factors. Let us denote by $y_1, y_2, y_3, y_4$, the 2-year,5-year, 10-year, and 30-year par coupon rates, respectively.

We shall assume that

$$P = f(y_1, y_2, y_3, y_4)$$

for some security with deterministic cash flows. The first-order approximation can be expressed as

$$\Delta P = \frac{\partial f}{\partial y_1}\Delta y_1 + \frac{\partial f}{\partial y_2}\Delta y_2 + \frac{\partial f}{\partial y_3}\Delta y_3 + \frac{\partial f}{\partial y_4}\Delta y_4.$$

The quantities

$$-\frac{\frac{\partial f}{\partial y_i}}{10,000}$$

can be thought of as the *DV01s* corresponding to the $i^{th}$ key rate.

The quantities

$$-\frac{\frac{\partial f}{\partial y_i}}{P}$$

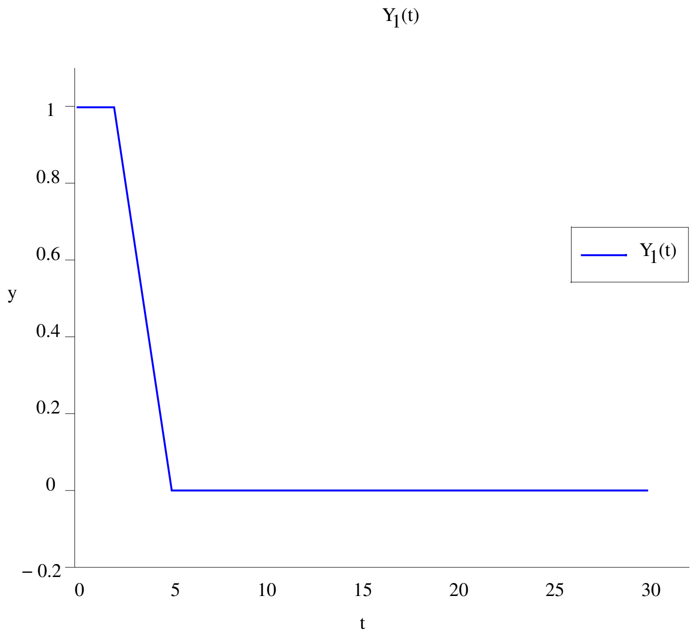
can be thought of as the *durations* corresponding to the $i^{th}$ key rate. The second-order approximation takes the form

$$\Delta P = \sum_{i=1}^{4} \frac{\partial f}{\partial y_i} \Delta y_i + \frac{1}{2} \sum_{i,j=1}^{4} \frac{\partial^2 f}{\partial y_i \partial y_j} \Delta y_i \Delta y_j.$$

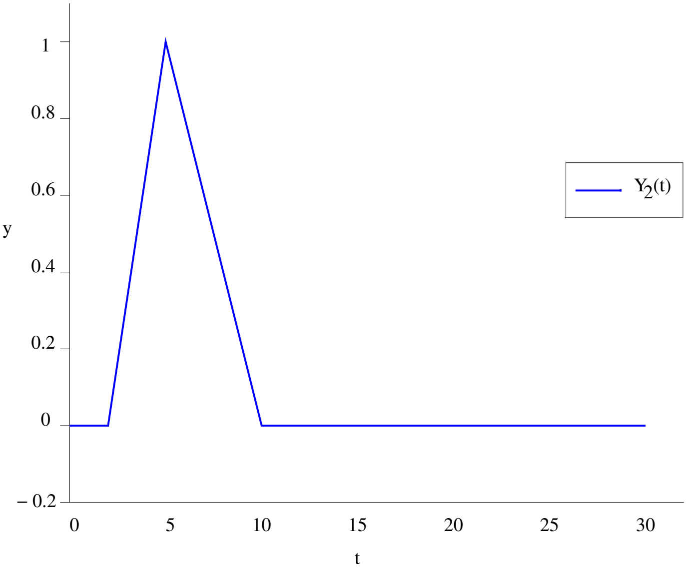
With multiple factors, duration is described by a vector and convexity is described by a matrix.

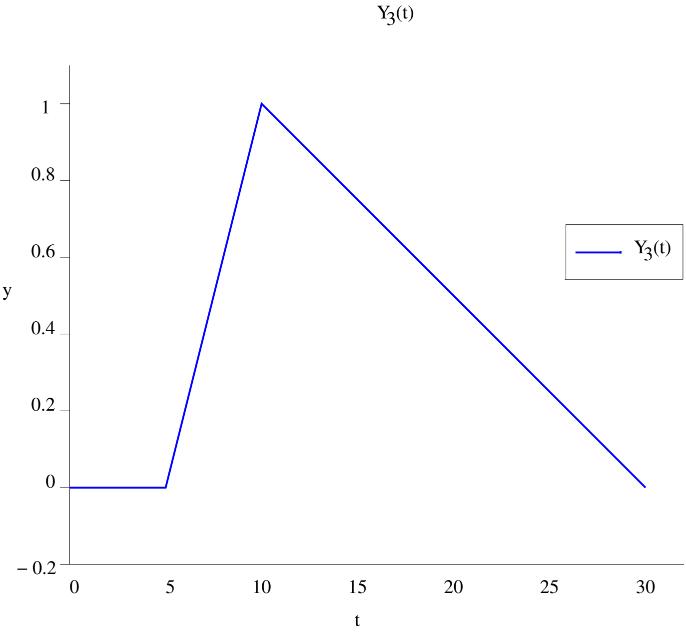
# Key Rate Perturbations

Let us assume that a change of one basis point in each individual key rate $y_i$ leads to a change in the par-coupon yield curve of $Y_i(t)$ basis points where the functions $Y_i$ are as shown in the 5 graphs that follow.

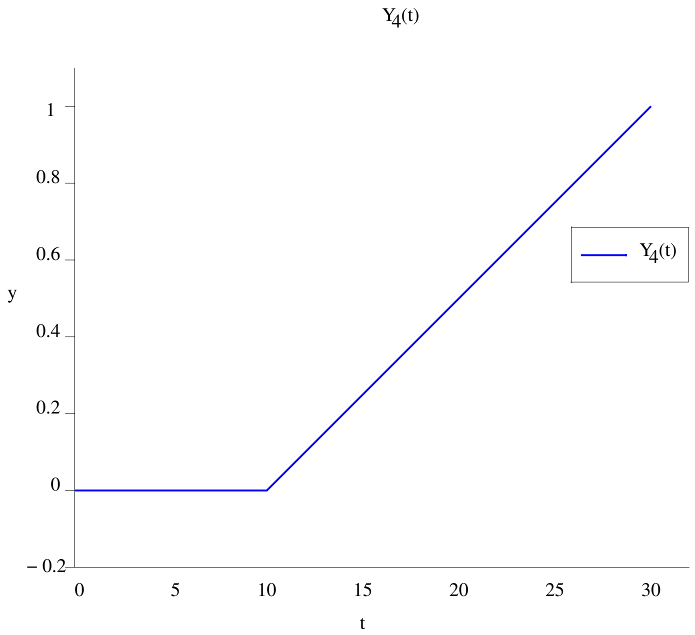

$Y_1(t)$
$Y_1(t)$
y
1
0.8
0.6
0.4
0.2
0
− 0.2
0
5
10
15
20
25
30
t

$Y_2(t)$

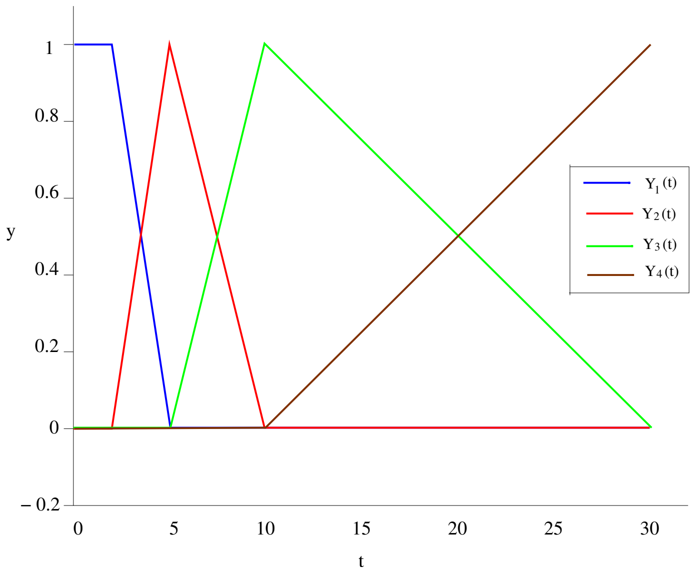




Y3(t)
1
0.8
0.6
0.4
0.2
0
– 0.2
y
0
5
10
15
20
25
30
t
Y3(t)



Y4(t)
1
0.8
0.6
0.4
0.2
0
− 0.2
y
0
5
10
15
20
25
30
t
Y4(t)



1
0.8
0.6
0.4
0.2
0
− 0.2
y
0
5
10
15
20
25
30
t
$Y_1(t)$
$Y_2(t)$
$Y_3(t)$
$Y_4(t)$

Let us denote the par-coupon yield for maturity $t$ by $y(t)$. We shall denote the observed values of the 2-, 5-, 10-, and 30-year yields by $y_1, y_2, y_3, y_4$. We also put

$$y_1^* = y(2), \quad y_2^* = y(5), \quad y_3^* = y(10), \quad y_4^* = y(30);$$

these are the *reference yield values*. Suppose that a change in the term structure occurs and we observe new values $y_1, y_2, y_3, y_4$ for the key rates. Then our "perturbed" par-coupon yield curve will be given by

$$\begin{aligned} y(t) \;+\; (y_1 - y_1^*)Y_1(t) \;+\; (y_2 - y_2^*)Y_2(t) \\ +(y_3 - y_3^*)Y_3(t) \;+\; (y_4 - y_4^*)Y_4(t). \end{aligned}$$

Since a yield curve completely determines the price of our security, the price can be expressed as

$$P = f(y_1, y_2, y_3, y_4).$$

Of course, this is not the exact price, because it is highly unlikely that the entire yield curve will have moved to fit the "perturbed" yield curve exactly, but with 4 key rates this should be a pretty good approximation.

The formula above uses the perturbed yield curve to compute the new price. Notice that

$$P^{original} = f(y_1^*, y_2^*, y_3^*, y_4^*).$$

# First-Order Approximation

Recall that the first-order approximation can be expressed as

$$\Delta P = \frac{\partial f}{\partial y_1}\Delta y_1 + \frac{\partial f}{\partial y_2}\Delta y_2 + \frac{\partial f}{\partial y_3}\Delta y_3 + \frac{\partial f}{\partial y_4}\Delta y_4.$$

In practice we would probably not compute the partial derivatives explicitly, because an explicit formula for the price would likely be too complicated. Instead, we would compute prices numerically and use difference quotients.

# Par Coupon yields and Discount Factors

The calculation of prices from par coupon yields is a bit involved, but easy to do on a spread sheet. Recall that the par coupon yield $y(\frac{n+1}{2})$ corresponding to maturity $\frac{n+1}{2}$ is given by

$$y(\frac{n+1}{2}) = \frac{2[1 - d(\frac{n+1}{2})]}{\sum_{i=1}^{n+1} d(\frac{i}{2})}.$$

Starting with a par-coupon yield curve $y(t)$, the discount factors can be computed recursively by

$$d\left(\frac{1}{2}\right) = \frac{1}{1 + \frac{y(\frac{1}{2})}{2}}; \quad d\left(\frac{n+1}{2}\right) = \frac{2 - y(\frac{n+1}{2}) \sum_{i=1}^{n} d(\frac{i}{2})}{2 + y(\frac{n+1}{2})}.$$

# Prices from Par Coupon Yields

For securities with deterministic cash flows, the prices are completely determined by the discount factors. We assume that a reference (or starting) yield curve $y(t)$ is given. Let us agree to use the key rates and perturbation functions (or "shock functions") described above. For any set of inputs, $y_1, y_2, y_3, y_4$, we can determine a new yield curve, which determines new discount factors. The new discount factors can be used to determine a new price.

**reference yield + perturbations $\rightarrow$ new yield curve**

**new yield curve $\rightarrow$ new discount factors**

**new discount factors $\rightarrow$ new price**

$$P = f(y_1, y_2, y_3, y_4).$$

# A 30-year Annuity

For purposes of illustration, let us analyze an annuity making payments of \$3,250 every 6 months for the next 30 years. Also, for purposes of illustration, let us assume that the par-coupon yield curve is currently flat at 5%; this implies that the spot-rate curve is flat at 5%. The assumption that the yield curve is initially flat is not significant here; it simply allows us to give the data in a simple way.

There is a spread sheet on Canvas (in a module called "Spreadsheets') that allows you to change the data in this example.

To determine the initial price of the annuity, we put

$$\lambda = \frac{1}{1 + \frac{.05}{2}} = .975609756.$$

Then, we have

$$P = 3,250 \sum_{i=1}^{60} \lambda^i = 3,250 \frac{\lambda - \lambda^{61}}{1 - \lambda} = 100,453.13.$$

Using the procedure outlined above, we compute the prices corresponding to a 1 BP change in each of the key rates individually:

$$f(.0501, .05, .05, .05) = 100,452.15,$$

$$f(.05, .0501, .05, .05) = 100,449.36,$$

$$f(.05, .05, .0501, .05) = 100,410.77,$$

$$f(.05, .05, .05, .0501) = 100,385.88$$

(There is no simple formula to compute these values. They are obtained by the procedure discussed above: compute new yields, then discount factors, then prices.)

We can now compute the DV01s and durations corresponding to each key rate. We compute these for the 10-year rate and summarize all results in a table.

$$DV01_3 = -\frac{f(.05, .05, .0501, .05) - f(.05, .05, .05, .05)}{10,000(.0001)} = 42.36$$

$$D_3 = \frac{10,000 DV01_3}{f(.05, .05, .05, .05)} = 4.217.$$

| Key Rate | DV01 | Duration |
|---|---|---|
| 2 year | .98 | .098 |
| 5 year | 3.77 | .375 |
| 10 year | 42.36 | 4.217 |
| 30 year | 67.25 | 6.695 |

We can perform a nice and simple check at this point. We can compute exactly the price of the annuity if the entire yield curve shifts by 1 BP and compare the answer to what we get from the first-order approximation if all key rates move up by 1 BP.

$1^{st}$-Order Approximation

$$\Delta P = -(.98 + 3.77 + 42.36 + 67.25) = -114.36,$$

$$P = 100,453.13 - 114.36 = 100,338.77.$$

Computing the exact price if the par-coupon yield is flat at 5.01% gives

$$100,338.81,$$

which agrees very well with the price we computed using the $1^{st}$ order approximation.

**Example 4.1** Compute the approximate price of the annuity assuming that the 2-year rate increases by 5 BP, the 5-year rate increases by 3 BP, the 10-year rate increases by 2 BP, and the 30-year rate drops by 1 BP. Using the key-rate DV01s we get

$$\Delta P = -(.98 \times 5 + 3.77 \times 3 + 42.36 \times 2 + 67.25 \times (-1)) = -33.68,$$

$$P = 100,453.13 - 33.68 = 100,385.77.$$

# Using the Key Rates to Construct a Hedge

Suppose that an agent has sold the annuity and wishes to hedge his short position using 2-, 5-, 10-, and 30-year par-coupon bonds. Here we assume that there are liquid par-coupon bonds of all 4 desired maturities. Let $F_1, F_2, F_3, F_4$ denote the face amounts of the 2-, 5-, 10-, and 30-year bonds used in the hedging portfolio. We choose the $F$'s so that DV01s match for each of the key rates. Therefore, we need to compute the DV01s of the 2-, 5-, 10-, and 30-year par-coupon bonds, assuming that the yields to maturity are 5%. These are summarized in the table below.

| Bond Maturity | DV01 (per 100 face) | Duration |
|---|---|---|
| 2 years | .018810 | 1.8810 |
| 5 years | .043760 | 4.3760 |
| 10 years | .077946 | 7.7946 |
| 30 years | .154543 | 15.4543 |

We can compute the DV01s of the par-coupon bonds using the formula that expresses the DV01 of a par bond in terms of face value, maturity, and yield to maturity. We can also compute them by using difference quotients.

Matching the DV01s for each of the key rates, we get

$$\frac{F_1(.018810)}{100} = .98, \quad \frac{F_2(.043670)}{100} = 3.77,$$

$$\frac{F_3(.077946)}{100} = 42.36, \quad \frac{F_4(.154543)}{100} = 67.25.$$

Solving for the face amounts gives $F_1 = \$5,209.99$, $F_2 = \$8,632.93$, $F_3 = \$54,345.32$, and $F_4 = \$43,515.40$.

# Some Comments on Hedging

**Remark 4.1**: The method of *bucket shifts* uses parallel shifts of a large number of “buckets” of forward rates. This technique is frequently used to hedge the risk in swaps.

**Remark 4.2**: A one-factor approach is very simple, but cannot protect against all possible moves of the spot rates. At the other extreme is complete *immunization* in which all cash flows are matched exactly. (Of course, this is not always feasible.) The multifactor approach is between these two extremes and can give excellent results.

**Remark 4.3**: For your benefit, I suggest that you read the material in Tuckman & Serrat on regression based hedging.

# Volatility-Weighted Hedging

We conclude our discussion of hedging for now with a simple, but useful idea, known as Volatility-Weighted Hedging. Suppose that an agent wishes to hedge the sale of a 20-year par coupon bond by purchasing a 30-year par coupon bond. The agent knows from experience that a change of 1 BP in the 30-year par coupon yield typically corresponds to a change of 1.1 BP in the 20-year par coupon yield.

If we denote by $DV01_{20}$ and $DV01_{30}$ the DV01s (per 100 face) of the 20- and 30-year par coupon bonds, then we want

$$F_{20} \times 1.1 \times \frac{DV01_{20}}{100} = F_{30} \times \frac{DV01_{30}}{100}.$$

# Example 4.2

Suppose that an agent who is short \$10,000,000 face of a 20-year par coupon bond wishes to hedge her interest rate risk by purchasing a 30-year par coupon bond. Let us assume that the 20-year par coupon yield is 5.8%, the 30-year par coupon yield is 6%, and the agent knows from experience that a 1 BP change in the 30-year par coupon rate typically corresponds to a 1.1 BP change in the 20-year par coupon rate. The DV01s per 100 face are given by

$$DV01_{20} = .117465, \quad DV01_{30} = .138378.$$

With $F_{20} = 10,000,000$, we find that

$$F_{30} = 10,000,000 \times 1.1 \times \frac{.117465}{.138378} = 9,337,575.$$

(A hedge based on assumption of parallel shifts in the par coupon rate would suggest the purchase of \$8,488,704 face of the 30-year bond.)

# Example 4.3

Suppose that an agent who is short \$10,000,000 face of a 20-year par coupon bond wishes to hedge his interest rate risk by investing equal amounts in a 10-year par coupon bond and a 30-year par coupon. Let us denote the par coupon yield for maturity $t$ by $y_{pc}(t)$. Assume that

$$y_{pc}(10) = 5.2\%, \quad y_{pc}(20) = 5.8\%, \quad y_{pc}(30) = 6\%.$$

Assume also that the agent knows from experience that every change of 1 BP in $y_{pc}(30)$ corresponds to a change of 1.1 BP in $y_{pc}(20)$ and a change of 1.2 BP in $y_{pc}(10)$. The DV01's per 100 face are given by

$$DV01_{10} = .077215, \quad DV01_{20} = .117465, \quad DV01_{30} = .138378.$$

# Example 4.3 (Cont.)

Let us denote the unknown face amount of the 10- and 30-year bonds by $F$. (These face amounts are equal because the prices of the two bonds are the same and they are both par bonds.)

