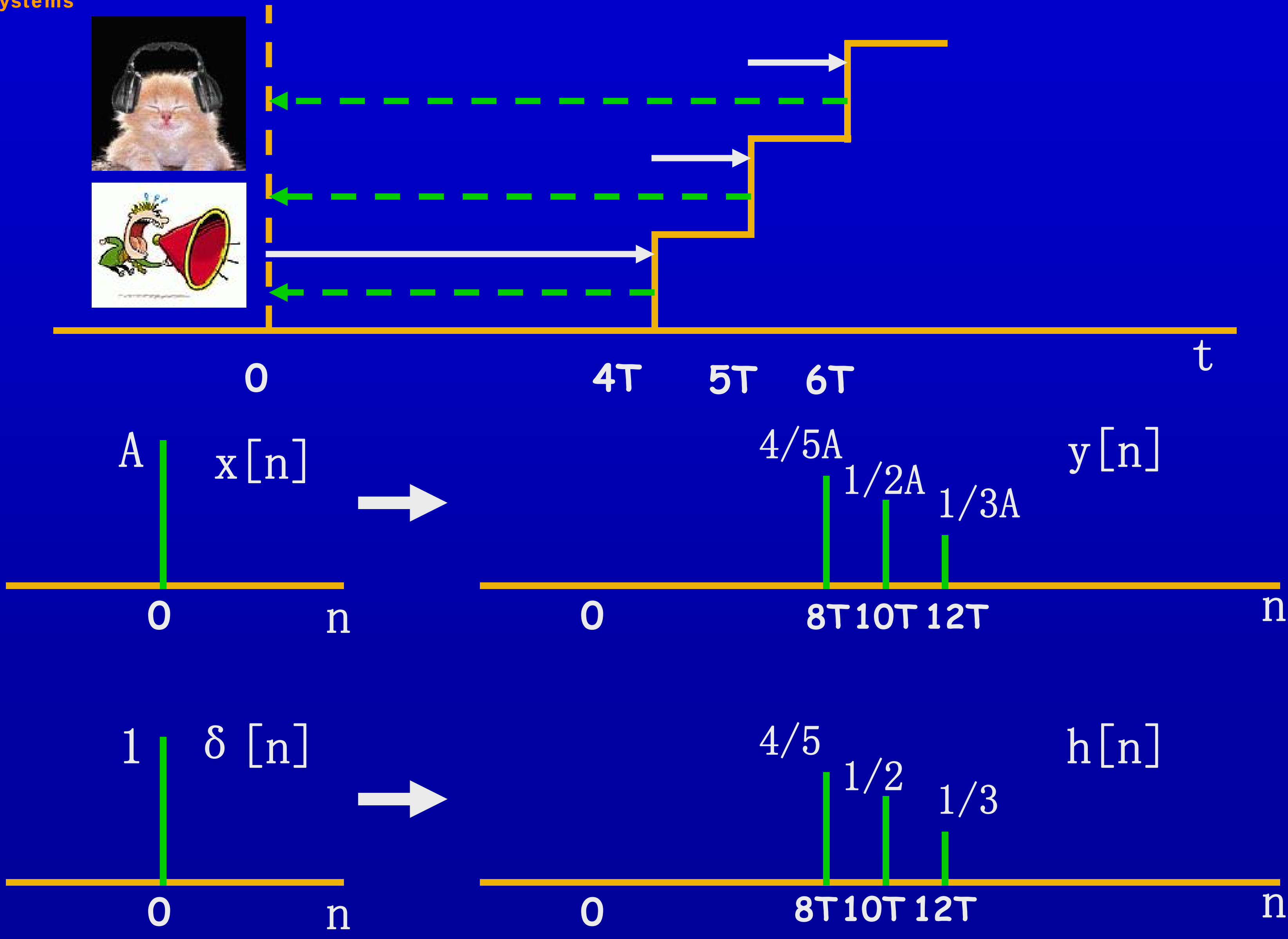
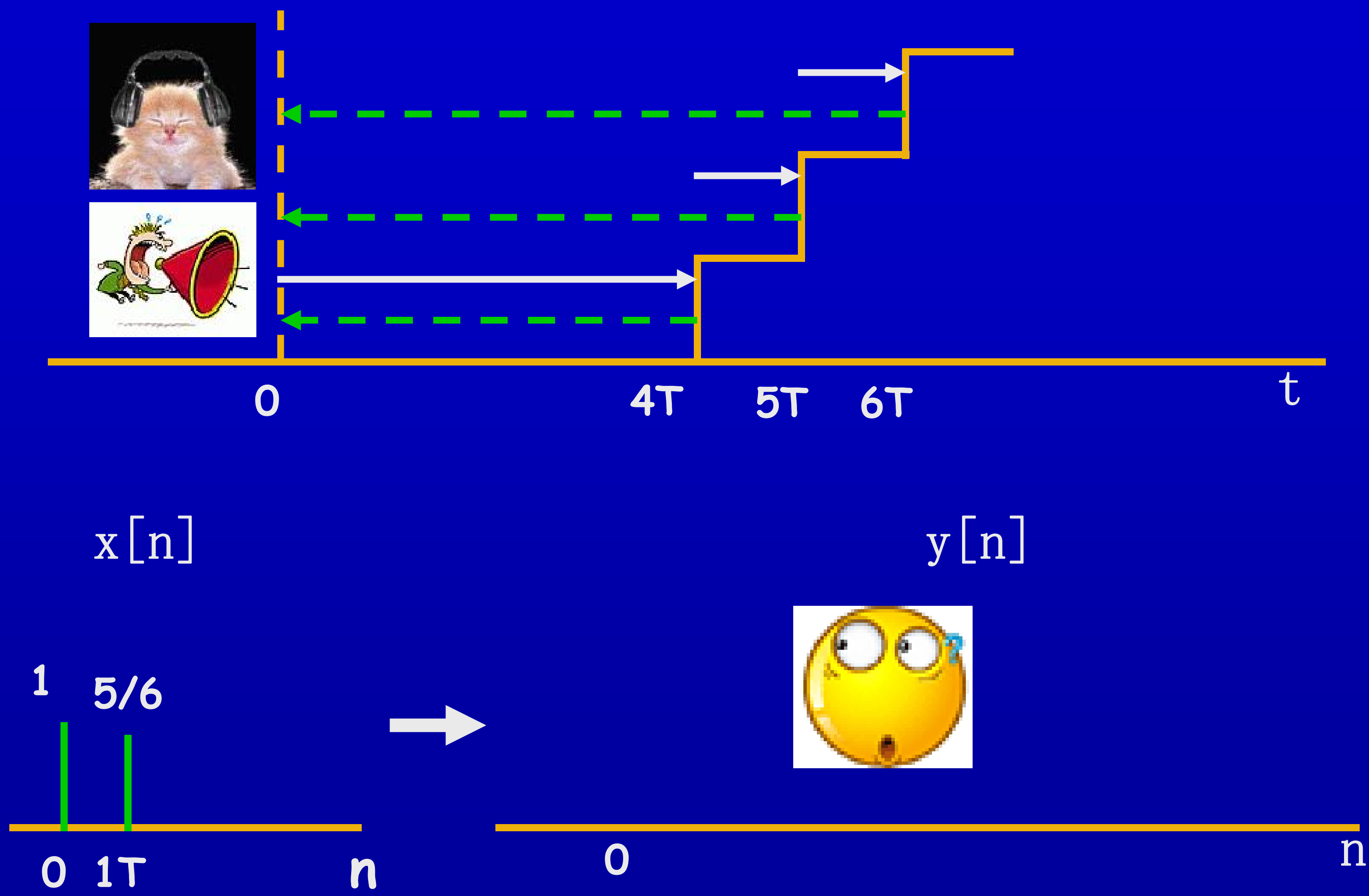
Chapter 2

Linear Time-Invariant Systems


0
4T
5T
6T
t
A
x[n]
0
n
4/5A
1/2A
1/3A
y[n]
0
8T 10T 12T
n
1
δ [n]
0
n
4/5
1/2
1/3
h[n]
0
8T 10T 12T
n


0
4T
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x[n]
y[n]
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n
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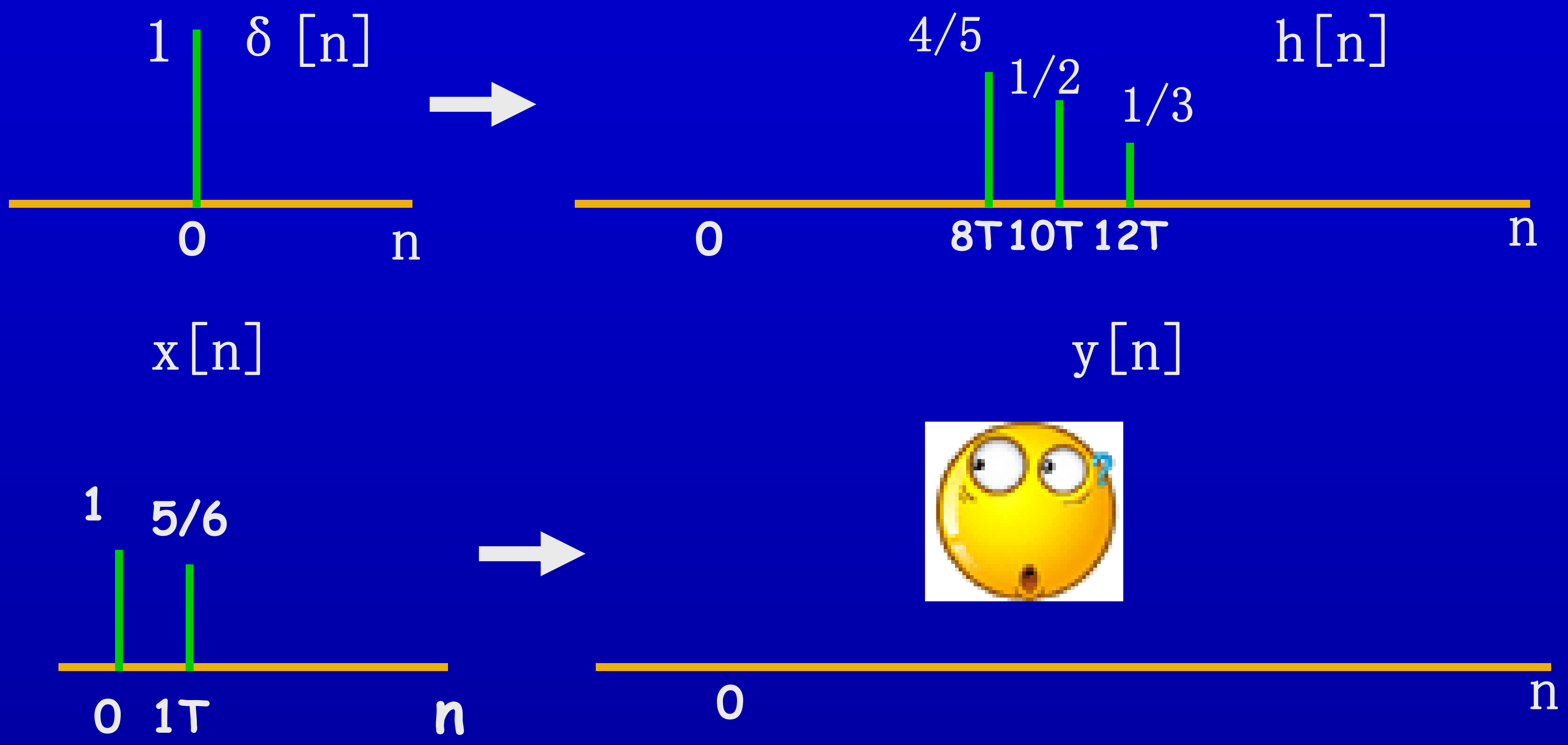
n



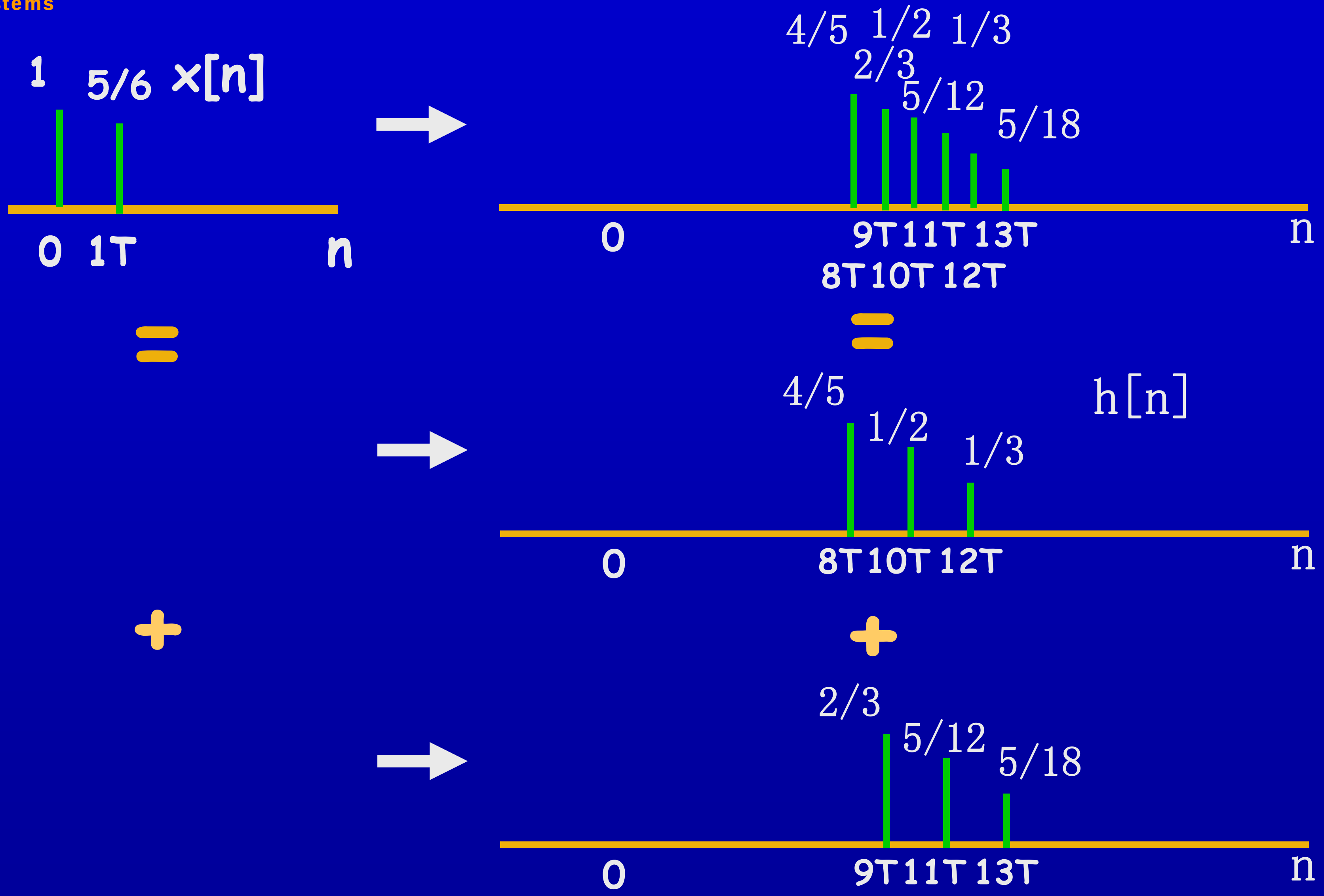
1
δ [n]
0
n
4/5
1/2
1/3
h[n]
0
8T 10T 12T
n
x[n]
y[n]
1
5/6
0 1T
n
0
n

