Homework 9

Due date: None. Extra credit if you turn it in by March 17, 2017.

Reading assignment: Chapter 5 of the text book.

Discrete-time Fourier Transform: If $x[n]$ is an absolutely summable discrete-time signal, its DTFT $\mathcal{X}(\omega)$ is given by the formula

$$\mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n},$$

and the signal can be recovered from the DTFT via the formula

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} \mathcal{X}(\omega)\, e^{j\omega n} d\omega.$$

Discrete-time Fourier Series. If $x[n]$ has period N, then it can be expanded as a sum of N discrete harmonic oscillators of period N:

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{j\frac{2\pi}{N}kn},$$

and the Fourier coefficients a_k are given by the formula:

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}.$$

Exercise 1. Let $x[n]$ be a discrete-time signal and N a positive finite integer such that $x[n] = 0$ for $n < -N$ and $n \geqslant N$. Let $x_p[n]$ be a periodic signal with period $2N$ such that $x_p[n] = x[n]$ for $-N \leqslant n < N$. Let a_k denote the DTFS coefficients of $x_p[n]$. Let $\mathcal{X}(w)$ denote the DTFT of $x[n]$. Find the relationship between a_k and $\mathcal{X}\left(\frac{2\pi}{2N} k\right)$. Explain how you can use your FFT code from homework 3 to rapidly compute $\mathcal{X}\left(\frac{2\pi}{2N} k\right)$ for $k = 0, \ldots, 2N-1$.

Exercise 2. This is a continuation of the previous exercise. Let $x[n] = \sqrt{1 - (n/10)^2}$ for $-10 \leqslant n \leqslant 10$, and 0 otherwise.

i. Let $N = 12$ and plot $x[n]$ and $x_p[n]$ to bring out the difference between the 2 signals.

ii. Let N=128 and use your FFT code from homework 4 to find $\mathcal{X}\left(\frac{2\pi}{2N} k\right)$ for $k = 0, \ldots, 2N-1$ and then plot $\left|\mathcal{X}\left(\frac{2\pi}{2N} k\right)\right|$.

iii. Let N=1024 and use your FFT code from homework 4 to find $\mathcal{X}\left(\frac{2\pi}{2N} k\right)$ for $k = 0, \ldots, 2N-1$ and then plot $\left|\mathcal{X}\left(\frac{2\pi}{2N} k\right)\right|$.

Approximate DTFT via DTFS. If $x[n]$ is decaying rapidly as $|n| \to \infty$, then the DTFT of $x[n]$ can be approximated by the finite sum:

$$\mathcal{X}(w) \sim \sum_{n=-N}^{N-1} x[n]\, e^{-j\omega n},$$

for some suitably large value of N.

Exercise 3. Let $x[n] = (1 + \sigma) e^{-n^2\sigma^2}$.

i. Make a representative plot of $x[n]$ for $\sigma = 10, 1$ and 0.1.

ii. Using the idea in the previous paragraph, use your FFT code from homework 4 to compute $\mathcal{X}(\omega)$ to reasonable amount of accuracy at more than 500 points in the interval $[0, 2\pi]$ for $\sigma = 10, 1$ and 0.1.

iii. Make an approximate plot of $|\mathcal{X}(\omega)|$ on the interval $[-\pi, \pi]$ for $\sigma = 10, 1$ and 0.1.

iv. Compare the plots of $x[n]$ and $|\mathcal{X}(\omega)|$. What conclusions can you draw?

Exercise 4. Find the discrete-time Fourier transforms of the following signals:

i. $x[n] = a^{|n|}$ for $0 < a < 1$.

ii. $x[n] = (-1)^n a^{|n|}$ for $0 < a < 1$.

iii. $x[n] = a^{|n|} + (-1)^n a^{|n|}$ for $0 < a < 1$.

iv. $x[n] = e^{j\frac{2\pi}{3}n} a^{|n|}$ for $0 < a < 1$.

v. $x[n] = e^{j\frac{2\pi}{3}2n} a^{|n|}$ for $0 < a < 1$.

vi. $x[n] = a^{|n|} + e^{j\frac{2\pi}{3}n} a^{|n|} + e^{j\frac{2\pi}{3}2n} a^{|n|}$ for $0 < a < 1$.

vii. For problems i, ii, iii and vi, plot $x[n]$ for $a = 1/2$. Also plot $|\mathcal{X}(\omega)|$ on the interval $[-\pi, \pi]$.