Matching the DV01 of the portfolio of the 10- and 30-year bond with the DV01 of the 20-year bond and solving for $F$ we find that

$$F = \frac{10,000,000(1.1)(.117465)}{.077215(1.2) + .138378} = 5,592,700.$$

# Principal Component Analysis (PCA)

An extremely important emperical approach to hedging is based on the idea of *principal components*. We will discuss the method in more detail during the last lecture and you will study it thoroughly in the Data Science classes.

The idea is to identify uncorrelated random variables that are statistically most significant in observed movements of the term structure, and use these random variables to construct hedges. It is interesting to note that the first principal component looks very much like a parallel shift and captures about 80% of the variance in the term structure. The first 3 principal components together typically capture about 95% of the variance in the term structure.

# Term Structure Models

In order to analyze fixed-income securities whose cash flows are not deterministic, we need to make a model for the evolution of interest rates.

There are two types of models: *short-rate models* and *whole-yield models*. We shall discuss the distinctions later.

We shall do a rigorous mathematical analysis in the context of binomial trees. If time permits, I will write down and briefly discuss continuous time versions of several of the most important models.

# A One-Period Binomial Pricing Model

We begin by discussing a simple one-period binomial pricing model. This should serve two purposes: (i) It should provide motivation for the way the multi-period model is formulated. (ii) Analysis of multiperiod problems can often be carried out by breaking things down into a sequence of one-period problems, so the one-period case will help solve multiperiod problems.

Suppose that there are two trading times (or dates) $t = 0$ and $t = 1$. We assume that there is a zero-coupon bond that will pay \$1 at time 1 (i.e., the bond has face value $F = 1$ and maturity $T = 1$.) Denote the price of this bond at time 0 by $B_{0,1}$. We assume that there is another basic (traded) security that has initial price $S_0 > 0$ and whose price $S_1 > 0$ at time 1 takes one of two possible different values, each with strictly positive probability.

# Sample Space & Probability Measure

Since there are two possibilities for the price of the risky asset at time 1, it is natural to take

$$\Omega = \{H, T\}$$

as the sample space. It is helpful to think of a random experiment in which a single coin is tossed and think of $H$ as corresponding to a head and $T$ as corresponding to a tail.

We write $\mathbb{P}(H)$ for the probability of a head and $\mathbb{P}(T)$ for the probability of a tail, i.e. we use the symbol $\mathbb{P}$ for the probability measure corresponding to the coin toss. We assume

$$\mathbb{P}(H) > 0, \quad \mathbb{P}(T) > 0,$$

but we do not necessarily require the coin to be "fair" (i.e., have equal probability for head and tail).

We are assuming that $S_1(H) \neq S_1(T)$. Without loss of generality, we assume that

$$S_1(H) > S_1(T).$$

(Indeed, if $S_1(T) > S_1(H)$, we could relabel the sides of the coin.)

The risky asset could, for example, be a zero-coupon bond that matures at some future date (such as time 2) that we are not considering in our model.

# Interest Rate

Let us denote the yield to maturity of the zero-coupon bond that matures at time 1 by $R_0$. Here we use the one-period compounding convention, so that

$$B_{0,1} = \frac{1}{1 + R_0}.$$

We can solve for $R_0$ to obtain

$$R_0 = \frac{1}{B_{0,1}} - 1.$$

A very simple, but important, observation is that we can describe the bond by giving the time-0 price, or by giving the interest rate (bond yield).

# Description of a Trading Strategy

In order to describe a trading strategy (or portfolio) in this model, we simply need to specify how much money is invested in the maturity-1 bond at time 0 and how much money is invested in the risky asset at time 0.

If we specify the total initial capital $X_0$ and the number of shares $\Delta_0$ of the risky asset purchased at time 0, then we can easily compute how much money is invested in the bond and how much money is invested in the risky asset. For consistency with what you will do in MPAP and Stochastic Calculus, I will describe trading strategies by giving the total initial capital and the number of shares of the risky asset purchased at time 0.

# Terminal Capital of a Strategy

Consider a strategy described by $(X_0, \Delta_0)$. (This means that the total initial capital is $X_0$ and that the portfolio will hold $\Delta_0$ shares of the risky asset.) Let us denote the capital of the strategy at time 1 by $X_1$. Notice that $X_1$ is a *random variable* on $\Omega$. The initial capital invested in the risky asset is $\Delta_0 S_0$ and the initial capital invested in the bond is $X_0 - \Delta_0 S_0$. The terminal capital is given by

$$\begin{aligned} X_1(\omega) &= (X_0 - \Delta_0 S_0)\tfrac{1}{B_{0,1}} + \Delta_0 S_1(\omega) \\ &= (X_0 - \Delta_0 S_0)(1 + R_0) + \Delta_0 S_1(\omega) \end{aligned}$$

for all $\omega \in \{H, T\}$.

# Arbitrage Strategies

By an *arbitrage strategy*, we mean a strategy with

(i) $X_0 = 0$,

(ii) $\mathbb{P}[X_1 \geq 0] = 1$,

(iii) $\mathbb{P}[X_1 > 0] > 0$.

It is clear that a strategy is an arbitrage if and only if

(i) $X_0 = 0$,

(ii') $X_1(T) \geq 0, \quad X_1(H) \geq 0$,

(iii') $X_1(T) + X_1(H) > 0$.

# Absence of Arbitrage

The model is free of arbitrage if and only if

$$(*) \qquad \frac{S_1(T)}{S_0} < \frac{1}{B_{0,1}} < \frac{S_1(H)}{S_0}.$$

If we put

$$u = \frac{S_1(H)}{S_0}, \quad d = \frac{S_1(T)}{S_0},$$

and recall that

$$R_0 = \frac{1}{B_{0,1}} - 1$$

then (*) is equivalent to

$$(**) \qquad d < 1 + R_0 < u.$$

Unless stated otherwise, we assume that (*) holds. Notice that (*) is also equivalent to

$$(***) \qquad \frac{S_1(T)}{S_0} < 1 + R_0 < \frac{S_1(H)}{S_0}.$$

# Replication and Pricing

Consider a derivative security that pays the amount $V_1(\omega)$ at time 1. Let us try to replicate the payoff of the security by a portfolio with initial capital $X_0$ and $\Delta_0$ shares of the risky asset. We need to find $X_0$ and $\Delta_0$ satisfying

$$(X_0 - \Delta_0 S_0)(1 + R_0) + \Delta_0 S_1(H) = V_1(H), \tag{1}$$

$$(X_0 - \Delta_0 S_0)(1 + R_0) + \Delta_0 S_1(T) = V_1(T). \tag{2}$$

# Replication and Pricing (Continued)

Subtracting (2) from (1), we obtain

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}. \tag{3}$$

Rearranging (1), we find that

$$X_0(1 + R_0) = V_1(H) + \Delta_0[S_0(1 + R_0) - S_1(H)]. \tag{4}$$

Substituting from (3) into (4), we obtain

$$X_0(1 + R_0) = \tilde{p}V_1(H) + \tilde{q}V_1(T), \tag{5}$$

# Risk-Neutral Probabilities

where

$$(6)\qquad \tilde{p} = \frac{S_0(1+R_0) - S_1(T)}{S_1(H) - S_1(T)}, \quad \tilde{q} = \frac{S_1(H) - S_0(1+R_0)}{S_1(H) - S_1(T)}$$

Observe that

$$(7)\qquad \tilde{p} > 0, \quad \tilde{q} > 0,$$

by virtue of (***). Moreover, we have

$$(8)\qquad \tilde{p} + \tilde{q} = 1.$$

We can define another probability measure $\tilde{\mathbb{P}}$ on $\Omega$ by

$$\tilde{\mathbb{P}}(H) = \tilde{p}, \quad \tilde{\mathbb{P}}(T) = \tilde{q}.$$

# Risk Neutral Probabilities (Continued)

The measure $\tilde{\mathbb{P}}$ is called a *risk-neutral measure* or *pricing measure*.

We conclude that $V$ is replicable and that the initial capital $X_0$ required to replicate $V$ is satisfies

$$X_0(1 + R_0) = \tilde{\mathbb{E}}[V_1].$$

(Here, $\tilde{\mathbb{E}}$ denotes expectation under the risk-neutral measure.)

The arbitrage-free time-0 price of $V$ is given by

$$V_0 = \tilde{\mathbb{E}}\left[\frac{V_1}{1 + R_0}\right].$$

# Example 4.4

Consider a one-period binomial model with $R_0 = .06$, $S_0 = 100$, $S_1(H) = 130$, and $S_1(T) = 94$.

Let $V$ be a put option on $S$ with strike price $K = 100$. Find the time-0 price $V_0$ of $V$.

We begin by finding the risk-neutral probabilities:

$$\tilde{p} = \frac{100(1.06) - 94}{130 - 94} = \frac{1}{3}, \quad \tilde{q} = \frac{130 - 100(1.06)}{130 - 94} = \frac{2}{3}.$$

# Example 4.4 (Continued)

The option payoff is given by

$$V_1(H) = 0, \quad V_1(T) = 6.$$

The time-0 price of the option is given by

$$V_0 = \frac{\frac{1}{3}(0) + \frac{2}{3}(6)}{1.06} = 3.77.$$

**Remark 4.4**: Notice that if instead of being given $R_0$ and price information for the risky asset, we were given $R_0$ and the risk-neutral measure, we could still compute the arbitrage-free price of any given payoff at time 1.

**Remark 4.5**: It is customary to define

$$u = \frac{S_1(H)}{S_0}, \quad d = \frac{S_1(T)}{S_0}.$$

In this case, the no arbitrage condition becomes

$$d < 1 + R_0 < u,$$

and the risk-neutral probabilities are given by

$$\tilde{p} = \frac{1 + R_0 - d}{u - d}, \quad \tilde{q} = \frac{u - 1 - R_0}{u - d}.$$

# $N$-Period Binomial Models

In the $N$-period binomial model, there will be $N+1$ trading dates $0, 1, 2, \cdots, N$. We shall write $B_{n,m}$ for the price at time $n$ of a ZCB that pays \$1 at time $m \geq n$. We shall write $R_n$ for the spot rate that will prevail at time $n$ for loans to be settled at time $n+1$ (one-period compounding). We can either model the bond prices $B_{n,n+1}$ or the interest rates $R_n$, because we have

$$B_{n,n+1} = \frac{1}{1+R_n}.$$

The interest rates $R_n$ are known at time $n$, but not earlier.

We shall describe the evolution of the interest rates in this course.

# Negative Interest Rates

**Remark 4.6**: Negative interest rates, although mathematically possible, have been considered questionable up until recently. However, the possibility of negative interest rates in and of itself does not imply arbitrage in the sense that the term is used in this course, MPAP, and Stochastic Calculus. This is because we do not allow an agent to hold cash in a self-financing strategy – all capital must be invested in the market. However, if you are allowed to hold cash without risk, then negative interest rates imply arbitrage. You should be aware of this point when you communicate with others, especially on interviews. Now, because of the economic situation in Europe and Japan, many banks are interested in models that allow negative interest rates.

# Short Rate Models

A risk-neutral measure (or pricing measure) is prescribed and the evolution of the short rates is described under the risk-neutral measure. Such models have no arbitrage (although they may give rise to negative interest rates). The spot rate curve is an output of the model, rather than an input. Typically there are parameters in the model that are chosen to try to match the spot-rate curve as closely as possible. One must be very careful in interpreting prices computed by the model – they are internally consistent in the sense that no arbitrage can occur by trading securities within the model. However, we need to worry about whether we can hedge a position in terms of securities that are traded in the actual market.

# Whole-Yield Models

The current spot-rate curve is an input to the model, and the random evolution of the spot-rate curve is modeled. Modeling the evolution of the spot-rate curve in such a way that arbitrage is not possible is a very complex issue. (Heath-Jarrow-Morton). A key idea is to use forward rates. Prices are again computed using a risk-neutral measure.

**Remark 4.7**: What I am calling short-rate models are often called *equilibrium models* and what I am calling whole-yield models are often called *arbitrage-free models*.

# Stochastic Analysis on Coin-Toss Space

Let $N$ be a fixed positive integer. There are $N+1$ times or dates, namely $0, 1, 2, \cdots, N$.

**Sample Space**: $\Omega = \{H, T\}^N$ – the set of all lists or strings of length $N$, with each entry in the list being $H$ or $T$. We imagine that a coin is being tossed $N$ times. "$H$" represents a head and "$T$" represents a tail. A typical element of $\Omega$ is denoted by $\omega = (\omega_1, \omega_2, \cdots, \omega_N)$. If $\omega_i = H$ then the $i^{th}$ toss is a head, while $\omega_i = T$ indicates that the $i^{th}$ toss is a tail. Sometimes the elements of $\Omega$ are referred to as *paths*. There are $2^N$ paths in $\Omega$, a fact that has important consequences concerning actual computations when $N$ is large.

**Probability Measures**: We shall equip $\Omega$ with one or more probability measures. Unless stated otherwise, we assume that each path has strictly positive probability.

**Binomial Product Measures**: A probability measure on $\Omega$ is said to be a *binomial product measure* (BPM) if there exist numbers $a, b > 0$ with $a + b = 1$ such that

$$\mathbb{P}(\omega) = a^{\#H(\omega)} b^{\#T(\omega)} \text{ for all } \omega \in \Omega.$$

Here $\#H(\omega)$ and $\#T(\omega)$ represent the number of heads and tails, respectively, in the string $\omega$. We refer to $a$ as the *probability of heads* and to $b$ as the *probability of tails*.

**Remark 4.8**: The term "binomial product measure" is not standard, but I think that is very useful. Binomial product measures correspond to independent coin tosses. If $\mathbb{P}$ is a BPM, then for each $n \in \{1, 2, \cdots, N\}$ we have

$$\mathbb{P}[\{\omega \in \Omega : \omega_n = H\}] = a,$$

$$\mathbb{P}[\{\omega \in \Omega : \omega_n = T\}] = b.$$

# Adapted Processes & Information

In financial markets, more information becomes available as time evolves. When we build models, it is crucial to ensure that prices of securities at time $n$ and decisions made at time $n$ (such as the number of shares to hold in a portfolio) are based solely on the information available at time $n$. This leads to the notion of an *adapted process*. In coin-toss space, the information available at time $n$ is represented by the partial string $(\omega_1, \omega_2, \cdots, \omega_n)$.

**Def**: Let $n \in \{0, 1, \cdots, N\}$ be given. A random variable $Y$ on $\Omega$ is said to be time-$n$ measurable provided that

$$Y(\omega) = Y(\omega_1, \cdots, \omega_n, \mu_{n+1}, \cdots, \mu_N)$$

for all $\omega \in \Omega$ and all $\mu_{n+1}, \cdots, \mu_N \in \{H, T\}$. (In other words, the value of $Y(\omega)$ depends only on the first n entries in the string $\omega$. $Y$ is completely determined by the information available at time $n$.)

For a time-$n$ measurable random variable $Y$, we write $Y(\omega_1, \cdots, \omega_n)$ in place of $Y(\omega)$ when convenient.

**Def**: Let $k, m \in \{0, 1, \cdots, N\}$ with $k \leq m$ be given. A list $(Y_n)_{k \leq n \leq m}$ of random variables on $\Omega$ is said to be an adapted process provided that $Y_n$ is time-$n$ measurable for every $n$ with $k \leq n \leq m$.

# Random Walk

For each $j \in \{1, 2, \cdots, N\}$, define the random variable $X_j$ on $\Omega$ by

$$X_j(\omega) = \begin{cases} 1 & \text{if } \omega_j = H \\ \\ -1 & \text{if } \omega_j = T, \end{cases}$$

and define the adapted process $(M_n)_{0 \leq n \leq N}$ by

$$M_0 = 0, \quad M_n = \sum_{j=1}^{n} X_j, \quad n = 1, \cdots, N$$

# Random Walk (Cont.)

**Remark 4.9**: It is very useful to observe that

$$M_n(\omega_1, \cdots, \omega_n) = \#H(\omega_1, \cdots, \omega_n) - \#T(\omega_1, \cdots, \omega_n).$$

Here $\#H(\omega_1, \cdots, \omega_n)$ and $\#T(\omega_1, \cdots, \omega_n)$ represent the number of heads and tails seen up to time $n$. If the measure is a binomial product measure with probability of heads equal to $\frac{1}{2}$, then $(M_n)_{0 \leq n \leq N}$ is called a *symmetric random walk* and can be thought of as a discrete version of Brownian motion. In fact, Brownian motion can be obtained as a limit of appropriately scaled random walks.

# Binomial Short-Rate Models

A binomial short-rate model is characterized by an adapted process $(R_n)_{0 \le n \le N-1}$ with

$$R_n(\omega) > -1 \quad \text{for all } \omega \in \Omega,$$

called an *interest rate process*, and a probability measure $\tilde{\mathbb{P}}$ on $\Omega$, called a *pricing measure*. (Most people call $\tilde{\mathbb{P}}$ a *risk-neutral measure*. However, for now, I want to use the term "pricing measure" because the term "risk-neutral measure" seems to lead to confusion for some people.)

If \$1 is deposited in the bank at time $n$, the value of the account at time $n+1$ will be $\$1 \times (1 + R_n)$. The interest rate for borrowing is the same as the interest rate for investing.

# Computing Prices

The measure $\tilde{\mathbb{P}}$ is used to compute prices in the following way. Let $Y$ be a time-$m$ measurable random variable. The price at time 0 to receive a payment of amount $Y(\omega_1, \cdots, \omega_m)$ at time $m$ is given by

$$\tilde{\mathbb{E}}[D_m Y],$$

where

$$D_m = \frac{1}{(1+R_0)(1+R_1)\cdots(1+R_{m-1})}.$$

We make the convention that $D_0 = 1$.

# Computing Prices (Cont.)

The process $(D_n)_{0\le n\le N}$ is called the *discount process*. It is a *predictable* process, because the value of $D_n$ can be determined from the information available at time $n-1$.

For $0 \le n \le m \le N$, the price at time $n$ to receive a payment of amount $Y(\omega_1, \cdots, \omega_m)$ at time $m$ is computed via the formula

$$\frac{1}{D_n}\tilde{\mathbb{E}}_n[D_m Y].$$

Here $\tilde{\mathbb{E}}_n$ denotes the conditional expectation under the pricing measure given the information at time $n$.

