Chapter 9

The Laplace Transform

9.0 Introduction

Convolution——

Analysis the system in time domain

Fourier transform——

Analysis the system in frequency domain

Signals & Systems

9.0 Introduction

$$h(t) = e^{t}u(t) \not\rightarrow H(jw)$$

- View of energy

  finite energy(square integrable)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

Laplace transform

Analysis the system in S domain

9.1 The Laplace transform

$$x(t) = e^{st} \longrightarrow \boxed{H(S)} \longrightarrow y(t) = H(S)e^{st}$$

$$\int_{-\infty}^{+\infty} h(t)e^{-st}dt$$

Laplace transform: $H(S) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt$

$$h(t) \xleftrightarrow{LT} H(S)$$

Fourier transform: $h(t) \xleftrightarrow{FT} H(S)\Big|_{S=jw}$

The relationship between the Laplace transform and the Fourier transform

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- $F\{x(t)\} = X(S)\big|_{S=jw}$

  Fourier transform is a particular form of Laplace transform

- If $S = \sigma + jw$

$$X(S) = X(\sigma + jw) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + jw)t}dt$$

$$= \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-jwt}dt$$

$$F\{x(t)e^{-\sigma t}\} = X(S)$$

Laplace transform is Fourier transform of $x(t)e^{-\sigma t}$

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Example 9.1

$$x(t)=e^{-at}u(t)$$

(1) *if* $a>0$, determine Fourier transform

(2) determine Laplace transform

Answer :

$$(1)\ X(jw)=\int_{-\infty}^{+\infty}e^{-at}u(t)e^{-jwt}dt=\frac{1}{jw+a}$$

$$(2)\ X(S)=\int_{-\infty}^{+\infty}e^{-at}u(t)e^{-st}dt=\int_{0}^{+\infty}e^{-(s+a)t}dt$$

$$=\frac{1}{S+a}\qquad \mathrm{Re}\{S\}>-a$$

$$e^{-at}u(t)\xleftrightarrow{LT}\frac{1}{S+a}\qquad \mathrm{Re}\{S\}>-a$$

$$if\ \ a=0,u(t)\xleftrightarrow{LT}\frac{1}{S}\qquad \mathrm{Re}\{S\}>0$$

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Example 9.2

$x(t) = -e^{-at}u(-t)$ **determine Laplace transform**

$$X(S) = \int_{-\infty}^{+\infty} -e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

$$= \frac{1}{S+a} \qquad \mathrm{Re}\{S\} < -a$$

$$-e^{-at}u(-t) \xleftrightarrow{LT} \frac{1}{S+a} \qquad \mathrm{Re}\{S\} < -a$$

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{S+a} \qquad \mathrm{Re}\{S\} > -a$$

Region of Convergence (ROC)

- ROC——the range of values of S for which integral in $X(S)=\int_{-\infty}^{+\infty}x(t)e^{-st}dt$ converges

- Laplace transform includes:

  (1)the algebraic expression

  (2)ROC

- The representation of ROC——Complex plane (S-plane)

$$-e^{-at}u(-t)\xleftrightarrow{LT}\frac{1}{S+a}\qquad \mathrm{Re}\{S\}<-a$$

$$e^{-at}u(t)\xleftrightarrow{LT}\frac{1}{S+a}\qquad \mathrm{Re}\{S\}>-a$$

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Example 9.3

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

determine Laplace transform of $x(t)$

$$e^{-2t}u(t) \xleftrightarrow{LT} \frac{1}{S+2} \qquad \mathrm{Re}\{S\} > -2$$

$$e^{-t}u(t) \xleftrightarrow{LT} \frac{1}{S+1} \qquad \mathrm{Re}\{S\} > -1$$

$$X(S) = \frac{3}{S+2} - \frac{2}{S+1} \qquad \mathrm{Re}\{S\} > -1$$

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Example 9.4

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$$

determine Laplace transform of $x(t)$

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{S+a} \qquad \mathrm{Re}\{S\} > -\mathrm{Re}\{a\}$$

