stability analysis

Nature of system response for various pole locations in s-plane-

Response of any linear time invariant system depends on its poles. Every pole has its own contribution in the system response. The total response of the system is addition of response of its all poles. Hence, if any pole is contributing unstable response then the ~~r~~ system response is also unstable. The poles of the system can be classified as

1. single real pole
2. Multiple real poles (repeated)
3. Complex conjugate pole pair
4. Repeated complex conjugate pole pairs.

- Let us consider these pole types one by one.

1. Single real pole-

Let the pole is at $s=a$, then its transfer function is $G(s) = \frac{1}{s-a}$. Then its time domain impulse response is,

$$g(t) = e^{at}$$

case-1 $a<0$ (pole in left half of s-plane)

for $a<0$, $g(t)$ decreases as time increases and at $t=\infty$, $g(t)=0$. This response is stable in nature.

Case-2 $a>0$ (pole in right half of s-plane)

For $a>0$, $g(t)$ increases as time increases and at $t=\infty$, $g(t)=\infty$. This response is unstable in nature.

case-3 $a=0$ (pole at origin)

For $a=0$, $g(t)=1$ for all t. The $g(t)$ is bounded but it does not approach to zero and response is marginally stable.

2. Complex conjugate pole pair–

Let the pole is at $s = a \pm jb$, then the corresponding transfer function is,

$$G(s) = \frac{1}{(s-a+jb)(s-a-jb)}$$

Its contribution to system response is,

$$g(t) = \frac{1}{b} e^{at} \sin bt$$

Case–1 $a<0$ (complex conjugate pole pair in left half)

For $a<0$, as time t increases, e^{at} decreases, hence the system response is ~~decrea~~ sine wave with decreasing amplitude. This response is stable in nature.

case–2 $a>0$ (complex conjugate pole pair in right half)

For $a>0$, as time t increases, e^{at} increases, hence the system response is sine wave with increasing amplitude. This response is unstable in nature.

case–3 $a=0$ (complex conjugate pole pair on imaginary axis)

For $a=0$, $g(t) = \frac{1}{b} \sin bt$ which is a sine wave

with constant amplitude. This response is marginally stable.

3. Multiple (repeated) real poles–

Let the poles (two) are at $s=a$ then the transfer function is, $G(s) = \frac{1}{(s+a)^2}$

Its contribution to system response is,

$$g(t) = te^{at}$$

Scanned by TapScanner

case-1 $a<0$ (poles in left half of s-plane)

For $a<0$, $g(t)$ decreases as time t increases and goes to zero at $t=\infty$. This response is stable.

case-2 $a>0$ (poles in right half of s-plane)

For $a>0$, $g(t)$ increases as time t increases and goes to infinity at $t=\infty$. This response is unstable.

case-3 $a=0$ (poles at origin)

For $a>0$, $g(t)=t$. Thus it increases with time and is unstable.

4. Repeated complex conjugate pole pair-

Let the poles are at $s=a\pm jb$. Then the transfer function is,

$$G(s) = \frac{1}{(s-a+jb)^2(s-a-jb)^2}$$

case-1 $a<0$ (poles in left half of s-plane)

For $a<0$, the response is ~~dim~~ ~~decreasing~~ sinusoidal with decreasing amplitude and it is stable.

case-2 $a>0$ (poles in right half of s-plane)

For $a>0$, the response is sinusoidal with increasing amplitude and it is unstable.

case-3 $a=0$ (~~po~~ poles on imaginary axis)

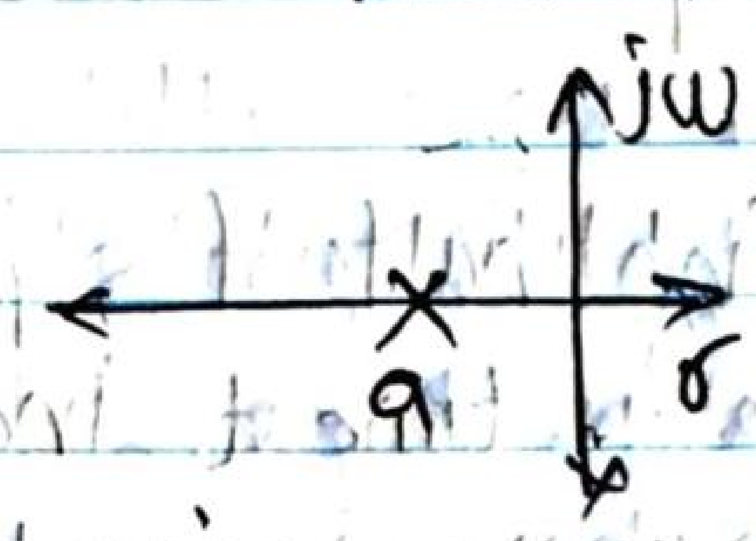
For $a=0$, the response is sinusoidal with increasing amplitude and it is unstable.

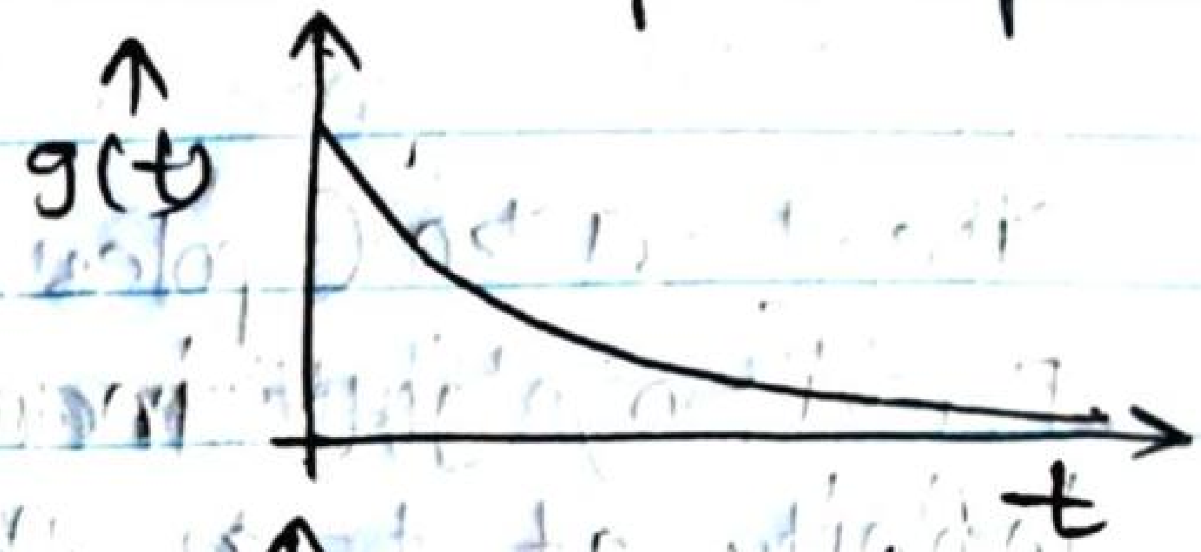
~~This pep~~ The responses for various pole locations can be sketched as,

