Continuous-time standard functions:

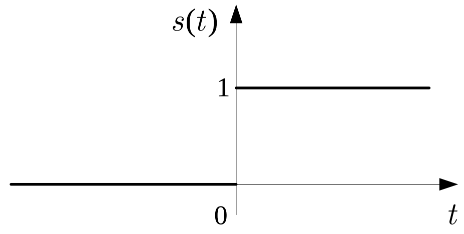
- unit step function
- impulse function/Dirac-Impulse
- complex-valued exponential function

Discrete-time standard sequences:

- unit step sequence
- unit impulse sequence
- complex-valued exponential sequence

The ***unit step function*** (or Heaviside step function) $s(t)$ is defined as

$$s(t) = \begin{cases} 0 & t < 0 \\ 0.5 & t = 0 \\ 1 & t > 0 \end{cases} . \tag{1.1}$$



A wide-variety of functions can be represented as linear combination of step functions:

$$x(t) = x(-\infty) + \int_{-\infty}^{\infty} s(t-\tau)\dot{x}(\tau)\,\mathrm{d}\tau \tag{1.2}$$

Often also following definition is used

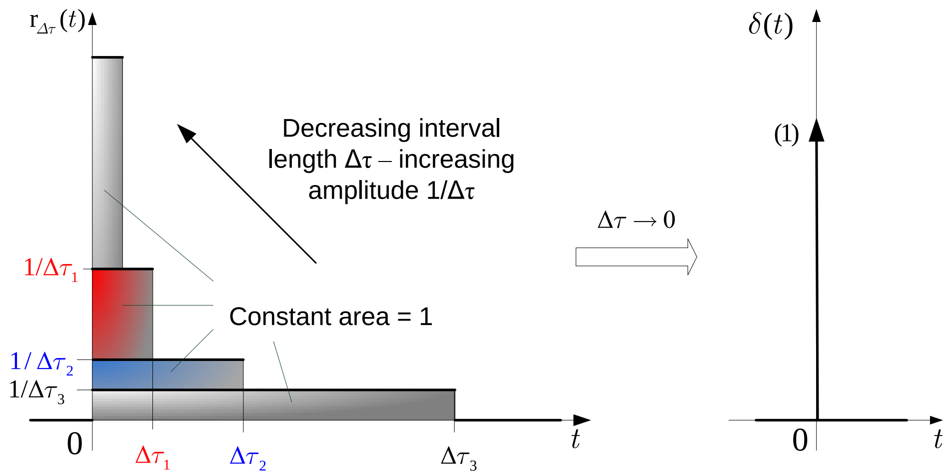
$$\epsilon(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} . \tag{1.3}$$

The ***impulse function*** $\delta(t)$ - a so-called generalized function or *distribution* - is defined by the following relation which shall be fulfilled for every continuous function $f(t)$:

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,\mathrm{d}\tau = f(t). \tag{1.4}$$

This relation is referred to as ***masking property*** of the unit impulse function.

$\delta(t)$ can be considered as limit value of the rectangular function $\mathrm{r}_{\Delta\tau}(t)$ for vanishing values $\Delta\tau$:



Every continuous function $x(t)$ can be represented as linear combination (integral) of time-shifted unit impulse functions $\delta(t-\tau)$ with weights $x(\tau)\,\mathrm{d}\tau$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau \tag{1.5}$$

- Area

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \quad \text{bzw.} \quad \int_{-\infty}^{\infty} a\delta(t)\,\mathrm{d}t = a \tag{1.6}$$

- Integration

$$\int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = s(t) \quad \text{bzw.} \quad \delta(t) = \frac{\mathrm{d}s(t)}{\mathrm{d}t} \tag{1.7}$$

- Linear combination of two impulse functions

$$a\delta(t) + b\delta(t) = (a+b)\delta(t) \tag{1.8}$$

- Scaling of time axes/argument

$$\int_{-\infty}^{\infty} \delta(at)\,\mathrm{d}t = \frac{1}{|a|} \quad \text{bzw.} \quad \delta(at) = \frac{1}{|a|}\delta(t) \tag{1.9}$$

- Masking for $t = 0$

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,\mathrm{d}t = x(0) \tag{1.10}$$

