Karlsruher Institut für Technologie

Saalübung MGR + FDM

Guten Morgen!

Modellbildung und Simulation
Kapitel 7: Systeme mit verteilten Parametern

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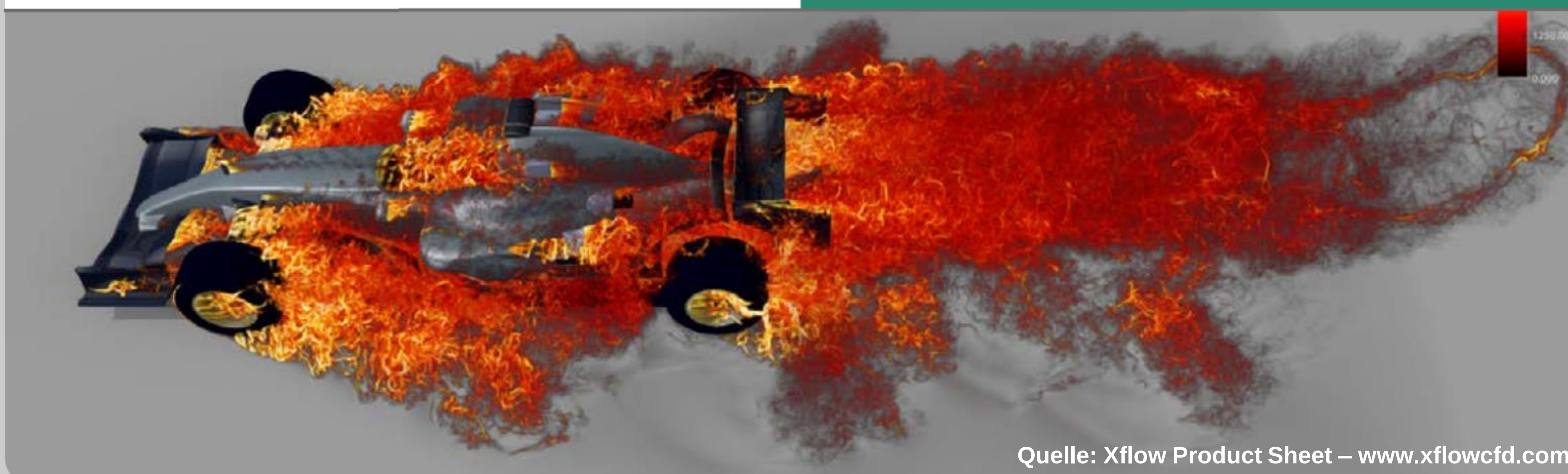
ITS



Quelle: Xflow Product Sheet – www.xflowcfd.com

Saalübung: gew. Residuen und FDM

Aufgabe 1: Methode der gewichteten Residuen

$$\frac{du}{dx} + u = 0 \qquad (\mathcal{L}u = 0) \quad , \qquad 0 \le x \le 1$$

$\left.\right\}$ exakte Lösung $u(x) = e^{x}$

AB (RB) $u(x=0) = 1$

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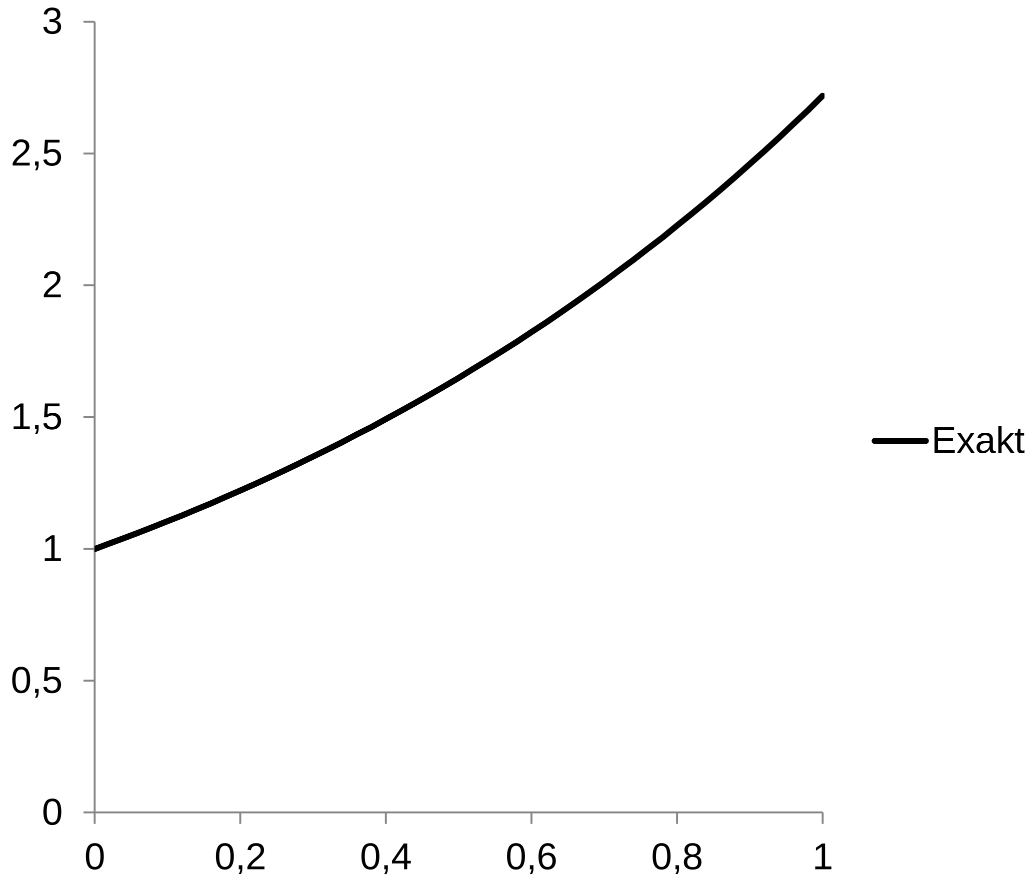
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Exakt

