Chapter 7

Sampling

7.0 Introduction

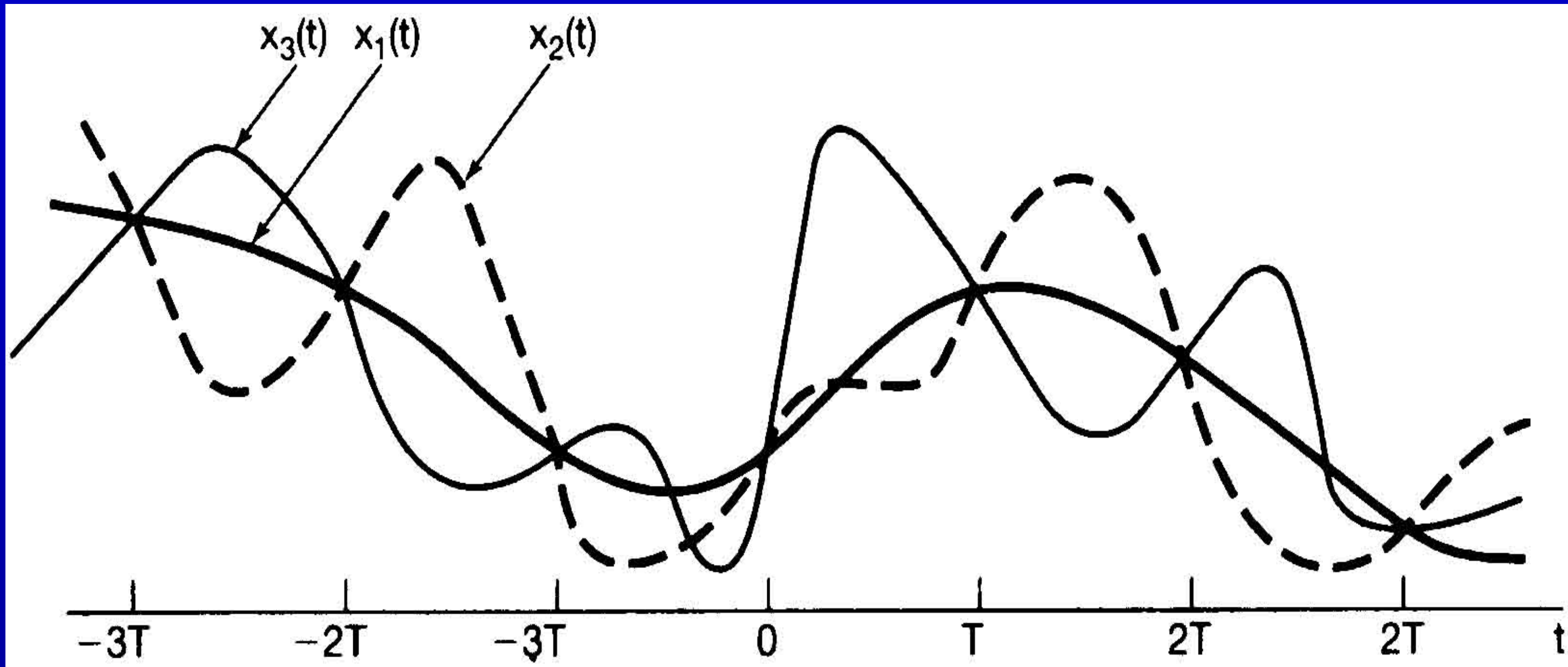
- A signal can be recovered from its samples under some condition

- Sampling is a bridge between continuous-time signal and discrete-time signal

Signals & Systems

7.1 Representation of a continuous-time signal by its samples: the sampling theorem

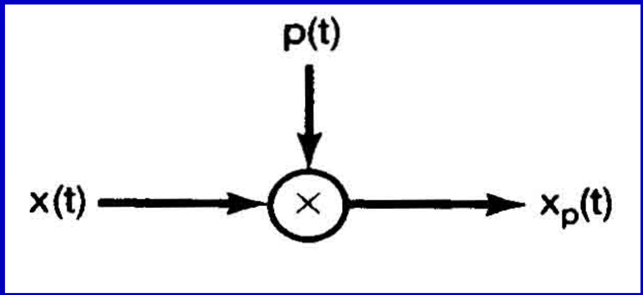
- The samples of the signals



the samples of the signals may be same

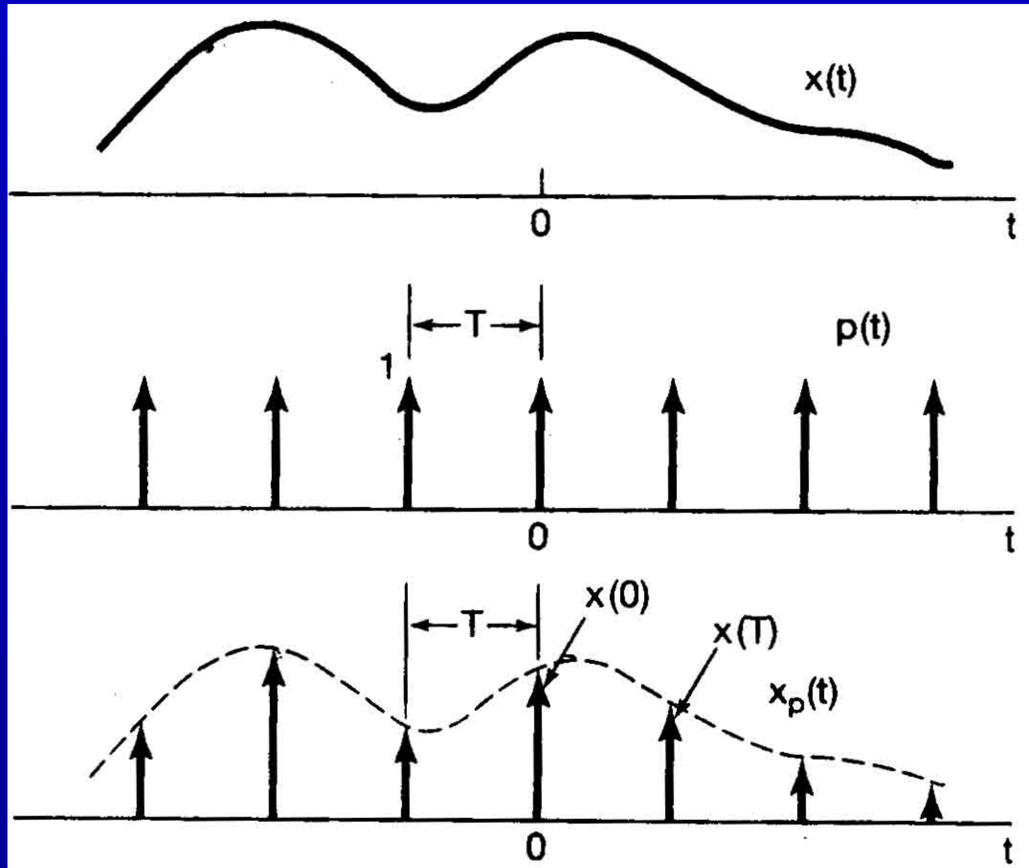
can not be recovered

7.1.1 Impulse-train sampling



$$p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$

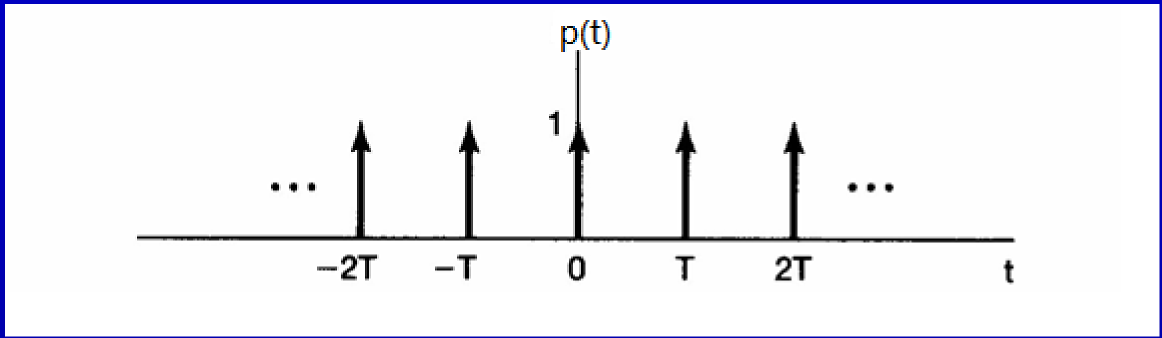
$$x_p(t)=\sum_{n=-\infty}^{+\infty}x(nT)\delta(t-nT)$$