- Multiplication with a *continuous* function $x(t)$

$$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau) \tag{1.11}$$

- Convolution

$$x(t) * \delta(t-\tau) = \int_{-\infty}^{\infty} x(\zeta)\delta(t-\zeta-\tau)\,\mathrm{d}\zeta = x(t-\tau) \tag{1.12}$$

The masking property in Eq. (1.4) leads to the ***continuous-time convolution*** which is for arbitrary functions $f(t)$ and $g(t)$ defined as

$$\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau = f(t) * g(t) = g(t) * f(t) = \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, \mathrm{d}\tau \qquad (1.13)$$

The ***complex exponential function*** is defined as:

$$x(t) = X\,\mathrm{e}^{st}, \tag{1.14}$$

where

- complex amplitude $X = \hat{X}\,\mathrm{e}^{\mathrm{j}\varphi}, \quad \hat{X} \in \mathbb{R}_0^+, \varphi \in \mathbb{R}$,
- complex frequency $s = \sigma + \mathrm{j}\omega, \qquad \sigma, \omega \in \mathbb{R}$.
- angular frequency $\omega = 2\pi f = 2\pi/T$,
- frequency f,
- period T.

- Functions of exponential order $x(t)$ can be represented as (Laplace-transform)

$$x(t) = \frac{1}{2\pi \mathrm{j}} \int_{\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} X(s)\, \mathrm{e}^{st}\, \mathrm{d}s,\ X(s) \in \mathbb{C}, \tag{1.15}$$

- Absolutely integrable functions/continuous functions $x(t)$ with a finite number of discontinuities can be represented as (Fourier-Transform)

$$x(t) = \int_{-\infty}^{\infty} X(f)\, \mathrm{e}^{\mathrm{j}2\pi f t}\, \mathrm{d}f,\ X(f) \in \mathbb{C}, \tag{1.16}$$

- Bounded periodic functions $x(t) = x(t+T)$ with a finite number of discontinuities can be represented as (Fourier-series)

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, \mathrm{e}^{\mathrm{j}k2\pi f t},\ X_k \in \mathbb{C},\ f = 1/T \tag{1.17}$$

The ***unit impulse sequence*** is defined as

$$\delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \in \mathbb{Z}\backslash\{0\} \end{cases} . \tag{1.18}$$

The discrete-time masking property is given as:

$$x[k] = \sum_{\kappa=-\infty}^{\infty} x[\kappa]\delta[k-\kappa] = x[k] * \delta[k]. \tag{1.19}$$

The masking property in Eq. (1.19) leads to the ***discrete-time convolution*** which is for arbitrary sequences $f[k]$ and $g[k]$ defined as

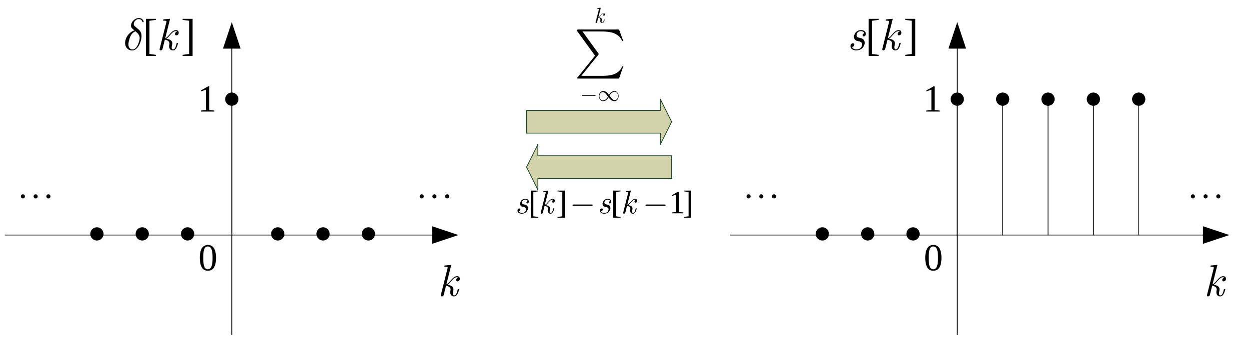
$$\sum_{\kappa=-\infty}^{\infty} f[\kappa]\, g[k-\kappa] = f[k] * g[k] = g[k] * f[k] = \sum_{\kappa=-\infty}^{\infty} f[k-\kappa]\, g[\kappa] \tag{1.20}$$

The ***unit step sequence*** is defined as

$$s[k] = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases} . \tag{1.21}$$