$$\tilde{u}(x) = \sum_{i=0}^{2} a_i x^i = a_0 x^0 + a_1 x^1 + a_2 x^2$$

$$\tilde{u}(x=0) \overset{!}{=} 1 \rightarrow a_0 = 1$$

$$r(x) = \frac{d\tilde{u}}{dx} - \tilde{u} = \frac{d(1 + a_1 x + a_2 x^2)}{dx} - (1 + a_1 x + a_2 x^2) =$$

$$\mathcal{L}u \rightarrow \mathcal{L}\tilde{u}$$

$$= -1 + a_1(1-x) + a_2(2x - x^2) = 0$$

Falsch

MGR: $\int_\Omega r(x) \cdot W(x)\, dx \overset{!}{=} 0$

K_I $\Rightarrow W = r(x_i) = 0$

$x_1 = 0{,}5 \ , \ x_2 = 1$

G1: $x = 0{,}5$

$$-1 + a_1\left(1 - \tfrac{1}{2}\right) + a_2\left(2 \cdot \tfrac{1}{2} - \left(\tfrac{1}{2}\right)^2\right) = 0$$

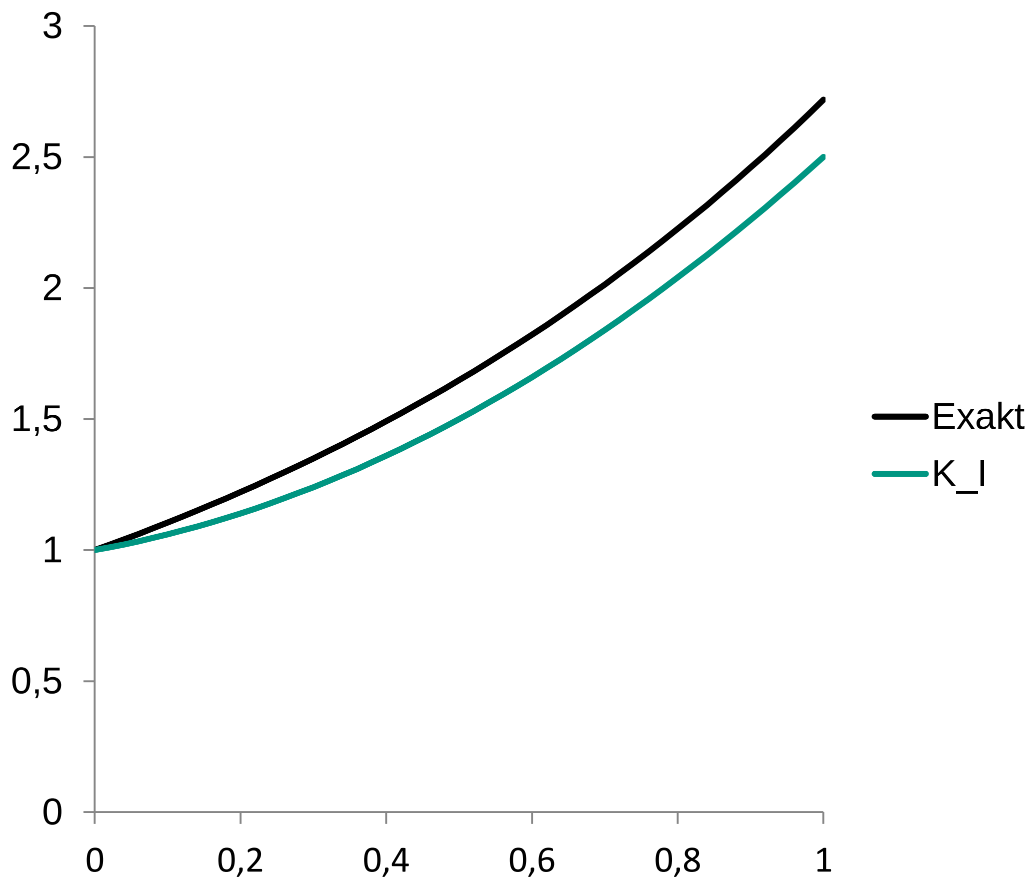
G2: $x = 1$

$$-1 + a_1(1-1) + a_2(2 \cdot 1 - 1^2) = 0$$

$G2: \; a_2 = 1$

$\rightarrow G1: \; a_1 = \frac{1}{2}$

$$\Rightarrow \quad \tilde{u} = 1 + \frac{1}{2} x + 1 \cdot x^2$$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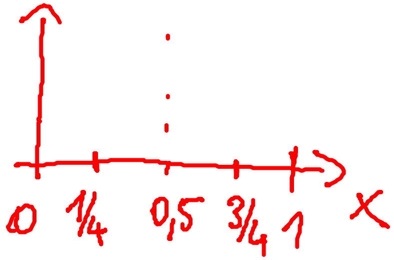
K_I

K-11



G3: $x = 1/4$

$$-1 + a_1\left(1 - \frac{1}{4}\right) + a_2\left(2 \cdot \frac{1}{4} - \left(\frac{1}{4}\right)^2\right) = 0$$

$$-1 + 3/4\, a_1 + \frac{7}{16} a_2 = 0$$

G4: $x = 3/4$

$$-1 + a_1\left(1 - 3/4\right) + a_2\left(2 \cdot 3/4 - \left(\frac{3}{4}\right)^2\right) = 0$$

