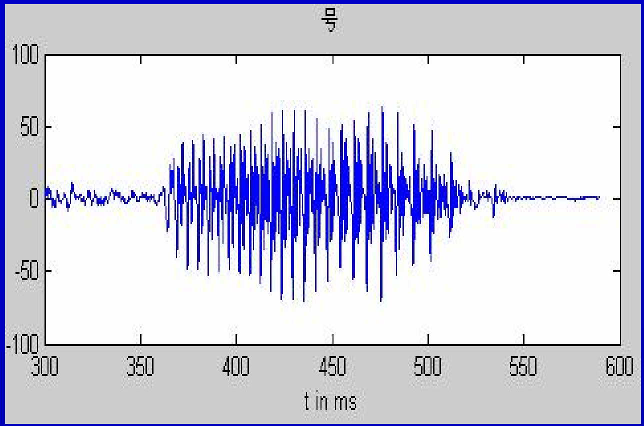
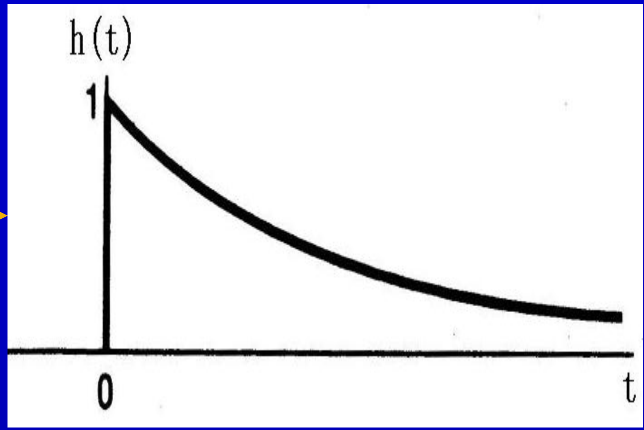
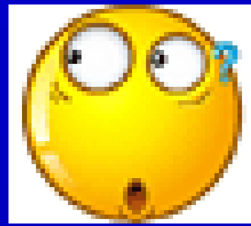
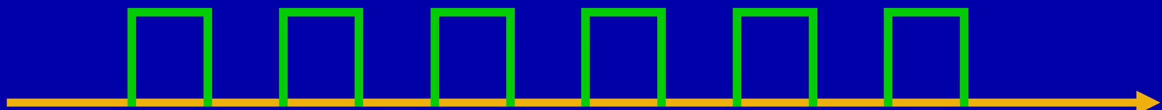
Chapter 3

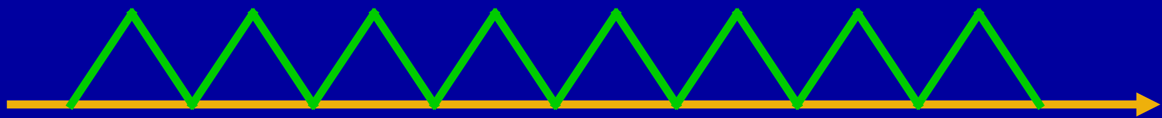
Fourier Series Representation of Periodic Signals


100
50
0
-50
-100
300
350
400
450
500
550
600
t in ms


h(t)
1
0
t







3.2 The response of LTI systems to complex exponentials

Jean Baptiste Joseph Fourier

(March 21, 1768 - May 16, 1830)

French mathematician and physicist

1807,periodic signal could be represented by sinusoidal series.

1829,Dirichlet provided precise conditions.

1960s,Cooley and Tukey discovered fast Fourier transform.

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(kw_0 t + \theta_k)$$

Signals & Systems

Eigenfunction and Eigenvalue

$$x(t) \xrightarrow{\text{LTI}} y(t) = A x(t)$$

eigenfunction　　　　eigenvalue

$$e^{st}、\quad z^{n}$$

$$e^{st} \rightarrow H(S)e^{st} \qquad z^{n} \rightarrow H(z)z^{n}$$

Example 3.1

$$If \quad y(t) = x(t-3)$$

$$(1) \quad x(t) = e^{j2t}$$

$$Determine \quad y(t)$$

Example

The convolution interal $y(t)=e^{2t}*e^{-2t}u(t)$ may be ( )

(a) $\frac{1}{4}e^{2t}u(t)$ (b) $\frac{1}{4}e^{2t}$

(c) $\frac{1}{4}e^{-2t}u(t)$ (d) $\frac{1}{4}e^{-2t}$

Example

$$x(t)=a_1e^{s_1t}+a_2e^{s_2t}+a_3e^{s_3t}$$

$$a_1e^{s_1t}\rightarrow a_1H(s_1)e^{s_1t}=y_1(t)$$

$$a_2e^{s_2t}\rightarrow a_2H(s_2)e^{s_2t}=y_2(t)$$

$$a_3e^{s_3t}\rightarrow a_3H(s_3)e^{s_3t}=y_3(t)$$

$$y(t)=y_1(t)+y_2(t)+y_3(t)$$

$$=a_1H(s_1)e^{s_1t}+a_2H(s_2)e^{s_2t}+a_3H(s_3)e^{s_3t}$$

Example 3.1

$$If \quad y(t) = x(t-3)$$

$$x(t) = \cos(4t)$$

$$Determine \quad y(t)$$

Conclusion

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{s_k t} \rightarrow y(t)=\sum_{k=-\infty}^{+\infty} a_k H(s_k) e^{s_k t}$$

$$x[n]=\sum_{k=-\infty}^{+\infty} a_k z_k{}^n \rightarrow y[n]=\sum_{k=-\infty}^{+\infty} a_k H(z_k) z_k{}^n$$

let $s = jw$ *and* $z = e^{jw}$

so $x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jw_k t}$ $\qquad x[n]=\sum_{k=-\infty}^{+\infty} a_k e^{jw_k n}$

3.3 Fourier series representation of continuous-time periodic signals

- A set of basic signals——Harmonically related complex exponentials

- Fourier series representation of a continuous-time periodic signals

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continuous-time periodic signals

$x(t)=x(t+nT)$

$x_0(t)=x_0(t+T)$ $\omega_0=2\pi/T$

$x_1(t)=x_1(t+T/2)=x_1(t+2\cdot T/2)$ $\omega_1=2\omega_0$

$x_2(t)=x_2(t+T/3)=x_2(t+3\cdot T/3)$ $\omega_1=3\omega_0$

$x(t)=x_0(t)+x_1(t)\ )+x_2(t)$

$=x_0(t+T)+x_1(t+2\cdot T/2)+x_2(t+3\cdot T/3)$

$=x_0(t+T)+x_1(t+T)+x_2(t+T)$

$=x(t+T)$

$\ldots\ e^{-j3w_0t}\ e^{j-2w_0t}\ e^{j-w_0t}\ e^{j0w_0t}\ e^{jw_0t}\ e^{j2w_0t}\ e^{j3w_0t}\ \ldots$

Signals & Systems

3.3.1 Linear combinations of harmonically related complex exponentials

➢ Harmonically related complex exponentials

$$\phi_k(t) = e^{jkw_0t} = e^{jk(2\pi/T)t} \qquad k = 0, \pm 1, \pm 2, \ldots$$

frequency: w_0
period: T

The signals are a set of basic signal

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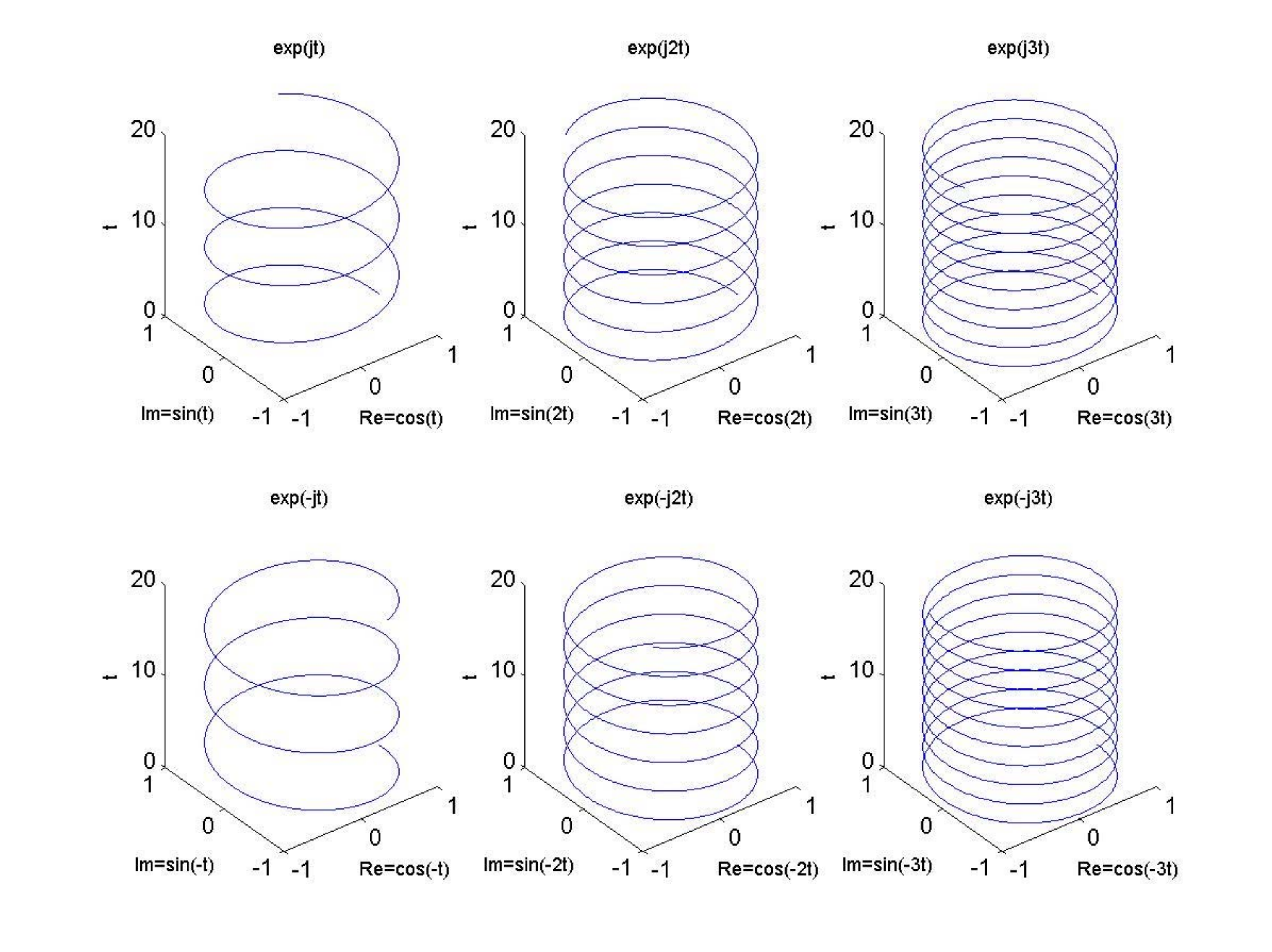

A linear combination of harmonically related complex exponentials

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$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}=\sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$k=0$ *the dc or constant components*

$k=\pm 1$ *fundamental components or*

first harmonic components

$k=\pm 2$ *the second harmonic components*

$\vdots$

$k=\pm n$ *the Nth harmonic components*

$$a_k=A_k e^{j\theta_k}$$

Example 3.2

$x(t)$ *is a periodic signal, fundamental frequency is* 2π

$$x(t)=\sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

$$a_0 = 1 \qquad a_1 = a_{-1} = \frac{1}{4}$$

$$a_2 = a_{-2} = \frac{1}{2} \qquad a_3 = a_{-3} = \frac{1}{3}$$

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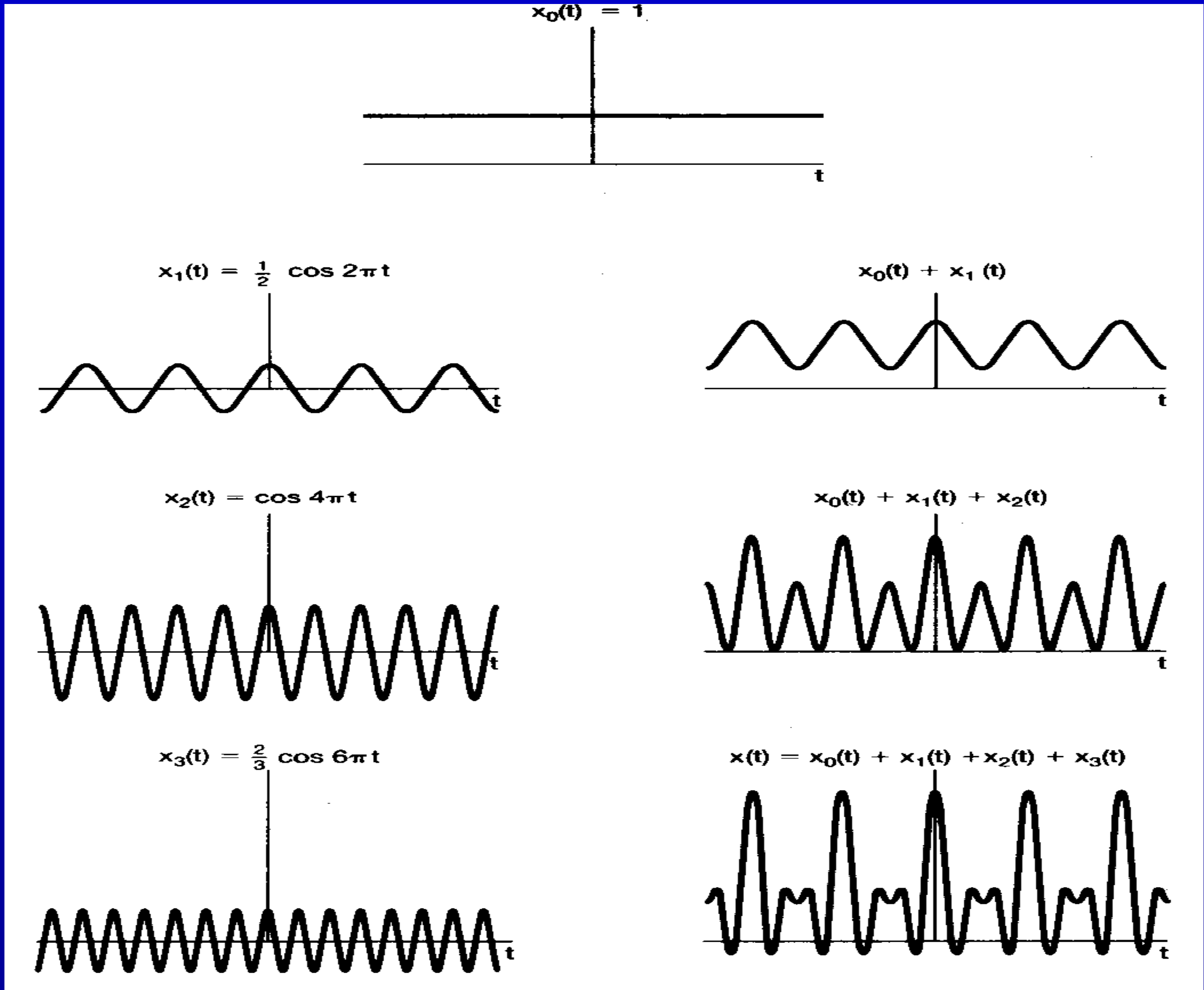
Example 3.2

$x(t)=1+1/2\cos 2\pi t+\cos 4\pi t+2/3\cos 6\pi t$



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An useful property

If x(t) is real, then

$$a_k = A_k e^{j\theta_k}$$

$$x(t) = a_0 + \sum_{k=1}^{+\infty} 2Re\left[A_k e^{j(kw_0t+\theta_k)}\right]$$

$$= a_0 + 2\sum_{k=1}^{+\infty} A_k \cos(kw_0t + \theta_k)$$

$$a_k = B_k + jC_k$$

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty} [B_k \cos kw_0t - C_k \sin kw_0t]$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}=\sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

$$a_k=\frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt=\frac{1}{T}\int_T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}dt$$

a_k Fourier series coefficients or the spectral coefficients

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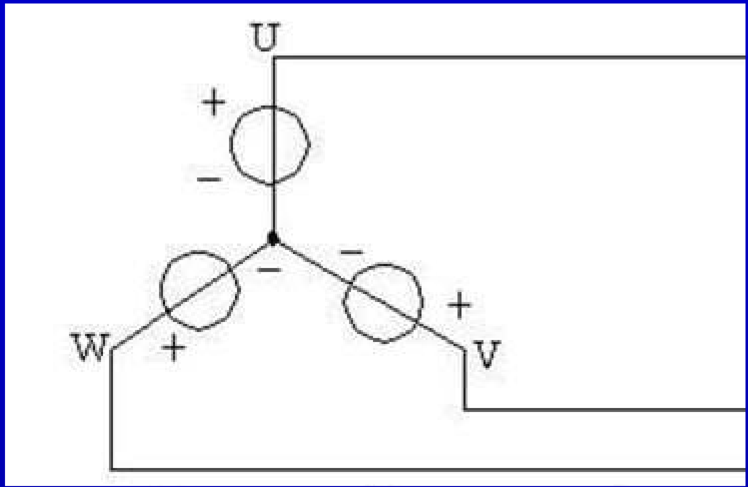
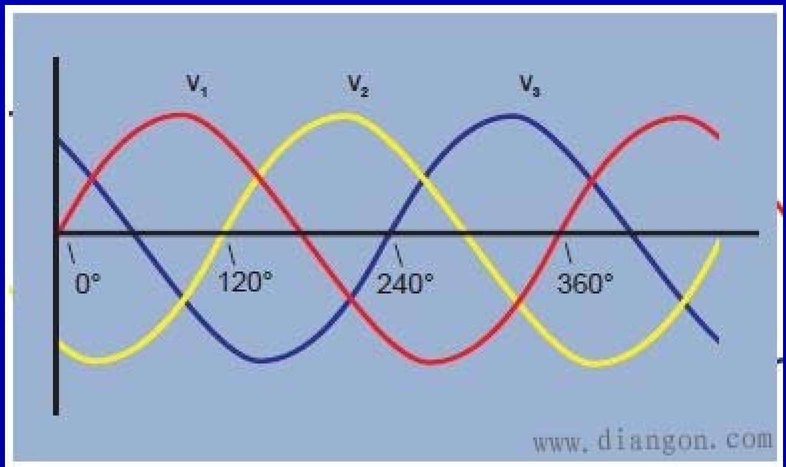
Example