$$\cos 3t = \frac{e^{j3t} + e^{-j3t}}{2}$$

$$x(t) = [e^{-2t} + \frac{e^{-(1-3j)t}}{2} + \frac{e^{-(1+3j)t}}{2}]u(t)$$

$$X(S) = \frac{1}{S+2} + \frac{1}{2}\left(\frac{1}{S+(1-3j)}\right) + \frac{1}{2}\left(\frac{1}{S+(1+3j)}\right) \qquad \mathrm{Re}\{S\} > -1$$

$$= \frac{2S^2 + 5S + 12}{(S+2)(S^2 + 2S + 10)}$$

Pole-Zero plot

- Laplace transform maybe a ratio of polynomials

$$X(S)=\frac{N(S)}{D(S)}$$

numerator

denominator

- Poles —— the roots of D(S)

  Zeros —— the roots of N(S)

- The representation of poles and zeros

  ——Pole-Zero plot

- Another representation of Laplace transform

  ——Pole-Zero plot and ROC

- The order of pole or zero

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Example 9.5

$$x(t)=\delta(t)-\frac{4}{3}e^{-t}u(t)+\frac{1}{3}e^{2t}u(t)$$

determine Laplace transform of $x(t)$

$$\delta(t)\xleftrightarrow{LT}1$$

$$e^{-t}u(t)\xleftrightarrow{LT}\frac{1}{S+1}\qquad \mathrm{Re}\{S\}>-1$$

$$e^{2t}u(t)\xleftrightarrow{LT}\frac{1}{S-2}\qquad \mathrm{Re}\{S\}>2$$

$$X(S)=1-\frac{4}{3}\frac{1}{S+1}+\frac{1}{3}\frac{1}{S-2}\qquad \mathrm{Re}\{S\}>2$$

$$=\frac{(s-1)^2}{(S+1)(S-2)}\qquad \mathrm{Re}\{S\}>2$$

9.2 The ROC for Laplace transform

❖Property 1 —— the ROC of X(S) consists of strips parallel to the jw-axis in the s-plane

$$\int_{-\infty}^{+\infty} \left| x(t) \right| e^{-\sigma t} dt < \infty$$

❖Property 2 —— for rational Laplace transforms, the ROC does not contain any poles

$$X(S)\Big|_{s=pole} \longrightarrow \infty$$

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Example 9.6

$$x(t)=\begin{cases} e^{-at} & 0<t<T \\ 0 & otherwise \end{cases}$$

determine Laplace transform of $x(t)$

$$X(S)=\int_0^T e^{-at}e^{-st}dt=\frac{1}{S+a}\left[1-e^{-(S+a)T}\right]$$

ROC is the entire s-plane

$$\lim_{S\to -a}X(S)=\lim_{S\to -a}\left[\frac{\frac{d}{ds}\left(1-e^{-(S+a)T}\right)}{\frac{d}{ds}(S+a)}\right]=\lim_{S\to -a}Te^{-aT}e^{-ST}=T$$

- Property 3 —— if $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane

- Property 4 —— if $x(t)$ is right sided, and if the line $Re\{s\}=\sigma_0$ is in the ROC, then all values of S for which $Re\{s\}>\sigma_0$ will also be in the ROC

❖Property 5 —— if x(t) is **left sided**, and if the line Re{s}= σ_0 is in the ROC, then all values of S for which Re{s}< σ_0 will also be in the ROC

❖Property 6 —— if x(t) is **two sided**, and if the line Re{s}= σ_0 is in the ROC, then the ROC will consist of a strip in the s-plane that include the line Re{s}= σ_0

Example 9.7

$$x(t) = e^{-b|t|}$$

determine Laplace transform of $x(t)$

$$x(t) = e^{-bt}u(t) + e^{bt}u(-t)$$

$$e^{-bt}u(t) \xleftrightarrow{LT} \frac{1}{S+b} \qquad \mathrm{Re}\{S\} > -b$$

$$e^{bt}u(-t) \xleftrightarrow{LT} \frac{-1}{S-b} \qquad \mathrm{Re}\{S\} < b$$

$$X(S) = \begin{cases} \dfrac{1}{S+b} - \dfrac{1}{S-b} & -b < \mathrm{Re}\{s\} < b \quad \text{for b>0} \\ \text{Laplace transform doesn't exist} & \text{for } b \le 0 \end{cases}$$