# Conditional Expectations

If the pricing measure is a binomial product measure with probability of heads $\tilde{p}$ and probability of tails $\tilde{q}$ then the conditional expectation at time $n$, $\tilde{\mathbb{E}}_n[W](\omega_1, \omega_2, \cdots, \omega_n)$ of a random variable $W$ is given by

$$\sum_{\omega_{n+1}, \cdots, \omega_N \in \{H,T\}} W(\omega) \tilde{p}^{\# H(\omega_{n+1}, \cdots, \omega_N)} \tilde{q}^{\# T(\omega_{n+1}, \cdots, \omega_N)}.$$

Observe that

$$\tilde{\mathbb{E}}_0[W] = \tilde{\mathbb{E}}[W],$$

and

$$\tilde{\mathbb{E}}_N[W] = W.$$

# Some Properties of Conditional Expectations

Assume that $\tilde{\mathbb{P}}$ is a binomial product measure with probablity of heads equal to $\tilde{p}$ and probability of tails equal to $\tilde{q}$. Let $X$ and $Y$ be random variables on $\Omega = \{H, T\}^N$, $c_1$ and $c_2$ be constants, and $n$ be an integer with $0 \leq n \leq N$. Then we have

(i) (Linearity) $\tilde{\mathbb{E}}_n[c_1 X + c_2 Y] = c_1 \tilde{\mathbb{E}}_n[X] + c_2 \tilde{\mathbb{E}}_n[Y]$

(ii) $\tilde{\mathbb{E}}_n[X]$ is time-$n$ measuarble.

(iii) (One-Step Ahead Property) If $X$ is time-$n+1$ measurable, then

$$\tilde{\mathbb{E}}_n[X](\omega_1, \cdots, \omega_n) = \tilde{p} X(\omega_1, \cdots, \omega_n, H) + \tilde{q} X(\omega_1, \cdots, \omega_n, T)$$

# Some Properties of Conditional Expectation (Cont.)

(iv) (Taking Out a Known Quantity) If $X$ is time-$n$ measurable then

$$\tilde{\mathbb{E}}_n[XY] = X\tilde{\mathbb{E}}_n[Y]$$

(v) (Iterated Conditioning) If $n \leq m \leq N$ then

$$\tilde{\mathbb{E}}_n[Y] = \tilde{\mathbb{E}}_n[\tilde{\mathbb{E}}_m[Y]]$$

**Remark 4.10**: Notice that the discount factor for time $n$ is given by

$$d(n) = \tilde{\mathbb{E}}[D_n].$$

Given an interest rate process and a pricing measure we can compute the zero-coupon yield curve (spot-rate curve).

**Remark 4.11**: By the $1^{st}$ fundamental theorem of asset pricing (to be treated in detail in MPAP), a binomial pricing model will be arbitrage free if and only if there is a probability measure $\tilde{\mathbb{P}}$ on $\Omega$ such that for all integers $m, n$ with $0 \leq n \leq m$ and every time-$m$ measurable random variable $Y$, the price at time $n$ to receive a single payment of amount $Y$ at time $m$ is given by

$$\frac{1}{D_n}\tilde{\mathbb{E}}_n[D_m Y].$$

The measure $\tilde{\mathbb{P}}$ need not be a binomial product measure. (We need only the easy direction of the theorem here.)

# A Quick Look at Four Important Binomial Models

We put

$$\Delta R_n = R_{n+1} - R_n.$$

**Ho-Lee Model**:

$$\Delta R_n = \lambda_{n+1} + \sigma X_{n+1}$$

$$R_n = R_0 + \sum_{i=1}^{n} \lambda_i + \sigma M_n.$$

**Remark 4.12**: The Ho-Lee Model can be written in the form

$$R_n(\omega_1, \cdots, \omega_n) = a_n + b \cdot \#H(\omega_1, \cdots, \omega_n),$$

where

$$a_n = R_0 - \sigma n + \sum_{i=1}^{n} \lambda_i, \quad b = 2\sigma.$$

We shall frequently use the more general form

$$R_n(\omega_1, \cdots, \omega_n) = a_n + b_n \cdot \#H(\omega_1, \cdots, \omega_n).$$

Notice that both of these lead to recombining trees for the short rates. This model provides a great deal of flexibility in fitting the current term structure.

**Binomial Version of Vasicek Model**: $0 < k < 1$

$$\Delta R_n = k(\theta - R_n) + \sigma X_{n+1}$$

The solution of this difference equation is given by

$$R_n = (1-k)^n (R_0 - \theta) + \theta + \sigma \sum_{j=1}^{n} (1-k)^{n-j} X_j.$$

**Binomial Version of Hull-White Model**: $0 < k < 1$

$$\Delta R_n = k(\theta_{n+1} - R_n) + \sigma_{n+1} X_{n+1}$$

One can also let $k$ depend on $n$.

**Black-Derman-Toy Model**:

$$\Delta(\ln R_n) = -\frac{\Delta\sigma_n}{\sigma_n}(\ln\theta_{n+1} - \ln R_n) + \sigma_{n+1}X_{n+1}.$$

After some work, this can be rewritten as

$$R_n(\omega_1, \cdots, \omega_n) = a_n b_n^{\#H(\omega_1,\cdots,\omega_n)}.$$

Notice that this leads to a recombining tree for the short rates.

# Carnegie Mellon University
# MSCF Program
# 46-956 Introduction to Fixed Income

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Mini 1

Lecture Notes for Week 5

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# Binomial Short Rate Models (Continued)

$N+1$ **Dates**: $n = 0, 1, \cdots, N$

**Sample Space**: $\Omega = \{H, T\}^N$

**Interest Rate Process**: $(R_n)_{0 \le n \le N-1}$ – adapted (values of the process at time $n$ depend only on the information available at time n).

**Pricing Measure**: $\tilde{\mathbb{P}}$ (risk-neutral measure)

**Discount Process**: $(D_n)_{0 \le n \le N}$ – predictable (values of the process at time $n$ are determined by the information available at time $n-1$):

$$D_0 = 1; \quad D_n = \frac{1}{(1+R_0)(1+R_1)\cdots(1+R_{n-1})}.$$

# Risk-Neutral Pricing Formula

For a security that makes a single payment of amount

$$V_m(\omega_1, \cdots, \omega_m)$$

at a given date $m$, the prices $V_n$ of the security at the dates $n = 0, 1, \cdots, m$ are given by

$$V_n = \frac{1}{D_n}\tilde{\mathbb{E}}_n[D_m V_m].$$

In particular, the time-0 price is given by

$$V_0 = \tilde{\mathbb{E}}[D_m V_m].$$

# Random Walk

Recall that

$$X_j(\omega) = \begin{cases} 1 & \text{if } \omega_j = H \\ \\ -1 & \text{if } \omega_j = T, \end{cases}$$

$$M_0 = 0, \quad M_n(\omega) = \sum_{j=1}^{n} X_j(\omega), \ n = 1, 2, 3, \cdots, N.$$

For purposes of testing, I will specify a short rate model in one of the following the three ways.

- The short rates will be given by a Ho-Lee Model in the form

$$R_n = a_n + b_n \cdot \#H(\omega_1, \omega_2, \cdots, \omega_n).$$

- The short rates will be given by a Black-Derman-Toy model in the form

$$R_n = a_n b_n^{\#H(\omega_1, \omega_2, \cdots, \omega_n)}.$$

- The short rates will be specified explicitly.

In all cases the risk-neutral measure will be a binomial product measure with probability of heads equal to .5.

# General Binomial Models

When we discuss binomial interest rate models at an abstract (or general) level, we will not specify the size of a time step and the interest rate process $(R_n)_{0 \leq n \leq N-1}$ will always represent one-period rates. All bonds will have face values equal to \$1 unless specified otherwise.

An excellent reference is Chapter 6 of *Stochastic Calculus for Finance I* by Steve Shreve.

# Zero Coupon Bonds

For each $n, m \in \{0, 1, \cdots, N\}$ with $0 \leq n \leq m$, we let $B_{n,m}$ denote the price at time $n$ of a zero coupon bond with maturity $m$ and face value \$1.

Observe that

$$B_{n,m} = \frac{1}{D_n} \tilde{\mathbb{E}}_n[D_m],$$

where $(D_n)_{0 \leq n \leq N}$ is the discount process. In particular,

$$B_{n,n} = 1.$$

# Coupon Bonds

Given $q > -1$ and $0 \leq n \leq m \leq N$, we let $C^q_{n,m}$ denote the price at time $n$ of a coupon bond with maturity $m$, face value \$1 and one-period coupon rate $q$. If $n > 0$, we make the convention that $C^q_{n,m}$ is the price *after* the coupon payment is received at time $n$. Observe that

$$C^q_{n,m} = B_{n,m} + q \sum_{i=n+1}^{m} B_{n,i},$$

provided that $m \geq n + 1$.

# General Security with Adapted Cash Flows

Let $(A_n)_{1\leq n\leq m}$ be an adapted process. The price at time $0$ of a security that pays the amount $A_n$ at each of the times $n = 1, 2, \cdots, N$ is given by

$$\tilde{\mathbb{E}}\left[\sum_{n=1}^{m} D_n A_n\right].$$

# One-Period Forward Rates

For $0 \leq n \leq m \leq N-1$ we denote by $F_{n,m}$ the interest rate agreed upon at time $n$ for a loan initiated at time $m$ and settled at time $m+1$. If \$1 is the amount borrowed at time $m$ then the amount to be repaid at time $m+1$ is $1+F_{n,m}$. It follows from previous considerations that

$$F_{n,m} = \frac{B_{n,m}}{B_{n,m+1}} - 1.$$

Observe that $F_{m,m} = R_m$.

# Float Notes

A floating rate note with maturity $m$ pays the amount $R_{n-1}$ at each of the times $n = 1, 2, \cdots, m$ together with the face value of \$1 at maturity. We already "know" that the price of such a bond at time 0 must be \$1. Let us compute the price using the pricing measure. Notice that

$$D_n(1 + R_{n-1}) = D_{n-1}.$$

It follows that

$$\tilde{\mathbb{E}}[D_n(1 + R_{n-1})] = B_{0,n-1},$$

$$\tilde{\mathbb{E}}[D_n R_{n-1}] = B_{0,n-1} - B_{0,n}.$$

The arbitrage-free price of a float note at time 0 is given by

$$\tilde{\mathbb{E}}\left[D_m + \sum_{i=1}^{m} D_i R_{i-1}\right]$$

$$= B_{0,m} + \sum_{i=1}^{m}(B_{0,i-1} - B_{0,i}) = B_{0,0} = 1.$$

# Interest Rate Swaps, Caps, and Floors

**Def**: An *m-period interest rate swap* with fixed rate $K$ is a contract that pays $K - R_{n-1}$ at each of the times $n = 1, 2, \cdots, m$. The *m-period swap rate* $SR_m$ is the value of $K$ that makes the time 0 price of this contract equal to zero.

**Def**: An *m-period interest rate cap* with fixed rate $K$ is a contract that pays $(R_{n-1} - K)^+$ at each of the times $n = 1, 2, \cdots, m$.

**Def**: An *m-period interest rate floor* with fixed rate $K$ is a contract that pays the amount $(K - R_{n-1})^+$ at each of the times $n = 1, 2, \cdots, m$.

**Remark 5.2**: As usual we use the notation $x^+$ to denote the *positive part* of $x$, i.e.

$$x^+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \le 0, \end{cases}$$

Recall that

$$x^+ - (-x)^+ = x.$$

The time 0 prices of $m$-period swaps, caps, and floors with fixed rate $K$ are given by

$$Swap_m^K = \tilde{\mathbb{E}}\left[\sum_{i=1}^{m} D_i(K - R_{i-1})\right] = C_{0,m}^K - 1$$

$$Cap_m^K = \tilde{\mathbb{E}}\left[\sum_{i=1}^{m} D_i(R_{i-1} - K)^+\right]$$

$$Floor_m^K = \tilde{\mathbb{E}}\left[\sum_{i=1}^{m} D_i(K - R_{i-1})^+\right]$$

Since

$$K - R_{i-1} + (R_{i-1} - K)^+ = (K - R_{i-1})^+,$$

we deduce that

$$Swap_m^K + Cap_m^K = Floor_m^K.$$

This is similar to *put-call parity*.

# Caplets and Floorlets

**Remark 5.3**: A single payment of $(R_{n-1} - K)^+$ at time $n$ is called an *interest rate caplet*. A single payment of $(K - R_{n-1})^+$ at time $n$ is called an *interest rate floorlet*.

# More on Swaps

Observe that the $m$-period swap rate is simply the $m$-period par coupon rate (Assignment 2):

$$SR_m = \frac{1 - B_{0,m}}{\sum_{i=1}^{m} B_{0,i}}.$$

Observe also that

$$Swap_m^K = \left( K \sum_{i=1}^{m} B_{0,i} \right) + B_{0,m} - 1.$$

**Simple Exercise**: Show that

$$Swap_m^K = \sum_{i=1}^{m} B_{0,i}(K - F_{0,i-1}).$$

# Some Basic Remarks on Pricing

Consider a security that makes a single payment of amount $V_m(\omega_1, \cdots, \omega_m)$ at time $m$. For $0 \leq n < m$, the price $V_n$ of this security at time $n$ is given by

$$V_n = \frac{1}{D_n}\tilde{\mathbb{E}}_n[D_m V_m].$$

This is the same as

$$V_n = \tilde{\mathbb{E}}_n\left[\frac{V_m}{(1+R_n)\cdots(1+R_{m-1})}\right].$$

The term $(1+R_n)$ can be taken outside of the conditional expectation because it is known at time $n$. In particular, we have

$$V_n = \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}].$$

The prices $V_n$ for $n < m$ can be computed by backward induction:

$$V_n = \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}].$$

If $\tilde{\mathbb{P}}$ is a binomial product measure with probability of heads $\tilde{p}$ and probability of tails $\tilde{q}$ then

$$\tilde{\mathbb{E}}_n[V_{n+1}](w) = \tilde{p}V_{n+1}(w, H) + \tilde{q}V_{n+1}(w, T).$$

Here $w = (\omega_1, \cdots, \omega_n)$.

# An Extended Example

**Example 5.1**: Consider a 3-period Ho-Lee Model with $a_0 = .05$, $a_1 = .045$, $a_2 = .04$, $b_1 = b_2 = .01$, and $\tilde{\mathbb{P}}$ a BPM with probability of heads =.5 Observe that

$$R_0 = .05, \quad R_1(H) = .055, \quad R_1(T) = .045,$$

$$R_2(H,H) = .06, \quad R_2(H,T) = R_2(T,H) = .05, \quad R_2(T,T) = .04.$$

The discount process is given by $D_0 = 1$,

$$D_1 = \frac{1}{1.05}, \quad D_2(H) = \frac{1}{(1.05)(1.055)}, \quad D_2(T) = \frac{1}{(1.05)(1.045)},$$

$$D_3(H,H) = \frac{1}{(1.05)(1.055)(1.06)}, \quad D_3(H,T) = \frac{1}{(1.05)(1.055)(1.05)},$$

$$D_3(T,H) = \frac{1}{(1.05)(1.045)(1.05)}, \quad D_3(T,T) = \frac{1}{(1.05)(1.045)(1.04)}.$$

The prices of the zero-coupon bonds at time 0 are given by

$$B_{0,1} = \frac{1}{1.05} = .9523810,$$

$$B_{0,2} = \frac{1}{2}[\frac{1}{(1.05)(1.055)} + \frac{1}{(1.05)(1.045)}] = .9070500,$$

$$B_{0,3} = \frac{1}{4}[\frac{1}{(1.05)(1.055)(1.06)} + \frac{1}{(1.05)(1.055)(1.05)}$$

$$+ \frac{1}{(1.05)(1.045)(1.05)} + \frac{1}{(1.05)(1.045)(1.04)}] = .8639160.$$

The 3-period swap rate is given by

$$SR_3 = \frac{1 - B_{0,3}}{B_{0,1} + B_{0,2} + B_{0,3}} = .049969.$$

Let $V$ denote a European call option with exercise date 2 and strike price \$95 on a zero-coupon bond having face value \$100 and maturity 3. We shall compute the time 0 price of this option.

We first compute the bond prices

$$B_{2,3}(H,H) = \frac{1}{1.06} = .9433962, \quad B_{2,3}(T,T) = \frac{1}{1.04} = .9615385,$$

$$B_{2,3}(H,T) = B_{2,3}(T,H) = \frac{1}{1.05} = .9523810.$$

The option values at time 2 are given by

$$V_2(H,H) = 0, \quad V_2(T,T) = 96.15385 - 95 = 1.15385,$$

$$V_2(H,T) = V_2(T,H) = 95.23810 - 95 = .23810.$$

Using *backward induction* we find that

$$V_1(H) = \frac{1}{1.055}[.5(0) + .5(.23810)] = .112844,$$

$$V_1(T) = \frac{1}{1.045}[.5(.23810) + .5(1.15385)] = .666005,$$

$$V_0 = \frac{1}{1.05}[.5(.112844) + .5(.666005)] = .370880.$$

# Backward Induction

We can compute time-0 prices by computing the risk-neutral expected value of the (discount process times the payoff). The idea behind backward induction is that we can compute time-0 prices by using a one-period binomial model to go from time $N$ back to time $N-1$, then use another one-period binomial model to go from time $N-1$ back to time $N-2$, and continue back to time 0. Iterated conditioning ensures that the result is the same.

# Martingales

Recall that:

**Def**: An adapted process $(M_n)_{0 \le n \le m}$ is said to be a *martingale* under $\tilde{\mathbb{P}}$ provided that

$$M_n = \tilde{\mathbb{E}}_n[M_{n+1}], \quad n = 0, 1, 2, \cdots, m-1.$$

Martingales have many "magical" properties that are very useful in computational finance.

*For a security that makes a single payment of amount $V_m$ at time $m$, the discounted price process $(D_n V_n)_{0 \le n \le m}$ is a martingale under the risk-neutral measure.*

# Backward Induction Algorithm

Consider a security that makes a single payment of amount $V_m$ at time $m$.

Start at time $m$ and work backward by using

$$V_n = \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}],$$

for $n = m-1, m-2, \cdots, 1, 0.$

# Securities with Multiple Payments

Consider a security that makes payments $A_n(\omega_1, \cdots, \omega_n)$ at each of the times $n = 1, 2, \cdots, m$. We can determine the price at time 0 by viewing this as a string of $m$ securities each making one payment and pasting together the results. We can also do the following. Let $V_n$ be the price of the security at time $n$ after the payment $A_n$ has been made. We can use backward induction:

$$V_m = 0; \quad V_n = \frac{1}{1 + R_n} \tilde{\mathbb{E}}_n [A_{n+1} + V_{n+1}],$$

for $n = 0, 1, \cdots, m - 1$.

Notice that with multiple payments, the discounted price process will not be a martingale under $\tilde{\mathbb{P}}$.

**Remark 5.4**: It is, of course, possible to make the convention that $V_n$ represents the value of the security just *before* the payment at time $n$ is made. This will change the formulas in a "natural" way. Be careful to check and explain what convention is being employed. Also for coupon bonds, care must be exercised in talking about the value at maturity because the last coupon payment and the payment of the principal are sometimes treated as two separate payments and sometimes as one single payment.

# Computational Simplifications

Since there are $2^N$ paths in $\Omega$ we must try to organize our computations in an efficient manner when there are more than just a few periods. If we compute naively, then we may be doing the same thing over and over again many times.

It may be possible to take advantage of special structure in the interest rate tree an eliminate a significant number of computations.

# Number of Heads as a State Variable

A very important situation where much simplification is possible occurs when the the short rate at time $n$ depends only on $\omega$ only through the number of heads up to time $n$, i.e.

$$(\#H) \qquad R_n(\omega_1, \cdots, \omega_n) = r_n(\#H(\omega_1, \cdots, \omega_n)),$$

and the pricing measure is a binomial product measure.