Pole location | Impluse response

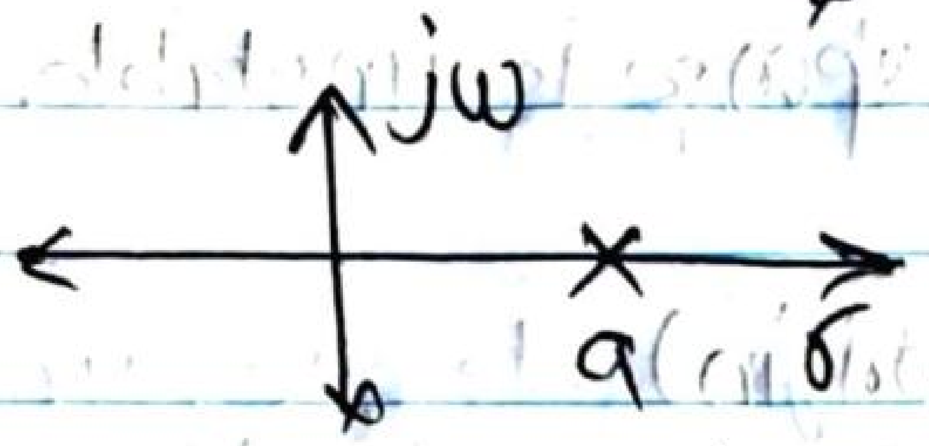
Pole at $s = a$

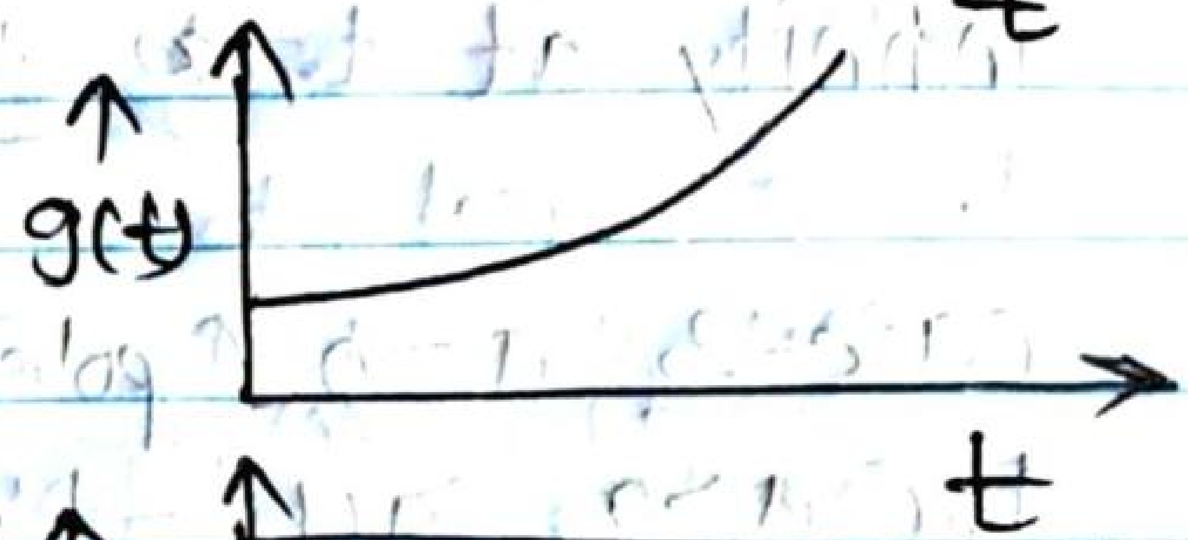
case I, $a < 0$



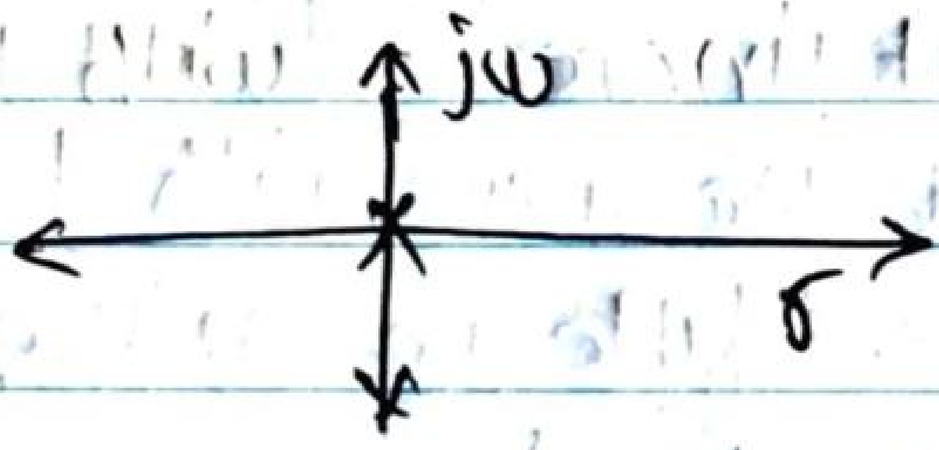


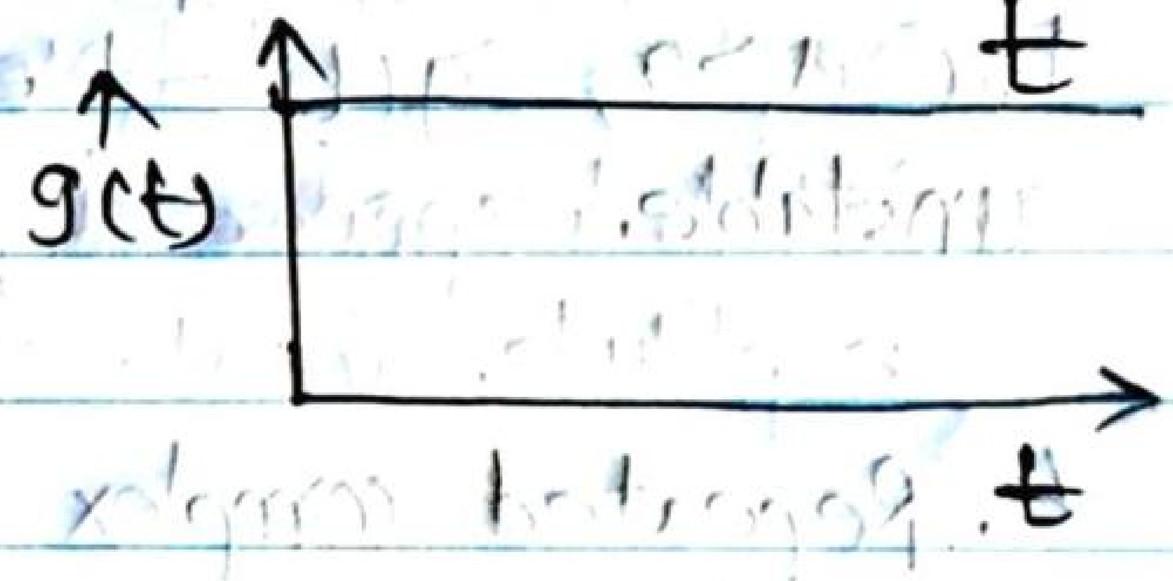
case 2, $a > 0$





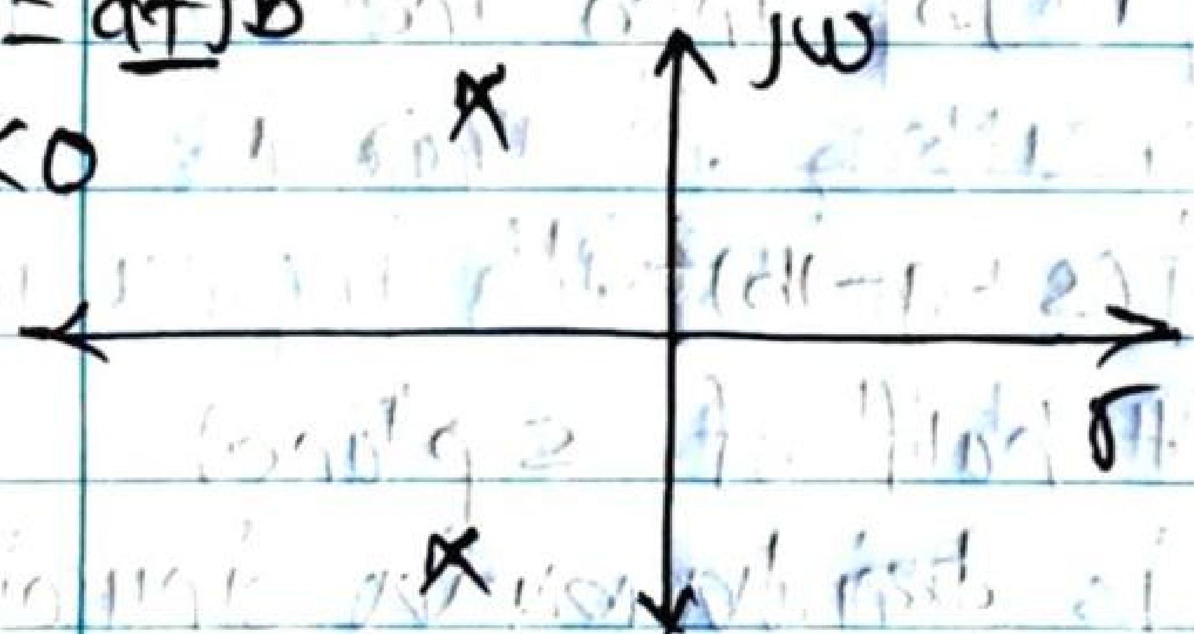
case 3, $a = 0$

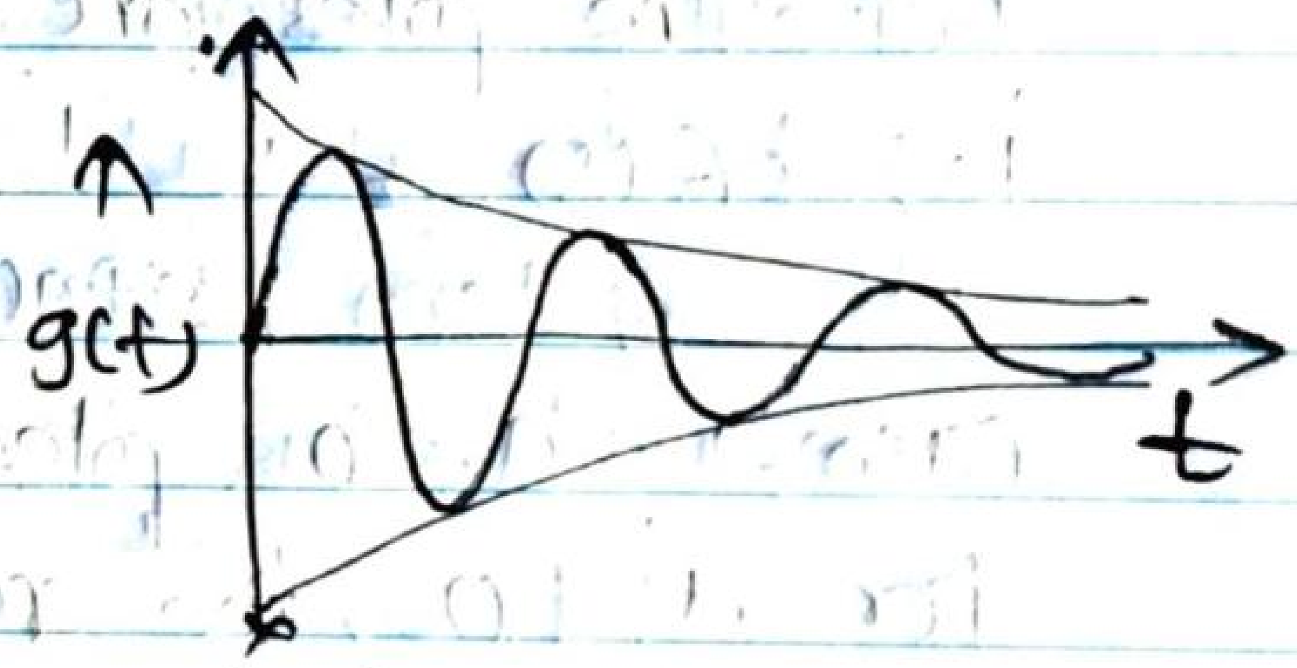




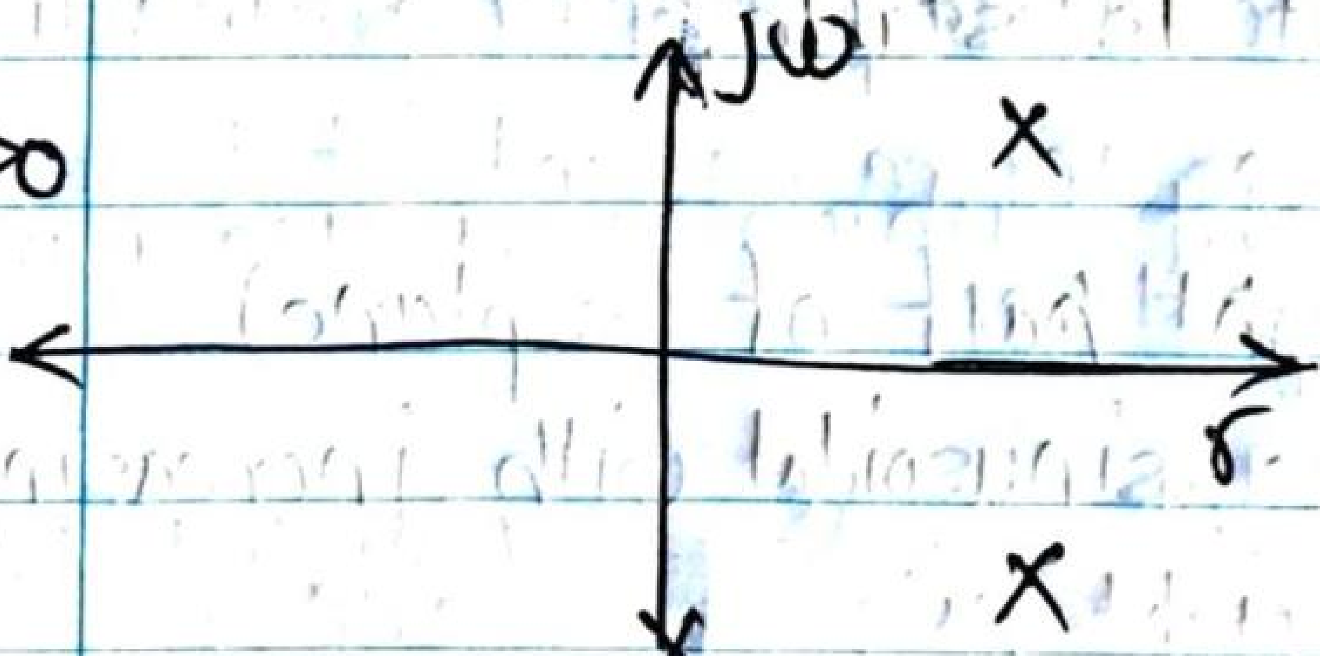
Pole at $s = a \pm jb$

case 1, $a < 0$

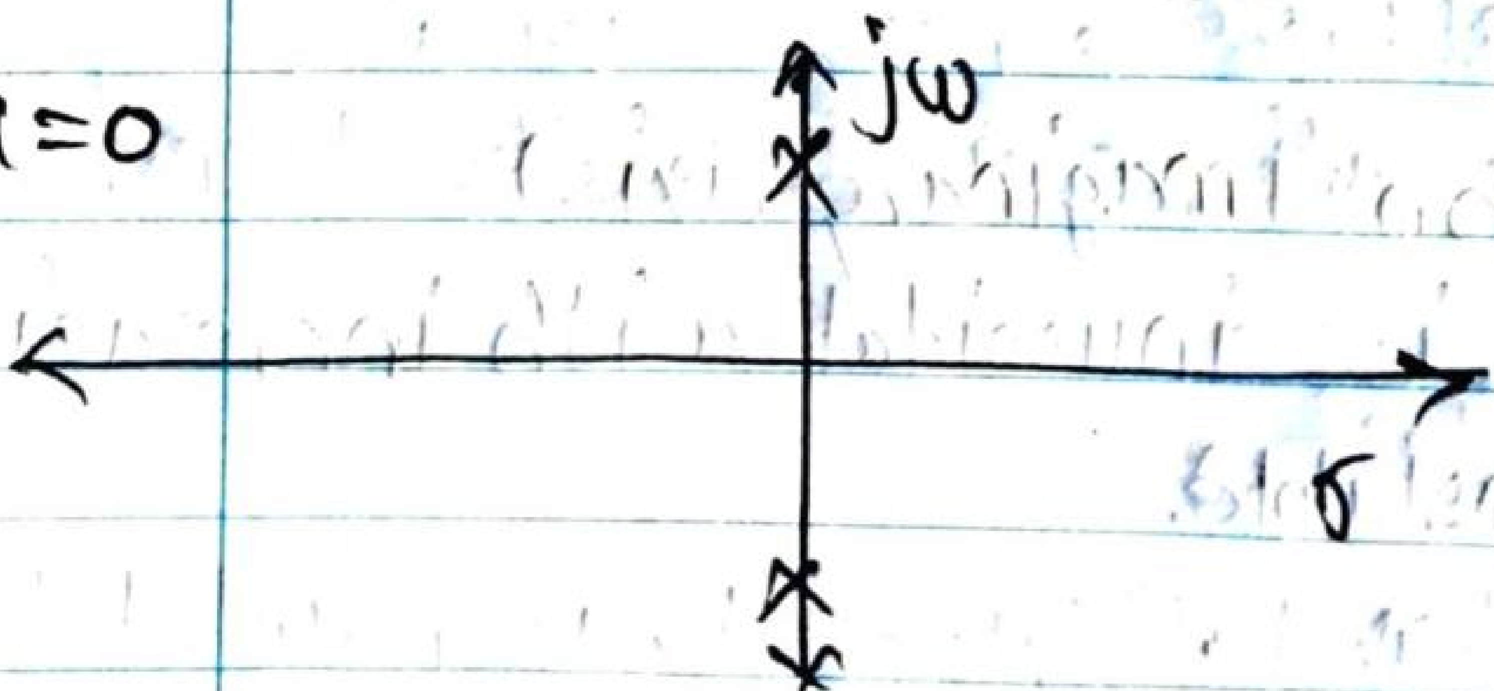


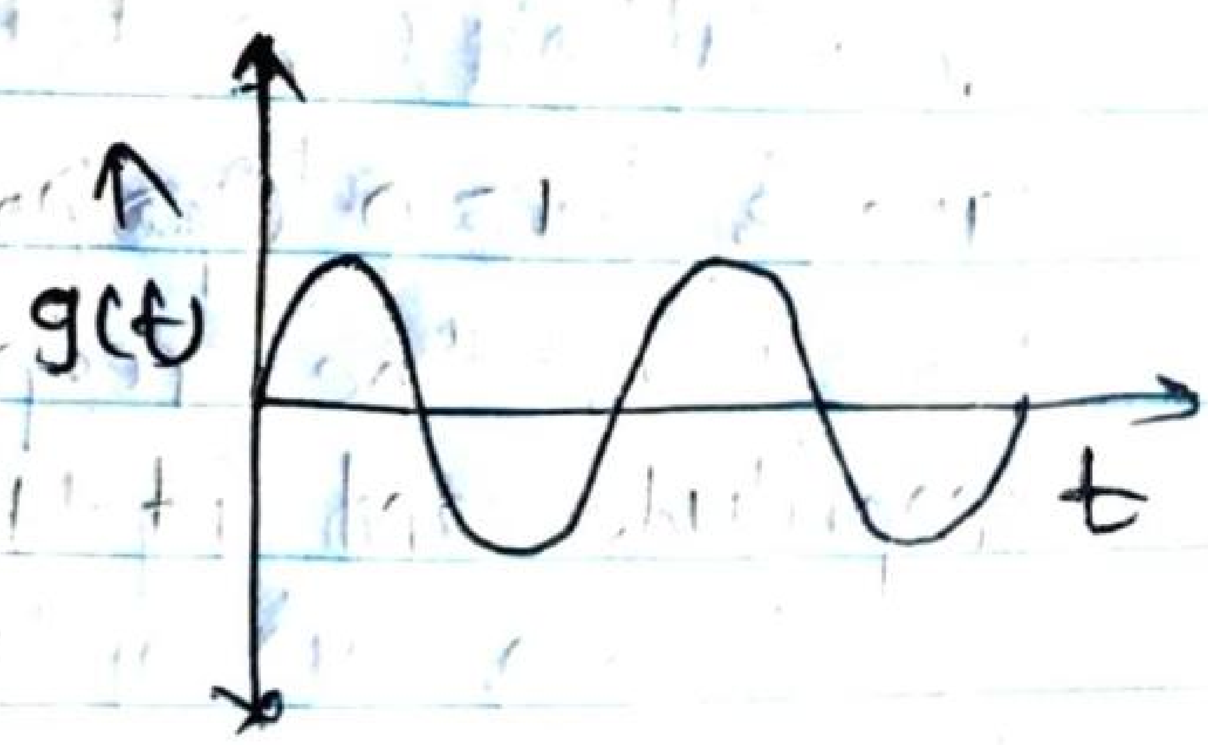


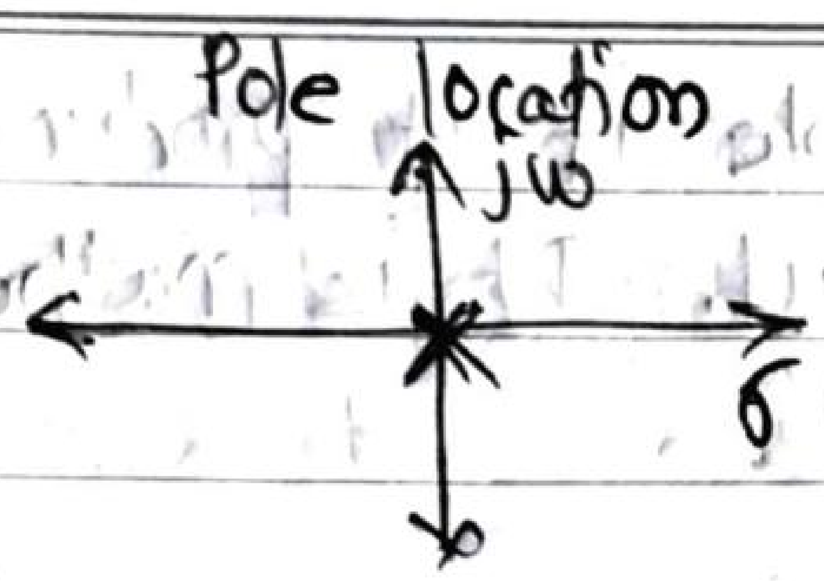
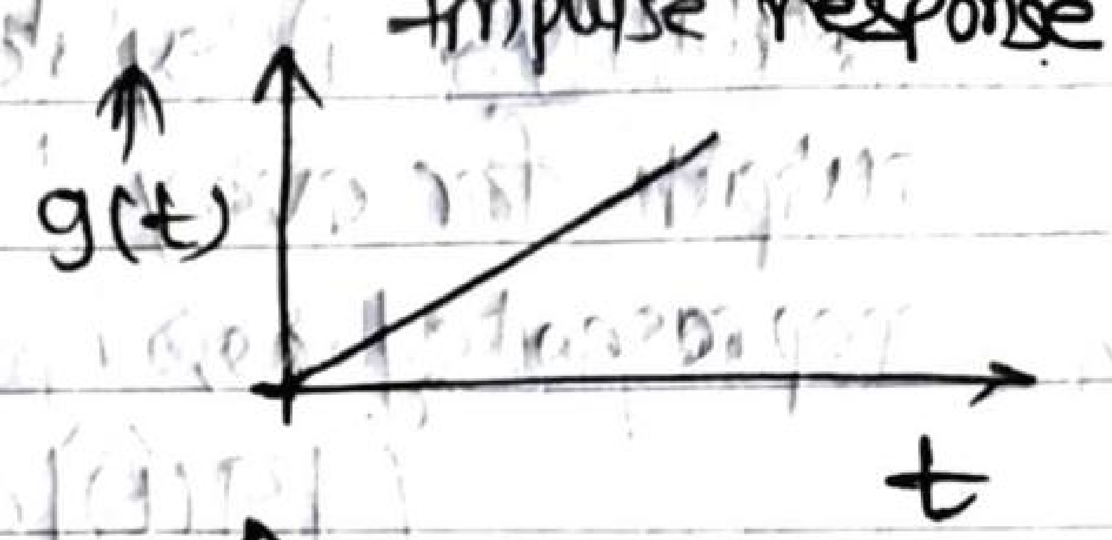
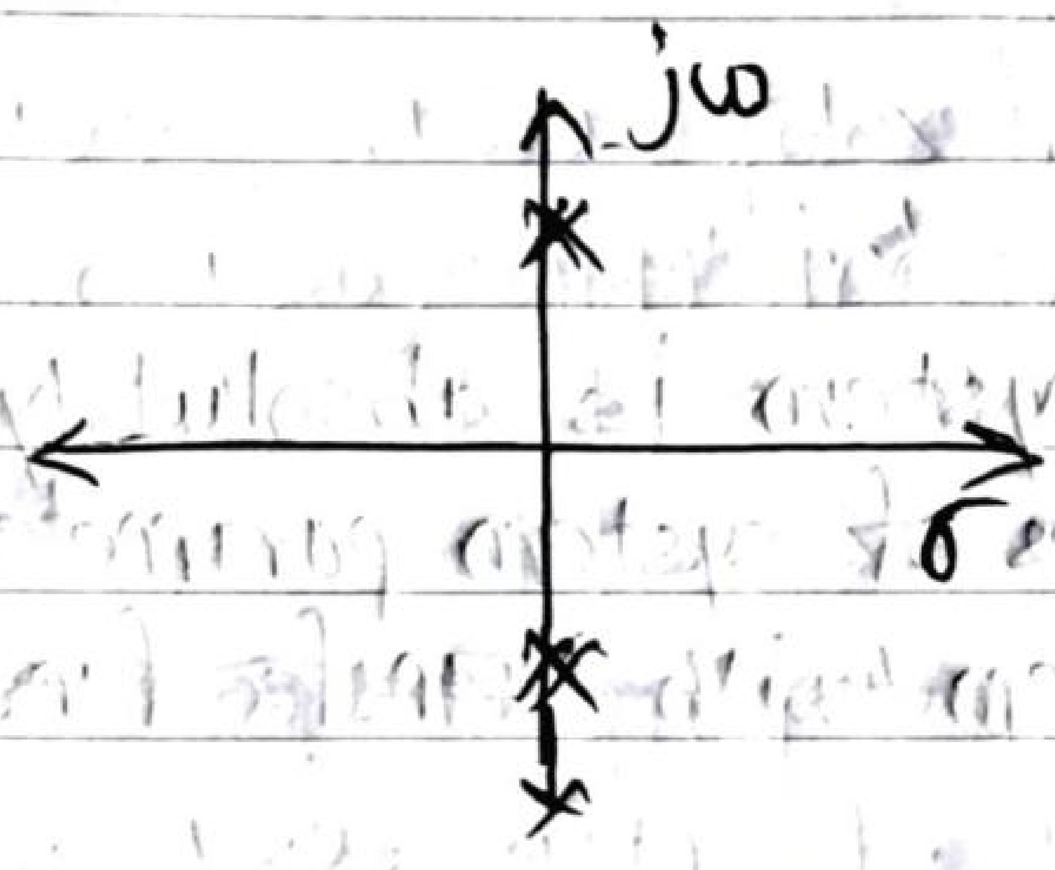
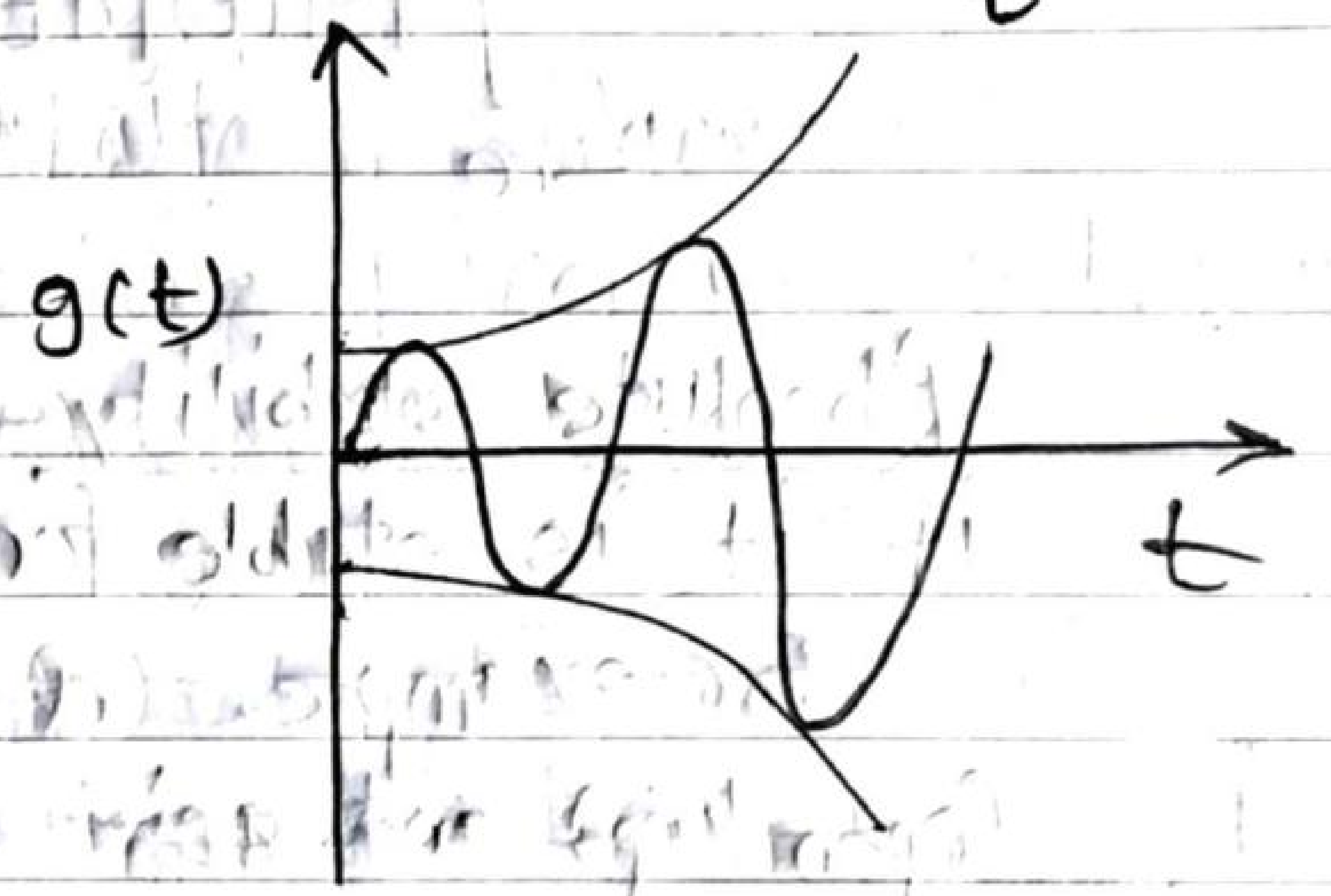
case 2, $a > 0$



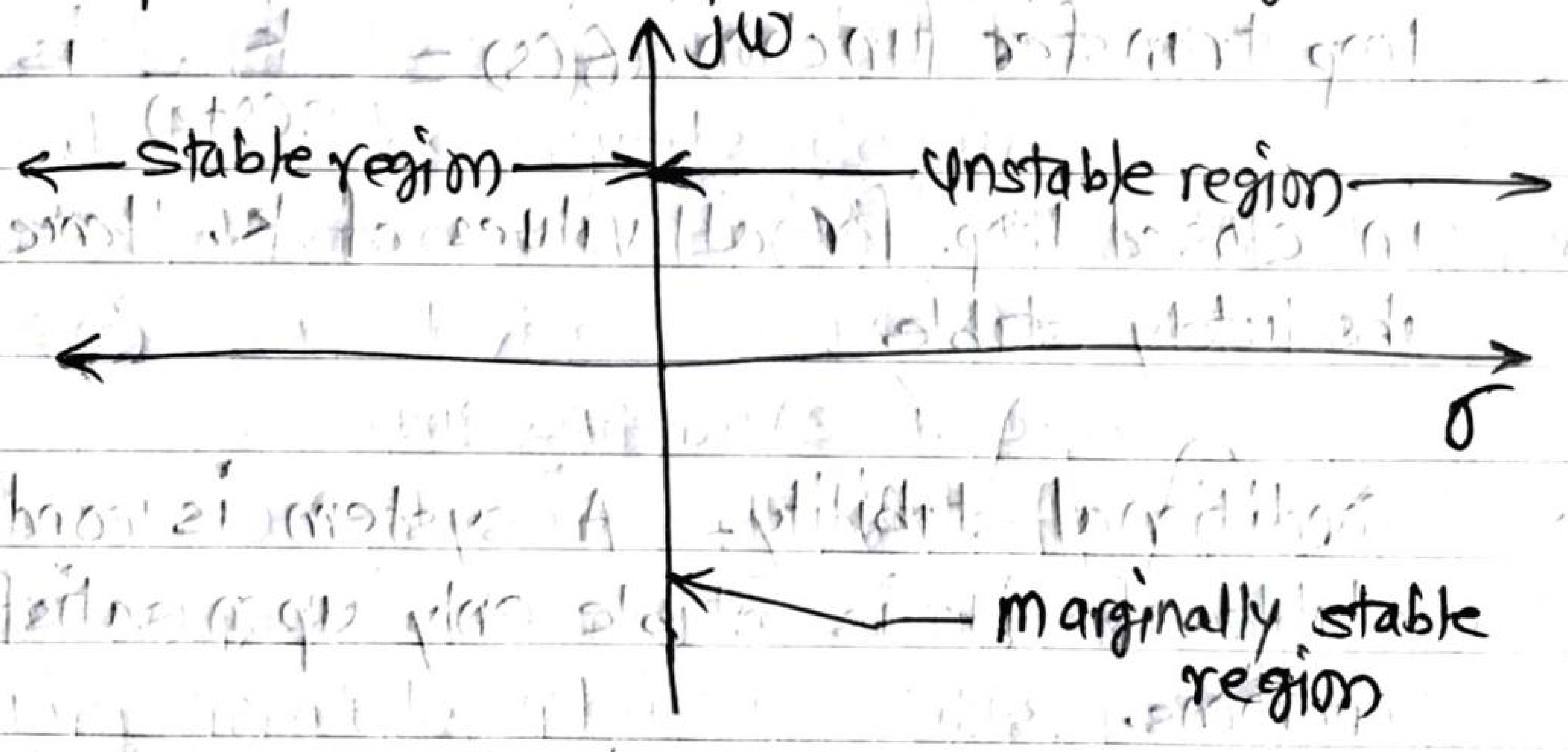
case 3, $a = 0$





	Pole location	Impulse response
Double Pole at origin		
Repeated complex poles on imaginary axis		

Thus s-plane can be devided into three regions as



stability based on pole locations

1. A system is stable if all its poles are in the left half of s-plane.
2. A system is unstable if
   a] Any of the pole is in the right half of s-plane
   b] Repeated poles are on the imaginary axis.
3. A system is marginally stable if non repeated poles are on the imaginary axis and remaining poles are in the left half of s-plane.