- Property 7 —— if the Laplace transform X(S) of x(t) is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of X(S) are contained in the ROC

- Property 8 —— if the Laplace transform X(S) of x(t) is rational

  if x(t) is right sided, the ROC is the region in the s-plane to the right of the rightmost pole

  if x(t) is left sided,the ROC is the region in the s-plane to the left of the leftmost pole

Example 9.8

$$X(S)=\frac{1}{(S+1)(S+2)}$$

determine possible ROCs

Pole: $S=-1,\ \ S=-2$

if x(t) is left sided, then ROC is $Re\{S\}<-2$

if x(t) is right sided, then ROC is $Re\{S\}>-1$

if x(t) is two sided, then ROC is $-2<Re\{S\}<-1$

9.3 The inverse Laplace transform

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- The representation of the inverse Laplace transform

$$X(S)=X(\sigma+jw)=F\{x(t)e^{-\sigma t}\}$$

$$=\int_{-\infty}^{+\infty}x(t)e^{-\sigma t}e^{-jwt}dt$$

$$\Rightarrow x(t)e^{-\sigma t}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\sigma+jw)e^{jwt}dw$$

$$\Rightarrow \qquad x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\sigma+jw)e^{(\sigma+jw)t}dw$$

$$because \quad ds=jdw$$

$$so \qquad x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(S)e^{St}ds$$

Partial-fraction expansion

$$X(S)=\sum_{i=1}^{m}\frac{A_i}{S+a_i}$$

$$e^{-bt}u(t)\overset{LT}{\longleftrightarrow}\frac{1}{S+b}\qquad \mathrm{Re}\{S\}>-b$$

$$-e^{-bt}u(-t)\overset{LT}{\longleftrightarrow}\frac{1}{S+b}\qquad \mathrm{Re}\{S\}<-b$$

Example 9.9

$$X(S)=\frac{1}{(S+1)(S+2)} \quad \mathrm{Re}\{S\}>-1$$

determine the inverse Laplace transform

$$X(S)=\frac{A}{S+1}+\frac{B}{S+2}$$

$$A=\left[(S+1)X(S)\right]\big|_{S=-1}=1$$

$$B=\left[(S+2)X(S)\right]\big|_{S=-2}=-1$$

$$x(t)=\left(e^{-t}-e^{-2t}\right)u(t)$$

Example 9.10

$$\mathrm{Re}\{S\}<-2$$

Example 9.11

$$-2<\mathrm{Re}\{S\}<-1$$

9.5 Properties of the Laplace transform

❖ Linearity

$$ax_1(t)+bx_2(t)\xleftrightarrow{LT}aX_1(S)+bX_2(S)$$

with ROC containing $R1\cap R2$

Example 9.13

$$x(t)=x_1(t)-x_2(t)$$

$$X_1(S)=\frac{1}{S+1} \qquad \mathrm{Re}\{S\}>-1$$

$$X_2(S)=\frac{1}{(S+1)(S+2)} \qquad \mathrm{Re}\{S\}>-1$$

determine X(S)

$$X(S)=\frac{1}{S+1}-\frac{1}{(S+1)(S+2)}=\frac{1}{S+2} \qquad \mathrm{Re}\{S\}>-2$$

Time shifting

If

$$x(t) \xleftrightarrow{LT} X(S) \quad ROC = R$$

then

$$x(t-t_0) \xleftrightarrow{LT} e^{-St_0} X(S) \quad ROC = R$$

Shifting in the s-domain

If

$$x(t) \xleftrightarrow{LT} X(S) \quad ROC = R$$

then

$$e^{S_0 t} x(t) \xleftrightarrow{LT} X(S-S_0) \quad ROC = R + \mathrm{Re}\{S_0\}$$