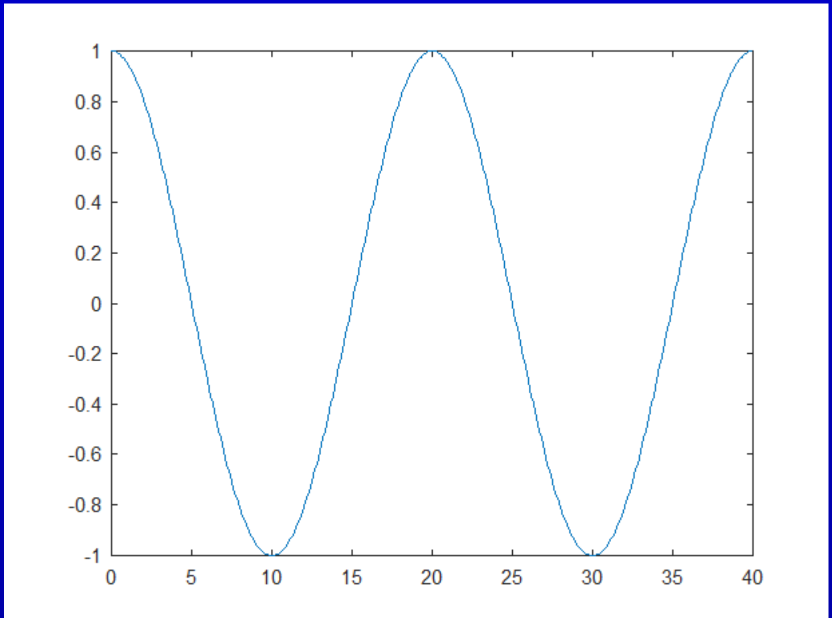
$$A\cos(\omega t)$$

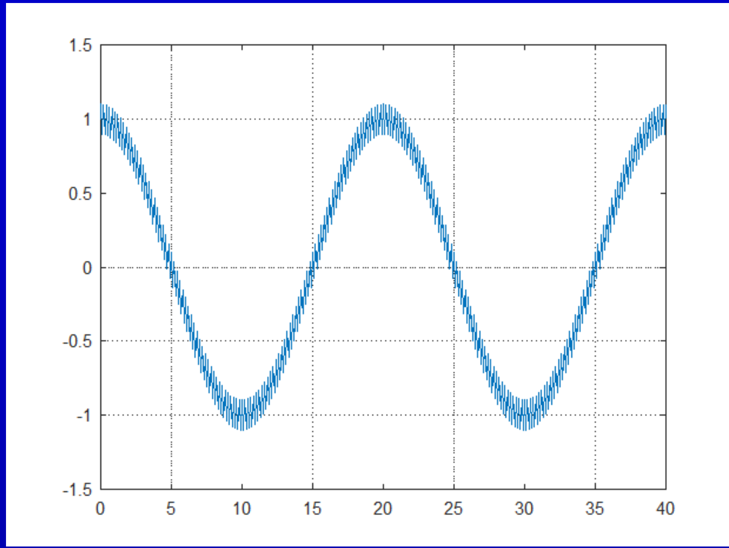
$$A\cos(\omega t-\frac{2\pi}{3})$$

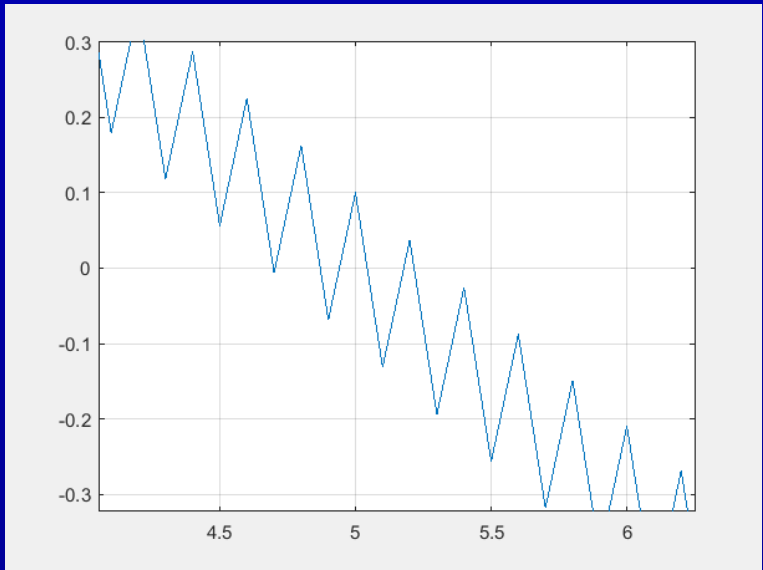
$$A\cos(\omega t-\frac{4\pi}{3})$$

$$A\cos(\omega t)-A\cos(\omega t-\frac{2\pi}{3})$$

Signals & Systems







Example 3.3

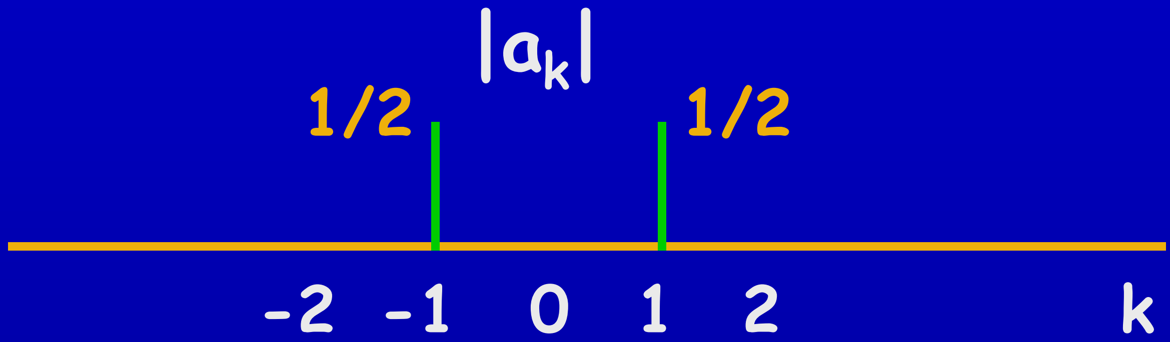
If $x(t)=\sin w_0 t$, determine the Fourier series coefficients

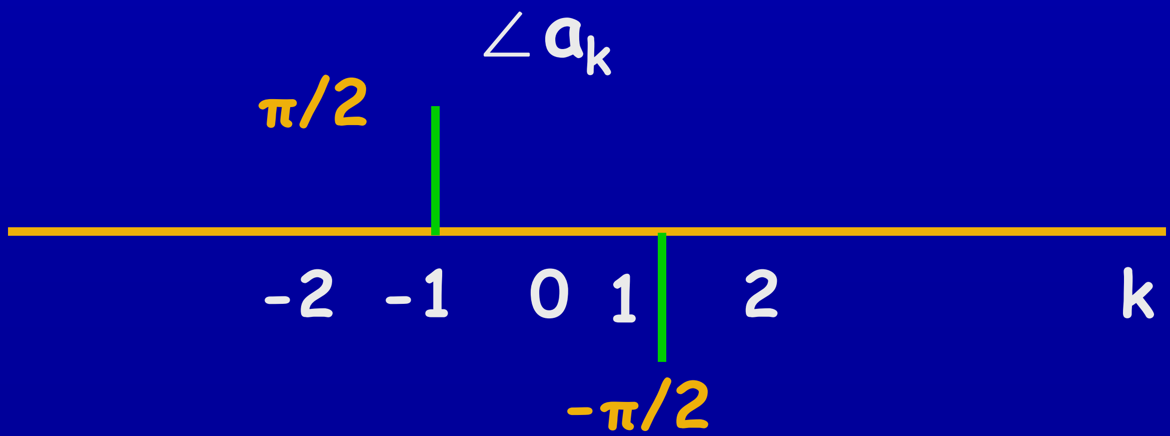
- $$a_k=\frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt=\frac{1}{T}\int_T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}dt$$
- $$x(t)=\sin w_0 t=\frac{1}{2j}(e^{jw_0 t}-e^{-jw_0 t})$$

$$a_1=\frac{1}{2j} \qquad a_{-1}=-\frac{1}{2j}$$

$$a_k=0 \qquad k\neq\pm 1$$

$$x(t) = \sin w_0 t = \frac{1}{2j}(e^{jw_0 t} - e^{-jw_0 t})$$

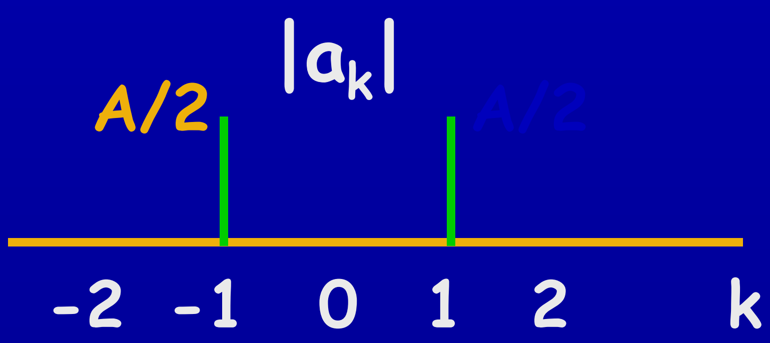


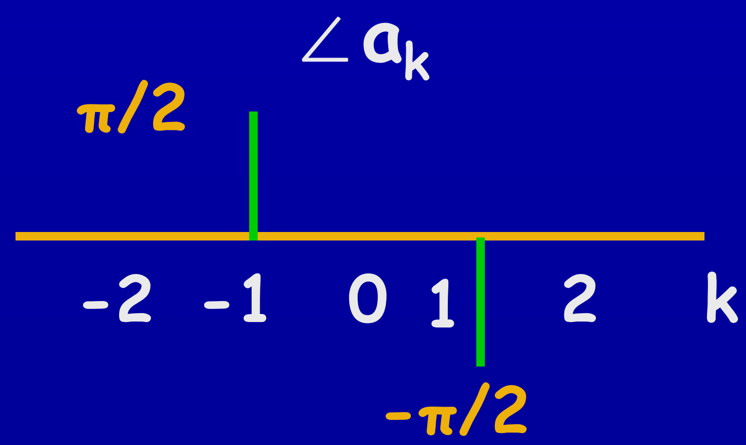


$$x(t) = A\sin w_1 t$$

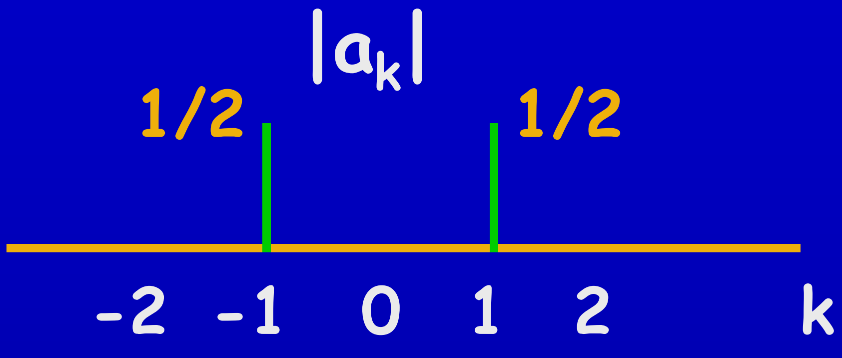
$$x(t) = A\sin w_1 t = \frac{A}{2j}(e^{jw_1 t} - e^{-jw_1 t})$$

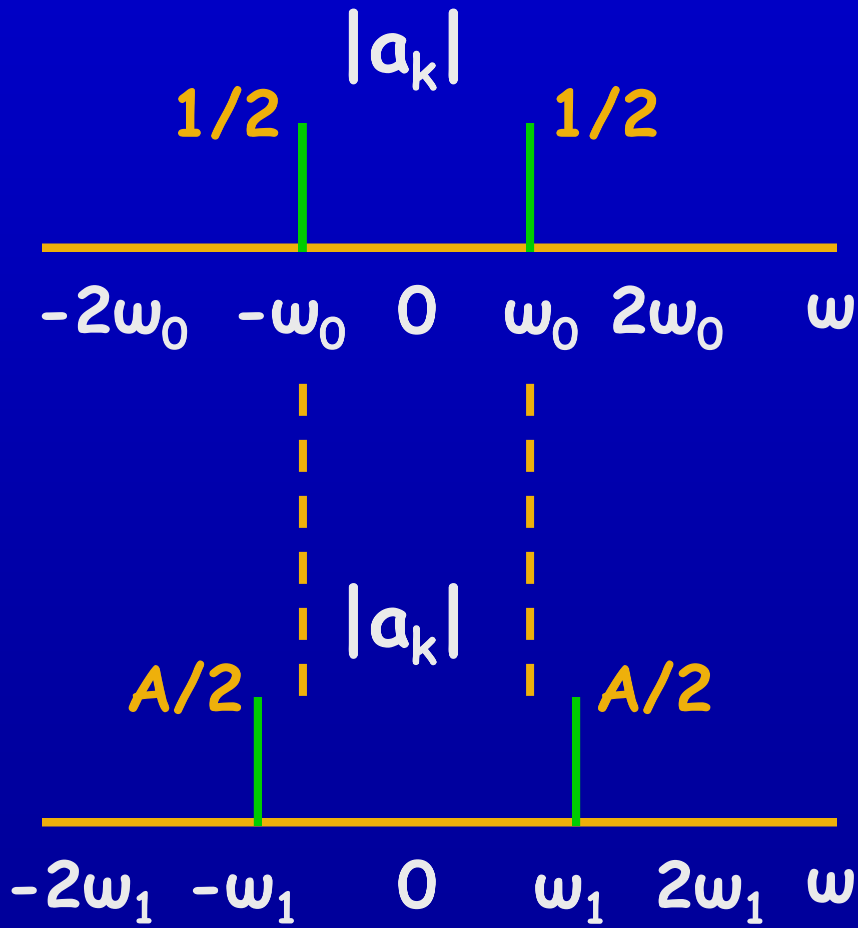
$$a_1 = \frac{A}{2j} \qquad a_{-1} = -\frac{A}{2j} \qquad a_k = 0 \qquad k \neq \pm 1$$



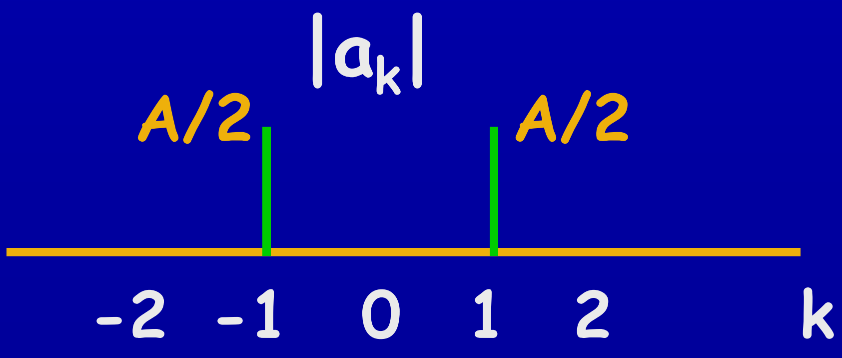


$$x(t) = \sin w_0 t$$





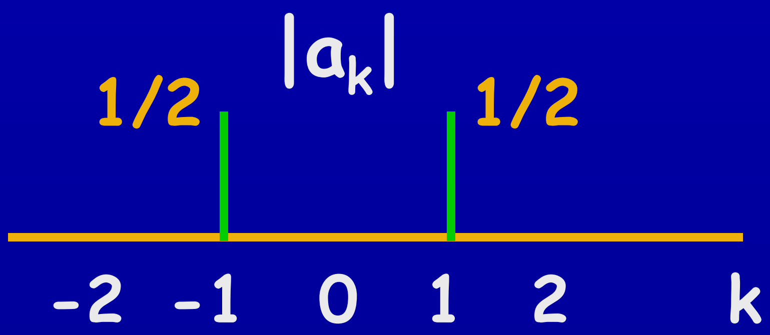
$$x(t) = A\sin w_1 t$$

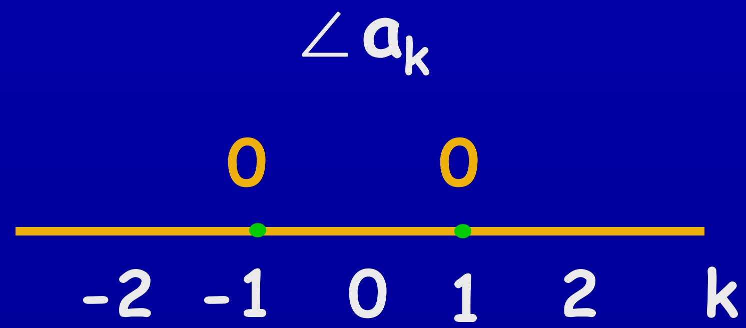


$$x(t) = \cos\omega_0 t = \sin(\omega_0 t + \frac{\pi}{2})$$

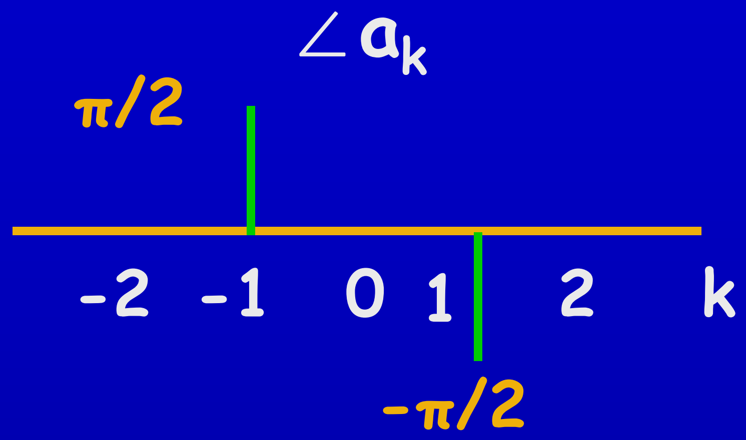
$$x(t) = \cos w_0 t = \frac{1}{2}(e^{jw_0 t} + e^{-jw_0 t})$$

$$a_1 = \frac{1}{2} \qquad a_{-1} = \frac{1}{2} \qquad a_k = 0 \qquad k \neq \pm 1$$