Stability- A system is stable if it produces bounded output for every bounded input. It is mathematically represented as,

$$\int_{-\infty}^{\infty} |g(t)|\,dt < \infty$$

Absolute stability- A system is absolutely stable if it is stable for values of system parameter.

~~For example, a system with transfer function (open loop) and~~

For example, a unity feedback system with open loop transfer function $G(s) = \frac{K}{s(s+2)}$ is stable in closed loop for all values of K. Hence system is absolutely stable.

Conditional stability- A system is conditionally stable if it is stable only upon satisfying certain conditions.

For example, a unity feedback system with open loop transfer function $G(s) = \frac{K}{s(s+2)(s+3)}$ is stable in closed loop for $0 < K < 30$ and is unstable for $K > 30$.

Relative stability- It is a measure of stability between two or more systems. A system with less settling time is relatively more stable. In other way, the system with poles away from imaginary axis (in the left half of s-plane) is more stable.

Routh Hurwitz stability criterion-

Necessary condition- The system with closed loop characteristic equation $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n = 0$ may be stable if all coefficients in the characteristic equation have same sign and no coefficient is zero, otherwise the system is not stable.

Sufficiency condition- The system is stable if all elements in the first column of Routh array have same sign. ~~If the elements in the first column of Routh~~ The Routh array is formed as shown below.

$$
\begin{array}{c|ccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \dots \\
s^{n-1} & a_1 & a_3 & a_5 & \dots & \\
s^{n-2} & b_1 & b_2 & b_3 & & \\
s^{n-3} & c_1 & c_2 & c_3 & & \\
\vdots & & & & & \\
s^2 & d_1 & a_n & & & \\
s^1 & e_1 & & & & \\
s^0 & a_n & & & &
\end{array}
$$

The first two rows are formed from the characteristic equation.

where, $b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$, $b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$

In this way all elements in third row are calculated.

Then, $c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$, $c_2 = \frac{b_1 a_5 - a_1 b_3}{b_2}$

This procedure is continued till $n+1^{th}$ row to complete the Routh array. If there are sign changes in the first column of Routh array the system is unstable and number of poles

Scanned by TapScanner

equal to number of sign changes are in right half of s-plane.

Example- consider a system with characteristic equation
$$Q(s) = s^4 + 4s^3 + 6s^2 + 2s + 3 = 0$$

soln The given characteristic equation is,
$$Q(s) = s^4 + 4s^3 + 6s^2 + 2s + 3 = 0$$

$\therefore$ The Routh array is,

s^4	1	6	3
s^3	4	2	
s^2	5.5	3	
s^1	-0.1818		
s^0	3		

$\therefore$ There are two sign changes in first column of Routh array, hence the system is unstable and two poles are in the right half of s-plane.

Example- Consider a system with characteristic equation
$$Q(s) = s^4 + 4s^3 + 6s^2 + 2s + 1 = 0$$

soln- The given characteristic equation is,
$$Q(s) = s^4 + 4s^3 + 6s^2 + 2s + 1 = 0$$

$\therefore$ The Routh array is,

s^4	1	6	1
s^3	4	2	
s^2	5.5	1	
s^1	1.2727		
s^0	1		

$\therefore$ There is no sign change in the first column of Routh array and the system is stable.

Special cases and their remedy-

Special case-1 When a zero appears in the first column of Routh array — In this case, the calculation of next rows of Routh array is not possible as all elements become ∞ and hence stability analysis is not possible. To overcome this problem there are two possible solutions.

Let the closed loop characteristic equation is,

$$Q(s) = s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

Then Routh array is,

$$\begin{array}{c|cc} s^3 & 1 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \frac{a_1 a_2 - a_3}{a_1} = 0 & \\ s^0 & \infty & \end{array}$$

if $a_1 a_2 - a_3 = 0$ then

To avoid this 0 is replaced by ε ~~and~~ such that $\varepsilon \to 0$ and further calculations are performed to form Routh array as,

$$\begin{array}{c|cc} s^3 & 1 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \varepsilon & \\ s^0 & a_3 & \end{array}$$

The investigation about sign change is done by replacing ε with 0. Then if there is sign change in the first column of Routh array the system is unstable and if there is no sign change in the first column of Routh array the system is marginally stable.

Example – Investigate the stability of system with characteristic equation $Q(s) = s^4 + 3s^3 + 2s^2 + 6s + 4 = 0$

Sol^n – The characteristic equation is,

$$Q(s) = s^4 + 3s^3 + 2s^2 + 6s + 4 = 0$$

The Routh array is,

s^4	1	2	4
s^3	3	6	
s^2	$0 \to \varepsilon$	4	
s^1	$\frac{6\varepsilon - 12}{\varepsilon}$		
s^0	4		

$$\lim_{\varepsilon \to 0} \frac{6\varepsilon - 12}{\varepsilon} = -\infty$$

$\therefore$ The sign in first column of Routh array are,

s^4	+	
s^3	+	
s^2	+	
s^1	−	2 sign changes
s^0	+	

As there are two sign changes, the system is unstable.

Another solution to this problem is to replace s in characteristic equation by $\frac{1}{z}$ and obtain new characteristic equation. However the remedy fails when the same problem occurs with newly formed equation.

Example- Investigate the stability of system with characteristic equation, $Q(s) = s^4 + 2s^3 + 2s^2 + 4s + 6 = 0$

Sol^n The characteristic equation is,

$$Q(s) = s^4 + 2s^3 + 2s^2 + 4s + 6 = 0$$

The Routh array is,

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 6 \\ s^3 & 2 & 4 & \\ s^2 & 0 & 6 & \end{array}$$

Further calculation of Routh array is not possible.

Replacing s by $\frac{1}{z}$ we get,

$$Q(z) = \frac{1}{z^4} + \frac{2}{z^3} + \frac{2}{z^2} + \frac{4}{z} + 6 = 0$$

$$\therefore \quad 6z^4 + 4z^3 + 2z^2 + 2z + 1 = 0$$

The Routh array is,

$$\begin{array}{c|ccc} z^4 & 6 & 2 & 1 \\ z^3 & 4 & 2 & \\ z^2 & -1 & 1 & \\ z^1 & 6 & & \\ z^0 & 1 & & \end{array}$$

$\therefore$ There are two sign changes in the first column of Routh array and hence the system is unstable.

~~Case~~ Special case-2 When all elements in a row of Routh array become zero. In this case also it is not possible to ~~form~~ complete the Routh array. In this case, the row just above the row with all zero elements is considered to form auxillary equation. This auxillary equation is differentiated w.r.t. 's' to get new auxillary equation of the row with all zeros and further calculations are performed to form Routh array. ~~The~~ if there is no sign change, the system is marginally stable otherwise system is unstable.

Example- consider the system with characteristic equation $Q(S) = s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 6s + 4 = 0$ and investigate stability.

sol^n $Q(s) = s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 6s + 4 = 0$

The Routh array is,

$$\begin{array}{c|cccc} s^6 & 1 & 5 & 8 & 4 \\ s^5 & 3 & 9 & 6 & \\ s^4 & 2 & 6 & 4 & \\ s^3 & 0 & 0 & & \end{array}$$

As all elements the row s^3 are 0, consider eq^n of row s^4 as, $A(s) = 2s^4 + 6s^2 + 4$

$\therefore \frac{dA(s)}{ds} = 8s^3 + 12s$

$\therefore$ Replace 0s in row s^3 with 8 and 12 to get Routh array

$$\begin{array}{c|cccc} s^6 & 1 & 5 & 8 & 4 \\ s^5 & 3 & 9 & 6 & \\ s^4 & 2 & 6 & 4 & \\ s^3 & 0 \rightarrow 8 & 0 \rightarrow 12 & & \\ s^2 & 3 & 4 & & \\ s^1 & 1.33 & & & \\ s^0 & 4 & & & \end{array}$$

$\therefore$ There is no sign change in the first column of Routh array and the system is marginally stable.

Scanned by TapScanner

Use of Routh stability criterion for ~~inverse~~ analysis of conditional stability-

Routh array can be formed for the characteristic equation with unknown constant k and its ~~value of~~ range for stable operation can be determined. This is much useful in controller tuning process by certain methods, in sketching root locus.

example- Consider a ~~cl~~ closed loop characteristic equation $Q(s) = s^4 + 10s^3 + 29s^2 + 20s + k = 0$. Determine the range of k for stability, value of k and frequency of oscillations when system is marginally stable.

sol^n The given characteristic eq^n is,

$$s^4 + 10s^3 + 29s^2 + 20s + k = 0$$

$$\begin{array}{c|ccc} s^4 & 1 & 29 & k \\ s^3 & 10 & 20 & \\ s^2 & 27 & k & \\ s^1 & \frac{540-10k}{27} & & \\ s^0 & k & & \end{array}$$

For the system to be system, all elements in the first column of Routh array must have same sign

$\therefore$ 1. $k > 0$

2. $\frac{540-10k}{27} > 0$

$\therefore 10k < 540$

$\therefore k < 54$

$\therefore$ Range of k for stable operation is $0 < k < 54$

At marginal stability,

$k = 54 = k_{mar}$

From row s^2

$27s^2 + k = 0$

$\therefore 27s^2 + 54 = 0$

$\therefore s^2 = -2$

$\therefore s = \pm j1.414$

$\therefore \omega_{mar} = 1.414$ rad/sec

Examples on Routh Hurwitz stability criterion–

1. Investigate the stability of system with characteristic equation $Q(s) = s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + 1 = 0$

Soln The given characteristic equation is,

$$Q(s) = s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + 1$$

∴ Routh array is,

s^5	1	10	5
s^4	5	10	1
s^3	8	4.8	
s^2	7	1	
s^1	3.6571		
s^0	1		

As there is no sign change in the first column of Routh array, the system is stable.

2. Investigate the stability of system with characteristic equation $Q(s) = s^5 + 3s^4 + 3s^3 + 7s^2 + 5s + 4 = 0$

Soln The given characteristic equation is,

$$Q(s) = s^5 + 3s^4 + 3s^3 + 7s^2 + 5s + 4 = 0$$

∴ Routh array is,

s^5	1	3	5
s^4	3	7	4
s^3	0.6667	3.6667	
s^2	-9.4991	4	
s^1	3.9474		
s^0	4		

Since there are two sign changes in the first column of Routh array, the system is unstable with 2 poles in the right half of s plane.

3. Investigate the stability of system with characteristic equation
$Q(s) = s^5 + 3s^4 + 5s^3 + 3s^2 + 5s + 3 = 0$

Sol^n The given characteristic equation is,
$Q(s) = s^5 + 3s^4 + 5s^3 + 3s^2 + 5s + 3 = 0$

The Routh array is,

s^5	1	5	5
s^4	3	3	3
s^3	4	4	
s^2	$0 \to \varepsilon$	3	
s^1	$\frac{4\varepsilon - 12}{\varepsilon}$		
s^0	3		

$$\lim_{\varepsilon \to 0} \frac{4\varepsilon - 12}{\varepsilon} = -\infty$$

∴ The sign pattern is

s^5	+	
s^4	+	
s^3	+	
s^2	+	1
s^1	−	
s^0	+	2

~~Hence~~ Hence there are two sign changes in the first column of Routh array and system is unstable.

4. Investigate the stability of system with characteristic equation
$Q(s) = s^5 + 6s^4 + 15s^3 + 30s^2 + 44s + 24 = 0$

Sol^n The given characteristic equation is,
$Q(s) = s^5 + 6s^4 + 15s^3 + 30s^2 + 44s + 24 = 0$

Scanned by TapScanner

The Routh array is,

s^5	1	15	44
s^4	6	30	24
s^3	10	40	
s^2	6	24	
s^1	$0 \to \epsilon$		
s^0	24		

$\therefore$ There is no sign change in the first column of Routh array and the system is marginally stable.

5. Investigate the stability of system with characteristic equation $Q(s) = s^5 + s^4 + 13s^3 + 13s^2 + 36s + 36 = 0$

Sol^n The given characteristic equation is,

$Q(s) = s^5 + s^4 + 13s^3 + 13s^2 + 36s + 36 = 0$

The Routh array is,

s^5	1	13	36
s^4	1	13	36
s^3	0	0	

$\therefore$ $A(s) = s^4 + 13s^2 + 36$

$\therefore \frac{d}{ds} A(s) = 4s^3 + 26s$

s^5	1	13	36
s^4	1	13	36
s^3	4	26	
s^2	6.5	36	
s^1	3.8462		
s^0	36		

As there is no sign change in the first column of Routh array the system is marginally stable

Scanned by TapScanner

6. Investigate stability of system with characteristic equation
$Q(s) = s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$

Sol^n The given characteristic equation is,
$Q(s) = s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$

s^6	1	8	20	16
s^5	2	12	16	
s^4	2	12	16	
s^3	0	0		

$\therefore A(s) = 2s^4 + 12s^2 + 16$

$\therefore \frac{d}{ds} A(s) = 8s^3 + 24s$

$\therefore$ Modified Routh array is,

s^6	1	8	20	16
s^5	2	12	16	
s^4	2	12	16	
s^3	8	24		
s^2	6	16		
s^1	2.6667			
s^0	16			

As there is no sign change in the first column of Routh array the system is marginally stable.

7. The unity feedback system has open loop transfer function
$G(s) = \dfrac{K}{s(s+2)(s+5)(s+10)}$. Determine the range of K for system stability, values of K and frequency of oscillations at marginal stability.

Sol^n The system has,
$G(s) = \dfrac{K}{s(s+2)(s+5)(s+10)}, \; H(s) = 0$

The closed loop characteristic equation is,

$$1 + G(s)H(s) = 0$$

$$\therefore 1 + \frac{k}{s(s+2)(s+5)(s+10)} = 0$$

$$\therefore s(s+2)(s+5)(s+10) + k = 0$$

$$\therefore s^4 + 17s^3 + 80s^2 + 100s + k = 0$$

The Routh array is,

s^4	1	80	k
s^3	17	100	
s^2	74.1176	k	
s^1	$\frac{7411.76-17k}{74.1176}$		
s^0	k		

For stability,

1. $k > 0$

2. $\frac{7411.76-17k}{74.1176} > 0$

$$\therefore k < 435.9862$$

$\therefore$ Range of k for stability is,

$$0 < k < 435.9862$$

$$\therefore K_{mar} = 435.9862$$

From row s^2

$$74.1176 s^2 + 435.9862 = 0$$

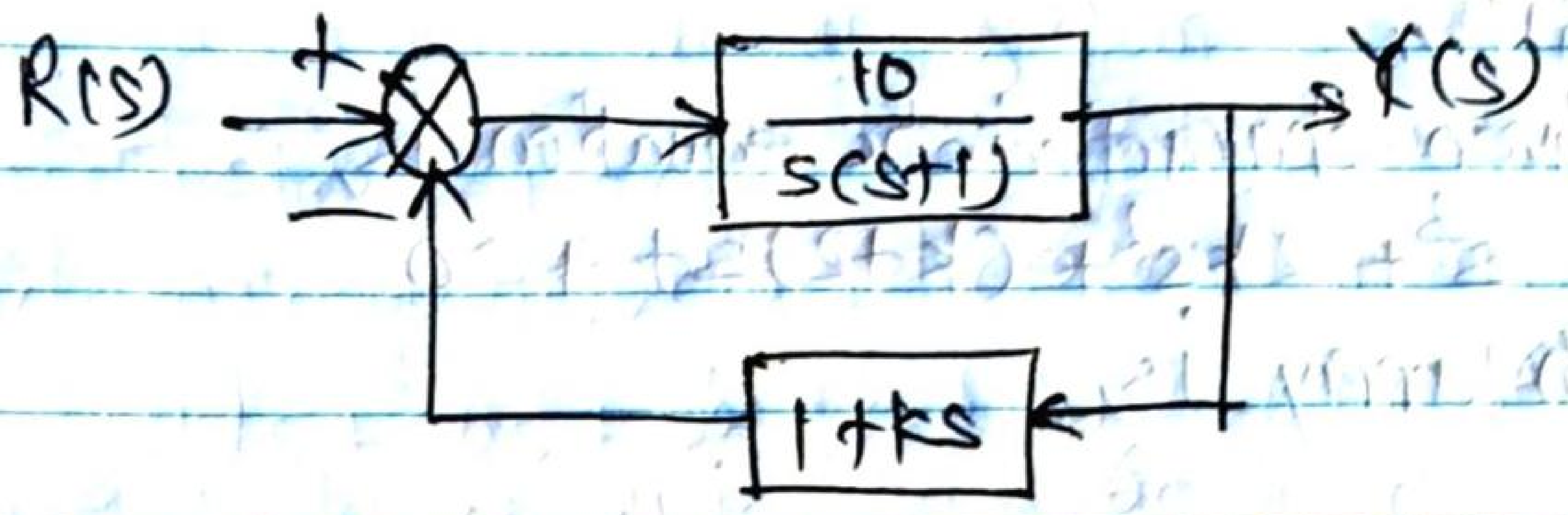
$$\therefore s^2 = -5.8824$$

$$\therefore s = \pm j\,2.4254$$

$$\therefore \omega_{mar} = 2.4254 \text{ rad/sec}$$

Scanned by TapScanner

8. For the system shown below, determine range of k for stability.



Sol^n For the given system,

$$G(s) = \frac{10}{s(s+1)}, \quad H(s) = 1+ks$$

∴ closed loop characteristic equation is,

$$1 + G(s)H(s) = 0$$

$$\therefore 1 + \frac{10(1+ks)}{s(s+1)} = 0$$

$$\therefore s^2 + s + 10ks + 10 = 0$$

∴ The Routh array is,

s^2	1	10
s^1	$1+10k$	
s^0	10	

∴ For stability,

$$1 + 10k > 0$$

$$\therefore k > -0.1$$

∴ Range of k is $-0.1 < k < \infty$

9. For the system with characteristic equation $Q(s) = s^3 + 2ks^2 + (k+2)s + 4 = 0$, find range of k for stability.

Sol^n The given characteristic equation is,

$$Q(s) = s^3 + 2ks^2 + (k+2)s + 4 = 0$$

$\therefore$ Routh array is,

$$\begin{array}{c|cc} s^3 & 1 & k+2 \\ s^2 & 2k & 4 \\ s^1 & \dfrac{(k+2)2k-4}{2k} & \\ s^0 & 4 & \end{array}$$