$$-1 + 1/4\, a_1 + \frac{15}{16} a_2 = 0$$

z. B. $G3 - 3 \cdot G4$

$$\Rightarrow a_2 = \frac{16}{19}$$

$$a_2 \rightarrow G3 \Rightarrow a_1 = \frac{16}{19}$$

$$\tilde{u} = 1 + \frac{16}{19}x + \frac{16}{19}x^2$$

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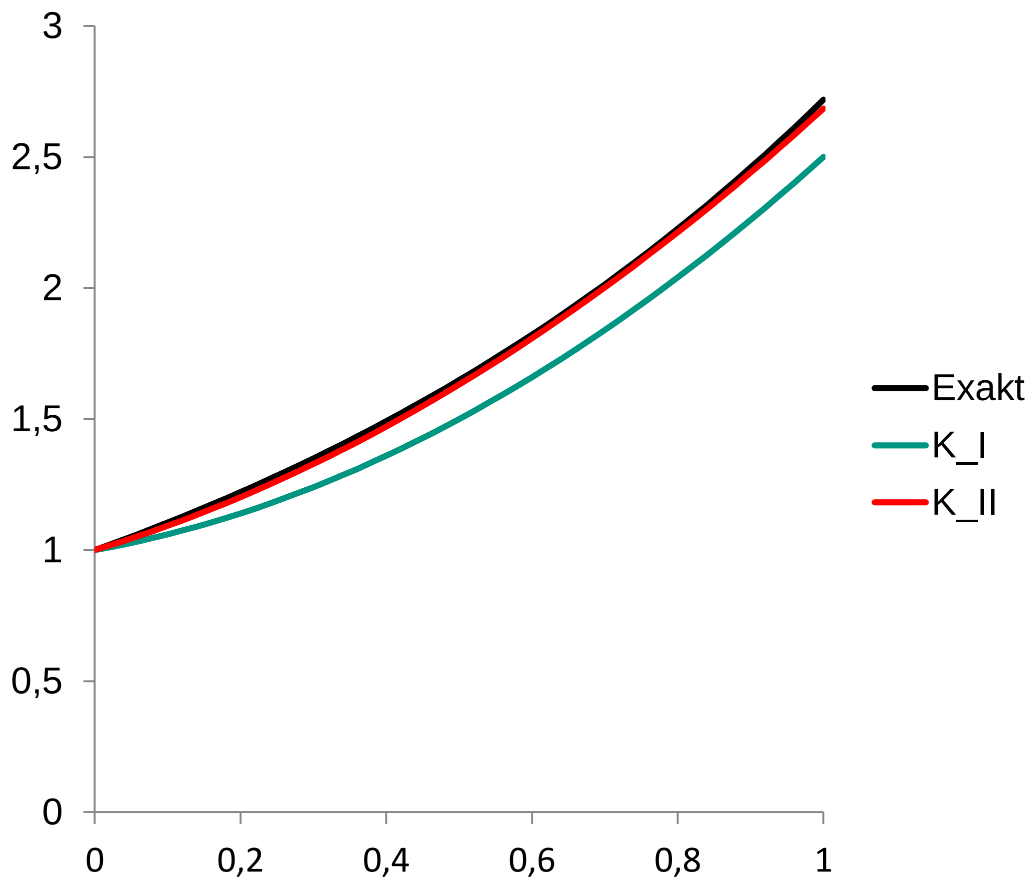
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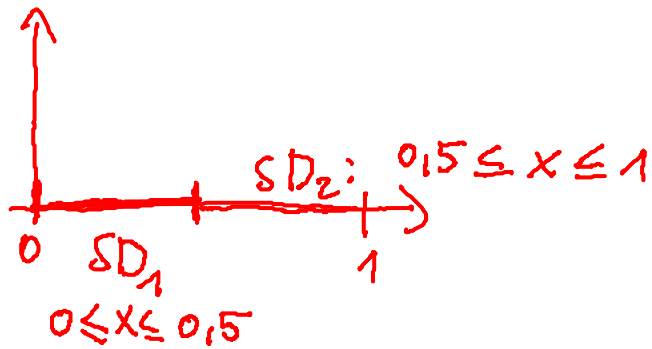
K_I
K_II

SD:



$$\int_{SD_i} r(x)\,dx = 0$$

G5: SD1

$$\int_0^{0,5} \left[-1 + a_1(1-x) + a_2(2x - x^2)\right] dx = 0$$

$$\left[-x + a_1\left(x - \frac{x^2}{2}\right) + a_2\left(2 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right)\right]_0^{0,5} = 0$$

