工程数学

- 微分方程与数学模型
- 常微分方程求解
- 定解问题建立
- 定解问题求解

塔科马海峡大桥(Tacoma Narrows Bridge位于美国华盛顿州)曾经是世界上第三长的悬索桥.

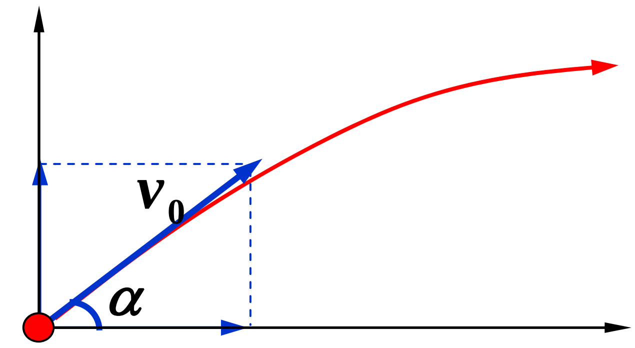
第一座大桥全长1524米,绰号舞动的格蒂,1940年7月1日通车,四个月后戏剧性地被微风摧毁.

——使得空气动力学和共振实验成为建筑工程学的必修课。

重建的大桥于1950年通车,2007年新的平行桥通车.

以后所有的桥梁, 无论是整体还是局部, 都必须通过严格的数学分析和风洞测试.

抛射体的数学模型



伽里略模型

微分方程

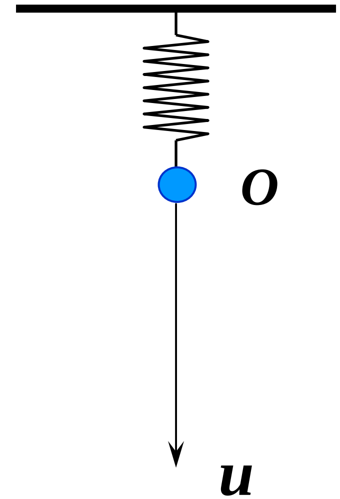
$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \dfrac{d^2y}{dt^2} = -g \end{cases}$$

初始条件：

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \qquad \begin{cases} \left.\dfrac{dx}{dt}\right|_{t=0} = v_0 \cos\alpha \\ \left.\dfrac{dy}{dt}\right|_{t=0} = v_0 \sin\alpha \end{cases}$$

微分方程——包含自变量、未知函数以及未知函数的导数组成的等式

简谐振动的数学模型



牛顿第二定律: $F = m\,a$

a—加速度;F—合外力;m—物体质量

虎克定律: $F = -k\,u(t)$

F—弹力;k—弹性系数; $u(t)$—弹簧伸长

$$m\,a = -k\,u(t) \quad \Rightarrow \quad m\frac{d^2u}{dt^2} = -ku(t)$$

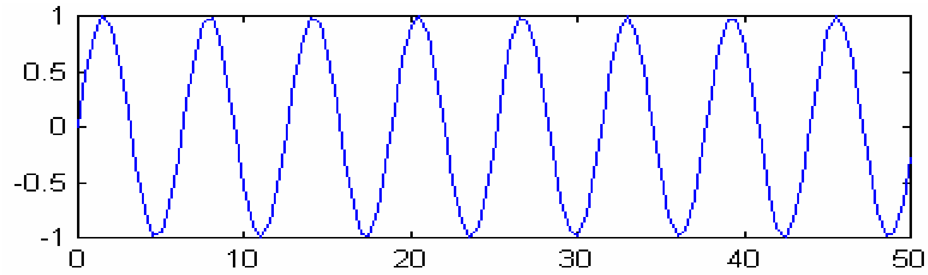
$$\Rightarrow \quad \frac{d^2u}{dt^2} + \omega^2 u(t) = 0 \qquad (\ \omega^2 = k/m\ )$$