Time Invariance

$$if\ \ x[n] \to y[n], then\ \ x[n-n_0] \to y[n-n_0]$$

Linearity

$$ax_1[n]+bx_2[n] \to ay_1[n]+by_2[n]$$


1
5/6
x[n]
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2/3
5/12
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9T 11T 13T
8T 10T 12T
n
=
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4/5
1/2
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h[n]
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n
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2/3
5/12
5/18
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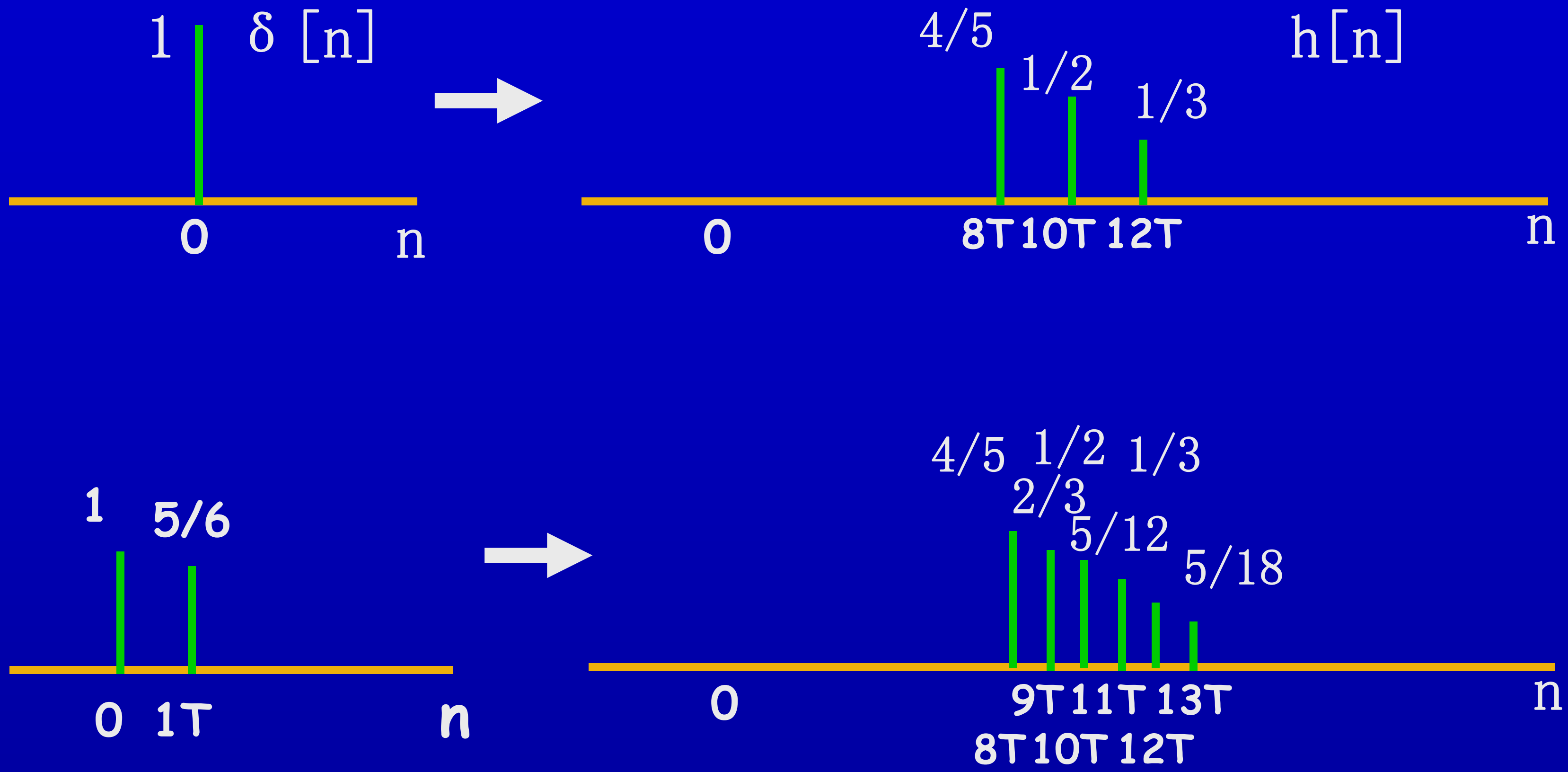
9T 11T 13T
n


1
δ[n]
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4/5
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h[n]
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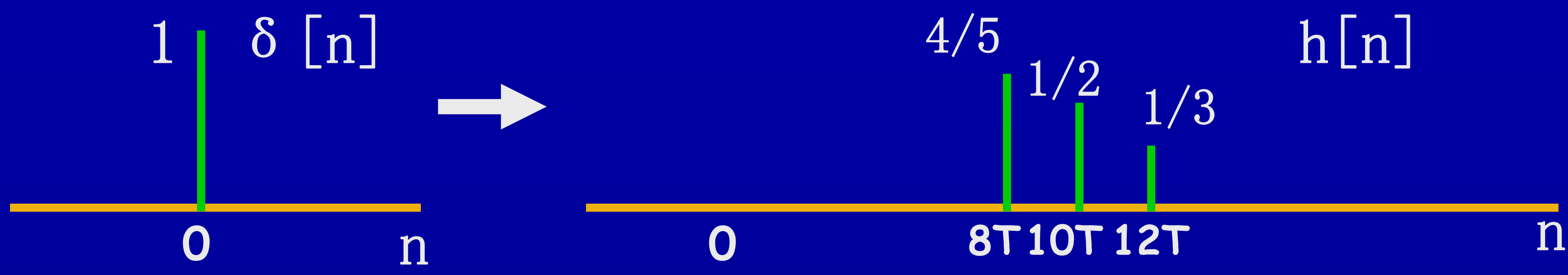
8T 10T 12T
n
1
5/6
0 1T
n
4/5 1/2 1/3
2/3
5/12
5/18
0
9T 11T 13T
8T 10T 12T
n

The Discrete-Time Unit Impulse Response

x[n] → LTI SYSTEM → y[n]

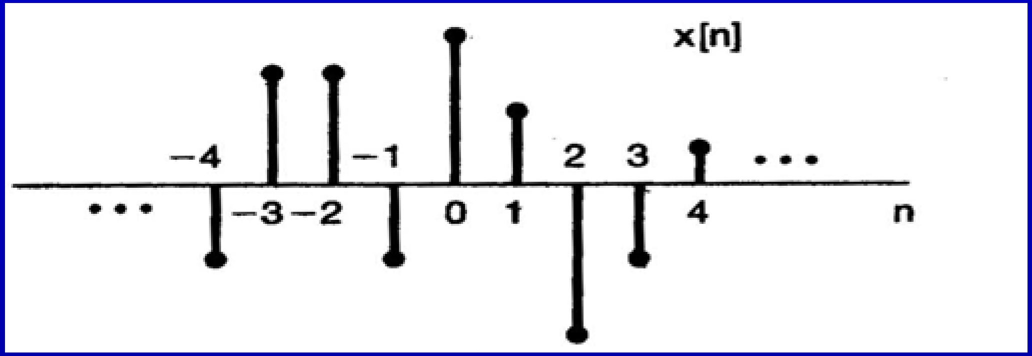
δ[n] → LTI SYSTEM → h[n]

h[n] -- unit impulse response



2.1 Discrete-Time LTI Systems: The Convolution Sum

2.1.1 The Representation Discrete-Time Signals In Term of Impulse



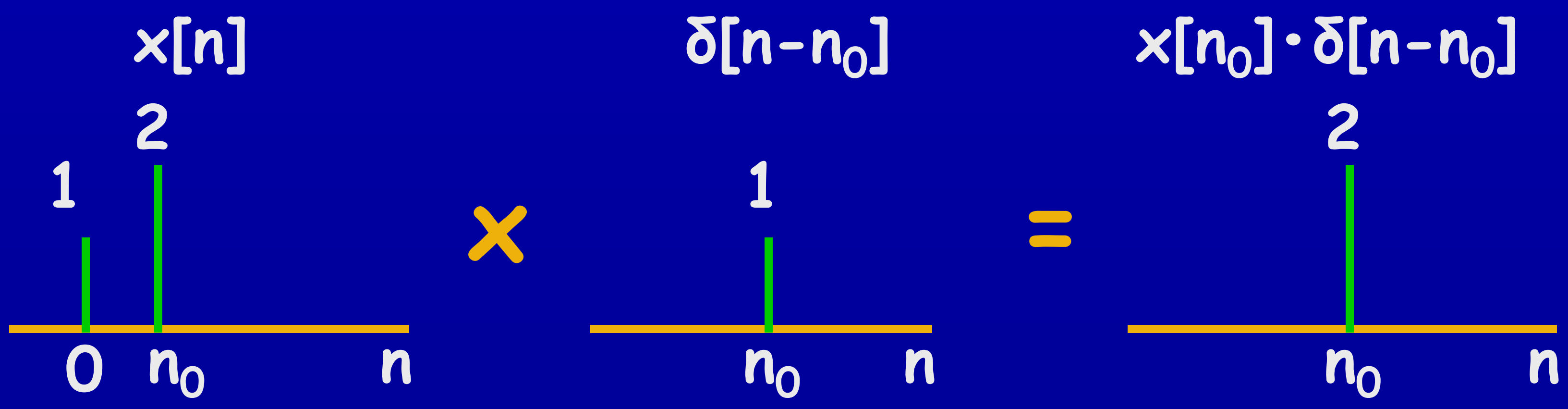
2.1 Discrete-Time LTI Systems: The Convolution Sum

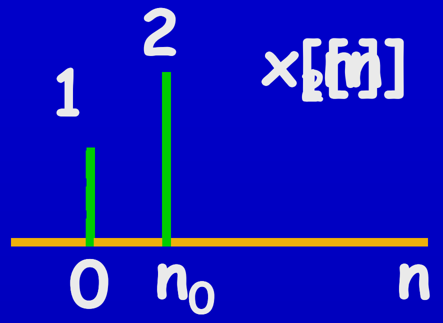
2.1.1 The Representation Discrete-Time Signals In Term of Impulse

Sampling Property

$$x[n]\cdot\delta[n]=x[0]\cdot\delta[n]$$

$$x[n]\cdot\delta[n-n_0]=x[n_0]\cdot\delta[n-n_0]$$





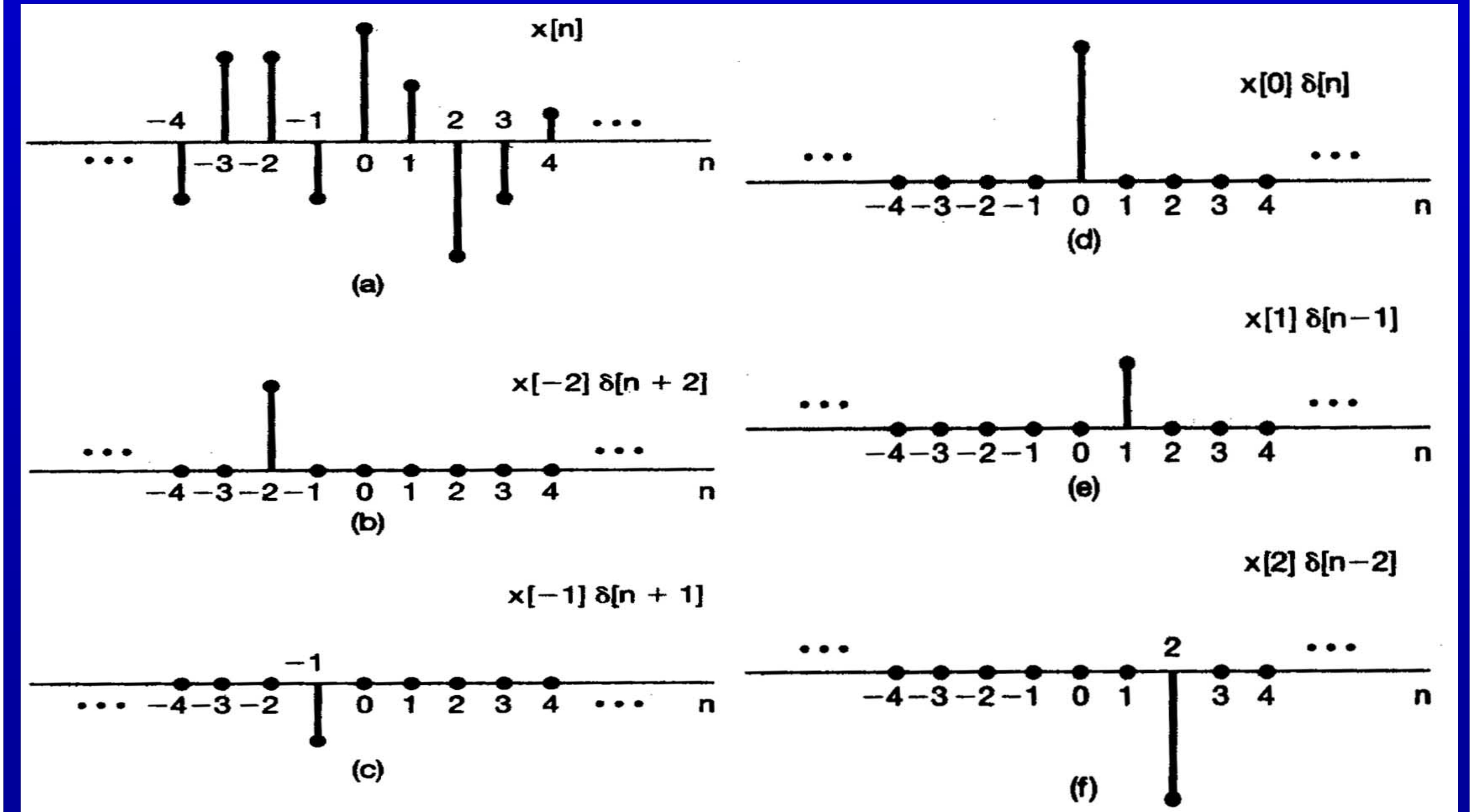
$$x[n] = x[0]\cdot\delta[n] + x[n_0]\cdot\delta[n-n_0]$$

$$=$$

$$x_1[n] = x[n]\cdot\delta[n] = x[0]\cdot\delta[n]$$

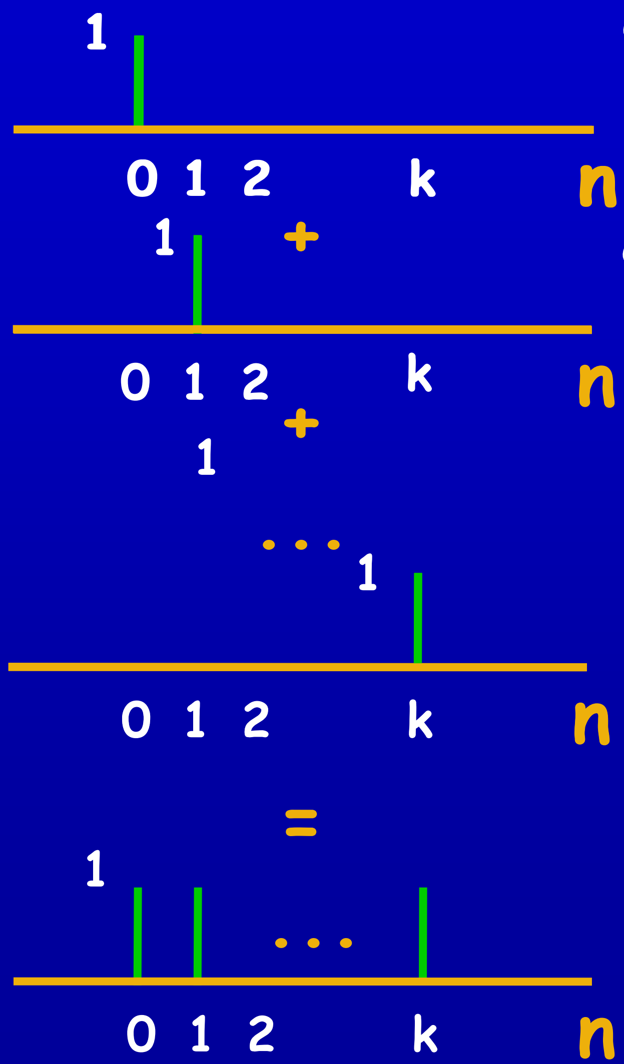
$$+$$

$$x_2[n] = x[n]\cdot\delta[n-n_0] = x[n_0]\cdot\delta[n-n_0]$$


x[n]
(a)
x[−2] δ[n + 2]
(b)
x[−1] δ[n + 1]
(c)
x[0] δ[n]
(d)
x[1] δ[n−1]
(e)
x[2] δ[n−2]
(f)
n

