① Taking the equation for $\ddot{\alpha}$ ①

$$mu^2\ddot{\alpha} + k\dot{\alpha} + mgd(\sin\alpha) - mu^2\omega^2(\sin\alpha\cos\alpha) = 0$$

$$\ddot{\alpha} = \frac{-k\dot{\alpha}}{mu^2} \mp \frac{mgd\sin\alpha}{mu^2} + \frac{mu^2\omega^2}{mu^2}(\sin\alpha\cos\alpha) = 0$$

$$\ddot{\alpha} = \frac{-k\dot{\alpha}}{mu^2} - \frac{g}{d}\sin\alpha + \omega^2\sin\alpha\cos\alpha$$

② State vector

$$x = \begin{pmatrix} \alpha \\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \dot{x} = \begin{pmatrix} \dot{\alpha} \\ \ddot{\alpha} \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}.$$

$$\dot{x} = \begin{pmatrix} x_2 \\ -g/u\sin x_1 + \omega^2\sin x_1\cos x_1 \end{pmatrix} \quad \Bigg| \quad \begin{matrix} x_2 = \dot{x}_1 \\ k = 0 \end{matrix}$$

③ Linearising about $x_{01} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\dot{x}(t) = f_0|_{x_{01}} + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \Bigg| (x - x_{01}) \Bigg|_{x = x_{01}}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -g/d\cos x_1 + \omega^2(\cos^2 x_1 - \sin^2 x_1) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ \omega^2 - g/u & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \Bigg| \quad x_1 = 0$$

System matrix $A = \begin{pmatrix} 0 & 1 \\ \omega^2 - g/l & 0 \end{pmatrix}$

general solution of an ode $x = re^{\lambda t}$

④ Eigen Value analysis

$$r\lambda e^{\lambda t} = A r e^{\lambda t}$$

$$(A - \lambda I) r e^{\lambda t} = 0$$

$$(A - \lambda I) r = 0$$

$$\det(A - \lambda I) = 0$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ \omega^2 - g/l & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\omega^2 - g/l) = 0$$

$$\lambda_{1,2} = \pm\sqrt{\omega^2 - g/l}$$

⑤ ~~Soluti~~ Exact Solution of the Ode

Applying ~~linear~~ Principle of superposition

$$X = \begin{pmatrix} r_{11} \\ r_{21} \end{pmatrix} e^{\lambda_1 t} + \begin{pmatrix} r_{21} \\ r_{22} \end{pmatrix} e^{\lambda_2 t} \quad \text{--- (1)}$$

Finding the Eigen vectors

1. $(A - \lambda_1 I)r = 0$

$$\begin{pmatrix} -\lambda_1 & 1 \\ \omega^2 g l d & -\lambda_1 \end{pmatrix} \begin{pmatrix} r_{11} \\ r_{12} \end{pmatrix} = 0$$

$$-\lambda_1 r_{11} + r_{12} = 0$$

$$\boxed{\therefore r_{12} = \lambda_1 r_{11}}$$

$$\Rightarrow r_1 = \begin{pmatrix} r_{11} \\ \lambda_1 r_{11} \end{pmatrix} = r_{11} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} = r_{11} \begin{pmatrix} 1 \\ (\sqrt{\omega^2 - g l d}) \end{pmatrix}$$

2. $(A - \lambda_2 I)r = 0$

$$\begin{pmatrix} -\lambda_2 & 1 \\ \omega^2 - g l d & -\lambda_2 \end{pmatrix} \begin{pmatrix} r_{21} \\ r_{22} \end{pmatrix} = 0$$

$$-\lambda_2 r_{21} + r_{22} = 0$$

$$r_{22} = \lambda_2 r_{21}$$

$$\Rightarrow r_2 = \begin{pmatrix} r_{21} \\ \lambda_2 r_{21} \end{pmatrix} = r_{21} \begin{pmatrix} 1 \\ -(\sqrt{\omega^2 - g l u}) \end{pmatrix}$$

Substituting Eigenvectors and Eigenvalues in eqn ①

$$x(t) = r_{11} \begin{pmatrix} 1 \\ \sqrt{\omega^2 - g/l} \end{pmatrix} e^{(\sqrt{\omega^2 - g/l})t} + r_{21} \begin{pmatrix} 1 \\ -\sqrt{\omega^2 - g/l} \end{pmatrix} e^{(-\sqrt{\omega^2 - g/l})t}$$

$$\begin{pmatrix} \pi/6 \\ 0 \end{pmatrix} = r_{11} \begin{pmatrix} 1 \\ \sqrt{\omega^2 - g/l} \end{pmatrix} e^{(\sqrt{\omega^2 - g/l})t} + r_{21} \begin{pmatrix} 1 \\ -\sqrt{\omega^2 - g/l} \end{pmatrix} e^{(-\sqrt{\omega^2 - g/l})t}$$

$r_{11} + r_{21} = \pi/6 \quad \therefore \; r_{11} = r_{21} = \pi/12$

$r_{11} = r_{21}$

$$x(t) = \pi/12 \begin{pmatrix} 1 \\ \sqrt{\omega^2 - g/l} \end{pmatrix} e^{(\sqrt{\omega^2 - g/l})t} + \pi/12 \begin{pmatrix} 1 \\ -\sqrt{\omega^2 - g/l} \end{pmatrix} e^{(-\sqrt{\omega^2 - g/l})t}$$

$$\dot{x}(t) = \frac{\pi}{12} \left(\sqrt{\omega^2 - g/l}\right) \begin{pmatrix} 1 \\ \sqrt{\omega^2 - g/l} \end{pmatrix} e^{(\sqrt{\omega^2 - g/l})t} + \pi/12 \left(-\sqrt{\omega^2 - g/l}\right) \begin{pmatrix} 1 \\ -\sqrt{\omega^2 - g/l} \end{pmatrix} e^{(-\sqrt{\omega^2 - g/l})t}$$