Modeling and Simulation

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Exercise Sheet Nr. 12

Topic: **Numerical Methods for Solving Partial Differential Eqs.: Finite-Element-Method**

Starting point: Variation principle

$$\delta\Pi = 0$$

Discretization of displacement field:

$$\begin{aligned} u_i^{approx}(x,y,z) &= \mathbf{N}_i(x,y,z)\mathbf{u}_i \\ \mathbf{N}(x,y,z) &:= \text{Interpolation matrix} \\ \mathbf{u} &:= \text{Node displacements} \end{aligned}$$

Discretization, distortion and stress fields (small distortions, linear elasticity):

$$\begin{aligned} \varepsilon_i &= \frac{1}{2}\left(\nabla u_i^{approx} + (\nabla u_i^{approx})^T\right) = \frac{1}{2}\left(\nabla \mathbf{N}_i \mathbf{u}_i + (\nabla \mathbf{N}_i \mathbf{u}_i)^T\right) := \mathbf{B}_i \mathbf{u}_i \\ \sigma_i &= \mathbf{C}_i \varepsilon_i = \mathbf{C}_i \mathbf{B}_i \mathbf{u}_i \end{aligned}$$

Calculation of the element stiffness

$$\mathbf{K}_i = \int_V \mathbf{B}_i^T \mathbf{C}_i \mathbf{B}_i \mathrm{d}V$$

Transferring the element stiffness $\mathbf{K_i}$ (local KOS!) in global stiffness matrix $\mathbf{K_{Ges}}$ (global KOS!). System equation for the special case of static:

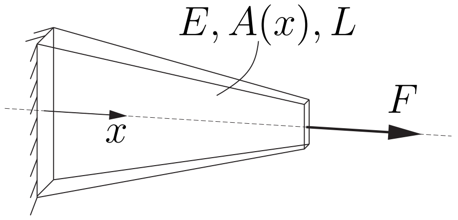
$$\begin{aligned} \mathbf{f} = \begin{pmatrix} \mathbf{f_a} \\ \mathbf{f_b} \end{pmatrix} &= \begin{pmatrix} \mathbf{K_{Ges}^{11}} & \mathbf{K_{Ges}^{12}} \\ \mathbf{K_{Ges}^{21}} & \mathbf{K_{Ges}^{22}} \end{pmatrix} \begin{pmatrix} \mathbf{u_a} \\ \mathbf{u_b} \end{pmatrix} = \mathbf{K_{Ges}} \mathbf{u} \\ \mathbf{u_a} &:= \text{known displacement boundary conditions} \\ \mathbf{f_a} &:= \text{unknown reaction forces} \\ \mathbf{u_b} &:= \text{unknown displacements} \\ \mathbf{f_b} &:= \text{known external forces acting} \end{aligned}$$

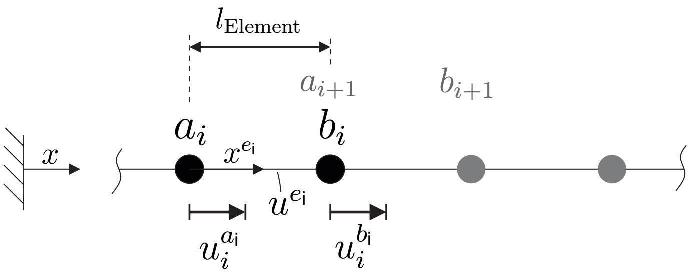
Solving the system equation

$$\begin{aligned} \mathbf{K_{Ges}^{22}} \mathbf{u_b} &= \left(\mathbf{f_b} - \mathbf{K_{Ges}^{21}} \mathbf{u_a}\right) \\ \mathbf{f_a} &= \mathbf{K_{Ges}^{11}} \mathbf{u_a} + \mathbf{K_{Ges}^{12}} \mathbf{u_b} \end{aligned}$$