Sifting Property

$$x[n] = \cdots x[-2]\delta[n+2] + x[-1]\delta[n+1]$$
$$+ x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$
$$= \sum_{k=-\infty}^{k=+\infty} x[k]\cdot\delta[n-k]$$



$$\delta[n]$$

$$\delta[n-1]$$

$$\delta[n-k]$$

$$u[n] = \sum_{k=0}^{+\infty} \delta[n-k]$$

$$u[n] = \sum_{k=0}^{+\infty} \delta[n-k]$$

$$u[-n-1] = \sum_{k=-\infty}^{-1} \delta[n-k]$$

2.1.2 The Discrete-Time Convolution Sum

$$x[n]=\sum_{k=-\infty}^{k=+\infty} x[k]\cdot\delta[n-k]$$

$$\downarrow \textit{linear}, \textit{time}\ \text{invariant}$$

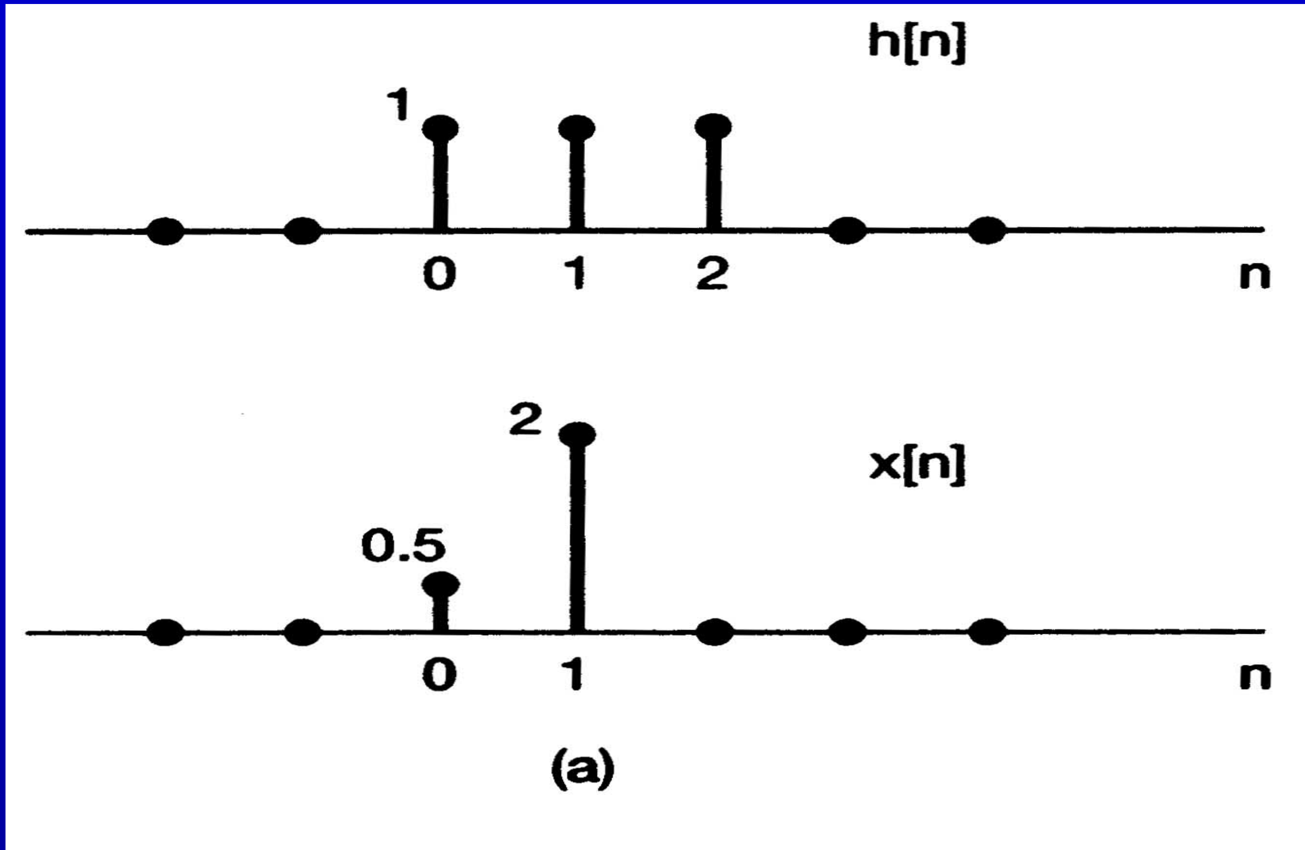
$$\textit{let}\ \ \delta[n-k]\rightarrow h[n-k]$$

$$y[n]=\sum_{k=-\infty}^{k=+\infty} x[k]\cdot h[n-k]$$

LTI system is completely characterized by its response of the unit impulse

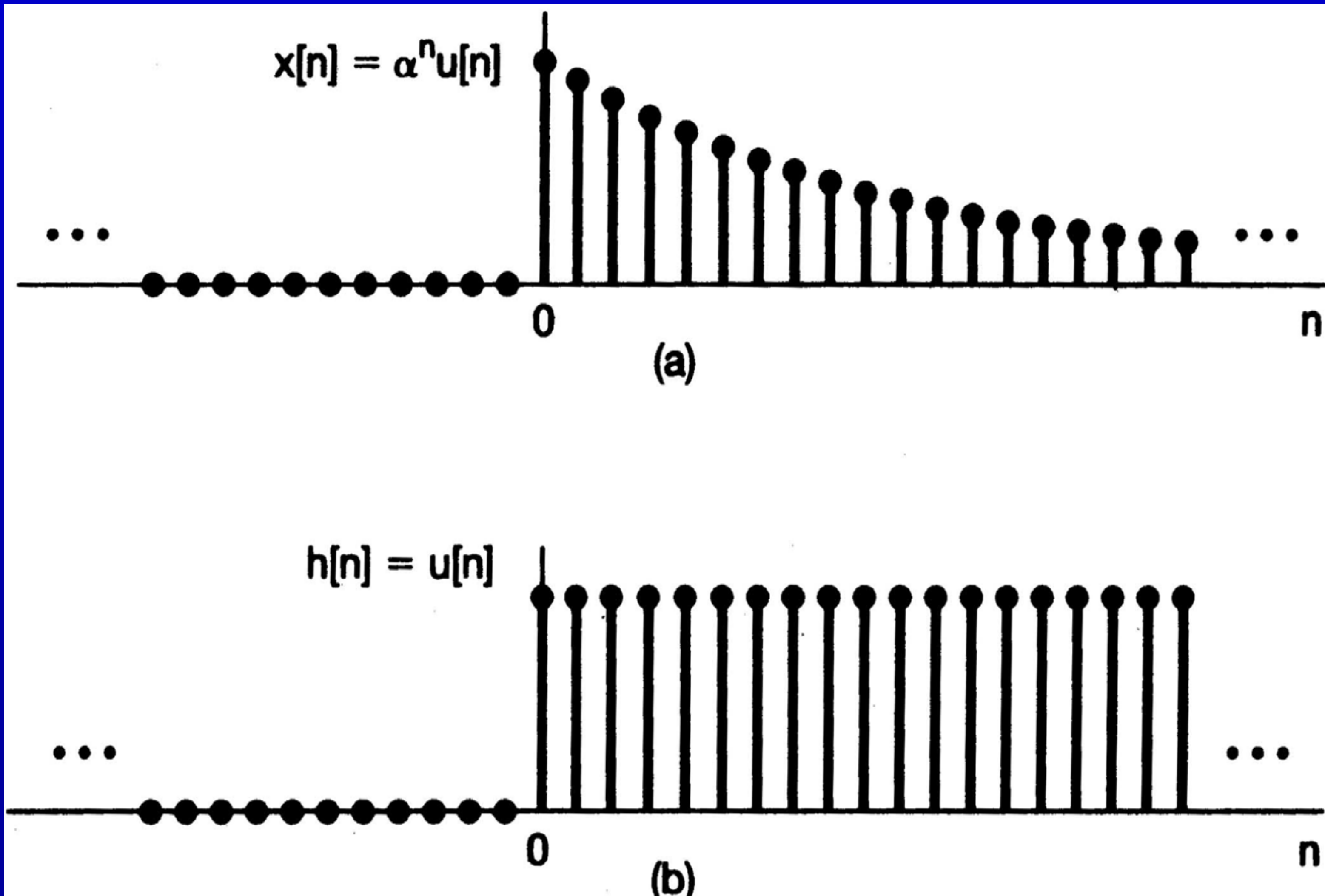
$$y[n]=x[n]*h[n]$$ *Convolution Sum*

Example 2.1、2.2

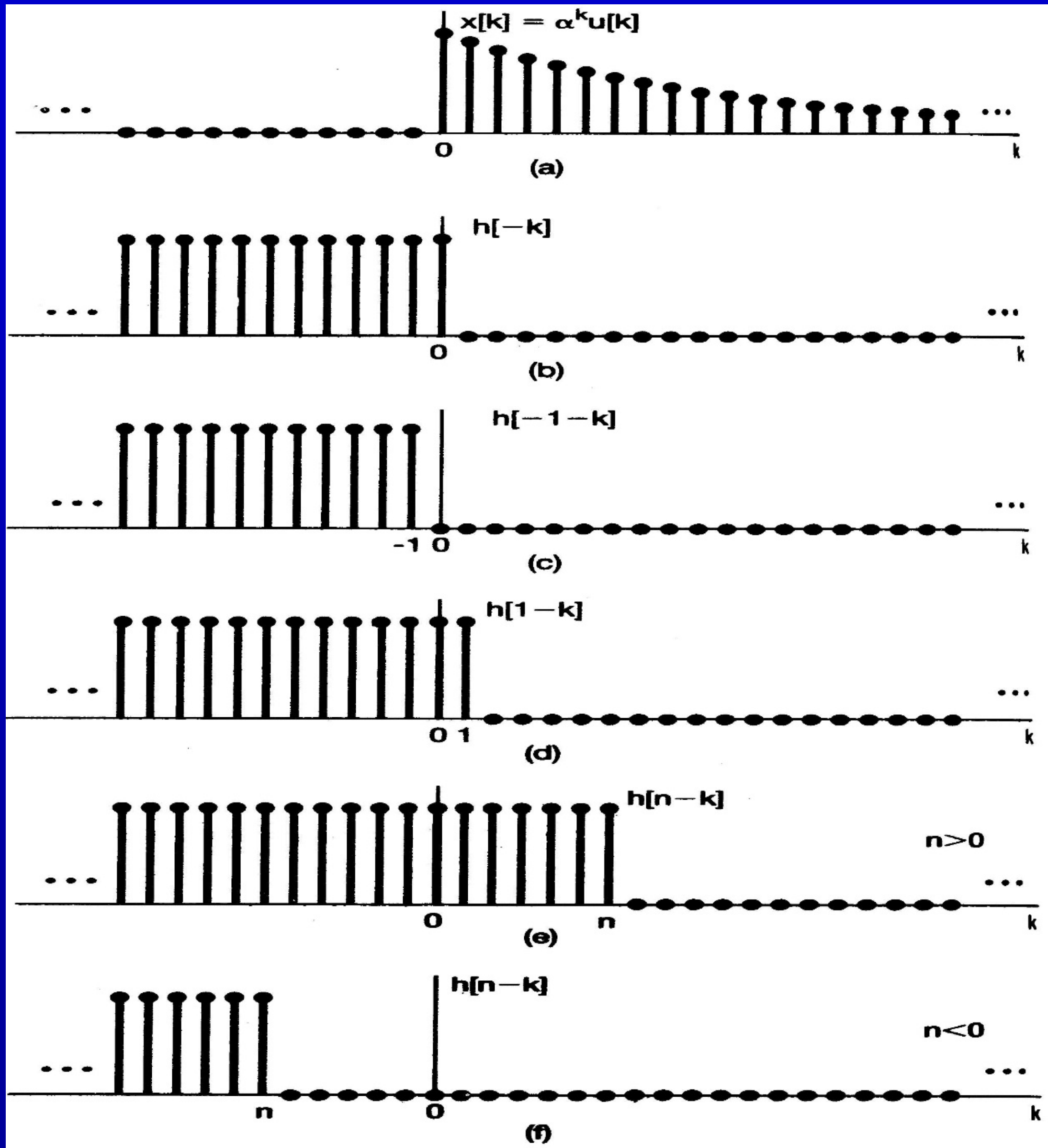


Determine y[n]