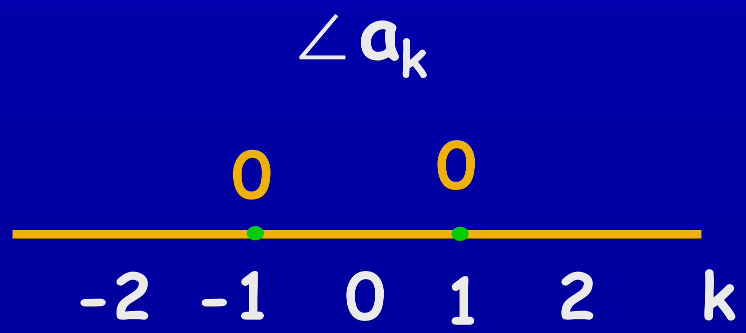




$$x(t) = \sin w_0 t$$



$$x(t) = \cos\omega_0 t = \sin(\omega_0 t + \frac{\pi}{2})$$



Example 3.4

If $x(t) = 1 + \sin w_0 t + 2\cos w_0 t + \cos(2w_0 t + \frac{\pi}{4})$
determine the Fourier series coefficients

$$a_0 = 1 \quad a_1 = 1 - \frac{1}{2}j \quad a_{-1} = 1 + \frac{1}{2}j$$

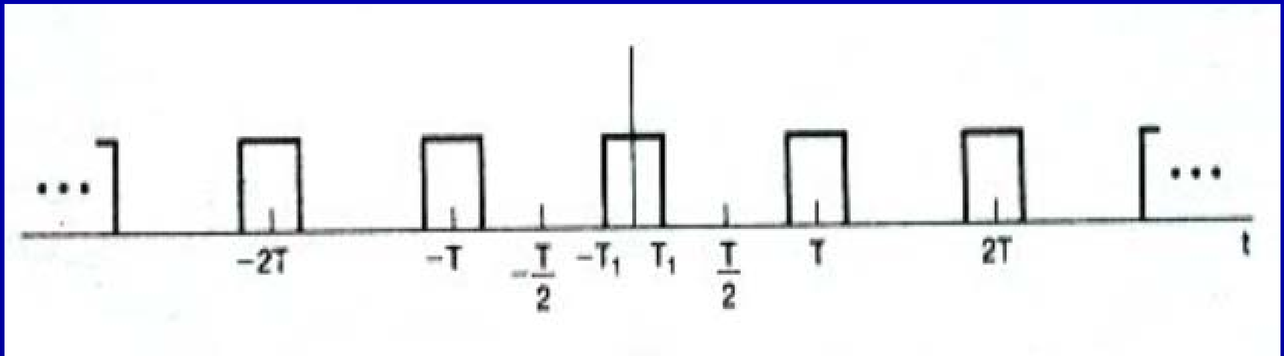
$$a_2 = \frac{\sqrt{2}}{4}(1 + j) \quad a_{-2} = \frac{\sqrt{2}}{4}(1 - j)$$

$$a_k = 0 \quad |k| > 2$$

Example 3.5

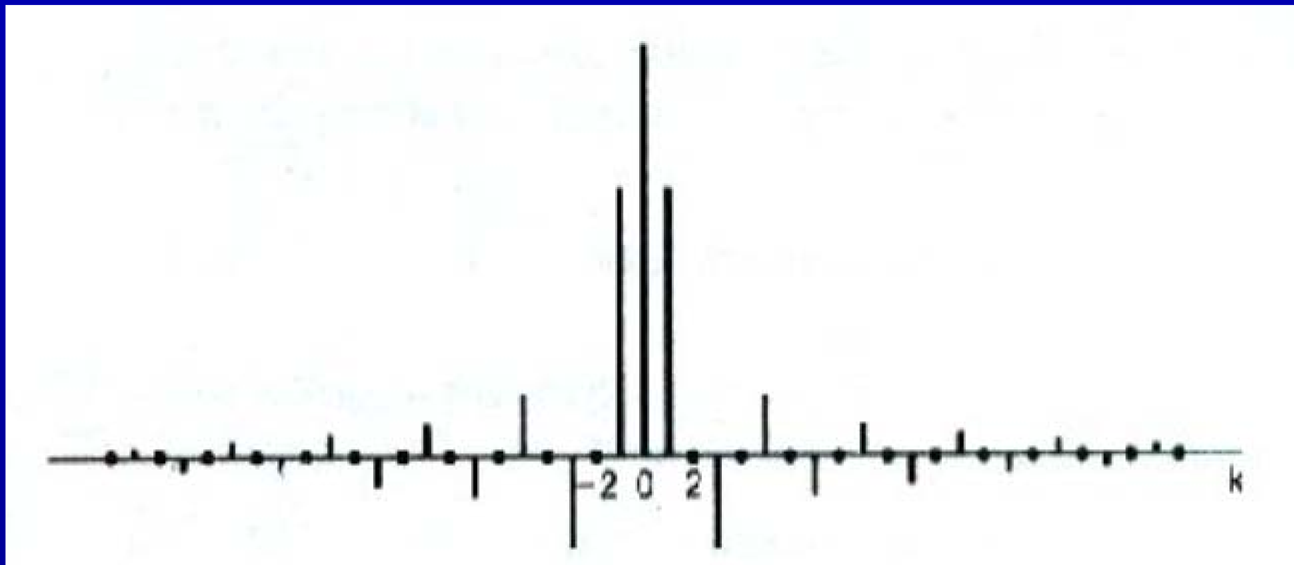
$$x(t)=\begin{cases}1 & |t|<T_1\\ 0 & T_1<|t|<\dfrac{T}{2}\end{cases}$$

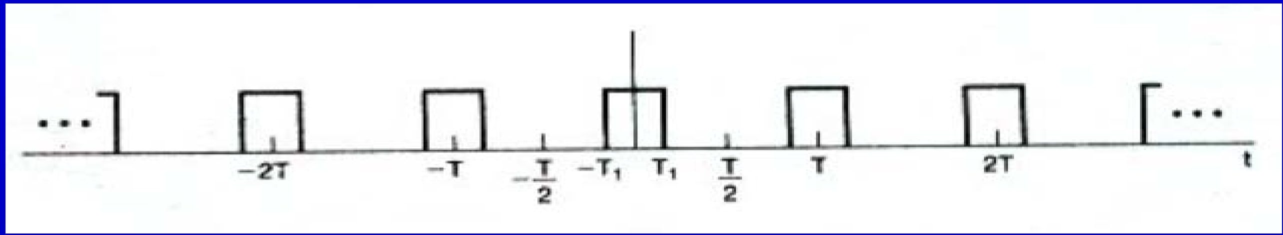
determine the Fourier series coefficients

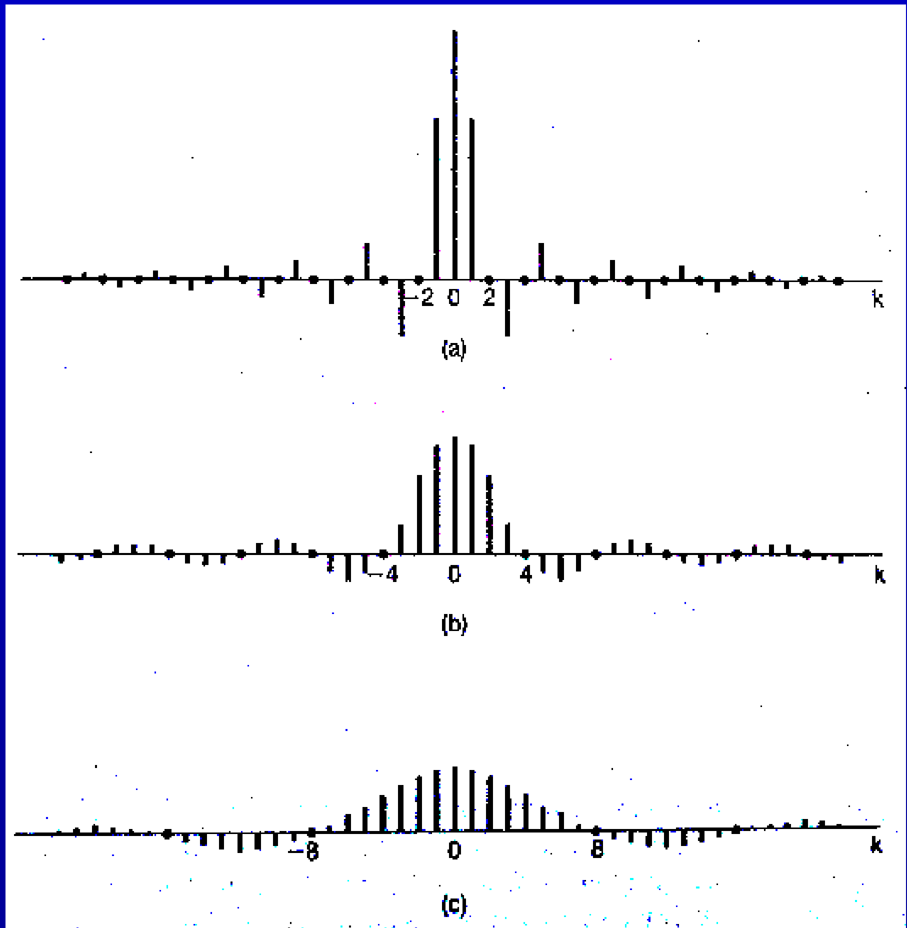


$$k = 0 \quad a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$k \neq 0 \quad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0 t} dt = \frac{2\sin\left(kw_0 T_1\right)}{kw_0 T} = \frac{\sin\left(kw_0 T_1\right)}{k\pi}$$







(a)$T=4T_1$

(b)$T=8T_1$

(c)$T=16T_1$

Example

let x(t) be a periodic signal with fundamental frequency w_0, and Fourier series coefficients a_k,

the Fourier series coefficients of $\dfrac{dx(t-1)}{dt}$ is (   )

(a) $a_k e^{-jkw_0}$

(b) $jkw_0 a_k e^{jkw_0}$

(c) $-jkw_0 a_k e^{-jkw_0}$

(d) $jkw_0 a_k e^{-jkw_0}$ ✓

3.4 Convergence of The Fourier Series

- Fourier's view
- Euler and Lagrange's view
- Our view

Two different classes of conditions

- The signal which has finite energy over a single period

$$\int_T \left| x(t) \right|^2 dt < \infty$$

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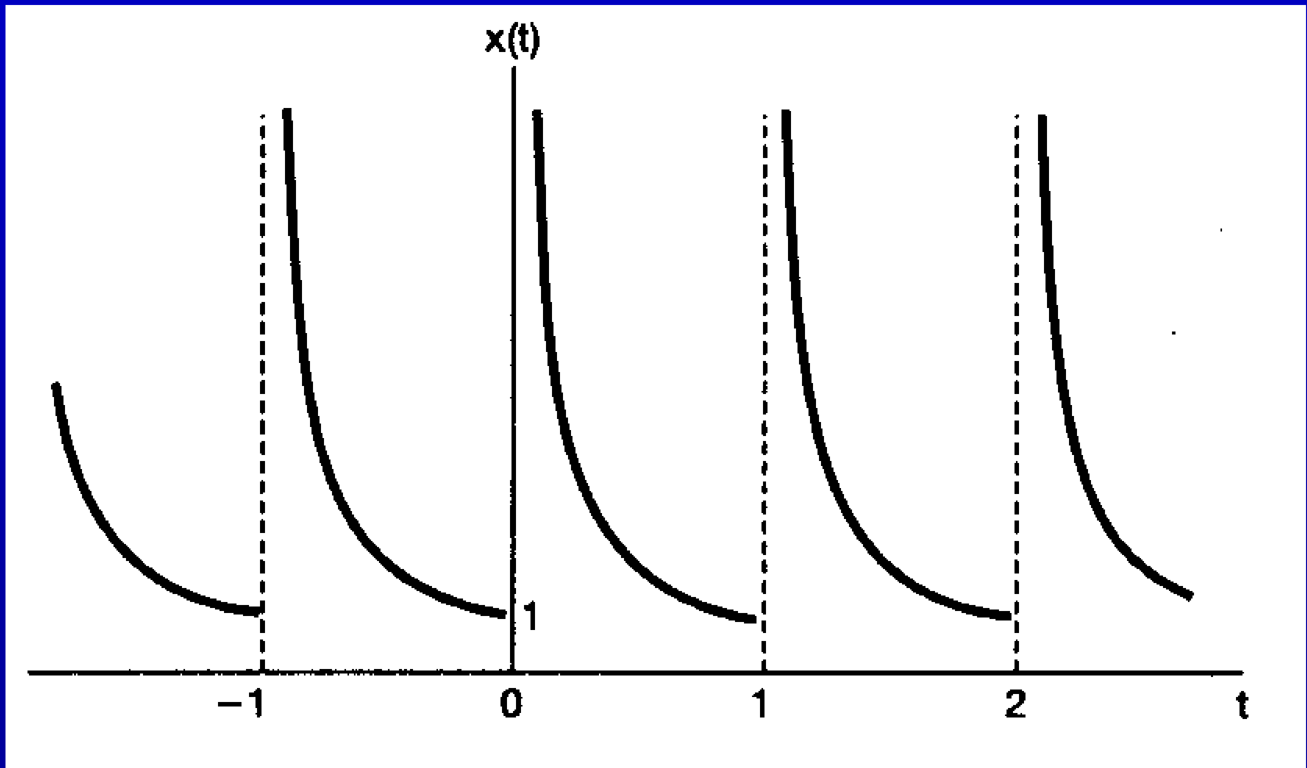
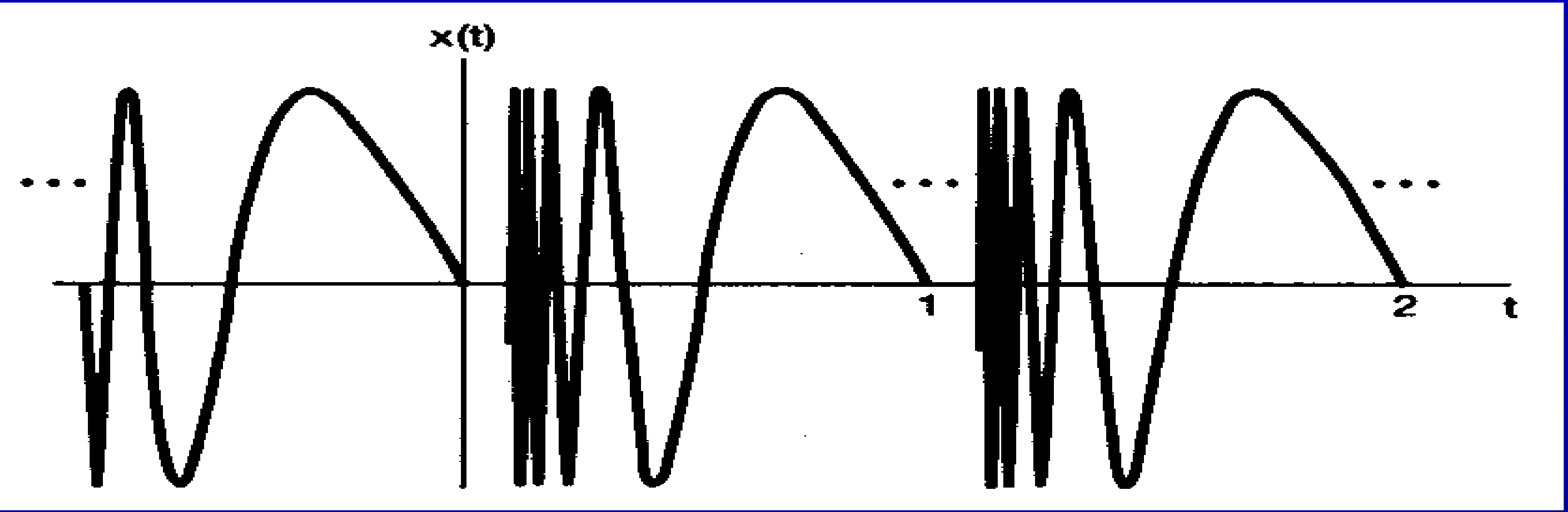
Dirichlet conditions

1、 $\int_T |x(t)| dt < \infty$



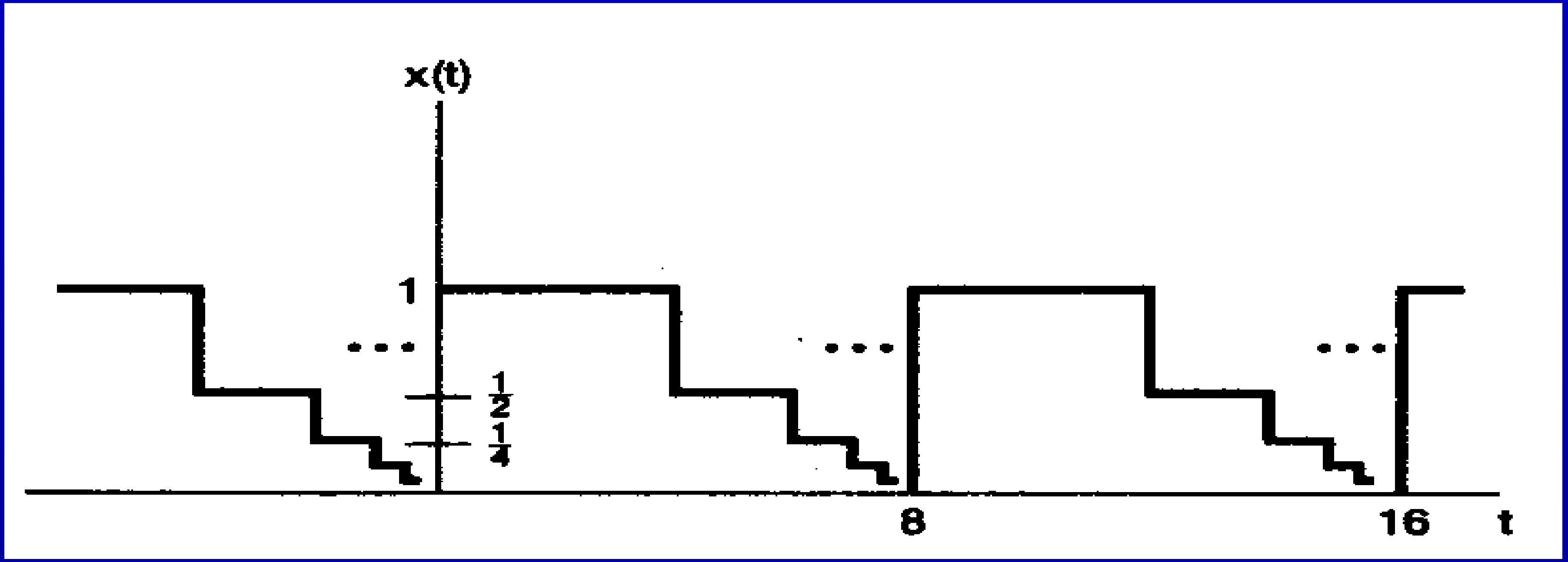
Dirichlet conditions

- 2、x(t) are no more than a finite number of maxima and minima during any single period of the signal



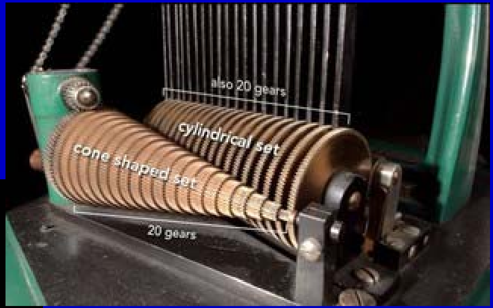
Dirichlet conditions

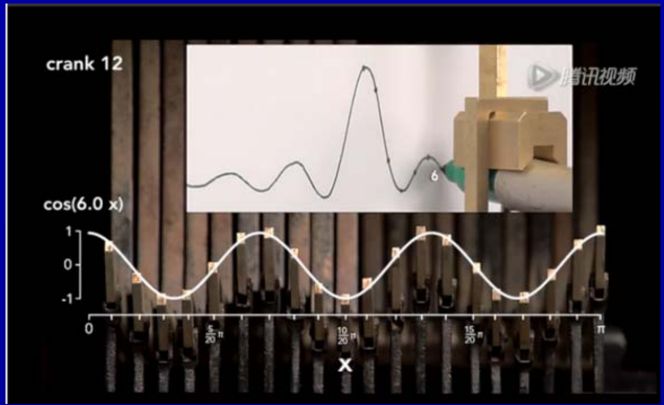
3、In any finite interval of time, x(t) are only a finite number of discontinuities and each of these discontinuities is finite.