Exercise 1 – homework –

Matlabfile: fem_Bearbeitungsfile.m





A cantilever bar with conic rectangular cross-section (left figure) is given. The following data are given:

$$F = 20 \text{ [kN]}$$
$$E = 70 \text{ [GPa]}$$
$$L = 1 \text{ [m]}$$
$$A(x) = 0.001 - 0.0009x \text{ [m}^2\text{]}$$

1. Write a system of differential equation for $u(x)$ as a function of the given parameter using Hooke's law $\sigma(x) = E\varepsilon(x)$ for small strains. Determine all the required boundary conditions.
2. Solve the differential equation using an appropriate Matlab-solver.
3. The structure is discretized by the first-order linear elements with the nodes a_i and b_i (right figure). The node a_i and. b_i is shifted by $u_i^{a_i}$ and $u_i^{b_i}$. Set up an appropriate approach for the displacement $u^{e_i}(x^{e_i})$ for the ith element. Define the interpolation matrix $\mathbf{N}(x^{e_i})$, such that $u^{e_i}(x^{e_i}) = \mathbf{N}(x^{e_i})\mathbf{u}$. The total number of elements M can be chosen freely.
   Furthermore, the function of area $A(x)$ must be discretized. Calculate an averaged area for each element using the end points $A_i = \frac{A_{i,a_i} + A_{i,b_i}}{2}$.
4. Determine the matrices $\mathbf{B}(x^{e_i})$ and $\mathbf{C}$ which derived from the discretized strain $\varepsilon_i(x^{e_i}) = \mathbf{B}(x^{e_i})\mathbf{u}$ and the stress $\sigma_i(x^{e_i}) = \mathbf{C}\mathbf{B}(x^{e_i})\mathbf{u}$ for the ith element.
5. Find the stiffness matrix $\mathbf{K}_i = \int_0^{V_i} \mathbf{B}^T(x^{e_i})\mathbf{C}\mathbf{B}(x^{e_i})\, \mathrm{d}V$ for the ith element.
6. Find the total stiffness matrix

$$\mathbf{K}_{Ges} = \begin{pmatrix} \mathbf{K_1^{a_1a_1}} & \mathbf{K_1^{a_1b_1}} & 0 & \dots & 0 \\ \mathbf{K_1^{b_1a_1}} & \mathbf{K_1^{b_1b_1}} + \mathbf{K_2^{a_2a_2}} & \mathbf{K_2^{a_2b_2}} & \dots & 0 \\ 0 & \mathbf{K_2^{b_2a_2}} & \mathbf{K_2^{b_2b_2}} + \mathbf{K_3^{a_3a_3}} & & \\ \vdots & & & \ddots & \\ 0 & & \dots & & \mathbf{K_M^{b_Mb_M}} \end{pmatrix}$$

   with $\mathbf{K_i} = \begin{pmatrix} \mathbf{K_i^{a_ia_i}} & \mathbf{K_i^{a_ib_i}} \\ \mathbf{K_i^{b_ia_i}} & \mathbf{K_i^{b_ib_i}} \end{pmatrix}$ for $i = 1 \dots M$ elements.
7. Determine the displacement vector $\mathbf{u} = \begin{pmatrix} \mathbf{u_a} & \mathbf{u_b} \end{pmatrix}^T$ and force vector $\mathbf{f} = \begin{pmatrix} \mathbf{f_a} & \mathbf{f_b} \end{pmatrix}^T$ the vector components $\mathbf{u_a}$ and $\mathbf{f_b}$, where $\mathbf{u_a}$ the known displacement boundary conditions, $\mathbf{f_a}$ the corresponding unknown reaction forces, $\mathbf{u_b}$ the unknown displacements and $\mathbf{f_b}$ die known acting external forces.
8. Calculate the unknown vectors $\mathbf{u_b}$ and $\mathbf{f_a}$. What is the displacement $u(x = L)$? Finally, describe the displacement and the stress diagram for the exact solution (subtask 2) and for

the FEM solution over the length of the bar graphically. Adapt the number of element M for the FEM so that your FEM solution agrees well with the exact solution.
Hint: Use the *Matlab* command *stairs( )* instead of *plot( )* in order to represent the stress.