Example 2.3



Signals & Systems



Signals & Systems

Example 2.4

$$x[n]=\begin{cases}1 & 0\le n\le 4\\ 0 & otherwise\end{cases}$$

$$h[n]=\begin{cases}\alpha^{n} & 0\le n\le 6\\ 0 & otherwise\end{cases}\qquad \alpha>1$$

determine $y[n]$

Example 2.5

$$x[n] = 2^n u[-n] \quad and$$

$$h[n] = u[n]$$

determine $y[n]$

Example

$$x[n] * \delta[n]$$

$$x[n] * \delta[n-n_0]$$

Example

$$x[n] = u[n+1] - 2u[n-1] + u[n-2]$$

$$h[n] = \delta[n+1] + \delta[n-1]$$

determine $y[n] = x[n] * h[n]$

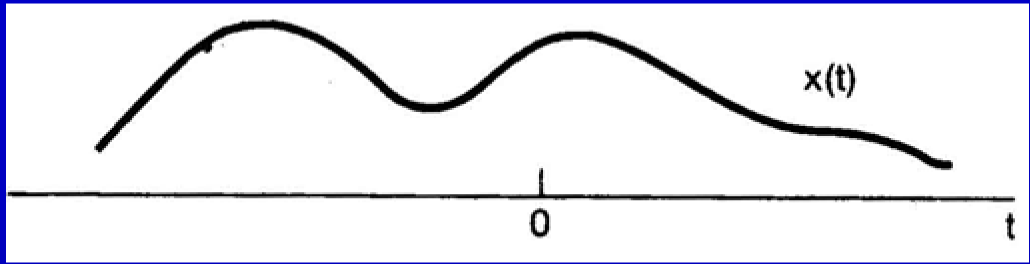
$a)\, y[n] = \{1, 0, 1, 0, -1\}, n = -2, -1, 0, 1, 2$

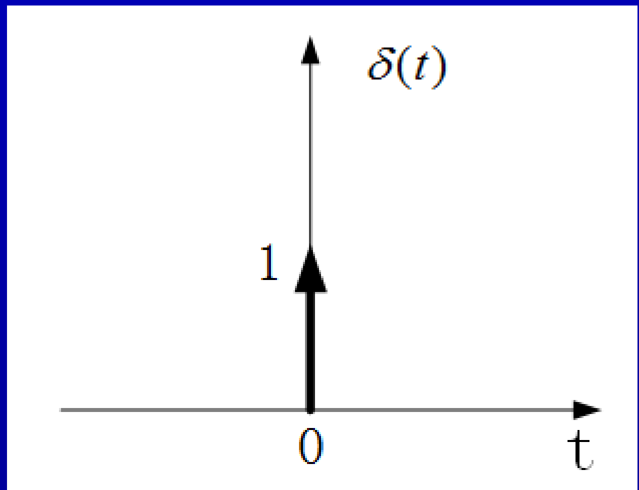
√ $b)\, y[n] = \{1, 1, 0, 1, -1\}, n = -2, -1, 0, 1, 2$

$c)\, y[n] = \{1, 1, 0, 1, -1\}, n = -1, 0, 1, 2, 3$

$d)\, y[n] = \{1, 0, -1, 0, -1\}, n = -2, -1, 0, 1, 2$

2.2 Continuous-Time LTI System: The Convolution Integral

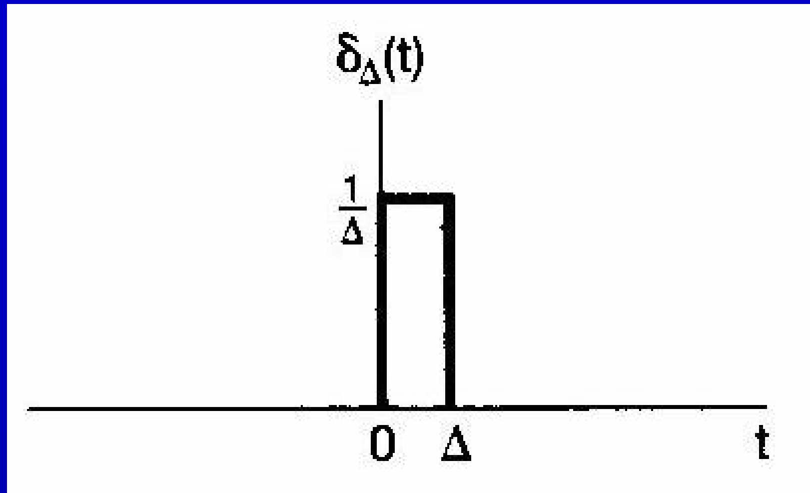


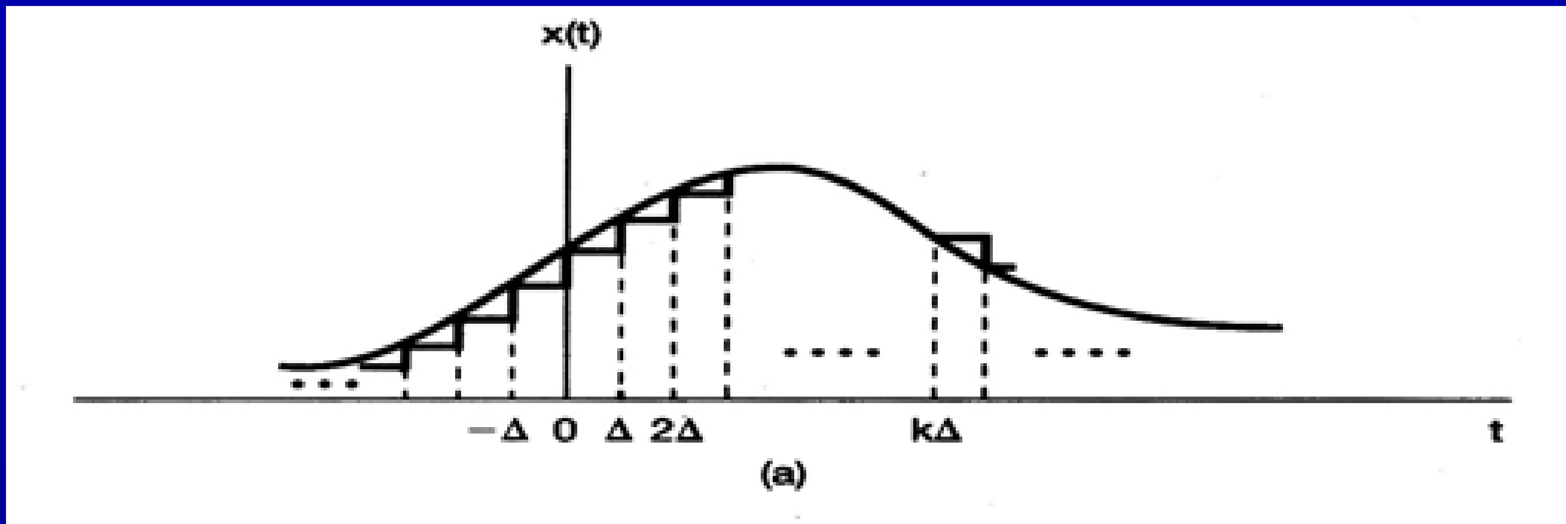


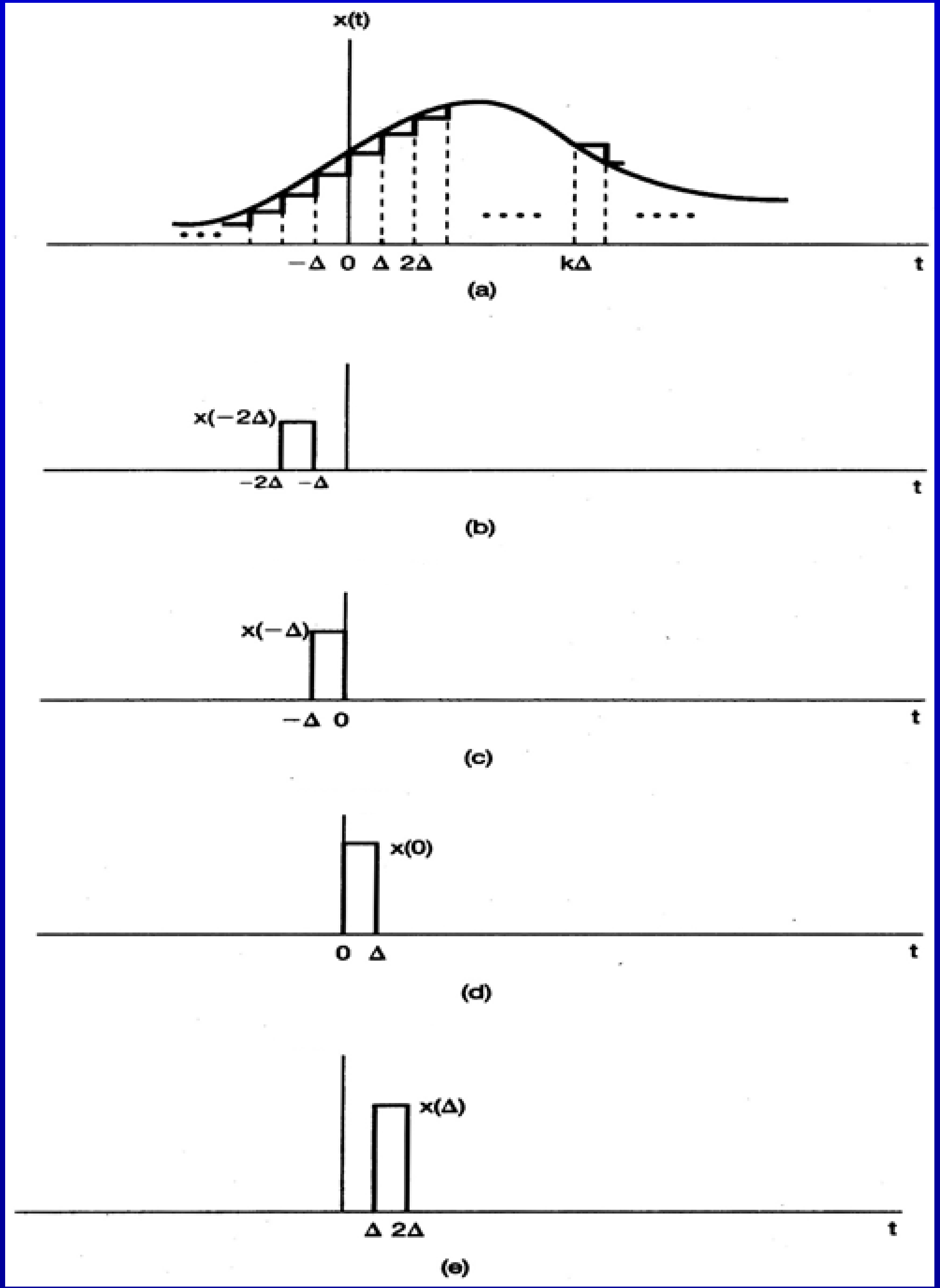
2.2.1 The Representation Continuous-Time Signals In Term Of Impulse

Signals & Systems

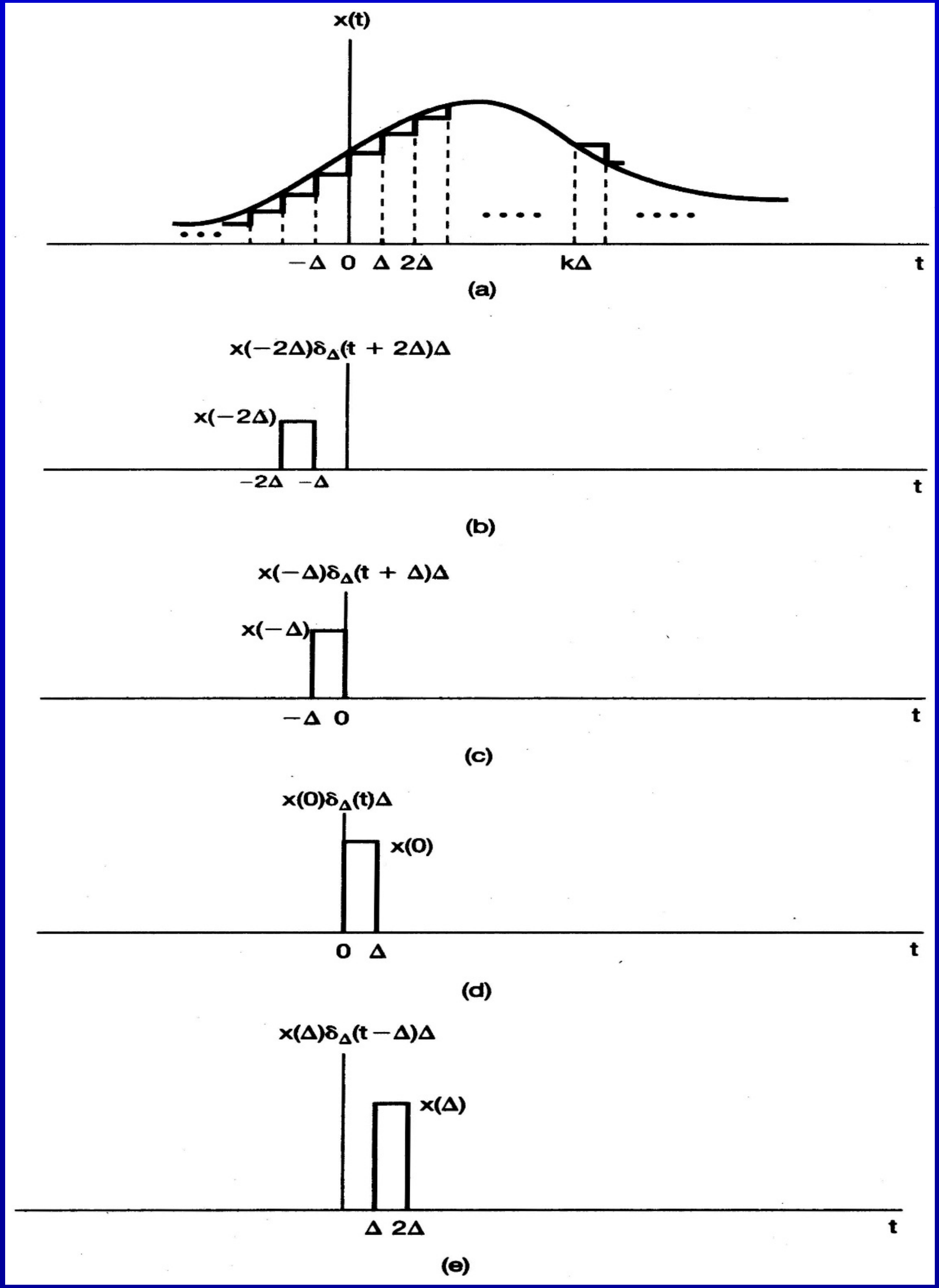
$$\delta_\Delta(t) = \begin{cases} \dfrac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & otherwise \end{cases}$$






x(t)
−Δ 0 Δ 2Δ
kΔ
t
(a)
x(−2Δ)
−2Δ −Δ
t
(b)
x(−Δ)
−Δ 0
t
(c)
x(0)
0 Δ
t
(d)
x(Δ)
Δ 2Δ
t
(e)