$$\vdots$$

$$-\frac{1}{2} + \frac{3}{8} a_1 + \frac{5}{26} a_2 = 0$$

G6: SD_2

$$\int_{0,5}^{1} \left[-1 + \ldots \right] dx = 0$$

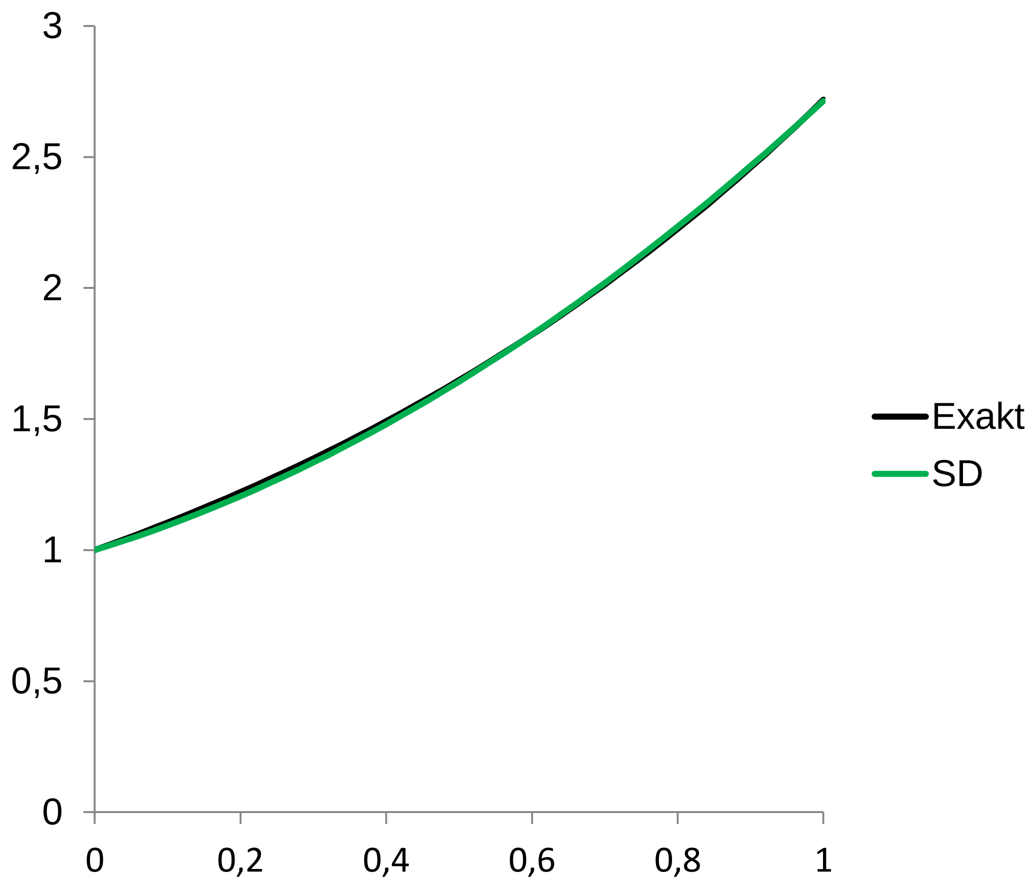
$$\left[\quad \right]_{0,5}^{1}$$

$$\Rightarrow \quad -\frac{1}{2} + \frac{1}{8} a_1 + \frac{11}{24} a_2 = 0$$

