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EE290T High-dim Data Analysis with Low-dim Model.

Exercise 1 - The Spark

Definition:

krank(A) is the largest number r, such that every subset of r columns of A is linearly independent.

Spark(A): $\text{Spark}(A) = \min_{d \neq 0, Ad=0} \|d\|_0$

1° proof: $\text{Spark}(A) \geq \text{krank}(A)+1$

s: if $\text{spark}(A) < \text{krank}(A)+1$: $\text{spark}(A) \leq \text{krank}(A)$

$Ad=0$ $\|d\|_0 \leq \text{krank}(A)$ set $\|d\|_0 = k \leq \text{krank}(A)$

set A has n columns

$[a_1, a_2 \dots a_n] \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = 0$ $d_{i_1} \dots d_{ik}$ means $d_{ij} > 0$

$\Downarrow$

$[a_{i_1}, \dots a_{ik}] \begin{bmatrix} d_{i_1} \\ \vdots \\ d_{ik} \end{bmatrix} = 0 \Rightarrow [a_{i_1} \dots a_{ik}]$ is Linear independent

$\Downarrow$

$k > \text{krank}(A)$

conflict with the hypothesis: $k \leq \text{krank}(A)$

so. $\text{spark}(A) \geq \text{krank}(A)+1$

2° proof: $\text{Spark}(A) \leq \text{krank}(A)+1$

let A has n columns $[a_1, a_2 \dots a_n]$ satisfy:

$[a_1, a_2 \dots a_n] \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = 0$

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because krank(A) is the largest num of A is linear dependent ∴

Exist $m = \text{krank}(A) + 1$

$[a_{i_1}, a_{i_2} \dots a_{i_m}]$ which are linear dependent

∴ Exist $[d_{i_1} \dots d_{i_m}]$ not zero. satisfy

$$a_i d_{i_1} + a_i d_{i_2} \dots d_{i_m} d_{i_m} = 0$$

if we set $d = [d_{i_1}, d_{i_2} \dots d_{i_m}, \underbrace{0\,0\,0 \dots}_{n-m}]$

$$Ad = 0 \Rightarrow \text{spark}(A) \le m = \text{krank}(A) + 1$$

based on 1° and 2° $\text{spark}(A) = \text{krank}(A) + 1$

$$\text{spark}(A) = \text{krank}(A) + 1$$

2. Compute the subdifferentials

1)

Solution: For the l_∞ norm $f(x) = |x|_\infty = \max_{i \le n} |x^{(i)}|$

we have e_i ar the orths of the cononical basis of $\mathbb{R}^n$

$$\partial f(x) = \text{Conv}\{\{e_i \mid i \in I_+(x)\} \cup \{-e_j \mid j \in I_-(x)\}\}$$

$$I_+(x) = \{i : x^{(i)} = |x|_\infty\}, \quad I_-(x) = \{i : -x^{(i)} = |x|_\infty\}$$

In particular, $\partial f(0) = B_1(0, 1) = \{x \in \mathbb{R}^n \mid |x|_1 \le 1\}$

2) $f(x) = \sum_{j=1}^{n} \|Xe_j\|_2 = \|X_1\|_2 + \|X_2\|_2 + \dots \|X_n\|_2$

$= \sqrt{x_{1,1}^2 + x_{2,1}^2 + \dots x_{n,n}^2} + \sqrt{x_{1,2}^2 + \dots + x_{n,2}^2} + \dots + \sqrt{x_{1,n}^2 + x_{2,n}^2 + \dots x_{n,n}^2}$

for $\|x\|_2$ $\quad f_0(x) = \|X\|_2$

$\partial f_0(x) = \{\frac{1}{\|x\|_2} x\}$ if $x \neq 0$. $\quad \partial f(x) = \{g \mid \|g\|_2 \leq 1\}$ if $x = 0$

$\partial f(x) = \sum_{j=1}^{n} \{\frac{1}{\|x_j\|_2} x_j$ if $x_j \neq 0$, $\{g \mid \|g\|_2 \leq 1\}$ if $x_j = 0\}$