A constant-ratio approximation algorithm for a class of hub-and-spoke network design problems and metric labeling problems: star metric case

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Abstract

Transportation networks frequently employ hub-and-spoke network architectures to route flows between many origin and destination pairs. In this paper, we deal with a problem, called the *single allocation hub-and-spoke network design problem*. In the single allocation hub-and-spoke network design problem, the goal is to allocate each non-hub node to exactly one of given hub nodes so as to minimize the total transportation cost. The problem is essentially equivalent to another combinatorial optimization problem, called the *metric labeling* problem. The metric labeling problem was first introduced by Kleinberg and Tardos [29] in 2002, motivated by application to segmentation problems in computer vision and energy minimization problems in related areas.

In this paper, we deal with the case where the set of hubs forms a star, which is called the *star-star hub-and-spoke network design problem*, and the *star-metric labeling problem*. This model arises especially in telecommunication networks in the case where set-up costs of hub links are considerably large or full interconnection is not required. We propose a polynomial-time randomized approximation algorithm for these problems, whose approximation ratio is less than 5.281. Our algorithms solve a linear relaxation problem and apply dependent rounding procedures.

1 Introduction

Design of efficient networks is desired in transportation systems, such as telecommunications, delivery services, and airline operations, and is one of the extensively studied topics in operations research field. Transportation networks frequently employ *hub-and-spoke network* architectures to route flows between many origin and destination pairs. A transportation network with many origins and destinations requires a huge cost, and hub-and-spoke networks play an important role in reducing transportation costs and set-up costs. Hub facilities work as switching points for flows in a large network. Each non-hub node is allocated to exactly one of the hubs instead of assigning every origin-destination pair directly. Using hub-and-spoke architecture, we can construct large transportation networks with fewer links, which leads to smart operating systems (see Figure 1).
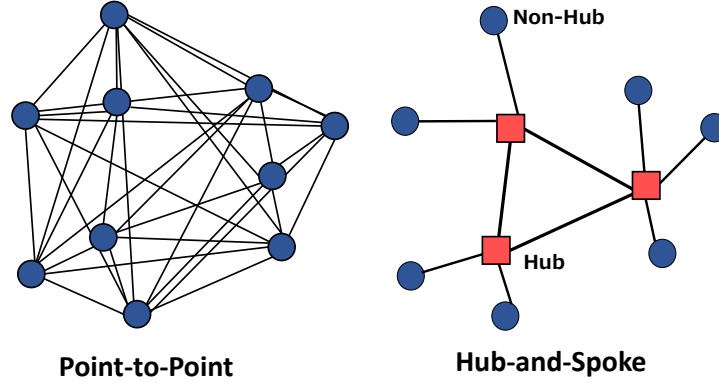
Figure 1: Point-to-Point vs. Hub-and-Spoke

1.1 Single Allocation Hub-and-Spoke Network Desing Problem

In real transportation systems, the location of hub facilities is often fixed because of costs for moving equipment on hubs. In that case, the decision of allocating non-hubs to hubs is much important for an efficient transportation. In this study, we discuss the situation where the location of the hubs is given, and deal with a problem, called a *single allocation hub-and-spoke network design problem*, which aims to minimize the total transportation cost.

Formally, the input consists of an h-set H of hubs, an n-set N of non-hubs, non-negative cost per unit flow $c(i, j) = c(j, i)$ for each pair $\{i, j\} \in H^2$, and $c(p, i)$ for each ordered pair $(p, i) \in N \times H$. Additionally, we are given $w(p, q)$ which denotes a non-negative amount of flow from non-hub p to another non-hub q. The task is to find an assignment $f : N \to H$, that maps non-hubs to hubs minimizing the total transportation cost $Q(f)$ defined below. The transportation cost corresponding to a flow from non-hub p to non-hub q is defined by $w_{pq}(c(p, f(p)) + c(f(p), f(q)) + c(f(q), q))$. Thus

$$Q(f) = \sum_{(p,q) \in N^2} w(p, q) \left(c(p, f(p)) + c(q, f(q)) + c(f(p), f(q)) \right),$$

and the goal is to find an assignment that minimizes the total transportation cost.

When the number of hubs is equal to two, there exist polynomial time exact algorithm [25, 39]. Sohn and Park [40] proved NP-completeness of the problem even if the number of hubs is equal to three. In the case where the given matrix of costs between hubs is a *Monge matrix*, there exists a polynomial-time exact algorithm [16]. Iwasa et al. [27] proposed a simple deterministic 3-approximation algorithm and a randomized 2-approximation algorithm under the assumptions that $c_{ij} \leq c_{pi} + c_{pj}$ $(\forall (i, j, p) \in H^2 \times N)$ and $c_{ij} \leq c_{ik} + c_{kj}$ $(\forall (i, j, k) \in H^3)$. They also proposed a (5/4)-approximation algorithm for the special case where the number of hubs is three. Ando and Matsui [2] deal with the case in which all the nodes are embedded in a 2-dimensional plane and the transportation cost of an edge per unit flow is proportional to the Euclidean distance between the

end nodes of the edge. They proposed a randomized $(1 + 2/\pi)$-approximation algorithm. In the previous our work [33], we proposed $2(1 + \frac{1}{h})$-approximation algorithm for the case where the set of hubs forms a cycle.