Assume that $(\#H)$ holds and that $\tilde{\mathbb{P}}$ is a binomial product measure. For a security that makes a single payment $V_m$ at time $m$ where $V_m(\omega_1, \cdots, \omega_m)$ can be expressed as function of $\#H(\omega_1, \cdots, \omega_m)$ alone, there are functions $v_n : \{0, 1, \cdots, n\} \to \mathbb{R}$ such that price of the security at time $n$ is given by

$$V_n(\omega_1, \cdots, \omega_n) = v_n(\#H(\omega_1, \cdots, \omega_n)),$$

for all $n = 0, 1, \cdots, m-1$. The functions $v_n$ can be computed by backward induction:

$$(SP) \qquad v_n(k) = \frac{1}{1 + r_n(k)}[\tilde{p}v_{n+1}(k+1) + \tilde{q}v_{n+1}(k)].$$

Here $\tilde{p}$ and $\tilde{q}$ are the risk-neutral probabilities of heads and tails.

**Example 5.2**: Consider a 5-period Black-Derman-Toy Model with $a_n = .05 - .005n$ for $n = 0, \cdots, 4$ and $b_n = 1.3$ for $n = 1, \cdots, 4$, and $\tilde{\mathbb{P}}$ is binomial product measure with probability of heads equal to .5. Recall that

$$R_n(\omega_1, \cdots, \omega_n) = (.05 - .005n)(1.3)^{\# H(\omega_1, \cdots, \omega_n)}.$$

Let us compute the prices of a zero-coupon bond with maturity 5 and an interest rate caplet $V$ that makes a single payment of $V_5 = 1,000(R_4 - .05)^+$ at time 5.

We shall make use of the fact thats (#H) holds, $\tilde{\mathbb{P}}$ is a binomial product measure, and that the prices of the securities at time 4 can be expressed as functions of $\#H(\omega_1, \cdots, \omega_4)$. We shall use $\#H$ as our state variable in the backward induction.

We begin by noting that

$$B_{4,5} = \frac{1}{1 + R_4,}$$

$$V_4 = \frac{1,000(R_4 - .05)^+}{1 + R_4}.$$

We use $g_n$ and $v_n$ for the prices of the bond and the caplet, respectively, at time n.

We summarize the time 4 interest rates and prices in a table.

# Time-4 Interest Rates and Prices

| $\#H(\omega_1, \cdots, \omega_4)$ | $r_4$ | $g_4$ | $v_4$ |
|---|---|---|---|
| 0 | .03 | .970874 | 0 |
| 1 | .039 | .962464 | 0 |
| 2 | .0507 | .951746 | .666223 |
| 3 | .06591 | .938166 | 14.9262 |
| 4 | .085683 | .921079 | 32.8669 |

We compute the time 3 values using the formulas

$$g_3(k) = \frac{1}{1 + r_3(k)}[(.5)g_4(k+1) + (.5)g_4(k)],$$

$$v_3(k) = \frac{1}{1 + r_3(k)}[(.5)v_4(k+1) + (.5)v_4(k)].$$

| $\#H(\omega_1, \omega_2, \omega_3)$ | $r_3$ | $g_3$ | $v_3$ |
|---|---|---|---|
| 0 | .035 | .933980 | 0 |
| 1 | .0455 | .915452 | .3186145 |
| 2 | .05915 | .892183 | 7.360819 |
| 3 | .076895 | .863243 | 22.19023 |

# Interest Rates and Prices at Times 2 and 1

| $\#H(\omega_1, \omega_2)$ | $r_2$ | $g_2$ | $v_2$ |
|---|---|---|---|
| 0 | .04 | .88915 | .1531800 |
| 1 | .052 | .859142 | 3.649921 |
| 2 | .0676 | .822137 | 13.839930 |

| $\#H(\omega_1)$ | $r_1$ | $g_1$ | $v_1$ |
|---|---|---|---|
| 0 | .045 | .836503 | 1.81967 |
| 1 | .0585 | .794180 | 8.26162 |

# Time-0 Prices

The time 0 prices are given by

$$B_{0,5} = g_0 = \frac{1}{1.05}[(.5)g_1(1) + .(5)g_1(0)] = .776516,$$

$$V_0 = v_0 = \frac{1}{1.05}[(.5)v_1(1) + (.5)v_1(0)] = 4.80061.$$

# A Useful Variation

Assume that $(\#H)$ holds and that $\tilde{\mathbb{P}}$ is a binomial product measure. Consider a security that makes a single payment of amount $V_m$ at time m, where $V_m$ can be expressed as a function of $\#H(\omega_1, \cdots, \omega_{m-1})$. The price of this security at time $m-1$ is given by

$$V_{m-1} = \frac{V_m}{1 + R_{m-1}}.$$

This means that for $n = 1, 2, \cdots, m-1$, there are functions $v_n : \{0, 1, \cdots, n\} \to \mathbb{R}$ such that

$$V_n(\omega_1, \cdots, \omega_n) = v_n(\#H(\omega_1, \cdots, \omega_n)),$$

and we can use (SP) to compute $v_0, v_1, \cdots, v_{m-2}$.

# Securities with Multiple Payments

Assume that $(\#H)$ holds and that $\tilde{\mathbb{P}}$ is a binomial product measure. Consider a security that pays $A_n$ at each of the times $n = 1, 2, 3, \cdots, m$, where each $A_n$ can be expressed as a function of either $\#H(\omega_1, \cdots, \omega_n)$ or $\#H(\omega_1, \cdots, \omega_{n-1})$. Let $V_n$ denote the price of the security at time $n$, just after the payment $A_n$. Then there exist functions $v_n : \{0, 1, \cdots, n\} \to \mathbb{R}$ such that

$$V_n(\omega_1, \cdots, \omega_n) = v_n(\#H(\omega_1, \cdots, \omega_n)),$$

for all $n = 0, 1, \cdots, m - 1$.

# Securities with Multiple Payments (Continued)

We record below the algorithm for the case when each payment can be expressed as a function of the number of heads one period earlier. In particular, we consider a security that pays the amount $a_n(\#H(\omega_1, \cdots, \omega_{n-1}))$ at each of the times $n = 1, 2, \cdots, m$. We write $v_n(k)$ for the price of the security at time $n$, after the payment has been made, and given that the number of heads up to time $n$ is $k$. We initialize with $v_m = 0$, and for $n = m-1, m-2, \cdots, 1, 0$ we have

$$v_n(k) = \frac{a_{n+1}(k)}{1 + r_n(k)} + \frac{1}{1 + r_n(k)}[\tilde{p} v_{n+1}(k+1) + \tilde{q} v_{n+1}(k)].$$

# American & Bermudan Options

An *American derivative security* or *American option* is characterized by an expiration date $m \in \{0, 1, 2, \cdots, N\}$ and an adapted process $(G_n)_{0 \le n \le m}$ called the *intrinsic value process*. Here we assume that $G_n \ge 0$ for all $n = 0, 1, 2, \cdots, m$ At each date $n = 0, 1, \cdots, m$, assuming that the option has not been exercised yet, the holder decides, based on the information available at time $n$, whether to exercise now and collect $G_n$ or to wait. The option cannot be exercised more than once.

A complete theory of such securities involves the notion of a *stopping time* and results about martingales, submartingales, and supermartingales composed with stopping times. This is treated in MPAP. Here we just give a brief discussion of the backward induction algorithm.

If we arrive at time $m$ and the security has not been exercised, we cannot wait any longer. The value $V_m$ is simply the intrinsic value, i.e.

$$V_m = G_m.$$

If we are sitting at an earlier time $n$, we compute the "value of waiting" and compare this with the intrinsic value. The higher of these two values is taken to be the value $V_n$ of the security at time $n$, i.e.

$$V_n = \max\left\{G_n, \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}]\right\}.$$

It is optimal to exercise the option at the smallest time $n$ such that $V_n = G_n$.

**Remark 5.5**: Assume that $\tilde{\mathbb{P}}$ is a binomial product measure. If the values of $R_n$ and $G_n$ both depend only on $n$ and the number of heads up to time $n$ we can use the number of heads up to time $n$ as a state variable in the backward induction procedure.

**Remark 5.6**: It is well known that if interest rates are positive, then it is not optimal to exercise an American call on a stock that does not pay dividends early. However, for a dividend-paying stock or a coupon bond, it may well be optimal to exercise an American call early.

# Bermudan Options

A *Bermudan option* has a (nonempty) set $\mathcal{E} \subset \{0, 1, \cdots, N\}$ of possible exercise dates and an intrinsic value process $(G_n)_{0 \leq n \leq N}$. Again, we assume here that $G_n \geq 0$. The option can be exercised once, at any date $n \in \mathcal{E}$. A simple (and obvious) modification of the American backward induction algorithm works for Bermudan options. Let $m$ be the largest element of $\mathcal{E}$. We set $V_m = G_m$. For earlier dates $n$: If $n$ is not a possible exercise date, then

$$V_n = \frac{1}{1 + R_n} \tilde{\mathbb{E}}_n[V_{n+1}];$$

if $n$ is an exercise date then

$$V_n = \max \left\{ G_n, \frac{1}{1 + R_n} \tilde{\mathbb{E}}_n[V_{n+1}] \right\}.$$

**Example 5.3**: Consider a 3-period model in which $\tilde{\mathbb{P}}$ is a binomial product measure with probability of heads equal to .5 and

$$R_0 = .06, \quad R_1(H) = .066, \quad R_1(T) = .052,$$

$$R_2(H,H) = .08, \quad R_2(H,T) = R_2(T,H) = .07, \quad R_2(T,T) = .05.$$

Let us compute the price at time 0 of an American call option with expiration date 3 and strike price \$1,000 on a par-coupon bond with maturity 3 and face value \$1,000. (Here the option is to buy the bond for \$1,000 just after the coupon has been paid.)

It is straightforward to check that

$$B_{0,1} = .943396226, \quad B_{0,2} = .890875776, \quad B_{0,3} = .834670563.$$

The par-coupon rate is given by

$$q = \frac{1 - B_{0,3}}{B_{0,1} + B_{0,2} + B_{0,3}} = .061945671.$$

Let $W_n$ denote the price of the bond at time n just after the coupon has been paid. (Note that $W_0 = W_3 = 1,000$.)

# Time-2 Bond Prices

$$W_2(H,H) = \frac{1061.945671}{1.08} = 983.2830283,$$

$$W_2(H,T) = W_2(T,H) = 992.4725893,$$

$$W_2(T,T) = 1,011.376829.$$

# Time-1 Bond Prices

$$\begin{aligned} W_1(H) &= \tfrac{1}{1.066}[(.5)(983.280283) + (.5)(992.4725893) + 61.9456706] \\ &= 984.8237399, \end{aligned}$$

$$W_1(T) = 1,011.283631.$$

# Intrinsic Values of the Call

The intrinsic values of the call option are given by

$$G_0 = 0, \quad G_1(H) = 0, \quad G_1(T) = 11.283631,$$

$$G_2(H,H) = G_2(H,T) = G_2(T,H) = 0, \quad G_2(T,T) = 11.376829,$$

$$G_3 = 0.$$

# Time-2 Call Prices

Using the backward induction algorithm,

$$V_3 = 0,$$

$$V_2(H,H) = V_2(H,T) = V_3(T,H) = 0,$$

$$V_2(T,T) = \max\{11.376829, 0\} = 11.376829.$$

# Time-1 and Time-0 Call Prices

$$V_1(H) = 0,$$

$$\begin{aligned} V_1(T) &= \max\left\{11.283631, \tfrac{1}{1.052}[(.5)(0) + (.5)(11.376829)]\right\} \\ &= 11.283631, \end{aligned}$$

$$V_0 = \max\left\{0, \frac{1}{1.06}[(.5)(0) + (.5)(11.283631)]\right\} = 5.322467453.$$

# Optimal Exercise Policy

Notice that in this example the optimal exercise strategy is to exercise the call at time 1 if $\omega_1 = T$. If $\omega_1 = H$ then the call will always be worthless.

# Forwards & Futures

Forward contracts and futures contracts are both designed to lock in a price now for purchase of an asset in the future.

A *forward contract* is an agreement between two parties concerning the sale of an asset at a future date $m$, called the *delivery date*. The party taking the *short position* agrees to sell the asset at time $m$ at a set price $K$, called the *delivery price*. The party taking the *long position* agrees to buy the asset at the price $K$ at time $m$. The delivery price is chosen so that at the time contract is made, the value of both positions is zero.

For $n \leq m$ we define $\mathrm{For}_{n,m}$ to be the value of K that makes the price of both positions on the contract zero at time $n$.

The amount of the asset to be sold on the delivery date must be specified as part of the contract.

**Remark 5.7**: Although the delivery price is chosen so that the value of both positions on forward contract is zero initially, as time evolves, the values of the long and short positions will generally both be nonzero. (However, they must sum to zero.)

Let us assume that the asset in question is a security and that the price of the security at time $m$ is $P_m(\omega_1, \cdots, \omega_m)$. Then we must have

$$\mathrm{For}_{n,m} = \frac{\tilde{\mathbb{E}}_n[D_m P_m]}{\tilde{\mathbb{E}}_n[D_m]}.$$

**Remark 5.8**: We are not assuming here that $\tilde{\mathbb{E}}_n[D_m P_m]$ is equal to $D_n P_n$.

**Remark 5.9**: If the security pays no dividends or coupons, then $D_n P_n = \tilde{\mathbb{E}}_n[D_m P_m]$, where $P_n$ is the price of the security at time $n$ and we have

$$\mathrm{For}_{n,m} = \frac{P_n}{B_{n,m}}.$$

There are important practical difficulties with forward contracts. In particular, it may be difficult to find a party wanting to take the counterposition for the same quantity of the security on the same delivery date. Moreover, there is serious risk of default.

*Futures contracts* correct both of these difficulties and offer certain other advantages as well. Investors do not make contracts with one another, but with a central exchange. Before taking a futures position, an investor must open a *margin account*. All investors with contracts for delivery of a given asset at date $m$ have the same delivery price.

This price is adjusted every day; if it goes up, investors with a long position have money deposited in their margin accounts, and investors with short positions have money deducted from their margin accounts. If the balance of an investor's account becomes too low, the investor receives a *margin call* and must either close out their position or add more money to the account. The delivery price is adjusted in such a way such that gains or losses in an investor's position are reflected in the margin account – an investor who closes out the account early does not owe any money nor is he owed any money.

**Def**: An *m-futures process* is an adapted process $(\mathrm{Fut}_{n,m})_{0\le n\le m}$ such that

(*i*)
$$\mathrm{Fut}_{m,m} = P_m,$$

(*ii*)
$$\tilde{\mathbb{E}}_n\left[\sum_{i=n}^{m-1} D_{i+1}(\mathrm{Fut}_{i+1,m} - \mathrm{Fut}_{i,m})\right] = 0.$$

for $n = 0, 1 \cdots, m-1$.

An investor with a long position receives $\mathrm{Fut}_{k+1,m} - \mathrm{Fut}_{k,m}$ on day $k+1$ for $k = 0, 1, \cdots, m-1$.

Condition (ii) says that at each date $n$, the value of the future stream of adjustments to the margin account is zero.

**Proposition**: There is exactly one adapted process satisfying (i) and (ii); it is given by

$$(Fut) \qquad \mathrm{Fut}_{n,m} = \tilde{\mathbb{E}}_n[P_m].$$

Before proving this very important proposition, we recall two basic properties of conditional expectations.

# Two Important Properties of Conditional Expectations

Let $n, m$ be integers with $0 \leq n, m \leq N$ and let $X$ and $Y$ be random variables on $\Omega$

- Taking out (or putting in) something known: If $X$ is time-$n$ measurable (i.e. known at time n) then

$$\tilde{\mathbb{E}}_n[XY] = X \cdot \tilde{\mathbb{E}}_n[Y].$$

- Iterated Conditioning: If $0 \leq n \leq m$ then

$$\tilde{\mathbb{E}}_n\left[\tilde{\mathbb{E}}_m[Y]\right] = \tilde{\mathbb{E}}_n[Y].$$

**Remark 5.10**: Iterated Conditioning is frequently called the *tower property* of conditional expectations. In MPAP, it is called the *chain rule*.

# Proof of the Proposition on Futures Processes

**Proof**: Let us put

$$\mathrm{Fut}_{n,m} = \tilde{\mathbb{E}}_n[P_m].$$

We need to show that this process satisfies (i) and (ii) of the proposition.

Since $P_m$ is known at time $m$ (i.e. is time-$m$ measurable), we have

$$\mathrm{Fut}_{m,m} = \tilde{\mathbb{E}}_m[P_m] = P_m,$$

and (i) is satisfied.

To verify (ii), let $n \in \{0, 1, \cdots, m-1\}$ be given.

# Proof of Proposition (Cont.)

Then for each $i \in \{n, \cdots, m-1\}$ we have

$$
\begin{aligned}
\tilde{\mathbb{E}}_n[D_{i+1}(\mathrm{Fut}_{i+1,m} - \mathrm{Fut}_{i,m})] &= \tilde{\mathbb{E}}_n[D_{i+1}(\tilde{\mathbb{E}}_{i+1}[P_m] - \tilde{\mathbb{E}}_i[P_m])] \\
&= \tilde{\mathbb{E}}_n[\tilde{\mathbb{E}}_i[D_{i+1}(\tilde{\mathbb{E}}_{i+1}[P_m] - \tilde{\mathbb{E}}_i[P_m]]) \\
&= \tilde{\mathbb{E}}_n[D_{i+1}(\tilde{\mathbb{E}}_i[\tilde{\mathbb{E}}_{i+1}[P_m]] - \tilde{\mathbb{E}}_i[P_m])] \\
&= \tilde{\mathbb{E}}_n[(D_{i+1}(\tilde{\mathbb{E}}_i[P_m] - \tilde{\mathbb{E}}_i[P_m]))] \\
&= 0.
\end{aligned}
$$

Summing up from $i = 1$ to $n + 1$, we see that (ii) is satisfied.

# Proof of Proposition (Cont.)

Assume now that (i) and (ii) hold. We shall show that

$$(M) \qquad \mathrm{Fut}_{k,m} = \tilde{\mathbb{E}}_k[\mathrm{Fut}_{k+1,m}]$$

for $k = 0, 1, \cdots, m-1$. It follows from (i) and (M) that (Fut) holds. The idea is actually quite simple: we "strip off" terms from (ii) one at a time and show that each individual term is zero.

Let $k \in \{0, 1, \cdots, m-1\}$ be given. We put $n = k+1$ in (ii) and subtract the result from (ii) with $n = k$ and apply $\tilde{\mathbb{E}}_k$ to the result to obtain

$$\tilde{\mathbb{E}}_k[D_{k+1}(\mathrm{Fut}_{k+1,m} - \mathrm{Fut}_{k,m})] = 0.$$

Since $D_{k+1}$ is known at time $k$ (and is not zero) we conclude (M) holds. $\square$

# Carnegie Mellon University
# MSCF Program
# 46-956 Introduction to Fixed Income

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Mini 1

Lecture Notes for Week 6


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# Forwards & Futures Continued (Binomial Framework)

- Underlying Asset is a Security
- Delivery Date: $m$
- Price of Underlying at time $m$ is $P_m(\omega_1, \cdots, \omega_m)$

# Long Forward Position

- Contract initiated at time $n \le m$
- No payments made or received until time $m$
- Value of long position at time $m$ is $P_m - \mathrm{For}_{n,m}$.