z.B. $G5 - 3 \cdot G6 \Rightarrow a_2 = \frac{6}{7}$

$a_2 \rightarrow G5 \Rightarrow a_1 = \frac{6}{7}$

$$\Rightarrow \quad \tilde{u} = 1 + \frac{6}{7}x + \frac{6}{7}x^2$$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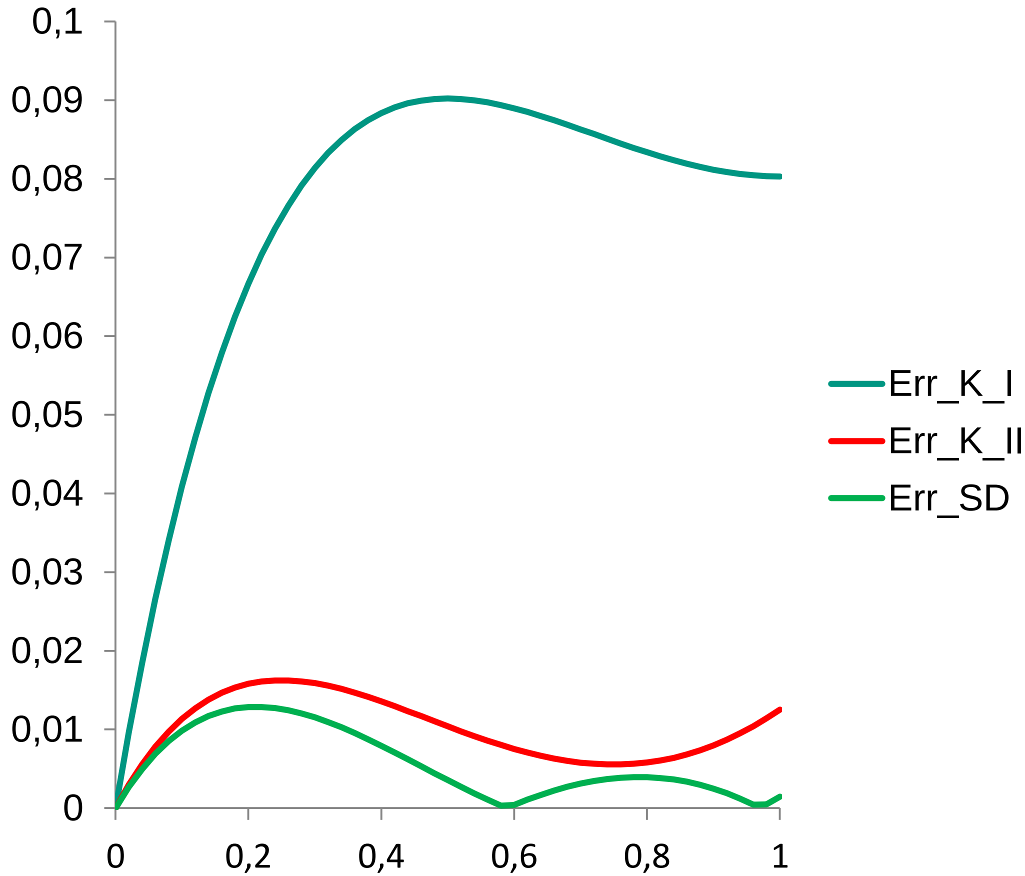
1
Exakt
SD

