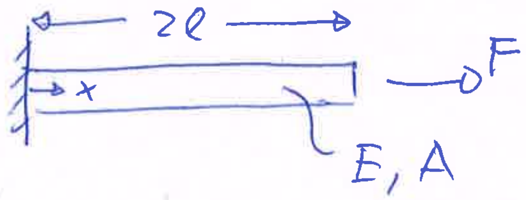
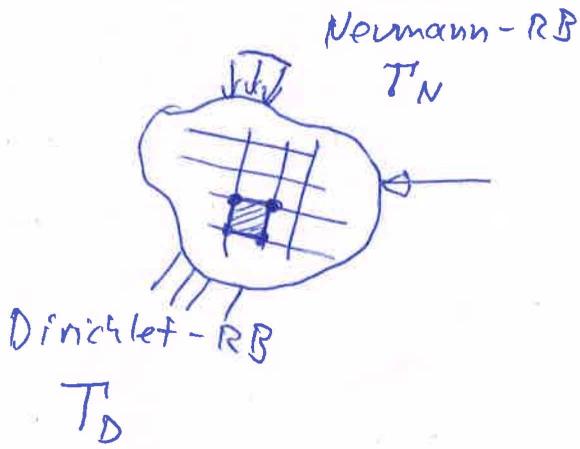
Modellbildung und Simulation: Klausurvorbereitung FEM





1 Starke Form:

$$(EA\,u_x)_x = 0 \quad , \quad EA\,u_x(L) = F \quad , \quad u(0) = 0$$

Schwache Form: mit Testfunktion $w(x)$
↳ verschwindet auf Γ_D

$$\int_0^L (EA\,u_x)_x\, w(x)\, dx \overset{\text{p.I.}}{=} \left[EA\,u_x\,w\right]_0^L - \int_0^L EA\,u_x\,w_x\,dx$$

$$0 = EA\,u_x(L)\,w(L) - EA\,u_x(0)\,\underbrace{w(0)}_{\to 0} - \int_0^L EA\,u_x\,w_x\,dx$$

$$\Rightarrow \int_0^L EA\,u_x\,w_x\,dx = F \cdot w(L)$$

FEM

$$\int_0^L EA\,u_x\,w_x\,dx = \int_0^{l_e} EA\,u_x\,w_x\,dx + \dots + \int_{(n-1)l_e}^{n\,l_e} EA\,u_x\,w_x\,dx$$

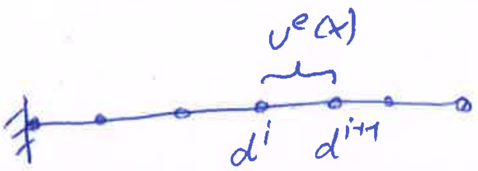
Einsetzen von:

$$u^e(x) = \vec{N}^e(x)\,\vec{d}^e = N^e(x)\,\underline{\underline{L}}^e\,\vec{d}$$

$$w^e(x) = \vec{N}^e(x)\,\vec{w}^e = N^e(x)\,\underline{\underline{L}}^e\,\vec{w}$$

$$\frac{du^e(x)}{dx} = \vec{B}^e(x)\,\vec{d}^e = \vec{B}^e(x)\,\underline{\underline{L}}^e\,\vec{d}$$

$$\frac{dw^e(x)}{dx} = \vec{B}^e(x)\,\underline{\underline{L}}^e\,\vec{w}$$



$$u^e(x) = \begin{pmatrix} N_1^e & N_2^e \end{pmatrix} \begin{pmatrix} d_i \\ d_{i+1} \end{pmatrix}$$

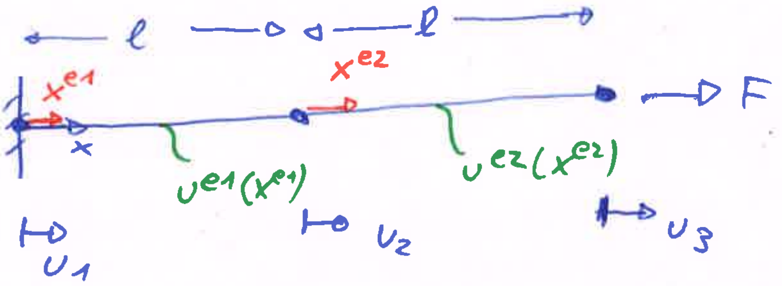
$$\int_0^L EA\, u_x w_x \, dx = \sum_e \int_{\Omega_e} EA\, u_x w_x\, dx$$

$$\approx \sum_e \int_{\Omega_e} EA \left(\vec{B}^e \underline{L}^e \vec{d}\right)\left(\vec{B}^e \underline{L}^e \vec{w}\right) dx$$

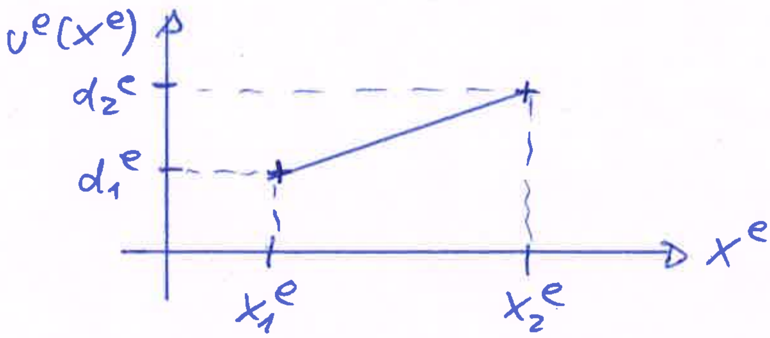
$$= \vec{w}^T \underbrace{\sum_e \underline{L}^{eT} \underbrace{\int_{\Omega} \left(EA\, \vec{B}^{eT} \vec{B}^e\right) dx}_{\underline{k}^e}\, \underline{L}^e}_{\underline{K}}\, \vec{d} = \vec{w}^T \vec{f}$$

$$\overset{\forall \vec{w}}{\Longrightarrow} \quad \underline{K}\, \vec{d} = \vec{f}$$