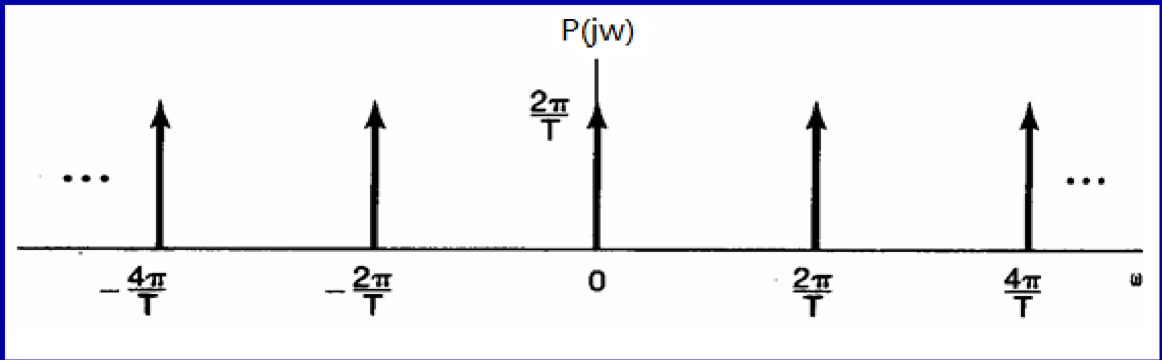


Signals & Systems

The spectrum of samples

$$p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)\xleftrightarrow{FT}\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_s)=P(j\omega)$$





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The spectrum of samples

$$p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)\xleftrightarrow{FT}\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_s)=P(j\omega)$$

$$x_p(t)=x(t)p(t)\xleftrightarrow{FT}\frac{1}{2\pi}X(jw)*P(jw)=X_P(j\omega)$$

$$X_P(j\omega)=\frac{1}{2\pi}\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_s)*X(jw)$$

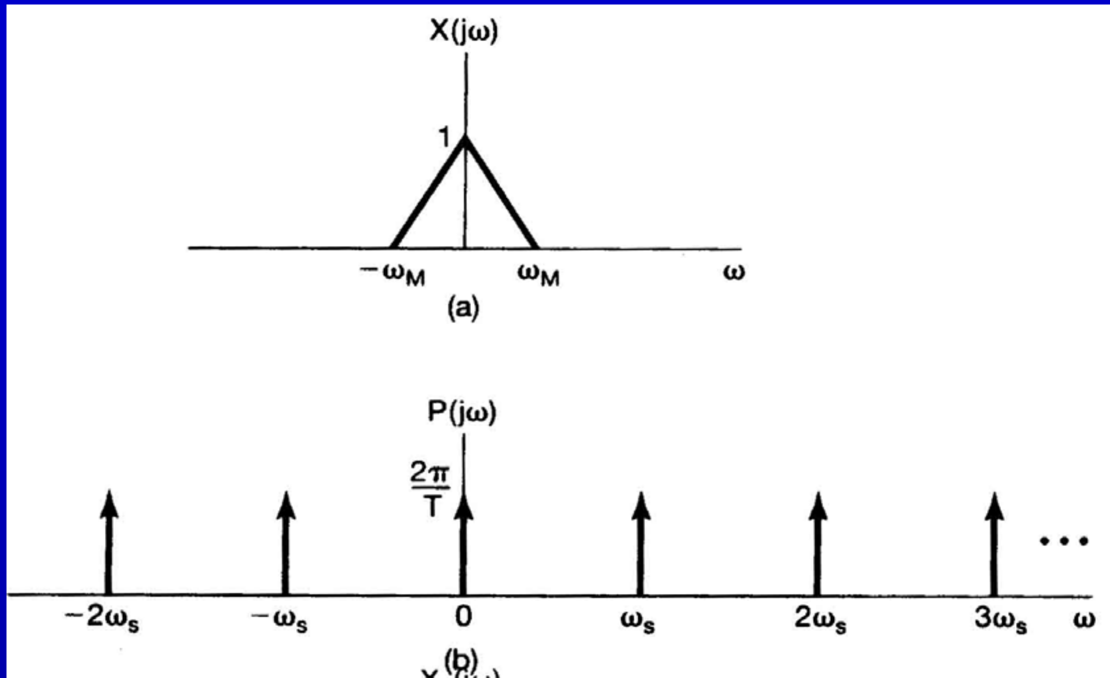
$$=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X\left(j(\omega-k\omega_s)\right)$$

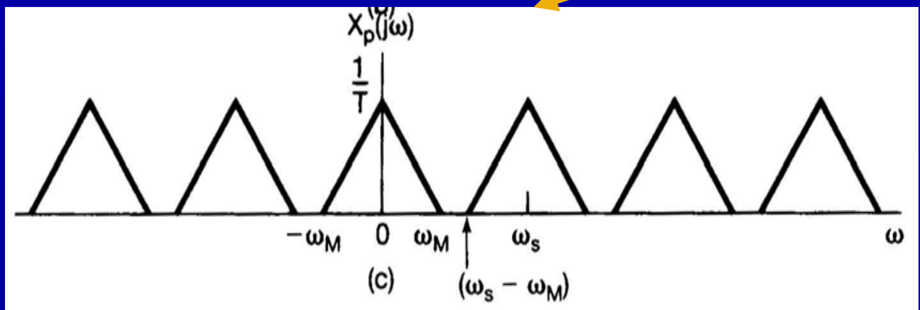
$$w_s=\frac{2\pi}{T}\rightarrow sampling\quad frequency$$