1.2 Metric Labeling Problem

In 2002, Kleinberg and Tardos [29] introduced the *metric labeling* problem, motivated by applications to segmentation problems in computer vision and energy minimization problems in related areas. A variety of heuristics that use classical combinatorial optimization techniques have developed in these fields ([7, 8, 32, 35] for example). A single allocation hub-and-spoke network design problem includes a class of the metric labeling problem. The metric labeling problem captures a broad range of classification problems and has connections to Markov random field. In such classification problems, the goal is to assign labels to some given set of objects minimizing the total cost of labeling.

Formally, the metric labeling problem takes as input an n-vertex undirected graph $G(V, E)$ with a nonnegative weight function w on the edges, a set L of labels with metric distance function $d : L \times L \to R$ associated with them, and an assignment cost $c(v, a)$ for each vertex $v \in V$ and label $a \in L$. The output is an assignment for every object $v \in V$ to a label $a \in L$. Given a solution $f : V \to L$ to the metric labeling, the quality of labeling $Q(f)$ is based on the contribution of two sets of terms.

Vertex labeling cost: For each object $v \in V$, this cost is denoted by $c(v, f(v))$. A vertex labeling cost $c(v, a)$ express an estimate of its likelihood of having each label $a \in L$. These likelihoods are observed from some heuristic preprocessing of the data. For example, suppose the observed color of pixel (i.e., object) v is white; then the cost $c(v, black)$ should be high while $c(v, white)$ should be low.

Edge separation cost: For each edge $e = \{u, v\} \in E$, the cost is denoted by $w(\{u, v\}) \cdot d(f(u), f(v))$. The weights of the edges express a prior estimate on relationships among objects; if u and v are deemed to be related, then we would like them to be assigned close or identical labels. A distance $d(a, b)$ for $a, b \in L$ represents how similar label a and b are. For example, $d(white, black)$ would be large while $d(orange, yellow)$ would be small. If we assign label a to object u and label b to object v, then we pay $w(\{u, v\})d(a, b)$ as the edge separation cost.

Thus,

$$Q(f) = \sum_{u \in V} c(u, f(u)) + \sum_{\{u,v\} \in E} w(\{u, v\})d(f(u), f(v)),$$

and the goal is to find a labeling $f : V \to L$ minimizing $Q(f)$. Due to the simple structure and variety of applications, the metric labeling has received much attention since its introduction by Kleinberg and Tardos [29].

In case the number of labels is two, the problem can be solved precisely in polynomial-time. The first approximation algorithm for the metric labeling problem was shown by Kleinberg and Tar-

dos [29], and its approximation ratio is $O(\log k \log\log k)$, where k denotes the number of labels. This algorithm uses the probabilistic tree embedding tequnique [5]. Using the improved representation of metrics as combination of tree metrics by Fakcharoenphol, Rao, and Talwar [24], its approximation ratio was improved to $O(\log k)$, which is the best general result to date. Constant-ratio approximations are known for some special cases [4, 11, 16, 29].

1.3 Contributions

We deal with the a single assignment hub-and-spoke network design problem where the given set of hubs forms a star, and corresponding problem is called the *star-star hub-and-spoke network design problems* and *star-metric labeling problems*. In this case, each hub is only connected to a unique depot. When all the transportation cost per unit flow between the depot and each hub are same, this problem is equivalent to the *uniform labeling* problem (all distances of labels are equal to 1) introduced in [29] which is still NP-hard. For star-metric case, using the result of [31] for planer graphs, there exists $O(1)$-approximation algorithm [29]. The previous $O(1)$-approximation ratio is at least 6. We proposed an improved approximation algorithm for star-metric case, and the approximation ratio is $\min\{\frac{r-1}{\log r}\left(2 + \frac{r^2+1}{r^2-1}\right) | r > 1\} (\approx 5.2809$ at $r \approx 1.91065)$. Our results give an important class of the metric labeling problem and hub-and-spoke network design problems, which has a polynomial time approximation algorithm with a constant approximation ratio. In case where set-up costs of hub links are considerably large, incomplete networks can be used instead of full interconnection among hub facilities. The star structures, that we discuss in this paper, frequently arise in especially telecommunication networks [34].

1.4 Related Work

Approximation Results for Metric Labeling Problems. Gupta and Tardos [26] considered an important case of the metric labeling problem, in which the metric is the *truncated linear metric* where the distance between i and j is given by $d(i,j) = \min\{M, |i-j|\}$. Chekuri et al. [16] proposed $(2 + \sqrt{2})$-approximation algorithm for the truncated linear metric, which is best known result.

In the case where the metric d on a set of labels L is a planar metric, there exists $O(\log \text{diam } G')$-approximation to the problem from the result [31] and [29], where $G' = (L, E, w)$ denote the weighted connected graph. Konjevod et al. [31] showed that for any positive integer s, the metric of G without a $K_{s,s}$ minor can be probabilistically approximated by a special case of tree metric, called *r-hierarchically well separated tree* (*r-HST*) with distortion $O(\log \text{diam } G)$. Kleinberg and Tardos [29] gave a constant ratio approximation algorithm to the metric labeling for the case where the metric d on a set of labels is the r-HST metric. Then $O(\log \text{diam } G')$-approximation was guaranteed by combining these results for this case.