Gibbs phenomenon

Albert Michelson





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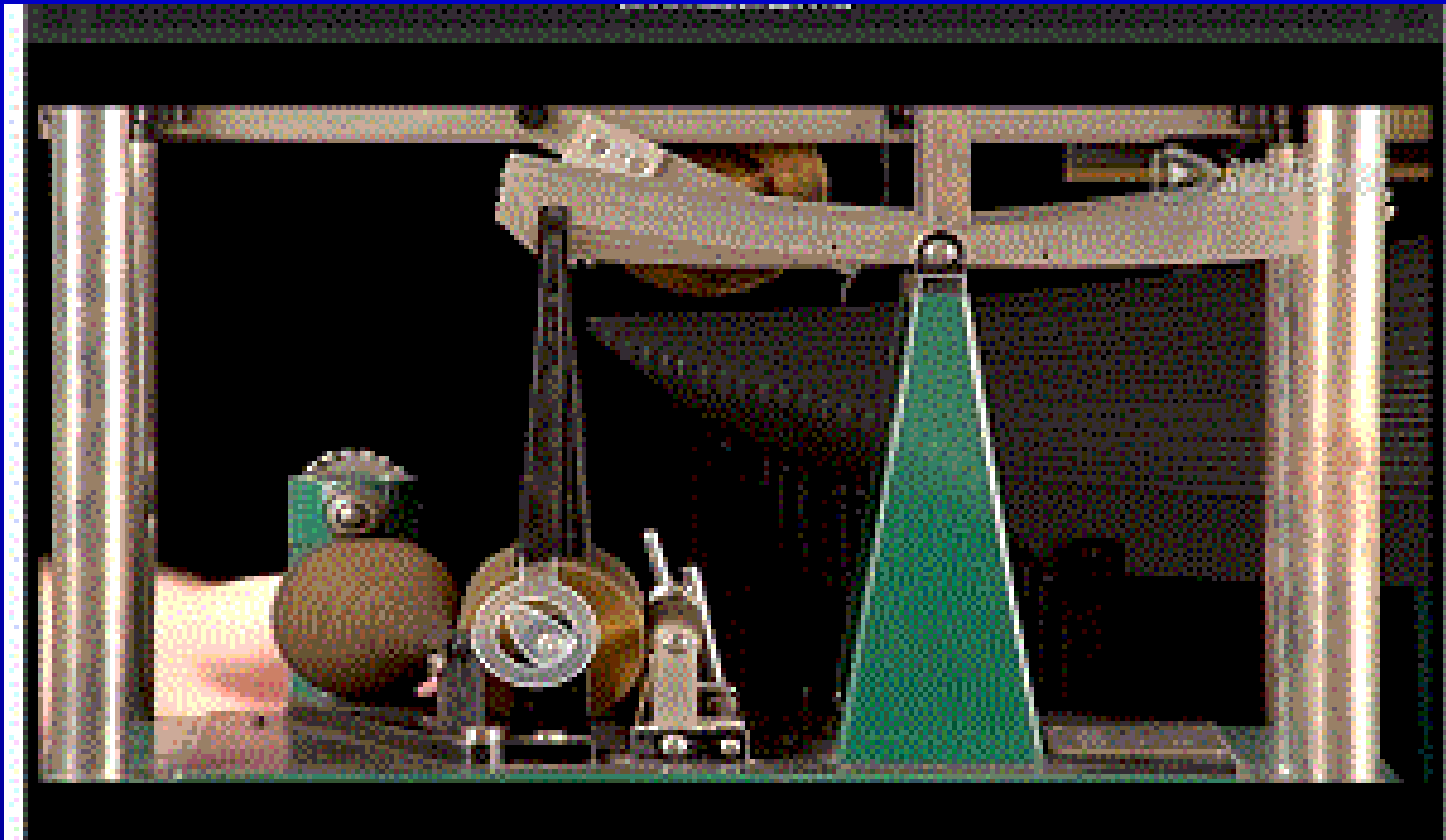

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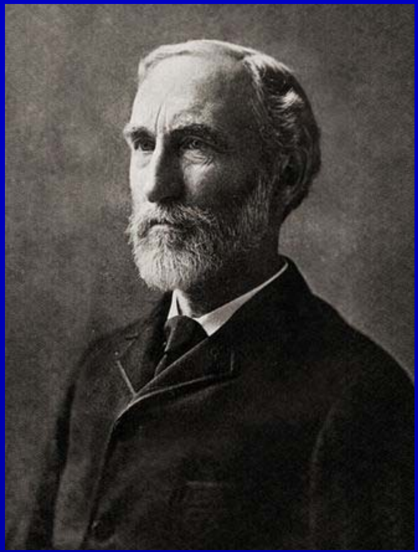
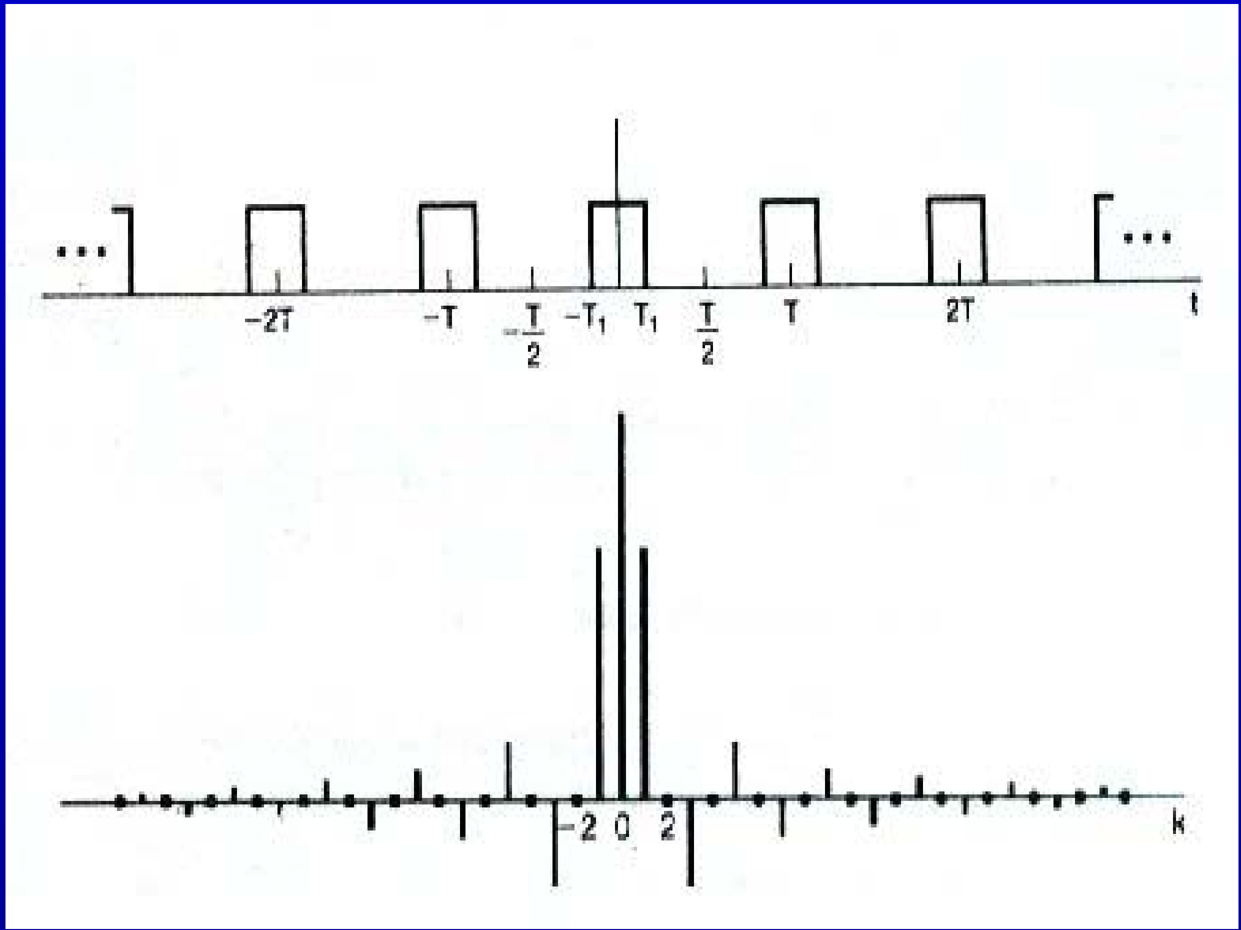
Gibbs phenomenon



Josiah Willard Gibbs



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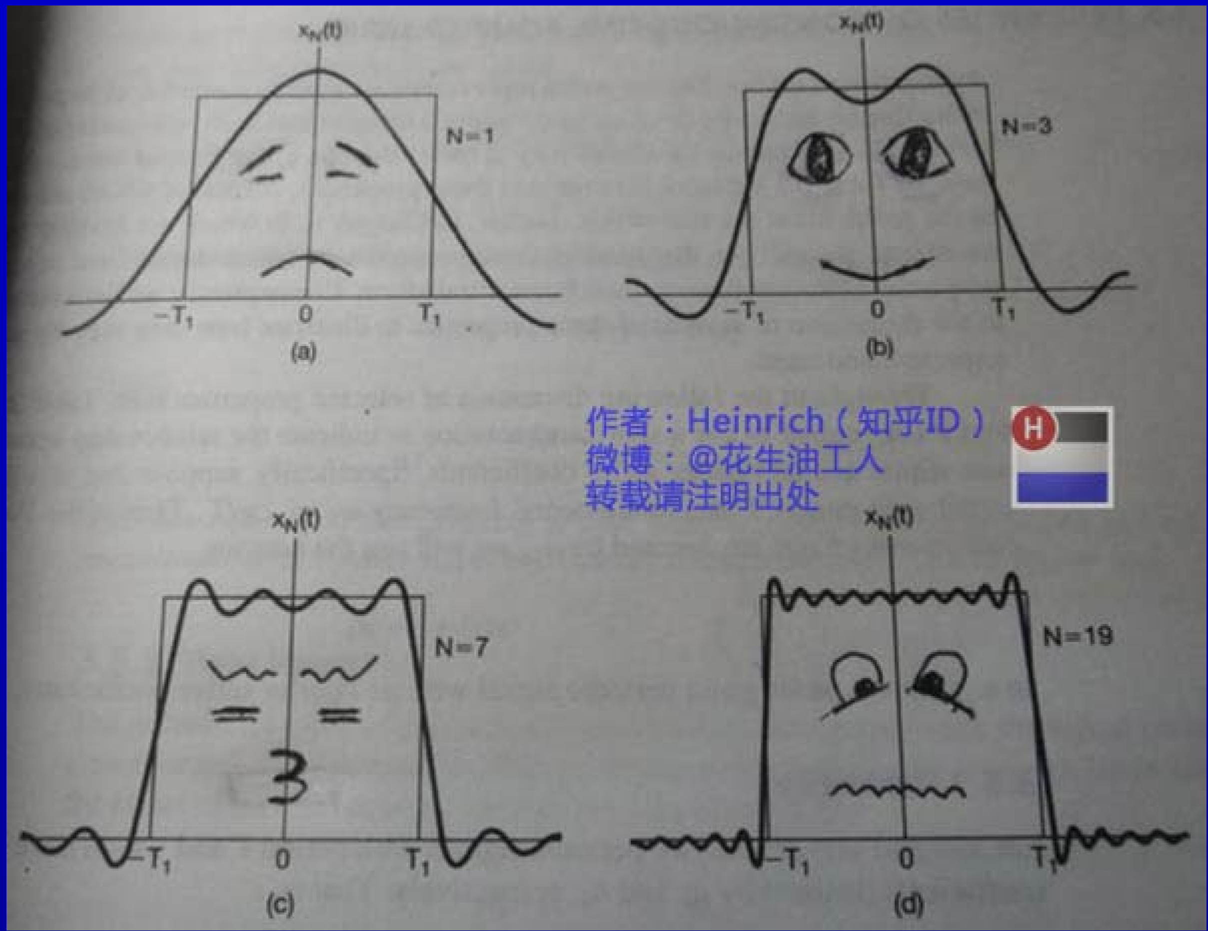

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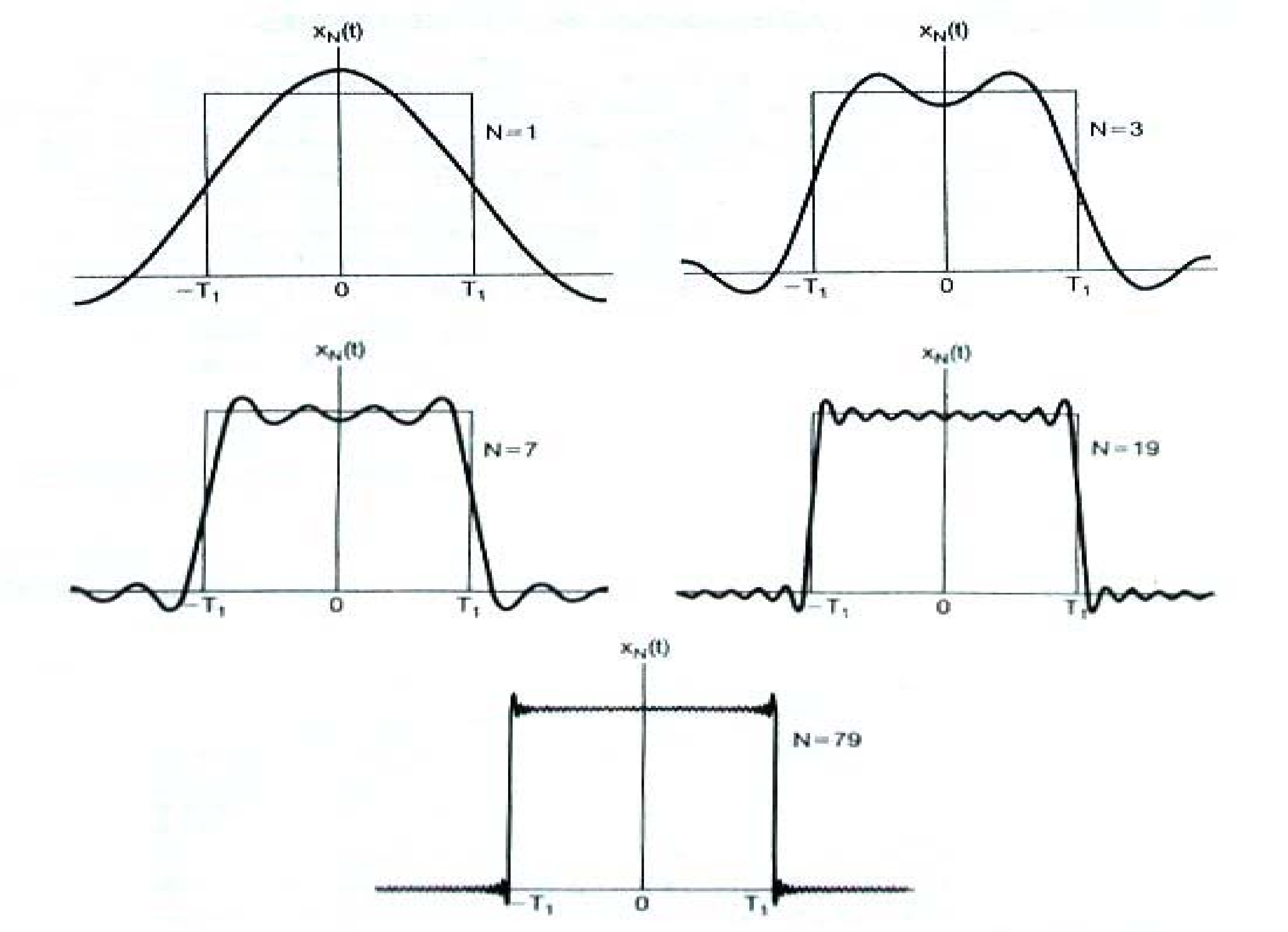
Gibbs phenomenon



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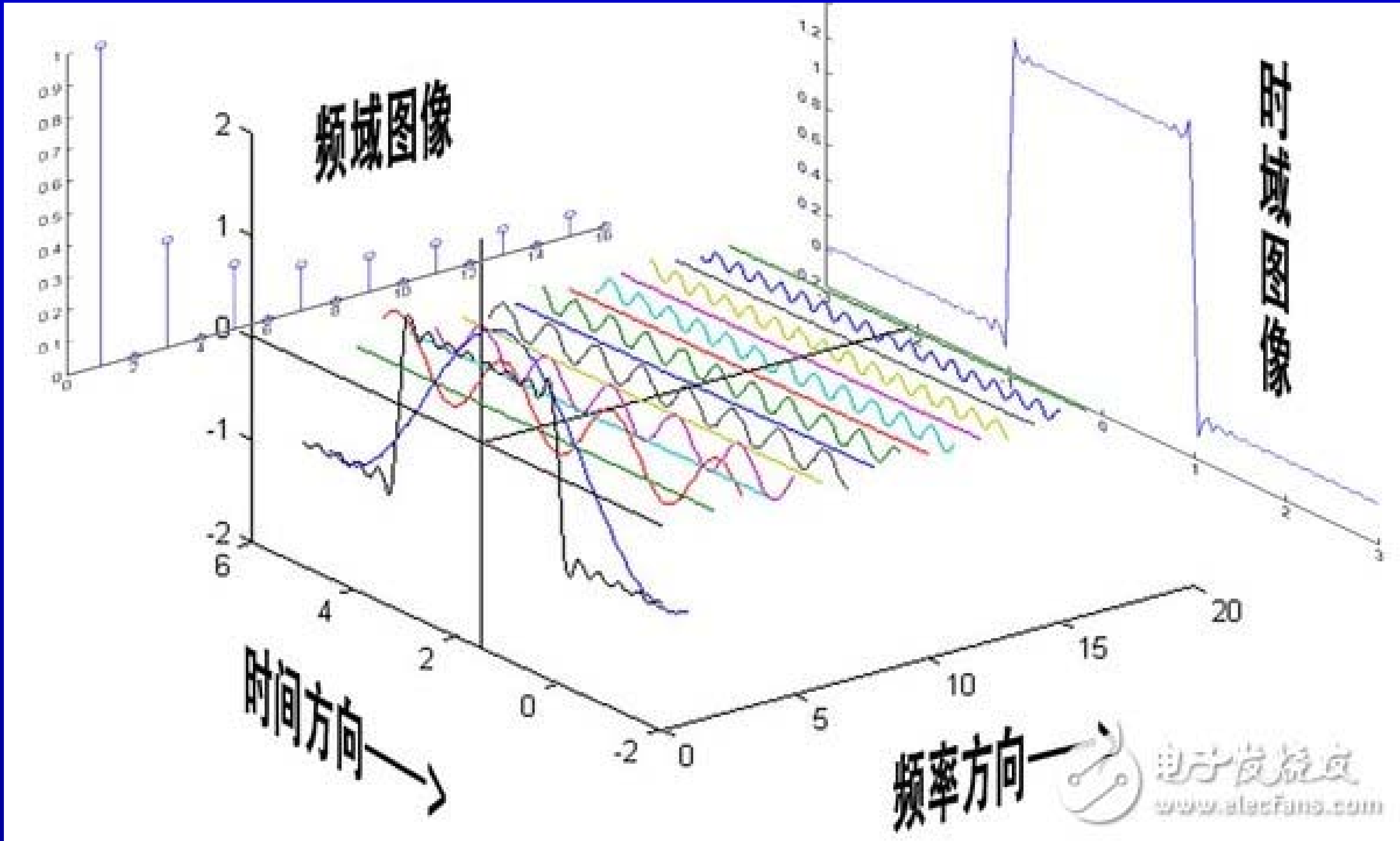