$\boxed{2}$



Linearer Ansatz: $\quad u^e(x^e) = a_0 + a_1 x^e$

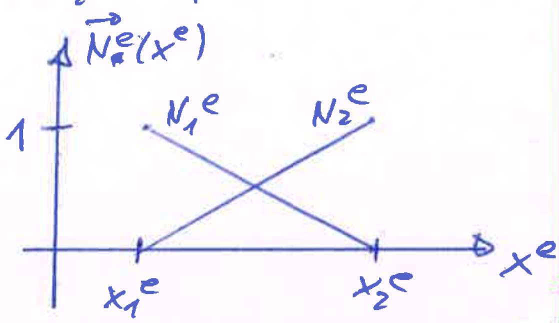


$$u^e(x_1^e) = d_1^e$$

$$u^e(x_2^e) = d_2^e$$

$$u^e(x^e) = \frac{x_2^e - x^e}{x_2^e - x_1^e}\, d_1^e + \frac{x^e - x_1^e}{x_2^e - x_1^e}\, d_2^e$$

$$= \underbrace{\begin{pmatrix} \underbrace{\frac{x_2^e - x^e}{x_2^e - x_1^e}}_{N_1^e} & \underbrace{\frac{x^e - x_1^e}{x_2^e - x_1^e}}_{N_2^e} \end{pmatrix}}_{\vec{N}^e} \begin{pmatrix} d_1^e \\ d_2^e \end{pmatrix}$$



1. Element

$$u^{e1} = \begin{pmatrix} \frac{\ell - x^{e1}}{\ell} & \frac{x^{e1}}{\ell} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{\ell - x^{e1}}{\ell} & \frac{x^{e1}}{\ell} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\frac{du^{e1}}{dx^{e1}} = \underbrace{\frac{1}{\ell}\begin{pmatrix} -1 & 1 \end{pmatrix}}_{\vec{B}_1^e} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\vec{B}_1^e = \frac{1}{\ell}\begin{pmatrix} -1 & 1 \end{pmatrix}$$

$$\vec{B}_2^e = \vec{B}_1^e \quad \text{(analog)}$$

$$\underline{\underline{K}}^e = \int_0^\ell EA\, \vec{B}^{eT}\, \vec{B}^e \, dx^e$$

$$\underline{\underline{K}}_1^e = \int_0^\ell EA \frac{1}{\ell}\begin{pmatrix}-1\\1\end{pmatrix}\frac{1}{\ell}\begin{pmatrix}-1 & 1\end{pmatrix} dx^{e1} = \frac{EA}{\ell^2}\int_0^\ell \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} dx^{e1}$$

$$= \frac{EA}{\ell}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \underline{\underline{K}}_2^e$$

$$\underline{\underline{K}} = \sum_e \underline{\underline{L}}^{eT}\, \underline{\underline{K}}^e\, \underline{\underline{L}}^e \qquad \underline{\underline{L}}_1^e = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

$$\underline{\underline{L}}_2^e = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$= \frac{EA}{\ell}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}$$

$\boxed{3}$ $$\underline{\underline{K}}\,\vec{d} = \vec{f}$$

$$\begin{array}{cc} \underline{\underline{K}}_{11} & \underline{\underline{K}}_{12} \end{array} \qquad \vec{u}_a \qquad\qquad \vec{f}_a$$

$$\begin{pmatrix} \frac{EA}{\ell} & -\frac{EA}{\ell} & 0 \\ -\frac{EA}{\ell} & \frac{2EA}{\ell} & -\frac{EA}{\ell} \\ 0 & -\frac{EA}{\ell} & \frac{EA}{\ell} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} N \\ 0 \\ F \end{pmatrix}$$

$$\begin{array}{cc} \underline{\underline{K}}_{21} & \underline{\underline{K}}_{22} \end{array} \qquad \vec{u}_b \qquad\qquad \vec{f}_b$$

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$$\underline{\underline{K}}_{11}\,\vec{u}_a + \underline{\underline{K}}_{12}\,\vec{u}_b = \vec{f}_a$$

$$\underline{\underline{K}}_{21}\,\vec{u}_a + \underline{\underline{K}}_{22}\,\vec{u}_b = \vec{f}_b$$

$$\vec{u}_b = \underline{\underline{K}}_{22}^{-1}\left(\vec{f}_b - \cancel{\underline{\underline{K}}_{21}\,\vec{u}_a}\right) \overset{u_1=0}{=} \frac{\ell}{EA}\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0\\ F\end{pmatrix}$$

$$= \frac{F\ell}{EA}\begin{pmatrix}1\\ 2\end{pmatrix}$$

$$\vec{f}_a = \cancel{\underline{\underline{K}}_{11}\,\vec{u}_a} + \underline{\underline{K}}_{12}\,\vec{u}_b = -\frac{EA}{\ell}\,\frac{F\ell}{EA} = -F$$