Between the unit impulse sequence and unit step sequence holds following relationship:

$$s[k] = \sum_{\kappa=-\infty}^{k} \delta[\kappa] \Leftrightarrow \delta[k] = s[k] - s[k-1].$$

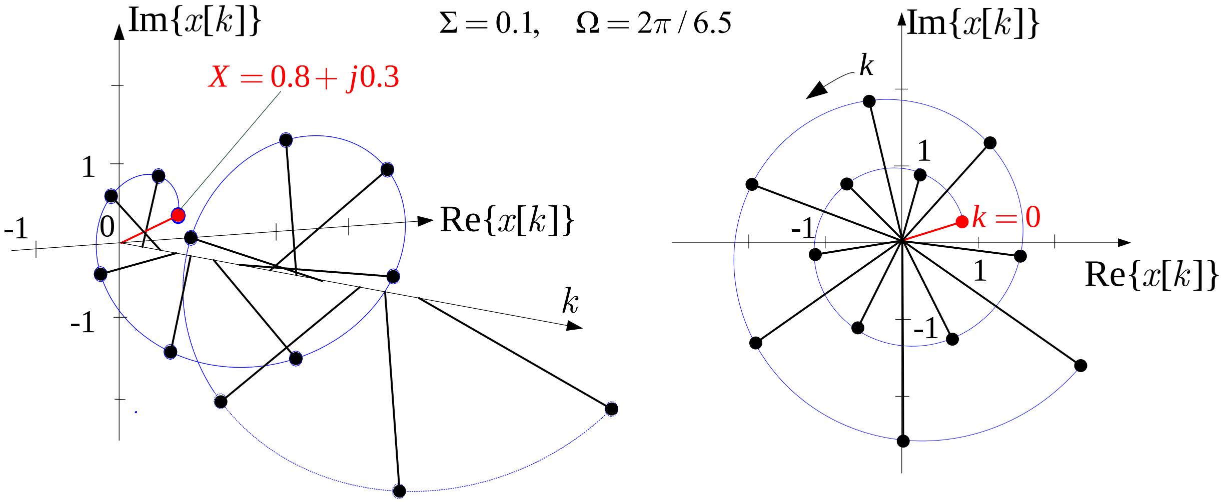


The ***complex exponential sequence*** is defined as

$$x[k] = X\ \mathrm{e}^{(\Sigma + \mathrm{j}\Omega)k} = X\ \left(r\ \mathrm{e}^{\mathrm{j}\Omega}\right)^k = X\ z^k, \tag{1.22}$$

where

- X: complex amplitude
- $\Sigma = \ln(r)$: attenuation
- Ω: circular frequency
- $z = r \cdot \mathrm{e}^{\mathrm{j}\Omega}$



For the unit amplitude ***periodic complex exponential sequence*** with $X = 1, r = 1$ follows:

$$e^{j\Omega k} = e^{j(\Omega+2\pi)k}. \tag{1.23}$$

Every shift of the circular frequency Ω by an integer multiple of 2π yields the same sequence:

$$e^{-j\pi k} = e^{j\pi k} = ... = e^{j(\pi+n2\pi)k},\ \Omega = -\pi + n \cdot 2\pi, n \in \mathbb{N}.$$

(This ambiguity is the reason for aliasing if the sampling frequency is not chosen large enough and violates the sampling theorem.)

- A sequence of exponential order $x[k]$ can be represented as (z-transform)

$$x[k] = \frac{1}{2\pi \mathrm{j}} \oint_C X(z) z^{k-1} \, \mathrm{d}z, \; X(z) \in \mathbb{C}, \tag{1.24}$$

- An absolutely summable sequence $x[k]$ can be represented as (Discrete-time Fourier-Transform)

$$x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(\mathrm{e}^{\mathrm{j}\Omega}\right) \, \mathrm{e}^{\mathrm{j}\Omega k} \, \mathrm{d}\Omega, \tag{1.25}$$

- M-periodic sequences $x[k] = x[k+M]$ can be represented as (Fourier series or Discrete Fourier-Transform)