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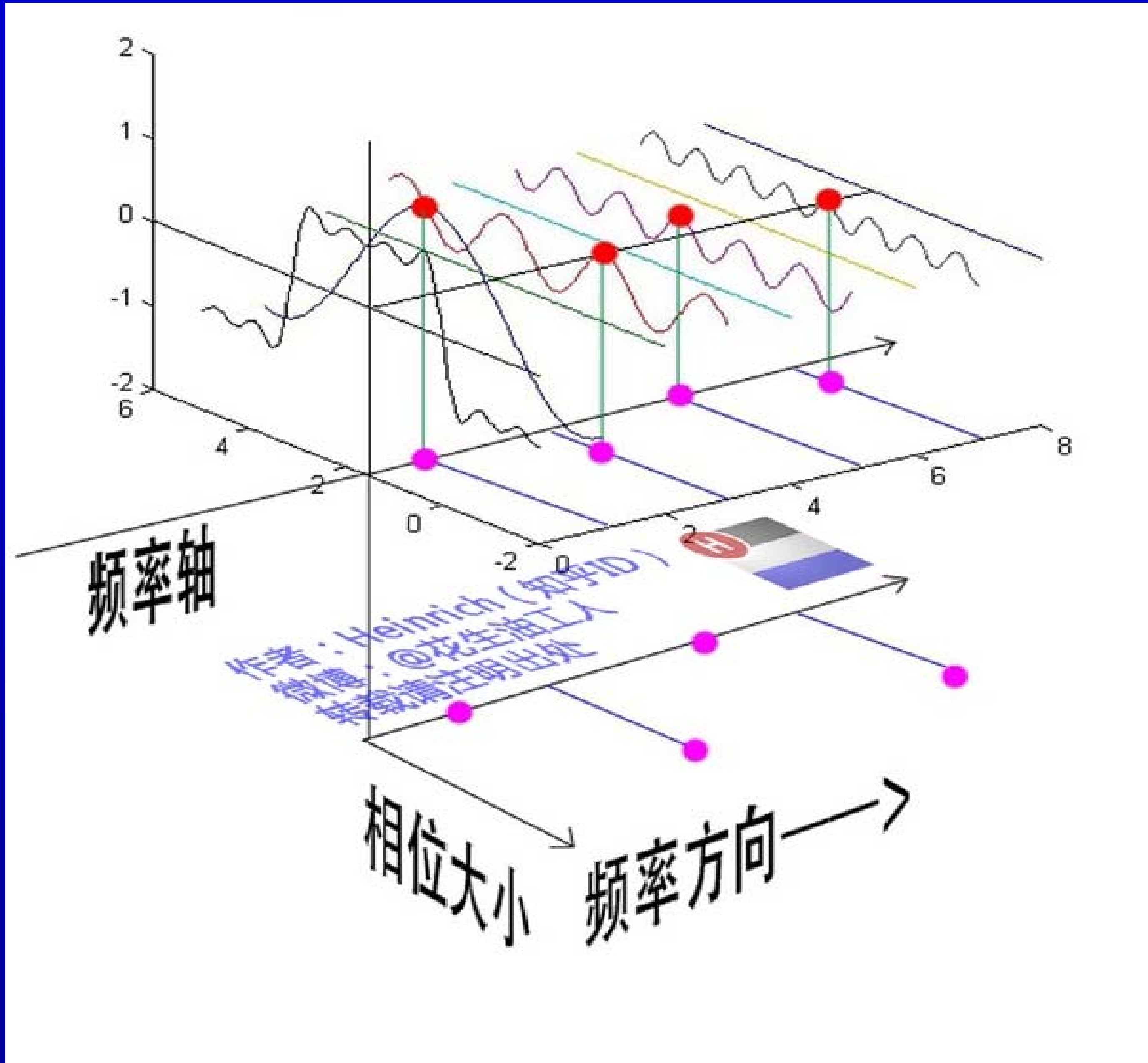

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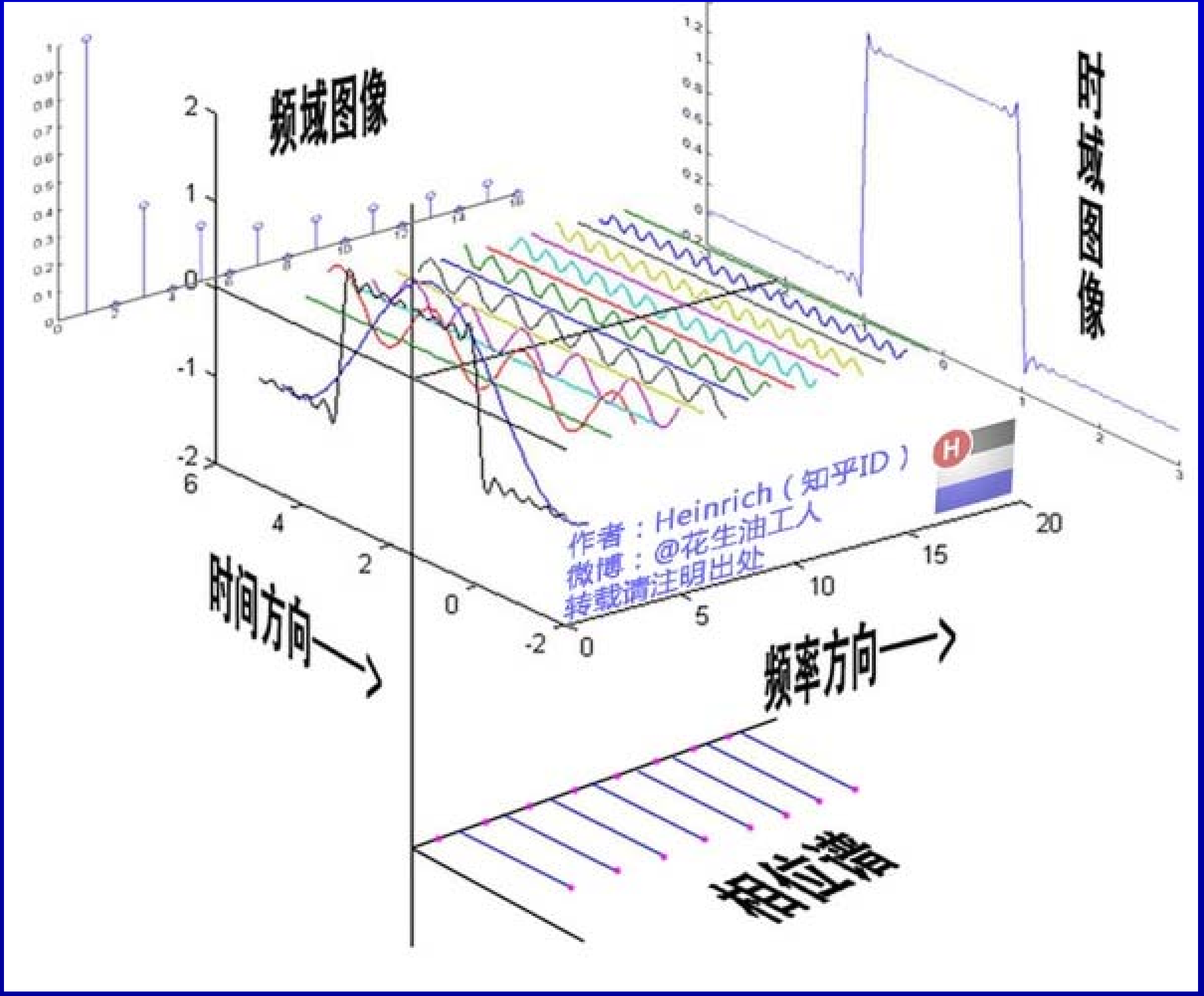


3.6 Fourier Series Representation of Discrete-Time Periodic Signals

$$\ldots e^{-j\frac{2\pi}{3}n} \quad e^{j0n} \quad e^{j\frac{2\pi}{3}n} \quad e^{j2\frac{2\pi}{3}n} \quad e^{j3\frac{2\pi}{3}n} \quad e^{j4\frac{2\pi}{3}n} \ldots$$

$$x[n] = \sum_{k} a_k e^{jk\omega_0 n}$$

$$|\omega_0| N = 2\pi m$$

$$\frac{2\pi}{N} = \frac{|\omega_0|}{m}$$

$$x[n] = a_1 e^{j1\frac{2\pi}{3}n} + a_2 e^{j2\frac{2\pi}{3}n} + a_3 e^{j3\frac{2\pi}{3}n} + a_4 e^{j4\frac{2\pi}{3}n}$$

$$x[n] = \sum_{k=<3>} a_k e^{jk\omega_0 n} \xrightarrow{N=3} x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

3.6 Fourier Series Representation of Discrete-Time Periodic Signals

- Harmonically related complex exponentials

$$\phi_k[n] = e^{jkw_0 n} = e^{jk(2\pi/N)n} \quad k = 0, \pm 1, \pm 2, \ldots$$

Frequency: w_0

period: N

$\phi_k[n]$ *is a periodic signal*

$$\phi_k[n] = \phi_{k+rN}[n]$$

$$x[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jkw_0 n}$$

$$= \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

k should take on the value about a period

x[n] is a periodic signal,and period is N

Determination of the Fourier series representation of a periodic signal

$$x[n]=\sum_{k=\langle N\rangle} a_k e^{jkw_0 n}=\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k=\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jkw_0 n}=\frac{1}{N}\sum_{n=\langle N\rangle} x[k]e^{-jk(2\pi/N)n}$$

$$a_k=a_{k+rN}$$

a_k Fourier series coefficients or the spectral coefficients

Example

$$\ldots e^{-j\frac{2\pi}{3}n} \quad e^{j0n} \quad e^{j\frac{2\pi}{3}n} \quad e^{j2\frac{2\pi}{3}n} \quad e^{j3\frac{2\pi}{3}n} \quad e^{j4\frac{2\pi}{3}n} \ldots$$

$$x[n] = a_1 e^{j1\frac{2\pi}{3}n} + a_2 e^{j2\frac{2\pi}{3}n} + a_3 e^{j3\frac{2\pi}{3}n}$$

$$x[n] = \qquad a_2 e^{j2\frac{2\pi}{3}n} + a_3 e^{j3\frac{2\pi}{3}n} + a_4 e^{j4\frac{2\pi}{3}n}$$

$$N = 3 \qquad a_1 = a_4 \qquad e^{jk\omega_0 n} = e^{j(k+rN)\omega_0 n}$$

$$a_k = a_{k+rN}$$

Matrix representation

$$x[n]=\sum_{k=\langle 3\rangle} a_k e^{jkw_0 n}=\sum_{k=\langle 3\rangle} e^{jkw_0 n} a_k$$

$$x[0]=\sum_{k=\langle 3\rangle} e^{jkw_0 0} a_k$$

$$x[1]=\sum_{k=\langle 3\rangle} e^{jkw_0 1} a_k$$

$$x[2]=\sum_{k=\langle 3\rangle} e^{jkw_0 2} a_k$$

$$\begin{array}{ccc} & & \begin{array}{ccc} k=0 & k=1 & k=2 \end{array} & \\ \begin{pmatrix} x[0] \\ x[1] \\ x[2] \end{pmatrix} & = & \begin{pmatrix} e^{j0\omega_0 0} & e^{j1\omega_0 0} & e^{j2\omega_0 0} \\ e^{j0\omega_0 1} & e^{j1\omega_0 1} & e^{j2\omega_0 1} \\ e^{j0\omega_0 2} & e^{j1\omega_0 2} & e^{j2\omega_0 2} \end{pmatrix} & \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \end{array}$$

$$X=D_N A$$

$$A=D_N{}^{-1}X \qquad D_N{}^{-1}=\frac{1}{N}D_N{}^{*}$$

$$a_k=\frac{1}{N}\sum_{n=\langle N\rangle} e^{-jkw_0 n} x[n]$$

N is odd

$$\ldots e^{-j\frac{2\pi}{3}n} \quad e^{j0n} \quad e^{j\frac{2\pi}{3}n} \quad e^{j2\frac{2\pi}{3}n} \quad e^{j3\frac{2\pi}{3}n} \quad e^{j4\frac{2\pi}{3}n} \ldots$$

$$N = 3$$

$$x[n] = a_{-1}e^{-j1\frac{2\pi}{3}n} + a_0 e^{j0n} + a_1 e^{j1\frac{2\pi}{3}n} \qquad a_{-k} = a_k^{\ *}$$

$$x[n] = \sum_{k=<3>} a_k e^{jk\omega_0 n} = a_0 + 2A_1 \cos(1\cdot\omega_0 n + \theta_1)$$

N is odd

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = a_0 + 2\sum_{k=1}^{\frac{N-1}{2}} A_k \cos(k\cdot\omega_0 n + \theta_k)$$

N is even

$$N = 4 \quad \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$\ldots e^{-j3\frac{\pi}{2}n} \quad e^{-j2\frac{\pi}{2}n} \quad e^{-j\frac{\pi}{2}n} \quad e^{j0n} \quad e^{j\frac{\pi}{2}n} \quad e^{j2\frac{\pi}{2}n} \quad e^{j3\frac{\pi}{2}n} \ldots$$

$$x[n] = a_{-1}e^{j-\frac{\pi}{2}n} + a_0 e^{j0n} + a_1 e^{j\frac{\pi}{2}n} + a_2 e^{j2\frac{\pi}{2}n} \qquad a_{-k} = a_k^*$$

$$x[n] = a_0 + 2A_1\cos(1 \cdot \frac{\pi}{2}n + \theta_1) + A_2\cos(\pi n + \theta_2)$$

N is even

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = a_0 + 2\sum_{k=1}^{\frac{N}{2}} A_k \cos(k \cdot \omega_0 n + \theta_k)$$

Example 3.10

If $x[n] = \sin w_0 n$, determine the Fourier series Coefficients, $2\pi / w_0$ is a integer N

$$x[n] = \frac{1}{2j}\left(e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}\right)$$

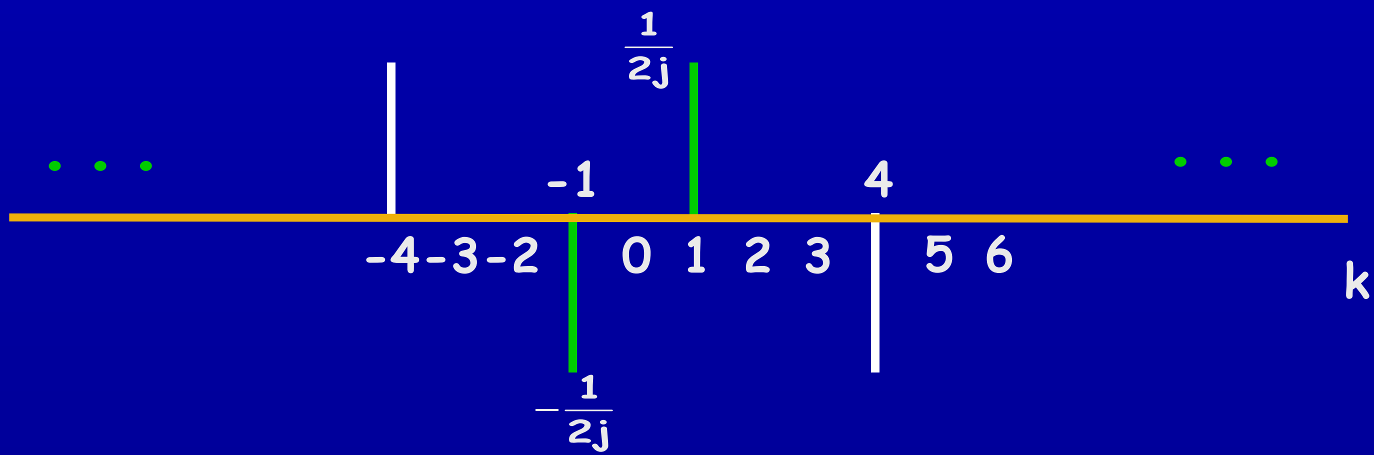
$$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j}$$

$$a_{1+rN} = \frac{1}{2j} \qquad a_{-1+rN} = -\frac{1}{2j}$$

$$x[n] = \sin\left(\frac{2\pi}{5}\right)n \qquad N = \frac{2\pi}{\frac{2\pi}{5}} = 5$$

$$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j}$$

$$a_{1+rN} = \frac{1}{2j} \qquad a_{-1+rN} = -\frac{1}{2j}$$



Example 3.11

If $x[n]=1+\sin(\frac{2\pi}{N})n+3\cos(\frac{2\pi}{N})n+\cos(\frac{4\pi}{N}n+\frac{\pi}{2})$
determine the Fourier series coefficients

$$a_0=1 \quad a_1=\frac{3}{2}-\frac{1}{2}j \quad a_{-1}=\frac{3}{2}+\frac{1}{2}j$$

$$a_2=\frac{1}{2}j \qquad a_{-2}=-\frac{1}{2}j$$

Signals & Systems

Example 3.12

$$x[n]=\begin{cases}1 & |n|\le N_1 \\ 0 & N_1<|n|<\dfrac{N}{2}\end{cases}$$

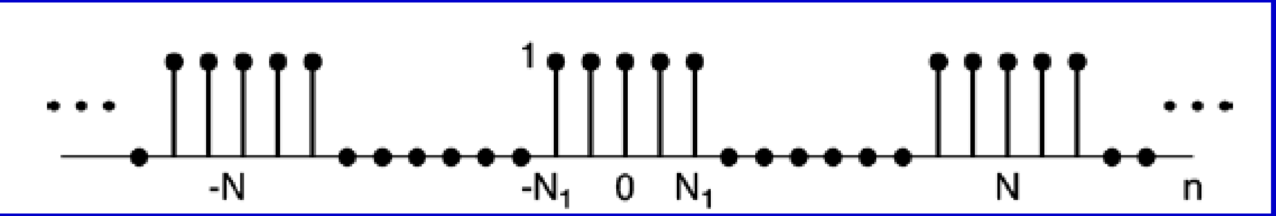
determine the Fourier series coefficients