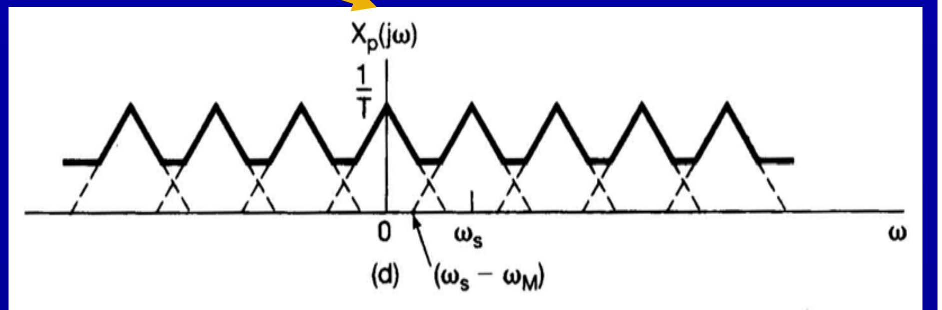
Sampling is a shift of the spectrum

The spectrum of samples

Signals & Systems





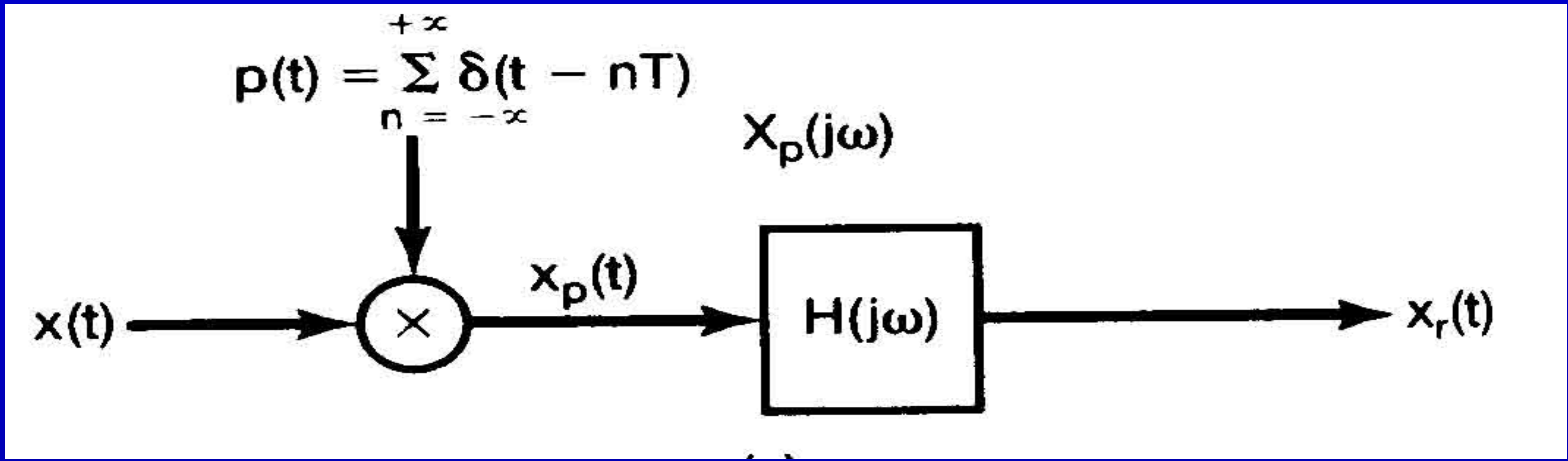


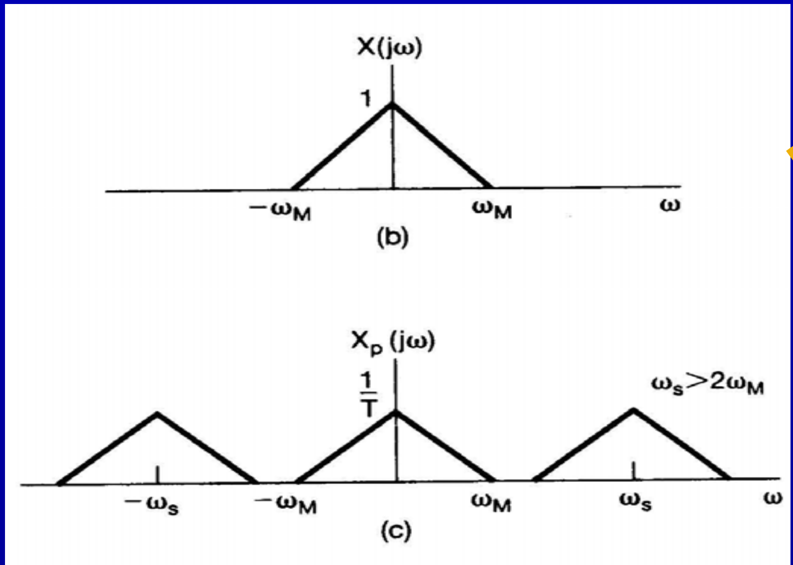
$\omega_s > 2\omega_M$, no overlap

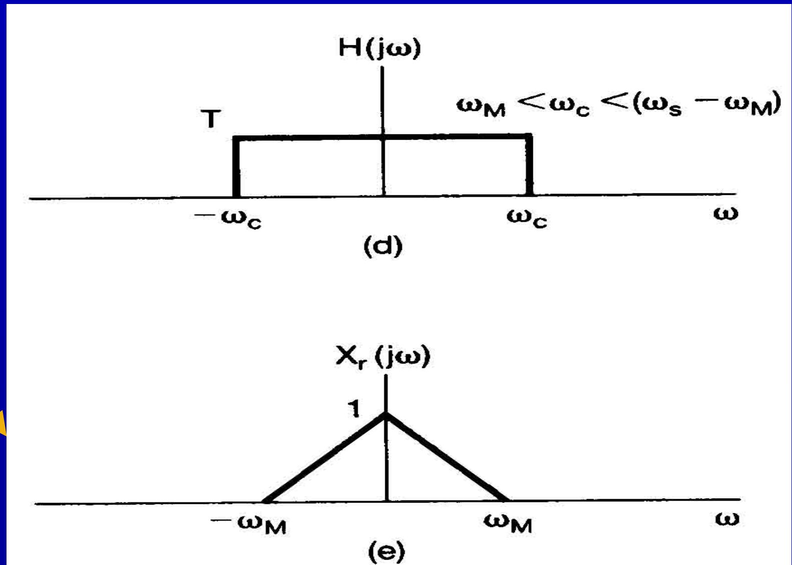
$\omega_s < 2\omega_M$, overlap exists

Recover a signal from its samples

Signals & Systems





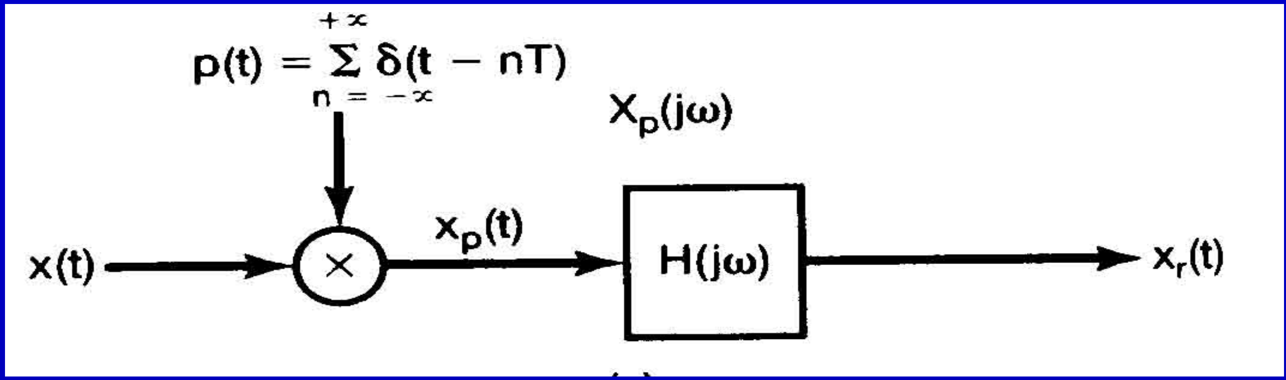


Sampling Theorem

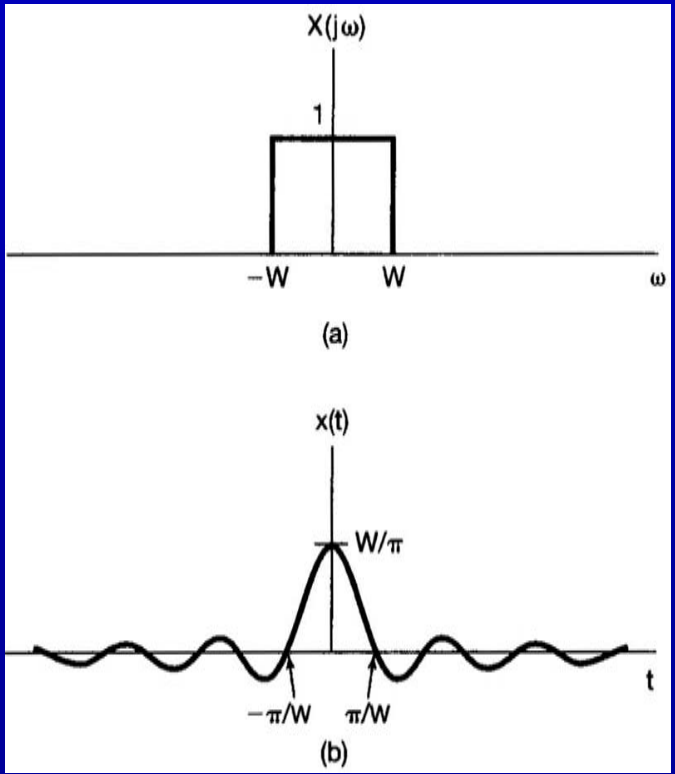
Let $x(t)$ be a band-limited signal with $X(j\omega)=0$ for $|\omega|>\omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n=0,\pm1,\pm2,\ldots$, if

$$\omega_s > 2\omega_M \quad \text{where} \quad \omega_s = \frac{2\pi}{T}$$

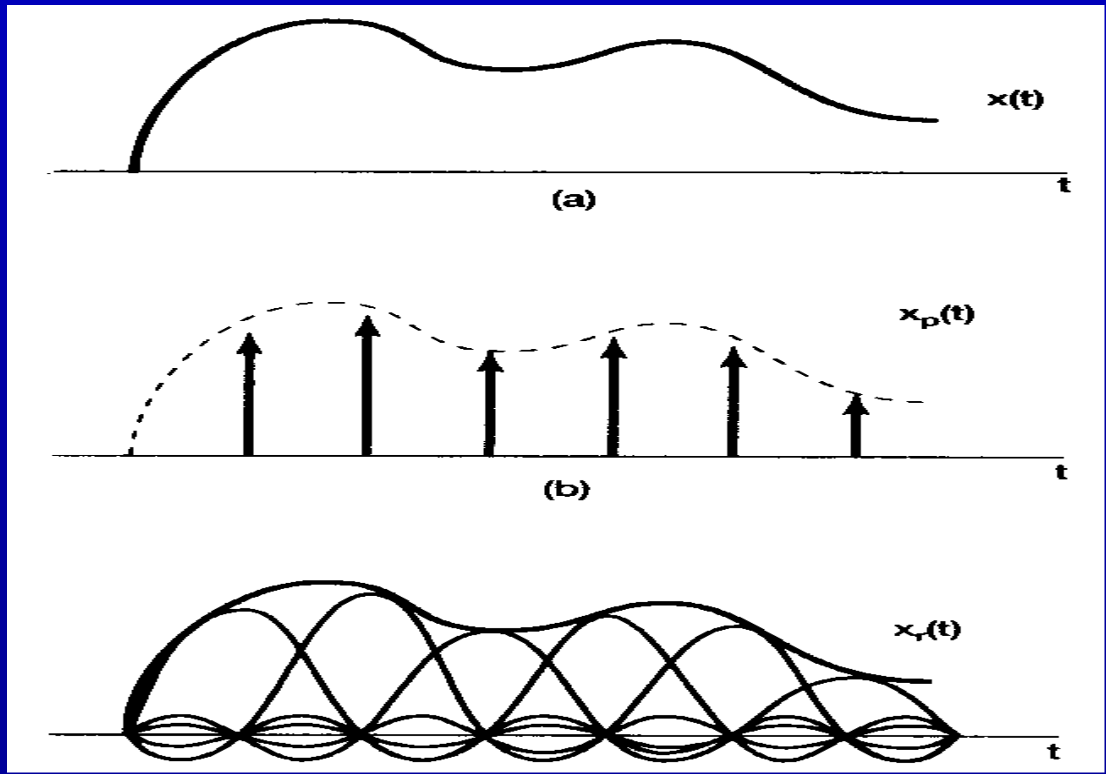
ω_M Nyquist frequency



$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$
$X_p(j\omega)$
x(t)
$x_p(t)$
H(jω)
$x_r(t)$