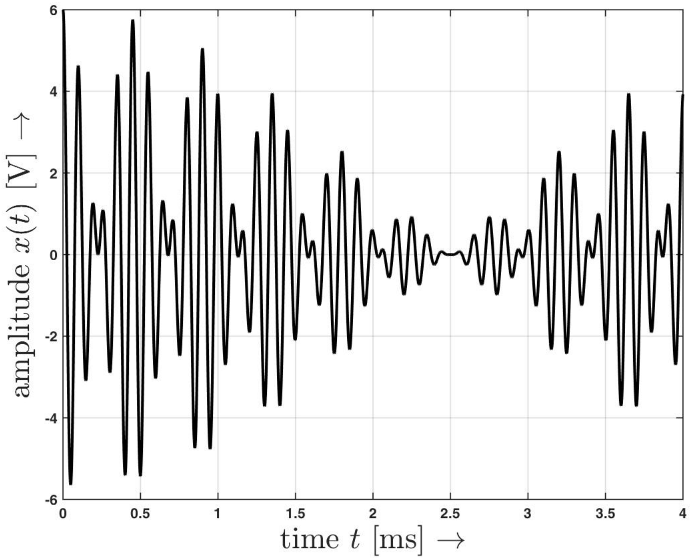
$$x[k] = \sum_{m=0}^{M-1} X_m \, \mathrm{e}^{\mathrm{j}2\pi km/M}, \; X_m \in \mathbb{C}. \tag{1.26}$$

  or

$$x[k] = \frac{1}{M} \sum_{m=0}^{M-1} X(m) \, \mathrm{e}^{\mathrm{j}2\pi km/M}, \; X(m) \in \mathbb{C}. \tag{1.27}$$

A ***signal*** is a function or sequence representing information

Radio signal as example of 1D signal $x(t)$:



Picture as example of 2D sequence $x[k, \ell]$

Energy and power are defined predominantly by assuming that the signal is normalized to a specific unit. For a real eletrotechnical system this would e.g. mean that signal amplitudes are measured in [V] (Volt) with reference to a standard resistance with value $R = 1\Omega$.

The energy E_x of a continuous-time signal $x(t)$ is:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t. \tag{1.28}$$

The energy E_x of a discrete-time signal $x[k]$ is:

$$E_x = \sum_{k=-\infty}^{\infty} |x[k]|^2. \tag{1.29}$$

Signals for which these integrals/sums exist are also referred to as "energy-limited" signals.

The integral/sum from $-\infty$ to ∞ however doesn't converge for many, e.g. periodic, signals. The signal energy has to be replaced by the average signal power in these cases.

The power P_x of a continuous-time signal $x(t)$ is:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, \mathrm{d}t. \tag{1.30}$$

The power P_x of a discrete-time signal $x[k]$ is:

$$P_x = \lim_{K\to\infty} \frac{1}{K+1} \sum_{-K/2}^{K/2} |x[k]|^2. \tag{1.31}$$

In a mathematical context the power equals the second moment.

The calculation of the signal power simplifies for the class of *periodic* signals:

The power P_x of a periodic continuous-time signal $x(t) = x(t+T)$ with period T is:

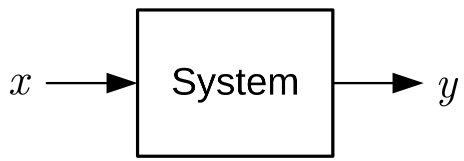
$$P_x = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, \mathrm{d}t. \tag{1.32}$$

The power P_x of a periodic dicrete-time signal $x[k] = x[k+K]$ with period K is:

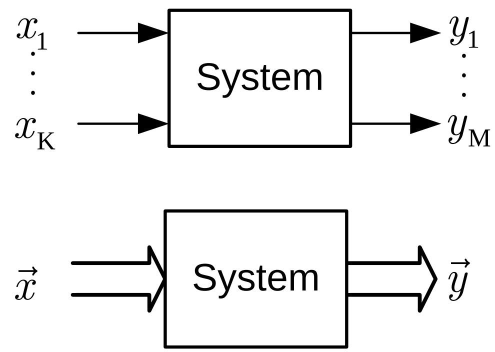
$$P_x = \frac{1}{K} \sum_{k=0}^{K-1} |x[k]|^2. \tag{1.33}$$

A ***system*** is a process or operator which transforms signals and connects them with each other.