❖ Time scaling

If

$$x(t) \xleftrightarrow{LT} X(S) \quad ROC = R$$

then

$$x(at) \xleftrightarrow{LT} \frac{1}{|a|} X(\frac{S}{a}) \quad ROC \quad R_1 = aR$$

When

$$a = -1$$

then

$$x(-t) \xleftrightarrow{LT} X(-S) \quad ROC \quad R_1 = -R$$

❖ Conjugation

If

$$x(t) \xleftrightarrow{LT} X(S) \quad ROC = R$$

then

$$x^*(t) \xleftrightarrow{LT} X^*(S^*) \quad ROC = R$$

❖ Convolution property

$$x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(S)X_2(S)$$

ROC containing $R1 \cap R2$

Differentiation in the time domain

If

$$x(t) \xleftrightarrow{LT} X(S) \quad ROC = R$$

then

$$\frac{dx(t)}{dt} \xleftrightarrow{LT} SX(S) \quad ROC \text{ containing } R$$

Differentiation in the S-domain

If

$$x(t) \xleftrightarrow{LT} X(S) \quad ROC = R$$

then

$$-tx(t) \xleftrightarrow{LT} \frac{dX(S)}{dS} \quad ROC=R$$

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Example 9.14

$$x(t) = te^{-at}u(t)$$

determine the Laplace transform

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{S+a} \quad \mathrm{Re}\{S\} > -a$$

$$te^{-at}u(t) \xleftrightarrow{LT} -\frac{d}{dS}\left[\frac{1}{S+a}\right] = \frac{1}{(S+a)^2} \quad \mathrm{Re}\{S\} > -a$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{(S+a)^n} \quad \mathrm{Re}\{S\} > -a$$

Example 9.15

$$X(S)=\frac{2S^2+5S+5}{(S+1)^2(S+2)} \qquad \mathrm{Re}\{S\}>-1$$

determine the inverse Laplace transform

$$X(S)=\frac{2}{(S+1)^2}-\frac{1}{S+1}+\frac{3}{S+2} \qquad \mathrm{Re}\{S\}>-1$$

$$e^{-at}u(t)\xleftrightarrow{LT}\frac{1}{S+a} \qquad \mathrm{Re}\{S\}>-a$$

$$te^{-at}u(t)\xleftrightarrow{LT}\frac{1}{(S+a)^2} \qquad \mathrm{Re}\{S\}>-a$$

$$x(t)=[2te^{-t}-e^{-t}+3e^{-2t}]u(t)$$

❖ Integration in the time domain

If

$$x(t) \xleftrightarrow{LT} X(S) \quad ROC = R$$

then

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{LT} \frac{1}{S} X(S)$$

$$ROC \text{ containing } R \cap \{\mathrm{Re}\{S\} > 0\}$$

❖ The initial- and final-value theorems

If $x(t) = 0$ *for* $t < 0$ *and* $x(t)$ *contains no impulses or higher order* singularities *at the origin*

then

$$x(0^+) = \lim_{S \to \infty} SX(S)$$

$$\lim_{t \to \infty} x(t) = \lim_{S \to 0} SX(S)$$

Example 9.16

$$X(S)=\frac{2S^2+5S+12}{(S^2+2S+10)(S+2)} \qquad \mathrm{Re}\{S\}>-1$$

determine $x(0^+)$ and $\lim_{t\to\infty} x(t)$

Answer:

- $x(t)=e^{-2t}u(t)+e^{-t}(\cos 3t)u(t)$

- $x(0^+)=\lim_{S\to\infty} SX(S)=2$

  $\lim_{t\to\infty} x(t)=\lim_{S\to 0} SX(S)=0$

9.7 Analysis and characterization of LTI system using the Laplace transform

- The relationship between the properties of system and $H(S)$
- How can get the $X(S)$ or $H(S)$ or $Y(S)$

Causality

The ROC associated with the system function for a causal system is a right-half plane

For a system with a rational system function , causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole