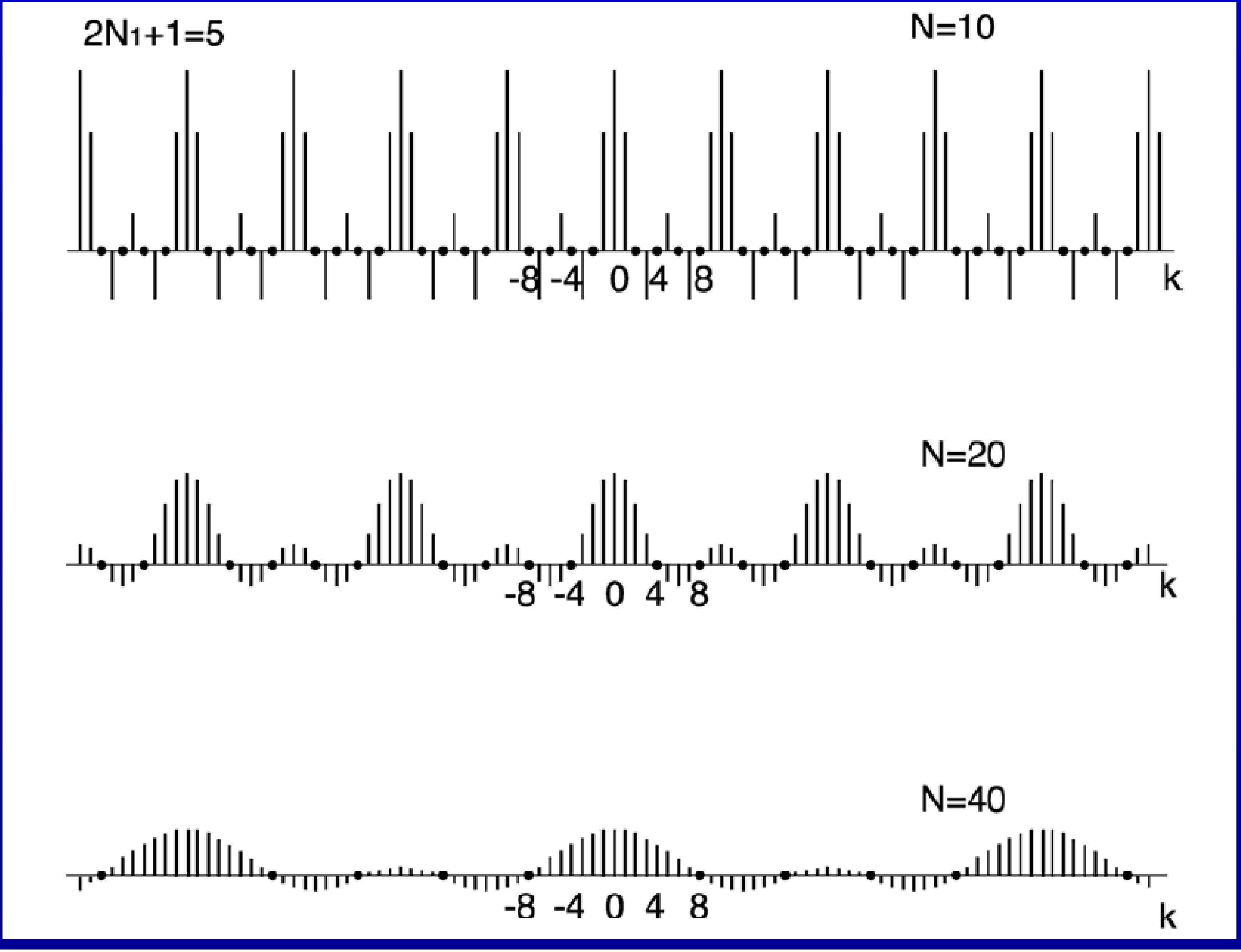
$$a_k=\frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

$$k\neq 0,\pm N,\pm 2N\cdots \quad a_k=\frac{1}{N}\frac{\sin\left[2\pi k(N_1+1/2)/N\right]}{\sin(\pi k/N)}$$

$$k=0,\pm N,\pm 2N\cdots \quad a_k=\frac{2N_1+1}{N}$$

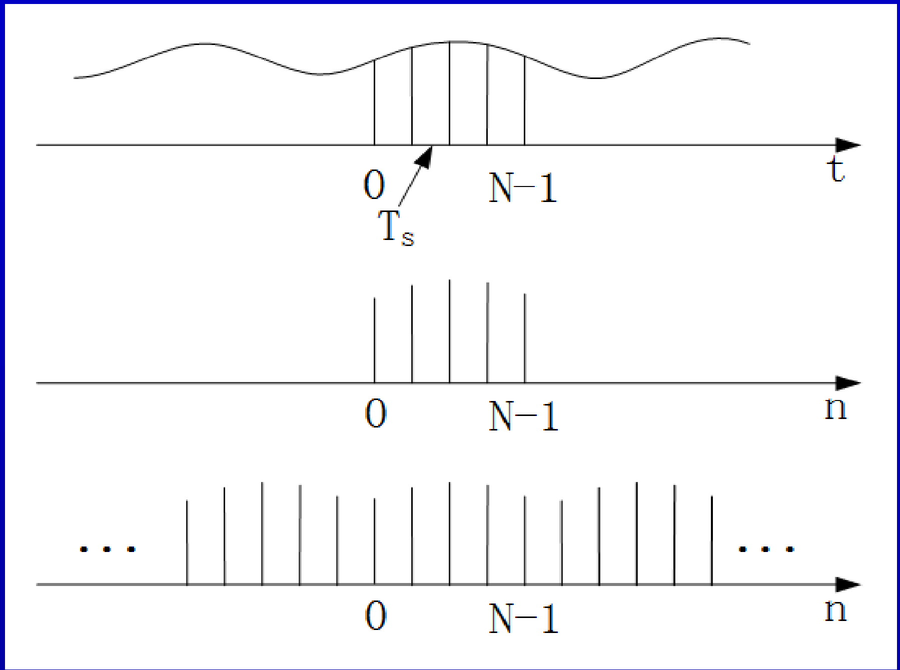
Signals & Systems

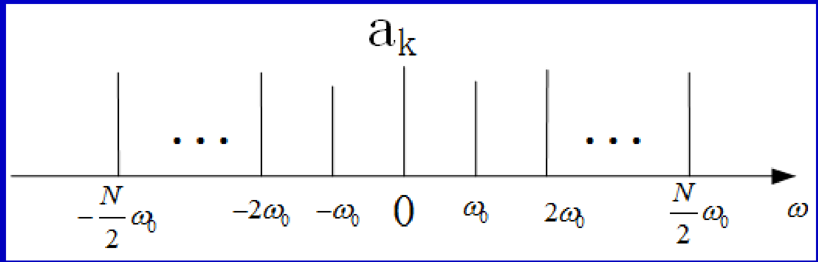




Signals & Systems

The relationship between CT and DT periodic signal





N is even

$$x[n] = a_0 + 2\sum_{k=1}^{\frac{N}{2}} A_k \cos(k \cdot \omega_0 n + \theta_k)$$

$$\omega_0 = \frac{2\pi}{N \cdot T_S} = \frac{2\pi \cdot F_S}{N}$$

The difference between FS

- The difference of basic signals
- The difference of Fourier series representation coefficients
- The difference of convergence

3.8 Fourier Series and LTI Systems

- Definition

$$\underline{x(t)} \rightarrow y(t) = \underline{A}x(t)$$

eigenfunction eigenvalue

- Two Eigenfunction

$$e^{st} \rightarrow H(S)e^{st}$$

$$z^{n} \rightarrow H(z)z^{n}$$

System function and frequency response

- System function

$$H(s)=\int_{-\infty}^{+\infty}h(t)e^{-st}dt$$

$$H(z)=\sum_{k=-\infty}^{+\infty}h[k]z^{-k}$$

- Frequency response

$$H(s)\big|_{s=jw}=\int_{-\infty}^{+\infty}h(t)e^{-jwt}dt=H(jw)$$

$$H(z)\big|_{z=e^{jw}}=\sum_{k=-\infty}^{+\infty}h[k]e^{-jwk}=H(e^{jw})$$

The response of LTI systems to CT periodic signal

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} \quad \text{is} \quad \text{periodic}$$

$$e^{jwt} \rightarrow H(jw)e^{jwt}$$

$$y(t)=\sum_{k=-\infty}^{+\infty} \underline{a_k H(jkw_0)} e^{jkw_0 t}$$

Fourier series coefficients for y(t)

Signals & Systems

The response of LTI systems to DT periodic signal

$$x[n]=\sum_{k=\langle N\rangle} a_k e^{jkw_0 n} \quad \text{is} \quad \text{periodic}$$

$$e^{jwn} \rightarrow H\left(e^{jw}\right)e^{jwn}$$

$$y[n]=\sum_{k=\langle N\rangle} \underline{a_k H\left(e^{jkw_0}\right)} e^{jkw_0 n}$$

Fourier series coefficients for y[n]

Signals & Systems

Example 3.16

$$x(t)=1+\frac{1}{2}\cos 2\pi t+\cos 4\pi t+\frac{2}{3}\cos 6\pi t$$

$$h(t)=e^{-t}u(t) \quad \text{determine} \quad y(t)$$

$$H(jw)=\int_{-\infty}^{+\infty} e^{-t}u(t)e^{-jwt}dt=\frac{1}{1+jw}$$

$$y(t)=\sum_{k=-3}^{+3} b_k e^{jk2\pi t} \qquad b_k = a_k H(jk2\pi)$$

$$b_0=1 \qquad b_1=\frac{1}{4}\left(\frac{1}{1+j2\pi}\right) \qquad b_{-1}=\frac{1}{4}\left(\frac{1}{1-j2\pi}\right)$$

$$b_2=\frac{1}{2}\left(\frac{1}{1+j4\pi}\right) \qquad b_{-2}=\frac{1}{2}\left(\frac{1}{1-j4\pi}\right)$$

$$b_3=\frac{1}{3}\left(\frac{1}{1+j6\pi}\right) \qquad b_{-3}=\frac{1}{3}\left(\frac{1}{1-j6\pi}\right)$$

Example 3.17

$$x[n]=\cos\left(\frac{2\pi n}{N}\right), h[n]=a^n u[n], \text{determine } y[n], -1<a<1$$

$$H\left(e^{jw}\right)=\sum_{n=-\infty}^{+\infty} a^n u[n] e^{-jwn}=\frac{1}{1-ae^{-jw}}$$

$$x[n]=\frac{1}{2}e^{j(2\pi/N)n}+\frac{1}{2}e^{-j(2\pi/N)n}$$

$$y[n]=\frac{1}{2}\left(\frac{1}{1-ae^{-j2\pi/N}}\right)e^{j(2\pi/N)n}$$

$$+\frac{1}{2}\left(\frac{1}{1-ae^{j2\pi/N}}\right)e^{-j(2\pi/N)n}$$

Notice

If the LTI system is stable, then H(jw) and $H(e^{jw})$ are finite

$$H(jw) = \int_{-\infty}^{+\infty} h(t)\, e^{-jwt} dt$$

$$H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-jwn}$$

For example:

$$h(t) = e^{-t}u(t) \quad h[n] = a^n u[n] \quad (|a| < 1)$$

$$h(t) = e^{t}u(t) \quad h[n] = a^n u[n] \quad (|a| > 1)$$

3.9 Filtering

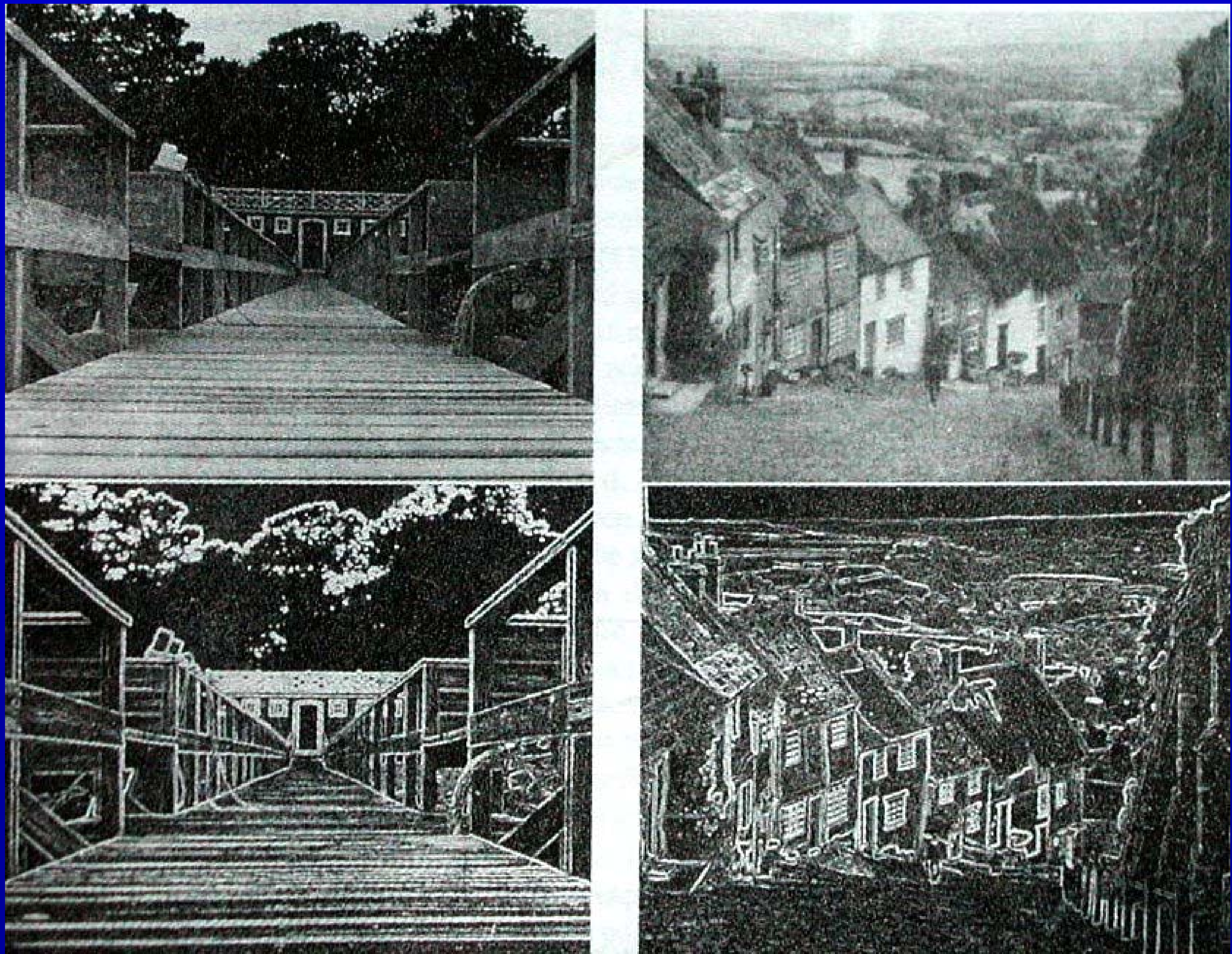
- **Filter** —— change the relative amplitudes of the frequency components or eliminate some frequency components entirely

- **Classify** —— frequency-shaping filter
  frequency-selective filter

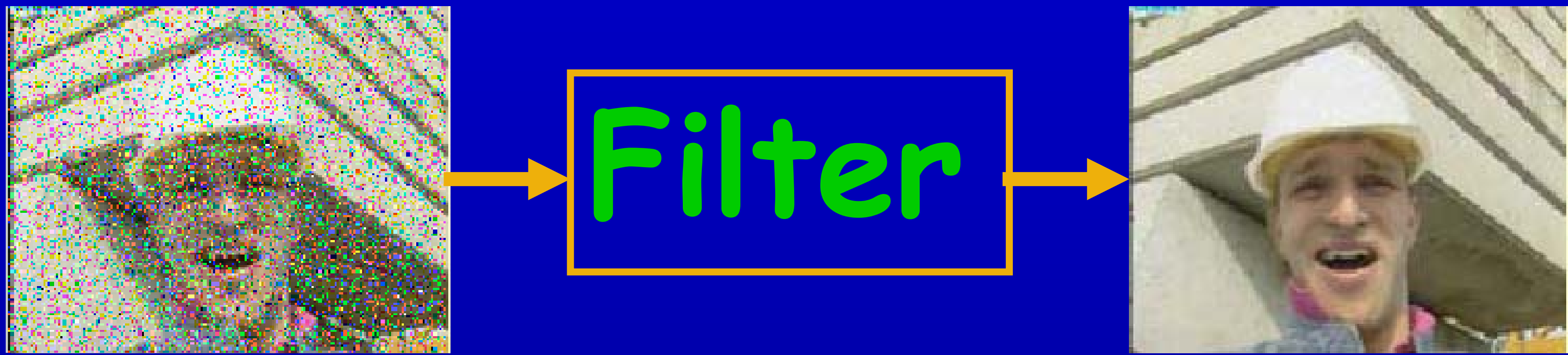
3.9.1 Frequency-Shaping Filters

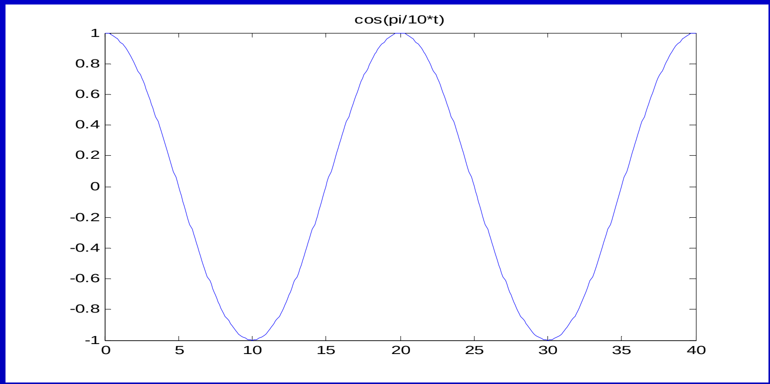
- Audio systems
  - modify the relative amounts of low-frequency energy and high-frequency energy
- Differentiating filter
  - $H(jw) = jw$
  - enhancing rapid variations or transitions
  - enhance edges in picture processing