Signals & Systems



Signals & Systems

Sifting property

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x[k\Delta]\cdot\delta_{\Delta}(t-k\Delta)\cdot\Delta$$

$$\downarrow \Delta \to 0$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\cdot\delta(t-\tau)\cdot d\tau$$

$$u(t) = \int_{0}^{+\infty} \delta(t-\tau)d\tau$$

2.2.2 The Continuous-Time Unit Impulse Response And The Convolution Integral

Signals & Systems

$$\hat{x}(t)=\sum_{k=-\infty}^{k=+\infty} x(k\Delta)\cdot\delta_{\Delta}(t-k\Delta)\cdot\Delta$$

$$\downarrow \textit{linear time } \text{invariant},$$

$$\textit{let } \delta_{\Delta}(t-k\Delta)\rightarrow \hat{h}(t-k\Delta)$$

$$\hat{y}(t)=\sum_{k=-\infty}^{k=+\infty} x(k\Delta)\cdot \hat{h}(t-k\Delta)\cdot\Delta$$

$$\downarrow \Delta\rightarrow 0$$

$$y(t)=\lim_{\Delta\rightarrow 0}\sum_{k=-\infty}^{k=+\infty} x(k\Delta)\cdot \hat{h}(t-k\Delta)\cdot\Delta$$

$$=\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t)$$

Convolution integral

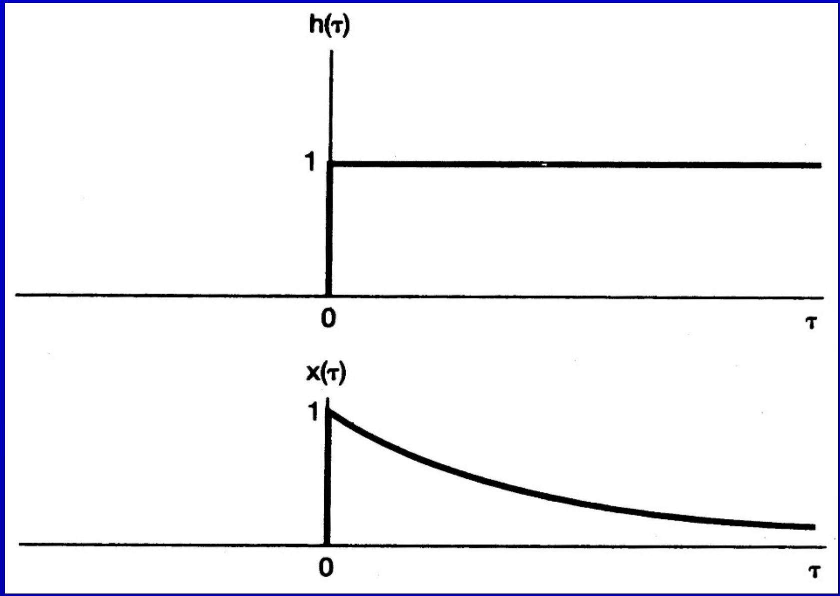
Example 2.6

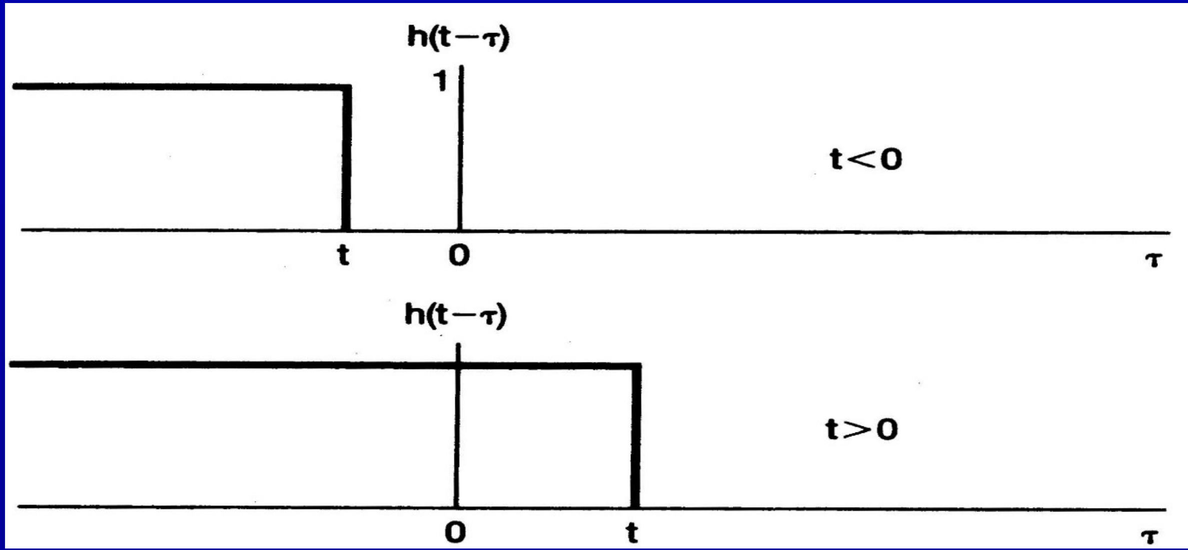
Signals & Systems

$$x(t) = e^{-at}u(t) \quad a>0$$

$$h(t) = u(t)$$

determine $y(t)$



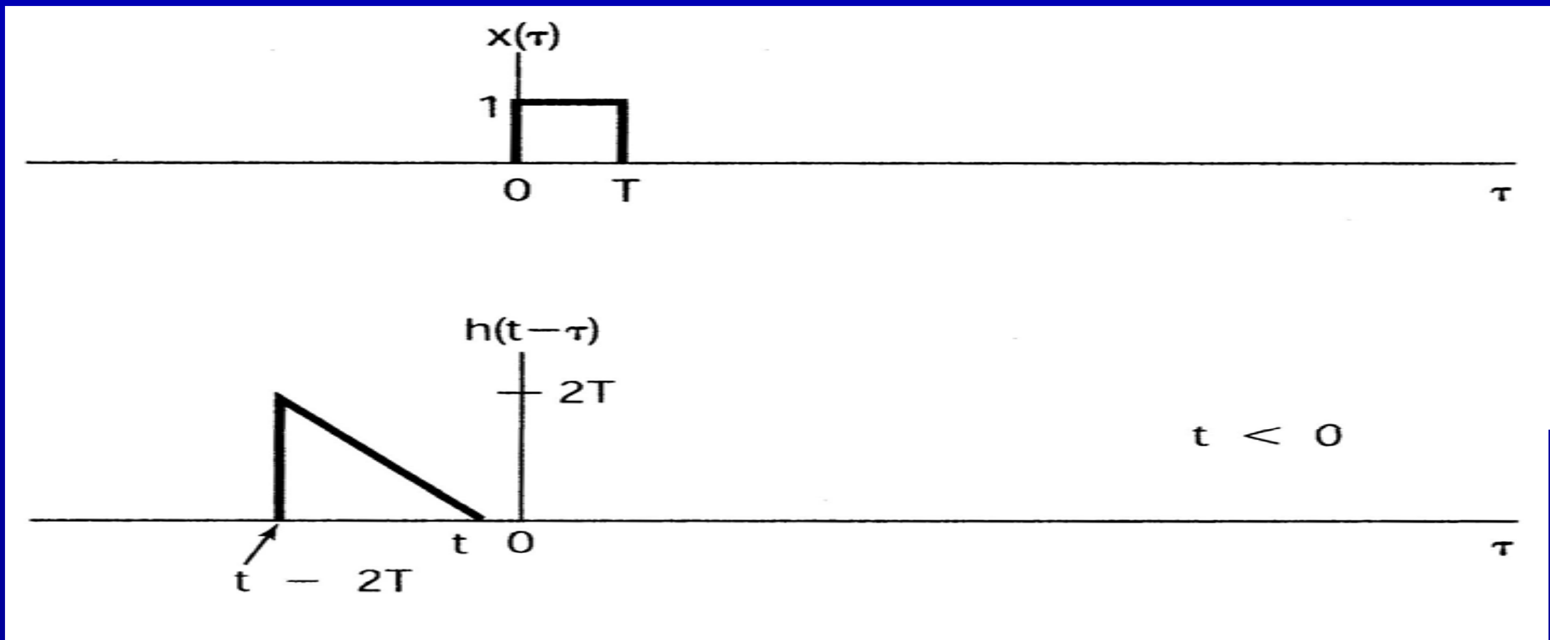


Signals & Systems

Example 2.7

$$x(t)=\begin{cases}1 & 0<t<T\\ 0 & otherwise\end{cases} \quad and \quad h(t)=\begin{cases}t & 0<t<2T\\ 0 & otherwise\end{cases}$$

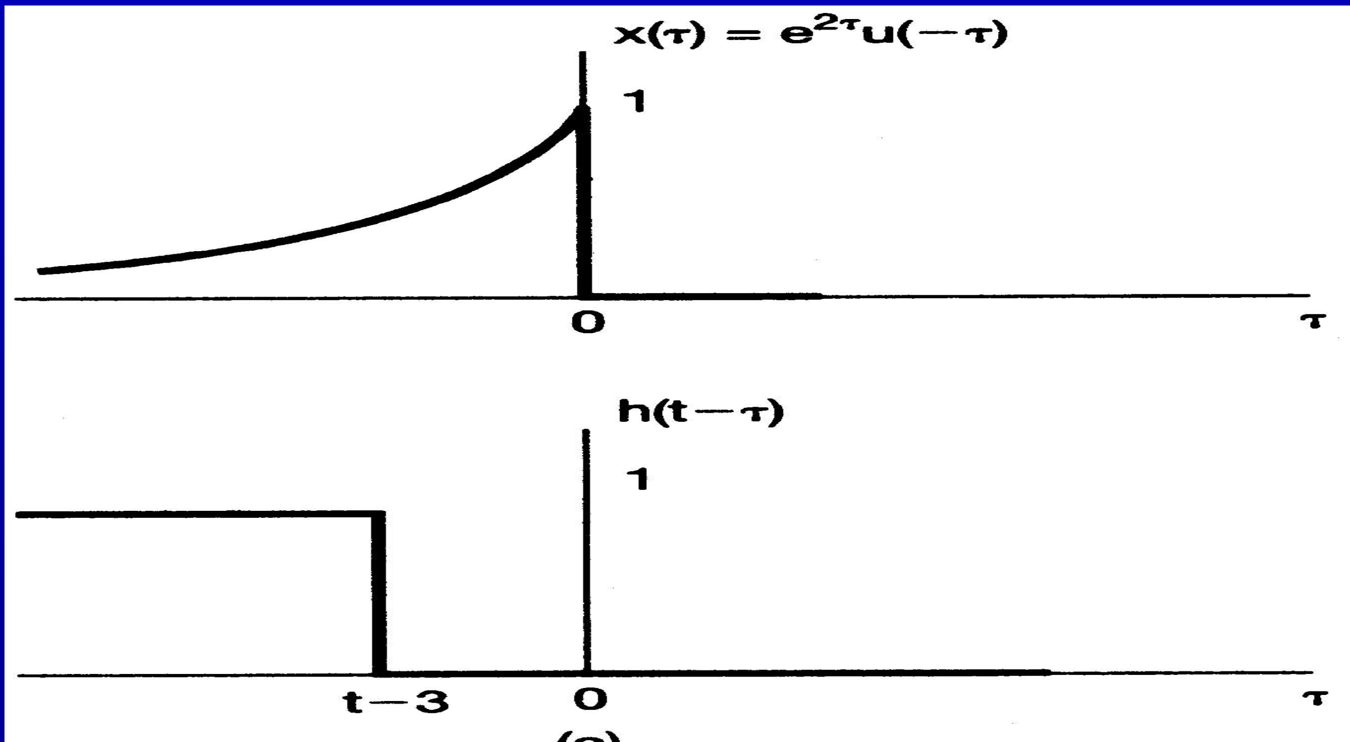
determine $y(t)$



Example 2.8

$$x(t) = e^{2t}u(-t) \quad and \quad h(t) = u(t-3)$$

determine $y(t)$



Example

$$x(t) * \delta(t)$$

$$x(t) * \delta(t - t_0)$$

Example

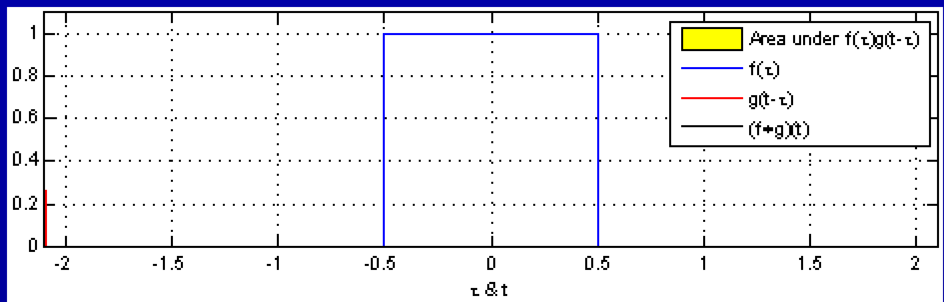
1、$y(t)=[u(t)-u(t-1)]*[u(t/2)-u(t/2-1)]=?$

$a)\, tu(t)-(t-1)u(t-1)-2(t-2)u(t-2)+(t-3)u(t-3)$

$b)\, tu(t)-2(t-1)u(t-1)-(t-2)u(t-2)+(t-3)u(t-3)$

$c)\, tu(t)-(t-1)u(t-1)-(t-2)u(t-2)+(t-3)u(t-3)$ ✓

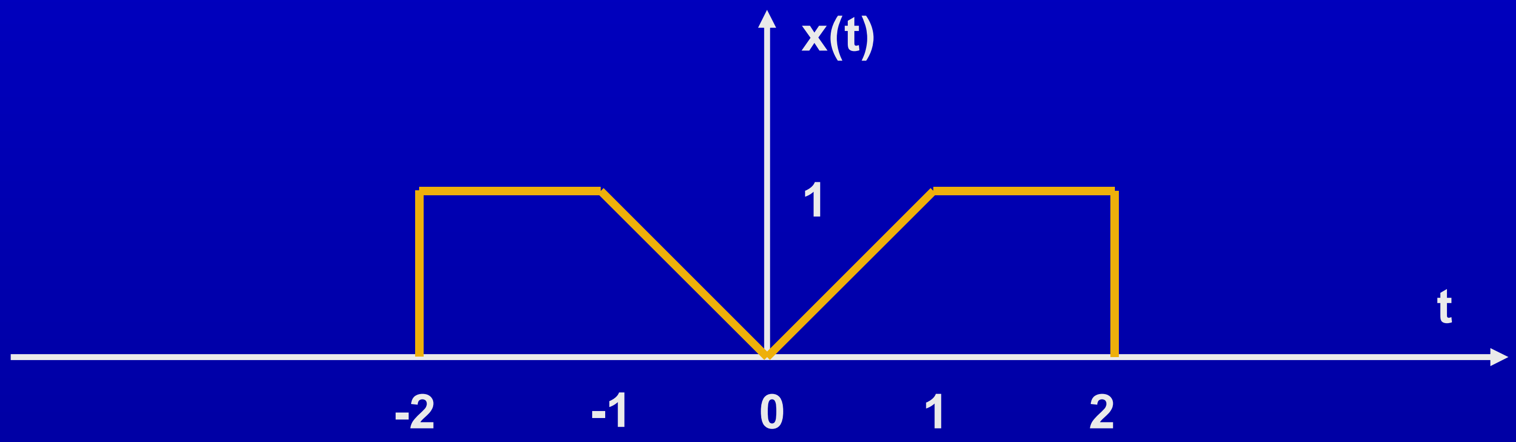
$d)\, tu(t)-(t-1)u(t-1)-(t-2)u(t-2)-(t-3)u(t-3)$