one input, one output

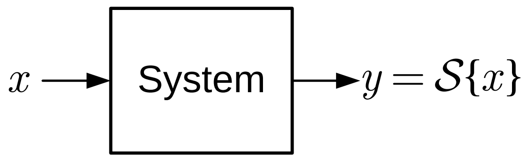


multiple inputs, multiple outputs



The transformation of an input signal x into an output signal y conducted by a system can mathematically be described by means of the operator $\mathcal{S}$:

$$y(t) = \mathcal{S}\{x(t)\} \quad \text{or} \quad y[k] = \mathcal{S}\{x[k]\}. \tag{1.34}$$



"Principle of superposition" (continuous-time)

Let $x_1(t)$ and $x_2(t)$ be two distinct input-signals and $y_1(t)$ and $y_2(t)$ be the corresponding output-signals of a discrete-time system

$$y_1(t) = \mathcal{S}\{x_1(t)\}, \qquad y_2(t) = \mathcal{S}\{x_2(t)\}, \tag{1.35}$$

Iff the ***system is linear***, then any linear combination $x(t) = ax_1(t) + bx_2(t)$ of $x_1(t), x_2(t)$ yields the same linear combination of $y_1(t)$ and $y_2(t)$ as output-signal

$$ay_1(t) + by_2(t) = \mathcal{S}\{ax_1(t) + bx_2(t)\}. \tag{1.36}$$

"Principle of superposition"(discrete-time)

Let $x_1[k]$ and $x_2[k]$ be two distinct input-signals and $y_1[k]$ and $y_2[k]$ be the corresponding output-signals of a discrete-time system

$$y_1[k] = \mathcal{S}\{x_1[k]\}, \qquad y_2[k] = \mathcal{S}\{x_2[k]\}, \tag{1.37}$$

Iff the ***system is linear***, then any linear combination $x[k] = ax_1[k] + bx_2[k]$ of $x_1[k], x_2[k]$ yields the same linear combination of $y_1[k]$ and $y_2[k]$ as output-signal

$$ay_1[k] + by_2[k] = \mathcal{S}\{ax_1[k] + bx_2[k]\}. \tag{1.38}$$

Let the input-output-relation of a continuous-time system be described by $y(t) = \mathcal{S}\{x(t)\}$:
The system is ***time-invariant***, if for every time shift τ holds

$$y(t) = \mathcal{S}\{x(t)\} \quad \Rightarrow \quad y(t-\tau) = \mathcal{S}\{x(t-\tau)\}, \quad \tau \in \mathbb{R} \tag{1.39}$$

Let the input-output-relation of a discrete-time system be described by $y[k] = \mathcal{S}\{x[k]\}$.
The system is ***time-invariant***, if for every time shift κ holds

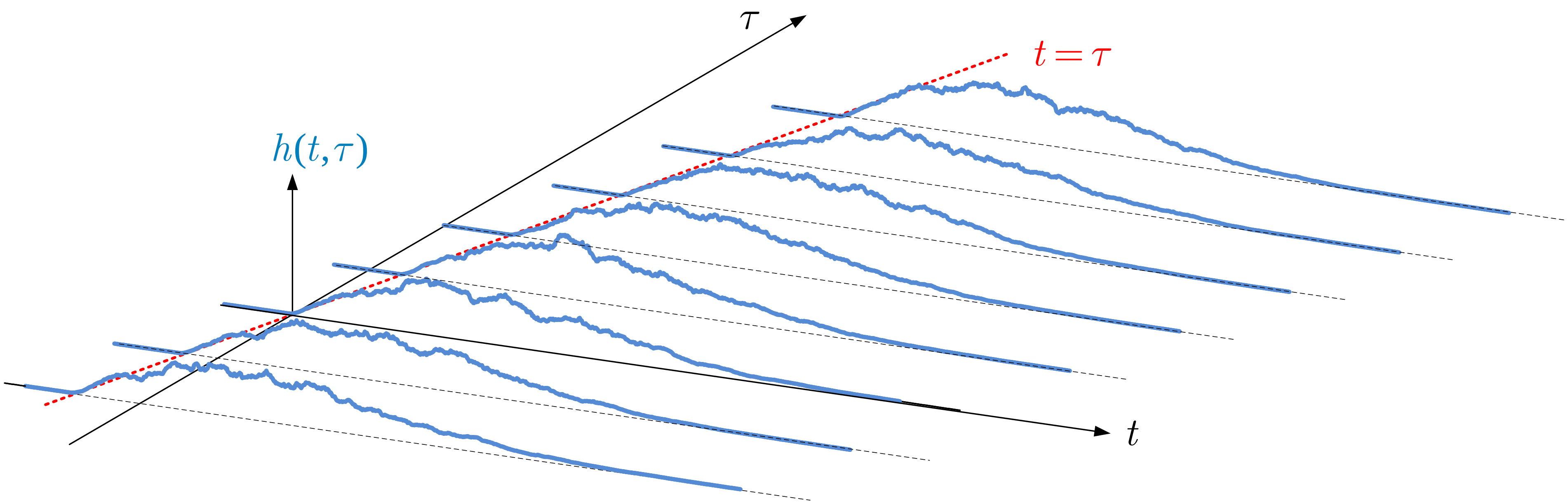
$$y[k] = \mathcal{S}\{x[k]\} \quad \Rightarrow \quad y[k-\kappa] = \mathcal{S}\{x[k-\kappa]\}, \quad \kappa \in \mathbb{Z}. \tag{1.40}$$