$\therefore$ For the system to be stable,

1. $2k > 0$

$\therefore k > 0$

2. $\dfrac{(k+2)2k-4}{2k} > 0$

$\therefore (k+2)(2k) - 4 > 0$

$\therefore 2k^2 + 4k - 4 > 0$

$\therefore k^2 + 2k - 2 > 0$

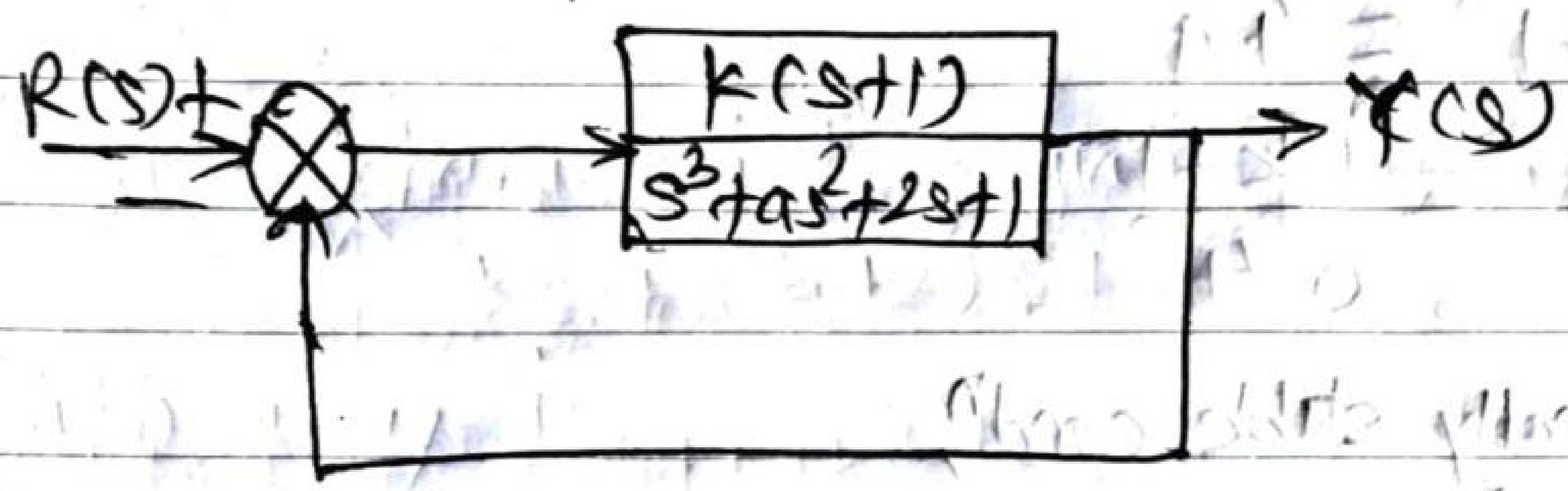
$\therefore (k - 0.732)(k + 2.732) > 0$

$\therefore k > 0.732, \; k > -2.732$

$\therefore$ The range of k for stability is,

$0.732 < k < \infty$

10. For the system shown below find the values of k and a such that the system oscillates with frequency 2 rad/sec.



Sol^n For the given system,

$$G(s) = \frac{k(s+1)}{s^3+as^2+2s+1}, \quad H(s) = 1$$

∴ The characteristic equation is,

$$1 + \frac{k(s+1)}{s^3+as^2+2s+1} = 0$$

$$\therefore s^3 + as^2 + 2s + 1 + ks + k = 0$$

$$\therefore s^3 + as^2 + (2+k)s + 1 + k = 0$$

∴ Routh array is,

s^3	1	$k+2$
s^2	a	$k+1$
s^1	$\frac{a(k+2)-(k+1)}{a}$	
s^0	$k+1$	

~~For stabili~~ From row s^2

$$as^2 + k + 1 = 0$$

$$\therefore s^2 = -\left(\frac{k+1}{a}\right)$$

$$\therefore s = \pm j\sqrt{\frac{k+1}{a}}$$

$$\therefore \sqrt{\frac{k+1}{a}} = 2$$

$\therefore \frac{k+1}{a} = 4$

$\therefore k+1 = 4a$

$a = \frac{k+1}{4}$

At marginally stable condⁿ,

$\therefore \cancel{\left(\frac{k+1}{4}\right)(k+2)}$

$a(k+2) - (k+1) = 0$

$\therefore \left(\frac{k+1}{4}\right)(k+2) - (k+1) = 0$

$\therefore k^2 + 3k + 2 - 4(k+1) = 0$

$\therefore k^2 - k - 2 = 0$

$\therefore (k-2)(k+1) = 0$

$\therefore k = 2, -1$ ($\because k = -1$ violets condition)

$k = 2$

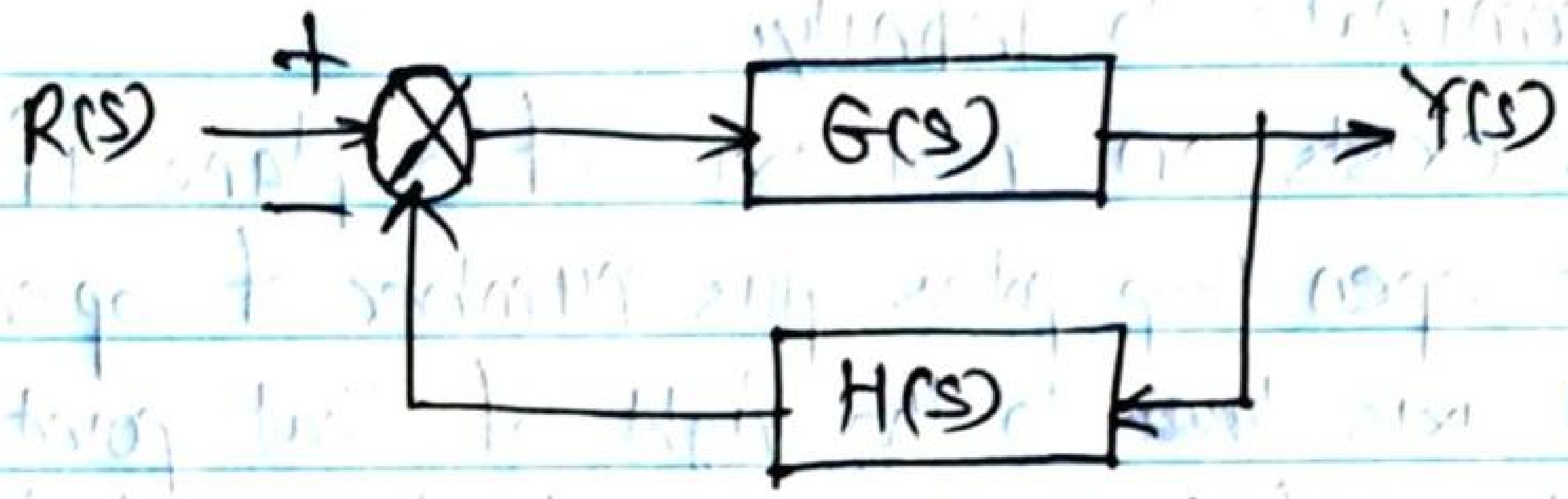
$\therefore a = \frac{k+1}{4} = \frac{3}{4} = 0.75$

Scanned by TapScanner

Root locus- Root locus is a locus of closed loop poles as system gain k is varied from 0 to ∞.

Angle and magnitude condition of root locus-

Consider a system with forward path function $G(s)$ and feedback $H(s)$.



The closed loop poles of this system are the roots of closed loop characteristic equation

$$1+G(s)H(s)=0$$

$$\therefore G(s)H(s)=-1$$

$G(s)H(s)$ is a complex quantity and can be represented as $|G(s)H(s)|\angle G(s)H(s)$.

~~$\therefore |G(s)H(s)| \angle G(s)H(s) = 1 \angle \pm 2(n+1)180^\circ$, n=0,1,2,...~~

$$\therefore |G(s)H(s)|=1$$

$$\angle G(s)H(s) = \pm(2n+1)180^\circ, \; n=0,1,2,\ldots$$

The above two equations are called as magnitude condition and angle condition respectively.

Any point in s-plane is on the root locus if and only if it satisfies these two conditions.

Properties of root locus (construction rules)

1. Root locus is symmetric about real axis of s-plane.
2. Root locus starts at open loop pole with $k=0$ and terminates either at open loop zero or infinity with $k=\infty$. If system has 'P' poles and 'Z' zeros, then Z branches of root locus terminate to open loop zeros and $P-Z$ branches terminate to infinity.
3. Root locus exists on real axis of s-plane if and only if number of open loop poles plus number of open loop zeros on real axis to the right of that point is odd.
4. The branches of root locus terminating at infinity travels along the asymptotes with angle,

$$\theta = \frac{(2n+1)\,180^\circ}{P-Z}, \quad n = 0, 1, 2, \ldots, P-Z-1$$

with number of asymptotes $= P-Z$

5. The asymptotes start on real axis at a point centroid and terminate at infinity. The centroid is given by,

$$\sigma = \frac{\Sigma \text{ real part of poles} - \Sigma \text{ real part of zeros}}{P-Z}$$

6. The root locus leaves real axis at a point called breakaway point. The breakaway points are roots of $\frac{dk}{ds}=0$. It should be noted that all roots of $\frac{dk}{ds}=0$ are not breakaway points.
7. The root locus intersects the imaginary axis at a point where the point on imaginary axis equals the frequency at marginal stability of system.

~~8. The complex poles and zeros depart / arrive~~

8. The root locus departs from complex open

loop poles at an angle

$$\phi_d = 180^\circ - \phi$$

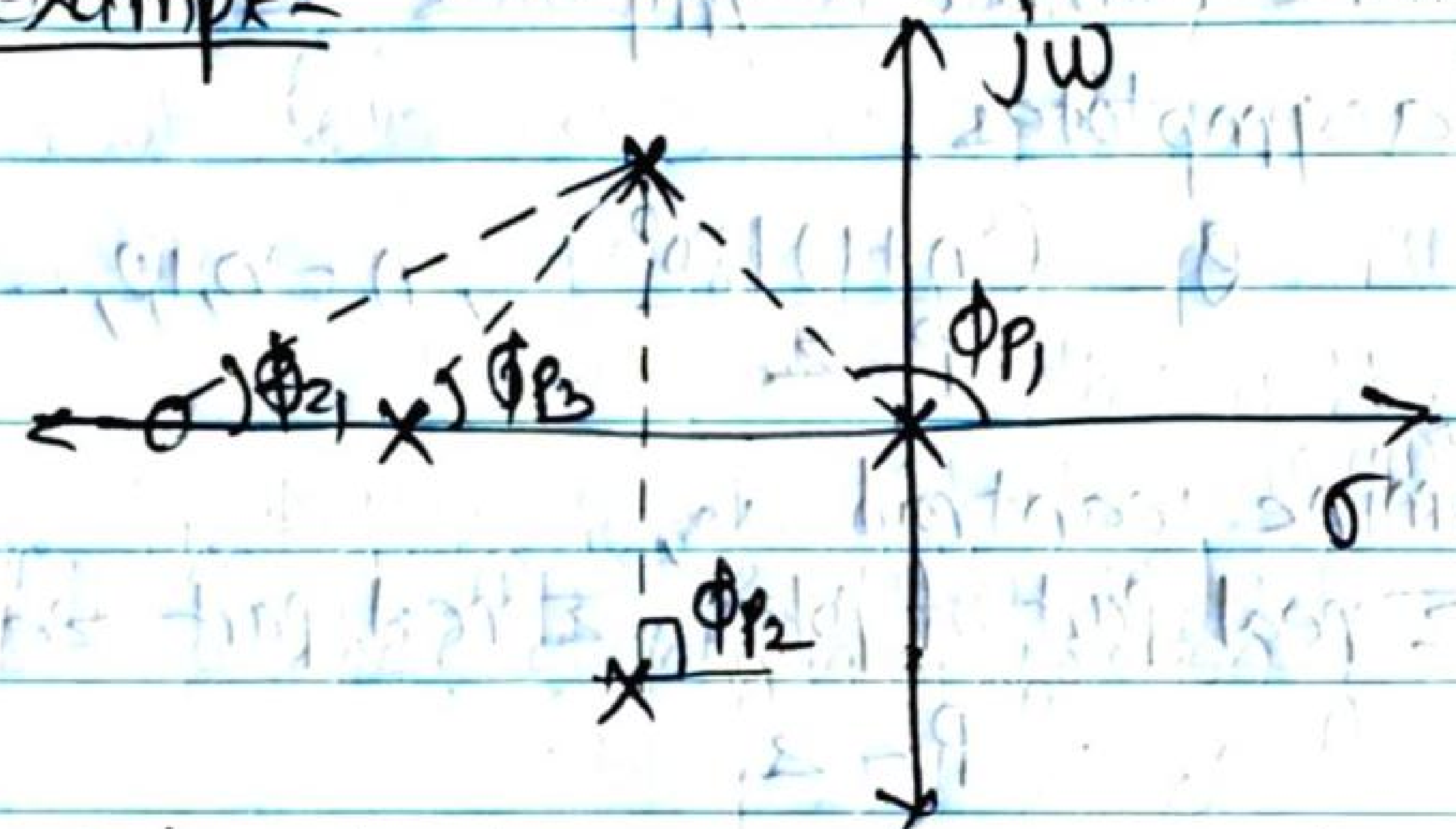
Where ϕ_d is angle of departure

$$\phi = \Sigma\phi_p - \Sigma\phi_z$$

where ϕ_p is angle made by poles about pole under consideration

ϕ_z is angle made by zeros about pole under consideration.

For example-



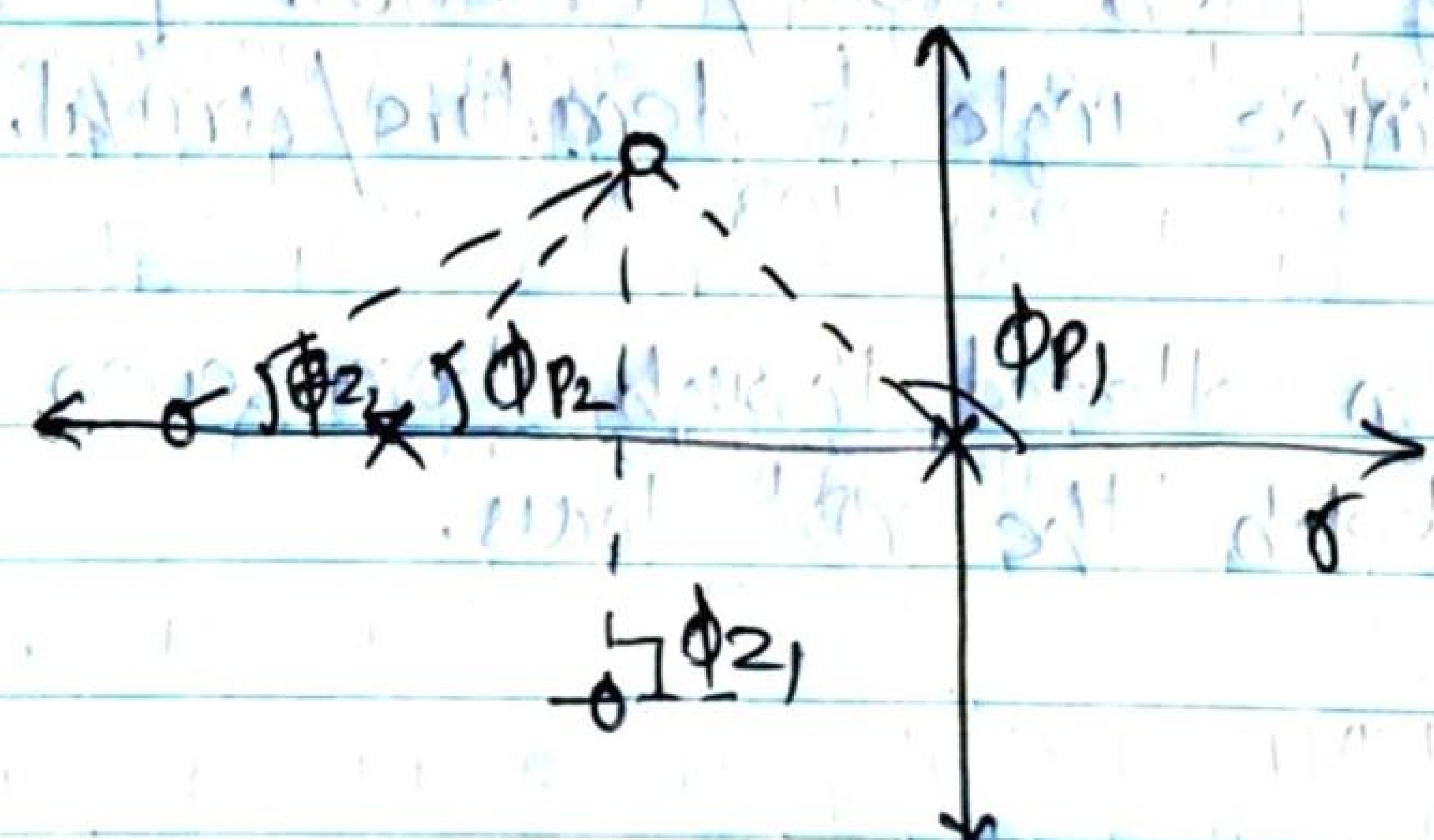
$$\therefore \phi = \phi_{p_1} + \phi_{p_2} + \phi_{p_3} - \phi_{z_1}$$

similarly, root locus arrive at complex open loop zero with an angle given by,

$$\phi_a = 180 + \phi$$

where ϕ_a is angle of arrival

For example -



$$\therefore \phi = \phi_{p_1} + \phi_{p_2} - (\phi_{z_1} + \phi_{z_2})$$

Scanned by TapScanner

Steps in sketching root locus-

step-1 - For the given system transfer function, determine number of open loop poles, open loop zeros and their locations. Draw pole-zero plot.

step-2 - From pole zero plot, check existence of root locus on real axis.

step-3 - Determine number asymptotes as $P-Z$ and angles of asymptotes,

$$\theta = \frac{(2n+1)180^\circ}{P-Z}, \quad n = 0,1,2,\ldots,P-Z-1$$

step-4 - Determine centroid as,

$$\sigma = \frac{\Sigma \text{ real part of poles} - \Sigma \text{ real part zeros}}{P-Z}$$

step-5 - Determine breakaway points as follows

The closed loop characteristic eqn is $1+G(s)H(s)=0$

From this equation, obtain equation for K and then breakaway points are from the roots of $\frac{dK}{ds}=0$.

step-6 - Determine intersection of root locus with imaginary axis ($j\omega$- crossover) from closed loop characteristic equation using Routh stability criterion.

step-7 - Determine angle of departure / arrival.