X(jω)
1
−W
W
ω
(a)
x(t)
W/π
−π/W
π/W
t
(b)

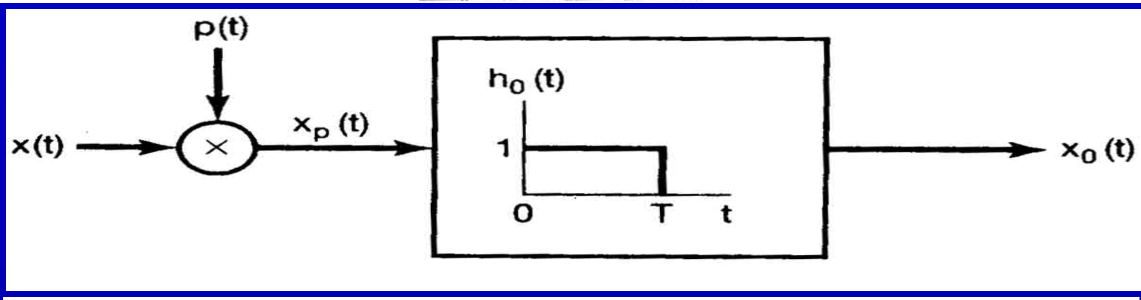


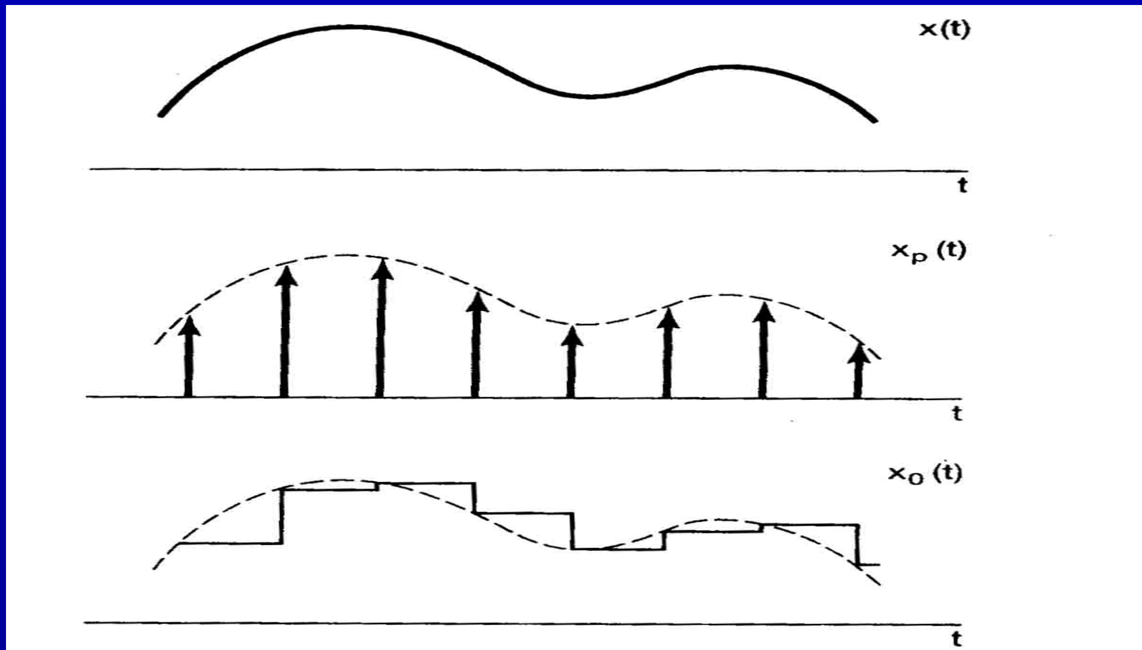
x(t)
t
(a)
$x_p(t)$
t
(b)
$x_r(t)$
t

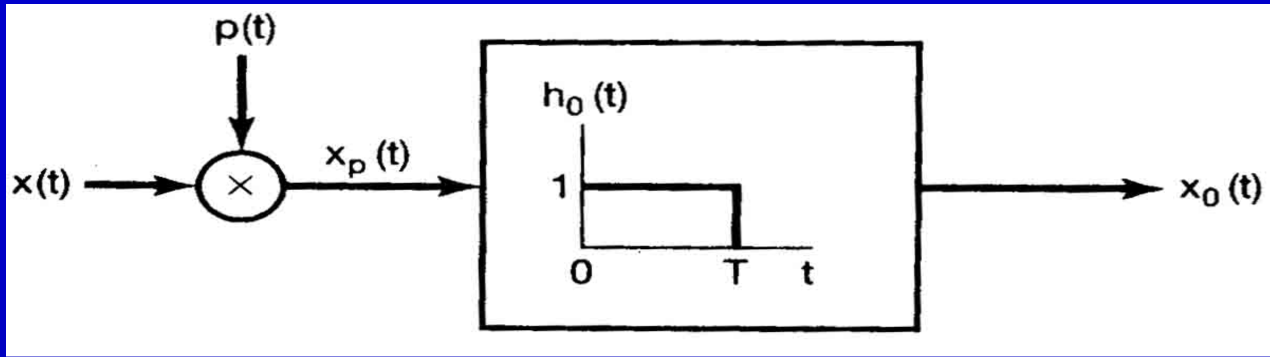
The realization of sampling and recover

- The generation and transmission of impulse-train are difficult
- The realization of ideal filter is impossible

7.1.2 Sampling with a Zero-Order Hold





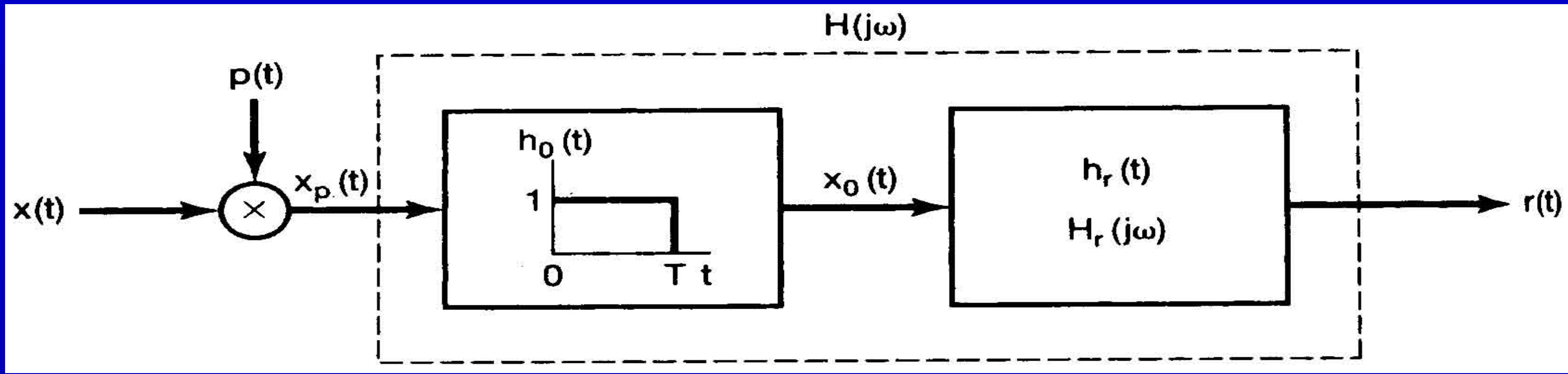


$$x_0(t) = x_p(t) * h_0(t) = \left(x(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t - kT) \right) * h_0(t)$$