Inapproximability Results. Chuzhoy and Naor [17] showed that there is no polynomial time

Table 1: Existing approximation algorithms for metric labeling problems

Metric	App. Ratio
general	$O(\log k)$ [24, 29]
planar graph	$O(\log \operatorname{diam} G')$ [29, 31]
truncked linear	$2 + \sqrt{2}$ [16]
uniform	2 [29]

approximation algorithm with a constant ratio for the metric labeling problem unless $P = NP$. Moreover, they proved that the problem is $\Omega((\log |V|)^{1/2-\delta})$-hard to approximate for any constant $\delta \, satisfying \, 0 < \delta < 1/2$, unless $NP \subseteq DTIME(n^{poly(\log n)})$ (i.e. unless NP has quasi-polynomial time algorithms).

In 2011, Andrew et al. [3] introduced *capacitated metric labeling*, in which there are additional restrictions that each label i receives at most l_i nodes. They proposed a polynomial-time, $O(\log |V|)$-approximation algorithm when the number of labels is fixed and proved that it is impossible to approximate the value of an instance of capacitated metric labeling to within any finite ratio, unless $P = NP$.

Hub Location Problems. *Hub location problems* (*HLPs*) consist of locating hubs and designing hub networks so as to minimize the sum of set-up costs and transportation costs. HLPs are formulated as a quadratic integer programming problem by O'Kelly [36] in 1987. Since O'Kelly proposed HLPs, hub location has been studied by researchers in different areas such as location science, geography, operations research, regional science, network optimization, transportation, telecommunications, and computer science. Many researches on HLPs have been done in various applications and there exists several reviews and surveys (see [1, 12, 15, 18, 30, 37] for example). In case where the location of the hubs is given, the remaining subproblem is essentially equal to the *single allocation hub-and-spoke network design problem* mentioned in previous subsections.

Fundamental HLPs assume a full interconnection between hubs. Recently, several researches consider incomplete hub networks which arise especially in telecommunication systems (see [1, 10, 13, 14] for example). These models are useful when set-up costs of hub links are considerably large or full interconnection is not required. That motivated us to consider a single allocation hub-and-spoke network design problem where the given set of hubs forms a star (see Figure 2). There are researches which assume that hub networks constitute a particular structure such as a line [22], a cycle [21], tree [19, 20, 23, 28, 38, 41], a star [34, 42, 43].
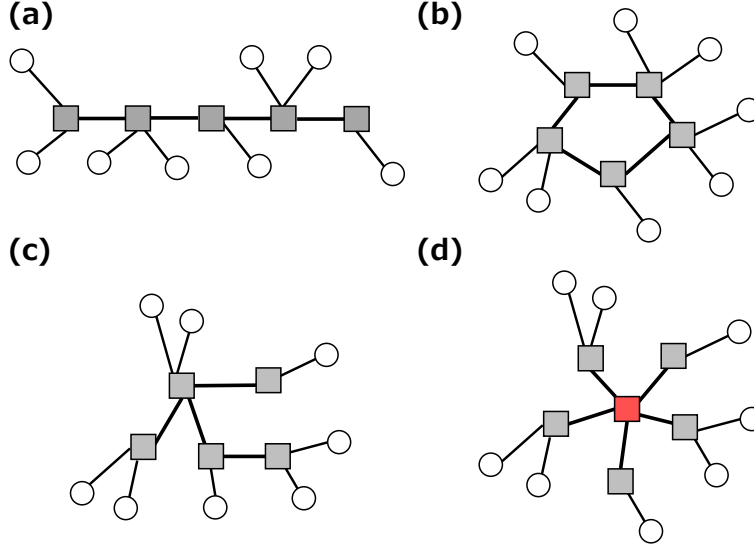
Figure 2: Structure of (a) line-star, (b) cycle-star, (c) tree-star, and (d) star-star

1.5 Paper Organization

This paper is structured as follows In Section 2, we provide a problem formulation. In Section 3, we describe an approximation algorithm. In Section 4, we analyze the approximation ratio of our algorithm.

2 Problem Formulation

Let $H = \{1, 2, \ldots, h\}$ be a h (≥ 3)-set of hub nodes and let $N = \{p_1, p_2, \ldots, p_n\}$ be a n-set of non-hub nodes. This paper deals with a single assignment hub network design problem which assigns each non-hub node to exactly one hub node. We discuss the case in which the set of hubs forms a star, and the corresponding problem is called the *star-star hub-and-spoke network design problem* and/or *star-metric labeling problem*. More precisely, we are given a unique depot, denoted by 0, which lies at the center of hubs. Each hub $i \in H$ connects to the depot and doesn't connect to other hubs. Let ℓ_i be the transportation cost per unit flow between the depot and a hub i. In our setting, we assume that $0 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_h$ and $\ell_i \in \mathbb{Z}$ for all $i \in H$. Then for each pair of hub nodes $(i, j) \in H$, c_{ij} denotes the transportation cost per unit flow between hub i and hub j and it satisfies that $c_{ij} = \ell_i + \ell_j$. We assume $c_{ii} = 0$ for all $i \in H$. For each ordered pair $(p, i) \in N \times H$, c_{pi} denotes a non-negative cost per unit flow on an undirected edge $\{p, i\}$. We denote a given non-negative amount of flow from a non-hub p to another non-hub q by w_{pq} (≥ 0). Throughout this paper, we assume that $w_{pp} = 0$ ($\forall p \in N$). We discuss the problem for finding an assignment of non-hubs to hubs which minimizes the total transportation cost defined below.