# Long Futures position

- Contract initiated at time $j \leq m$
- Nothing is paid to enter the contract
- Long position receives $\mathrm{Fut}_{k+1,m} - \mathrm{Fut}_{k,m}$ at time $k+1$ for each $k = j, \cdots, m-1$
- $\mathrm{Fut}_{m,m} = P_m$
- Position can be closed out at no cost at any time

# Forward & Futures Prices

$$\mathrm{For}_{n,m} = \frac{\tilde{\mathbb{E}}_n[D_m P_m]}{\tilde{\mathbb{E}}_n[D_m]}$$

$$\mathrm{Fut}_{n,m} = \tilde{\mathbb{E}}_n[P_m]$$

**Remark 6.1**: If the underlying security pays no dividends or coupons then $\tilde{\mathbb{E}}_n[D_m P_m] = D_n P_n$ and

$$\mathrm{For}_{n,m} = \frac{P_n}{B_{n,m}}.$$

# Forward vs Futures Prices

It is natural to ask about the relationship between $\mathrm{For}_{n,m}$ and $\mathrm{Fut}_{n,m}$. Examining the formulas for forward and futures prices, we obtain the following result.

**Remark 6.2**:

(a) If $D_m$ and $P_m$ are uncorrelated under $\tilde{\mathbb{P}}$ then $\mathrm{For}_{0,m} = \mathrm{Fut}_{0,m}$.

(b) If $D_m$ and $P_m$ are positively correlated under $\tilde{\mathbb{P}}$ then $\mathrm{For}_{0,m} > \mathrm{Fut}_{0,m}$.

(c) If $D_m$ and $P_m$ are negatively correlated under $\tilde{\mathbb{P}}$ then $\mathrm{For}_{0,m} < \mathrm{Fut}_{0,m}$.

(Using conditional correlations we can obtain similar results for $n > 0$.)

**Example 6.1**: Consider a 3-period model in which $\tilde{\mathbb{P}}$ is a binomial product measure with probability of heads equal to .5 and

$$R_0 = .06, \quad R_1(H) = .066, \quad R_1(T) = .052,$$

$$R_2(H,H) = .08, \quad R_2(H,T) = R_2(T,H) = .07, \quad R_2(T,T) = .05.$$

Let us compute the forward prices and the futures prices for delivery at time 2 of a par coupon bond with face value 1,000 and maturity 3. (The bond is to be delivered ex-coupon at time 2.) This is the par coupon bond from Example 5.3. Let $W_n$ denote the price of the bond at time $n$ just after the coupon is paid.

Recall that

$$W_2(H,H) = 983.283, \quad W_2(T,T) = 1,011.377,$$

$$W_2(H,T) = W_2(T,H) = 992.473.$$

We find that

$$\tilde{\mathbb{E}}[D_2] = .890876, \quad \tilde{\mathbb{E}}[D_2 W_2] = 886.3747,$$

$$\mathrm{For}_{0,2} = \frac{886.3747}{.890876} = 994.948,$$

$$\mathrm{Fut}_{0,2} = \tilde{\mathbb{E}}[W_2] = 994.901.$$

Let us also find the forward price for the bond using replication. To replicate the long position, we purchase the bond and short $\mathrm{For}_{0,2}$ face of zero-coupon bonds maturing at time 2. Notice that by holding the bond we receive two coupon payments (61.946 each) that a holder of a long forward position is not entitled to. Therefore, we must also short 61.945671 face of a ZCB maturing at time 1 and 61.945671 face of a ZCB maturing at time 2. The initial capital of this strategy is

$$1,000 - \mathrm{For}_{0,2} B_{0,2} - 61.945671(B_{0,1} + B_{0,2}).$$

Setting this expression equal to zero and solving for $\mathrm{For}_{0,2}$ yields

$$\mathrm{For}_{0,2} = \frac{1,000 - 61.945671(.943396226 + .890875776)}{.890875776} = 994.94768.$$

**Example 6.2**: Consider a 3-period model in which $\tilde{\mathbb{P}}$ is a binomial product measure with probability of heads equal to .5 and

$$R_0 = .06, \quad R_1(H) = .066, \quad R_1(T) = .052,$$

$$R_2(H,H) = .08, \quad R_2(H,T) = R_2(T,H) = .07, \quad R_2(T,T) = .05.$$

Consider a security that pays $10,000R_2$ at time 3. To compute $\mathrm{For}_{0,3}$, we observe that

$$\begin{aligned} \tilde{\mathbb{E}}[D_3 R_2] &= \tilde{\mathbb{E}}[D_3(1+R_2)] - \tilde{\mathbb{E}}[D_3] \\ &= \tilde{\mathbb{E}}[D_2] - \tilde{\mathbb{E}}[D_3] \\ &= B_{0,2} - B_{0,3}. \end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\mathrm{For}_{0,3}}{10{,}000} &= \frac{\tilde{\mathbb{E}}[D_3 R_2]}{\tilde{\mathbb{E}}[D_3]} \\
&= \frac{B_{0,2} - B_{0,3}}{B_{0,3}} \\
&= \frac{B_{0,2}}{B_{0,3}} - 1 \\
&= F_{0,2},
\end{aligned}$$

where $F_{0,2}$ is the forward interest rate agreed upon at time 0 for a one-period loan initiated at time 2.

Using the the bond prices from Example 5.3, we find that

$$\mathrm{For}_{0,3} = 673.382.$$

The initial futures price is given by

$$\begin{aligned}\mathrm{Fut}_{0,3} &= \tilde{\mathbb{E}}[10,000R_2] \\ &= \frac{1}{4}[800 + 700 + 700 + 500] \\ &= 675\end{aligned}$$

Let us look at the cash flows for associated with a long forward position initiated at time 0 and held until time 3, and a long futures position initiated at time 0 and held until time 3 for this security.

Some very simple computations give

$$\mathrm{Fut}_{1,3}(H) = 750, \quad \mathrm{Fut}_{1,3}(T) = 600,$$

$$\mathrm{Fut}_{2,3}(H,H) = 800, \quad \mathrm{Fut}_{2,3}(T,T) = 500,$$

$$\mathrm{Fut}_{2,3}(H,T) = \mathrm{Fut}_{2,3}(T,H) = 700.$$

# Cash Flows for Long Forward Position

The table below shows the cash flows for a long forward position entered into at time 0 for delivery at time 3 and held until time 3.

| $\omega$ | **time 0** | **time 1** | **time 2** | **time 3** |
|---|---|---|---|---|
| HHH | 0 | 0 | 0 | 126.618 |
| HHT | 0 | 0 | 0 | 126.618 |
| HTH | 0 | 0 | 0 | 26.618 |
| HTT | 0 | 0 | 0 | 26.618 |
| THH | 0 | 0 | 0 | 26.618 |
| THT | 0 | 0 | 0 | 26.618 |
| TTH | 0 | 0 | 0 | (173.382) |
| TTT | 0 | 0 | 0 | (173.382) |

# Cash Flows for Long Futures Position

The table below shows the cash flows for a long futures position entered into at time 0 for delivery at time 3 and held until time 3.

| $\omega$ | **time 0** | **time 1** | **time 2** | **time 3** |
|---|---|---|---|---|
| HHH | 0 | 75 | 50 | 0 |
| HHT | 0 | 75 | 50 | 0 |
| HTH | 0 | 75 | (50) | 0 |
| HTT | 0 | 75 | (50) | 0 |
| THH | 0 | (75) | 100 | 0 |
| THT | 0 | (75) | 100 | 0 |
| TTH | 0 | (75) | (100) | 0 |
| TTT | 0 | (75) | (100) | 0 |

**Remark 6.3**: The reason for all the zeros at time 3 for the futures position is that for this security we have

$$\mathrm{Fut}_{3,3}(\omega_1, \omega_2, \omega_3) = \mathrm{Fut}_{2,3}(\omega_1, \omega_2).$$

**Exercise**: What happens if the delivery date is changed to time 2 (and the underlying still pays at time 3)? What happens if the underlying pays $10,000R_2$ at time 2 and the delivery date is time 2? This will be part of Assignment 6.

# Value of Long Forward Position

The table below shows the values for a long forward position entered into at time 0 for delivery at time 3 and held until time 3.

| $\omega$ | **time 0** | **time 1** | **time 2** | **time 3** |
|---|---|---|---|---|
| HHH | 0 | 66.658 | 117.239 | 126.618 |
| HHT | 0 | 66.658 | 117.239 | 126.618 |
| HTH | 0 | 66.658 | 24.877 | 26.618 |
| HTT | 0 | 66.658 | 24.877 | 26.618 |
| THH | 0 | (66.658) | 24.877 | 26.618 |
| THT | 0 | (66.658) | 24.877 | 26.618 |
| TTH | 0 | (66.658) | (165.126) | (173.382) |
| TTT | 0 | (66.658) | (165.126) | (173.382) |

# Time-3 Total Value of Long Futures Position

The table below shows the value at time 3 of the long futures position entered into at time 0 for delivery at time 3 and held until time 3, assuming that payments received at times 1 and 2 are invested in the bank account (at the short rate) until time 3.

| $\omega$ | **time 3** |
|---|---|
| HHH | 140.36 |
| HHT | 140.36 |
| HTH | 32.0465 |
| HTT | 32.0465 |
| THH | 22.577 |
| THT | 22.577 |
| TTH | (187.845) |
| TTT | (187.845) |

**Example 6.3**: Consider a 3-period Ho-Lee Model with $a_0 = .05$, $a_1 = .045$, $a_2 = .04$, and $b_1 = b_2 = .01$. The risk-neutral measure is a BPM with probability of heads equal to .5. Consider a zero-coupon with face value 100 and maturity 3. We are interested in forward and futures prices for delivery of the bond at time $m = 2$.

Which do we expect to be larger, $\mathrm{For}_{0,2}$ or $\mathrm{Fut}_{0,2}$? How we expect

$\mathrm{For}_{1,2}$ to be related to $\mathrm{Fut}_{1,2}$?

You are asked to compute these numbers on Assignment 6. You should get

$$\mathrm{For}_{0,2} = 95.24458, \quad \mathrm{Fut}_{0,2} = 95.24242,$$

$$\mathrm{For}_{1,2}(H) = \mathrm{Fut}_{1,2}(H) = 94.78886, \; \mathrm{For}_{1,2}(T) = \mathrm{Fut}_{1,2}(T) = 95.69598.$$

# Callable and Putable Bonds

Consider a general $N$-period binomial interest rate mode with interest rate process $(R_n)_{0\le n\le N-1}$. Let $F, q > 0$ and $m \in \{1, 2, \cdots, N\}$ be given.

Let $C^q_{n,m}$ denote the price at time $n$ of a coupon bond with maturity $m$ and face value \$1 (after the coupon is received).

Let $\mathcal{E} \subset \{1, 2, \cdots, m-1\}$ be a set of possible "exercise dates", and $(F_n)_{n\in\mathcal{E}}$ be a set of possible or call or put prices.

We consider a bond with maturity $m$, face value $F$, and coupon rate $q$.

The bond is *callable* with call dates $\mathcal{E}$ and call prices $(F_n)_{n\in\mathcal{E}}$ provided that each time $n \in \mathcal{E}$ (assuming that the bond has not already been called), the issuer has the right to pay the bond holder $Fq + F_n$ and is then relieved of the obligation to make any future payments. (The amount $Fq$ represents the coupon payment due at time $n$ and the amount $F_n$ is a payment that is made in place of paying $F$ at maturity.)

# Callable and Putable Bonds (Continued)

The bond is *putable* with put dates $\mathcal{E}$ and put prices $(F_n)_{n \in \mathcal{E}}$ provided that at each time $n \in \mathcal{E}$ (assuming that the bond has not already been put), the holder can sell the bond back to the issuer at the price $F_n$ after the coupon has been paid. (In other words, the holder receives the amount $Fq + F_n$ at time $n$ and no further payments.)

In both the callable and the putable case, if the optionality feature of the bond is exercised at time $n$, the holder of the bond receives $Fq + F_n$ at time $n$ and no future payments. The difference is that with a callable bond the issuer of the bond has the option to exercise and with a putable bond the holder of the bond has the option to exercise.

In order to understand how to compute the prices of callable and putable bonds, it is very useful to observe that

$$x - (x - K)^+ = \min\{x, K\},$$

and

$$x + (K - x)^+ = \max\{x, K\}.$$

Let $U$ be a Bermudan derivative security with exercise dates $\mathcal{E}$ and intrinsic values

$$G_n^{(U)} = (FC^q_{n,m} - F_n)^+,$$

and $W$ be a Bermudan derivative security with exercise dates $\mathcal{E}$ and intrinsic values

$$G_n^{(W)} = (F_n - FC^q_{n,m})^+.$$

# Callable and Putable Bonds (Cont.)

Let $V_n^c$ be the price at time $n$ of a callable bond with face value $F$, maturity $m$, coupon rate $q$, call dates $\mathcal{E}$, and call prices $(F_n)_{n\in\mathcal{E}}$, and let $V_n^p$ be the price at time $n$ of a putable bond with face value $F$, maturity $m$, coupon rate $q$, put dates $\mathcal{E}$, and put prices $(F_n)_{n\in\mathcal{E}}$. The prices of the callable and putable bonds at time $n$ are the prices after the coupon has been paid.

Then, for all $n \in \{0, 1, 2, \cdots, m\}$, we have

$$V_n^c = FC_{n,m}^q - U_n, \quad V_n^p = FC_{n,m}^q + W_n.$$

# Callable and Putable Bonds (Cont.)

We can also use backward induction more directly to price the callable and putable bond as follows:

If $n \notin \mathcal{E}$, and $n \leq m-1$, then

$$V_n^c = \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}^c + Fq],$$

$$V_n^p = \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}^p + Fq].$$

If $n \in \mathcal{E}$ then

$$V_n^c = \min\left\{F_n, \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}^c + Fq]\right\},$$

$$V_n^p = \max\left\{F_n, \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}^p + Fq]\right\}.$$

# Mortgages

Mortgages are loans secured by property. Until the 1970's banks made mortgage loans and held them until maturity.

In the 1970's *securitization* of mortgages began. Banks with limited capital available could make mortgage loans because these loans could be sold quickly and efficiently. This created new securities for investors.

Individual mortgages are grouped in pools and packaged in *mortgage-backed securities* (MBS). There are several different kinds of MBS. Before discussing them, we need to look at individual mortgages.

# Mortgages (Continued)

The most standard type of mortgage in the US is a *fixed rate* mortgage with *level payments*. (The most common maturities are 15 years and 30 years.) An individual borrows an amount $P$ and agrees to pay the bank the same amount $A$ every month for the next $T$ years.

Since almost all mortgages involve monthly payments, it is customary to use monthly compounding.

The *mortgage rate* $y$ is defined to be the yield to maturity of the mortgage, computed using the monthly compounding convention (and assuming that there are no prepayments).

# Mortgages (Cont.)

If we put

$$\lambda = \frac{1}{1 + \frac{y}{12}},$$

then

$$P = A \sum_{i=1}^{12T} \lambda^i = A\lambda \frac{1 - \lambda^{12T}}{1 - \lambda}.$$

# Mortgages (Cont.)

The mortgage rate is used to break each monthly payment down into an interest component and a principal component. This is very important in practice because mortgage interest is tax deductible. Let $B(n)$ denote the outstanding principal balance

after the $n^{th}$ payment has been made. (Note that $B(0) = P$, $B(12T) = 0$.) The interest component of the $n + 1^{st}$ payment is

$$B(n)\frac{y}{12},$$

and the principal component of this payment is

$$A - B(n)\frac{y}{12}.$$

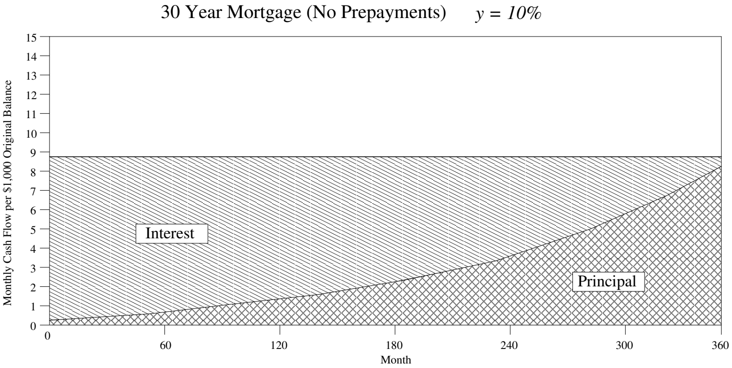
# Mortgages (Cont.)

The outstanding principal balance is reduced by the principal component of the payment.

Near the beginning of a mortgage, each payment is mostly interest; near the end of a mortgage the payments are mostly principal.

The outstanding principal balance immediately after the $k^{th}$ payment has been made is given by

$$A \sum_{i=1}^{12T-k} \lambda^i.$$



30 Year Mortgage (No Prepayments) y = 10%
Monthly Cash Flow per $1,000 Original Balance
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
0
60
120
180
240
300
360
Month
Interest
Principal

# Mortgages (Cont.)

Mortgages are "priced fairly" at the time of origination. Over time, as interest rates change, the present value of the remaining payments may be greater or less than the principal outstanding.

An important provision of most mortgages is that they allow *prepayment* of the outstanding principal balance at any time, relieving the homeowner of the obligation to make any future payments.

If interest rates have fallen, homeowners may decide to prepay their current mortgages and refinance at a lower rate. Of course, mortgages are sometimes prepaid even when rates have risen (for example when a home is sold). In order to model securities obtained by pooling mortgages, we need to model prepayments.

# Mortgage-Backed Securities

Individual mortgages are grouped together in pools and packaged as *mortgage-backed securities* (MBS). These securities fall into two basic classes:

I. Simple Pass Throughs
II. Mortgage Derivatives

In a pass through, each investor is entitled to receive a pro rata share of all cash flows (principal and interest). This is essentially equivalent to holding a mortgage, but it is averaged over a large pool.

# Mortgage Derivatives

Derivative mortgage securities divide principal and interest cash flows and allocate them to two or more classes of investors in a non pro rata fashion. The two most common types of mortgage derivatives are:

(i) Collateralized Mortgage Obligations (CMOs)

(ii) Stripped MBS: Interest Only (IOs) and Principal Only (POs)

# Prepayment Consequences

A naive “fair loan” treatment of a mortgage is not appropriate because of the possibility of prepayment. In practice homeowners, pay a higher mortgage rate (as opposed to paying the rate appropriate for a non-prepayable mortgage and receiving less than the face amount at the time of the loan.)

At origination of the loan, the present value of the future cash flows minus the value of the prepayment option equals the initial principal amount. The mortgage rate that satisfies this condition in the current interest rate environment is called the *current coupon rate* or *current mortgage rate*.

# Negative Convexity of Pass Through MBS

**Remark 6.4**: An important characteristic securities for which principal can be prepaid removing the obligation to make future interest payments is that prepayment is more likely when current rates are low and reinvestment is undesirable. This manifests itself in *negative convexity*. (For a PO strip, there are no interest payments, so that prepayment does not reduce the total amount of money received. Prepayments for a PO strip are good for the holder because the money is received earlier – future payments are not reduced – they are made earlier.)