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Example 9.17

$$h(t) = e^{-t}u(t)$$

determine $H(S)$ *and analysis the ROC*

$$H(S) = \frac{1}{S+1} \qquad \mathrm{Re}\{S\} > -1$$

The system is causal , so the ROC of H(S) is a right-half plane

Example 9.18

$$h(t) = e^{-|t|}$$

determine $H(S)$ *and analysis the ROC*

$$H(S) = \frac{-2}{S^2 - 1} \qquad -1 < \mathrm{Re}\{S\} < 1$$

The system is not causal , the ROC of H(S) is not a right-half plane

Example 9.19

$$H(S)=\frac{e^{S}}{S+1} \qquad \mathrm{Re}\{S\}>-1$$

determine $h(t)$

$$e^{-t}u(t)\xleftrightarrow{LT}\frac{1}{S+1} \quad \mathrm{Re}\{S\}>-1$$

$$e^{-(t+1)}u(t+1)\xleftrightarrow{LT}\frac{e^{S}}{S+1} \quad \mathrm{Re}\{S\}>-1$$

$$h(t)=e^{-(t+1)}u(t+1)$$

ROC is a right-half plane , but the system is not causal , unless the system is rational

Stability

An LTI system is stable if and only if the ROC of its system function H(S) includes the entire jw-axis[i.e. Re{S}=0]

Example 9.20

$$H(S)=\frac{S-1}{(S+1)(S-2)}$$

analysis the stability of system

A causal system with rational system function H(S) is stable if and only if all of the poles of H(S) lies in the left-half of the s-plane —— i.e. , all of the poles have negative real parts

Example 9.21

$$h_1(t) = e^{-t}u(t) \quad h_2(t) = e^{2t}u(t)$$

***analysis* the stability by using Laplace transform**

$$H_1(S) = \frac{1}{S+1} \qquad \mathrm{Re}\{S\} > -1$$

$$H_2(S) = \frac{1}{S-2} \qquad \mathrm{Re}\{S\} > 2$$

LTI system characterized by linear constant-coefficient differential equations

The method of determine h(t) by differential equations

- $$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\left(\sum_{k=0}^{N} a_k S^k\right) Y(S) = \left(\sum_{k=0}^{M} b_k S^k\right) X(S)$$

$$H(S) = \frac{\left(\sum_{k=0}^{M} b_k S^k\right)}{\left(\sum_{k=0}^{N} a_k S^k\right)}$$

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Example 9.23

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

determine h(t) of the system

$$SY(S) + 3Y(S) = X(S)$$

$$H(S) = \frac{Y(S)}{X(S)} = \frac{1}{S+3}$$

$$h(t) = \begin{cases} e^{-3t}u(t) & \mathrm{Re}\{S\} > -3 \\ -e^{-3t}u(-t) & \mathrm{Re}\{S\} < -3 \end{cases}$$

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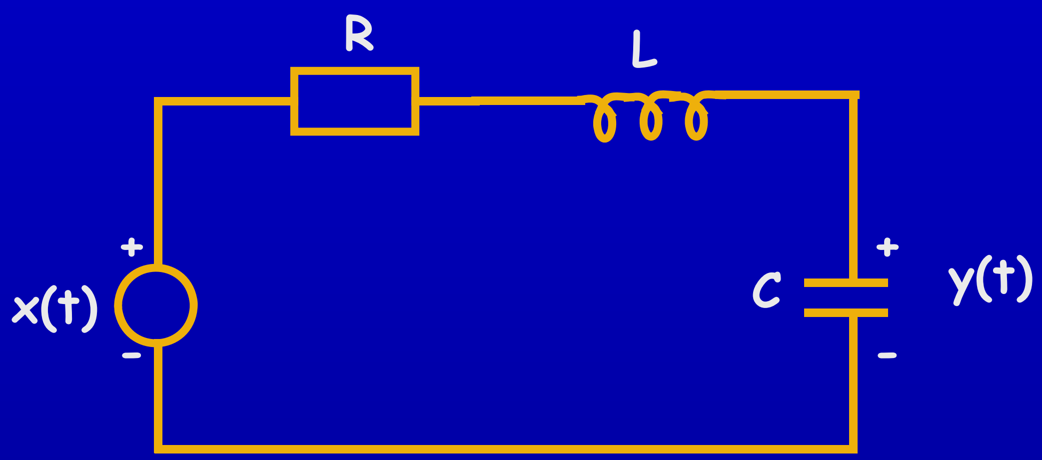
Example 9.24