	$x = \frac{1}{4} = 0,25$	$x = \frac{1}{2} = 0,5$	$x = \frac{3}{4} = 0,75$	$x = 1$
Exakt e^x	**1,2840**	**1,6487**	**2,1170**	**2,7183**
K I	$\frac{19}{16} = 1,1875$	$\frac{3}{2} = 1,5$	$\frac{31}{16} = 1,9375$	$\frac{5}{2} = 2,5$
ε	$0,0965$	$0,1487$	$0,1795$	$0,2183$
K II	$\frac{24}{19} = 1,2632$	$\frac{31}{19} = 1,6316$	$\frac{40}{19} = 2,1053$	$\frac{51}{19} = 2,6842$
ε	$0,0208$	$0,0171$	$0,0177$	$0,0341$
SD	$\frac{71}{56} = 1,2679$	$\frac{23}{14} = 1,6429$	$\frac{119}{56} = 2,125$	$\frac{19}{7} = 2,7143$
ε	$0,0161$	$0,0058$	$0,008$	$0,004$

Relativer Fehler

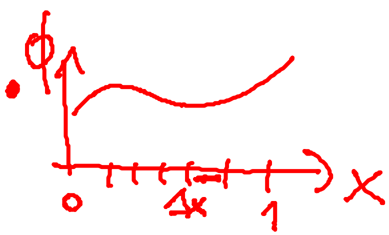


Aufgabe 2: FDM

$$\phi = \begin{pmatrix} \rho \\ u \\ v \\ w \\ E \end{pmatrix}$$

$$\left(\frac{\partial(\rho\phi)}{\partial t}\right) + \frac{\partial(\rho v_j \phi)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\Gamma \frac{\partial \phi}{\partial x_j}\right) + q_\phi$$

- Koeff.s = Konst. $\Rightarrow \rho v_j \frac{\partial \phi}{\partial x}$, $\Gamma\left(\frac{\partial^2 \phi}{\partial x^2}\right)$

- 

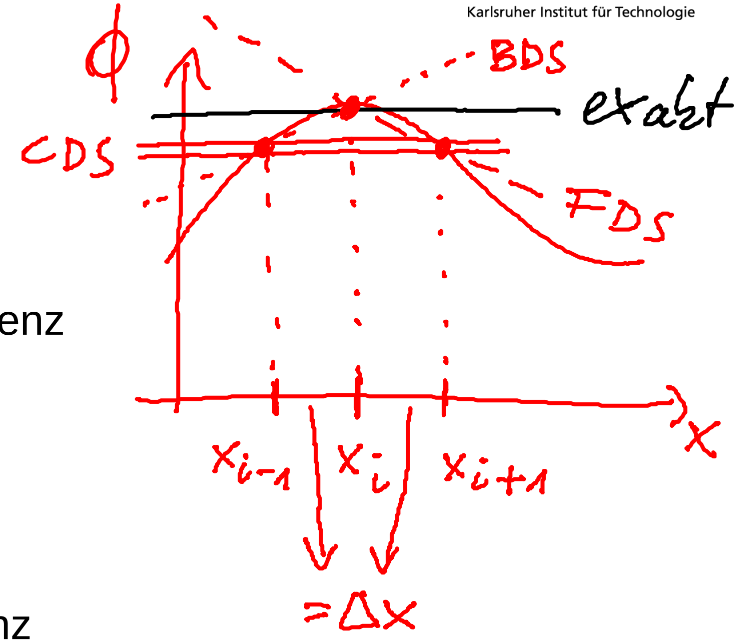
$\Delta x_i = x_i - x_{i-1}$ $\Longrightarrow \Delta x = \text{konst.} \Rightarrow$ äquidist.

$$\phi(x) = \phi(x_i) + (x - x_i)\left(\frac{\partial \phi}{\partial x}\right)_i + \frac{(x - x_i)^2}{2!}\left(\frac{\partial^2 \phi}{\partial x^2}\right)_i +$$

$$\frac{(x - x_i)^3}{3!}\left(\frac{\partial^3 \phi}{\partial x^3}\right)_i + \cdots + \frac{(x - x_i)^n}{n!}\left(\frac{\partial^n \phi}{\partial x^n}\right)_i + H$$

FDM - Beispielformeln

Beispiel für einfache Differenzenformeln:



$$\left(\frac{\partial \phi}{\partial x}\right)_i = \frac{\phi_i - \phi_{i-1}}{x_i - x_{i-1}} + \varepsilon$$

Rückwärtsdifferenz

BDS ....