Example

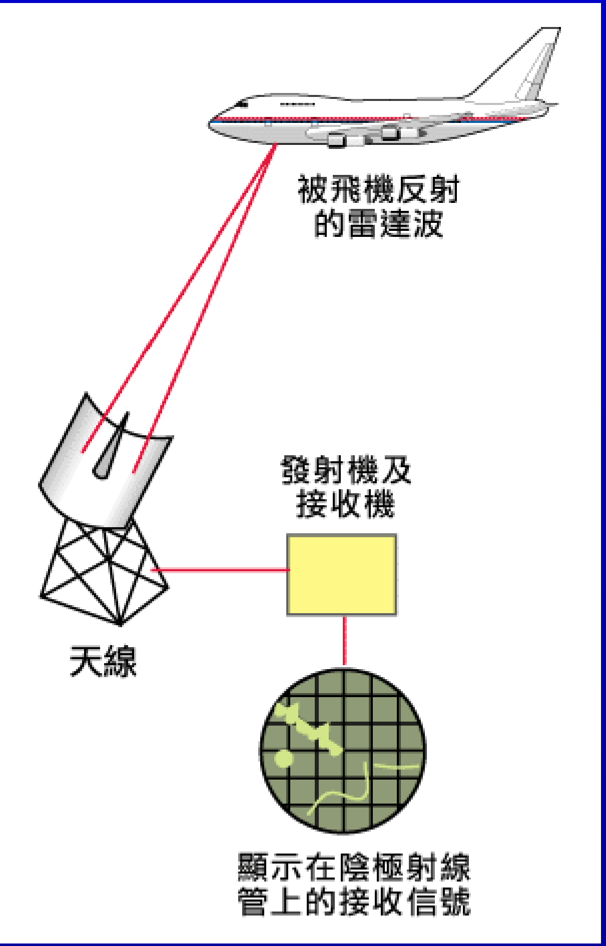
$$h(t) = u(t-1) - u(t-2),$$

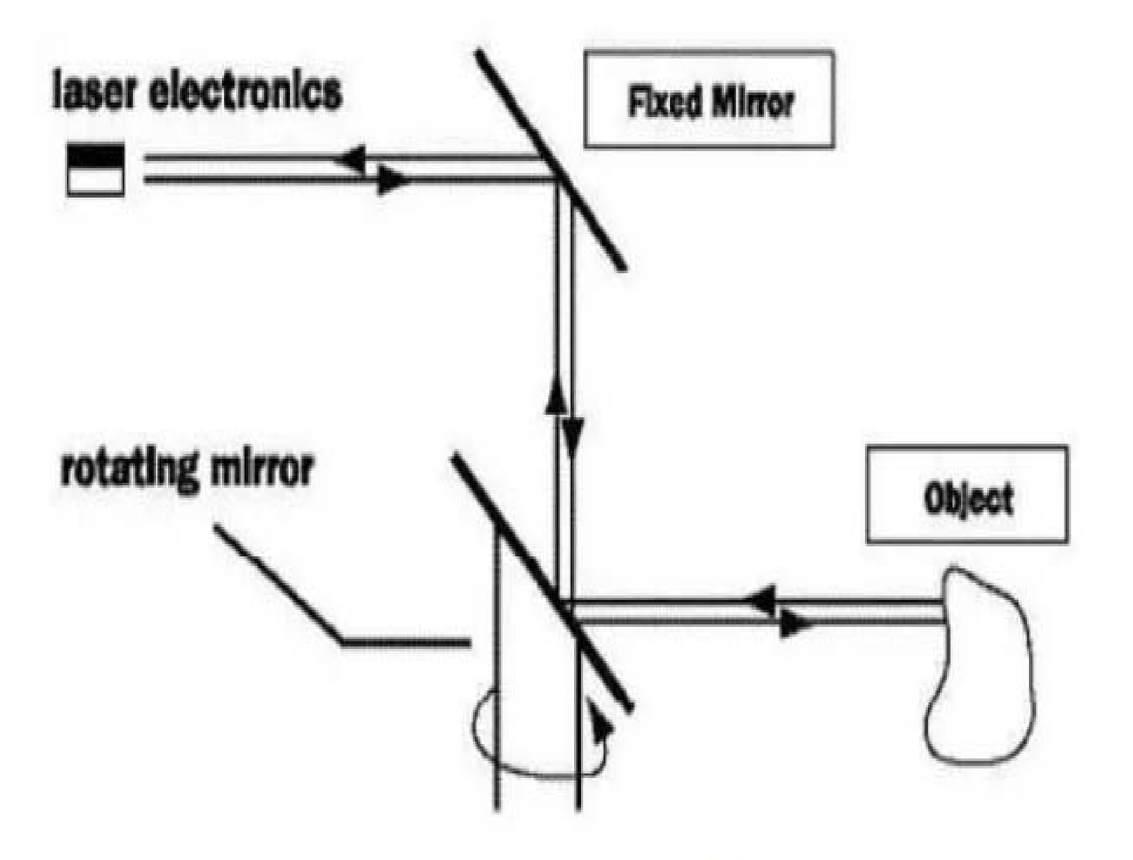
$$x(t) \text{ as below, the output } y(t)\big|_{t=0} = 1$$



Signals & Systems

Application





激光雷达工作原理（图片来自Velodyne）

Signals & Systems

Application

Signals & Systems

Application

探测主机

部分探测天线

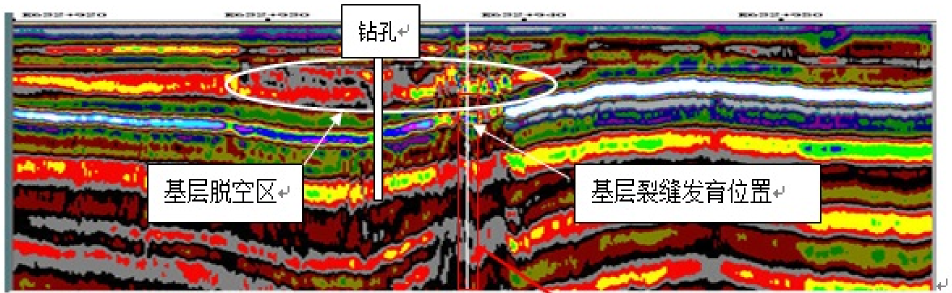


图 1 郑洛高速郑州段 K632+920～K632+956 雷达剖面图（500MHz、4.0m）

路基土探测

路面检测

车载路面状况连续检测

图 2 钻孔结果 （面层下部破碎且存在空洞）

图 3 开挖显示的基层裂缝

2.3 Properties Of LTI System

Example 2.9

$$h[n] = \begin{cases} 1 & n = 0,1 \\ 0 & otherwise \end{cases}$$

LTI system

$$y[n] = x[n] + x[n-1]$$

non - LTI system

$$y[n] = \begin{cases} \{x[n] + x[n-1]\}^2 \\ \max\{x[n], x[n-1]\} \\ x^2[n] + x^2[n-1] \end{cases}$$

☺LTI system are completely determined by its impulse response

$$y[n] = x[n] * h[n] \qquad \textit{Convolution Sum}$$

$$y(t) = x(t) * h(t) \qquad \textit{Convolution Integeral}$$

☺The unit impulse response of a nonlinear system does not completely characterize the behavior of the system

2.3.1 The Commutative Property

$$x[n]*h[n]=h[n]\ *x[n]=\sum_{k=-\infty}^{+\infty}h[k]x[n-k]$$

$$x(t)*h(t)=h(t)\ *x(t)=\int_{-\infty}^{+\infty}h(\tau)x(t-\tau)d\tau$$

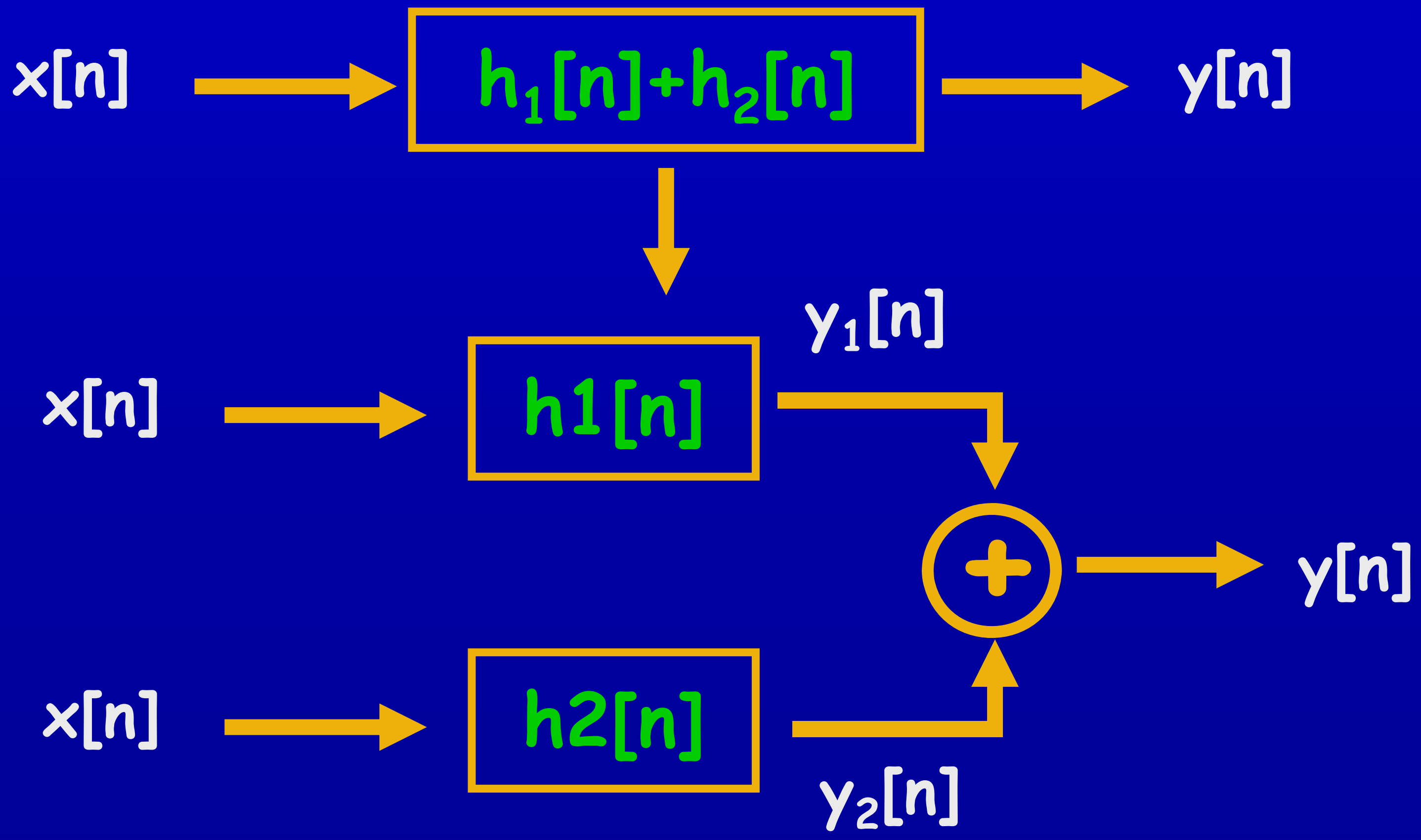
Application:

- ✓ Signal ←→System
- ✓ One form for computing convolution may be easier.

2.3.2 The Distributive Property

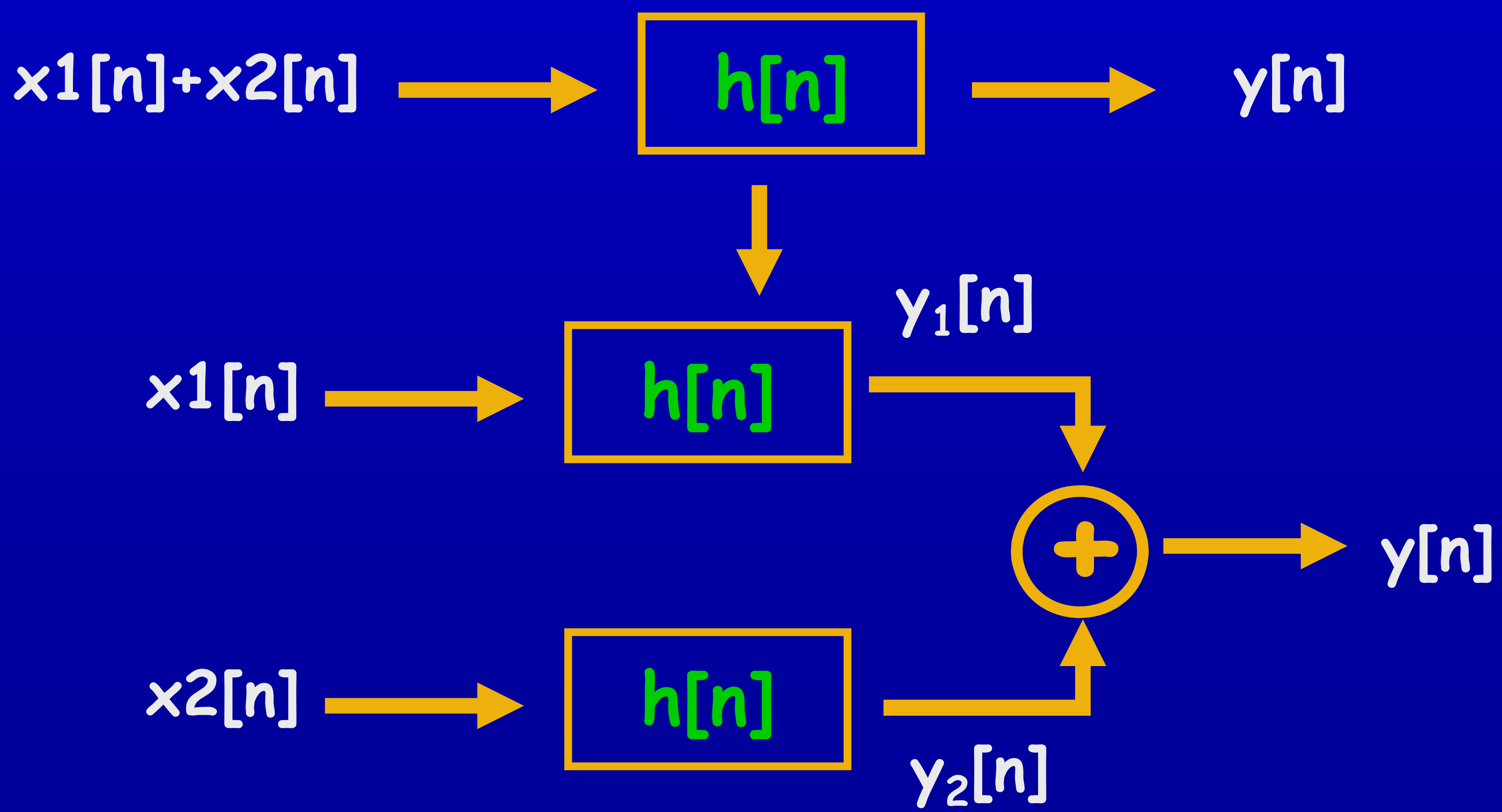
$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$$