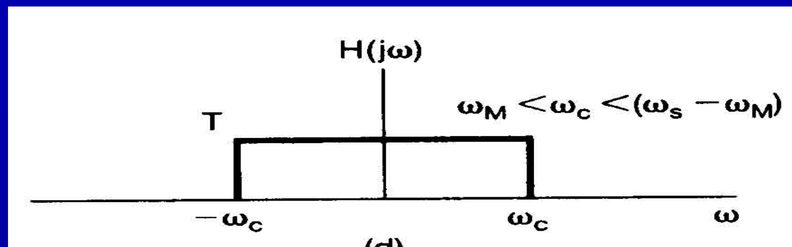
$$= \left(\sum_{k=-\infty}^{+\infty} x(kT) \cdot \delta(t - kT) \right) * h_0(t)$$

$$= \sum_{k=-\infty}^{+\infty} x(kT) \cdot h_0(t - kT)$$

Recover a signal from its zero-order hold samples



If



Then $x(t) = r(t)$

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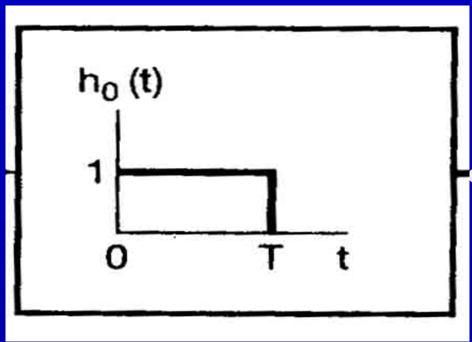
Determine $h_r(t)$

$$H(jw) = H_0(jw)\,H_r(jw)$$



$$\xleftrightarrow{FT} H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$$

$$H_r(jw) = \frac{e^{jwT/2}}{\dfrac{2\sin(wT/2)}{w}}\,H(jw)$$

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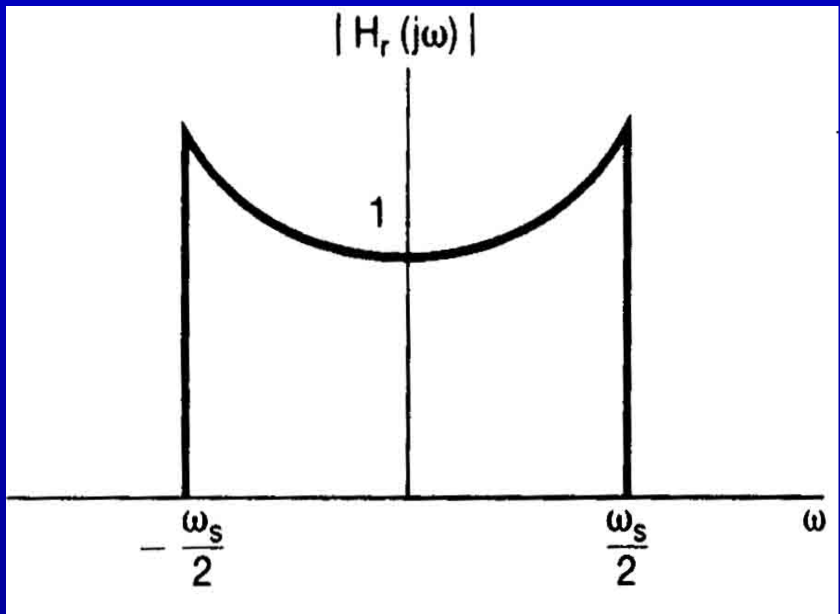
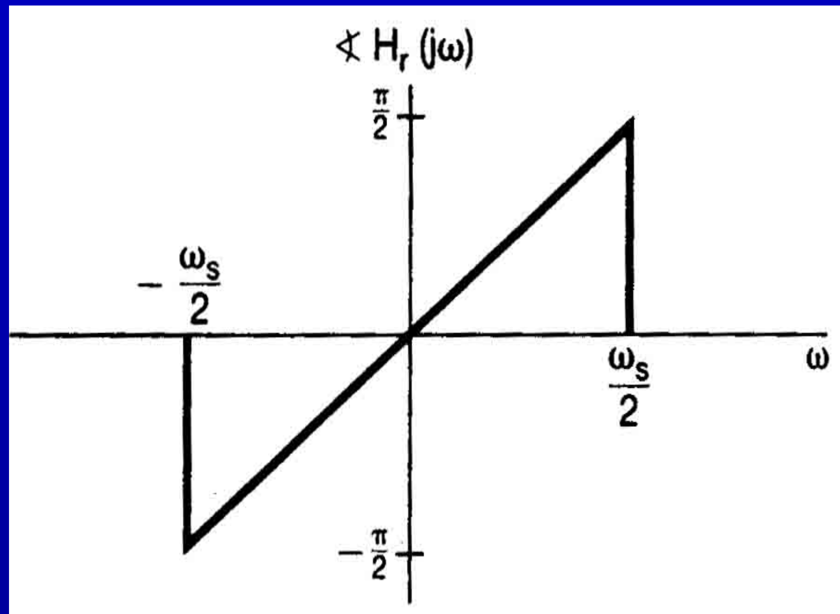
Determine $h_r(t)$

$$H_r(jw) = \frac{e^{jwT/2}}{\dfrac{2\sin(wT/2)}{w}} H(jw)$$

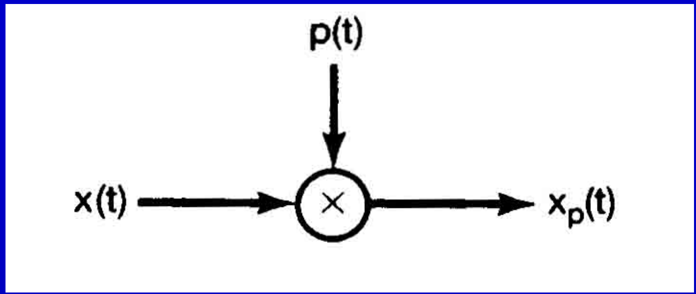




$$w_c = \frac{w_s}{2}$$

Signals & Systems

7.3 The Effect of Undersampling: Aliasing

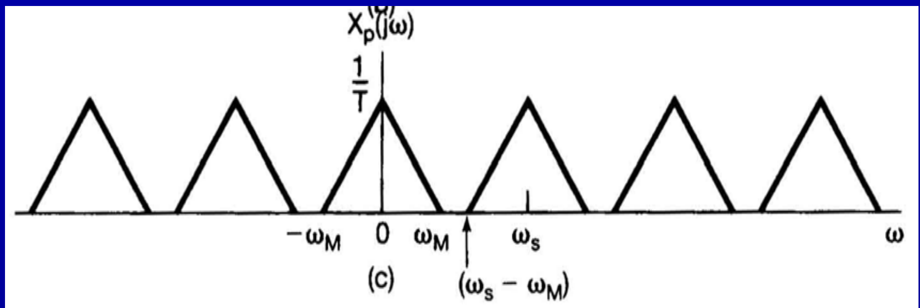


$$X_p(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X\big(j(\omega-k\omega_s)\big)$$

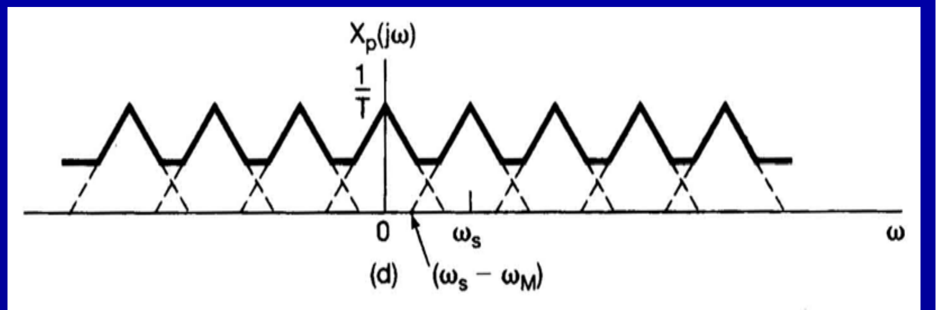
$w_S > 2w_M, x(t)$ can be recovered by its samplers