The information collected through above seven step is enough to sketch the root locus.

Examples on root locus-

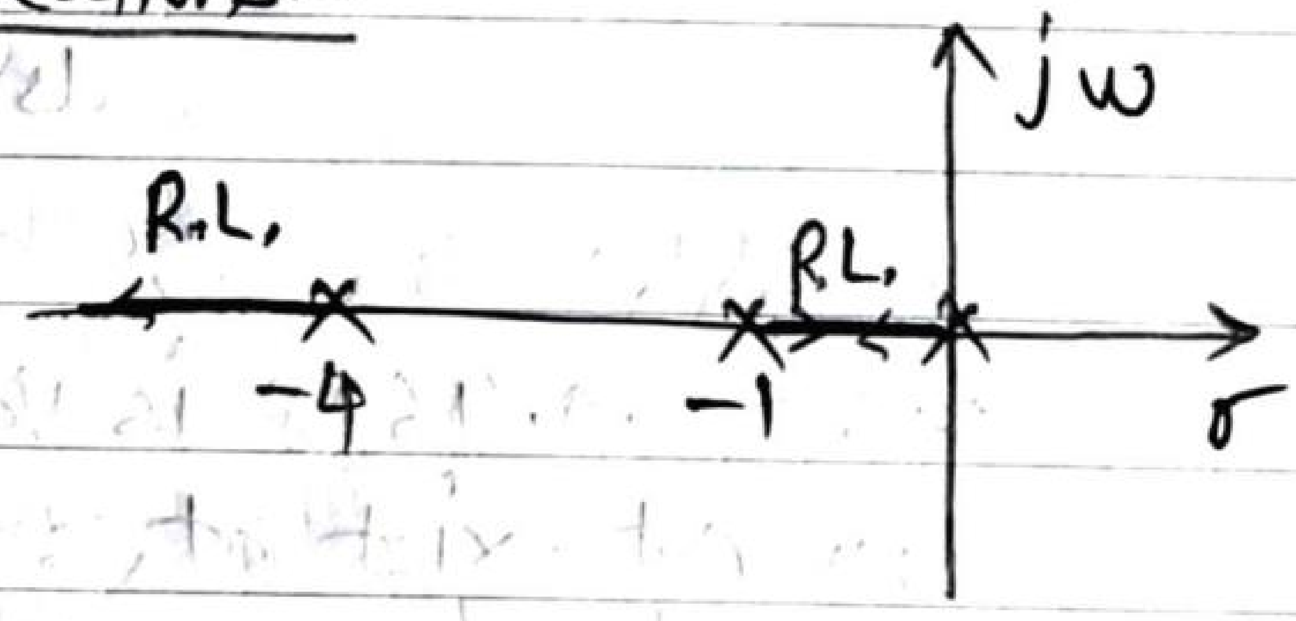
1. Sketch root locus of the unity feedback system with open loop transfer function $G(s) = \frac{K}{s(s+1)(s+4)}$.

Soln The given system has $G(s) = \frac{K}{s(s+1)(s+4)}$, $H(s) = 1$

No. of poles, zeros and their locations-

$P = 3$, $Z = 0$

Poles at $s = 0, -1, -4$



Existance of root locus on real axis-

Root locus exists on real axis between $s = 0$ & -1 and $s = -4$ & $-\infty$.

No of asymptotes and their angles-

No. of asymptotes $= P - Z = 3$

Angles $\theta = \frac{(2n+1)180^\circ}{P-Z}$, $n = 0, 1, \ldots, P-Z-1$

$$\theta_1 = \frac{180^\circ}{3} = 60^\circ$$

$$\theta_2 = \frac{(3)(180^\circ)}{3} = 180^\circ$$

$$\theta_3 = \frac{(5)(180^\circ)}{3} = 300^\circ$$

Centroid-

$$\sigma = \frac{\Sigma \text{ real part of poles} - \Sigma \text{ real part of zeros}}{P-Z}$$

$$= -\frac{5}{3} = -1.6667$$

Scanned by TapScanner

Breakaway points-

The characteristic equation is,

$$1+\frac{k}{s(s+1)(s+4)}=0$$

$\therefore$, $s(s+1)(s+4)+k=0$

$\therefore$, $k=-[s^3+5s^2+4s]$

$\therefore$, $\frac{dk}{ds}=-[3s^2+10s+4]=0$

$\therefore$, $s=-0.4648, -2.8685$

$\therefore$ $s=-0.4648$ is valid breakaway point because root locus does not exist at $s=-2.8685$.

Intersection with imaginary axis-

The characteristic equation is,

$$s^3+5s^2+4s+k=0$$

$\therefore$, The Routh array is

s^3	1	4
s^2	5	k
s^1	$\frac{20-k}{5}$	
s^0	k	

For stability,

1. $k>0$
2. $\frac{20-k}{5}>0$

$\therefore$ $k<20$

$\therefore$ Range of k for stability is

$0<k<20$

$\therefore$ $k_{max}=20$

From row s^2

$5s^2+20=0$

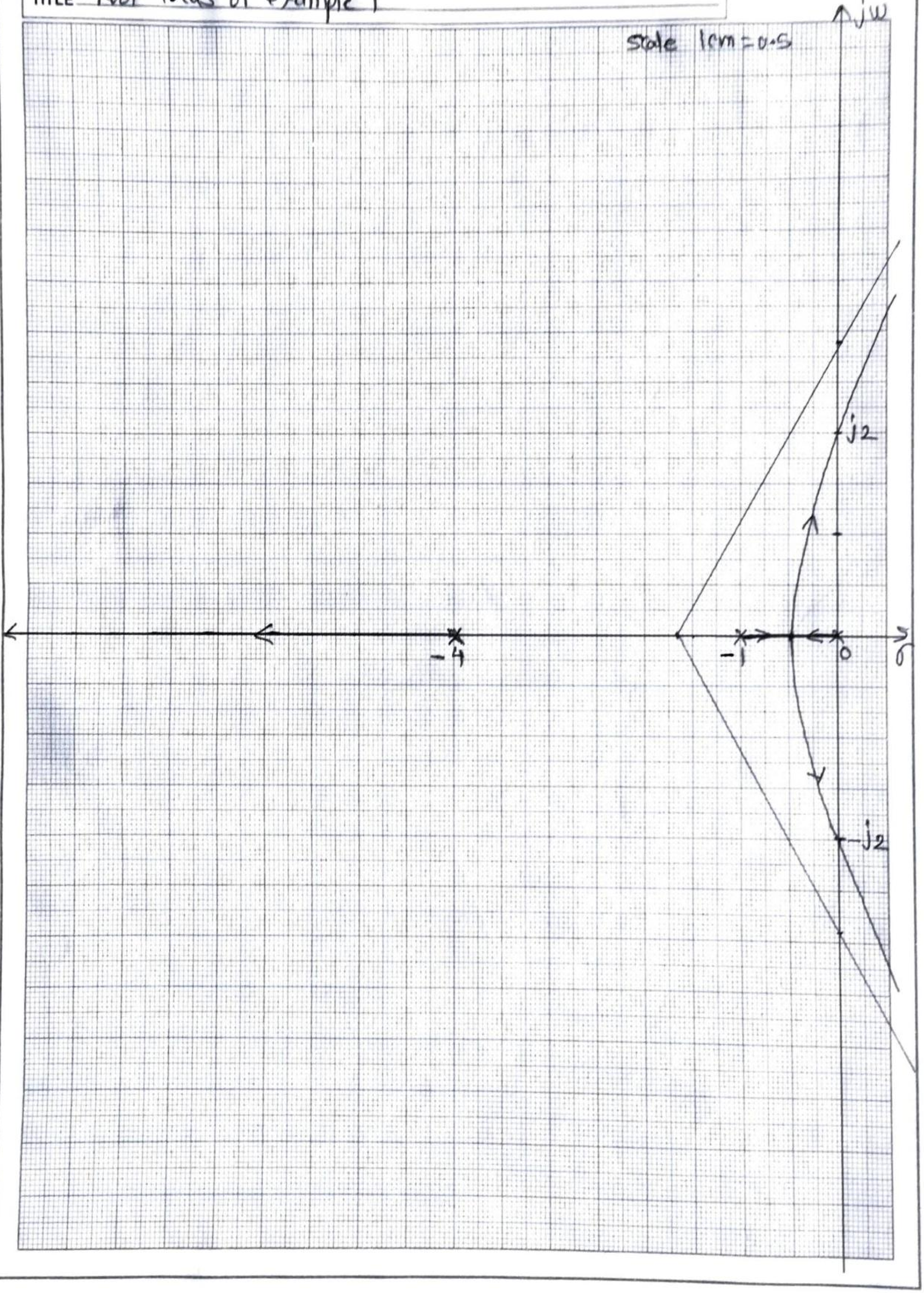
$\therefore$, $s^2=-4$

$\therefore$ $s=\pm j2$

The complete root locus is as sketched on the graph

Scanned by TapScanner

TITLE Root locus of example 1



2. Sketch root locus of the system with open loop transfer function

$$G(s)H(s) = \frac{K}{s(s+1)(s+3)(s+5)}$$

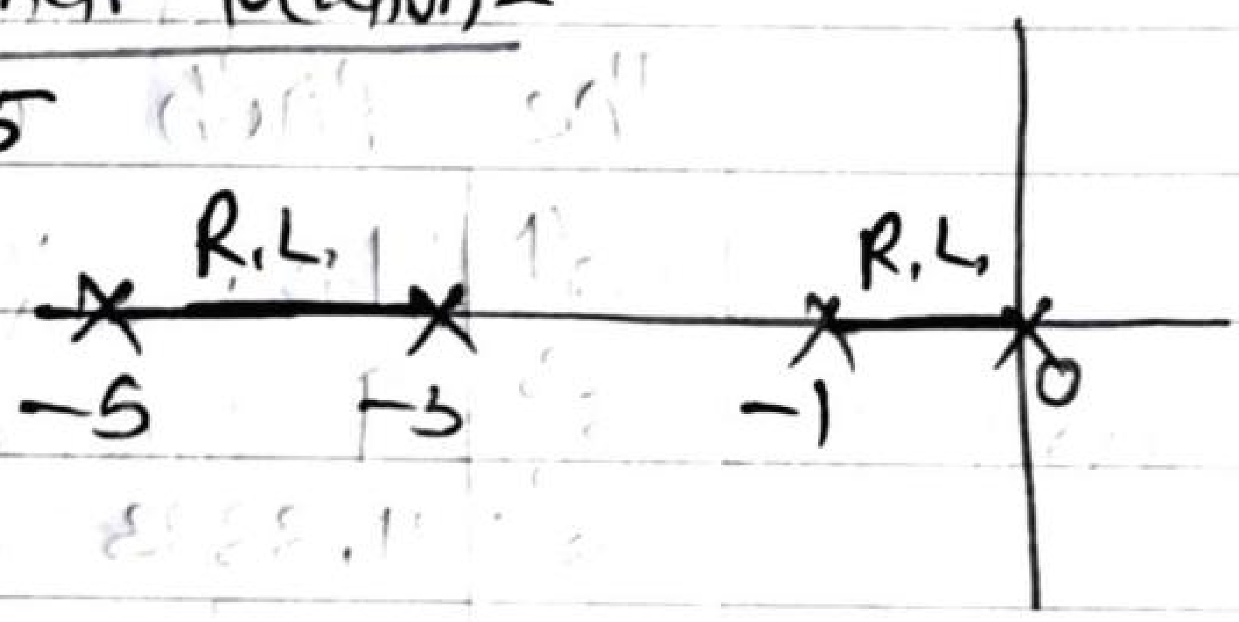
sol^n The system has,

$$G(s)H(s) = \frac{K}{s(s+1)(s+3)(s+5)}$$

No. of open loop poles, zeros and their location-

$P = 4$, poles at $s = 0, -1, -3, -5$

$z = 0$



Existance of root locus on real axis-

Root locus exists on real axis between $s = 0, -1$ and $s = -3, -5$.

No. of asymptotes and their angles-

No. of asymptotes $= P - z = 4$

$$\theta = 45^\circ, 135^\circ, 225^\circ, 315^\circ.$$

Centroid-

$$\sigma = \frac{-1-3-5}{4} = -2.25$$

Break away points-

The closed loop characteristic equation is,

$$1 + \frac{K}{s(s+1)(s+3)(s+5)} = 0$$

$\therefore\ s(s+1)(s+3)(s+5) + K = 0$

$$\therefore K = -[s(s+1)(s+3)(s+5)]$$

$$= -[s^4 + 9s^3 + 23s^2 + 15s]$$

$$\therefore \frac{dK}{ds} = -[4s^3 + 27s^2 + 46s + 15] = 0$$

$$\therefore s = -0.4258, -2.07, -4.25$$

Since root locus does not lie between $s=-1$ and $s=-3$, the point $s=-2.07$ is not valid. The valid breakaway points are $s=-0.4285, -4.25$.

Intersection with imaginary axis -

The closed loop characteristic equation is,

$$s^4+9s^3+23s^2+15s+K=0$$

The Routh array is,

s^4	1	23	K
s^3	9	15	
s^2	21.3333	K	
s^1	$\frac{320-9K}{21.3333}$		
s^0	K		

For stability

1. $K>0$
2. $320-9K<0$

$\therefore K<35.5556$

$\therefore K_{mar}=35.5556$

From row s^2,

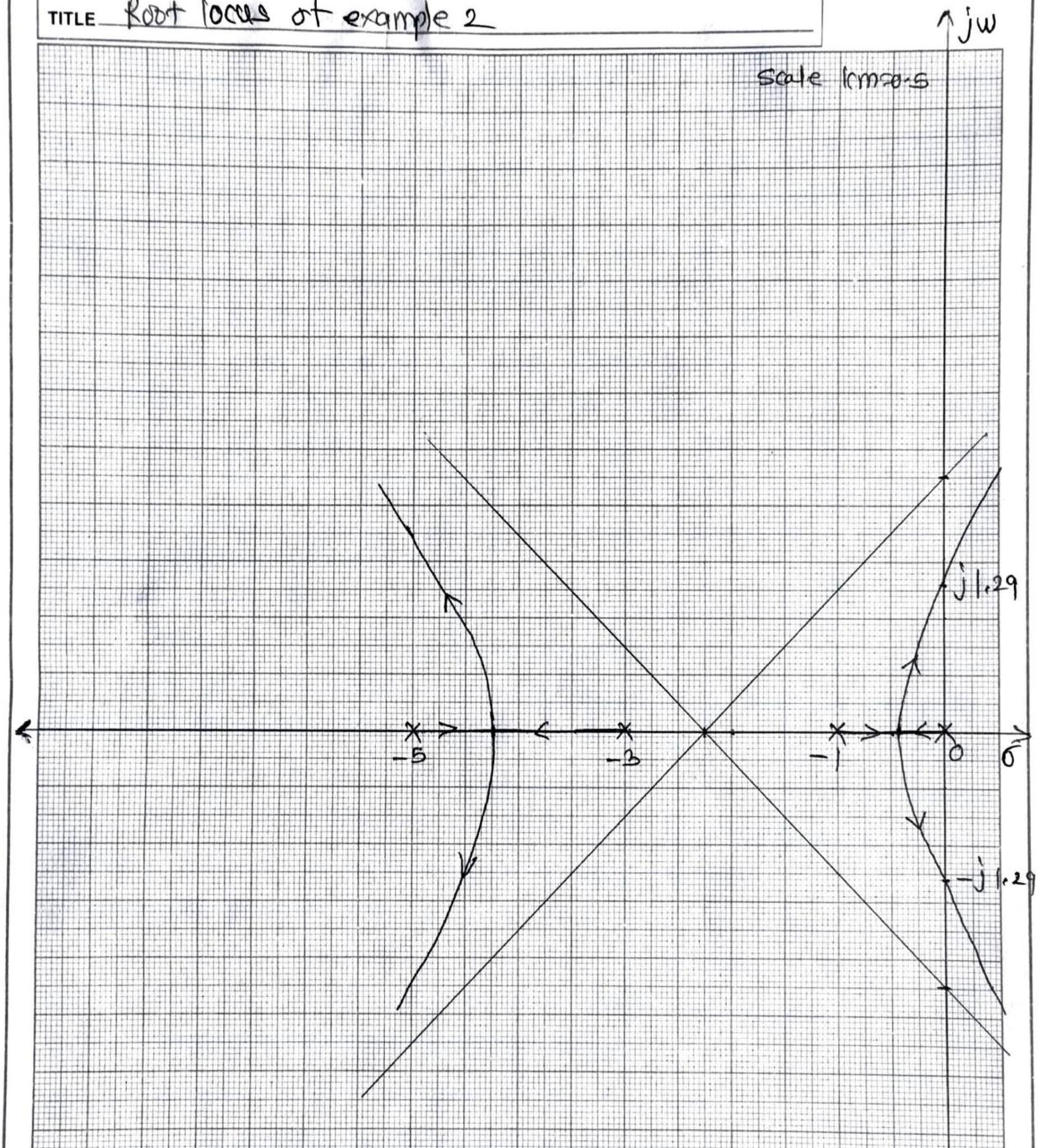
$$21.3333s^2+35.5556=0$$

$$s^2=-1.6667$$

$$\therefore s=\pm j1.2910$$

Using above information, the root locus is sketched on the graph.

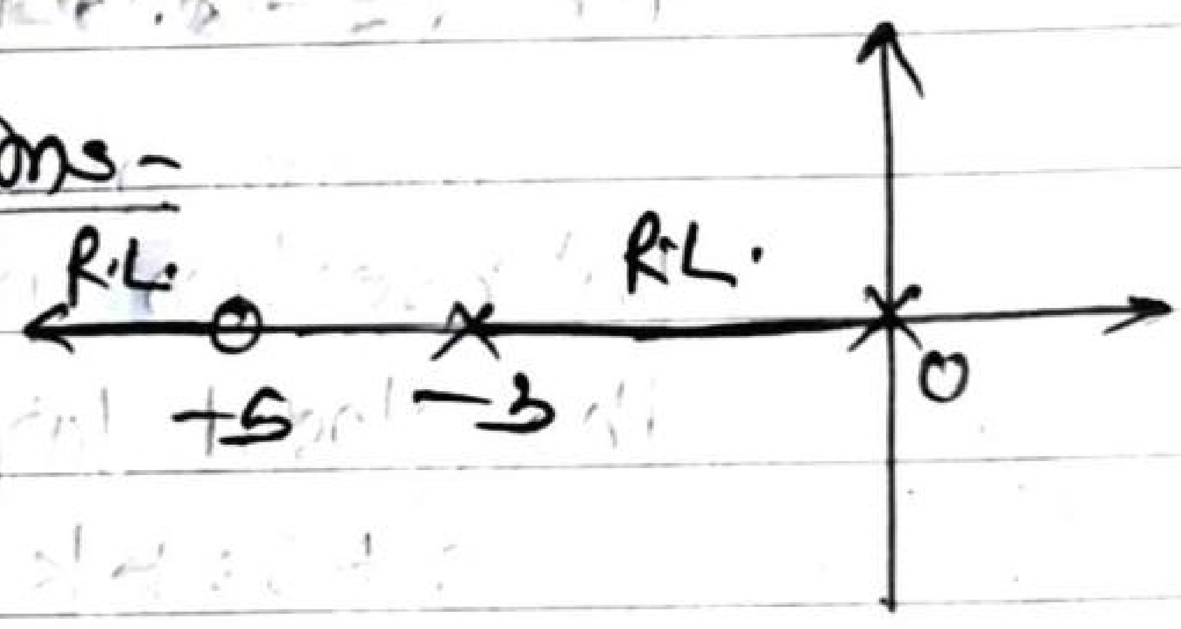
TITLE Root locus of example 2



3. Sketch root locus of the system with open loop transfer function $G(s)H(s) = \dfrac{k(s+5)}{s(s+3)}$

sol^n- The system is described by,

$$G(s)H(s) = \frac{k(s+5)}{s(s+3)}$$



No. of poles, zeros and their locations-

$P = 2$, poles at $s = 0, -3$

$Z = 1$, zero at $s = -5$

Existance of root locus on real axis-

The root locus exists on real axis between $s = 0, -3$ and $s = -5, -\infty$.