一般形式 $$\frac{d^2u}{dt^2} + p(t)\frac{du}{dt} + q(t)u(t) = f(t)$$

谐振动

$$\frac{d^2u}{dt^2}+\omega^2u(t)=0$$

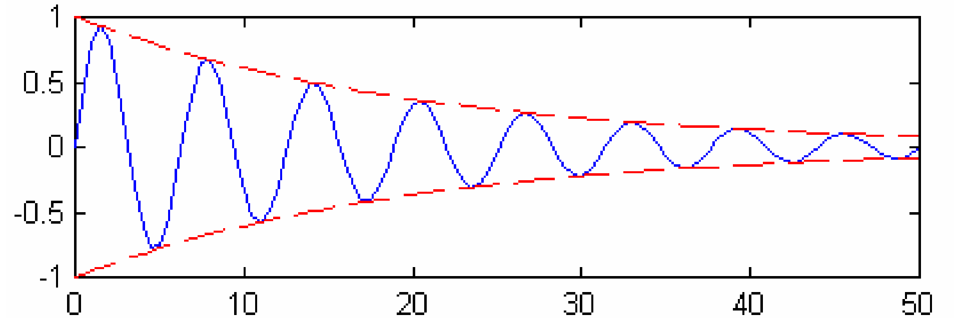
周期 $T=2\pi/\omega$



$u_1=\sin t \qquad (\omega=1)$

小阻尼振动

$$\frac{d^2u}{dt^2}+2\varepsilon\frac{du}{dt}+\omega^2u(t)=0$$

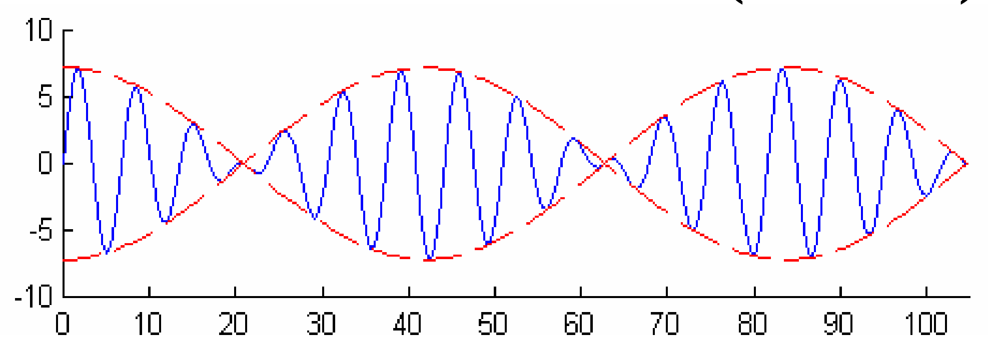


$u_2=\exp(-0.05t)\sin t \qquad (\varepsilon=0.05)$

无阻尼强迫振动

$$\frac{d^2u}{dt^2}+\omega^2u(t)=p\sin\omega_0 t$$

$(\omega=1,\omega_0=0.85,p=1)$



$$u_3=\frac{p}{1-\omega_0^2}(\sin t+\sin\omega_0 t)$$

人口增长模型I (Thomas Robert Malthus)

$$\frac{dy}{dx} = r\ y, \qquad y(x_0) = y_0$$

$$\rightarrow \quad \frac{dy}{y} = r\,dx \quad \rightarrow \quad \ln y = r\ x + C \quad \rightarrow \quad \cdots\cdots\cdot$$

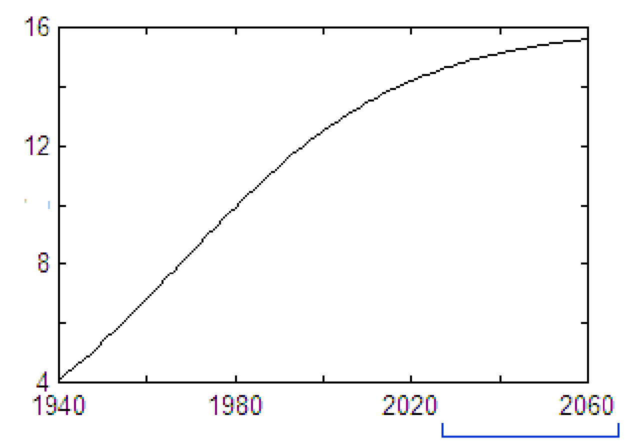
$$\rightarrow \quad y(x) = y_0 \exp[r(x - x_0)]$$

