Karlsruhe Institute of Technology

Modeling and Simulation

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Exercise Sheet Nr. 6

Topic: **Numerical Integration of Ordinary Differential Equations**

Numerical integration of initial value problem

$$\boxed{\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t),\ \mathbf{x}(t_0) = \mathbf{x}_0}$$

Discretization of a continuous model using

- explicit method $\boxed{\mathbf{x}(t_{n+1}) = \mathbf{g}(t_0, t_1, \dots, t_n)}$ vs. implicit method $\boxed{\mathbf{x}(t_{n+1}) = \mathbf{g}(t_0, t_1, \dots, t_{n+1})}$
- one-step method $\boxed{\mathbf{x}(t_{n+1}) = \mathbf{g}(t_n)}$ vs. multistep method $\boxed{\mathbf{x}(t_{n+1}) = \mathbf{g}(t_0, t_1, \dots, t_n)}$
- stepsize $\boxed{h = t_{n+1} - t_n}$
- notation: $\boxed{\mathbf{x}(t_n) = \mathbf{x}_n, \quad \mathbf{x}(t_{n+1}) = \mathbf{x}_{n+1}}$

Integration method

- Euler: explicit $\boxed{\mathbf{x}_{n+1} = \mathbf{x}_n + h\,\mathbf{f}(\mathbf{x}_n, t_n)}$ vs. implicit $\boxed{\mathbf{x}_{n+1} = \mathbf{x}_n + h\,\mathbf{f}(\mathbf{x}_{n+1}, t_{n+1})}$
- Predictor-corrector method
  1. Heun's method (2. order)

$$\begin{aligned} \mathbf{k}_n^{(1)} &= h\mathbf{f}(\mathbf{x}_n, t_n) \\ \mathbf{k}_n^{(2)} &= h\mathbf{f}(\mathbf{x}_n + \mathbf{k}_n^{(1)}, t_n + h) \\ \Longrightarrow \mathbf{x}_{n+1} &= \mathbf{x}_n + \frac{1}{2}\left(\mathbf{k}_n^{(1)} + \mathbf{k}_n^{(2)}\right) \end{aligned}$$

  2. Runge-Kutta-Method (4. order)

$$\begin{aligned} \mathbf{k}_n^{(1)} &= h\mathbf{f}\left(\mathbf{x}_n, t_n\right) \\ \mathbf{k}_n^{(2)} &= h\mathbf{f}\left(\mathbf{x}_n + \mathbf{k}_n^{(1)}/2, t_n + h/2\right) \\ \mathbf{k}_n^{(3)} &= h\mathbf{f}\left(\mathbf{x}_n + \mathbf{k}_n^{(2)}/2, t_n + h/2\right) \\ \mathbf{k}_n^{(4)} &= h\mathbf{f}\left(\mathbf{x}_n + \mathbf{k}_n^{(3)}, t_n + h\right) \\ \Longrightarrow \mathbf{x}_{n+1} &= \mathbf{x}_n + \frac{1}{6}\left(\mathbf{k}_n^{(1)} + 2\mathbf{k}_n^{(2)} + 2\mathbf{k}_n^{(3)} + \mathbf{k}_n^{(4)}\right) \end{aligned}$$

Error analysis and stepsize control

- Local discretization error $l(h)$: difference between the exact and the approximated differential quotient
- Global discretization error $e(h)$: maximum error between the approximated and the exact solution after n steps
- Optimal stepsize with a given error δ: $\boxed{h_{opt} = \left(\frac{2^p-1}{2^p}\frac{\delta}{\mathbf{x}_{n+1}-\tilde{\mathbf{x}}_{n+1}}\right)^{\frac{1}{p}} h}$ with order p (Euler: $p=1$, Heun: $p=2$, Runge-Kutta: $p=4$)

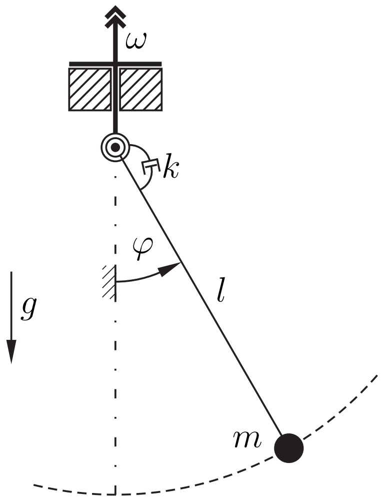
Exercise 1 – homework –

Matlabfile: Pendel_Bearbeitungsfile.m

The differential equation for the rotating pendulum shown in the figure beyond is given by

$$ml^2\ddot{\varphi} + k\dot{\varphi} + mgl\sin(\varphi) - ml^2\omega^2\sin(\varphi)\cos(\varphi) = 0.$$

The initial conditions are given by $\varphi(t=0) = \frac{\pi}{6}$ und $\dot{\varphi}(t=0) = 0$. The solution of the equation shall be obtained by different integration methods.