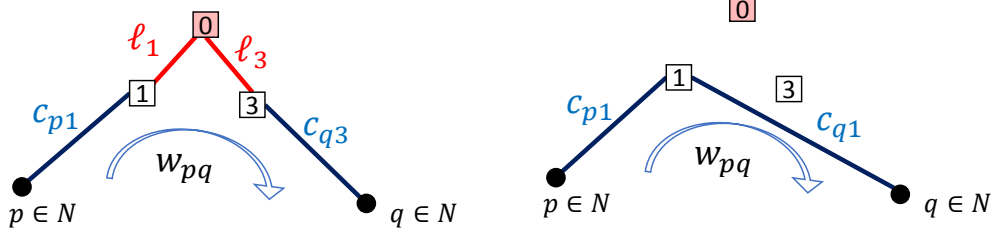When non-hub p and non-hub q ($p \neq q$) are assigned to hub i and hub j, respectively, an amount

Figure 3: Examples of transportation

of flow w_{pq} is sent along a path $((p, i), (i, 0), (0, j), (j, q))$. In the rest of this paper, a matrix $C = (c_{ij})$ defined above is called the *cost matrix* and/or the *star-metric matrix*. The transportation cost corresponding to a flow from the origin $p \in N$ to destination $q \in N$ is defined by $w_{pq}(c_{pi} + c_{ij} + c_{qj})$.

In case where $h = 3$ and $\ell_1 = \ell_2 = \ell_3 = 1$, the corresponding problem is equivalent to the problem where a 3-set of hubs forms a complete graph and C satisfies that $c_{12} = c_{23} = c_{31} = 2$. Thus the star-star hub network design problem is NP-hard [40].

Now we formulate our problem as 0-1 integer programming. First, we introduce a 0-1 variable x_{pi} for each pair $\{p, i\} \in N \times H$ as follows:

$$x_{pi} = \begin{cases} 1 & (p \in N \text{ is assigned to } i \in H), \\ 0 & (\text{otherwise}). \end{cases}$$

Since each non-hub is connected to exactly one hub, we have a constraint $\sum_{i \in H} x_{pi} = 1$ for each $p \in N$. Then, the star-star hub network design problem (star-metric labeling problem) can be formulated as follows:

$$\text{SHP:} \quad \text{min.} \quad \sum_{(p,q) \in N^2,\ p \neq q} w_{pq} \left(\sum_{i \in H} c_{pi} x_{pi} + \sum_{j \in H} c_{qj} x_{qj} + \sum_{k \in H} \ell_k |x_{pk} - x_{qk}| \right)$$

$$\text{s. t.} \quad \sum_{i \in H} x_{pi} = 1 \qquad\qquad (\forall p \in N),$$

$$x_{pi} \in \{0, 1\} \qquad\qquad (\forall \{p, i\} \in N \times H).$$

Next we describe a linear relaxation problem. By substituting non-negativity constraints of the variables x_{pi} ($\forall \{p, i\} \in N \times H$) for 0-1 constraints in SHP and replace $|x_{pk} - x_{qk}|$ with Z_{pqk}, we

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obtain the following a linear relaxation problem denoted by LRP.

$$\text{LRP:} \quad \text{min.} \quad \sum_{(p,q)\in N^2,\ p\neq q} w_{pq}\left(\sum_{i\in H} c_{pi}x_{pi} + \sum_{j\in H} c_{qj}x_{qj} + \sum_{k\in H} \ell_k Z_{pqk}\right)$$

$$\text{s. t.} \quad \sum_{i\in H} x_{pi} = 1 \qquad\qquad (\forall p \in N),$$

$$0 \le x_{pi} \qquad\qquad (\forall\{p,i\} \in N \times H),$$

$$- Z_{pqk} \le x_{pk} - x_{qk} \le Z_{pqk} \quad (\forall(p,q) \in N^2, \forall k \in H).$$

We can solve LRP in polynomial time by employing an interior point algorithm.

3 Algorithm

We now design an approximation algorithm. The approach is proceeded as follows:

Step 1. Choose $\lambda \in [0,1)$ uniformly at random and classify the hubs under $\kappa_{\max} + 1$ classes according to **Definition 1**.

Step 2. Solve the linear relaxation problem LRP and obtain an optimal solution x^*.

Step 3. Find a partition of non-hubs by **Algorithm 1**.