人口增长模型II ((Logistic Equation)

$$\frac{dy}{dx} = r\ y(1 - y/K),$$

$$\rightarrow \quad \frac{dy}{y(1 - y/K)} = r\,dx, \quad \rightarrow \quad \cdots\cdots\cdot$$

$$y(x) = \frac{K}{1 + \exp(-rx - C)}$$



二阶常系数齐次线性常微分方程

$$y'' + py' + qy = 0$$

辅助方程 $m^2 + pm + q = 0$

两相异实根 $m_1 \neq m_2$ $\rightarrow$ $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

两相等实根 $m_1 = m_2 = m$ $\rightarrow$ $y = (C_1 + C_2 x)e^{mx}$

两共轭复根 $m_{1,2} = \alpha \pm \beta i$ $\rightarrow$

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

积分公式

Green（**Green**）公式

$$\oint_L p(x,y)dx + q(x,y)dy = \iint_D [q_x(x,y) - p_y(x,y)]dxdy$$

斯托克斯（**Strokes**）公式

$$\oint_L p(x,y,z)dx + q(x,y,z)dy + r(x,y,z)dz = \oiint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p & q & r \end{vmatrix}$$

$$\oint_L \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot d\vec{s}$$

高斯（Gauss）公式

$$\oiint\limits_S p(x,y,z)dydz + q(x,y,z)dzdx + r(x,y,z)dxdy = \iiint\limits_V (p_x + q_y + r_z)dxdydz$$

$$\oiint\limits_S \vec{A}\cdot d\vec{s} = \iiint\limits_V \nabla\cdot\vec{A}dv = \iiint\limits_V div\,\vec{A}dv$$

$$\oiint\limits_S \vec{A}\cdot d\vec{s} = \oiint\limits_S \vec{A}\cdot\vec{n}ds = \oiint\limits_S \nabla u\cdot\vec{n}ds = \oiint\limits_S \frac{\partial u}{\partial\vec{n}}ds$$

常用算子

$$Df(x) = f'(x)$$

$$\nabla = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$$

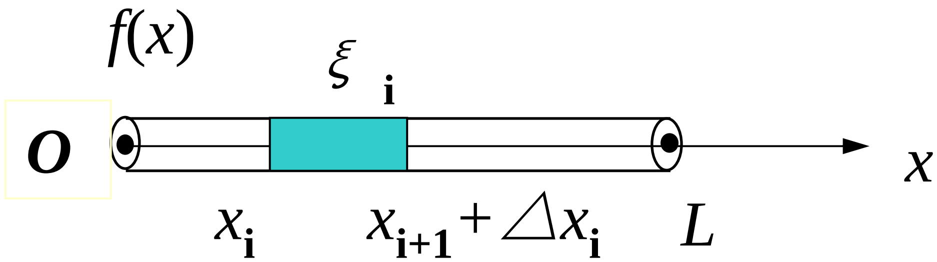
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