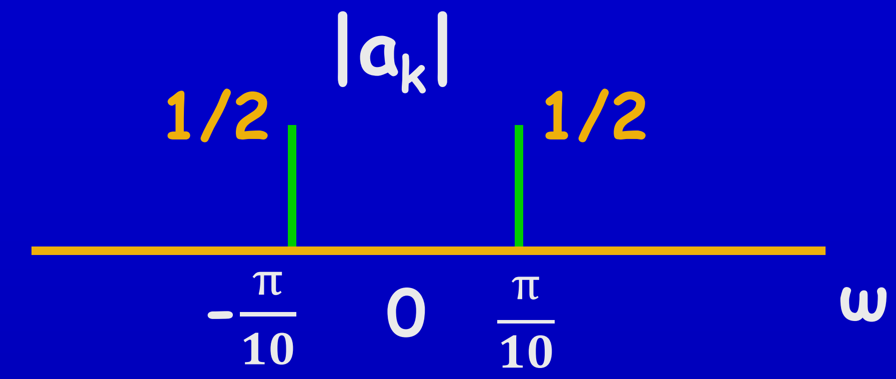
Frequency-Shaping Filters

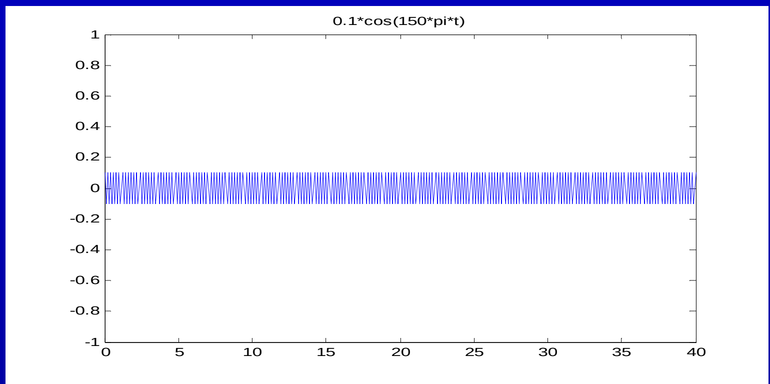


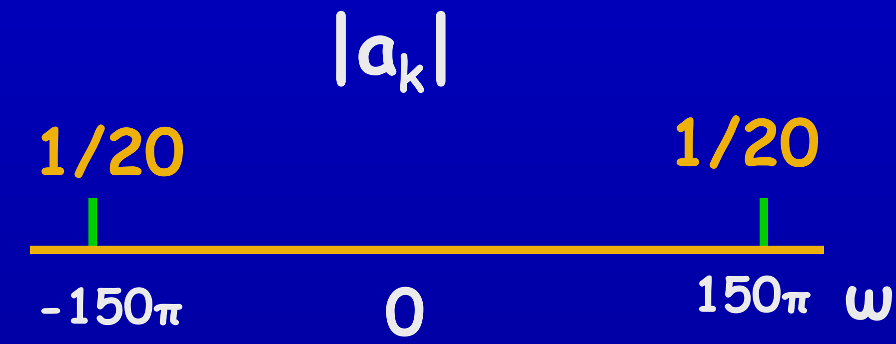
3.9.2 Frequency-Selective Filters

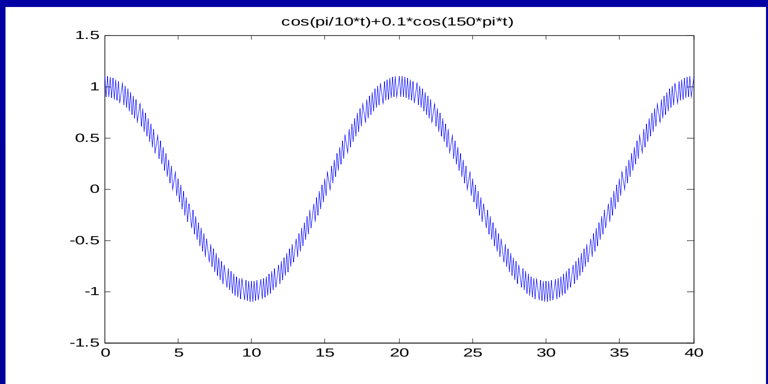


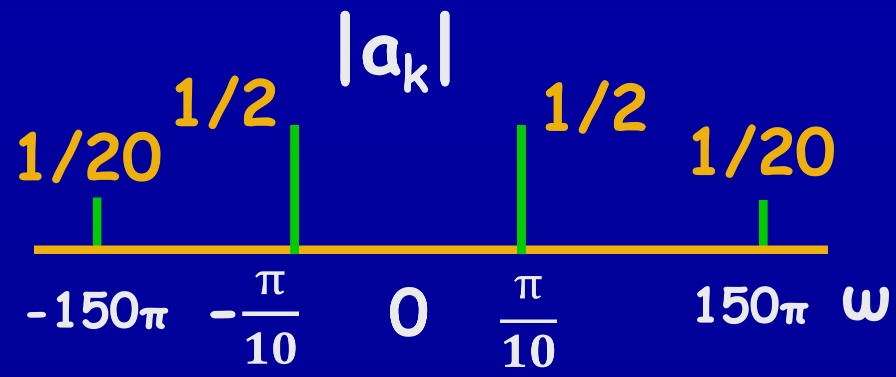

cos(pi/10*t)


|a_k|
1/2
1/2
-π/10
0
π/10
ω


0.1*cos(150*pi*t)

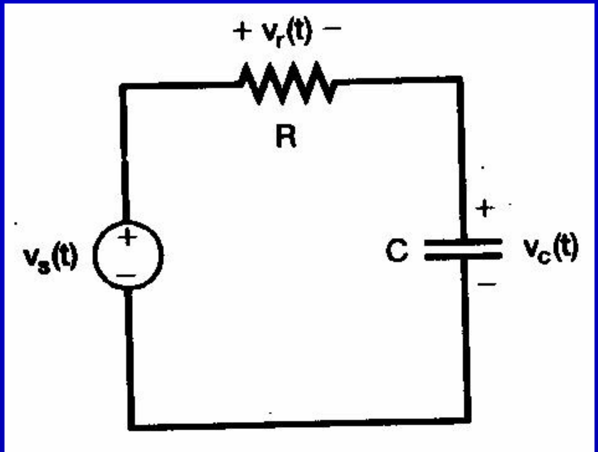

|a_k|
1/20
1/20
-150π
0
150π
ω

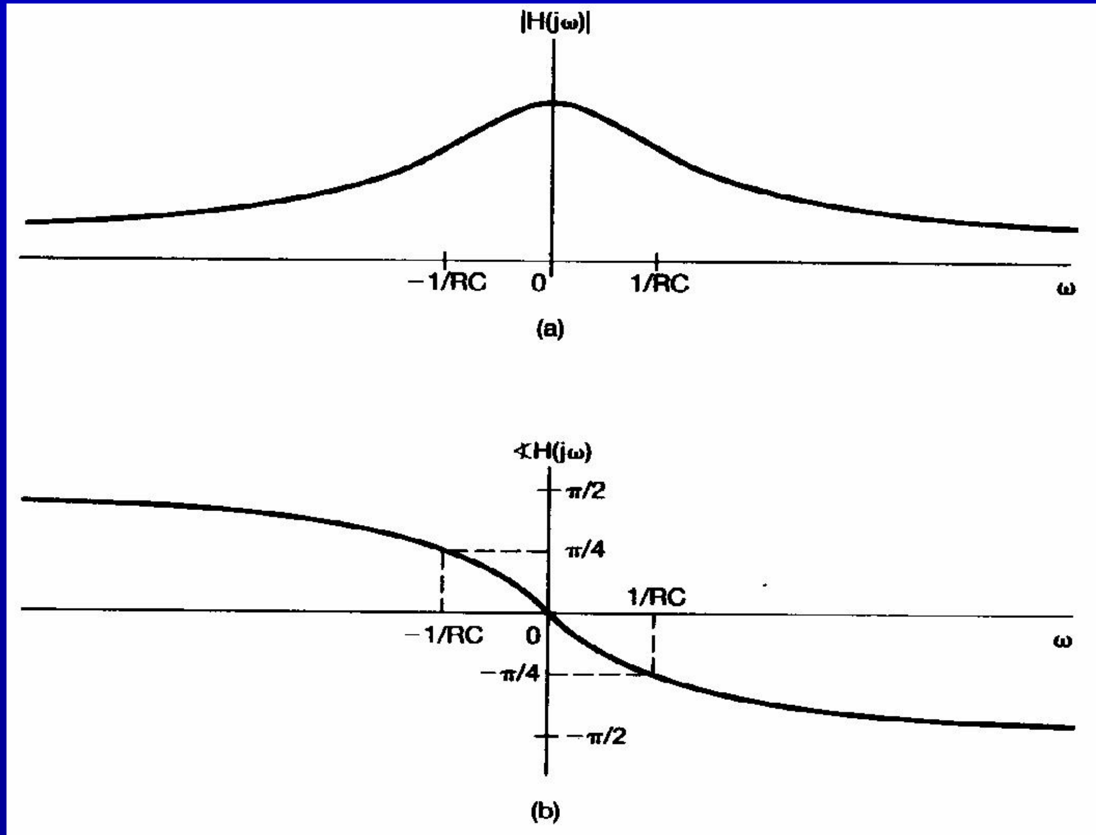

cos(pi/10*t)+0.1*cos(150*pi*t)


|a_k|
1/20
1/2
1/2
1/20
-150π
-π/10
0
π/10
150π
ω

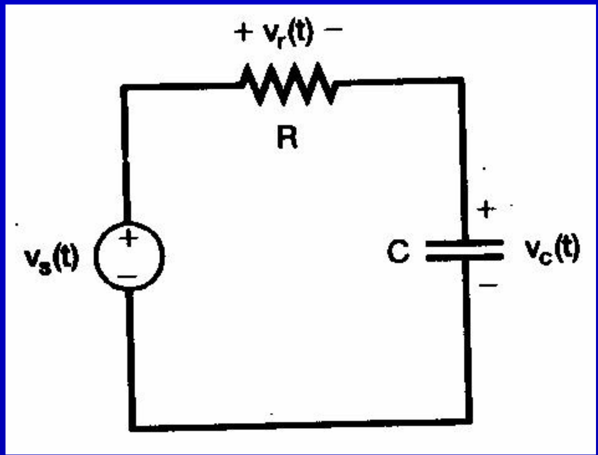
Signals & Systems

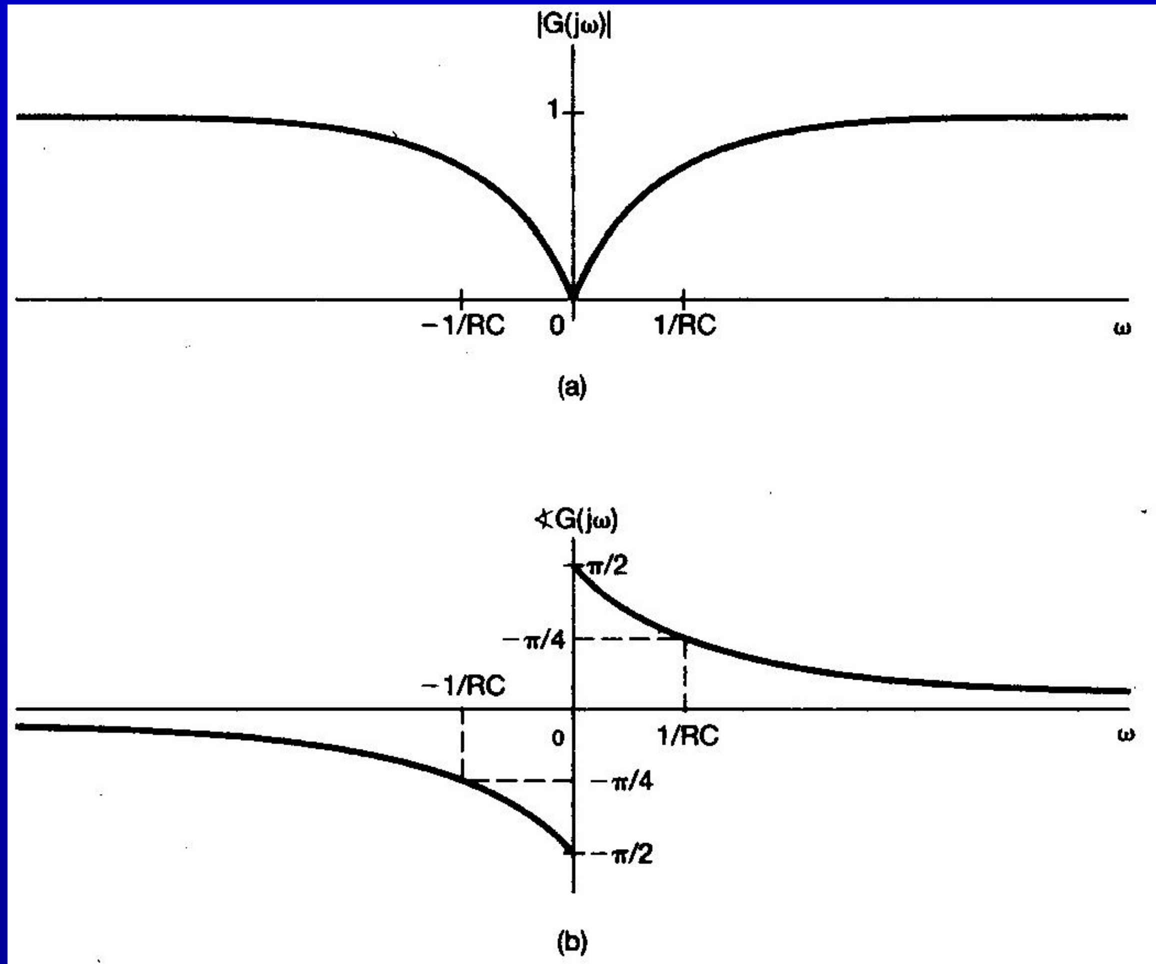
RC LOW PASS





RC HIGH PASS

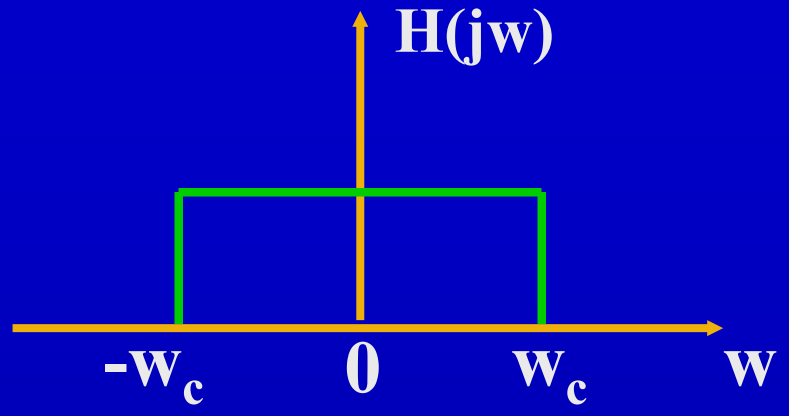




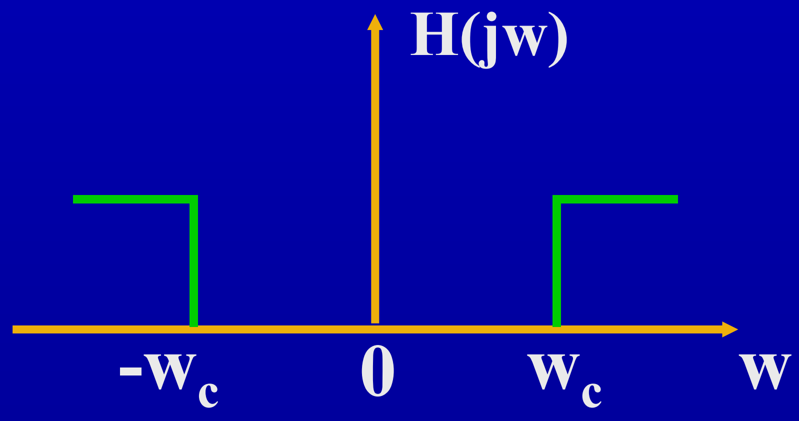
- **Application**
  - filter the noise
  - communication system
- **Classify**
  - low-pass filter
  - high-pass filter
  - band-pass filter
  - band-stop filter
- **Concepts**
  - cutoff frequency
  - passband and stopband