Step 4. Assign each non-hub to a hub by **Algorithm 2**.

Now, we describe our algorithm precisely. In Step 1, we classify the set of hubs according to the distance between each hub and the depot (see Figure 4). We assign each hub to a class. This classification is based on the following definition.

Definition 1. *For any $\lambda \in [0,1)$, we say that hub i belongs to class κ if and only if $\ell_i\ (\ge 1)$ satisfies the inequality $r^{\max\{(\kappa-2)+\lambda,0\}} \le \ell_i < r^{(\kappa-1)+\lambda}$ and hub i belongs to class 0 if and only if $\ell_i = 0$, where κ is a non-negative integer.*
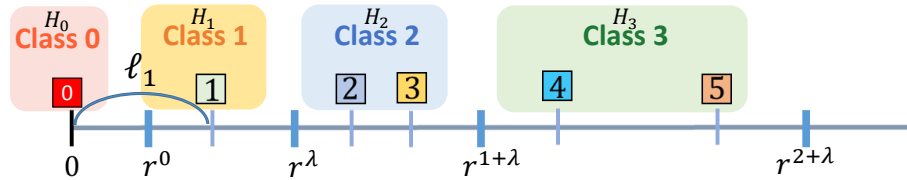

Figure 4: Classification of Hubs

Before we describe the details of later steps, we introduce some notations. Let $\alpha(\lambda, i)$ be the class of hub $i \in H$. We denote a subset of integers $\{0, 1, 2, ..., \kappa_{\max}\}$ by $[\kappa_{\max}]$ where $\kappa_{\max} =$

Algorithm 1: Classify each non-hub into a class

Require: An optimal solution $\boldsymbol{x}^*$ of LRP and a total order π of the hubs.

Ensure: A partition of non-hubs $N_0, N_1, \cdots, N_{\kappa_{\max}}$.

1: Set $N_i = \emptyset$ $(\forall i \in [0, 1, \ldots, \kappa_{\max}])$
2: Generate a random variable U which follows a uniform distribution defined on $[0, 1)$.
3: **for** $p \in N$ **do**
4:    Insert non-hub p into a subset $N_{\alpha(\lambda, \pi(i))}$, where $i \in \{1, 2, \ldots, |H|\}$ is the minimum number that satisfies $U < x_{p\pi(1)} + x_{p\pi(2)} + \cdots + x_{p\pi(i)}$.
5: **end for**
6: **return** $N_0, N_1, \cdots, N_{\kappa_{\max}}$.

$\max_{i \in H} \alpha(\lambda, i)$. Let H_κ be a subset of hubs that belongs to class $k \in [\kappa_{\max}]$ i.e. $H_\kappa = \{i \in H \mid \alpha(\lambda, i) = \kappa\}$. Let $\beta(p)$ be the class that non-hub $p \in N$ belongs to. We denote a subset of non-hubs that belong to class $\kappa \in [\kappa_{\max}]$ by N_κ i.e. $N_\kappa = \{p \in N \mid \beta(p) = \kappa\}$.

In Step 2, we solve the linear relaxation problem LRP formulated in the previous subsection. We use an optimal solution $\boldsymbol{x}^*$ in **Algorithm 1** and **Algorithm 2**.

In Step 3, we assign each non-hubs to a class of hubs defined by **Definition 3**. For example, if non-hub p_1 belongs to the subset N_3 obtained by **Algorithm 1**, p_1 will be assigned to one of hubs in H_3 defined by **Definition 3**. Given an optimal solution of LRP and a total order of the hubs, **Algorithm 1** outputs a partition of non-hubs, $N_0, N_1, \cdots, N_{\kappa_{\max}}$.

Here, a total order of hubs depends on labels of classes. For example, the total order of hubs in Figure 5 is $(5, 4, 1, 3, 2)$. The order of class labels π' is $(\kappa_{\max}, , \ldots, 4, 2, 0, 1, 3, \ldots, \kappa_{\max} - 1)$ when $\kappa_{\max}$ is an even number, and $(\kappa_{\max} - 1, \ldots, 4, 2, 0, 1, 3, \ldots, \kappa_{\max})$ when $\kappa_{\max}$ is an odd number. The order of class labels in Figure 5 is $(2, 0, 1, 3)$ for example. For each non-hub p, we place x_{pi} for $i \in H_\kappa$ in the order of class labels. Then, the total order of hubs π is defined as $\big($any order of hubs in $H_{\pi'(1)}$, any order of hubs in $H_{\pi'(2)}, \ldots$, any order of hubs in $H_{\pi'(\kappa_{\max}+1)}\big)$ in our rounding scheme, where $\pi'(i)$ denotes the i-th element of π'.

In Step 4, we decide an assignment from non-hubs to hubs using rounding technique. In **Algorithm 2**, we perform a rounding procedure for each subset of non-hubs. For a subset $N_\kappa \subseteq N$, we first choose hub $i \in H_\kappa$ and $U \in [0, 1)$ uniformly at random. Then, if $U \leq x_{pi}^*$, we assign non-hub p to hub i (see Figure 6). Until all the non-hubs are assigned to one of hubs, we continue this procedure. Note that in each phase we can set the upper bound of U to the maximum value of x_{pi} of remained non-hubs.