微分中值定理

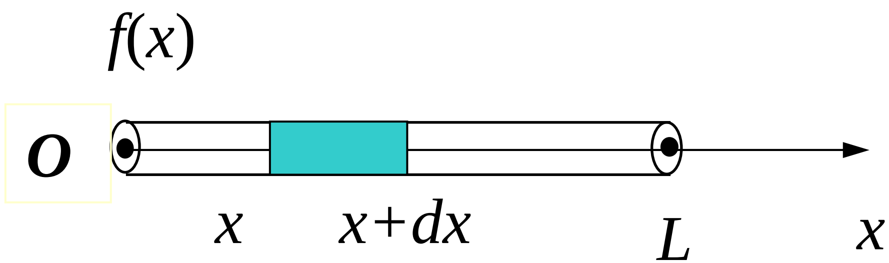
$$f(b) - f(a) = f'(\xi)(b-a)$$

$$u(b,t) - u(a,t) = u_x(\xi,t)(b-a)$$

细金属丝的质量



$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$



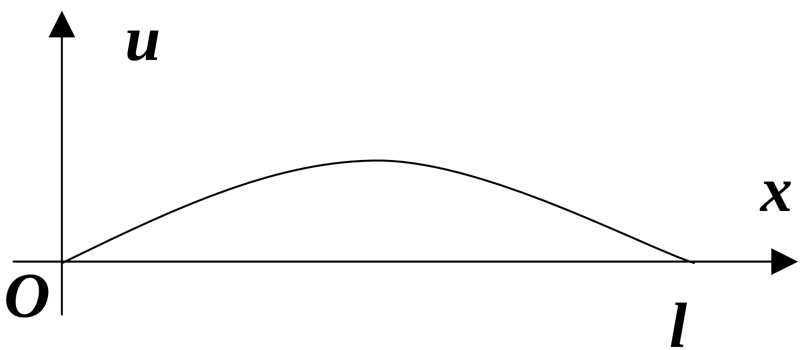
$$\sum_{i=0}^{L} f(x)dx$$

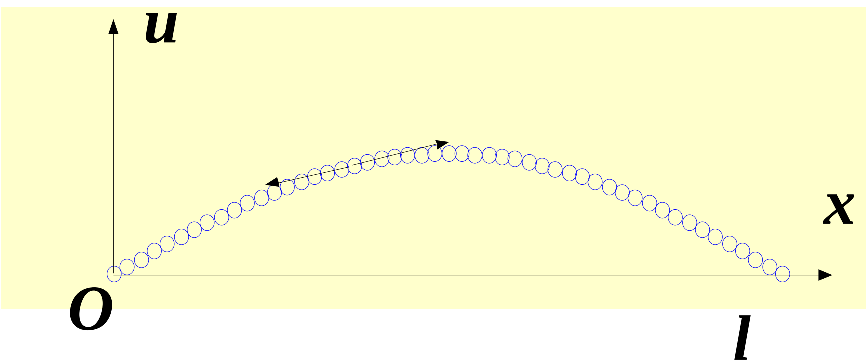
$$\sum_{i=1}^{100} i^2 = 1 + 2^2 + 3^2 + \cdots + 100^2$$

$$\int_0^L f(x)dx$$

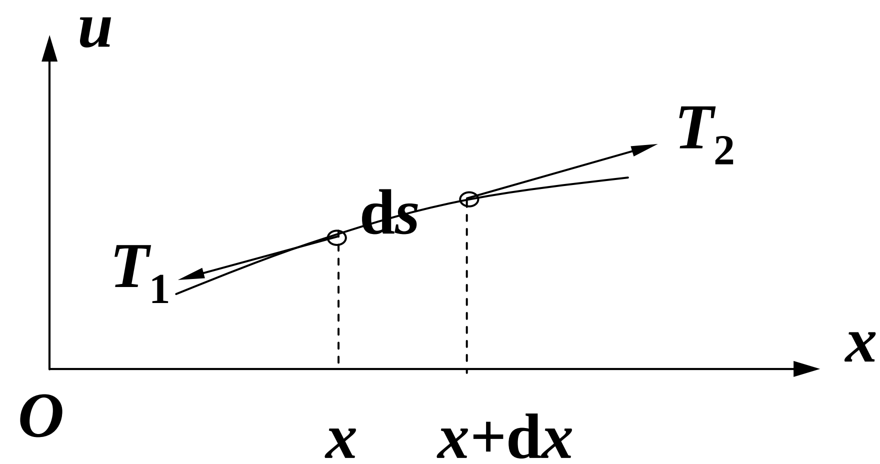
弦的横向振动问题

一根均匀柔软的细弦线, 一端固定在坐标原点, 另一端沿 x 轴拉紧固定在 x 轴上的 l 处， 受到扰动， 开始沿 x 轴(平衡位置)作微小横振动(细弦线上各点运动方向垂直于x 轴). 试建立细弦线上任意点位移函数 $u(x, t)$ 所满足的规律 .





设细弦上各点线密度为 ρ，细弦上质点之间相互作用力为张力 $T(x, t)$



水平合力为零 ➔ $T_2\cos\alpha_2 - T_1\cos\alpha_1 = 0$

$$T_2\cos\alpha_2 = T_1\cos\alpha_1 = T_0$$

铅直合力：$F=ma$ ➔ $T_2\sin\alpha_2 - T_1\sin\alpha_1 = \rho ds u_{tt}$

$$T_2 = T_0/\cos\alpha_2, T_1 = T_0/\cos\alpha_1$$

$$T_0[\tan\alpha_2 - \tan\alpha_1] = \rho ds u_{tt}$$

$$T_0[u_x(x+dx,t)-u_x(x,t)]=\rho\, ds\, u_{tt}$$

$$ds=\sqrt{1+u_x^2}\,dx \approx dx \quad \Rightarrow \quad \frac{T_0}{\rho}\frac{u_x(x+dx,t)-u_x(x,t)}{dx}\approx u_{tt}$$