Determine the H(S) of the system



Example 9.25

if the input to an LTI system is

$$x(t) = e^{-3t}u(t)$$

the output is

$$y(t) = [e^{-t} - e^{-2t}]u(t)$$

determine $H(S)$ ***and differential equation***

$$\mathrm{X(S)} = \frac{1}{\mathrm{S+3}} \quad \mathrm{Re}\{S\} > -3$$

$$Y(S) = \frac{1}{(S+1)(S+2)} \quad \mathrm{Re}\{S\} > -1$$

$$H(S) = \frac{Y(S)}{X(S)} = \frac{S+3}{(S+1)(S+2)} = \frac{S+3}{S^2+3S+2} \quad \mathrm{Re}\{S\} > -1$$

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)$$

Example 9.26

An LTI system include the information as below:

1. the system is causal

2. the system function is rational and has only two poles, at $s=-2$ and $s=4$

3. If $x(t)=1$, then $y(t)=0$

4. the value of the impulse response at $t=0^{+}$ is 4

Example 9.27

A stable and causal system , H(S) is rational , contains a pole at s=-2 , and does not have a zero at the origin , please determine the statements below:

$a)\ F\{h(t)e^{3t}\}\ converges$

$b)\ \int_{-\infty}^{+\infty} h(t)dt = 0$

$c)\ th(t)\ is\ the\ impulse\ response\ of\ a\ causal\ and\ stable\ system$

$d)\ dh(t)/dt\ contains\ at\ least\ one\ pole\ in\ its\ Laplace\ transform$

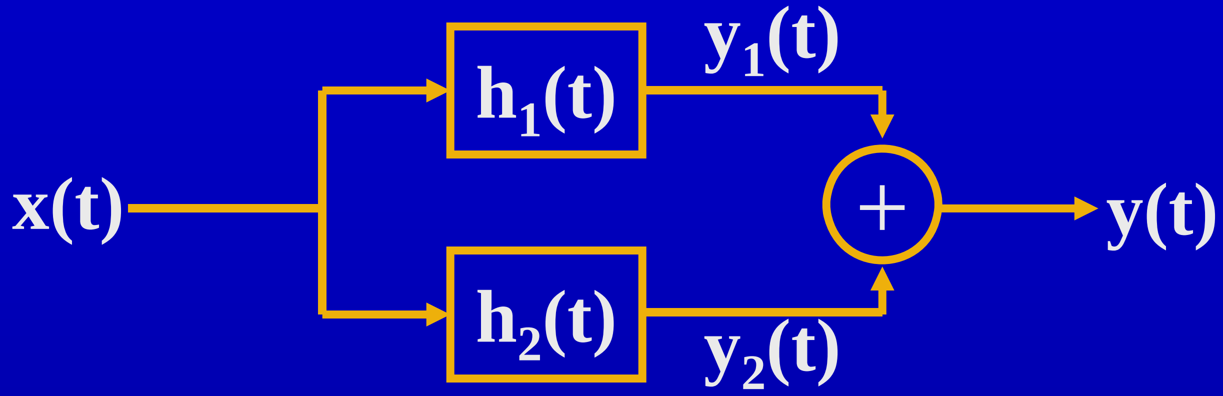
$e)\ h(t)\ has\ finite\ duration$

$f)\ H(S) = H(-S)$

$g)\ \lim_{S\to\infty} H(S) = 2$

9.8 System function algebra and block diagram representations

Parallel interconnection

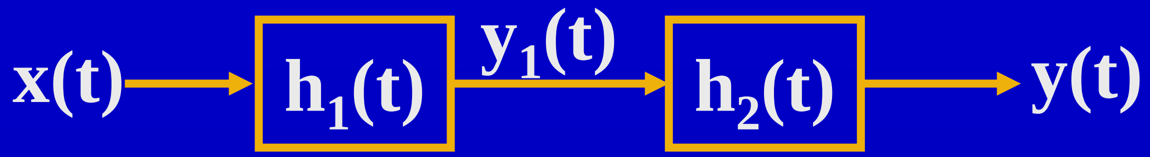


$$y(t)=y_1(t)+y_2(t)=h_1(t)*x(t)+h_2(t)*x(t)$$
$$=[h_1(t)+h_2(t)]*x(t)$$

we can get :