Iff a system is not time-invariant, then it is ***time-variant***.

For general parameters system characteristics are referred to as shift-invariant or shift-variant.

The response of a linear continuous-time system to an excitation with a unit impulse $\delta(t-\tau)$ is referred to as **impulse response** $h(t,\tau)$

$$h(t,\tau) = \mathcal{S}\{\delta(t-\tau)\}. \tag{1.41}$$



Note: The system is exited with an impulse at time instant $t=\tau$ and the impulse response is in general a function of the actual time t and the exitation time τ!

Using that $x(t)$ can be written as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau,$$

the reaction $y(t)$ of the system on the input $x(t)$ is given as for a ***linear time-variant*** (LTV) system with impulse response $h(t,\tau)$

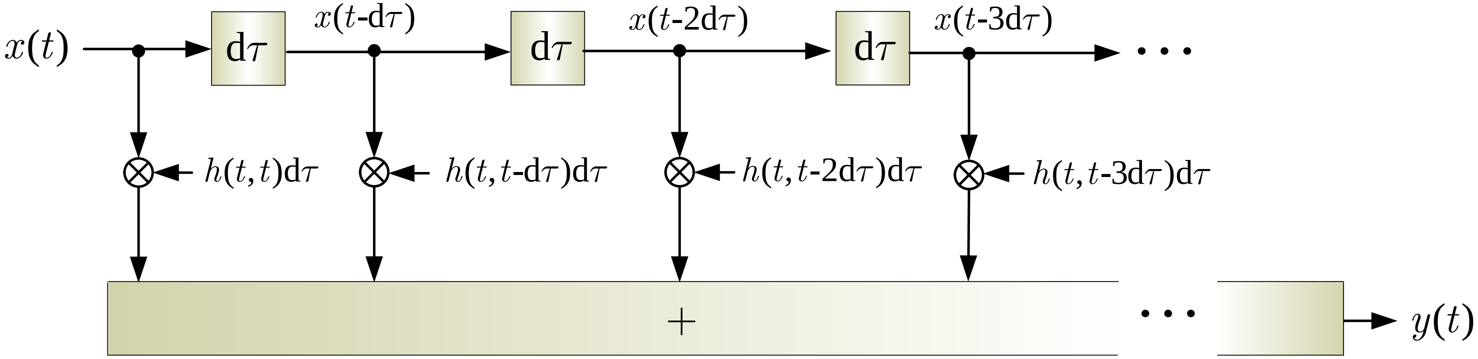
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t,\tau)\,\mathrm{d}\tau. \tag{1.42}$$

For the real-world case of causal systems with

$$h(t,\tau) = 0 \quad \forall t < \tau \tag{1.43}$$

and causal signals the output signal is calculated as

$$y(t) = \int_{0}^{t} x(\tau)h(t,\tau)\,\mathrm{d}\tau. \tag{1.44}$$



For ***linear time-invariant*** (LTI) systems the impulse response only relies on the time difference between exitation time and actual time