$\Rightarrow$ $u_{tt}= a^2\, u_{xx}$ 其中 $\dfrac{T_0}{\rho}=a^2$

一维波动方程： $u_{tt} = a^2\, u_{xx}$

考虑有恒外力密度$f(x，t)$作用时，可以得到一维波动方程的非齐次形式

$$u_{tt} = a^2\, u_{xx} + f(x, t)$$

弦振动问题微分方程

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

定解条件

初始条件——历史状态:

$$u(x,t)|_{t=0} = \varphi(x), \quad u_t(x,t)|_{t=0} = g(x)$$

边界条件——边界状态,

$$u(x,t)|_{x=0} = 0, \quad u(x,t)|_{x=l} = 0$$

波动方程求解的分离变量法

$$\begin{cases} u_{tt} = a^2 u_{xx}, (0 < x < l, t > 0) \\ u\big|_{x=0} = 0, u\big|_{x=l} = 0 \\ u\big|_{t=0} = \varphi(x), u_t\big|_{t=0} = \psi(x) \end{cases}$$

付里叶1768～1830

分离变量法. 设 $u(x, t) = T(t) \cdot X(x)$ ➔

$$u_{tt} = a^2 u_{xx} \quad ➔ \quad T''(t)\, X(x) = a^2 T(t)\, X''(x)$$

$$\frac{T''}{a^2 T} = \frac{X''}{X} \quad ➔ \quad \frac{T''}{a^2 T} = \frac{X''}{X} = -\lambda$$

常微分方程 $\quad T'' + \lambda a^2 T = 0 \qquad X'' + \lambda X = 0$

边界条件: $u\big|_{x=0}=0, u\big|_{x=l}=0$

$$\begin{aligned} T(t)\cdot X(0)&=0 \\ T(t)\cdot X(l)&=0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X(0)&=0 \\ X(l)&=0 \end{aligned}$$

固有值问题: $\begin{cases} X''+\lambda X=0,\ \ 0<x<l \\ X(0)=0, X(l)=0 \end{cases}$

确定非零函数 $X(x)$ 和数 λ

解 μ 的二次方程： $\mu^2+\lambda=0$

$$\mu_1=\sqrt{-\lambda} \qquad \mu_2=-\sqrt{-\lambda}$$

(1) $\lambda<0$ $\begin{cases} X''+\lambda X=0, \ 0<x<l \\ X(0)=0, X(l)=0 \end{cases}$

通解: ➔ $X=Ae^{\sqrt{-\lambda}x}+Be^{-\sqrt{-\lambda}x}$

边界条件: ➔ $\begin{bmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}l} & e^{-\sqrt{-\lambda}l} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

二阶线性方程组系数矩阵行列式 $\begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}l} & e^{-\sqrt{\lambda}l} \end{vmatrix}\neq 0$

➔ $A=0$, $B=0$, 问题只有零解.

(2) $\lambda = 0$

$$\begin{cases} X'' + \lambda X = 0, \ 0 < x < l \\ X(0) = 0, X(l) = 0 \end{cases}$$

通解: $X(x) = Ax + B$

边界条件➔ $\begin{cases} B = 0 \\ A\,l + B = 0 \end{cases}$ $A = B = 0$

问题只有零解

(3) $\lambda > 0$ $\mu^2 + \lambda = 0$ $\Rightarrow$ $\mu_1 = i\sqrt{\lambda}$, $\mu_2 = -i\sqrt{\lambda}$

通解: $X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

边界条件: ➔ $\begin{bmatrix} 1 & 0 \\ \cos\sqrt{\lambda}l & \sin\sqrt{\lambda}l \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