4 Analysis of Approximation Ratio

In this subsection, we show that our algorithm obtains a $\min\{\frac{r-1}{\log r}\left(2 + \frac{r^2+1}{r^2-1}\right) | r > 1\} \approx 5.2809$-approximate solution for any instance.

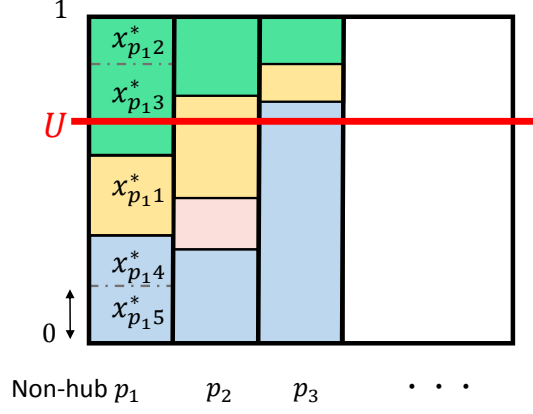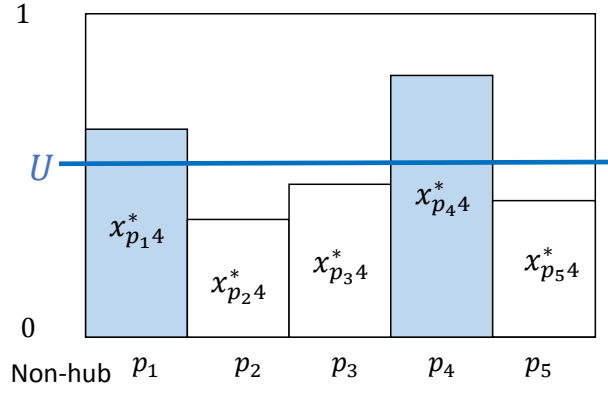Figure 5: Dependent rounding procedure to classify each non-hub into a class

Figure 6: Non-hub p_1 and p_4 are assigned to hub 4 by this phase, where $H_\kappa = \{4,5\}$ and $N_\kappa = \{p_1, p_2, p_3, p_4, p_5\}$

Algorithm 2: Assign each non-hub to a hub

Require: An optimal solution $\boldsymbol{x}^*$ of LRP and $\kappa_{\max} + 1$ subsets of non-hubs $N_0, \ldots, N_{\kappa_{\max}}$.
Ensure: An assignment from non-hubs to hubs X.
1: **for** $\kappa = 0, 1, \ldots, \kappa_{\max}$ **do**
2:     Initialize $S \leftarrow N_\kappa$
3:     **while** $|S| > 0$ **do**
4:         Choose hub $i \in H_\kappa$ uniformly at random.
5:         Choose $U \in [0, 1)$ uniformly at random.
6:         **for** $p \in S$ **do**
7:             **if** $U \le x_{pi}^*$ **then** $X_{pi} = 1, X_{pj} = 0$ $(\forall j \in H_\kappa \setminus \{i\})$
8:             $S \leftarrow S \setminus \{p\}$.
9:         **end for**
10:     **end while**
11: **end for**
12: **return** X.

Notation. We introduce some notations that we use throughout this subsection. Let $\alpha(\lambda, i)$ be the class of hub $i \in H$. For any $i \in H$, let define $u(\lambda, i) = r^{(\alpha(\lambda,i)-1)+\lambda}$ if $\ell_i \ge 1$, $u(\lambda, i) = 0$ if $\ell_i = 0$, where r is a real number satisfying $r > 1$, i.e.,

$$u(\lambda, i) = \begin{cases} r^{\alpha(\lambda,i)+\lambda-1} & (\ell_i \ge 1), \\ 0 & (\ell_i = 0). \end{cases}$$

Let define a cost $\hat{c}_{ij}$ for each pair $\{i, j\} \in H^2$ as follows:

$$\hat{c}_{ij} = \begin{cases} |u(\lambda, i) - u(\lambda, j)| & (\alpha(\lambda, i) = \alpha(\lambda, j) \pmod 2), \\ u(\lambda, i) + u(\lambda, j) & \text{(otherwise)}. \end{cases}$$

Remark. A metric defined by $\hat{C} = \hat{c}_{ij}$ becomes a line metric (see Figure 7). Thus the matrix $\hat{C}$ is a Monge matrix.
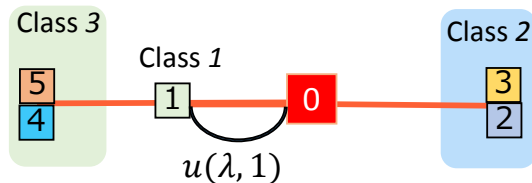

Figure 7: A metric defined by $\hat{C}$

Now we start with the following lemma.