$\Delta x = x_i - x_{i-1}$

$$\left(\frac{\partial \phi}{\partial x}\right)_i = \frac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i} + \varepsilon$$

Vorwärtsdifferenz

FDS — — —

$x_i - x_{i+1} = -\Delta x$

$$\left(\frac{\partial \phi}{\partial x}\right)_i = \frac{\phi_{i+1} - \phi_{i-1}}{x_{i+1} - x_{i-1}} + \varepsilon$$

Zentraldifferenz

CDS ═══

$x_{i+1} - x_i = \Delta x$

ε stellt den Abbruchfehler dar

$$G1:\ \phi(x_{i+1}) = \phi(x_i) + \frac{\Delta x}{1!}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\Delta x^2}{2!}\left(\frac{\partial^2 \phi}{\partial x^2}\right) + H$$

$$G2:\ \phi(x_{i-1}) = \phi(x_i) + \frac{-\Delta x}{1!}\left(\frac{\partial \phi}{\partial x}\right) + \frac{(-\Delta x)^2}{2!}\left(\frac{\partial^2 \phi}{\partial x^2}\right) + \frac{(-\Delta x)^3}{3!}\left(\frac{\partial^3 \phi}{\partial x^3}\right) + H$$

$$x - x_i \Rightarrow x_{i-1} - x_i \Rightarrow \Delta x = -(x_i - x_{i-1})$$

$$FDS:\ G1\quad \frac{\Delta x}{1!}\cdot\frac{\partial \phi}{\partial x} = \phi(x_{i+1}) - \phi(x_i) - H$$

$$\frac{\partial \phi}{\partial x} = \frac{\phi(x_{i+1}) - \phi(x_i)}{\Delta x} - H/\Delta x \quad (+\varepsilon)$$

$$H \sim \frac{\left(\frac{\Delta x^2}{2!}\frac{\partial^2 \phi}{\partial x^2}\right)}{\Delta x}$$

$$\Downarrow$$

$$\varepsilon \sim \Delta x \cdot \left(\frac{\partial^2 \phi}{\partial x^2}\right)$$

$$BDS:\ G2$$

$$\frac{\partial \phi}{\partial x} = \frac{\phi(x_i) - \phi(x_{i-1})}{\Delta x} + \varepsilon$$

$$\varepsilon \sim \Delta x$$

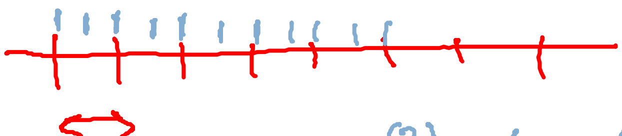
$$CDS:\ G1 - G2$$

CDS : $G_1 - G_2$

$$\phi(x_{i+1}) - \phi(x_{i-1}) = 0 + \frac{2\Delta x}{1!}\frac{\partial \phi}{\partial x} + 0 + \frac{\Delta x^3}{3!}\frac{\partial^3 \phi}{\partial x^3} + H$$

$$\frac{\partial \phi}{\partial x} = \frac{\phi(x_{i+1}) - \phi(x_{i-1})}{2\Delta x} - \frac{\Delta x^2}{2 \cdot 3!}\frac{\partial^3 \phi}{\partial x^3}$$

$\hookrightarrow \varepsilon \sim \Delta x^2$



$\Delta x^{(1)}$. $\Delta x^{(2)} = \frac{1}{2}\Delta x^{(1)}$

FDS, BDS : $\varepsilon^{(2)} = \frac{1}{2}\varepsilon^{(1)}$

CDS : $\varepsilon^{(2)} = \frac{1}{4}\varepsilon^{(1)}$

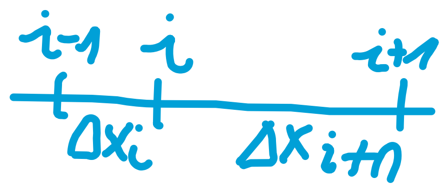
CDS: G1 − G2

$$\phi(x_{i+1}) - \phi(x_{i-1}) = 0 + \boxed{\frac{2\Delta x}{1!}\frac{\partial\phi}{\partial x} + 0} + \frac{\Delta x^3}{3!}\frac{\partial^3\phi}{\partial x^3} + H$$

$$\frac{\partial\phi}{\partial x} = \frac{\phi(x_{i+1}) - \phi(x_{i-1})}{2\Delta x} - \frac{\Delta x^2}{2\cdot 3!}\frac{\partial^3\phi}{\partial x^3}$$