# Factors Influencing the Value of MBS

- The current interest rate environment and investors feelings about future interest rate movements.
- The *weighted average maturity* WAM.
- The *weighted average coupon* WAC
- The speed of prepayments.

# Embedded Options in Bonds

When pricing bonds with embedded options, it is reasonable to assume that the issuers act in accordance with the basic mathematical principles of option pricing in an interest rate model – they will exercise an option if and only if the value of immediate exercise exceeds the discounted risk-neutral conditional expected value of holding.

In practice, homeowners do not behave as institutional investors. Of course, one reason for this is that a home is not just a *financial investment* but also used as a place to live. Also, even if a homeowner *is* motivated solely by financing considerations, he or she may not be equipped to make an optimal financial decision.

In addition to refinancing to get a better rate, mortgages are sometimes prepaid because the homeowner moves, gets divorced, or inherits money, or because of a disaster. This makes the valuation of MBS much more complicated than callable bonds, for example. (Some homeowners prepay a little bit extra each month and shorten their mortgage. This is known as *curtailment*. Curtailments generally represent a small percentage of prepayments.)

# Mortgage Rate

**Remark 6.5**: Even if prepayment decisions were based solely (and optimally) on financial considerations, it is still somewhat complicated to compute a mortgage rate from an interest rate model. It is similar to computing the coupon rate that will make a callable bond trade at par.

# Categories of Prepayments

For mortgages that were issued relatively recently, the main types of prepayments, in order of importance are

- Refinancing
- Turnover
- Defaults
- Curtailments

These effects are generally modelled separately.

# Simplistic Prepayment Assumptions

1. Twelve-Year Retirement (for 30-year loans): This model assumes that all mortgages are prepaid at the end of 12 years. This is the simplest and least useful model that we shall discuss.
2. Constant Monthly Mortality (Single Monthly Mortality): Assumes that the percentage of mortgages that will be prepaid each month is constant, say .5%. The monthly prepayment rate is denoted SMM. Usually the rate of prepayment is expressed as an annual rate called CPR (for *constant prepayment rate* or *conditional prepayment rate*). We note that

$$CPR = 1 - (1 - SMM)^{12}, \quad SMM = 1 - (1 - CPR)^{\frac{1}{12}}.$$

3. FHA Experience: The Federal Housing Administration has a large base of historical data on actual prepayments. The FHA data give the probability $x_n$ that a 30-year mortgage will survive to the end of year $n$, $n = 1, 2, \cdots, 30$.

The probability that the mortgage will be prepaid during year $n$ is

$$p_n = x_{n-1} - x_n.$$

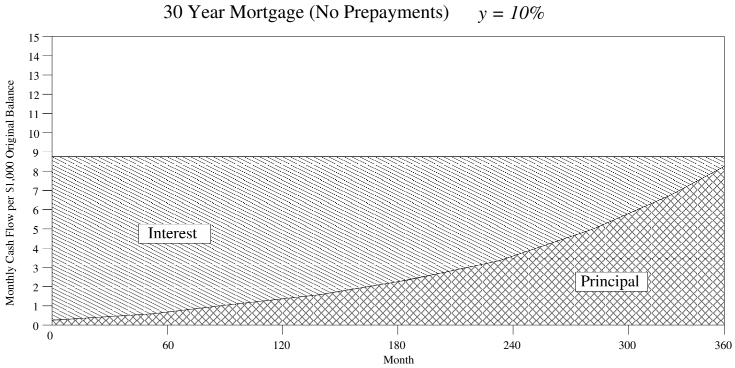
The conditional probability $y_n$ that the mortgage will survive through year $n$, given that it has survived through year $n-1$ is

$$y_n = \frac{x_n}{x_{n-1}}.$$

These probabilities can be converted to monthly survival probabilities. Use of these probabilities is referred to as 100% FHA experience.

4. PSA Standard Prepayment Model: The Public Securities Agency standard prepayment model assumes that an annual prepayment rate of .2% during the first month, .4% during the second month, .6% during the third month, increasing linearly until 6% during the $30^{th}$ month, and then holding level at 6%.

   The next few pages show some graphs of principal and interest payments for a 30-year mortgage with $y = 10\%$ under several prepayment scenarios.


30 Year Mortgage (No Prepayments) y = 10%
Monthly Cash Flow per $1,000 Original Balance
15
14
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Interest
Principal
0
60
120
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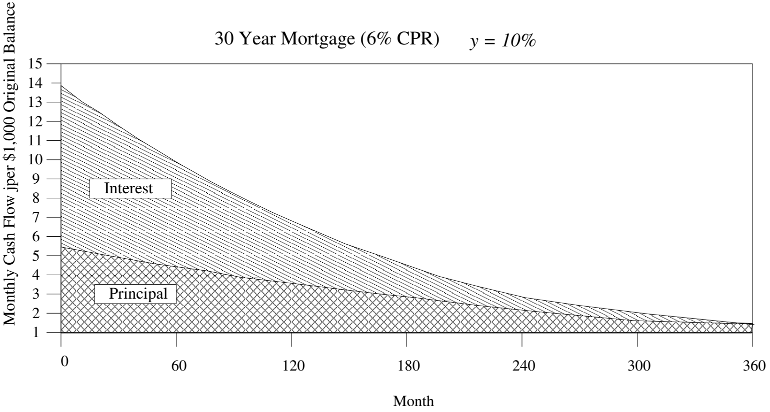
240
300
360
Month


30 Year Mortgage (6% CPR)
y = 10%
Monthly Cash Flow jper $1,000 Original Balance
15
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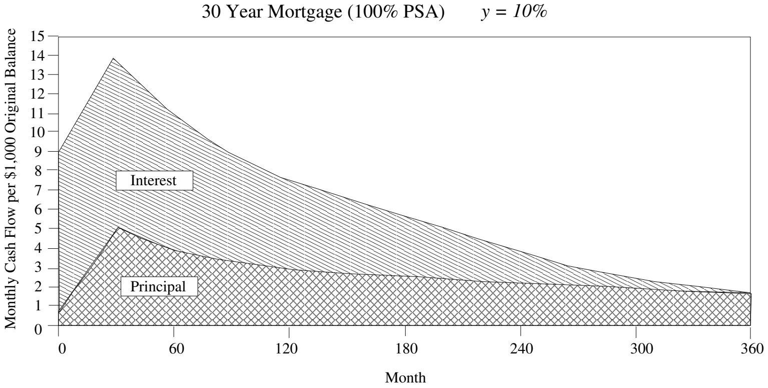
2
1
Interest
Principal
0
60
120
180
240
300
360
Month


30 Year Mortgage (100% PSA) y = 10%
Monthly Cash Flow per $1,000 Original Balance
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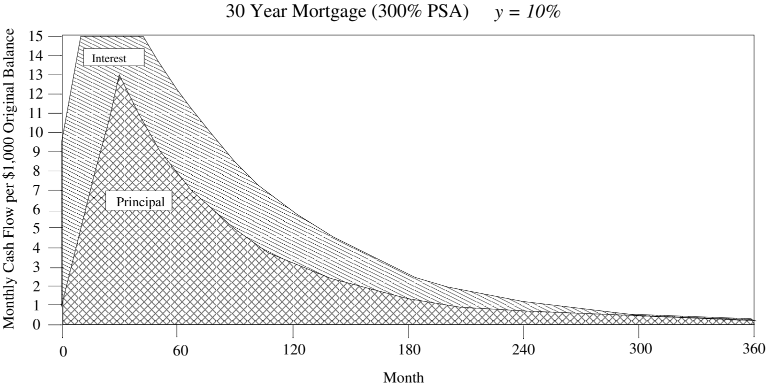
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
Interest
Principal
0
60
120
180
240
300
360
Month


30 Year Mortgage (300% PSA) y = 10%
Monthly Cash Flow per $1,000 Original Balance
Interest
Principal
Month
0
60
120
180
240
300
360
0
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12
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14
15

**Remark 6.6**: Most FHA mortgages are assumable (meaning that if the home is sold, the buyer can simply take over the current mortgage). Use of FHA data for other mortgages tends to underestimate the rate of prepayments.

**Remark 6.7**: The PSA prepayment model was introduced in 1985. It is really a *benchmark* that is used in the industry. Practitioners talk about 200% PSA and 300% PSA, etc. It is the annual rate that is multiplied by 2 or 3 in these cases. In 200% PSA the annualized rate levels out at 12%.

**Example 6.4**: Consider a pool of \$100 million of newly issued 30-year mortgages with mortgage rate $y = .09$. Let us compute the amount of interest and principal paid at time 1 month assuming 150% PSA prepayments.

We first compute the total payment due each month assuming no prepayments. Let $\lambda = (1 + .09/12)^{-1}$. Then we have

$$100,000,000 = A \sum_{1=i}^{360} \lambda^{i} = A \frac{\lambda}{1-\lambda}(1 - \lambda^{360}).$$

Solving for $A$ gives $A = 804,622.62$.

The amount of interest due at $t = 1$ month is

$$100,000,000 \times \left(\frac{.09}{12}\right) = 750,000.$$

The scheduled principal payment at $t = 1$ month is therefore

$$804,622.62 - 750,000 = 54,622.62.$$

To compute the prepayments at $t = 1$ month we first note that for month 1, the PSA standard is an annualized rate of .2%. This means that 150% PSA corresponds to an annualized rate of .3%. This is equivalent to a monthly rate of

$$SMM = 1 - (1 - .003)^{\frac{1}{12}} = .000250.$$

The prepayments are computed by applying the SMM to the remaining principal to obtain

$$.000250 \times (100,000,000 - 54,622.62) = 24,986.34.$$

The outstanding principal after the payments are received at $t = 1$ month is

$$100,000,000 - 54,622.62 - 24,986.34 = 99,920,391.04.$$

Models 1 thru 4 presented above are too simplistic to obtain quantitative results in practice. In particular, they do not account for the fact that there are more prepayments when current interest rates are low. The models that are actually used by financial institutions are largely proprietary and they are very complicated. One of the first pricing models for MBS to account for refinancing incentives is due to (Kenneth) Dunn & McConnell in 1981. A recent model due to Kalotay, Yang & Fabozzi divides homeowners who prepay to obtain a better rate into three categories. We shall do some examples with a model that accounts for increased prepayments when rates are low in a very simple way.

# Pass Through MBS

There are a lot of institutional details and conventions (accrued interest, etc.) involved with trading MBS. The *pool factor* $p_f(t)$ at time $t$ is defined by

$$p_f(t) = \frac{B(t)}{P},$$

where $B(t)$ is the outstanding principal balance at time $t$ and $P$ is the original balance.

There is a delay between the time that mortgage payments are made by homeowners and the time that holders of MBS receive the money. This delay is significant in analyzing the "true yield" of a MBS. (The actual delay varies from one agency to another. The agencies make significant income from the "float" earned between the time they collect cash flows and disburse them to investors.)

Investors typically get about 50 basis points less than the coupon rate of the pool.

# Ginnie Mae

**GNMA (Ginnie Mae)**: Government National Mortgage Agency (formed by US Congress in 1968). GNMA is a wholly owned government corporation within the US Dept of HUD. Guarantees FHA- and VA-based MBS. Pass-through securities are backed by the full faith and credit of the US government. Pools are very homogeneous. GNMA covers relatively low-income homes. Prepayments have been less volatile than other types of pass-throughs. GNMA does not make or purchase loans, nor does it issue or sell securities. It guarantees MBS issued by approved private lending institutions.

GNMA MBS are the only ones that are "officially" backed by the full faith and credit of the US government.

# Fannie Mae

**FNMA (Fannie Mae)**: Federal National Mortgage Association (founded as a government agency by the US Federal government in 1938). In 1968 it changed to a shareholder-owned corporation with a federal charter. After the credit crisis of 2007-2008, the government placed FNMA in conservatorship. Buys FHA, VHA, and other conventional loans which may have much higher value. Pools are more heterogeneous than GNMA. FNMA maintains a large mortgage portfolio and issues debt, providing liquidity in the mortgage market. FNMA guarantees the the full and timely payment of interest and principal and interest, but this guarantee is not backed by the full faith and credit of the US government. However, FNMA does have a significant credit with the US Treasury.

# Freddie Mac

**FHLMC (Freddie Mac)**: Federal Home Loan Mortgage Company (chartered by US government in 1970). Created by Congress – provides a link between mortgage lenders and capital markets. It was also a shareholder-owned corporation until the credit crisis of 2007-2008. It buys mortgages from banks and sells pass-through securities. Pools are more heterogeneous than GNMA. FHLMC guarantees the full and timely payment of interest and principal, but this guarantee is not officially backed by the full faith and credit of the US government. However, FHLMC does have a significant credit with the US Treasury.

**Remark 6.8**: There are also private label pass throughs that provide a secondary market for mortgages that do not qualify for agency pass throughs.

**Example 6.5**: Consider an investor who is holding \$50 million original par value of GNMA issued some time in the past and wishes to sell all of the shares now. The current pool factor is .87 and the current price is 90.625. Assume that the coupon rate is $y = .09$ and that settlement will take place 18 days into the month. Let us compute the transaction price.

$$P^{flat} = 50 \times (.87) \times (.9065) = 39.43275 \text{ million.}$$

The accrued interest is given by

$$AI = 50 \times (.87) \times (\frac{.09}{12}) \times (\frac{18}{30}) = .19575 \text{ million.}$$

The full price is given by

$$P^{full} = \$39,628,500.$$

# Binomial Mortgage Models

In a binomial world, we will consider $m$-period mortgages with a one-period mortgage rate $Y$. If the outstanding principal is $P$ and the payment each period is $A$ (assuming no prepayments) we have

$$P = A \sum_{i=1}^{m} \frac{1}{(1+Y)^i}.$$

# A Simple Prepayment Model

Consider a binomial interest rate model with interest rate process $(R_n)_{0 \leq n \leq N-1}$ and a mortgage pool with one-period coupon rate $Y$. The percentage of outstanding principal that is prepaid at time $n$ for $n = 1, 2, \cdots, N-1$ is assumed to be given by

$$\gamma_n = \alpha + \beta \cdot (Y - \mu - R_n)^+,$$

where $\alpha, \beta, \mu$ are given positive constants and $Y$ is the one-period mortgage rate for the pool. (If $P_n$ is the outstanding principal after the scheduled amount of principal is paid at time $n$, then $\gamma_n P_n$ is the amount of principal that is prepaid at time $n$.)

**Disclaimer**: This model is "not recommended for commercial use", but it has the right flavor.

# Burnout

The simple model just described does not account for *burnout*.

Prepayments for mortgage pools that were heavily refinanced in the past are not very sensitive to interest rate changes. The idea is that homeowners who would be inclined to refinance (and who would qualify to refinance) have already done so. This phenomenon is known as Burnout.

There is a very complicated “path dependence” for prepayments.

**Example 6.6**: Let us use the simple prepayment model to price a pass through security on a (previously issued) pool of mortgages having maturity 3 and one-period rate $Y = .10$. We assume that the risk-neutral measure is a BPM with probability of heads equal to .5, the interest rate process is given by

$$R_0 = .10, \quad R_1(H) = .12, \quad R_1(T) = .08,$$

$$R_2(H,H) = .14, \quad R_2(H,T) = R_2(T,H) = .10, \quad R_2(T,T) = .06,$$

and that $\alpha = .10$, $\beta = 10$, and $\mu = .01$.

We first compute the prepayment percentages at the various nodes:

$$\gamma_1(H) = \gamma_2(H, H) = \gamma_2(H, T) = \gamma_2(T, H) = .10,$$

$$\gamma_1(T) = .2, \quad \gamma_2(T, T) = .4.$$

We shall value the security per \$100 of principal outstanding at time 0.

The total payment due at each time $n = 1, 2, 3$ assuming no prepayments is $A = 40.21148$. This comes from the computation

$$100 = A(\lambda + \lambda^2 + \lambda^3),$$

with $\lambda = (1.1)^{-1}$.

At $n = 1$, the interest due is $10$, and the principal due is $30.21148$. This means that the outstanding principal (ignoring prepayments) will be $100 - 30.21148 = 69.78852.$ If the first toss is $H$ the amount of principal prepaid at $n = 1$ will be $.10 \times 69.78852 = 6.978852$ and the outstanding principal will become $62.80967$. If the first toss is $T$ the amount of principal prepaid at $n = 1$ will be $.2 \times 69.78852 = 13.95116$ and the outstanding principal will become $55.83736$. Notice that if the first toss is $H$ the holder of the MBS will receive $40.21148 + 6.978852$ at time $1$ (10 interest plus 37.19 principal) and if the first toss is $T$ the holder will receive $40.21148 + 13.95116$ (10 interest plus 44.17 principal).

The payments to be received at each date $n$, per \$100 of time-0 principal, are summarized in the table below. Here $I_n$ is the interest payment at time $n$ and $P_n$ is the principal payment at time $n$.

| $\omega$ | $I_1$ | $P_1$ | $I_2$ | $P_2$ | $I_3$ | $P_3$ |
|---|---|---|---|---|---|---|
| HHH | 10 | 37.19 | 6.28 | 33.20 | 2.96 | 29.61 |
| HHT | 10 | 37.19 | 6.28 | 33.20 | 2.96 | 29.61 |
| HTH | 10 | 37.19 | 6.28 | 33.20 | 2.96 | 29.61 |
| HTT | 10 | 37.19 | 6.28 | 33.20 | 2.96 | 29.61 |
| THH | 10 | 44.17 | 5.58 | 29.51 | 2.63 | 26.32 |
| THT | 10 | 44.17 | 5.58 | 29.51 | 2.63 | 26.32 |
| TTH | 10 | 44.17 | 5.58 | 38.28 | 1.76 | 17.55 |
| TTT | 10 | 44.17 | 5.58 | 38.28 | 1.76 | 17.55 |

Using backward induction, we find that the time-0 price per \$100 principal is $V_0 = \$99.89$.

It is interesting to note that

$$B_{0,1} = .90909, \quad B_{0,2} = .826720, \quad B_{0,3} = .75231,$$

which gives

$$40.21(B_{0,1} + B_{0,2} + B_{0,3}) = 100.05.$$

In other words, if prepayments were not allowed, the price would be \$100.05.

Observe that if the prepayment function takes interest rates into account, then the cash flows for a pass through will be "path dependent". For a large tree the computations can become unmanageable very quickly. Monte Carlo methods are often used.

It is interesting to note that in this example, if all mortgages were refinanced optimally (and there were no other prepayments) then the time-0 price of the pass through would be 99.17.

# Stripped MBS

The idea here is quite simple. The interest and principal payments are separated into two different securities called IOs and POs, respectively. We will compute IO and PO prices for the previous example shortly. IOs tend to lose value when rates fall and gain value when rates rise. (An IO strip is a nice example of a security with negative duration.) POs tend to increase in value when rates fall and lose value when rates rise. The prices of both types of securities are very sensitive to interest-rate changes.

**Remark 6.9**: There are also more complicated kinds of strips.

**Example 6.7**: Consider the pass through of Example 6.6, with the same interest rate tree and prepayment model. Let $V$ be an IO strip and $W$ be a PO strip.

In Assignment 6, you will be asked to compute the prices $V_0$ and $W_0$ of the IO and PO strips (per \$100 of time-0 principal.) You should get

$$V_0 = 15.91, \quad W_0 = 83.97.$$

It is interesting to note that if prepayments were not allowed, then the price of the IO strip would be 17.61 and the price of the PO strip would be 82.43.