$$x(t)*(h_1(t)+h_2(t)) = x(t)*h_1(t)+x(t)*h_2(t)$$



$$(x_1[n]+x_2[n])*h[n]=x_1[n]*h[n]+x_2[n]*h[n]$$

$$(x_1(t)+x_2(t))*h(t)=x_1(t)*h(t)+x_2(t)*h(t)$$



Example 2.10

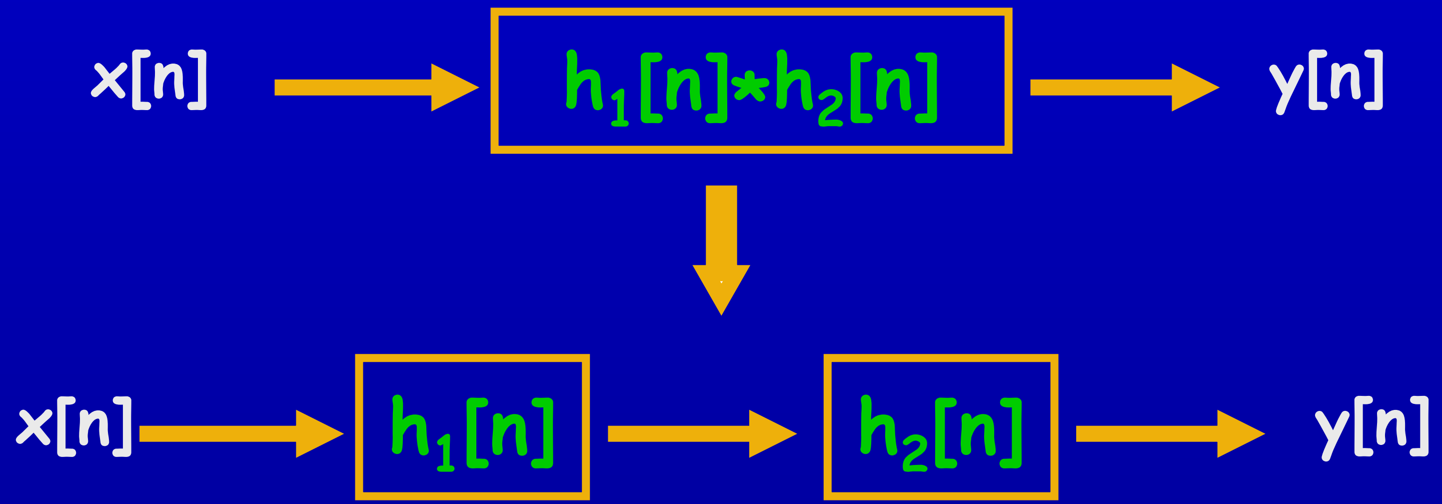
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$$

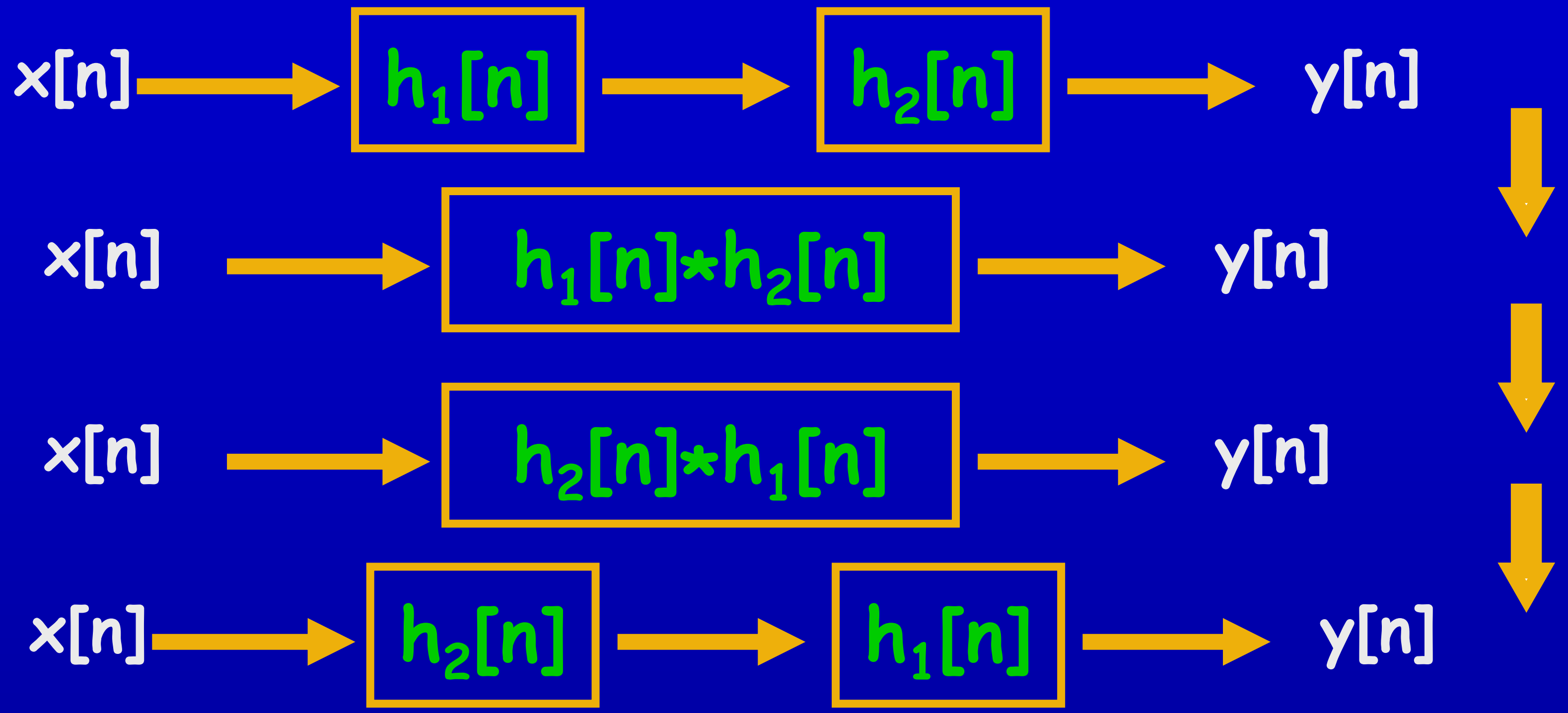
$$h[n] = u[n]$$

2.3.3 The Associative Property

$$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$$

$$x(t)*[h_1(t)*h_2(t)] = [x(t)*h_1(t)]*h_2(t)$$





Example $y_1[n] = x^2[n]$ $y_2[n] = 2x[n]$

Signals & Systems

2.3.4 LTI Systems with and without memory

✓ Discrete-time system without memory

$$h[n] = 0 \ for \ n \neq 0$$

$$h[n] = k\delta[n]$$

$$y[n] = kx[n]$$

$$eg. \ \ h[n] = \begin{cases} 1 & n = 0,1 \\ 0 & otherwise \end{cases} \qquad memory \ system$$

✓ Continuous-time system without memory

$$h(t) = k\delta(t)$$

$$y(t) = kx(t)$$

Example: y(t)=x(t)+x(t-1)

2.3.5 Invertibility of LTI System

$$x(t) \longrightarrow \underbrace{\boxed{h(t)} \xrightarrow{y(t)} \boxed{h_1(t)}}_{\delta(t)\ \ \textit{Identity System}} \longrightarrow w(t)=x(t)$$

$$h(t)*h_1(t)=\delta(t)$$

$$h[n]*h_1[n]=\delta[n]$$

Example 2.11

$$y(t)=x(t-t_0) \qquad h(t)=\delta(t-t_0)$$

Example 2.12

$$h[n]=u[n] \quad \textit{accumulator}$$

2.3.6 Causality of LTI System

$$h[n]=0 \quad for\ n<0$$

$$y[n]=\sum_{k=-\infty}^{+\infty}x[k]\cdot h[n-k]=\sum_{k=0}^{+\infty}h[k]\cdot x[n-k]$$

Example

$$accumulator \quad h[n]=u[n]$$

$$its\ inverse \quad h[n]=\delta[n]-\delta[n-1]$$

$$delay \quad h(t)=\delta(t-t_0) \quad t_0>0$$

$$y[n]=2x[n]+3$$

$$not\ linear, memoryless(causal)$$

Initial rest(初始松弛)

$$if \quad x[n]=0 \quad for\ n<n_0, then \quad y[n]=0 \quad for\ n<n_0$$

causal for a linear system ←→initial rest

Causal Signals

- ✓ The signal which is zero if n<0 or t<0
- ✓ Causality of an LTI system is equivalent to its impulse response being a causal signal.

Example

$$y[n] = 2x[n]$$

$$y[n] = 3x[n-4]$$

$$y[n] = x[n] - 7$$

2.3.7 Stability of LTI System

Signals & Systems

Every bounded input produces a bounded output.

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty \qquad \textit{absolutely summable}$$

$$\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty \qquad \textit{absolutely } \textbf{integrable}$$

$$if\ |x[n]| < B \ \ for\ all\ n$$

$$then\ |y[n]| = \left|\sum_{k=-\infty}^{+\infty} h[k]\cdot x[n-k]\right|$$

$$\le \sum_{k=-\infty}^{+\infty} |x[n-k]|\cdot|h[k]| \ \le B\sum_{k=-\infty}^{+\infty} |h[k]|$$

Example 2.13

determine the stabilities of the systems below:

$\delta[n-n_0]$、 $u[n]$

2.3.8 The Unit Step Response of LTI System

Signals & Systems

- Relationship of the impulse response and the unit step response

$$s[n] = u[n] * h[n] = h[n] * u[n]$$

$$s[n] = \sum_{k=-\infty}^{n} h[k] \qquad h[n] = s[n] - s[n-1]$$

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau \qquad h(t) = \frac{ds(t)}{dt}$$

- The unit step response can also used to characterize an LTI system. Since we can calculate the unit impulse response from it .

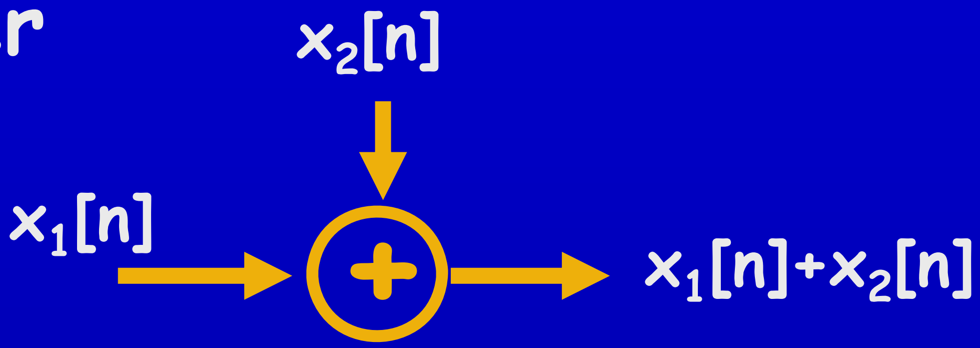
2.4.3 Block Diagram Representations of First-Order Systems Described by Differential and Difference Equations

The effect:

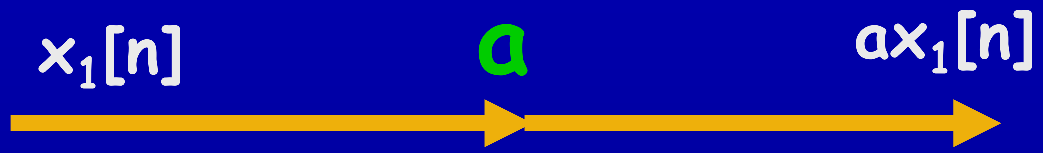
- ✓Pictorial representation.
- ✓Considerable value for the **simulation** or **implementation** of the system.