CT Filter

- low-pass filter

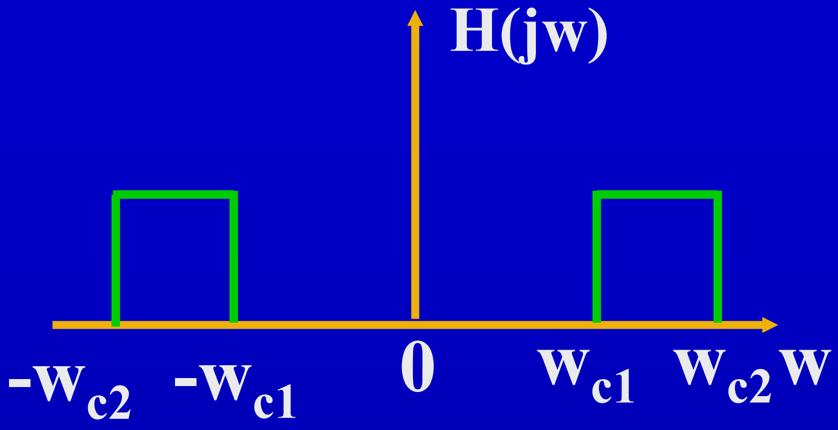


- high-pass filter

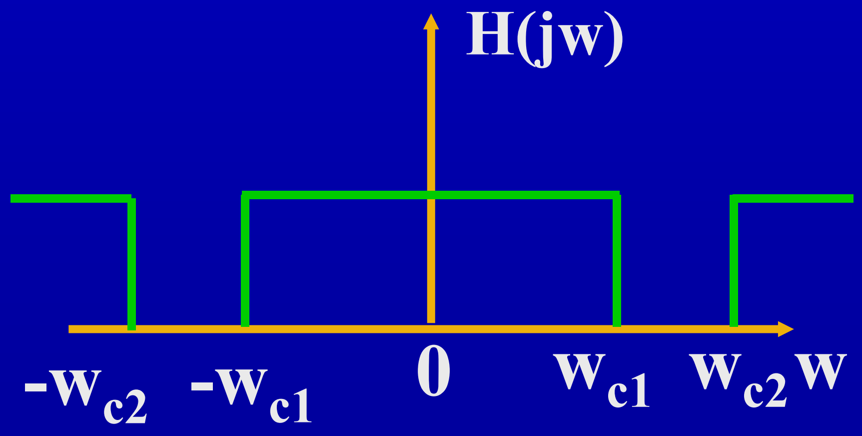


CT Filter

- band-pass filter



- band-stop filter



Signals & Systems

DT Filter

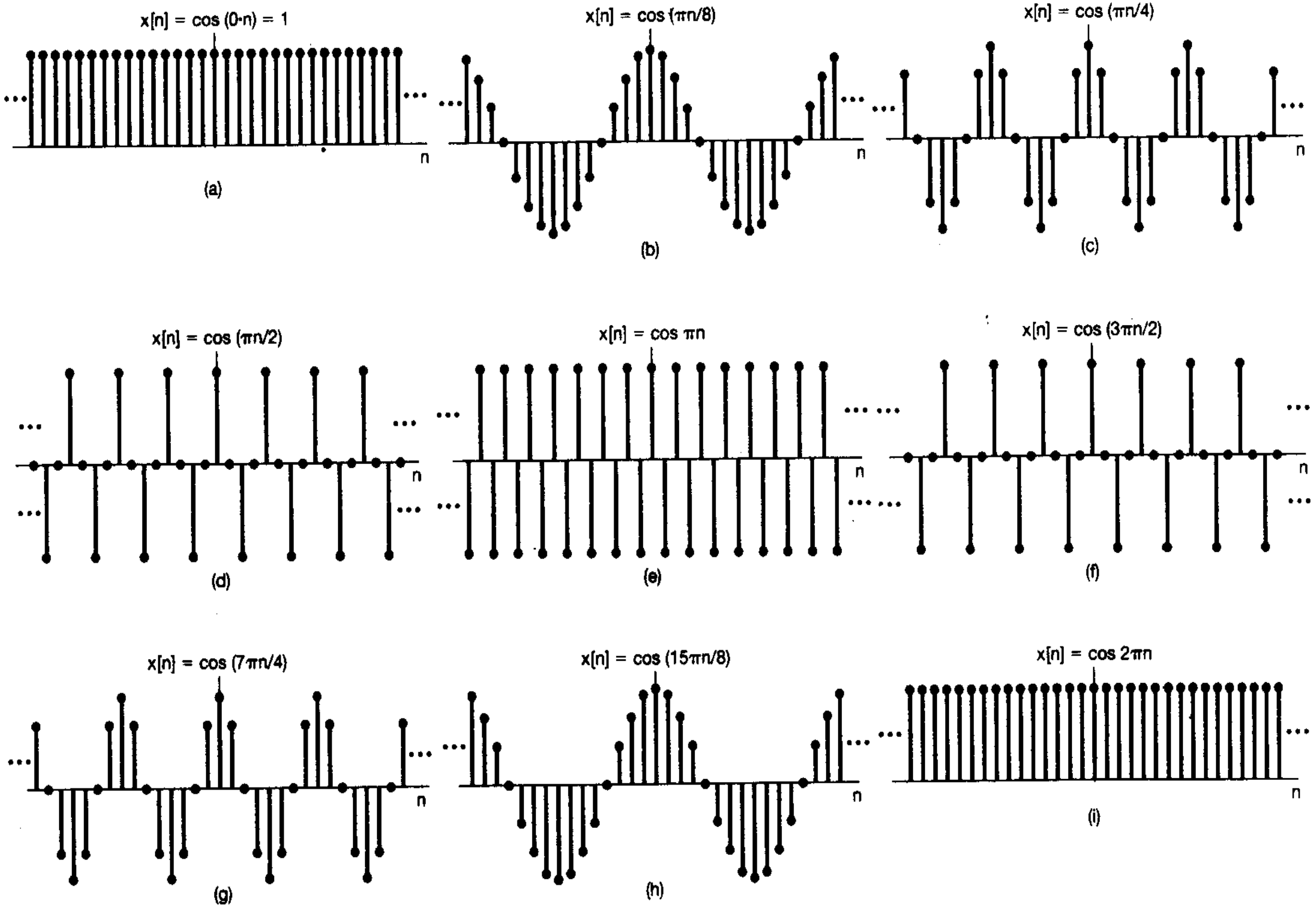
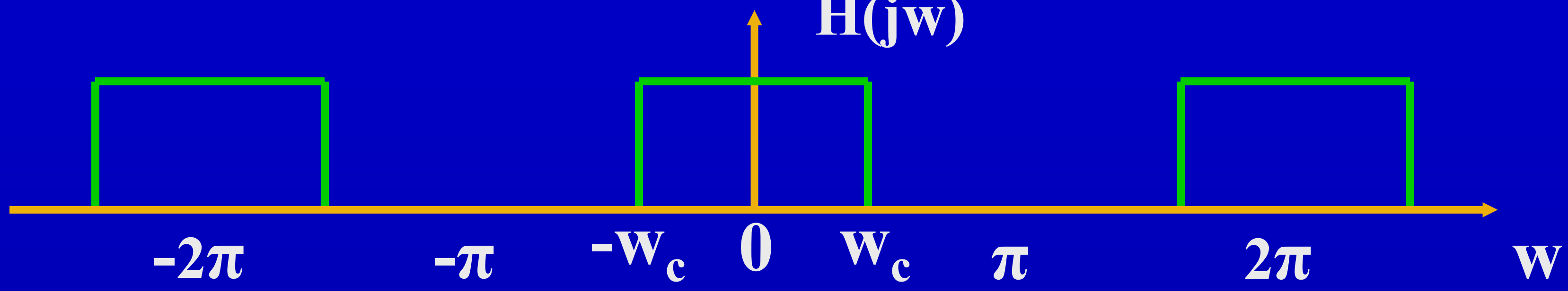


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

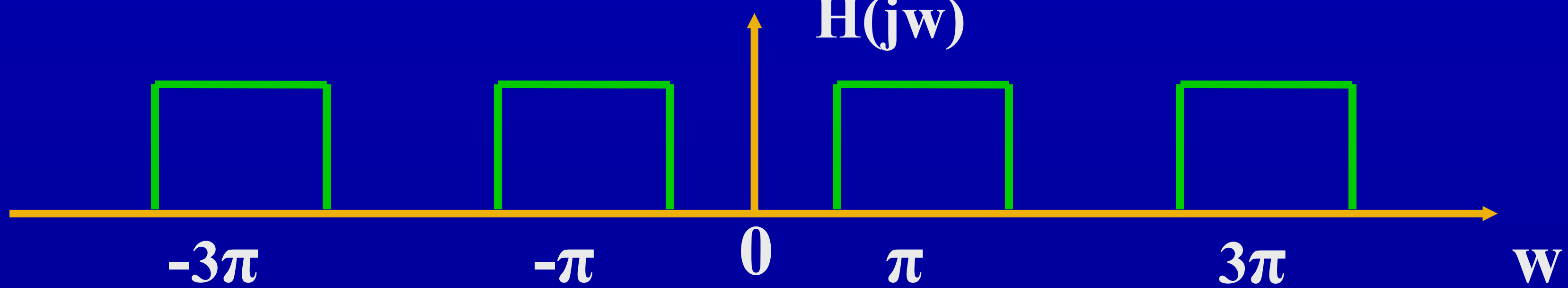
Signals & Systems

DT Filter

- low-pass filter



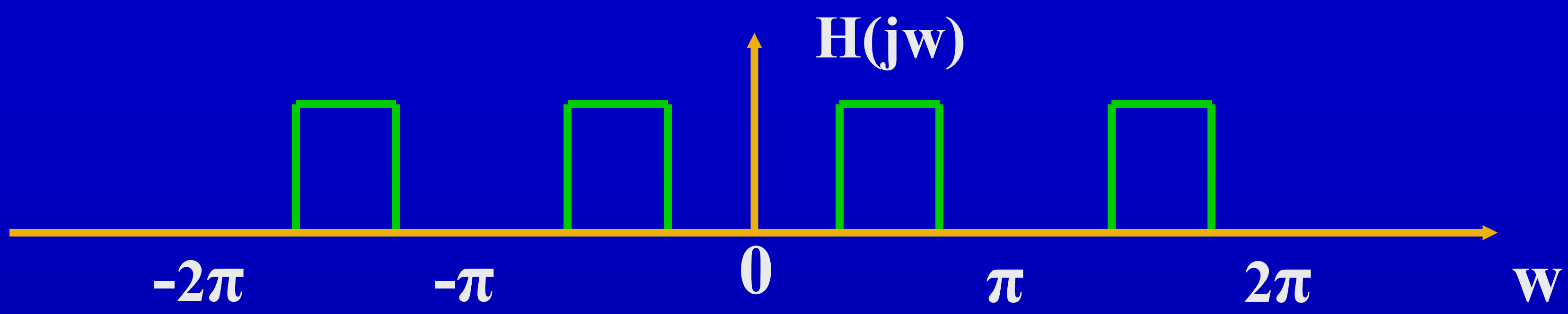
- high-pass filter



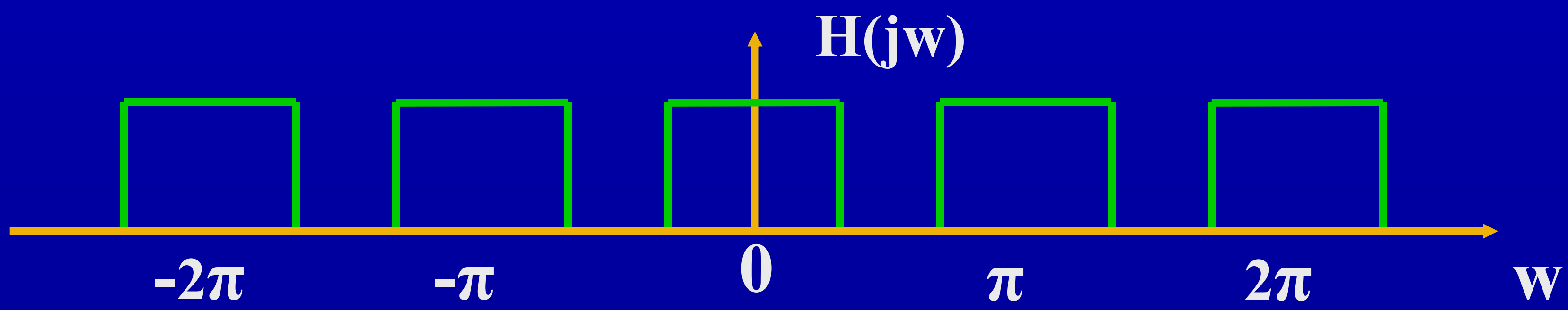
Signals & Systems

DT Filter

- band-pass filter

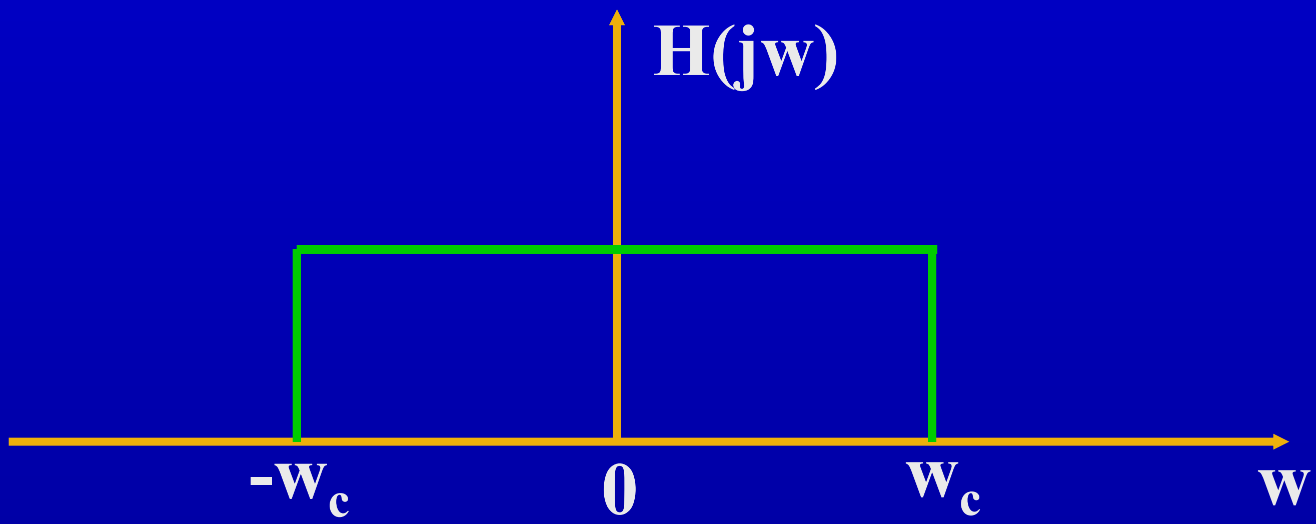


- band-stop filter



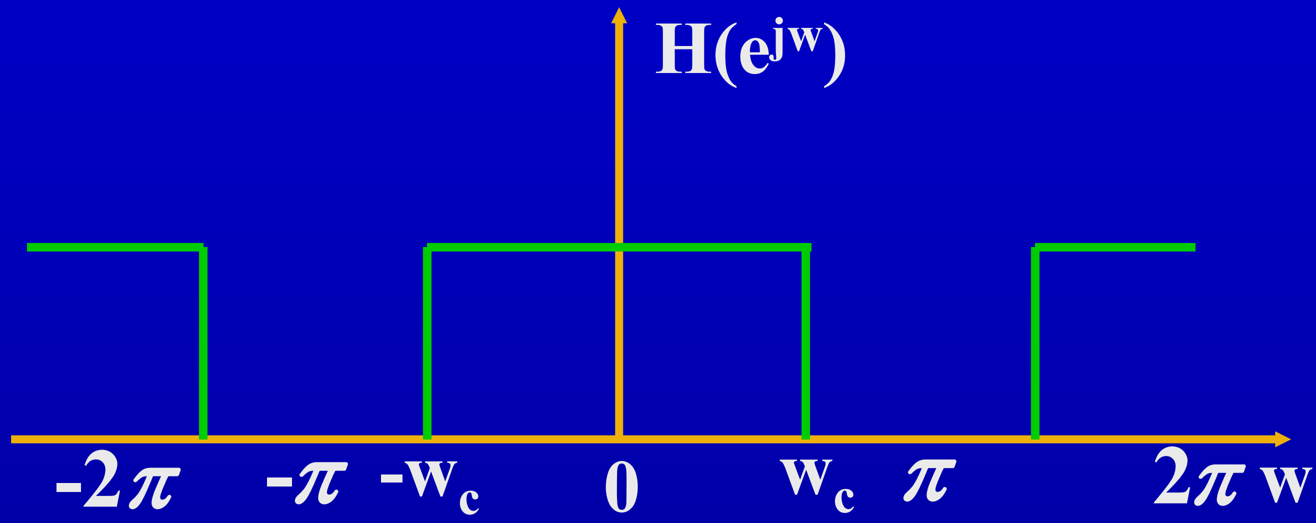
Signals & Systems

Low-pass filter(for CT)



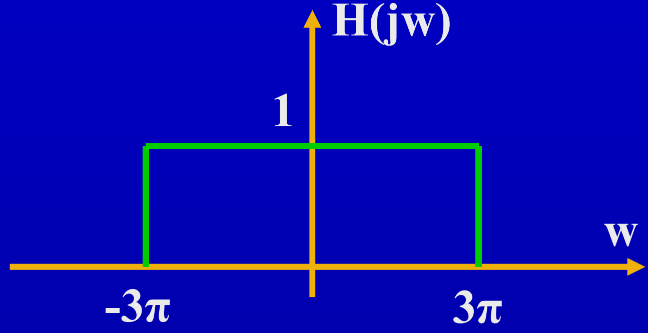
Signals & Systems

Low-pass filter(for DT)

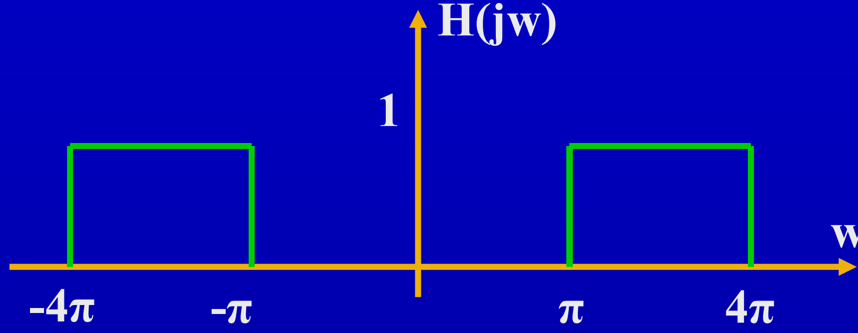


Example

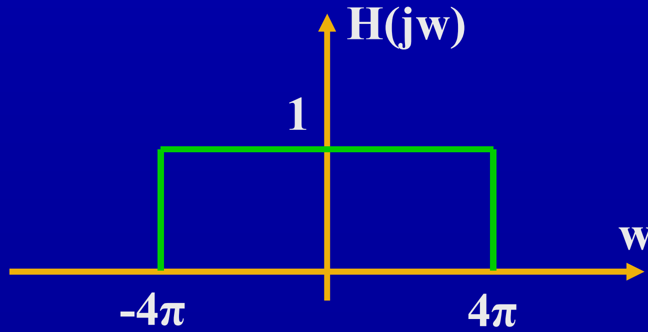
The output of system(   ) to the input x(t)=cos2πt+sin10t is periodic



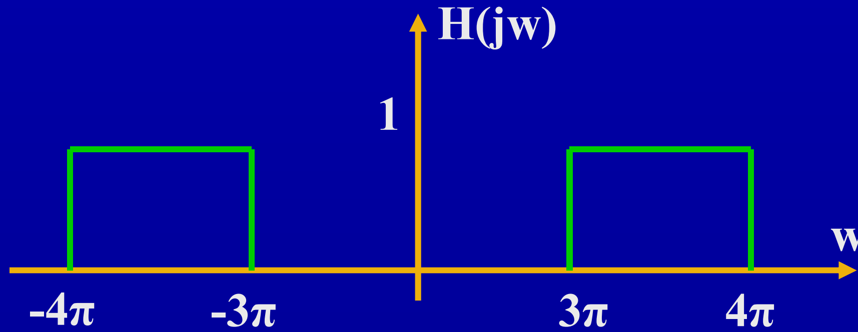
(a)



(b)



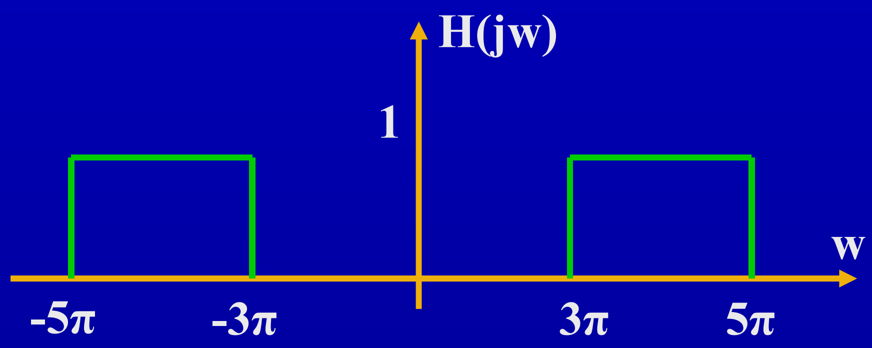
(c)



(d)

Example

Consider a CT LTI system which frequency response is illustrated in fig.3. when the input to this system is a periodic signal x(t) with fundamental period T=1 and FS coefficients a_k, it is found y(t)=x(t), the FS coefficients a_k must satisfy(   )



(a) $a_k \neq 0, |k| \neq 2$

(b) $a_k = 0, |k| \neq 2$ ✓

(c) $a_k = \begin{cases} \neq 0 & |k| < 2 \\ 0 & |k| \geq 2 \end{cases}$

(d) $a_k = \begin{cases} \neq 0 & |k| > 3 \\ 0 & |k| \leq 3 \end{cases}$