ECE 515, Fall 2007 Handout #3

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Singular Value Decomposition

We here give a complete description and proof for the Singular Value Decomposition (SVD) Theorem. First, recall that the inner product defined on $\mathbb{C}^n$ is $< x, y >= x^* y, \forall x, y \in \mathbb{C}^n$. We now introduce the following important lemma:

Lemma 1 *Let $A \in \mathbb{C}^{m\times n}$ and A^* be its conjugate transpose. We then always have:*

$$Nu(AA^*) = Nu(A^*), \quad Ra(AA^*) = Ra(A).$$

Proof: To prove $\mathrm{Nu}(AA^*) = \mathrm{Nu}(A^*)$, we have:
(a) $AA^*x = \theta \Rightarrow < x, AA^*x >= \|A^*x\|^2 = 0 \Rightarrow A^*x = \theta$, hence $\mathrm{Nu}(AA^*) \subseteq \mathrm{Nu}(A^*)$.
(b) $A^*x = \theta \Rightarrow AA^*x = \theta$, hence $\mathrm{Nu}(AA^*) \supseteq \mathrm{Nu}(A^*)$.

To prove $\mathrm{Ra}(AA^*) = \mathrm{Ra}(A)$, we first need to prove that $\mathbb{C}^n$ is a direct sum of $\mathrm{Ra}(A^*)$ and $\mathrm{Nu}(A)$. We prove this by showing that a vector x is in $\mathrm{Nu}(A)$ if and only if it is orthogonal to $\mathrm{Ra}(A^*)$: $x \in \mathrm{Nu}(A) \Leftrightarrow < A^*x, y >= 0, \forall y \in \mathrm{Ra}(A^*) \Leftrightarrow < x, Ay >= 0, \forall y$. Hence $\mathrm{Nu}(A)$ is exactly the subspace which is orthogonal supplementary to $\mathrm{Ra}(A^*)$ (sometimes denoted as $\mathrm{Ra}(A^*)^{\perp}$). Therefore $\mathbb{C}^n$ is a direct sum of $\mathrm{Ra}(A^*)$ and $\mathrm{Nu}(A)$. Let $\mathrm{Img}_A(S)$ denote the image of a subspace S under the map A. Then we have: $\mathrm{Ra}(A) = \mathrm{Img}_A(\mathbb{C}^n) = \mathrm{Img}_A(\mathrm{Ra}(A^*)) = \mathrm{Ra}(AA^*)$ (in the second equality we used the fact that $\mathbb{C}^n$ is a direct sum of $\mathrm{Ra}(A^*)$ and $\mathrm{Nu}(A)$.) ■

In the above A can be regarded as a linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$. In fact the same result still holds even if the domain of the linear map A is replaced by an infinite dimensional linear space with an inner product (*i.e.*, $\mathbb{C}^n$ is replaced by a Hilbert space). In that case, this lemma is also known as the *Finite Rank Operator Fundamental Lemma*, which will be useful for obtaining conditions of controllability and observability for LTI systems.

We are now ready to give a complete proof for the Singular Value Decomposition Theorem is:

Theorem 1 (Singular Value Decomposition) *Let $F = \mathbb{R}$ or $\mathbb{C}$. Let $A \in F^{m\times n}$ be a matrix of rank r. Then there exist matrices $U \in F^{m\times m}$ and $V \in F^{n\times n}$, and $\Sigma_1 \in \mathbb{R}^{r\times r}$ such that:*

1. *$V = [V_1 : V_2], V_1 \in F^{n\times r}$, satisfies:*
   V is unitary, i.e., $V^*V = I_{n\times n}$,
   $Ra(V_1) = Ra(A^*)$; the columns of V_1 form an orthonormal basis of $Ra(A^*)$
   $Ra(V_2) = Nu(A)$; the columns of V_2 form an orthonormal basis of $Nu(A)$
   The columns of V form a complete orthonormal basis of eigenvectors of A^*A.