No. of asymptotes and their angles-

No. of asymptotes $= P - Z = 1$

Angles, $\theta = 180^0$

Centroid-

$$\sigma = \frac{-3-(-5)}{1} = 2$$

Break away points-

The closed loop characteristic equation is,

$$1 + \frac{K(s+5)}{s(s+3)} = 0$$

$$\therefore s(s+3) + k(s+5) = 0$$

$$\therefore k = -\left[\frac{s(s+3)}{s+5}\right] = -\left[\frac{s^2+3s}{s+5}\right]$$

$$\therefore \frac{dk}{ds} = \frac{(s+5)(2s+3)-(s^2+3s)}{(s+5)^2} = 0$$

$$\therefore \frac{dk}{ds} = \frac{2s^2+13s+15-s^2-3s}{(s+5)^2} = 0$$

$$\therefore s^2+10s+15=0$$

$$\therefore s = -1.8377, -8.1623$$

Both points are valid. $s=-1.8377$ is breakaway point and $s=-8.1623$ is the breakin point.

Intersection with imaginary axis –

The closed loop characteristic equation is,

$$s^2+3s+ks+5k=0$$

The Routh array is,

s^2	1	$5k$
s^1	$3+k$	
s^0	$5k$	

For stability

1. $5k>0$

$\therefore k>0$

2. $k+3>0$

$\therefore k>-3$

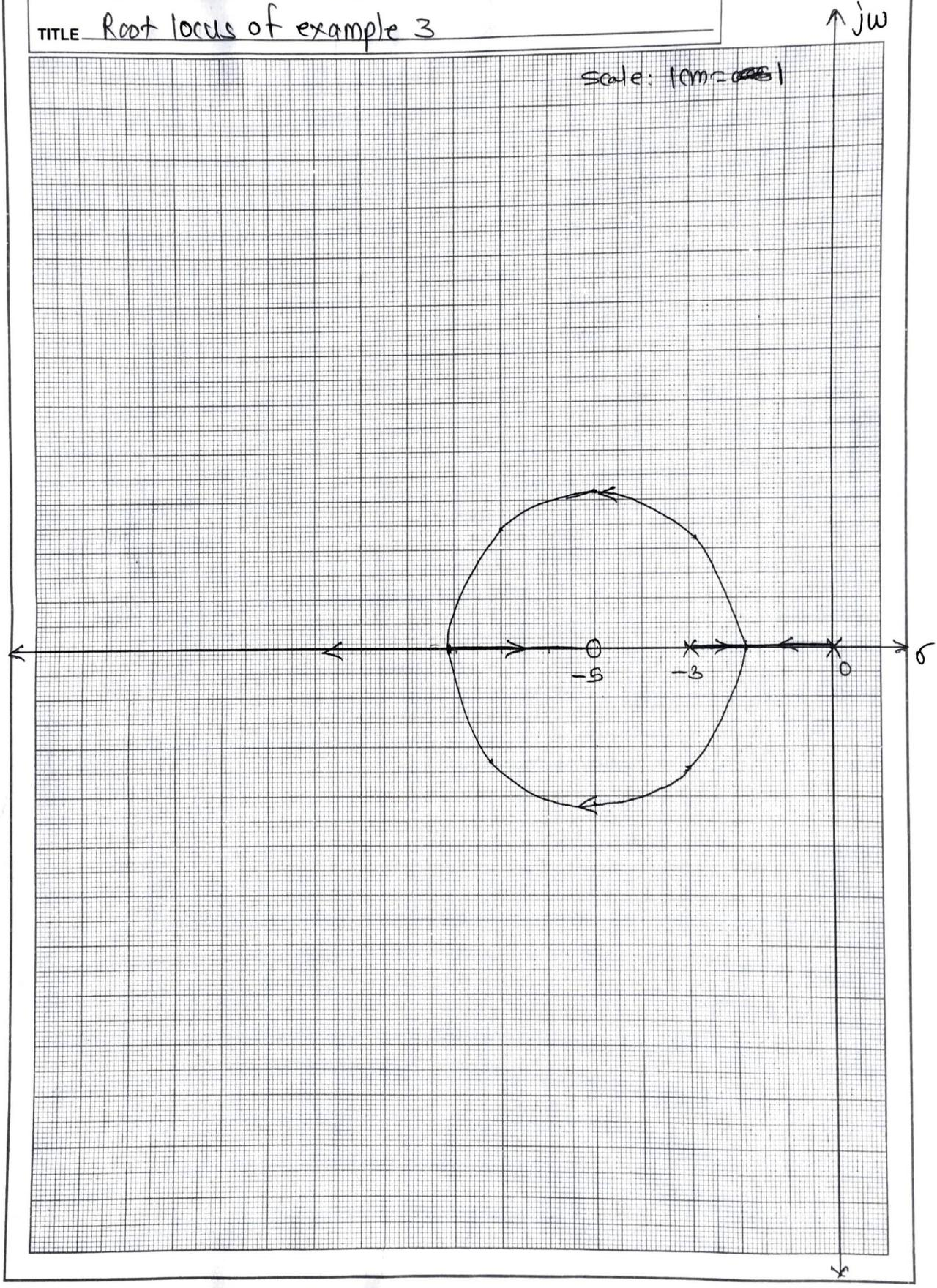
$\therefore$ Range of k is

$-3<k<\infty$

$\therefore$ System is stable for all $k>0$ and root locus will not intersect imaginary axis.

Based on the above information, the root locus is sketched on graph.

TITLE Root locus of example 3



4. Sketch root locus of system with open loop transfer function

$$G(s)\,H(s) = \frac{K(s+4)(s+6)}{s(s+2)}$$

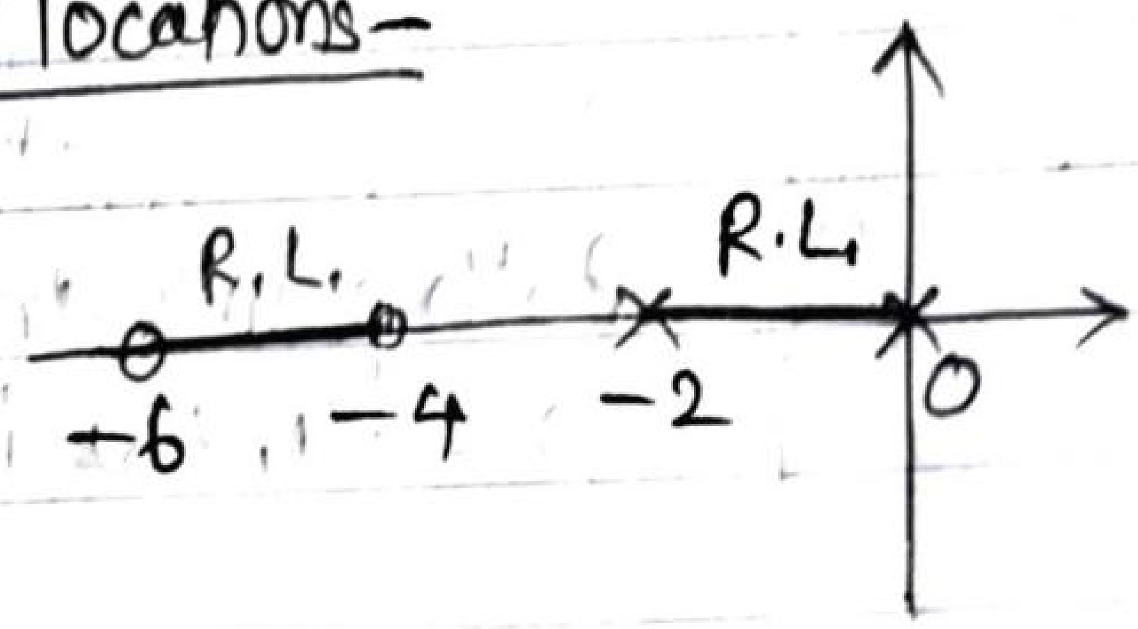
<u>Soln</u> The system has open loop transfer function,

$$G(s)\,H(s) = \frac{K(s+4)(s+6)}{s(s+2)}$$

<u>No. of open loop poles, zeros and their locations -</u>

$P = 2$, poles at $s = 0, -2$

$Z = 2$, zeros at $s = -4, -6$



<u>Existance of root locus on real axis -</u>

The root locus exists on real axis between $s = 0, -2$ and $s = -4, -6$

<u>No. of asymptotes and their angles -</u>

No. of asymptotes $= P - Z = 0$

$\therefore$ There are no asymptotes

<u>Centroid -</u>

As there is no asymptote, centroid is absent.

<u>Breakaway points -</u>

The closed loop characteristic equation is,

$$1 + \frac{K(s+4)(s+6)}{s(s+2)} = 0$$

$$\therefore s(s+2) + K(s+4)(s+6) = 0$$

$$\therefore K = -\left[\frac{s(s+2)}{(s+4)(s+6)}\right]$$

$$= -\left[\frac{s^2+2s}{s^2+10s+24}\right]$$

Scanned by TapScanner

$$\therefore \frac{dk}{ds} = -\left[\frac{(s^2+10s+24)(2s+2)-(s^2+2s)(2s+10)}{(s^2+10s+24)^2}\right] = 0$$

$$\therefore 2s^3+22s^2+68s+48-(2s^3+14s^2+20s) = 0$$

$$\therefore 2s^3+22s^2+68s+48-2s^3-14s^2-20s = 0$$

$$\therefore 8s^2+48s+48 = 0$$

$$\therefore s^2+6s+6 = 0$$

$$\therefore s = -1.2680, -4.7320$$

Both points are valid, $s=-1.268$ is breakaway point and $s=-4.732$ is break in point.

Intersection with imaginary axis-

The closed loop characteristic equation is,

$s^2+2s+k(s^2+10s+24)=0$

$\therefore$ Routh array is

s^2	$1+k$ $24k$
s^1	$2+10k$
s^0	$24k$

For stability,

1. $24k > 0$

$\therefore k > 0$

2. $2+10k > 0$

$\therefore k > -0.2$

3. $1+k > 0$

$\therefore k > -1$

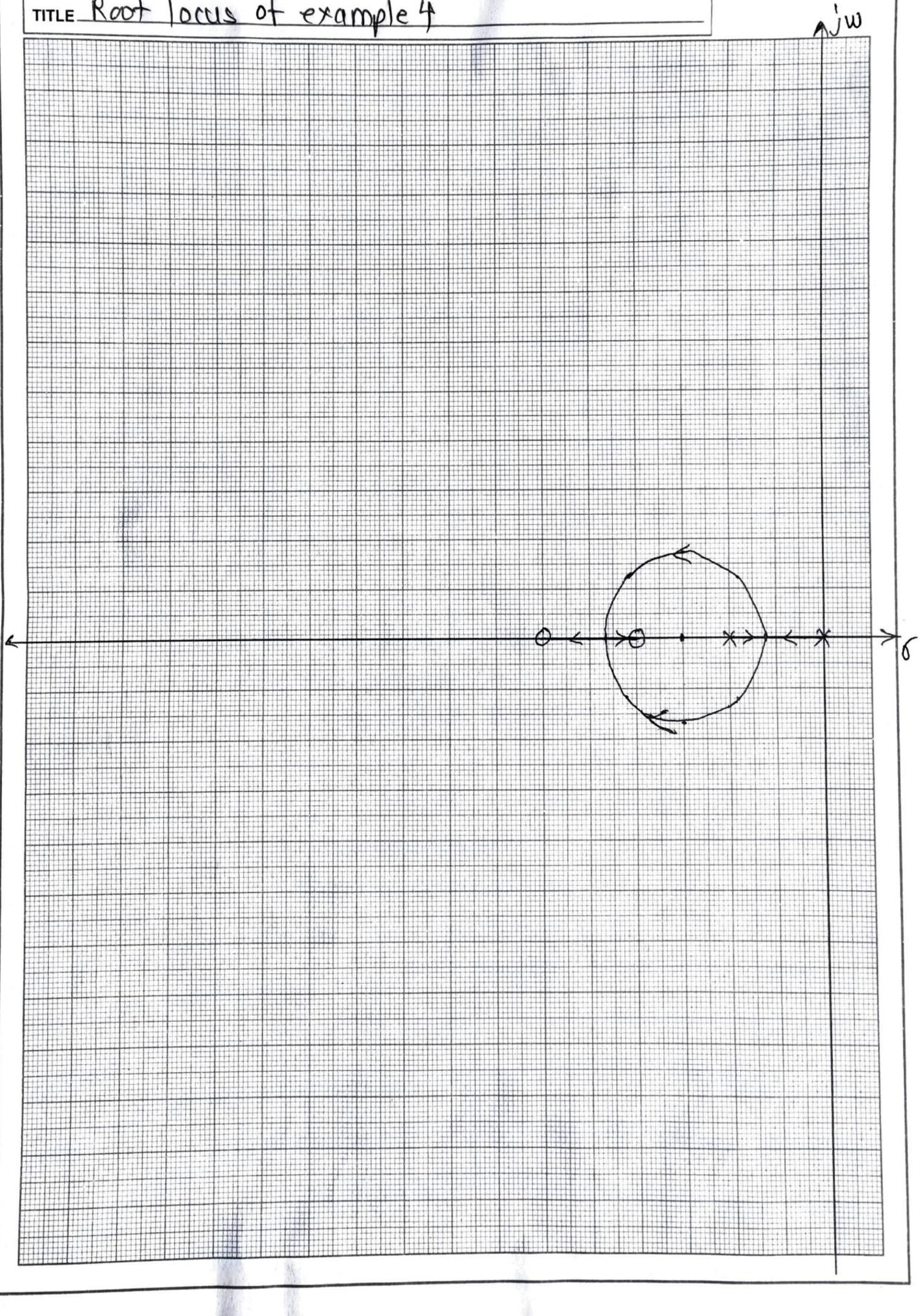
$\therefore$ Range of k is $-1 < k < \infty$

$\therefore$ system is stable for all $k>0$. Hence root locus will not intersect the imaginary axis.

Based on above information, the root locus is sketched on graph.

Scanned by TapScanner

TITLE Root locus of example 4



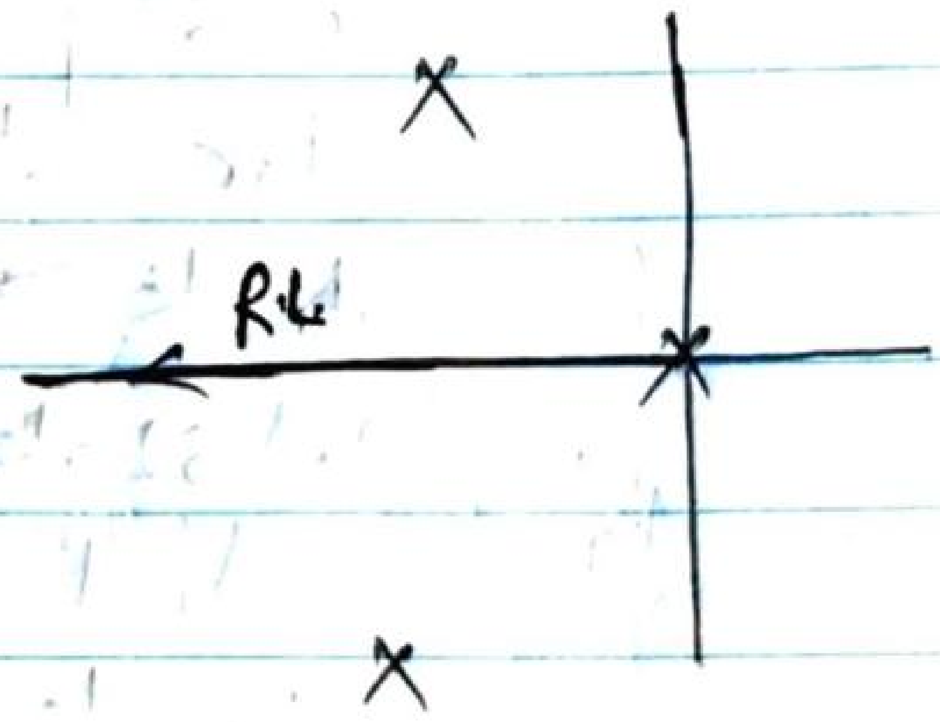
5. sketch root locus of system with open loop transfer function

$$G(s)H(s) = \frac{K}{s(s^2+4s+13)}$$

soln The given system has transfer function,

$$G(s)H(s) = \frac{K}{s(s^2+4s+13)}$$

No. of poles, zeros and their locations–



$P = 3$, poles at $s = 0, -2+j3, -2-j3$

$Z = 0$

Existance of root locus on real axis–

Root locus exists on entire left half ($s = 0$ to $-\infty$) on real axis.

No. of asymptotes and their angles–

No. of asymptotes $= P - Z = 3$

Angles, $\theta = 60^\circ, 180^\circ, 300^\circ$.

Centroid–

$$\sigma = \frac{-2-2}{3} = -1.33$$

Breakaway point–

As root locus exist on real axis in entire left half, the breakaway point does not exist.

