Home work 4

Due date: February 7, 2017, by 5p.m. at the home work box.

Reading assingment. Read chapter 2 of the text book. Go through all the problems at the end of chapter 2.

Filtering. This is the process of selectively amplifying certain frequencies in the signal.

Musical notes. In the Western equal-tempered scale, the note A_4 is defined to be the pure tone at 440Hz. The note A_5 is then defined to be at twice that frequency (880Hz) and A_3 is defined to be at half that frequency (220Hz). Similarly for higher and lower A notes. The full set of notes in one *octave* are: A, $A^{\#}$, B, C, $C^{\#}$, D, $D^{\#}$, E, F, $F^{\#}$, G, and $G^{\#}$. There are 12 of them (in this order of increasing frequencies). If f is the frequency of A in a certain scale (say A_4, in which case $f = 440$Hz), then we can get the corresponding $A^{\#}$ by multiplying f by $2^{1/12}$. To get the frequency for B we multiply by $2^{2/12}$ and so on. Here is a complete table:

$$\begin{aligned}
A &\to f\\
A^{\#} &\to 2^{1/12}f\\
B &\to 2^{2/12}f\\
C &\to 2^{3/12}f\\
C^{\#} &\to 2^{4/12}f\\
D &\to 2^{5/12}f\\
D^{\#} &\to 2^{6/12}f\\
E &\to 2^{7/12}f\\
F &\to 2^{8/12}f\\
F^{\#} &\to 2^{9/12}f\\
G &\to 2^{10/12}f\\
G^{\#} &\to 2^{11/12}f
\end{aligned}$$

Note that once you have the frequency of B_4 say, you can get the frequency of B_3 (half that of B_4) and B_5 (twice that of B_4), etc.. (For a complete list of the exact frequencies see, for example, `http://www.phy.mtu.edu/~suits/notefreqs.html`.)

Exercise 1. Everybody can sing?

A) Make a recording of yourself saying "Hello World" or anything else. Plot the absolute value of the Fourier coefficients of your signal, making sure that the horizontal axis of your plot is labelled in true Hertz. Mark 3 large peaks in your graph, and note down at which frequencies (in Hertz) they occur. Try to pick these peaks so that they are not spaced too closely. Also, watch out for aliasing. For each of these frequencies, figure out the nearest musical note they correspond to. Submit the graph with the 3 peaks that you chose and the corresponding musical notes clearly marked.

B) Your next job is to create a sound signal which only contains frequencies that are close to the 3 dominant notes that you noted in part A. Let us say they are C, $D^{\#}$ and G. (Of course your three notes may be different, in which case you should use those.) Then we want to create a *new* sound signal starting from your recording that pretty much contains only tones close to C, $D^{\#}$ and G (in all octaves); let us denote the Fourier series coefficients of this new signal by b_k. To do this take the Fourier coefficients of your original sound recording. For each Fourier coefficient a_k do the following. If the frequency corresponding to a_k is closer to the 3 dominant notes (C, or $D^{\#}$, or G, in our example) than any other musical note, set $b_k = a_k$. Otherwise set $b_k = 0$. (Note: in our example you must compare to C, $D^{\#}$, and G, in *all* the octaves.) Construct the new time-domain signal whose Fourier coefficients are the b_k's and record it as a sound file. What does it sound like when you play it back? Can you still recognize the words? Make a plot of the absolute values of b_k and a_k in the same graph, with the horizontal axis in true Hertz. The graphs should be identical, except for some frequencies where $|b_k|$ should be exactly 0. Submit your working code along with the final graphs.

Congratulations! You have made a filter. By fiddling with the rule for setting $b_k = 0$ you can create different effects.

Exercise 2. Discrete convolution. Assume that the input signal $x[n]$ is of length N, with $0 \leqslant n < N$, and that $x[n] = 0$ when $n < 0$ or $N \leqslant n$. Also assume that the impulse response $h[n]$ of the LTI system is of length K; that is $h[n] = 0$ for $n < 0$ and $K \leqslant n$. Write a function in your favorite programming language that takes $x[n]$ in an array of length N, $h[n]$ in an array of length K, and returns an array that contains the signal

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k].$$

What must be the length of the array that contains the output signal $y[n]$? To what time instant does the first element in the output array correspond? To what time instant does the last element in the output array correspond? Submit a printed copy of your code along with the rest of your answers.

Unit-step function. This is denoted by $u[n]$ and defined as follows:

$$u[n] = \begin{cases} 0, & n < 0, \\ 1, & n \geqslant 0. \end{cases}$$

Exercise 3. Let α and β denote complex numbers. Establish the following convolutions:

a) When $\alpha \neq \beta$

$$(\alpha^n u[n]) * (\beta^n u[n]) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\, u[n].$$

b)

$$(\alpha^n u[n]) * (\alpha^n u[n]) = (n+1)\alpha^n u[n].$$

c) When $|\alpha| < |\beta|$,

$$(\alpha^n u[n]) * (\beta^n u[-n-1]) = \frac{\beta^{n+1} u[-n-1] + \alpha^{n+1} u[n]}{\beta - \alpha}.$$

d) When $\alpha \neq \beta$

$$(n\,\alpha^n u[n]) * (\beta^n u[n]) = \left(\frac{\alpha\,\beta(\beta^n - \alpha^n)}{(\beta-\alpha)^2} - \frac{n\,\alpha^{n+1}}{\beta - \alpha} \right) u[n].$$

e)

$$(n\,\alpha^n u[n]) * (\alpha^n u[n]) = \frac{n(n+1)}{2}\, \alpha^n u[n].$$

f) When $|\alpha| < |\beta|$

$$(n\,\alpha^n u[n]) * (\beta^n u[-n-1]) = \frac{\alpha\beta^{n+1}}{(\beta-\alpha)^2}\, u[-n-1] + \frac{n\,\alpha^{n+1}}{(\beta-\alpha)}\, u[n] + \frac{\beta\alpha^{n+1}}{(\beta-\alpha)^2}\, u[n].$$

Add the above formulas to your cheat sheet.

Exercise 4. Establish the following properties of discrete convolutions:

a) $x[n] * h[n] = h[n] * x[n]$ (commutative).

b) $(\alpha\, x[n]) * h[n] = \alpha(x[n] * h[n])$ (scaling).

c) $(x_1[n] + x_2[n]) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$ (superposition).

d) $x[n] * \delta[n] = x[n]$ (sifting).

e) If $y[n]=x[n]*h[n]$ and $y_D[n]=x[n-n_0]*h[n]$, then $y_D[n]=y[n-n_0]$ (time delay).

f) If $y[n]=x[n]*h[n]$ and $y_R[n]=x[-n]*h[-n]$, then $y_R[n]=y[-n]$ (time reversal). *Note*: "Proving" this formula by making the substitution $n\to -n$ in the formula $y[n]=x[n]*h[n]$ is wrong! If it was right, then by making the substitution $n\to n-n_0$ we would "prove" the formula $y[n-n_0]=x[n-n_0]*h[n-n_0]$, which is clearly wrong.

Add the above formulas to your cheat sheet.

Exercise 5. Using your cheat sheets find explicit expressions for the following convolutions:

1. $\cos(\omega_0 n)\, u[n-1]*\sin(\omega_1 n+\varphi_1)\, u[n-2]$.
2. $\left(\frac{1}{2}\right)^{n+5}u[n-3]*\left(\frac{2}{3}\right)^{n-1}u[n+7]$.
3. If $y[n]=x[n]*h[n]$ express $x[n-n_0]*h[n-n_0]$ in terms of $y[n]$. *Warning*: It is *not* $y[n-n_0]$.