Question 1

Find a general solution of the following differential equation.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0 . \qquad \textbf{(4)}$$

Question 2

$$z^5 = \mathrm{i}, \quad z \in \mathbb{C}.$$

a) Solve the equation, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0$, $-\pi < \theta \le \pi$. **(5)**

b) Plot the roots of the equation as points in an Argand diagram. **(1)**

Question 3

a) Sketch the graph of $y = \text{arsech}\, x$, defined for $0 < x \leq 1$. **(3)**

b) Show clearly that

$$\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}.$$

(4)

c) Hence evaluate

$$\int_{\frac{1}{2}}^{1} \text{arsech}\, x \ dx.$$

Give the answer in the form $\lambda\left[2\pi - 3\ln\left(2+\sqrt{3}\right)\right]$, where λ is a rational number to be found. **(8)**

Question 4

By showing formally all the limiting processes evaluate the following integral

$$\int_0^{\frac{1}{4}\pi} \frac{1}{x} - \frac{\sin 2x}{1-\cos 2x} \, dx \, .$$

Give the answer in the form $\ln\left[\frac{\pi\sqrt{2}}{n}\right]$, where n is a positive integer to be found. **(8)**

Question 5

Consider the following infinite convergent series.

$$\frac{3}{1\times2}-\frac{5}{2\times3}+\frac{7}{3\times4}-\frac{9}{4\times5}+\frac{11}{5\times6}-....$$

a) Use the method of differences, to find the sum of this series. **(8)**

b) Verify the answer of part **(a)** by using a method based on the Maclaurin expansion of $\ln(1+x)$. **(8)**

Question 6

The following polar equations are given.

$$r_1 = \cos\theta, \quad 0 \le \theta \le \pi.$$

$$r_2 = \frac{1}{\cos\theta - \sin\theta}, \quad -\tfrac{1}{4}\pi \le \theta \le \tfrac{5}{4}\pi.$$

Find, in exact simplified form, the area of the **smaller** of the two finite regions, bounded by r_1 and r_2. **(8)**

Question 7

Use appropriate integration techniques to find an exact simplified value for

$$\int_{0}^{\frac{1}{4}\pi} \frac{10}{2-\tan x}\,dx\,. \qquad \textbf{(10)}$$

Question 8

$$f(x) = 2\arcsin\sqrt{x} - \arcsin(2x-1), \quad 0 \le x \le 1.$$

By considering $f'(x)$ sketch the graph of $f(x)$. **(8)**