Aufgabe 1

1) $\ddot{\varphi} + \frac{d}{m}\dot{\varphi} = -\frac{g}{L}\sin\varphi$

$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \varphi \\ \dot{\varphi} \end{pmatrix}$

$\ddot{\varphi} = -\frac{d}{m}\dot{\varphi} - \frac{g}{L}\sin\varphi$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{d}{m}x_2 - \frac{g}{L}\sin(x_1)$

$\underset{\sim}{f}(\underset{\sim}{x}) = \begin{pmatrix} x_2 \\ -\frac{d}{m}x_2 - \frac{g}{L}\sin(x_1) \end{pmatrix}$

2) $0 = x_2$

$0 = -\frac{d}{m}x_2 - \frac{g}{L}\sin(x_1)$

$(\dot{\underset{\sim}{x}} \overset{!}{=} 0)$

$\Rightarrow \quad x_2 = 0$

$\sin(x_1) = 0 \qquad x_1 = k\pi, \; k \in \mathbb{Z}$

$\underset{\sim}{x}_0 = \begin{pmatrix} k\pi \\ 0 \end{pmatrix}$

k gerade: untere Ruhelage

k ungerade: obere Ruhelage

3) Linearisierung um $\underset{\sim}{x}_{01} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\dot{\underset{\sim}{x}}(t) \approx \underset{\sim}{f}\Big|_{\underset{\sim}{x}_{01}} + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\Bigg|_{\underset{\sim}{x}_{01}} (\underset{\sim}{x} - \underset{\sim}{x}_{01})$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 & -\frac{d}{m} \end{pmatrix}\Bigg|_{\underset{\sim}{x}_{01}} (\underset{\sim}{x} - \underset{\sim}{x}_{01})$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{d}{m} \end{pmatrix}}_{\underline{\underline{A}}} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$\dot{\underset{\sim}{x}} \approx \underline{\underline{A}}\,\underset{\sim}{x}$

Ansatz: $\underset{\sim}{x} = \underset{\sim}{r}e^{\lambda t}$

$\underset{\sim}{r}\lambda e^{\lambda t} = \underline{\underline{A}}\,\underset{\sim}{r}e^{\lambda t} \rightsquigarrow (\underline{\underline{A}} - \lambda\underline{\underline{I}})\underset{\sim}{r}e^{\lambda t}$

$\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) \overset{!}{=} 0$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{g}{L} & -\lambda - \frac{d}{m} \end{vmatrix} = -\lambda\left(-\lambda - \frac{d}{m}\right) + \frac{g}{L} \overset{!}{=} 0$$

$$\lambda^2 + \frac{d}{m}\lambda + \frac{g}{L} = 0$$

$$-\frac{g}{L} + \left(\frac{d}{2m}\right)^2 = \left(\lambda + \frac{d}{2m}\right)^2$$

$$\lambda = -\frac{d}{2m} \pm \sqrt{\left(\frac{d}{2m}\right)^2 - \frac{g}{L}}$$

<u>Bedingung für Stabilität</u> :

- $\frac{d}{m} > 0$ ✓
- $\frac{g}{L} > 0$ ✓

$\Big\}$ asympt. Stabilität

Linearisierung um $\underset{\sim}{x}_{02} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$

$$\dot{\underset{\sim}{x}}(t) \approx f\Big|_{\underset{\sim}{x}_{02}} + \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 & -\frac{d}{m} \end{pmatrix}\Bigg|_{\underset{\sim}{x}_{02}} (\underset{\sim}{x} - \underset{\sim}{x}_{02})$$

$$\dot{\underset{\sim}{x}}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{g}{L} & -\frac{d}{m} \end{pmatrix}}_{\underline{\underline{A}}} (\underset{\sim}{x} - \underset{\sim}{x}_{02})$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} -\lambda & 1 \\ \frac{g}{L} & -\lambda - \frac{d}{m} \end{vmatrix} \overset{!}{=} 0 \quad \leadsto \quad \lambda = -\frac{d}{2m} \pm \sqrt{\left(\frac{d}{2m}\right)^2 + \frac{g}{L}}$$

$\rightarrow$ Instabilität, da $\frac{g}{L} > 0$ und damit positiver Realteil

4) $\boxed{\underset{\sim}{x}_{01}}$ $\triangleright \left(\frac{d}{2m}\right)^2 > \frac{g}{L}$ $\quad \lambda_2 < \lambda_1 < 0$ $\rightarrow$ stabiler Knoten

$\triangleright \left(\frac{d}{2m}\right)^2 < \frac{g}{L}$ $\quad$ stabiler Strudel bzw. Fokus (λ imaginär)

$\triangleright \left(\frac{d}{2m}\right)^2 = \frac{g}{L}$ $\quad$ stabiler Knoten ($\lambda_1 = \lambda_2 < 0$)