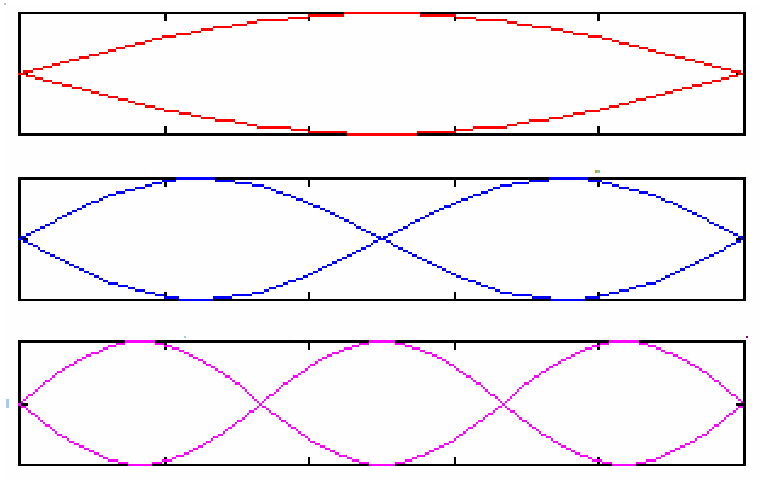
系数矩阵行列式为零 ➔ $\sin\sqrt{\lambda}l = 0$

➔ $\sin\sqrt{\lambda}l = 0$ ➔ $\sqrt{\lambda}l = n\pi$ (n=1,2,⋯)

固有值: $\lambda_n = \dfrac{n^2\pi^2}{l^2}$ (n=1,2,⋯)

固有函数:

$$X_n(x) = B_n \sin\frac{n\pi}{l}x$$



$$\lambda_n = \frac{n^2\pi^2}{l^2} = \omega_n^2$$ 代入方程 $$T'' + \lambda_n a^2 T = 0$$

通解: $$T_n(t) = C_n \cos\omega_n at + D_n \sin\omega_n at$$

弦振动方程的基本解:

$$u_n(x, t) = T_n(t)\, X_n(x)$$

$$= (a_n \cos\omega_n at + b_n \sin\omega_n at)\sin\omega_n x$$

$$= (a_n \cos\frac{n\pi at}{l} + b_n \sin\frac{n\pi at}{l})\sin\frac{n\pi x}{l}$$

叠加原理 ➔ $$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi at}{l}+b_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

方程初始条件：

$$u(x,0)=\varphi(x) \qquad \varphi(x)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{l}$$

$$u_t(x,0)=\psi(x) \qquad \psi(x)=\sum_{n=1}^{\infty}b_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

$$a_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi x}{l}dx$$

$$b_n=\frac{l}{n\pi a}\frac{2}{l}\int_0^l\psi(x)\sin\frac{n\pi x}{l}dx$$

固有函数 $\{X_n(x)\}$ 正交性证明

$$X_n'' + \lambda_n X_n = 0 \qquad X_m X_n'' + \lambda_n X_m X_n = 0$$

$$X_m'' + \lambda_m X_m = 0 \qquad X_n X_m'' + \lambda_m X_n X_m = 0$$

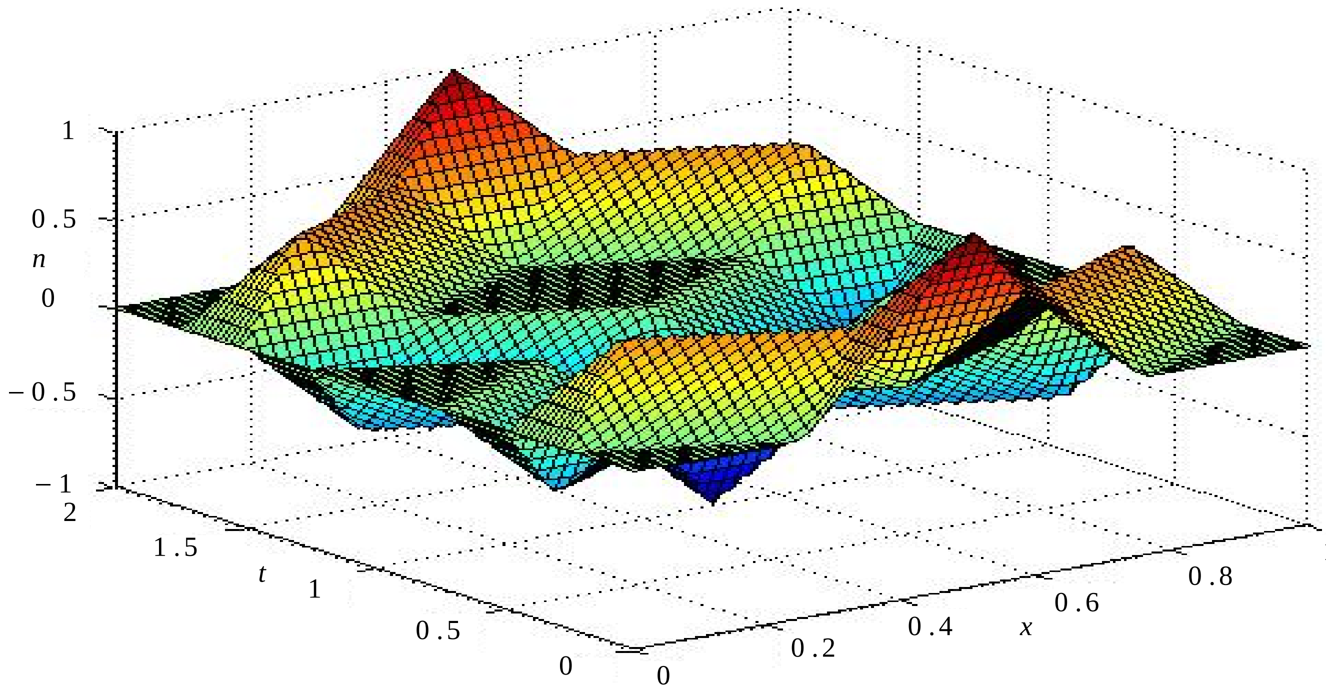
$$(\lambda_n - \lambda_m)\int_0^l X_m X_n dx = \int_0^l [X_n X_m'' - X_m X_n'']dx$$