Lemma 1. *Let $\boldsymbol{x}^*$ be an optimal solution of LRP. A vector of random variables X obtained by the proposed algorithm satisfies that $\Pr[X_{pi} = 1] = x_{pi}^*$ $(\forall p \in N, \forall i \in H)$.*

Proof.

$$\Pr[X_{pi} = 1] = \Pr[p \in N \text{ is classified into } N_{\alpha(\lambda,i)}]\Pr[p \in N \text{ is assigned to } i(\in H)]$$

$$= \left(\sum_{j:\alpha(\lambda,j)=\alpha(\lambda,i)} x_{pj}^* \right) \frac{x_{pi}^*/|H^{\alpha(\lambda,i)}|}{\sum_{j:\alpha(\lambda,j)=\alpha(\lambda,i)} x_{pj}^*/|H^{\alpha(\lambda,i)}|} = x_{pi}^*.$$

$\square$

Lemma 2. *For any pair of hubs $\{i,j\} \in H_\kappa \times H_{\kappa'}$, any real number $r > 1$, and any real number $\lambda \in [0,1)$, we have the inequality $u(\lambda,i) + u(\lambda,j) \leq \frac{r^2+1}{r^2-1}\hat{c}_{ij}$, where $\kappa, \kappa' \in [\kappa_{\max}]$ and $\kappa \neq \kappa'$.*

Proof. (Case i) $\kappa - \kappa' \equiv 0 \pmod 2$

In this case, it is obvious that

$$u(\lambda,i) + u(\lambda,j)$$

$$= \frac{u(\lambda,i) + u(\lambda,j)}{\max\{u(\lambda,i), u(\lambda,j)\} - \min\{u(\lambda,i), u(\lambda,j)\}} \max\{u(\lambda,i), u(\lambda,j)\} - \min\{u(\lambda,i), u(\lambda,j)\}$$

$$= \frac{\max\{u(\lambda,i), u(\lambda,j)\} + \min\{u(\lambda,i), u(\lambda,j)\}}{\max\{u(\lambda,i), u(\lambda,j)\} - \min\{u(\lambda,i), u(\lambda,j)\}} \hat{c}_{ij}$$

$$= \frac{r^2 \max\{u(\lambda,i), u(\lambda,j)\} + r^2 \min\{u(\lambda,i), u(\lambda,j)\}}{r^2 \max\{u(\lambda,i), u(\lambda,j)\} - r^2 \min\{u(\lambda,i), u(\lambda,j)\}} \hat{c}_{ij}.$$

Recall that $k \neq \kappa'$ and $k - \kappa' \equiv 0 \pmod 2$, and thus it holds that $r^2 \min\{u(\lambda,i), u(\lambda,j)\} \leq \max\{u(\lambda,i), u(\lambda,j)\}$ for any pair of hubs $\{i,j\} \in H_\kappa \times H_{\kappa'}$. Then we get

$$\frac{r^2 \max\{u(\lambda,i), u(\lambda,j)\} + r^2 \min\{u(\lambda,i), u(\lambda,j)\}}{r^2 \max\{u(\lambda,i), u(\lambda,j)\} - r^2 \min\{u(\lambda,i), u(\lambda,j)\}} \hat{c}_{ij}$$

$$\leq \frac{r^2 \max\{u(\lambda,i), u(\lambda,j)\} + \max\{u(\lambda,i), u(\lambda,j)\}}{r^2 \max\{u(\lambda,i), u(\lambda,j)\} - \max\{u(\lambda,i), u(\lambda,j)\}} \hat{c}_{ij} = \frac{r^2+1}{r^2-1}\hat{c}_{ij}.$$

(Case ii) $\kappa - \kappa' \equiv 1 \pmod 2$

From the definition, we have that

$$u(\lambda,i) + u(\lambda,j) = \hat{c}_{ij} \leq \frac{r^2+1}{r^2-1}\hat{c}_{ij}.$$

$\square$

Lemma 3. *Let X be a vector of random variables obtained by the proposed algorithm and let $\boldsymbol{x}^*$ be an optimal solution of LRP. For any pair of non-hubs $(p,q) \in N^2$ and any real number $\lambda \in [0,1)$,*

we have the following inequality

$$\mathrm{E}\left[\sum_{\kappa\in[\kappa_{\max}]}\sum_{(i,j)\in H_\kappa^2:i\neq j}(\ell_i+\ell_j)X_{pi}X_{qj}\right]\leq 2\sum_{i\in H}u(\lambda,i)|x_{pi}^*-x_{qi}^*|.$$

Proof. First, for any integer $\kappa\in[\kappa_{\max}]$, we show that

$$\mathrm{E}\left[\sum_{(i,j)\in H_\kappa^2:i\neq j}(\ell_i+\ell_j)X_{pi}X_{qj}\right]\leq 2\sum_{i\in H_\kappa}u(\lambda,i)|x_{pi}^*-x_{qi}^*|. \tag{4.1}$$

(Case i) $\kappa=0$

We can see that $\mathrm{E}\left[\sum_{(i,j)\in H_0^2:i\neq j}(\ell_i+\ell_j)X_{pi}X_{qj}\right]=0$ $(\because \forall(i,j)\in H_0^2,\ell_i=\ell_j=0)$ and $\sum_{i\in H_\kappa}u(\lambda,i)|x_{pi}^*-x_{qi}^*|=0$ $(\because \forall i\in H^0,u(\lambda,i)=0)$. Then we obtain the inequality (4.1) for this case.