Intersection with imaginary axis–

The closed loop characteristic equation is,

$$1 + \frac{K}{s(s^2+4s+13)} = 0$$

$$\therefore s^3 + 4s^2 + 13s + K = 0$$

Scanned by TapScanner

∴ Routh array is,

s^3	1	13
s^2	4	K
s^1	$\frac{52-K}{4}$	
s^0	K	

For stability

1. $K > 0$

2. $\frac{52-K}{4} > 0$

∴ $K < 52$

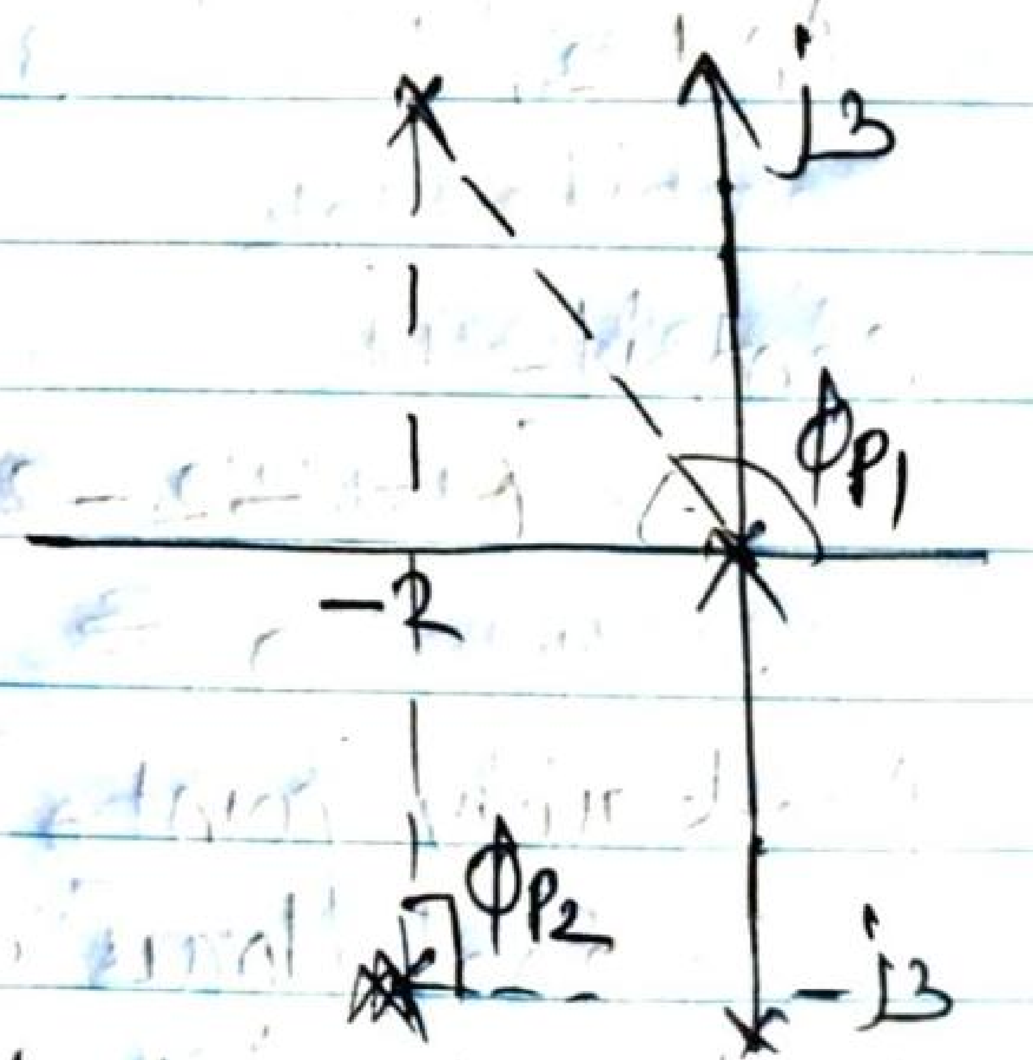
∴ $K_{mar} = 52$

From row s^2

$4s^2 + 52 = 0$

∴ $s^2 = -13$

∴ $s = \pm j\,3.6056$



Angle of departure-

For $s = -2 + j3$

$\phi_{p_1} = 180 - \tan^{-1}\frac{3}{2} = 123.69$

$\phi_{p_2} = 90^\circ$

∴ $\phi_p = 213.69^\circ$

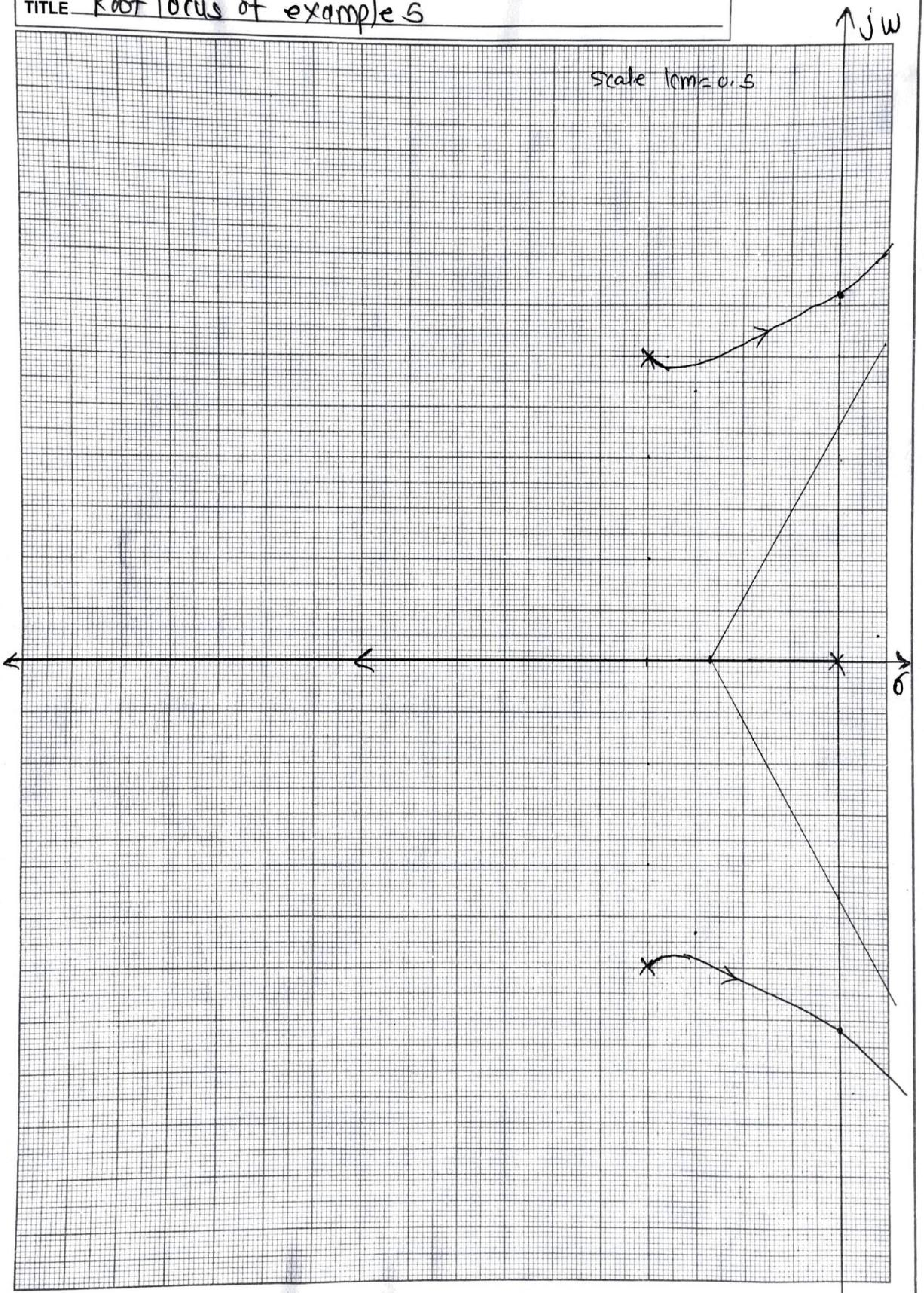
∴ $\phi_d = 180 - 213.69 = -33.69^\circ$

similarly for pole at $s = -2 - j3$

$\phi_d = 33.69^\circ$

From above information the complete root locus is sketched on graph paper.

TITLE Root locus of example 5

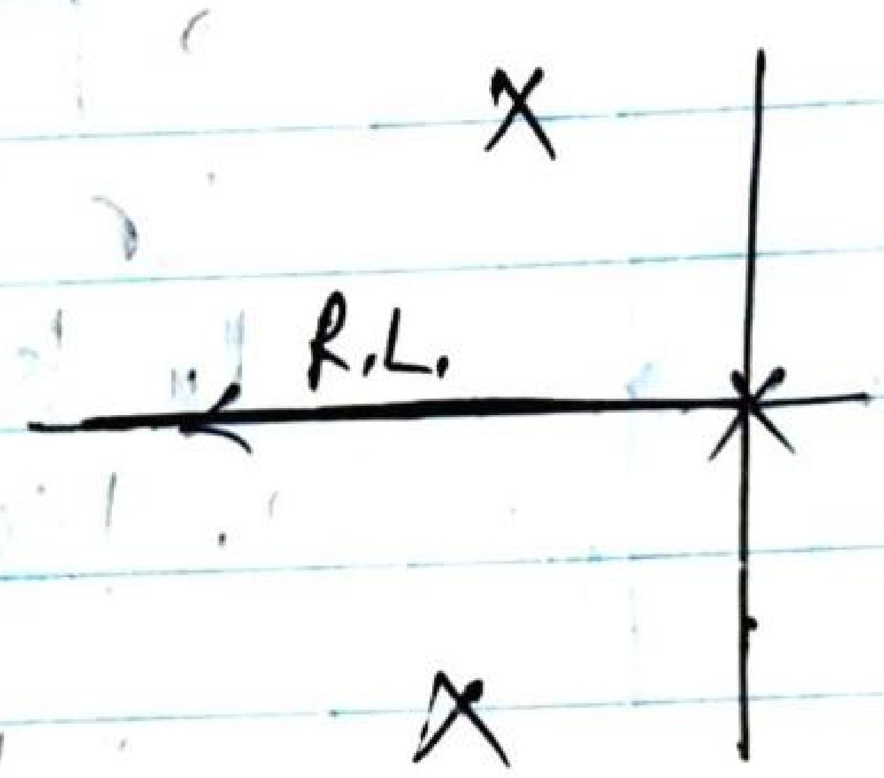


6. sketch root locus of system with open loop transfer function

$$G(s)H(s) = \frac{K}{s(s^2+6s+25)}.$$

sol^n The given system has open loop transfer function,

$$G(s)H(s) = \frac{K}{s(s^2+6s+25)}$$



No. of poles, zeros and their locations-

$P=3$, poles at $s=0, -3+j4, -3-j4$

$Z=0$

Existance of root locus on real axis-

The root locus exists on real axis between $s=0$ and $-\infty$.

No. of asymptotes and their angles-

No. of asymptotes $= P-Z = 3$

Angles, $\theta = 60°, 180°, 300°$.

Centroid-

$$\sigma = \frac{-3-3}{3} = -2$$

Break away points-

As root locus lies on real axis in entire left half of s-plane, break away point does not exist.

Intersection with imaginary axis-

The closed loop characteristic equation is,

$$1 + \frac{K}{s(s^2+6s+25)} = 0$$

$$\therefore s^3 + 6s^2 + 25s + K = 0$$

Scanned by TapScanner

Routh array is,

s^3	1	25
s^2	6	K
s^1	$\frac{150-K}{6}$	
s^0	K	

For stability,

1. $K > 0$
2. $\frac{150-K}{6} > 0$

$\therefore K < 150$

$\therefore$ Range of K is $0 < K < 150$

$\therefore K_{mar} = 150$

From row s^2

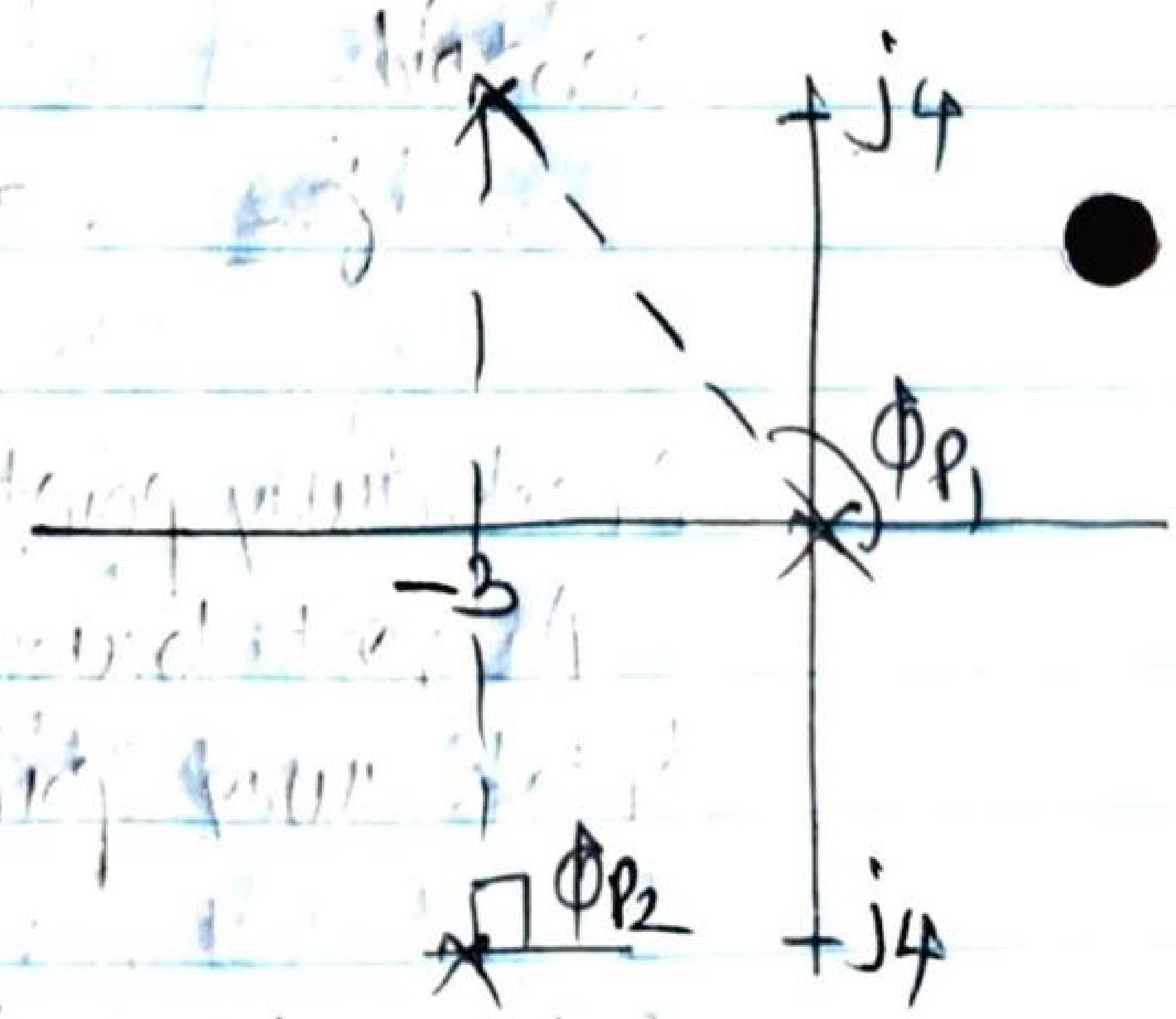
$$6s^2 + 150 = 0$$

$$\therefore s^2 = -25$$

$$s = \pm j5$$

Angle of departure-

For pole at $s = -3 + j4$



$$\phi_{P_1} = 180 - \tan^{-1}\frac{4}{3} = 126.87^\circ$$

$$\phi_{P_2} = 90^\circ$$

$$\therefore \phi_P = 216.87^\circ$$

$$\therefore \phi_d = 180 - 216.87 = -36.87^\circ$$

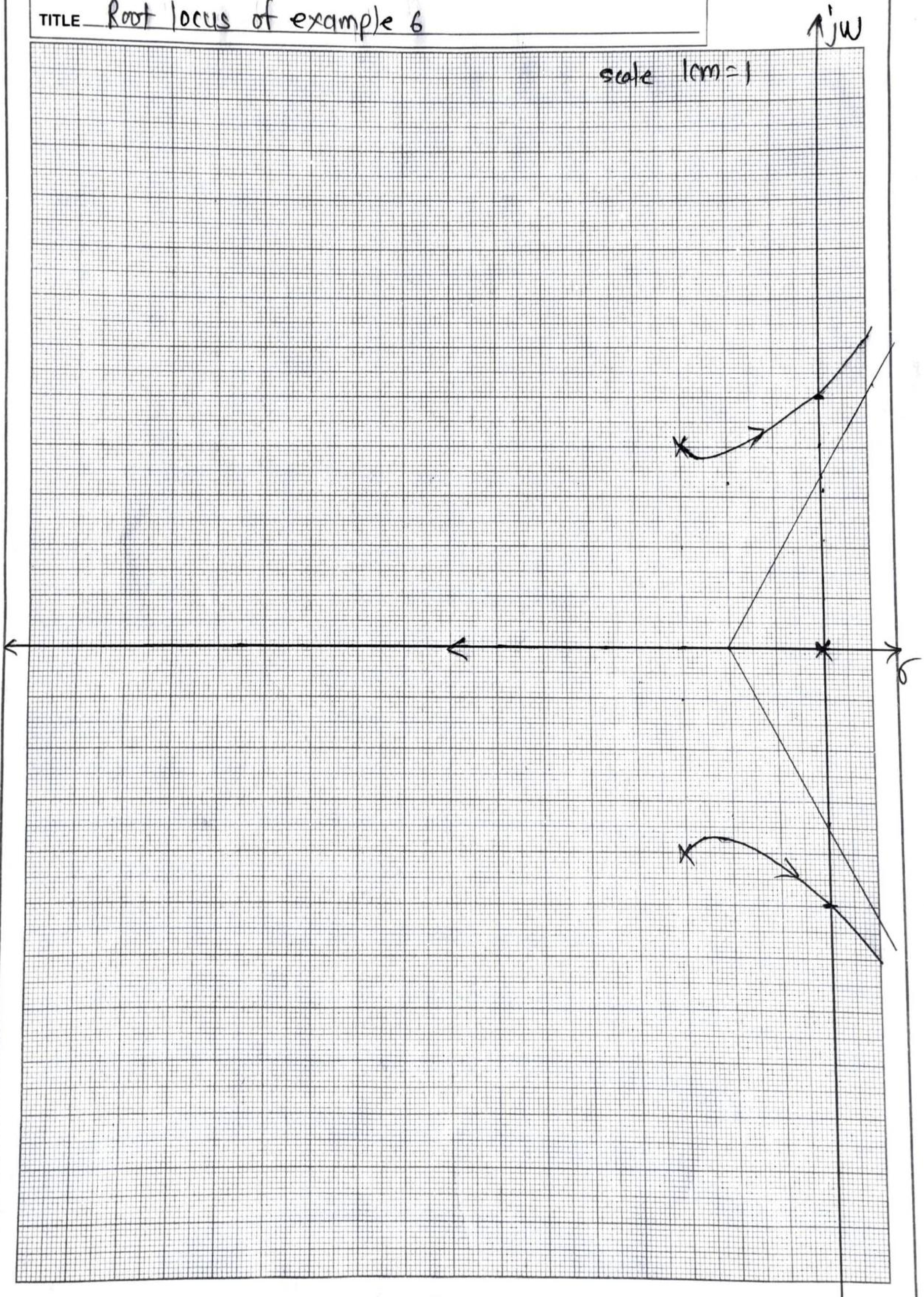
similarly for pole at $s = -3 - j4$

$$\phi_d = 36.87^\circ$$

Based on above information the root locus is sketched on graph.

Scanned by TapScanner

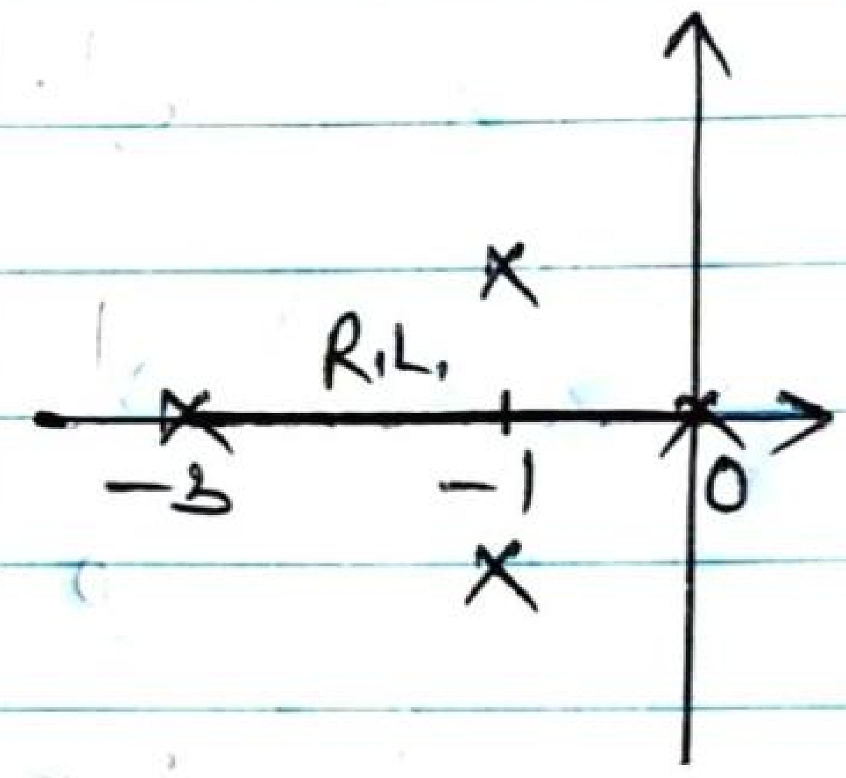
TITLE Root locus of example 6



7. sketch root locus of system with open loop transfer function $G(s)H(s) = \dfrac{K}{s(s+3)(s^2+2s+2)}$

<u>sol</u>^{n} The given system has open loop transfer function,

$$G(s)H(s) = \frac{K}{s(s+3)(s^2+2s+2)}$$



<u>No. of poles, zeros and their locations –</u>

$P = 4$, poles at $s = 0, -3, -1+j, -1-j$

$z = 0$

<u>Existance of root locus on real axis –</u>

Root locus exists on real axis between $s = 0$ and -3.