$\boxed{\underset{\sim}{x}_{02}}$ Sattelpunkt ($\lambda_2 < 0 < \lambda_1$)

5) s. Abbildung

6) $\left.\begin{array}{l} x_1 = -2\pi \\ x_2 = 2{,}5 \end{array}\right\}$ $f\begin{pmatrix} -2\pi \\ 2{,}5 \end{pmatrix} = \begin{pmatrix} 2{,}5 \\ -\frac{d}{m}2{,}5 - \cancel{\frac{g}{L}\sin(-2\pi)} \end{pmatrix}$

$$= \begin{pmatrix} 2{,}5 \\ -0{,}25 \end{pmatrix}$$

7) s. Abbildung

Aufgabe 2

1) $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$ $\qquad \ddot{y} = -\frac{k}{m}\dot{y} - \frac{c}{m}y - \mu g \, \mathrm{sgn}(\dot{y} - v_0)$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_2 - \frac{c}{m}x_1 - \mu g \, \mathrm{sgn}(x_2 - v_0)$$

2) $\dot{\underset{\sim}{x}} \overset{!}{=} 0 \quad \Rightarrow \quad x_2 = 0$

$$x_1 = -\frac{\mu m g}{c}\,\mathrm{sgn}(-v_0)$$

$\rightarrow 0 > v_0 \Rightarrow x_1 = -\frac{\mu m g}{c}$ $\qquad \underset{\sim}{x}_{01} = \begin{pmatrix} -\frac{\mu m g}{c} \\ 0 \end{pmatrix}$

$\rightarrow 0 < v_0 \Rightarrow x_1 = \frac{\mu m g}{c}$ $\qquad \underset{\sim}{x}_{02} = \begin{pmatrix} \frac{\mu m g}{c} \\ 0 \end{pmatrix}$

$\boxed{\dot{y} < v_0} \Rightarrow \mathrm{sgn}(\dot{y} - v_0) = -1$

Ruhelage $\underset{\sim}{x}_{02}$

3) $\dot{x}_1 = x_2$

$$\dot{x}_2 = -\frac{k}{m}x_2 - \frac{c}{m}x_1 + \mu g$$

$\xi_1 = x_1 - x_{10} = x_1 - \frac{\mu m g}{c} \Rightarrow x_1 = \xi_1 + \frac{\mu m g}{c}$

$\xi_2 = x_2 - x_{20} = x_2 \Rightarrow x_2 = \xi_2$

$$\Rightarrow \begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = -\frac{k}{m}\xi_2 - \frac{c}{m}\left(\xi_1 + \cancel{\frac{\mu m g}{c}}\right) + \cancel{\mu g} \end{cases}$$

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{k}{m} \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

4) $0 \overset{!}{=} \det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$

$$0 = \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{m} & -\lambda - \frac{k}{m} \end{vmatrix} \Rightarrow \lambda = -\frac{k}{2m} \pm \sqrt{\left(\frac{k}{2m}\right)^2 - \frac{c}{m}}$$

5) <u>Bedingung für Stabilität</u>: $\circ\ \frac{k}{m} > 0 \longrightarrow \boxed{k > 0}$

$\circ\ \frac{c}{m} > 0$ ✓ $\qquad \hookrightarrow$ asymptotische Stabilität

- $k < 0$ $\rightarrow$ immer instabil
- $k = 0$ $\rightarrow$ größter Eigenwert hat $Re(\lambda) = 0$
  ↳ Stabilität (Untersuchung von linearem System)

! Bei nichtlinearem System kann bei größtem Eigenwert mit Realteil 0 keine Aussage getroffen werden.

6) s. Lösungsblatt

<u>Aufgabe 3</u>

$$\dot{\psi} - \Omega(\gamma_1 + \gamma_2) = c_2 \gamma_2$$
$$\dot{\gamma}_1 + \dot{\gamma}_2 + \Omega\psi = -c_1 \gamma_2$$
$$\dot{\gamma}_1 + \Omega\psi = -k(\gamma_1 - \bar{\gamma})$$

1) $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \psi \end{pmatrix}$

$$\dot{\psi} = \Omega(\gamma_1 + \gamma_2) + c_2 \gamma_2$$
$$\dot{\gamma}_2 = -c_1 \gamma_2 + k(\gamma_1 - \bar{\gamma})$$
$$\dot{\gamma}_1 = -\Omega\psi - k(\gamma_1 - \bar{\gamma})$$

$$\dot{x}_1 = -\Omega x_3 - k x_1 + \bar{\gamma} k$$
$$\dot{x}_2 = k x_1 - c_1 x_2 - \bar{\gamma} k$$
$$\dot{x}_3 = \Omega x_1 + (\Omega + c_2) x_2$$