$w_S < 2w_M, x(t)$ can not be recovered by its samplers,

spectrum alias



$\omega_s > 2\omega_M$, no overlap



$\omega_s < 2\omega_M$, overlap exists

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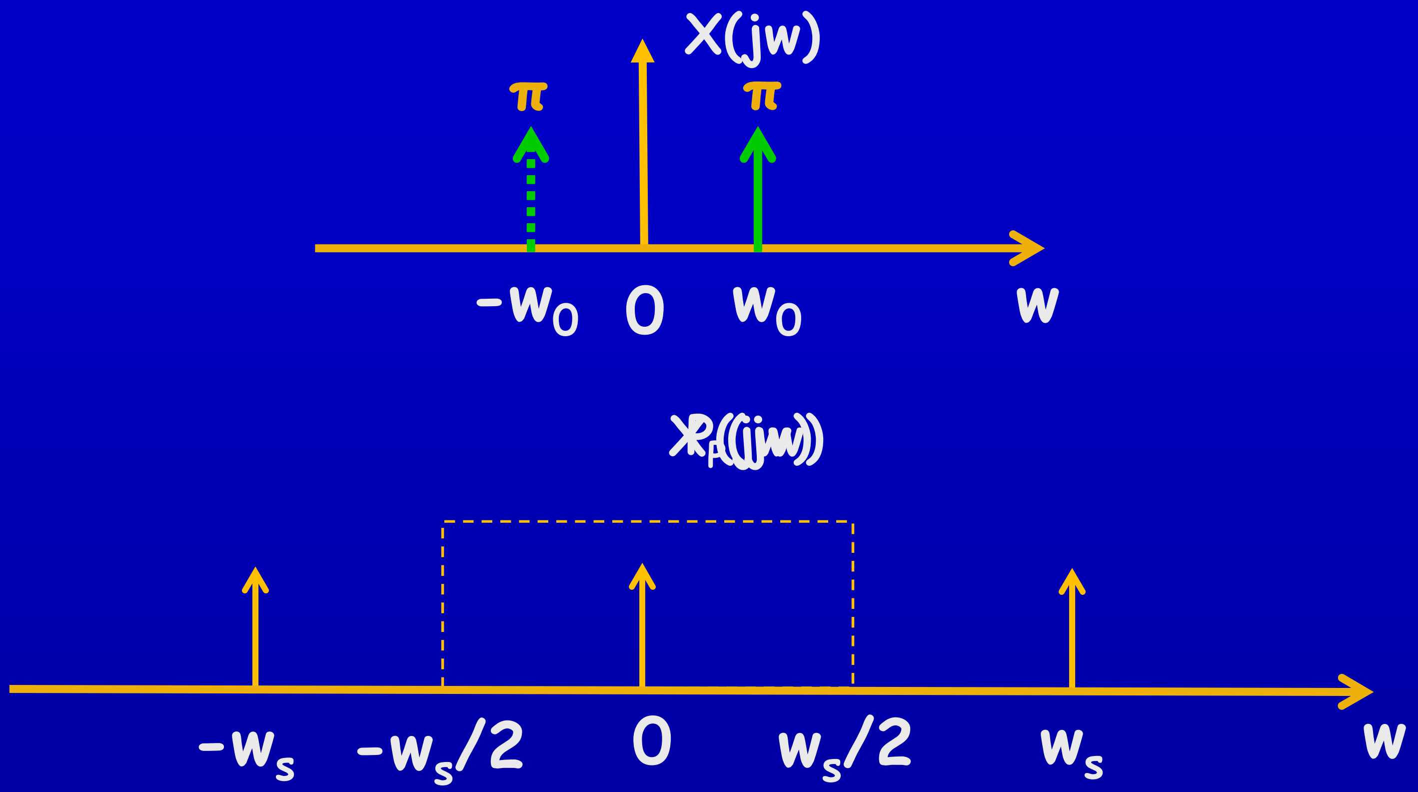
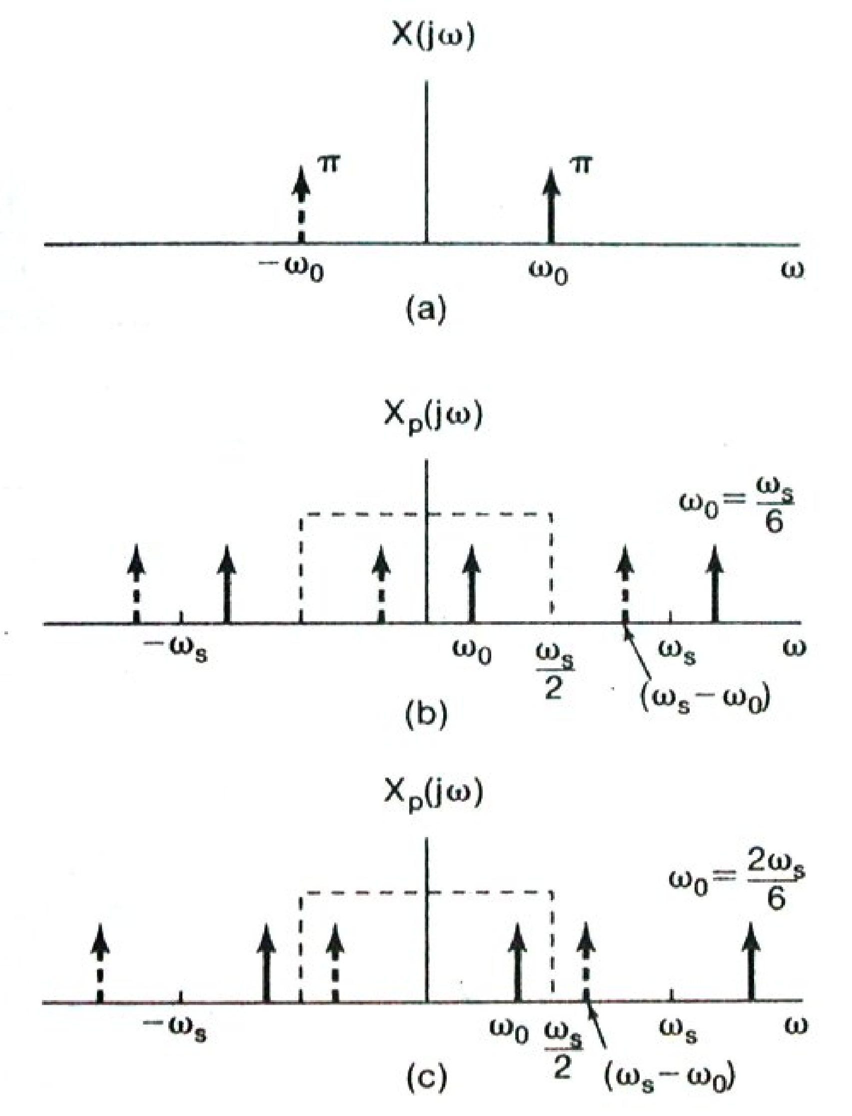
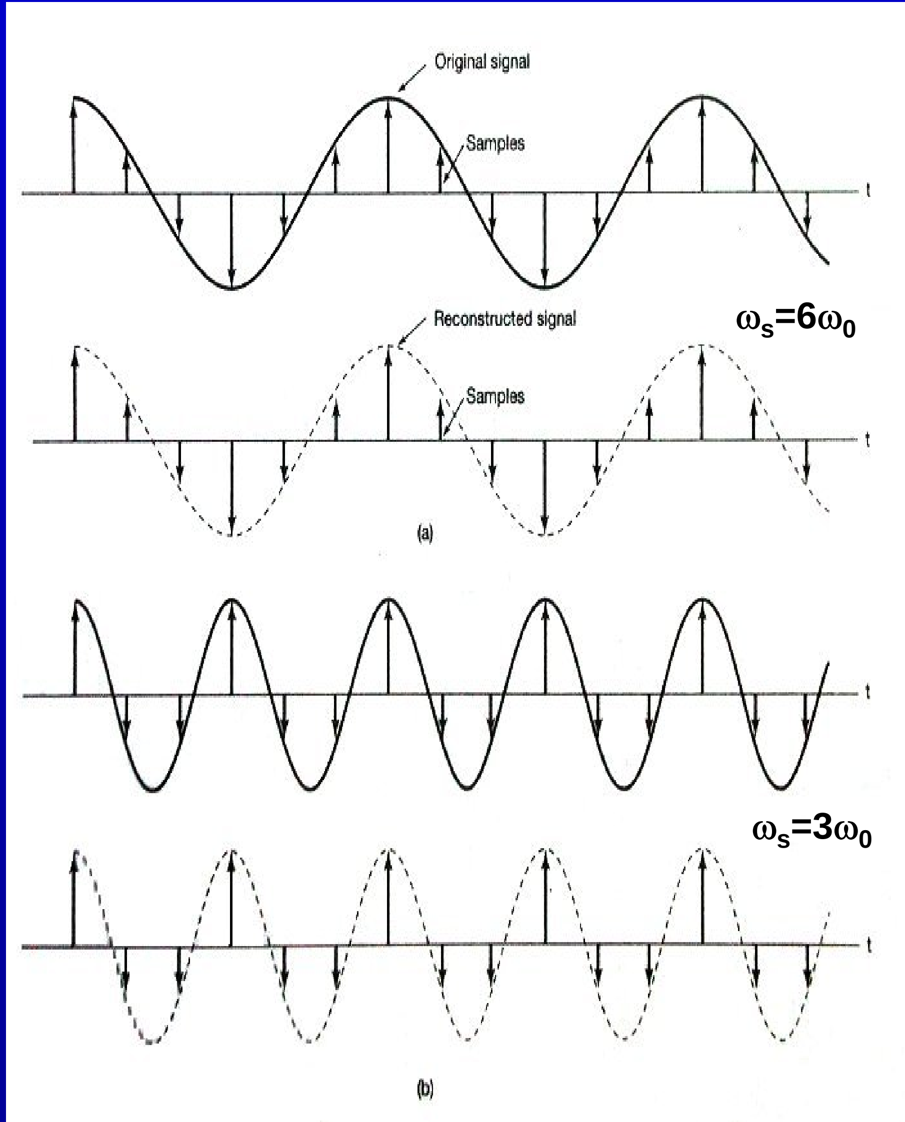
$\omega_s = 6\omega_0$