$$h(t)=h_1(t)+h_2(t)$$
$$H(S)=H_1(S)+H_2(S)$$

Series combination



$$y(t) = y_1(t) * h_2(t) = h_1(t) * x(t) * h_2(t)$$
$$= [h_1(t) * h_2(t)] * x(t)$$

we can get :

$$h(t) = h_1(t) * h_2(t)$$
$$H(S) = H_1(S) H_2(S)$$

Feedback interconnection

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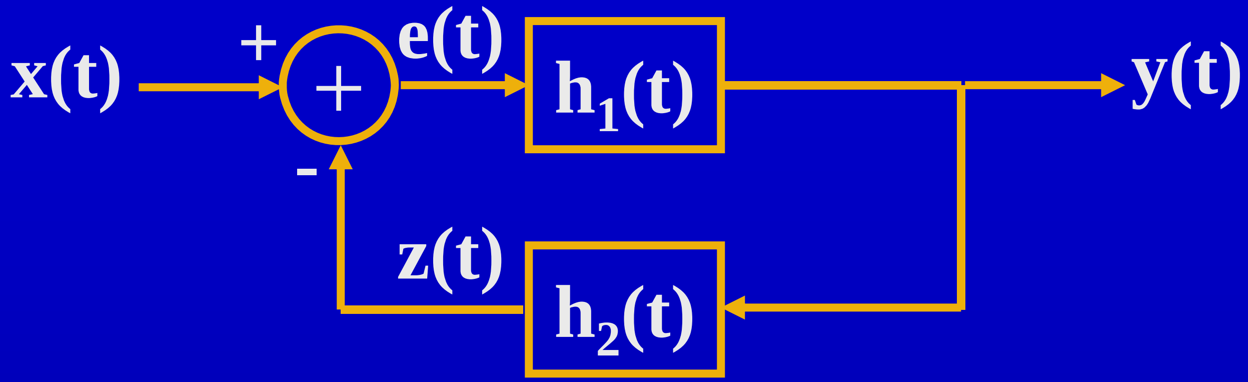
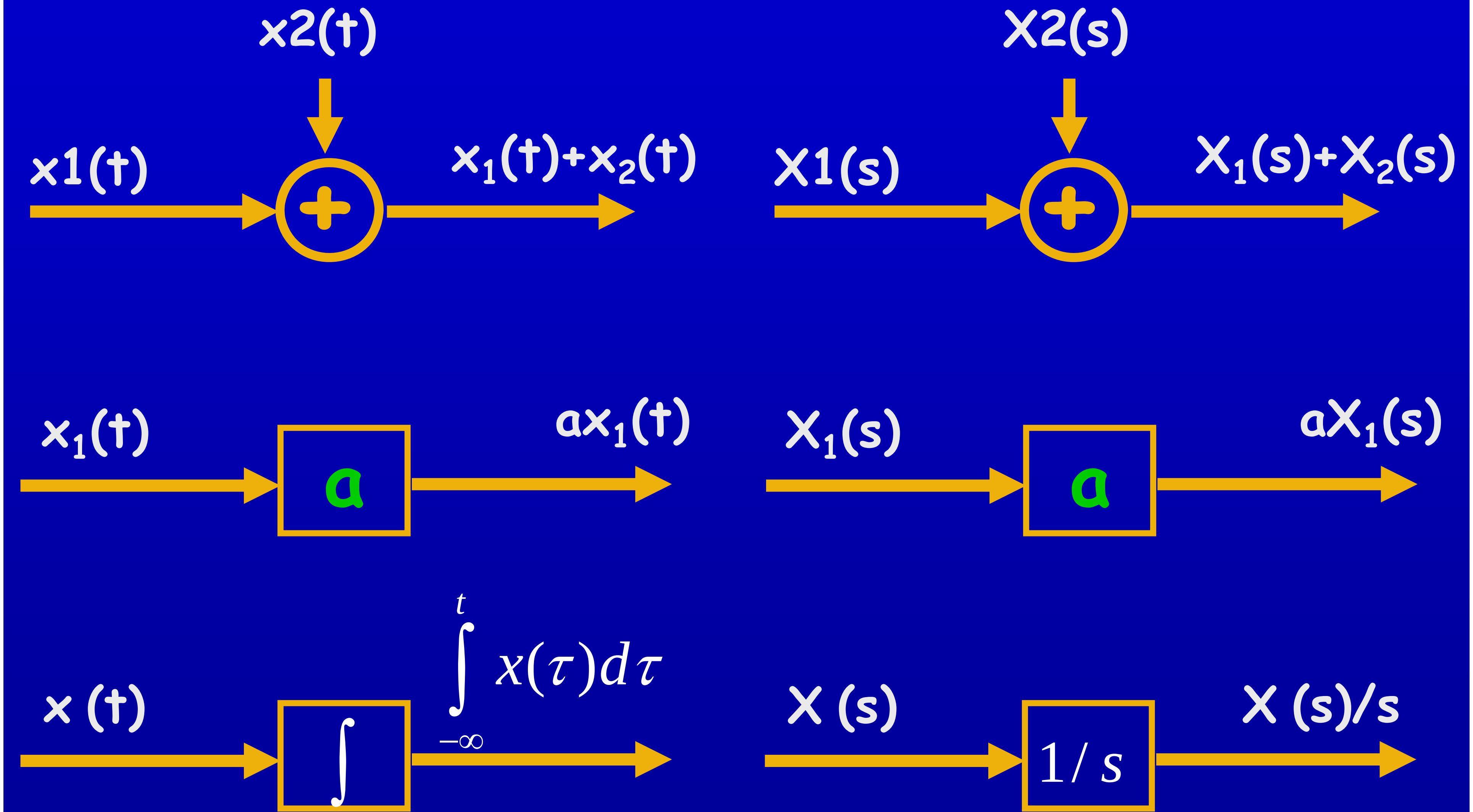