$$h(t, \tau) = h(t - \tau). \tag{1.45}$$

That is, a LTI system is fully characterized by the impulse response

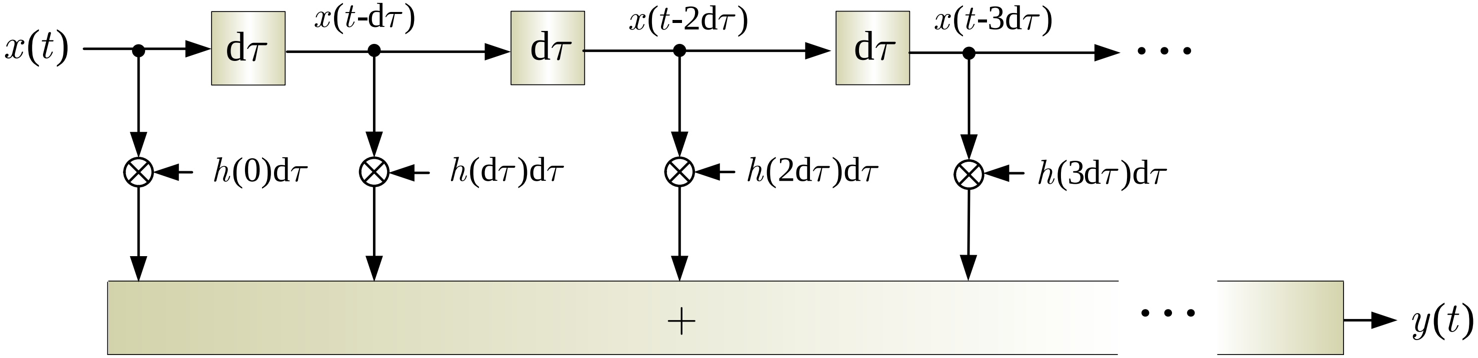
$$h(t) = \mathcal{S}\{\delta(t)\}. \tag{1.46}$$

and the input-output signal relation is given as

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau)\, \mathrm{d}\tau = \int_{-\infty}^{\infty} x(t - \tau) h(\tau)\, \mathrm{d}\tau = x(t) * h(t). \tag{1.47}$$

For causal signals and systems this simplifies to

$$y(t) = \int_{0}^{t} x(\tau) h(t - \tau)\, \mathrm{d}\tau = \int_{0}^{t} x(t - \tau) h(\tau)\, \mathrm{d}\tau. \tag{1.48}$$



The response of a linear discrete-time system to an excitation with a unit impulse $\delta[k-\kappa]$ is referred to as **impulse response** $h[k,\kappa]$

$$h[k,\kappa] = \mathcal{S}\{\delta[k-\kappa]\}. \tag{1.49}$$

Using that $x[k]$ can be written as $x[k] = \sum_{\kappa=-\infty}^{\infty} x[k]\delta[k-\kappa]$, the reaction $y[k]$ of a ***linear time-variant*** (LTV) system with impulse response $h[k,\kappa]$ on the input $x[k]$ is given as

$$y[k] = \sum_{\kappa=-\infty}^{\infty} x[\kappa]h[k,\kappa]. \tag{1.50}$$

For ***linear time-invariant*** (LTI) systems the impulse response only relies on the time difference between exitation time and actual time

$$h[k,\kappa] = h[k-\kappa] \tag{1.51}$$

and the input-output signal relation is given as

$$y[k] = \sum_{\kappa=-\infty}^{\infty} x[\kappa]h[k-\kappa] = \sum_{\kappa=-\infty}^{\infty} x[k-\kappa]h[\kappa] = x[k] * h[k]. \tag{1.52}$$

For discrete-time causal LTI systems and causal signals the summation limits are given as

$$y[k] = \sum_{\kappa=0}^{k} x[\kappa]h[k-\kappa] = \sum_{\kappa=0}^{k} x[k-\kappa]h[\kappa]. \tag{1.53}$$

Let $x(t)$ be a time-domain function, then its Fourier transform is defined as:

$$X(f) = \int_{-\infty}^{\infty} x(t)\ \mathrm{e}^{-\mathrm{j}2\pi f t}\, \mathrm{d}t. \tag{2.1}$$

Let the Fourier transform $X(f)$ be a function of the frequency f, then its corresponding time-domain function is defined as:

$$x(t) = \int_{-\infty}^{\infty} X(f)\ \mathrm{e}^{\mathrm{j}2\pi f t}\, \mathrm{d}f. \tag{2.2}$$

The correspondence between the pair of Fourier transforms $x(t)$ and $X(f) = \mathcal{F}\{x(t)\}$ is symbolized by:

$$x(t) \ \circ\!\!-\!\!\bullet\ X(f) = |X(f)|\ \mathrm{e}^{\mathrm{j}\arg\{X(f)\}}. \tag{2.3}$$