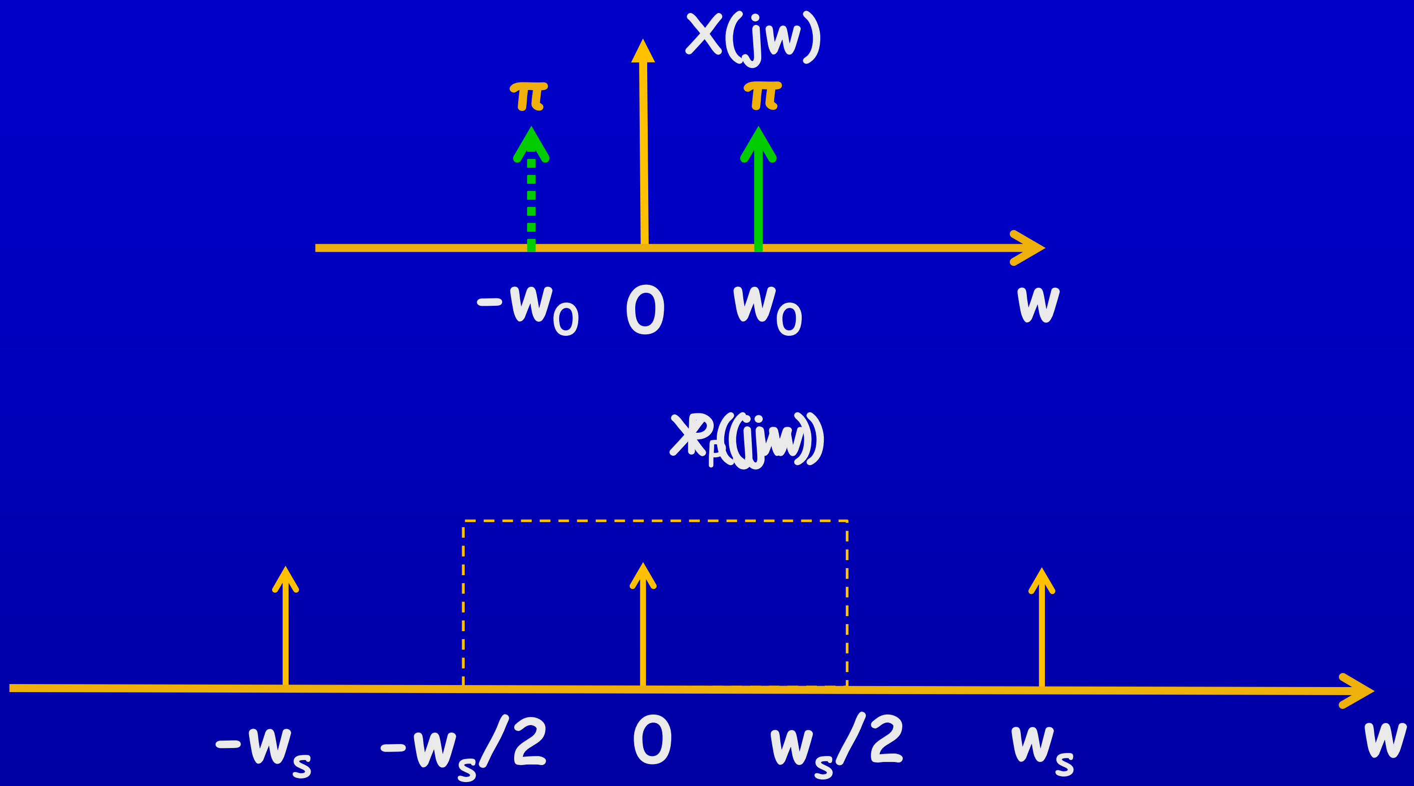



X(jω)
π
π
−ω0
ω0
ω
(a)
Xp(jω)
ω0 = ωs/6
−ωs
ω0
ωs/2
ωs
ω
(ωs − ω0)
(b)
Xp(jω)
ω0 = 2ωs/6
−ωs
ω0
ωs/2
ωs
ω
(ωs − ω0)
(c)


Original signal
Samples
t
ωs=6ω0
Reconstructed signal
Samples
t
(a)
t
ωs=3ω0
t
(b)