$\hookrightarrow \varepsilon \sim \Delta x^2$

Netz mit Streckung



$x - x_i$

G1: $x_{i-1} - x_i = -\Delta x_i$

G2: $x_{i+1} - x_i = \Delta x_{i+1}$

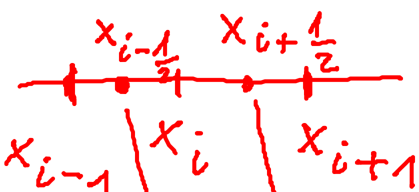
$$\phi(x_{i+1}) - \phi(x_{i-1}) = 0 + \frac{\Delta x_{i+1} + \Delta x_i}{1!}\frac{\partial\phi}{\partial x} + \underbrace{\frac{\Delta x_{i+1} - \Delta x_i}{2!}\frac{\partial^2\phi}{\partial x^2}}_{\varepsilon \approx} + H$$

$$\frac{\partial^2\phi}{\partial x^2} = ? \leftarrow G_1 + G_2$$

$$\phi(x_{i+1}) + \phi(x_{i-1}) = 2\phi(x_i) + 0 + \frac{\Delta x^2 + (-\Delta x)^2}{2!}\left(\frac{\partial^2\phi}{\partial x^2}\right) + 0 + H$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\phi(x_{i+1}) - 2\phi(x_i) + \phi(x_{i-1})}{\frac{2\,\Delta x^2}{2!}} - \frac{H}{\Delta x^2}$$

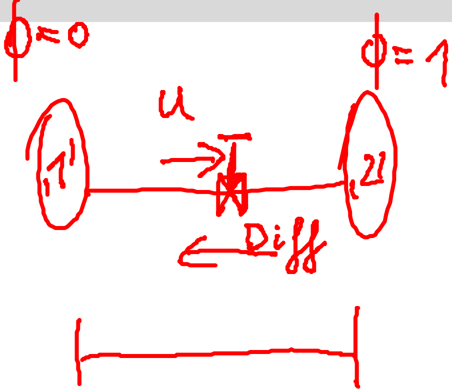
$$\varepsilon \approx \Delta x^2$$



$$\left(\frac{\partial\phi}{\partial x}\right)_{i+\frac{1}{2}}$$

$$\left(\frac{\partial\phi}{\partial x}\right)_{i-\frac{1}{2}}$$

$$\rightarrow \left(\frac{\partial\left(\frac{\partial\phi}{\partial x}\right)_{i\pm\frac{1}{2}}}{\partial x}\right)_i$$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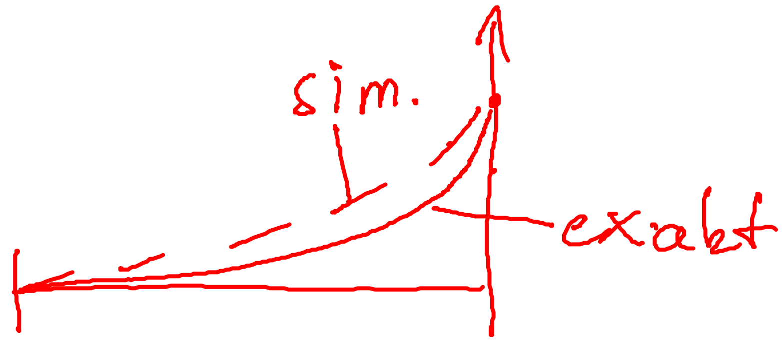
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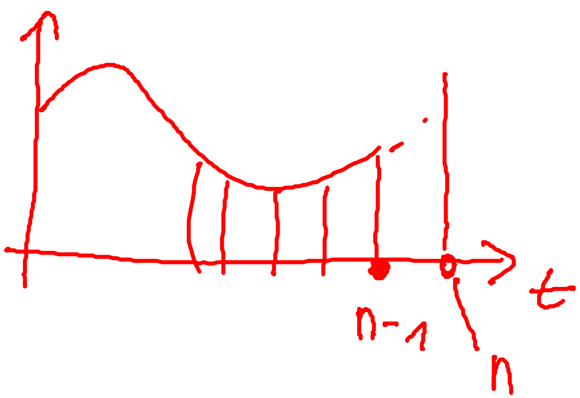
$$\frac{\partial \phi}{\partial t} = ?$$

$$\frac{\phi^n - \phi^{n-1}}{\Delta t} + \underbrace{\rho u \frac{\partial \phi}{\partial x} = \Gamma \frac{\partial^2 \phi}{\partial x^2}}_{RHS}$$



$$\phi^n = \phi^{n-1} + RHS \cdot \Delta t$$

ϕ^{n-1} ← explizit

ϕ^n ← implizit

CFL