$$\begin{cases} Y(S) = H_1(S)E(S) \\ E(S) = X(S) - Z(S) \\ Z(S) = H_2(S)Y(S) \end{cases}$$

we can get :

$$Y(S) = H_1(S)\left[X(S) - H_2(S)Y(S)\right]$$

$$H(S) = \frac{Y(S)}{X(S)} = \frac{H_1(S)}{1 + H_1(S)H_2(S)}$$


x2(t)
x1(t)
$x_1(t)+x_2(t)$
X2(s)
X1(s)
$X_1(s)+X_2(s)$
$x_1(t)$
a
$ax_1(t)$
$X_1(s)$
a
$aX_1(s)$
x (t)
$\int$
$\int_{-\infty}^{t} x(\tau)d\tau$
X (s)
$1/s$
X (s)/s

Example 9.28

A causal LTI system

$$H(S)=\frac{1}{S+3}$$

Example 9.29

A causal LTI system

$$H(S)=\frac{S+2}{S+3}$$

Example 9.30

A causal LTI system

$$H(S) = \frac{1}{(S+1)(S+2)}$$

- Cascade form
- Parallel form
- Direct form

Example 9.31

A system

$$H(S)=\frac{2S^2+4S-6}{S^2+3S+2}$$

- Direct form
- Cascade form
- Parallel form

Example

Consider an LTI system with input $x(t)=\delta(t)+e^{-3t}u(t)$ and output $y(t)=-e^{-t}u(-t)$

1、 H(S)＝？ sketch the pole-zero pattern, then indicate the ROC of H(S)

2、determine the h(t), is the system causal and stable?

3、x(t)=e^{-3t}, y(t)＝？

4、draw a block diagram

5、 determine the differential equation of this system