# Carnegie Mellon University
# MSCF Program
# 46-956 Introduction to Fixed Income

Fall 2018 Mini 1

## Lecture Notes for Week 7: Course Wrap-Up – Assorted Topics

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October 10 & 15

# What You Need to Know for the Final

I. Discount factors, interest rate basics, compounding conventions, (annual, semiannual, monthly, continuous, simple interest), spot rates, forward rates.

II. Securities with deterministic cash flows; zero-coupon bonds; coupon bonds; annuities; arbitrage (law of one price); replication; basic idea of flat prices, full prices, and accrued interest.

III. Bond yields; yield to maturity; premium, par, and discount bonds; floating rate bonds & swaps; inverse floaters; DV01; duration; convexity; sensitivity analysis; hedging; ideas behind key rates.

IV. Term structure modeling in a binomial world; one-period compounding; discount process; Ho-Lee & Black-Derman-Toy models (binomial product measures with probability of heads equal to .5); risk-neutral pricing formula; bond prices and yields; forward rates; swap rates.

V. Backward induction; using the number of heads as a state variable in situations where $R_n$ depends only on $n$ and the number of heads up until time $n$.

VI. Fixed-income derivatives (in a binomial world); swaps; caps; floors; options (European, Bermudan, and American); forwards and futures, MBS (on exam, I will give you $I_n$ and $P_n$ for each $n$ or ask you to compute them for one step).

# Elaboration on Topics

Here, for material pertaining to I,II, and III, we assume that all times are multiples of 6 months and semiannual compounding is employed. When we speak about yields, we assume that the securities in question make deterministic and nonnegative payments, with at least one payment being strictly positive.

- Discount Factors in Terms of Spot Rates and Forward Rates

$$\frac{1}{d(t)} = \left(1 + \frac{\hat{r}(t)}{2}\right)^{2t} = \left(1 + \frac{f(.5)}{2}\right)\left(1 + \frac{f(1)}{2}\right)\cdots\left(1 + \frac{f(t)}{2}\right)$$

- To replicate a forward loan, you buy one ZCB and sell another.
- A Coupon Bond = An Annuity + A ZCB.
- If $f(T + .5) > q$ then holding the face and coupon rate fixed and increasing the maturity from $T$ to $T + .5$ will decrease the price of a coupon bond.

- Big Coupon Bond = Small Coupon Bond + Annuity
- If the spot rate curve is flat at level $y$, then all securities have YTM $y$.
- A bond trades at par if and only if $q = y$. A bond trades above par if and only if $q > y$.
- The YTM is between the smallest and largest spot rates corresponding to payment times.
- If you have a long position on two securities, the YTM of the portfolio is between the YTMs of the components.
- If the yield curve is upward sloping then $y_a(t) \leq y_{pc}(t) \leq \hat{r}(t)$
- If $\hat{r}(t) \geq \hat{r}(t - .5)$ then $f(t) \geq \hat{r}(t)$ (Also works for $>$)
- If the spot rate curve is upward sloping then so is the annuity yield curve and the par coupon yield curve.

- The par-coupon yield for maturity $T$ satisfies

$$F = Fd(T) + F\frac{y_{pc}(T)}{2}\sum_{i=1}^{2T} d\left(\frac{i}{2}\right).$$

- DV01s add.
- The duration [convexity] of a portfolio is a weighted average of the durations [convexities] of the pieces.
- For a ZCB with maturity $T$ we have

$$D_{Mac} = T, \quad C \approx T^2$$

# Floating Rate Bonds and Swaps

- The price of a floating rate bond just before a coupon reset equals face value.
- At initiation, a plain vanilla receiver swap can be replicated by going long a par-coupon bond and short a floating rate bond.
- The swap rate for a plain vanilla swap is equal to the par-coupon rate.
- *You should be able to determine a swap rate for a customized swap.*
- You should understand inverse floaters as well.
- The time-0 price of a payment of $\frac{F}{2} r_{t-.5,t}$ st time $t$ is

$$F[d(t-.5)-d(t)] = \frac{F}{2} f(t) d(t).$$

# Some More Thoughts on One-Factor Models

*For coupon bonds, you should understand the impact of face value, coupon, maturity, and yield on the basic sensitivity measures.*

*Keep in mind that is really easy to do computations for perpetuities.*

*You should understand how to apply DV01, duration, and convexity to basic trades such as barbells-versus-bullets and steepeners, flatteners.*

# Multifactor Sensitivity Analysis and Hedging

*You should understand the ideas behind key rate shifts and be able to solve a problem if given a table of numerical values.*

*The key rate shifts are generally chosen for mathematical convenience. Sometimes this is very helpful, but movements in actual term structures cannot be expected to behave like the key rate shifts. In practice the key rates are generally correlated with one another.*

- Binomial Short Rate Models; one-period compounding; discount process; Ho-Lee & Black-Derman-Toy models (binomial product measures with probability of heads equal to .5); conditional expectations and martingales, risk-neutral pricing formula; bond prices and yields; forward rates; swap rates.

- Backward Induction; using the number of heads as a state variable in situations where $R_n$ depends only on $n$ and the number of heads up until time $n$.

- Interest Rate Swaps, Caps, and Floors

- European, Bermudan and American Options

  *You should be able to analyze a new type of American or Bermudan option to decide if early exercise could be beneficial.*

- Callable and Putable Bonds

- Forwards and Futures

$$\mathrm{For}_{n,m} = \frac{\tilde{\mathbb{E}}_n[D_m P_m]}{\tilde{\mathbb{E}}_n[D_m]}, \quad \mathrm{Fut}_{n,m} = \tilde{\mathbb{E}}_n[P_m]$$

- Basic Ideas of MBS: Pass through and IO strips, PO strips.

  *You should understand the ideas behind prepayment modelling.*

  *On the exam, I would give you $I_n$ and $P_n$ for each $n$ or ask you to compute them for one step. I will provide the PSA prepayment formula if needed.*

I want to go back to a list of securities from the first lecture.

# Some Important Examples of Fixed-Income Securities

- Zero-Coupon Bonds
- Coupon Bonds
- Annuities
- Inflation Protected Bonds
- Floaters and Inverse Floaters
- Callable and Putable Bonds
- Interest Rate Swaps, Caps, Floors, Swaptions
- Interest Rate Futures (especially Eurodollar Futures)
- Mortgage Backed Securities
- Bond Options
- Bond Futures
- Options on Bond Futures

# Short-Rate DV01 in Binomial Models

Consider a security $V$ that is being priced in an $N$-period binomial interest rate model. The model gives prices at the end of the first time step in two different interest rate environments. We can use these prices to approximate the DV01 of the security.

We define the short-rate DV01 by

$$DV01^{SR} = -\frac{V_1(H) - V_1(T)}{10,000(R_1(H) - R_1(T))}.$$

**Example 7.1**: Consider a 10-period Ho-Lee model with

$$R_n = R_0 - .005n + .01\#H_n, \quad R_0 = .06$$

Let $V$ be a coupon bond with face value $F = 10,000$, one-period coupon rate $q = .06$, and maturity 10, and let $W$ be an inverse floater with face value $F = 10,000$ and maturity 10 which pays at each time $n = 1, 2, \cdots, 10$ a coupon with one-period coupon rate

$$Q_n = .12 - R_{n-1}$$

with a cap of .085 and a floor of .035 on the coupon rate. (This is the inverse floater from Assignment 5 with a face value of 10,000 instead of 1,000.)

Since the price of each of these securities is very close to 10,000, the DV01s should be approximately equal to durations. The short-rate DV01s are based on prices at time 1, so they should be lower than the true DV01s by somewhere in the neighborhood of 10%. (9 periods left to maturity instead of 10.)

You should check that for the fixed-coupon bond $V$ we have

$$DV01^{SR} = 6.823.$$

The (theoretical) yield-based DV01 for this bond is about 7.4.

For the inverse-floater $W$ we have

$$DV01^{SR} = 13.516.$$

One can divide by price and multiply by 10,000 to define short-rate duration.

Here is another way to approximate DV01 using a spreadsheet. We can change $R_0$ to .0601, compute a new price and subtract it from the original price. Implementing this procedure, I found:

- $DV01 = 7.3838$ for $V$
- $DV01 = 14.585$ for $W$

## Dollar Duration and Dollar Convexity

Sometimes practitioners talk about *dollar duration* and *dollar convexity*. These are simply ordinary duration and convexity, multiplied by price, i.e.

$$D^{\$} = -\frac{dP}{dy}, \quad C^{\$} = \frac{d^2 P}{dy^2}.$$

# Some Remarks on Calibration

Calibration of term structure models to market data will be discussed in several future classes, including Financial Computing IV and Studies in Financial Engineering. For people who want to look at something in a binomial framework (Ho-Lee and BDT) before then, here are a few references that I think you can find on the web. There are also two review problems concerning calibration.

**An inappropriate model cannot be rescued by adding lots of parameters and making an exact fit to current bond prices!**

# References for Calibrating Binomial Models

- John C. Hull, Webpage, Technical Note 23 (Under *Options, Futures, and Other Derivatives*.)
- Simon Benninga & Zvi Wiener, *Binomial Term Structure Models*
- Markus Leippold & Zvi Wiener, *Algorithms Behind Term Structure Models of Interest Rates, I*
- Christopher Klose & Li Chang Yuan *Implementation of the Black-Derman-Toy Model*

It is often to easier calibrate *continuous time* term structure models because there are many nice explicit formulas for certain models that do not have clean analogs in the discrete time case. For example, in the continuous-time Ho-Lee model

$$dr_t = \lambda_t \, dt + \sigma \, dW_t,$$

there is a beautiful formula expressing prices of ZCBs (of all maturities) in terms of the volatility parameter $\sigma$ and the drift process $\lambda_t$. For a given value of $\sigma > 0$, one can always choose the drift process to match the current terms structure, assuming some smoothness of the rates as a function of maturity. The volatility can be found from historical data or from prices of interest rate caps (or caplets).

We can discretize a continuous time model to get a binomial version.

# PCA: Covariance Matrices

Let

$$X = \begin{pmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_N \end{pmatrix}$$

be a vector-valued random variable (with real-valued components), i.e. a random vector. The covariance matrix $\mathbb{C}(X)$ of $X$ is the $N \times N$ matrix having components

$$(\mathbb{C}(X))_{ij} = \mathrm{Cov}(X_i, X_j) = \mathrm{Cov}(X_j, X_i).$$

Recall that $\mathbb{C}(X)$ is symmetric and positive semidefinite, i.e.

$$\mathbb{C}(X)^\top = \mathbb{C}(X), \quad x^\top \mathbb{C}(X) x \geq 0 \text{ for all } x \in V^N.$$

Let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{pmatrix} \in V^N$$

be given. Recall that

$$x \cdot X = x_1 X_1 + x_2 X_2 + \cdots + x_N X_N,$$

and that

$$\mathrm{Var}[x \cdot X] = x^\top \mathbb{C}(X) x.$$

Let us put

$$f(x) = \mathrm{Var}[x \cdot X], \quad g(x) = \sum_{i=1}^{N} x_i^2, \quad \text{for all } x \in V^N.$$

Using Lagrange Multipliers to maximize $f(x)$ subject to $g(x) = 1$, we find that

$$\mathbb{C}(X)x = \lambda x,$$

for some $\lambda \in \mathbb{R}$. The maximum value will therefore be the largest eigenvalue of $\mathbb{C}(X)$ and will be attained when $x$ is an eigenvector corresponding to the largest eigenvalue. We talked about this in the second Linear Algebra Lecture.

The eigenvalues of $\mathbb{C}(X)$ are real and nonnegative. We order them so that

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N.$$

One of the most important results from basic linear algebra concerns existence of an orthonormal basis of eigenvectors for real symmetric matrices. Applying this result to our situation tells us that we may choose $e^{(1)}, e^{(2)}, \cdots, e^{(N)}$ such that

$$\mathbb{C}(X)e^{(i)} = \lambda_i e^{(i)} \text{ for } i = 1, 2, \cdots, N,$$

$$e^{(i)} \cdot e^{(j)} = \begin{cases} 1 \text{ if } i = j \\ \\ 0 \text{ if } i \neq j. \end{cases}$$

Let $C \in \mathbb{R}^{N \times N}$ be the matrix whose $j^{th}$ column is $e^{(j)}$, i.e.

$$C_{ij} = (e^{(j)})_i.$$

Then $C$ is invertible and

$$C^{-1} = C^{\top}.$$

Moreover $C\mathbb{C}(X)C^{\top}$ is a diagonal matrix. In fact, we have

$$C\mathbb{C}(X)C^{\top} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N).$$

In other words $\mathbb{C}(X)$ is a diagonal matrix with $C_{ii} = \lambda_i$. Let us put

$$Z = C^{\top}X.$$

Then we have

$$\mathbb{C}(Z) = C\mathbb{C}(X)C^\top = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N).$$

The components of $Z$ are uncorrelated and

$$\mathrm{Var}(Z_i) = \lambda_i.$$

We can recover $X$ from $Z$ through the formula

$$X = CZ.$$

It is useful to observe that

$$X = CZ = Z_1 e^{(1)} + Z_2 e^{(2)} + \cdots + Z_N e^{(N)}. \tag{1}$$

Since the $\lambda$'s are ordered from largest to smallest, the "most significant" terms appear earlier on the right-hand side of (1).

# Yield Curve Dynamics and Hedging with PCA

Using a one-factor hedge based on duration and convexity is not really adequate because changes in spot rates of different maturities are not perfectly correlated. (There is more than one source of randomness driving the evolution of the spot rate curve.) In practice, spot rates certainly do not always shift in parallel.

Although the idea of "key rates" can provide a much better hedge, the shift functions do not have a natural economic interpretation. Furthermore, the method of key rate shifts involves changing the key rates "independently" and in practice, changes in rates of different maturities are correlated. A technique called *principal component analysis* (PCA) is designed to identify the "most important" kind of yield curve shifts that are observed empirically.

Let $y(t, T)$ be some kind of yield (e.g. ZCB yield or par yield) that prevails at time $t$ for borrowing or investing between time $t$ and $T$.

Let us choose a set of relative maturities

$$T_1 < T_2 < \cdots < T_N.$$

Typical values of $N$ in practice are somewhere around 10. A reasonable choice of relative maturities might look like

$$.25, .5, 1, 2, 3, 5, 7, 10, 30.$$

We want to model a random vector $X$ where

$$X_j(t) = y(t + \Delta t, t + \Delta t + T_j) - y(t, t + T_j), \quad j = 1, 2, \cdots, N,$$

and $\Delta t$ is a small positive increment.

In order for this to be useful (and possible to implement), we assume that the distribution of $X_j(t)$does not depend on $t$. This assumption turns out to be reasonable. We cannot compute an actual covariance matrix for $X$, so we use a sample covariance matrix. To this end, we fix a small positive time increment $\Delta t > 0$ and choose times

$$t_0, t_1, t_2, \cdots, t_M,$$

with

$$t_k - t_{k-1} = \Delta t, \quad \text{for } k = 1, 2, \cdots, M.$$

We observe

$$\Gamma_j^{(k)} = y(t_k, t_k + T_j) - y(t_{k-1}, t_{k-1} + T_j)$$

for $j = 1, 2, \cdots, N$ and $k = 1, 2, \cdots, M$.

**Remark 7.1**: When banks implement PCA analysis $\Delta t$ is usually taken to be one day. They don't worry about weekends because the changes in rates from Friday to Monday are usually about the same order of magnitude as the changes between consecutive week days. (Weekend volatility of rates is usually low.) Banks often use somewhere around 3 months of past data when they perform PCA. (They recompute them frequently.) In textbook examples sometimes $\Delta t$ is as large as one month. In academic studies 10 years (or more) of data is sometimes used.

Let $V$ be a (symmetric) sample covariance matrix for $\Gamma^{(1)}, \Gamma^{(2)}, \cdots, \Gamma^{(M)}$. We order the eigenvalues of $V$ so that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N,$$

and let $e^{(1)}, e^{(2)}, \cdots e^{(N)}$ be an orthonormal basis of associated eigenvectors.

We let $C$ be the $N \times N$ matrix whose $j^{th}$ column is $e^{(j)}$ and put

$$Z^{(k)} = C^T \Gamma^{(k)}, \ k = 1, 2, \cdots, M.$$

The sample covariance matrix of

$$Z^{(1)}, Z^{(2)}, \cdots, Z^{(M)}$$

is diagonal and the sample variance of

$$Z_j^{(1)}, Z_j^{(2)}, \cdots, Z_j^{(M)}$$

is $\lambda_j$.

Since $C^\top = C^{-1}$ we have

$$\Gamma^{(k)} = CZ^{(k)},\ k = 1, 2, \cdots, M.$$

What this really says is that

$$\Gamma^{(k)} = Z_1^{(k)} e^{(1)} + Z_2^{(k)} e^{(2)} + \cdots + Z_N^{(k)} e^{(N)}. \tag{2}$$

Notice that the coefficients of the eigenvectors on the right-hand side of (2) having larger sample variances come earlier in the list. This indicates that if we want to make a model for yield changes having fewer than $N$ random factors, we should look at linear combinations of the first few eigenvectors.

In practice 3 or 4 terms are usually retained (or sometimes only 2). For definiteness, let us use 3 terms. The eigenvectors are called *principal components*.

Our model for changes in the yield for relative maturity $T_j$ over a time interval of length $\Delta t$ will be

$$z_1 e_j^{(1)} + z_2 e_j^{(2)} + z_3 e_j^{(3)}.$$

If we let $y_j^*$ denote the current value of the yield for relative maturity $T_j$ then our new yield for relative maturity $T_j$ will be

$$y_j = y_j^* + z_1 e_j^{(1)} + z_2 e_j^{(2)} + z_3 e_j^{(3)},\ j = 1, 2, \cdots, N. \tag{3}$$

Here $z_1, z_2, z_3$ are our interest rate factors.

# Hedging with Principal Components

Suppose we want to hedge a short position on a security $S$ with a portfolio of securities. Here's what we should do: Use (3) to express the price of the security $S$ in the form $f(z_1, z_2, z_3)$ and express the price of the portfolio in the form $g(z_1, z_2, z_3)$. The idea is to match "factor DV01s". In other words, want to have

$$\frac{\partial f}{\partial z_i}(0,0,0) = \frac{\partial g}{\partial z_i}(0,0,0), \ i = 1, 2, 3.$$

We will generally want to have the number of securities in the portfolio match the number of principal components we are using so that the number of equations matches the number of unknowns. Here the unknowns will typically be face values of bonds.

**Remark 7.2**: It often happens that yields of relative maturities that are not on the original list of sampled yields will be needed. In practice, values for these are obtained by some kind of interpolation procedure. In other words, one uses interpolation to produce 3 curves of yield shifts: $Y_1(T)$, $Y_2(T)$, $Y_3(T)$ having

$$Y_i(T_j) = (e^{(i)})_j.$$

If $y(T)$ is the current value of the yield for relative maturity, our model says that a short time in the future the yield for relative maturity will be given by

$$y(T) + z_1 Y_1(T) + z_2 Y_2(T) + z_3 Y_3(T).$$

**Remark 7.3**: The shapes of principal components in a given economy are somewhat stable over time, athough there can be dramatic changes in the shapes of the components during extreme economic conditions such as the 2008 financial crisis. As we have from the graphs shown in class, principal components from different areas of the world can look quite different.