(固有值问题边界条件) $= [X_n X_m' - X_m X_n']_0^l = 0$

$$\int_0^l X_m X_n dx = 0 \qquad (m \neq n)$$

$$\int_0^l X_m X_m dx = \int_0^l \sin^2 \frac{m\pi}{l} x dx \qquad (m \neq 0)$$

$$= \frac{1}{2}\int_0^l [1 - \cos\frac{2m\pi}{l} x]dx = \frac{l}{2}$$


1
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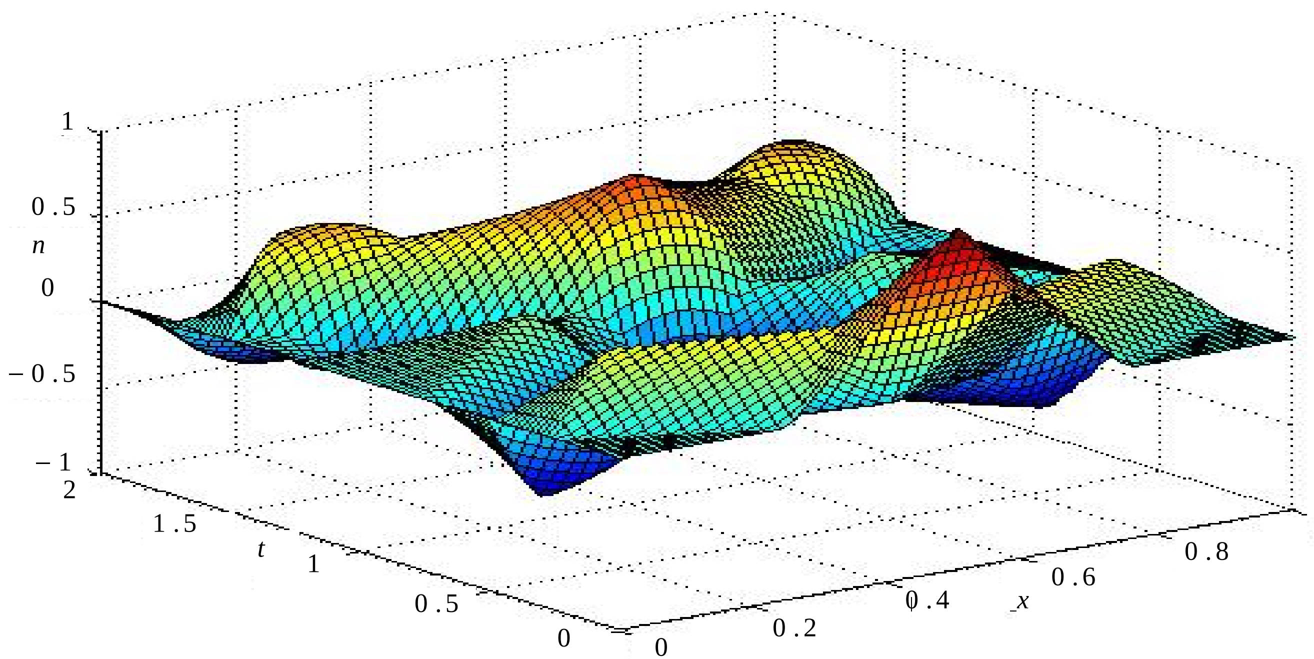
x
0.6
0.8
1