<u>No. of asymptotes and their angles –</u>

No. of asymptotes $= P - z = 4$

Angles, $\theta = 45^0, 135^0, 225^0, 315^0$

<u>centroid –</u>

$$\sigma = \frac{-3-1-1}{4} = -1.25$$

<u>Breakaway points –</u>

The closed loop characteristic equation is,

$$1 + \frac{K}{s(s+3)(s^2+2s+2)} = 0$$

$\therefore s(s+3)(s^2+2s+2) + K = 0$

$$\therefore K = -[s(s+3)(s^2+2s+2)]$$

$$= -(s^4+5s^3+8s^2+6s)$$

$$\therefore \frac{dK}{ds} = 4s^3+15s^2+16s+6 = 0$$

$\therefore s =$

Intersection with imaginary axis-

The closed loop characteristic equation is,

$$s^4+5s^3+8s^2+6s+k=0$$

s^4	1	8	k
s^3	5	6	
s^2	6.8	k	
s^1	$\frac{40.8-5k}{6.8}$		
s^0	k		

For stability

1. $k>0$
2. $\frac{40.8-5k}{6.8}>0$

$\therefore k<8.16$

$\therefore$ Range of k is $0<k<8.16$

$\therefore k_{mar}=8.16$

From row s^2

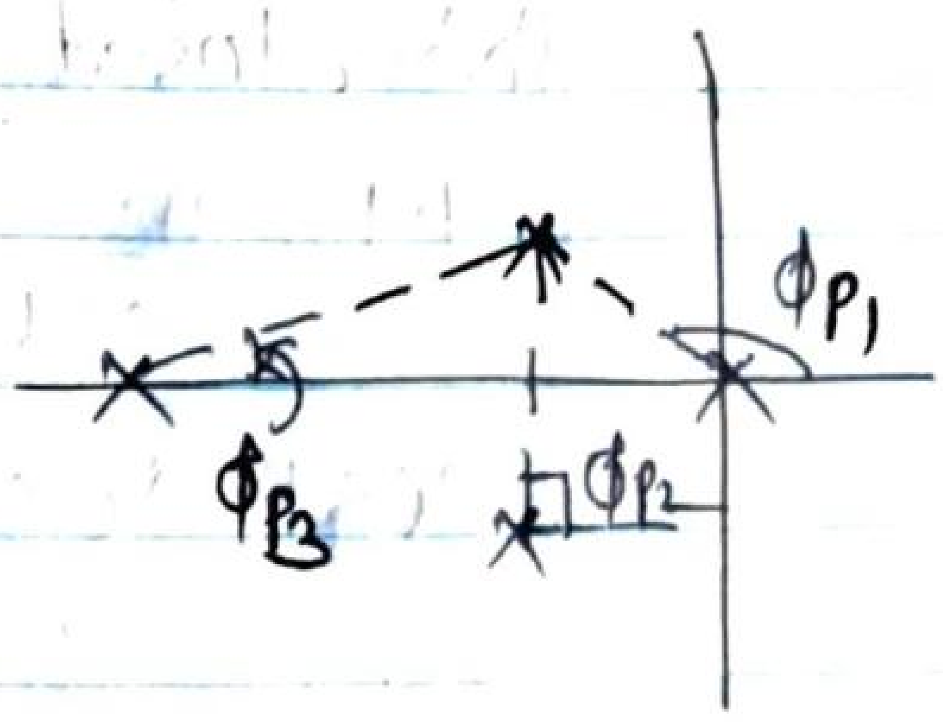
$$6.8s^2+8.16=0$$

$$\therefore s^2=1.2$$

$$\therefore s=\pm j1.0954$$

Angle of departure-

For pole at $s=-1+j$



$\phi_{p_1}=180-45=135^0$

$\phi_{p_2}=90^0$

$\phi_{p_3}=26.57^0$

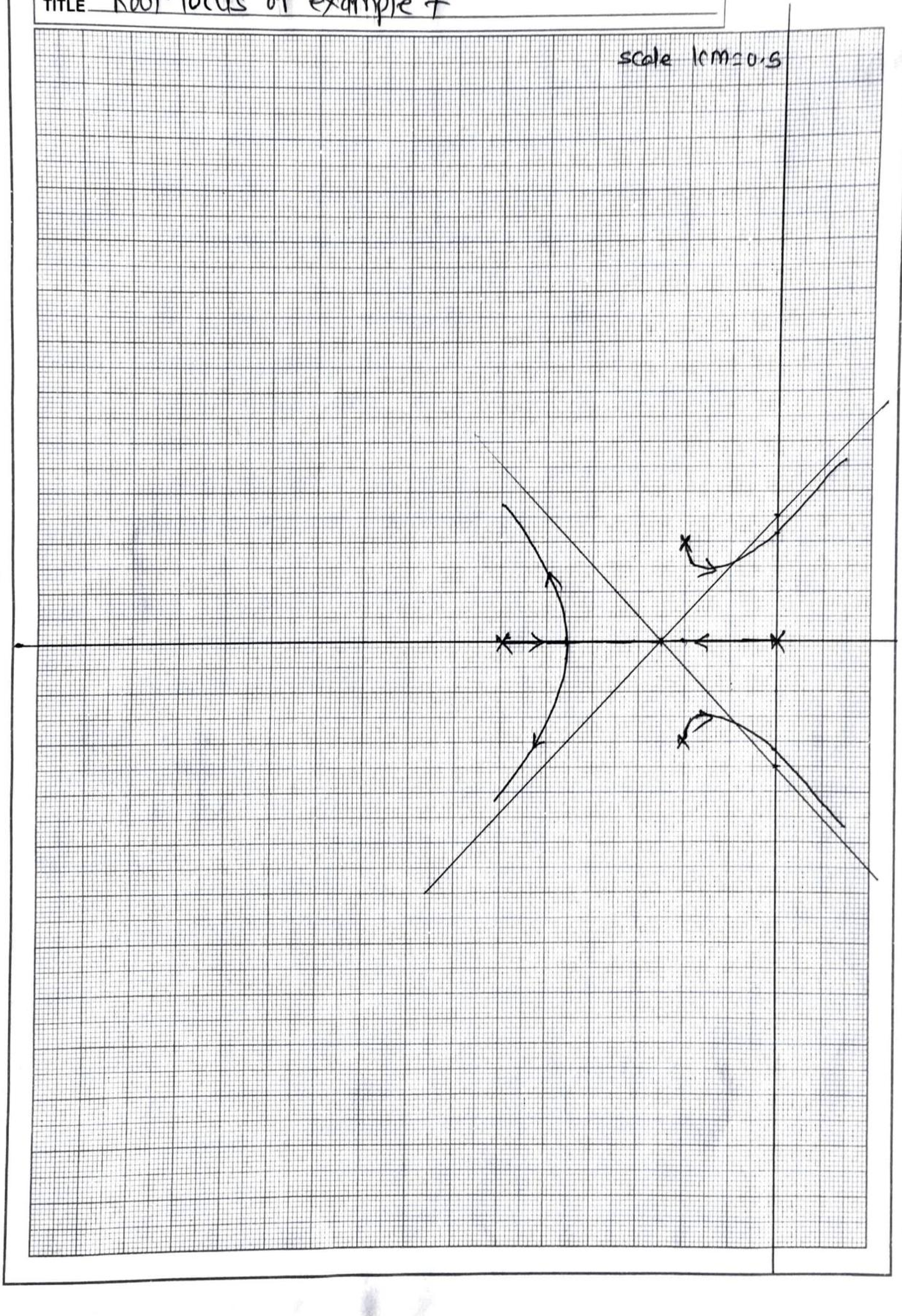
$\therefore \phi_p=251.57^0$

$\therefore \phi_d=180-251.57=-71.57^0$

similary for pole at $s=-1-j$, $\phi_d=71.57^0$

The root locus is sketched on graph.

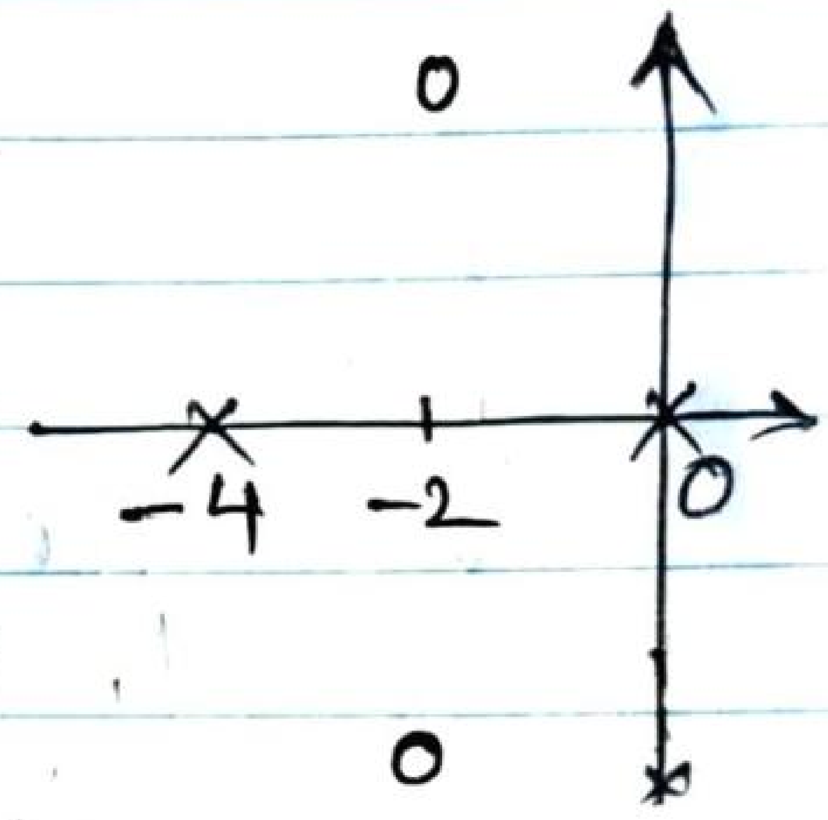
TITLE Root locus of example 7



8. sketch root locus of system with open loop transfer function $G(s)H(s) = \frac{k(s^2+4s+13)}{s(s+4)}$

<u>sol^n</u> The given system has open loop transfer function

$$G(s)H(s) = \frac{k(s^2+4s+13)}{s(s+4)}$$



<u>No. of poles, zeros and their location</u>

$P = 2$, poles at $s = 0, -4$

$Z = 2$, zeros at $s = -2+j3, -2-j3$

<u>Existance of root locus on real axis –</u>

Root locus exists on real axis between $s = 0$ & -4.

<u>No. of asymptotes and their angles, centroid –</u>

No. of asymptotes $= P - Z = 0$

$\therefore$ Asymptotes, their angles and centroid does not exist.

<u>Breakaway point –</u>

The closed loop characteristic equation is,

$$1 + \frac{k(s^2+4s+13)}{s(s+4)} = 0$$

$$\therefore s(s+4) + k(s^2+4s+13) = 0$$

$$\therefore k = -\left[\frac{s^2+4s}{s^2+4s+13}\right]$$

$$\therefore \frac{dk}{ds} = -\left[\frac{(s^2+4s+13)(2s+4) - (s^2+4s)(2s+4)}{(s^2+4s+13)^2}\right]$$

$$\therefore (2s+4)(13) = 0$$

$$\therefore 2s+4 = 0$$

$$\therefore s = -2$$

$s = -2$ is the breakaway point.

Intersection with imaginary axis—

The closed loop characteristic equation is,

$$s^2+4s+K(s^2+4s+13)=0$$

$\therefore$ Routh array is,

s^2	$1+K$	$13K$
s^1	$4+4K$	
s^0	$13K$	

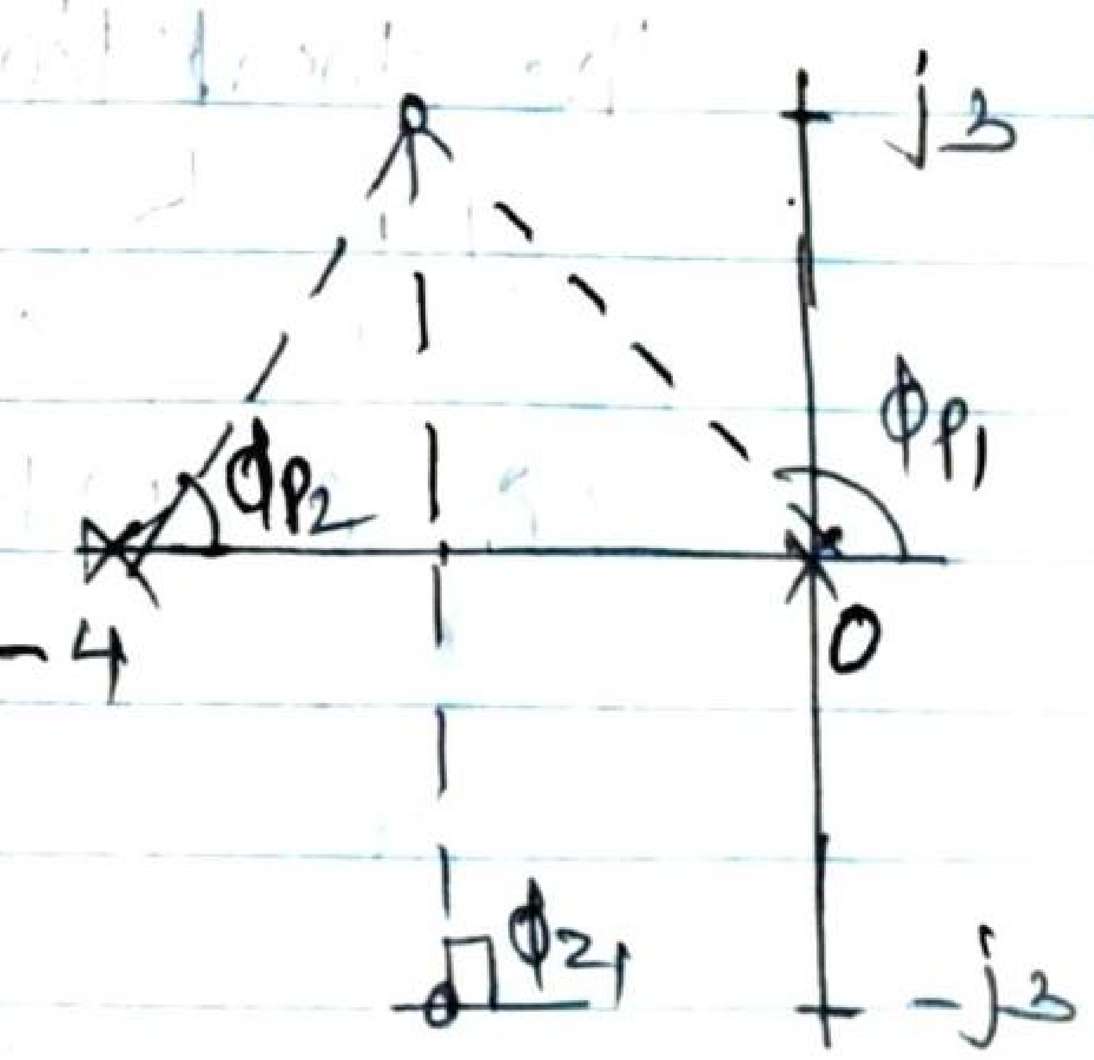
For stability

1. $13K>0$
   $\therefore K>0$
2. $4+4K>0$
   $\therefore K>-1$
3. $1+K>0$
   $\therefore K>-1$

$\therefore$ Range of K is $-1<K<\infty$

$\therefore$ System is stable for all $K>0$. Hence root locus will not intersect imaginary axis.

Angle of arrival—



For zero at $s=-2+j3$

$\phi_{P_1}=180-\tan^{-1}(3/2)=123.69^\circ$

$\phi_{P_2}=\tan^{-1}3/2=56.31^\circ$

$\therefore \phi_p=180^\circ$

$\phi_z=90^\circ$

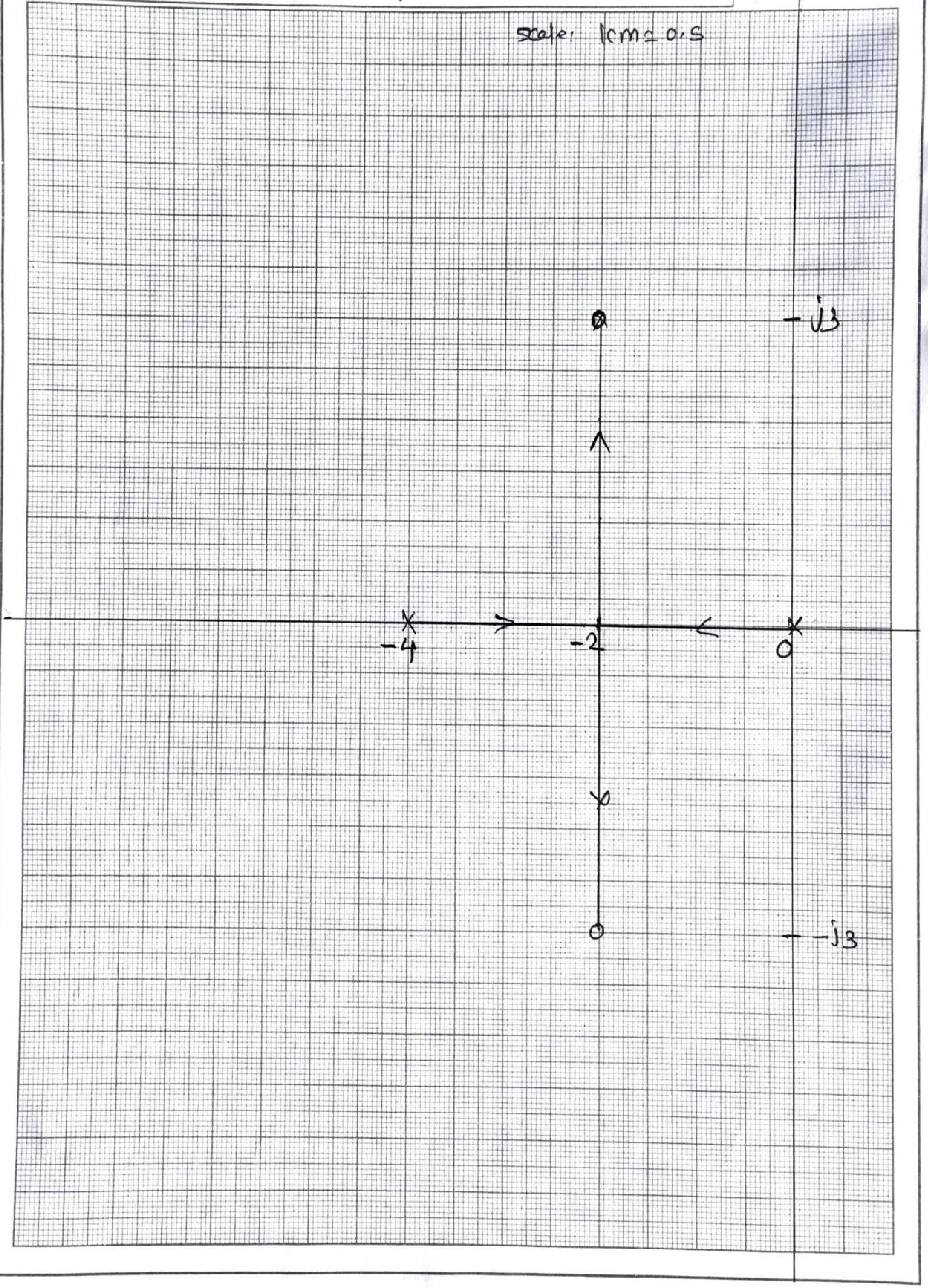
$\therefore \phi_a=-[180-(180-90)]=-90^\circ$

similarly for zero at $s=-1-j$

$\phi_a=90^\circ$

The root locus is sketched on graph.

TITLE Root locus of example 8



Scanned by TapScanner