Discrete-time system

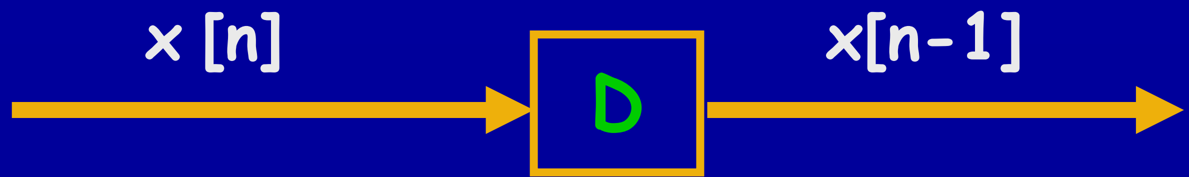
- adder



- multiplication by a coefficient



- unit delay

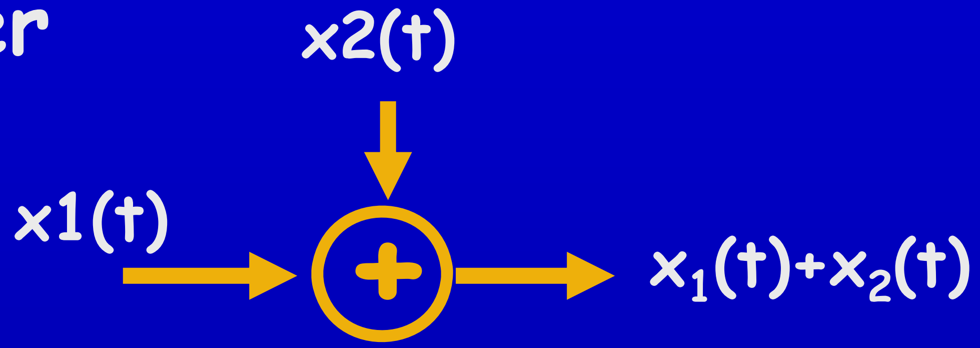


Example

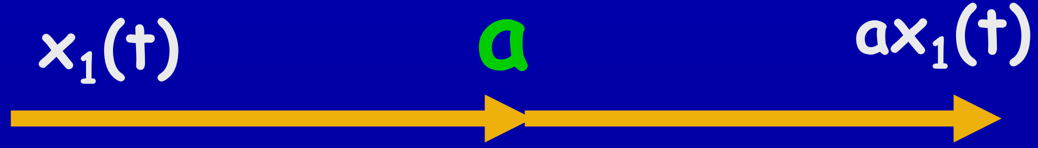
represent $y[n]=-ay[n-1]+bx[n]$

Continuous-time system

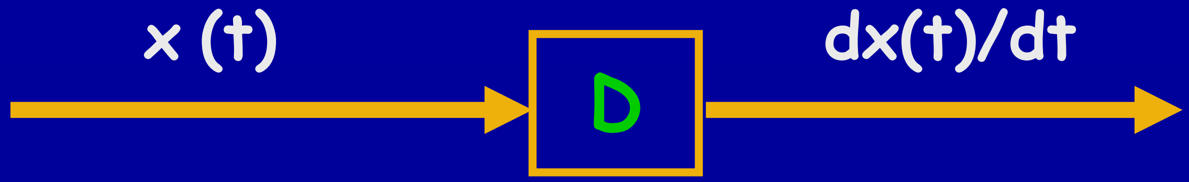
- adder



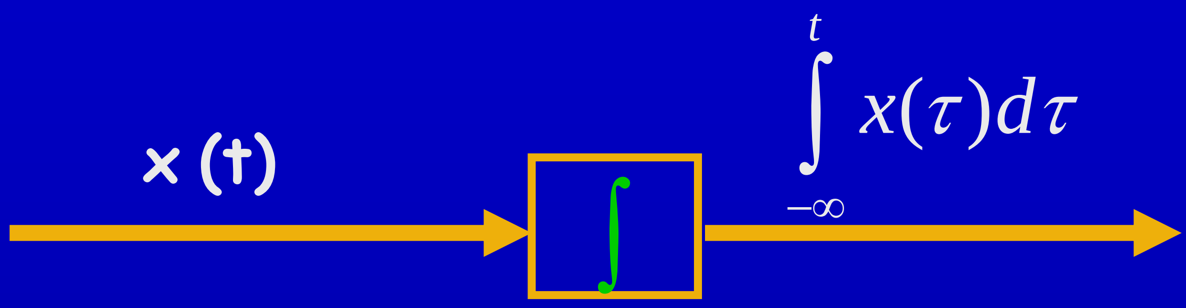
- multiplication by a coefficient



- differentiator



➢integrator



Example

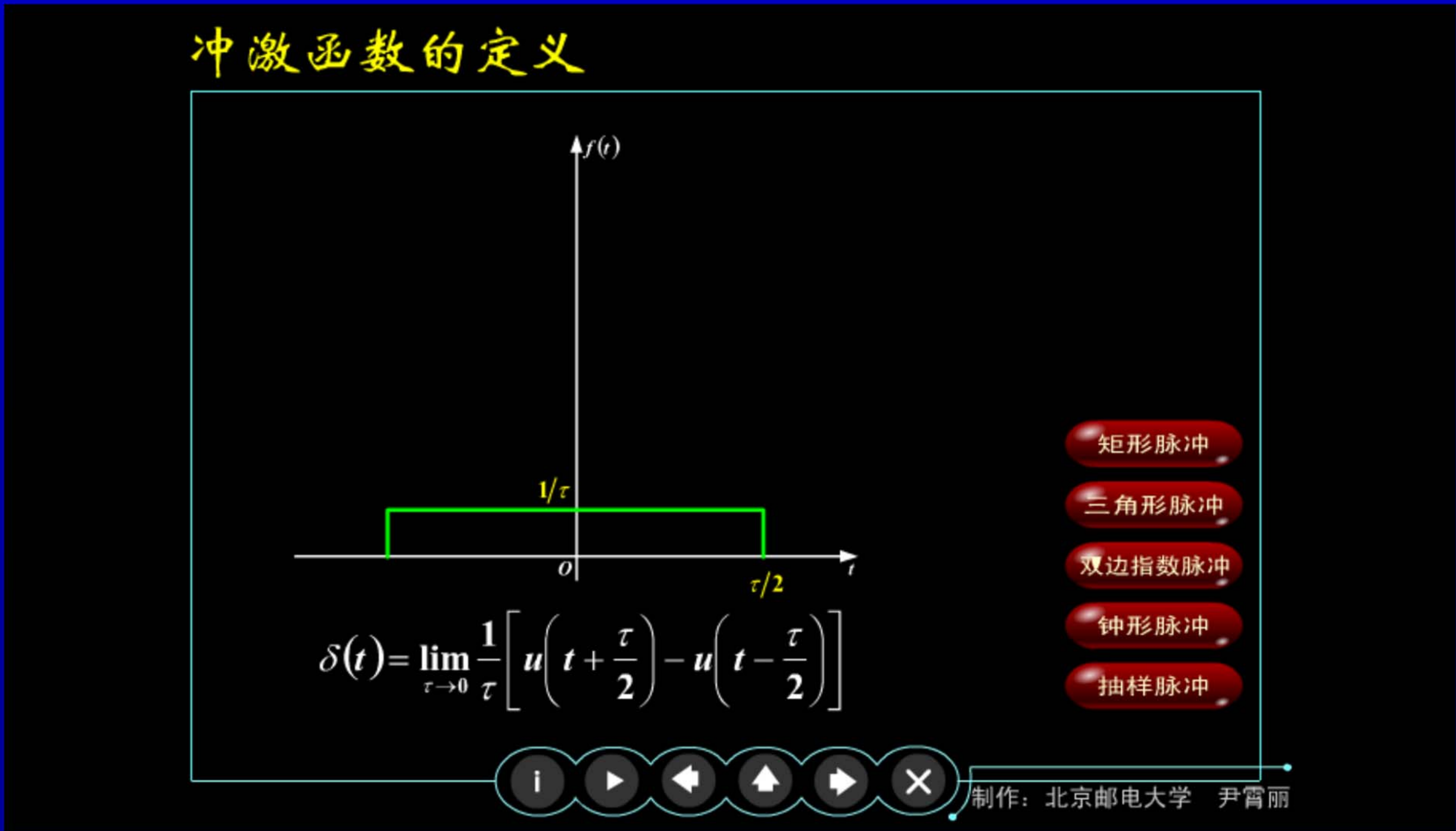
represent $$\frac{dy(t)}{dt} = bx(t) - ay(t)$$

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$

2.5 Singularity Functions

- Understand unit impulse function

  Idealization of a pulse :“short enough”



Signals & Systems

Singularity Functions

$$x(t) \longrightarrow \underbrace{\boxed{h(t)} \xrightarrow{y(t)} \boxed{h_1(t)}}_{\delta(t)\ \ \textit{Identity System}} \longrightarrow w(t) = x(t)$$

$$x(t) \longrightarrow \boxed{\delta(t)} \longrightarrow x(t)$$

$$\delta(t) \longrightarrow \boxed{\delta(t)} \longrightarrow h(t)$$

Singularity Functions

$$x(t) = x(t) * \delta(t)$$

$$=> \delta(t) = \delta(t) * \delta(t)$$

$$r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t) \to \delta(t)$$

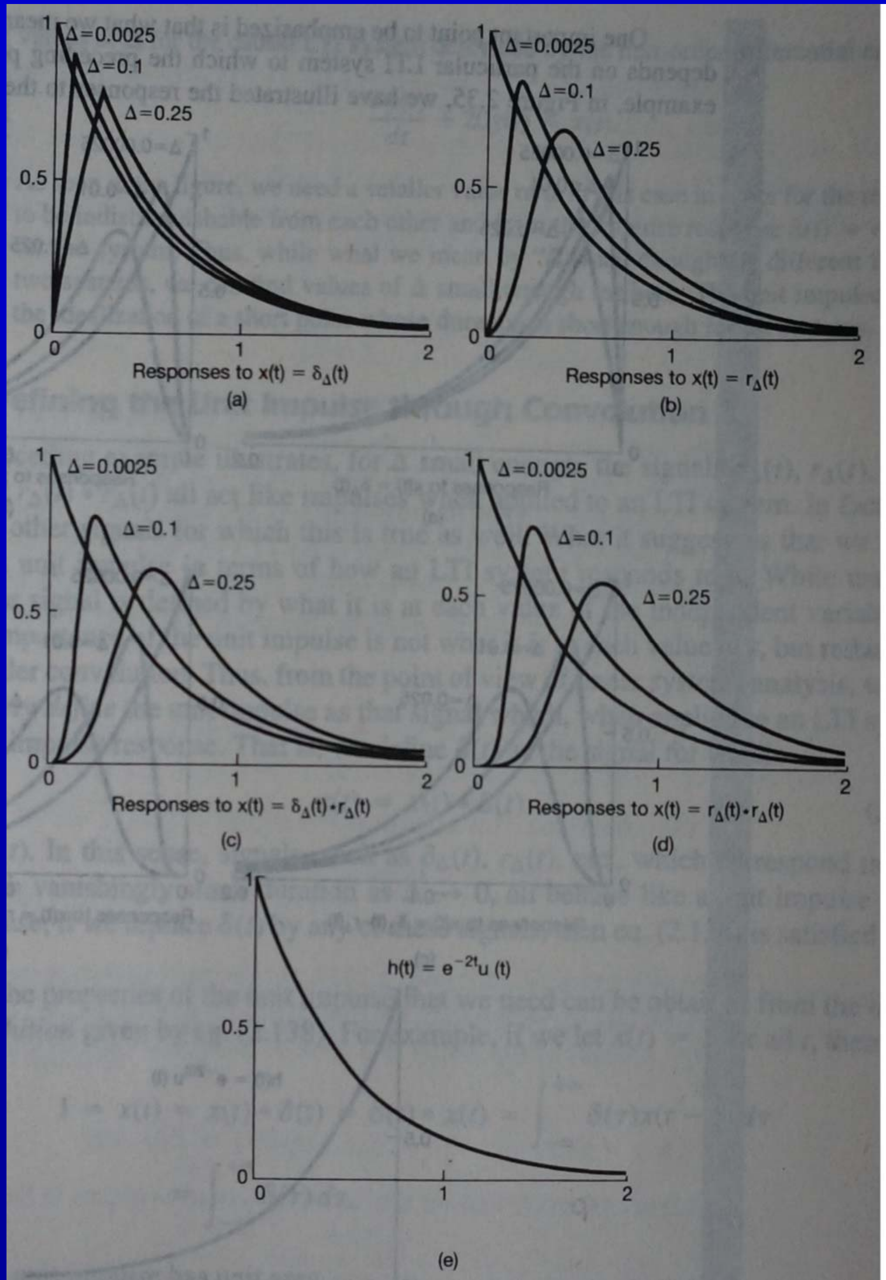
$$h(t) = \delta(t) \qquad u_0(t) \triangleq \delta(t)$$

Example 2.16

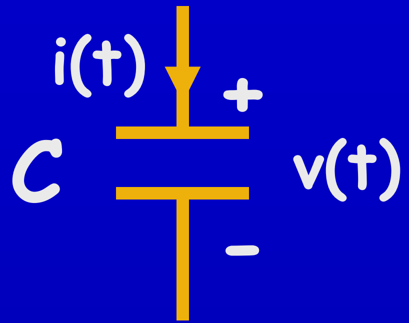
the response of $\frac{\mathrm{dy(t)}}{\mathrm{dt}} + 2y(t) = x(t)$

when inputs are $\delta_\Delta(t)$、$r_\Delta(t)$、$r_\Delta(t) * \delta_\Delta(t)$、$r_\Delta(t) * r_\Delta(t)$

Signals & Systems



2.5.3 Unit Doublets and Other Singularity Function



$$i(t) = C\frac{dv(t)}{dt}$$

$$\delta(t) \longrightarrow \boxed{dx(t)/dt} \longrightarrow u_1(t)=d\delta(t)/dt$$

- Unit Doublets $u_1(t)=d\delta(t)/dt$

$$x(t) \longrightarrow \boxed{dx(t)/dt} \longrightarrow x(t)*u_1(t)=x'(t)$$

➢K-th derivative of unit impulse

$$U_k(t) = U_1(t) * \cdots * U_1(t) \qquad k \ times$$

$$Operational\ Definition\ of\ U_k(t)$$

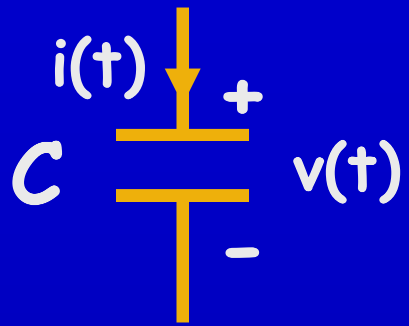
$$d^k x(t) / dt^k = x(t) * U_k(t) \quad k > 0$$

➢Unit doublets has zero area

$$\text{let } x(t) = 1 \text{ then}$$

$$dx(t)/dt = x(t) * \mathrm{U}_1(t)$$

$$= \int_{-\infty}^{+\infty} \mathrm{U}_1(\tau) x(t-\tau) d\tau = \int_{-\infty}^{+\infty} \mathrm{U}_1(\tau) d\tau = 0$$



$$v(t) = \frac{1}{c}\int_{-\infty}^{t} i(\tau)d\tau$$

$$x(t) \longrightarrow \boxed{\int_{-\infty}^{t} x(\tau)d\tau} \longrightarrow \int_{-\infty}^{t} x(\tau)d\tau$$

$$h(t) = u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad u_{-1}(t) \triangleq u(t)$$

$$u_{-k}(t) = \underbrace{u(t) * \cdots * u(t)}_{k\ times} = \int_{-\infty}^{t} u_{-(k-1)}(\tau)d\tau$$

$$u_1(t) = \frac{d\delta(t)}{dt} \qquad u_k(t) = \frac{d^k\delta(t)}{dt^k}$$

$$u_0(t) = \delta(t)$$

$$u_{-1}(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t) \qquad u_{-k}(t) = \int_{-\infty}^{t} u_{-(k-1)}(\tau)\,d\tau$$

$$u_{-1}(t) * u_1(t) = u_0(t) = \delta(t)$$

$$u_k(t) * u_r(t) = u_{k+r}(t)$$

Example 1

Consider an LTI system with unit impulse response

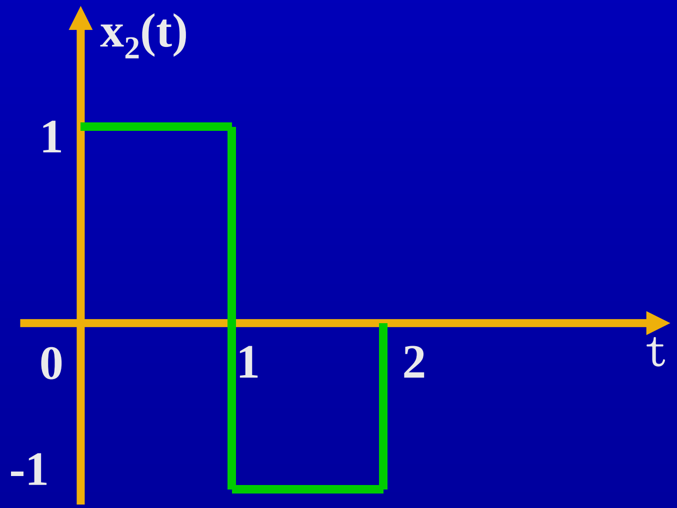
$h(t) = e^{-t}u(t)$, if the input $x(t) = \dfrac{d\delta(t)}{dt} + \delta(t)$,

the output $y(t)$ *is* $\delta(t)$

Example 2

Signals & Systems

Consider an LTI system whose response to the signal $x_1(t) = tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$ is the signal $y_1(t) = \cos tu(t)$. if input $x_2(t)$ is illustrated as below, the output $y_2(t)$ is



$$y_2(t) = -\sin tu(t) + \delta(t)$$

Signals & Systems

Example 3

Consider an LTI system with unit impulse response

$h(t) = \dfrac{d\delta(t)}{dt} + \delta(t)$, if the input $x(t)$ is illustrated as

below, the output $y(t)\big|_{t=3/2}$ is **0.5**