1
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t
1
0.5
0
0
0.2
0.4
x
0.6
0.8
1

例 求解波动方程初边值问题

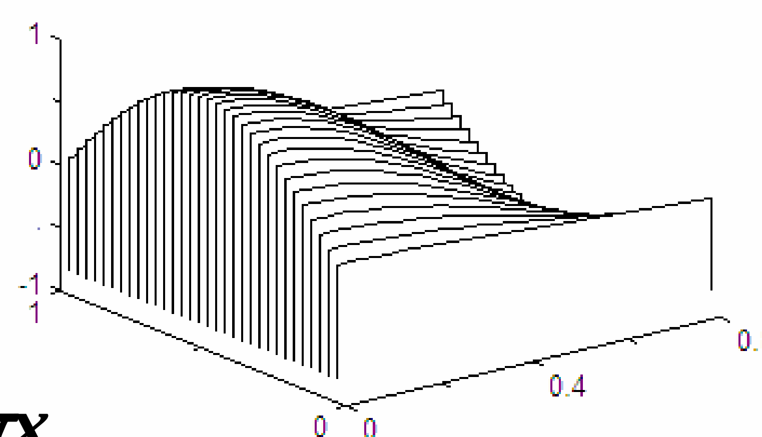
$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < 1, t > 0 \\ u(0,t) = 0, u(1,t) = 0, & t > 0 \\ u\big|_{t=0} = \sin \pi x, u_t\big|_{t=0} = 0, & 0 < x < 1 \end{cases}$$

固有值: $\lambda_n = n^2\pi^2 \quad (n=1,2,\cdots)$

固有函数:

$$X_n(x) = B_n \sin n\pi x$$

$$(n=1,2,\cdots)$$



解函数: $u(x,t) = \cos \pi t \sin \pi x$

例1 求解波动方程初边值问题

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < 1, t > 0 \\ u(0,t) = 0, u(1,t) = 0, & t > 0 \\ u\big|_{t=0} = \sin 2\pi x, u_t\big|_{t=0} = x(1-x), & 0 < x < 1 \end{cases}$$

固有值: $\lambda_n = n^2\pi^2 \quad (n=1,2,\cdots)$

固有函数: $X_n(x) = B_n \sin n\pi x \quad (n=1,2,\cdots)$

解函数: $u(x,t) = \sum_{n=1}^{\infty} (a_n \cos n\pi at + b_n \sin n\pi at) \sin n\pi x$

$$a_n = 2\int_0^l \varphi(x) \sin n\pi x dx = 2\int_0^l \sin 2\pi x \cdot \sin n\pi x dx$$

$$a_2 = 2\int_0^1 \sin^2 2\pi x dx = 1 \qquad a_n = 0, \ (n \neq 2)$$

$$b_n = \frac{2}{n\pi\, a}\int_0^1 \psi(x)\sin n\pi x dx \;= \frac{2}{n\pi\, a}\int_0^1 x(1-x)\sin n\pi x dx$$

$$= \frac{2}{n\pi\, a}\frac{2}{(n\pi)^2}\int_0^1 \sin n\pi x dx = \frac{2}{n\pi\, a}\frac{2}{(n\pi)^3}[1-\cos n\pi]$$

$$= \frac{4}{(n\pi)^4\, a}[1-(-1)^n]$$

$$u(x,t) = \sum_{n=1}^{\infty}(a_n \cos n\pi at + b_n \sin n\pi at)\sin n\pi x$$

$$= \cos 2\pi at \sin 2\pi x +$$

$$+\sum_{n=1}^{\infty}\frac{4}{(n\pi)^4 a}[1-(-1)^n]\sin n\pi at \sin n\pi x$$