2) $\overset{\circ}{\underset{\sim}{x}} \overset{!}{=} \underset{\sim}{0} \Rightarrow$ (I) $0 = \Omega x_1 + (\Omega + c_2) x_2$

(II) $0 = -k x_1 - \Omega x_3 + \bar{\gamma} k$

(III) $0 = k x_1 - c_1 x_2 - \bar{\gamma} k$ $\quad \Big] \cdot -\frac{\Omega}{k}$

$$0 = \left(\Omega + c_2 + \frac{c_1 \Omega}{k}\right) x_2 + \bar{\gamma} \cancel{k} \frac{\Omega}{\cancel{k}} \Rightarrow x_2 = -\frac{\bar{\gamma}\Omega}{\Omega + c_2 + \frac{c_1 \Omega}{k}}$$

$$\boxed{x_2 = -\frac{\bar{\gamma} k \Omega}{(\Omega + c_2)k + c_1 \Omega}}$$

<u>x_2 in (I)</u>

$$x_1 = +\frac{\Omega + c_2}{\cancel{\Omega}}\left(+\frac{\bar{\gamma} k \cancel{\Omega}}{(\Omega + c_2)k + c_1 \Omega}\right)$$

$$\boxed{x_1 = \frac{\bar{\gamma} k (\Omega + c_2)}{(\Omega + c_2)k + c_1 \Omega}}$$

<u>x_1 in II</u>

$$x_3 = \frac{1}{\Omega}(\bar{\gamma} k - k x_1) = \frac{k}{\Omega}\left(\bar{\gamma} - \frac{\bar{\gamma} k (\Omega + c_2)}{(\Omega + c_2)k + c_1 \Omega}\right)$$
$$= \frac{k}{\Omega}\left(\frac{\bar{\gamma}((\cancel{\Omega + c_2})k + c_1 \Omega) - \bar{\gamma} k (\cancel{\Omega + c_2})}{(\Omega + c_2)k + c_1 \Omega}\right)$$

$$\boxed{x_3 = \frac{\bar{\gamma} k c_1}{(\Omega + c_2)k + c_1 \Omega}}$$

$$\underset{\sim}{x}_0 = \frac{I k}{(\Omega + c_2)k + c_1 \Omega} \begin{pmatrix} \Omega + c_2 \\ -\Omega \\ c_1 \end{pmatrix}$$

3) Linearisierung um die Ruhelage $\underset{\sim}{x}_0$:

$$\dot{\underset{\sim}{x}} \approx \underset{\sim}{f}\Big|_{\underset{\sim}{x}_0} + \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}\right|_{\underset{\sim}{x}_0} (\underset{\sim}{x} - \underset{\sim}{x}_0)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} -k & 0 & -\Omega \\ k & -c_1 & 0 \\ \Omega & \Omega + c_2 & 0 \end{pmatrix}}_{\underline{\underline{A}}} (\underset{\sim}{x} - \underset{\sim}{x}_0)$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \overset{!}{=} 0$$

$$\begin{vmatrix} -k-\lambda & 0 & -\Omega \\ k & -c_1-\lambda & 0 \\ \Omega & \Omega + c_2 & -\lambda \end{vmatrix} = (-k-\lambda)(-c_1-\lambda)(-\lambda) - \Omega \Big[k(\Omega + c_2) - \Omega(-c_1-\lambda$$

$$0 = +(k+\lambda)(c_1\lambda + \lambda^2) + \Omega k(\Omega + c_2) + \Omega^2 (c_1 + \lambda)$$

$$= \underset{a_0 = 1}{\lambda^3} + \underbrace{(k + c_1)}_{a_1} \lambda^2 + \underbrace{(k c_1 + \Omega^2)}_{a_2} \lambda + \underbrace{\Omega^2 k + \Omega k c_2 + \Omega^2 c_1}_{a_3 > 0}$$

4) <u>Bedingungen für Stabilität:</u>

- $a_2 > 0$ ✓
- $a_1 a_2 - a_0 a_3 = (k + c_1)(k c_1 + \Omega^2) - (\Omega^2 k + \Omega k c_2 + \Omega^2 c_1) >$

$$0 < k^2 c_1 \cancel{+ k\Omega^2} + c_1^2 k + \cancel{c_1 \Omega^2} - \cancel{\Omega^2 k} - \Omega k c_2 - \cancel{\Omega^2 c_1}$$

$$0 < k^2 c_1 + c_1^2 k - \Omega k c_2$$

$$\boxed{0 < k c_1 + c_1^2 - \Omega c_2}$$

- $a_0 > 0$ ✓

5) Richtungsfeld bzw. Phasenraum 3-dimensional, $\underset{\sim}{f}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = I k \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$