**Remark 7.4**: It is customary to call the first principal component a *parallel shift*, the second principal component a *slope shift*, and the third principal component a *curvature shift*. This terminology is consistent with the data shown in class.

# Treasury Inflation Protected Securities

TIPS stands for *Treasury Inflation Protected Securities*. These are coupon bearing bonds in which the face value is adjusted for inflation They were first issued by the Treasury in 1997, although the idea of inflation protected bonds in the US dates back to colonial times. (The Massachusetts Bay Company issued such bonds in 1780!)

The are currently being issued with maturities of 5, 10, and 30 years. They have a (fixed) annual coupon rate $q$ and an "original" face value $F$. The face value is adjusted periodically to a new value $F^{adj}$ using the *Consumer Price Index* (CPI-U). In times of deflation, the adjusted face value will go down.

Coupons are paid every 6 months and are computed by

$$F^{adj}\frac{q}{2}.$$

# TIPS (Cont.)

At maturity, the bond holder receives

$$\max\{F, F^{adj}\}.$$

The repayment of principal at maturity has protection at both ends!

# US Treasury Floating Rate Notes

In 2014, the US Treasury began issuing 2-year Floating Rate Notes (FRNs). They pay coupons quarterly and also pay face value at maturity.

The coupons rate is the 3-month T-Bill rate plus a spread. The spread is determined at auction and remains constant throughout the life of the bond. The floating rate is reset every week. Interest is credited daily (based on the face value), but is paid once every three months.

# LIBOR

The *London Interbank Offered Rate* (LIBOR) is a rate at which banks are willing to lend to counterparties with credit comparable to strong banks (in a market which is not subject to US regulations). The rate varies with term and is quoted on an actual/360 basis. The official name is ICE LIBOR (ICE stands for Intercontinental Exchange); It used to be called BBA LIBOR (British Banker's Association). Many other rates are keyed off of LIBOR.

# LIBOR (Continued)

LIBOR is quoted for five different currencies: USD, EUR, GBP, JPY, CHF (and for a number of different maturities). The most important LIBOR rates are the 3-month and the 6-month USD rates. Many interest rate swaps use LIBOR for the floating rate. It was announced last year that LIBOR will stop being quoted in 2021 – this leads to some serious issues. There was a LIBOR Fixing scandal – you should read about. You should also read about what will replace LIBOR.

Last week, USD 3-month LIBOR was approximately 2.44% (at the end of the week); one year ago, it was 1.36%. two years ago it was .87%.

# SOFR

In 2014 the Federal Reserve Board and the Federal Reserve Bank of New York announced creation of the *Alternative Rates Reference Committee* (ARRC) to assess viable alternatives to LIBOR. In June 2017 the ARRC announce a broad Treasury Repo Rate, the *Secured Overnight Financing Rate* (SOFR) as its recommended alternative to LIBOR. SOFR has been quoted daily since April 3, 2018.

The SOFR value on October 11, 2018 was 2.17%.

SOFR is based on actual trades, rather than on "opinions". SOFR Futures are traded on the CME.

# TED Spread

LIBOR is not risk free. There is a spread, called the *TED spread* between 3-month LIBOR and the 3-month T-Bill rate. (TED stands for Treasury-Eurodollar.) Initially, it was computed using futures prices. However, since the CME dropped T-Bill futures, it is now computed as the difference between 3-month LIBOR and the 3-month T-Bill rate.

Until 2007, the spread between the 3-month LIBOR and T-Bill rates typically varied from about 10 bp to 50 bp (with the average being about 30 bp). On September 17, 2008, the TED spread exceeded 300 bp. On October 10, 2008, it reached 457 bp. On October 5, 2018 the TED spread was about 23 bp (2.41% - 2.18%). On October 5, 2017, the TED spread was 30 bp (1.35% - 1.05%). On October 5, 2016, the TED spread was 55 bp (87 bp - 32 bp). On October 14, 2015 the TED spread was 31.98 bp. On September 30, 2014, the TED spread was 21.51 bp. .

# Interest Rate Swaps in Practice

The market for interest rate swaps is HUGE. (Several hundred trillion dollars notional face!)

They are traded OTC. Common maturities in the US are 2, 3, 5, 7, 10, and 30 years.

The most common type of swap in the US pays fixed twice per year and floating 4 times per year. The floating rate is 3 month USD LIBOR.

An extremely important issue with swaps is that collateral must be posted. Collateral earns interest, but **NOT** at the floating rate (or at the fixed rate) for the swap. This means that "textbook" formulas will not apply exactly.

# Some Important Variants of Swaps

Some important variants of standard swaps include

- Amortizing Swaps (The notional principal is adjusted over time according to a schedule.)
- Zero Swaps (There is no exchange of payments until maturity.)
- Putable Swaps (The receiver of fixed has the right to cancel the swap before maturity.)
- Callable Swaps ( The payer of fixed has the right to cancel the swap before maturity.)

# Swaptions

A swaption is an option to enter a swap at a future date at a prescribed fixed rate. They are traded OTC.

There are receiver swaps and payer swaptions. Both types are available in American, Bermudan, and European flavors.

For American (and Bermudan) swaptions, sometimes the maturity date of the swap is prescribed and sometimes the length of time that payments will be exchanged after the option is exercised is prescribed (instead of the date of the final payment).

# Negative Swap Spreads

In theory, a swap rate based on LIBOR should be higher than the par-coupon yield for US Treasuries of the same maturity, because LIBOR is not risk-free. The swap rate minus the corresponding treasury rate is known as the *swap spread*. The 30-year swap spread turned negative for the first time in August 2008, after the collapse of Lehman Brothers. The 10-year swap spread turned negative on March 23, 2010 and the 7-year swap spread turned negative shortly thereafter. The 5-year swap spread became negative for the first time in October 2015. (You should read about this to prepare for interviews.)

# Swap Spreads (Continued)

Last week, swap spreads for maturities 2, 5, 7, 10, and 30 years were

$$18, \ 12.5, \ 6, \ 4.5, \ -10 \text{ bps}$$

One year ago, swap spreads for maturities of 2, 5, 7, 10, and 30 years were

$$26, \ 8, \ -2, \ -4.5, \ -32.5 \text{ bps}$$

On October 11, 2016, swap spreads for maturities of 2, 5, 7, 10, and 30 years were

$$23.5, \ 1.6, \ -14, \ -16, \ -56 \text{ bps}$$

On October 13, 2015, the 30-year swap spread was -34 bp, the 10-year spread was -3 bp, the 5-year spread was 4 bp, and the 2-year swap spread was 12 bp.

# Overnight Index Swaps

An Overnight Index Swap (OIS) is an interest rate swap in which the floating payments are calculated using the Fed Funds overnight rate. They have a wide range of maturities. When the maturity is less than one year, payments are exchanged only at maturity. For maturities longer than one year, payments are exchanged annually.

The fixed rate on such a swap is called the *OIS rate*.

OIS are used in practice to achieve some of the same objectives as Fed Fund Futures. OIS have a wider range of products available. Fed Funds Futures are only traded with maturities up to 3 years (and liquid only for smaller maturities). OIS are available OTC with maturities of 30 years or longer.

# Eurodollar Futures

Eurodollar futures contracts are extremely important. They are traded at the Chicago Mercantile Exchange (CME). Each contract has a *notional* or face value of \$1,000,000. However, in contrast to T-Bill futures, Eurodollar futures contracts are really futures contracts on an interest rate (rather than on a zero-coupon bond price). The settlement value of one contract with delivery date $T$ is the interest on a \$1,000,000 3-month LIBOR deposit. (The LIBOR rate used is the one for deposits between $T$ and $T + .25$. However the interest is received at time $T$ in this contract.) The (quoted) settlement price for Eurodollar futures (with delivery date $T$) per \$100 notional is

$$100(1 - L_{T,T+.25}),$$

where $L_{T,T+.25}$ is the (annualized) 90-day LIBOR rate that prevails at time $T$.

# Eurodollar Futures (Continued)

The change in price of one contract is 2,500 times the change in the quoted price. For example, if you are long one contract and the price goes up from 97.50 to 97.75, you will have

$$\$2,500 \times (.25) = \$625$$

credited to your margin account.

The DV01 of a Eurodollar Futures Contract is \$25.

It is useful to observe that a one basis point increase in Libor results in a .01 (1 cent) decrease in the quoted price. The tick size for Eurodollar futures contracts is usually one half of a basis point, which is equivalent to .005 in the quoted price, and this translates into \$12.50 per contract. During the final month of a contract, the tick size is reduced to one fourth of a basis point, which is equivalent to .0025 in the quoted price and \$6.25 in the value of a contract.

# Eurodollar Futures (Continued)

Each Eurodollar Futures Contract “implies” a futures value for 3 month LIBOR. This value will be close to, but not identical to, forward LIBOR for the same time interval. In fact, of several of the popular one-factor term structure models there is a formula for the difference:

$$L^{fut}_{T,T+.25} - L^{for}_{T,T+.25} = \frac{1}{2}\sigma^2 T(T + .25).$$

A typical value of $\sigma$ for 3 month LIBOR is about .01.

See Chapter 15 of Tuckman & Serrat for more information on Eurodollar Futures.

# Options on Eurodollar Futures

Options on Eurodollar futures are options to enter a Eurodollar futures contract at a prescribed strike price. They are traded on the CME and are American style options.

The strike price is quoted in the same manner as the quoted price for Eurodollar futures. Upon of exercise of an option at time $t$ the holder of the option is assigned a long position on the relevant Eurodollar futures contract and cash in the amount

$$G_t = 2,500(QP_{t,T} - K)^+.$$

If an option is not exercised by maturity and is in the money, the exchange will exercise it automatically on behalf of the holder.

# T-Bill Futures

The deliverable asset in a T-Bill futures contract is a T-Bill having 90 days until maturity.

These are not traded on the CME, but are traded on some smaller electronic exchanges.

# Treasury Note and Treasury Bond Futures

Treasury Note and Treasury Bond Futures are based on hypothetical bonds having 6% coupons. For this reason, current price quotes are above 100 per 100 face. Settlement of note and bond futures by delivery of a security at maturity is an extremely complex process. As maturity for a particular contract nears, the CME publishes a list of bonds that are acceptable to settle the contract. The bonds on the list have different coupons and maturities, so they are not equally valuable. Each bond has a conversion factor associated with it that indicates how much face of the bond is considered to be equivalent to one contract. If the system worked perfectly, agents with short positions would be indifferent to which bond they deliver. However, as maturity approaches, a bond on the list will emerge as the *cheapest to deliver* (CTD).

I suggest that you read the CME publication *Understanding Treasury Futures*.

# Treasury Note and Bond Futures (Continued)

- ZT – 2 year note futures, \$200,000 face per contract
- Z3N – 3 year, note futures \$200,000 face per contract
- ZF – 5 year note futures, \$100,000 face per contract
- ZN – 10 year note futures, \$100,000 face per contract
- TN – ultra 10 year note futures, \$100,000 face per contract
- ZB – (30 - year) bond futures, \$100,00 face per contract
- UB – ultra (30-year) bond futures, \$100,000 face per contract.

All prices are quoted per \$100 face.

# Fed Funds

Banks often find that they have cash balances to invest or deficits to finance. The market in which banks trade funds overnight to manage cash balances is called the *federal funds market* or *fed funds market*. Only banks can borrow or invest in this market. However, the interest rate in the fed funds market drives other rates. The Board of Governors of the Federal Reserve System sets monetary policy in the United States. An important component of their job is to target the fed funds rate at a level consistent with price stability and "overall economic well-being".

The Federal Reserve calculates and publishes a weighted average rate called the *fed funds effective rate*. For the most part the fed succeeds in keeping the average rate close to the target. Individual investors can trade in Fed Funds Futures. (See Chapter 15 of Tuckman & Serrat.)

# Corporate & Municipal Bonds

In practice there is quite a bit of interest in corporate bonds and municipal bonds. Corporate bonds are issued by corporations wanting to raise capital. They vary in quality depending on the strength of the issuing corporation and on certain features of the *bond indenture* (contract of terms). Some corporate bonds are traded on exchanges. Most are traded over the counter (OTC) in a network of bond dealers. In contrast with US Treasury securities, corporate and municipal bonds are subject to *credit risk* and *liquidity risk*.

# Corporate Bonds

Corporate bonds are rated by credit rating agencies. There is an extremely important distinction: *Investment Grade* versus *Non-Investment grade*.

Bonds rated AAA, AA, A, or BBB are considered to be investment grade, while bonds rated BB or below are considered to be non-investment grade.

Investment grade bonds are also called *high grade bonds*. Non-investment grade bonds are also called *high yield bonds* or *junk bonds*.

# Corporate Bonds (Continued)

There are several types of corporate bonds. *Debentures* are the most common; they are not backed by any specific collateral, but instead are backed by the company's full faith and credit. Some types of corporate bonds are backed by collateral such as property, equipment, or securities. Unsecured short-term loans from investors are known as *commercial paper* (CP). CP is sold at a discount to face value and is redeemed for face value at maturity. Typical maturities are from 30 to 90 days. The maximum maturity is 270 days (to avoid SEC registration).

# Municipal Bonds

Municipal Bonds are issued by agencies of state and local governments as well as enterprises with a public purposes, e.g. hospitals. For individual investors the chief attraction of munis is that the interest income is exempt from federal taxes (and possibly from state and local taxes as well). There are two main types of munis: *general obligation bonds* (GOs) and *revenue bonds*. General obligation bonds are backed by the full faith, credit, and taxing power of the issuer. Revenue bonds are backed by the cash flow of a specific project.

# Modeling Default Risk

The discussion here is taken from Chapter 11 of *Interest Rate Models* by Andrew Cairns. There are two basic approaches to modeling default risk in bonds:

- Structural Models
- Reduced-Form Models

Structural models attempt to model the actual causes of default. They are explicit models for a firm with debt and equity. Most models of this type are too simplistic to use in practice, but can shed insight into the nature of default and the interaction between equity holders and bond holders.

# Default Risk (Continued)

Reduced-form models are statistical models that make use only of observed market statistics, such as credit ratings (AAA, etc.). Of course, the credit ratings should reflect detailed company-specific information. The market statistics are used in conjunction with market data on default-free bonds to analyze defaultable bonds. Practitioners typically use reduced-form models.

# The Merton Model for Default

The earliest structural model for default was developed by Merton. Consider a company which has equity and has also issued debt in the form of zero-coupon bonds with maturity $T$ and face value $K$. The total value of the firm (equity and debt) at time $t$ is $\mathcal{F}(t)$. No dividends or coupons are paid. At time $T$ the bond holders are entitled to receive $K$; the remaining value of the company is distributed to the equity holders and the firm is wound up. If $\mathcal{F}(T) < K$ there is default: Bond holders will receive $\mathcal{F}(T)$ and equity holders will receive nothing.

An extension of Merton's model and a model for default in discrete-time are discussed in Chapter 11 of Cairns.

Two classic references on default are

- *Credit Risk: Modeling, Valuation, and Hedging* by Bielecki & Rutkowski
- *Credit Risk* by Duffie & Singleton

# Repurchase Agreements

A *repurchase agreement* or *repo* is an agreement in which the owner of securities agrees to sell them to a counterparty and buy them back at a slightly higher price a short time later. In essence, the counterparty is providing a collateralized loan. The *repo rate* (which can depend on the type of collateral) is generally only slightly higher than the T-bill rate. The most common type of repo is an *overnight repo* in which the rate is negotiated daily. Longer-term repos, known as *term repos* exist as well.

# Repos (Continued)

When structured properly, a repo presents very little risk for either party. The collateral is usually US Treasuries or government guaranteed MBS. Since it could happen that the borrower defaults and the market price of the securities used as collateral declines, there is often a *hair cut* requiring the borrower to deliver securities worth a bit more than the amount of the loan. With term repos, the borrower is sometimes allowed to withdraw collateral in advancing markets.

# Repos (Continued)

Very frequently, the borrowers in repos are financial institutions in the business of making markets in US Treasuries. For example, suppose that a mutual fund wants to sell \$100,000,000 face of a certain US Treasury security to a trading desk. The trading desk will buy the bonds and sell them to another client. However, until a buyer is found, the trading desk will *repo out* the securities. If a client does not emerge to buy the bonds, the trading desk will have to refinance the position again. A *reverse repo* is the opposite transaction, namely purchase of a security for cash with an agreement to sell it back for a predetermined price at a later date.

# Collateralized Mortgage Obligations

In a *collateralized mortgage obligation* (CMO) investors are divided into several classes and these classes receive interest and principal payments according to different rules. As a simple example, there might be three classes of investors, say A,B, and C. The same amount of principal is not necessarily attributed to each class. Each class will receive interest payments until it is paid off. Initially, all principal payments (scheduled and prepayments) are paid to class A, until this class is paid off. Then all principal payments are paid to class B, until this class is paid off. Once classes A and B are paid off, then C will start receiving principal payments. Notice that class A bears more prepayment risk than class B, and class C bears less prepayment risk than class B. This is an example of a *sequential CMO*

# Option Adjusted spread

Suppose you have an interest rate model that has been calibrated to the Treasury spot rate curve.

When using this model to price an interest rate derivative, such as a callable bond, the price produced by the model will generally not match exactly with the market price.

The shift of the Treasury curve that will make

$$P^{Model} = P^{Market}$$

is called the option adjusted spread, or OAS.

# OAS (Cont.)

There are numerous reasons why the OAS for a particular security might be different from zero, including an inappropriate choice of an interest rate model. It could also be that a security is trading at a low price because of low liquidity. This would lead to a positive OAS. In this case, the OAS could be interpreted as a measure of liquidity risk.

It is essential to keep in mind that different interest rate models will lead to different OAS for the same security.

# Some Useful References

- Tuckman, *Fixed Income Securities: Tools for Today's Markets*
- Veronesi, *Fixed Income Securities: Valuation, Risk, & Risk Management*
- Shreve, *Stochastic Calculus for Finance, Vols. I & II*
- Kerry Back, *A course in Derivative Securities*
- Hull, *Options Futures and Other Derivatives*
- Luenberger, *Investment Science*
- Andrew Cairns, *Interest Rate Models: An Introduction*
- Jarrow, *Modeling Fixed-Income Securities and Interest Rate Options*

# Some Useful References (Continued)


- Brigo & Mercurio *Interest Rate Models – Theory and Practice With Smile, Inflation, and Credit*
- Choudhury, *Fixed Income Securities and Derivatives Handbook: Analysis and Valuation*
- Fabozzi, *Handbook of Fixed Income Securities*
- Fabozzi, *Fixed Income Mathematics*
- Fabozzi, *Interest Rate, Term Structure, and Valuation Modeling*
- James & Webber, *Interest Rate Modeling*
- Sundaresan, *Fixed Income Markets and Their Derivatives*

# Useful References (Continued)

- Corb, *Interest Rate Swaps and other Derivatives*
- Burghardt, Belton, Lane, and Papa, *The Treasury Bond Basis*
- Burghardt, *The Eurodollar Futures and Options Handbook*
- Fabozzi, The Handbook of Mortgage-Backed Securities, 7th Edition