2. *$U = [U_1 : U_2], U_1 \in F^{m\times r}$, satisfies:*
   U is unitary, i.e., $U^*U = I_{m\times m}$,
   $Ra(U_1) = Ra(A)$; the columns of U_1 form an orthonormal basis of $Ra(A)$
   $Ra(U_2) = Nu(A^*)$; the columns of U_2 form an orthonormal basis of $Nu(A^*)$
   The columns of U form a complete orthonormal basis of eigenvectors of AA^*.

3. $\Sigma_1 = diag(\sigma_1, \sigma_2, \dots, \sigma_r) \in \mathbb{R}^{r\times r}$ *such that* $\sigma_1 \geq \sigma_2 \geq \cdots, \geq \sigma_r > 0$*.* $A \in F^{m\times n}$ *has a dyadic expansion:*

$$A = U_1 \Sigma_1 V_1^*, \quad \textit{or equivalently,} \quad A = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

where u_i, v_i *are the columns of* U_1 *and* V_1 *respectively.*

4. $A \in F^{m\times n}$ *has a singular value decomposition (SVD):*

$$A = U\Sigma V^*, \quad \textit{with} \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}.$$

Proof: 1. $A \in F^{m\times n}$ has rank r, hence the nonnegative (or, equivalently, positive semidefinite) Hermitian matrix A^*A has rank r according to the Lemma. It has n nonnegative eigenvalues σ_i^2 ordered as:

$$\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_r^2 > 0 = \sigma_{r+1}^2 = \cdots = \sigma_n^2$$

to which corresponds a complete orthonormal eigenvector basis $(v_i)_{i=1}^n$ of A^*A. This family of vectors (in F^n) form the columns of a unitary $n \times n$ matrix, say, V. From the lemma, $\mathrm{Ra}(A^*A) = \mathrm{Ra}(A^*)$ and $\mathrm{Nu}(A^*A) = \mathrm{Nu}(A)$, the properties listed in 1 follow.

2. Define a diagonal matrix $\Sigma_1 = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_r) \in \mathbb{R}^{r\times r}$. We then have $A^*AV_1 = V_1\Sigma_1^2$, hence $(AV_1\Sigma_1^{-1})^*(AV_1\Sigma_1^{-1}) = I_{r\times r}$. This defines a $m \times r$ matrix:

$$U_1 = AV_1\Sigma_1^{-1}. \tag{1}$$

Then $U_1^*U_1 = I_{r\times r}$. Since A^*A and AA^* both have exactly r nonzero eigenvalues, it follows that the columns of U_1 form an orthonormal basis for $\mathrm{Ra}(AA^*)$ and $\mathrm{Ra}(A)$. Thus the properties of U_1 listed in 2 hold. Now define an $m \times (m - r)$ matrix U_2 with orthonormal columns which are orthogonal to columns of U_1. Then $U = [U_1 : U_2]$ is clearly an unitary matrix. From the proof of the lemma, columns of U_2 form an orthonormal basis of $\mathrm{Nu}(A^*)$ or $\mathrm{Nu}(AA^*)$. Therefore, columns of U_2 are all the eigenvectors corresponding to the zero eigenvalue. Hence columns of U form a complete orthonormal basis of eigenvectors of AA^*. List 2 is then fully proven.

3. From the definition of U_1 in (1), we have:

$$A = U_1\Sigma_1V_1^*.$$

The dyadic expansion directly follows.

4. The singular value decomposition follows because:

$$A[V_1 : V_2] = [U_1\Sigma_1 : 0] = [U_1 : U_2]\Sigma \quad \Rightarrow \quad A = U\Sigma V^*.$$

■

After we have gone through all the trouble proving this theorem, you must know that SVD has become a numerical routine available in many computational softwares such as MATLAB. Within MATLAB, to compute the SVD of a given $m \times n$ matrix A, simply use the command "$[U, S, V] = SVD(A)$" which returns matrices U, S, V satisfying $A = USV^*$ (where S represents Σ as defined above).