In the following, damping is neglected $k = 0$. The pendulum is examined for small disturbances with respect to the equilibrium point $\varphi = \dot{\varphi} = 0$.

1. Linearize the differential equation and determine the system matrix.
2. Calculate the exact solution of the linearized system analytically (handwritten).

The parameters $l = 20\,\text{cm}$, $g = 9{,}81\,\frac{\text{m}}{\text{s}^2}$ and $\omega = \frac{1}{2}\sqrt{\frac{g}{l}}$ are given now. Furthermore use $n_{max} = 10^4$ discretization steps.

3. Calculate the solution of the linearized system with the explicit Euler's method.
4. Calculate the solution of the linearized system with the implicit Euler's method.
5. Calculate the solution of the linearized system with the Runge-Kutta method.
6. Determine the local discretization error $l(h)$ and the global discretization error $e(h)$ for all three integration methods.
7. Plot the time response of $\varphi(t)$ and $\dot{\varphi}(t)$ for all three integration methods as well as the exact solution. Is it possible to choose h so that they are numerically stable? What value for h can be assumed in this case? Illustrate for the chosen value of h the corresponding point in the figure of stable regions. Make sure that the stepsize is not unnecessarily small in order to save computing time.
8. Plot the local and global discretization error with respect to time.
9. Raise the rotational speed to $\omega = 2\sqrt{\frac{g}{l}}$ and simulate again. What happens now? Consider the eigenvalues obtained in task 2.

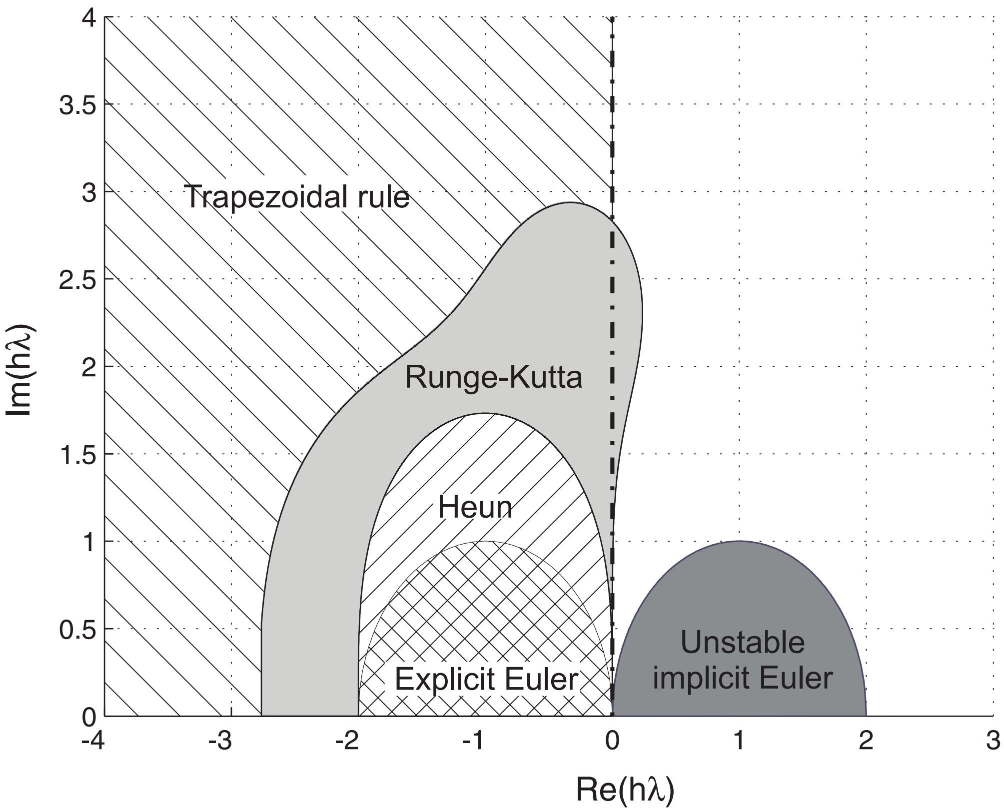


Abbildung 1: Stable regions of numerical integration methods

<u>Additional task (optional)</u>: Plot the direction field for the linearized system (*MATLAB: quiver*) and plot the exact and numerical solutions. Assess the quality of the integration methods for certain values of h. What happens with the exact solution using the modified rotational speed in task 9?