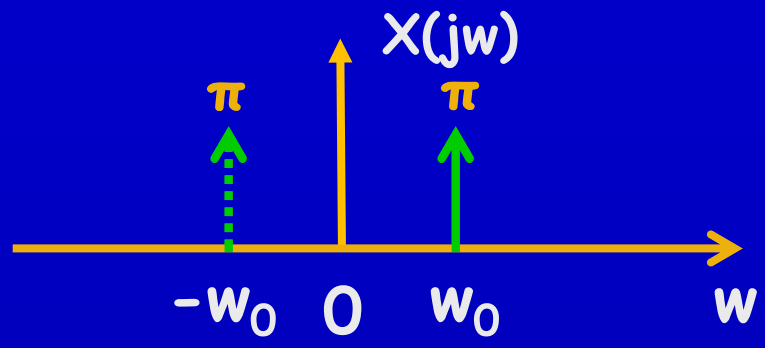
Signals & Systems

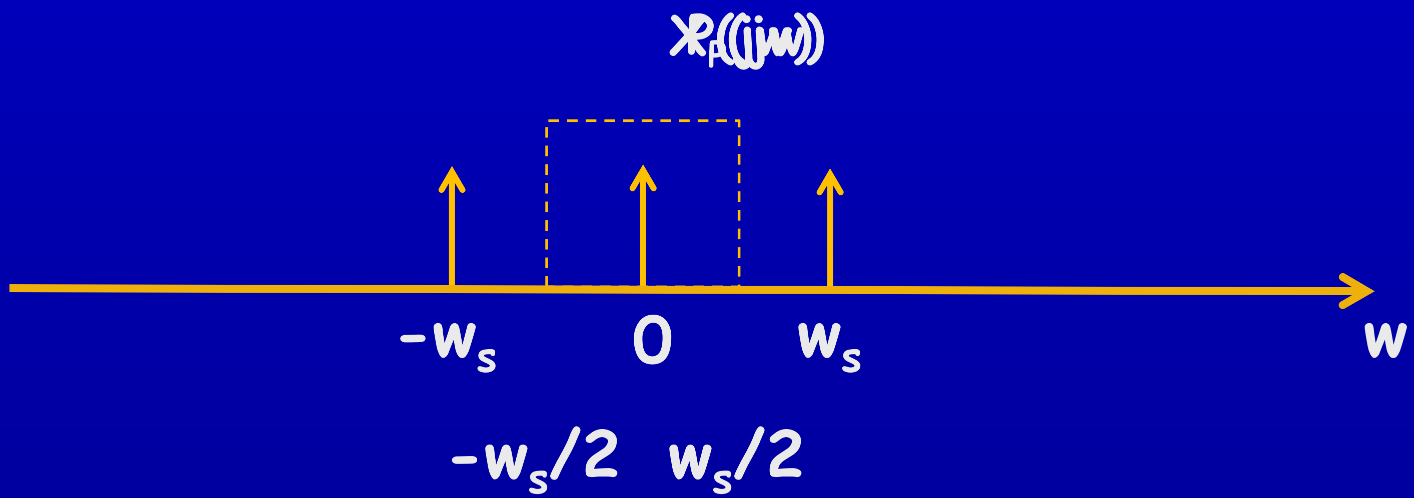
$\omega_s = 3\omega_0$

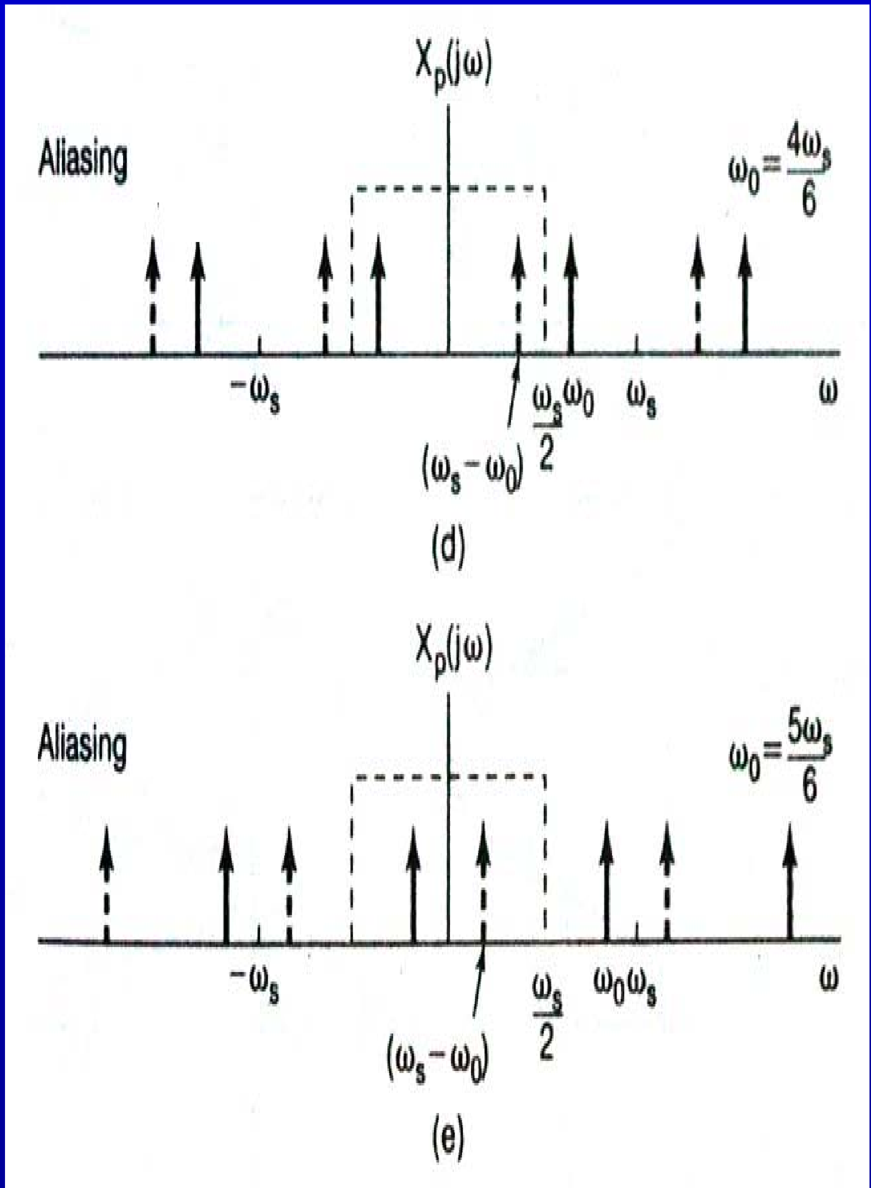


Signals & Systems

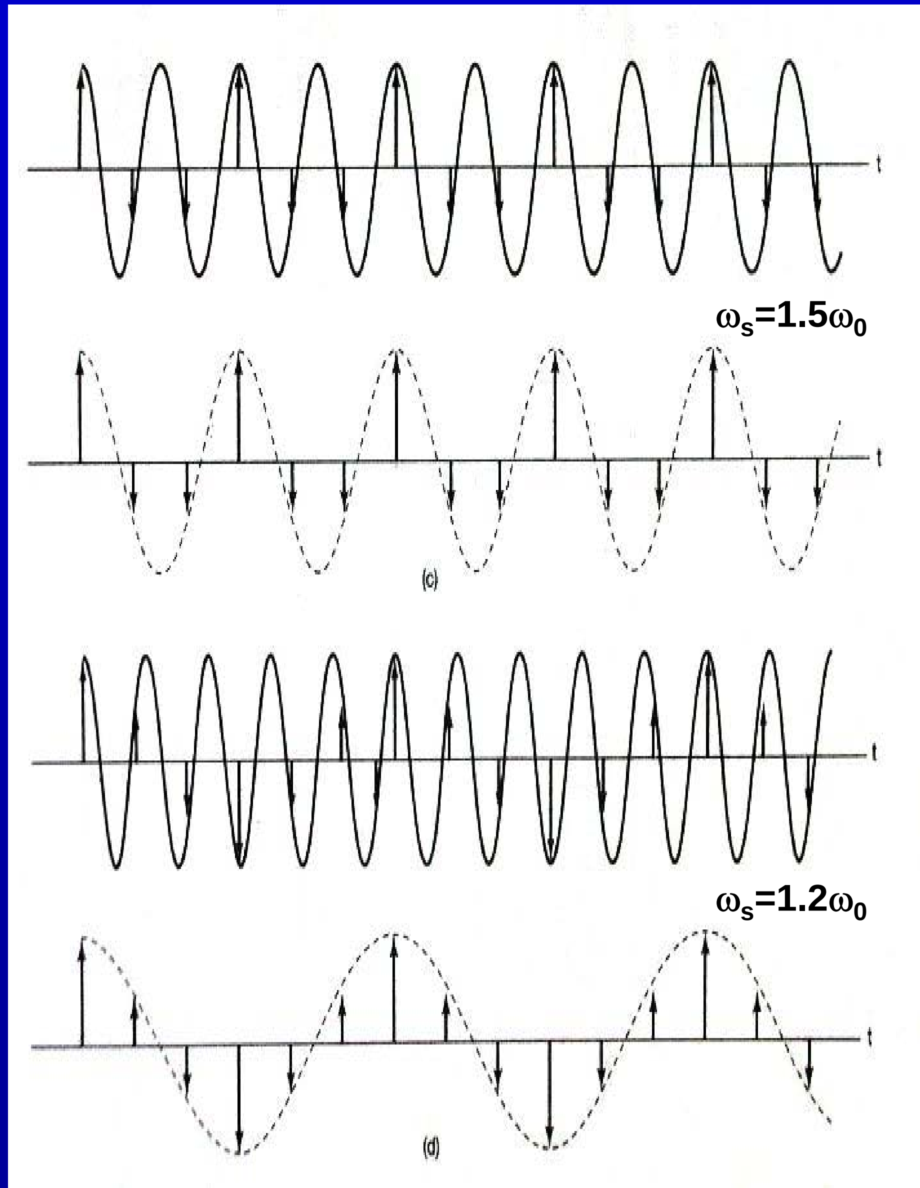
$$\omega_s = 1.5\omega_0$$







Aliasing
$X_p(j\omega)$
$\omega_0 = \frac{4\omega_s}{6}$
$-\omega_s$
$\frac{\omega_s}{2}$
ω_0
ω_s
ω
$(\omega_s - \omega_0)$
(d)
Aliasing
$X_p(j\omega)$
$\omega_0 = \frac{5\omega_s}{6}$
$-\omega_s$
$\frac{\omega_s}{2}$
ω_0
ω_s
ω
$(\omega_s - \omega_0)$
(e)



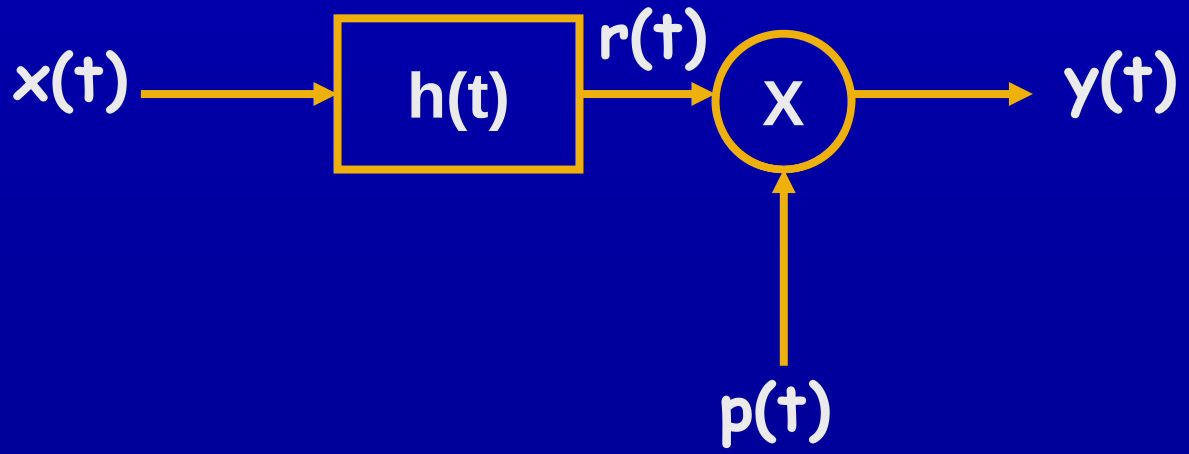
t
$\omega_s=1.5\omega_0$
t
(c)
t
$\omega_s=1.2\omega_0$
t
(d)

Example

Consider the system shown as below, with input $x(t)=u(t)-u(t-2)$

$h(t)=\dfrac{\sin w_c t}{\pi t}$, and $p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-\dfrac{n}{4})$

specify the range of values for w_c which ensure that $r(t)$ is recoverable from $y(t)$



Example

Consider the system shown as below, with input $x(t)=\left(\dfrac{\sin 10\pi t}{\pi t}\right)^2$

$h_1(t)=\dfrac{\sin 10\pi t}{\pi t}$, and $p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$

a) specify the range of values for T which ensure that $r_1(t)$ is recoverable from $r_2(t)$

b) if T=1/20 second, sketch the spectrum of $r_1(t)$ and $r_2(t)$

c) sketch and dimension a filter $H_2(jw)$ so that $y(t)=r_1(t)$