(Case ii) $\kappa\in\{1,2,\ldots,\kappa_{\max}\}$

In this case, it is easy to see that

$$\begin{aligned}\mathrm{E}\left[\sum_{(i,j)\in H_\kappa^2:i\neq j}(\ell_i+\ell_j)X_{pi}X_{qj}\right]&=\sum_{(i,j)\in H_\kappa^2:i\neq j}\left((\ell_i+\ell_j)\Pr[X_{pi}=X_{qj}=1]\right)\\ &\leq\sum_{(i,j)\in H_\kappa^2:i\neq j}2r^{\kappa+\lambda-1}\Pr[X_{pi}=X_{qj}=1]\ (\because \forall i\in H_\kappa,\ell_i\leq 2r^{\kappa+\lambda-1})\\ &=2r^{\kappa+\lambda-1}\sum_{(i,j)\in H_\kappa^2:i\neq j}\Pr[X_{pi}=X_{qj}=1]. \end{aligned} \tag{4.2}$$

We say that non-hub p and non-hub q are **separated** by a single phase in **Algorithm 2** if both p and q are unassigned before the phase and exactly one of p and q is assigned in this phase (See Figure 8). Note that even if p and q are separated by some phase, they may be assigned to a mutual hub later.
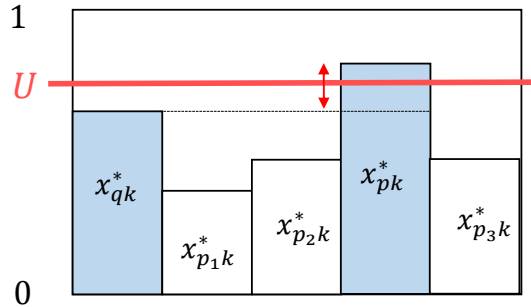

Figure 8: Non-hub p and non-hub q are separated in this phase.

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The probability $\sum_{(i,j)\in H_\kappa^2 : i\neq j} \Pr[X_{pi} = X_{qj} = 1]$ in the right-hand side of inequality (4.2) is the probability that both non-hub p and q are classified into N_κ by **Algorithm 1** and non-hub p and q are assigned to different hubs by **Algorithm 2**. This probability can be bounded by the probability that both non-hub p and q are classified into N_κ by **Algorithm 1** and non-hub p or q are separated by some phase in **Algorithm 2**. Then for any $\kappa \in \{1, 2, \ldots, \kappa_{\max}\}$, we have that

$$
\sum_{(i,j)\in H_\kappa, i\neq j} \Pr[X_{pi} = X_{qj} = 1] = \sum_{(i,j)\in H_\kappa^2 : i\neq j} \Pr[\beta(p) = \beta(q) = \kappa]\Pr[X_{pi} = X_{qj} = 1 \mid \beta(p) = \beta(q) = \kappa]
$$

$$
\leq \Pr[\beta(p) = \beta(q) = \kappa] \sum_{k\in H_\kappa} \frac{|x_{pk}^* - x_{qk}^*|/|H_\kappa|}{\max\{x_{pk}^*, x_{qk}^*\}/|H_\kappa|}.
$$

Thus, we obtain that

$$
2r^{\kappa+\lambda-1} \sum_{(i,j)\in H_\kappa^2 : i\neq j} \Pr[X_{pi} = X_{qj} = 1]
$$

$$
\leq 2r^{\kappa+\lambda-1}\Pr[\beta(p) = \beta(q) = \kappa]\frac{\sum_{i\in H_\kappa} |x_{pi}^* - x_{qi}^*|/|H_\kappa|}{\sum_{i\in H_\kappa} \max\{x_{pi}^*, x_{qi}^*\}/|H_\kappa|}
$$

$$
\leq 2r^{\kappa+\lambda-1}\Pr[\beta(p) = k]\frac{\sum_{i\in H_\kappa} |x_{pi}^* - x_{qi}^*|}{\sum_{i\in H_\kappa} x_{pi}^*}
$$

$$
= 2r^{\kappa+\lambda-1} \sum_{i\in H_\kappa} |x_{pi}^* - x_{qi}^*| \ (\because \Pr[\beta(p) = \kappa] = \sum_{i\in H_\kappa} x_{pi}^*)
$$

$$
= 2 \sum_{i\in H_\kappa} r^{\kappa+\lambda-1}|x_{pi}^* - x_{qi}^*| = 2 \sum_{i\in H_\kappa} u(\lambda, i)|x_{pi}^* - x_{qi}^*|.
$$

Then we have inequality (4.1) for this case. From inequality (4.1), we have the desired result:

$$
\mathrm{E}\left[\sum_{\kappa\in[\kappa_{\max}]} \sum_{(i,j)\in H_\kappa^2 : i\neq j} (\ell_i + \ell_j)X_{pi}X_{qj} \right] \leq 2\sum_{i\in H} u(\lambda, i)|x_{pi}^* - x_{qi}^*|.
$$

$\square$

Next, to show Lemma 6, we first describe Lemma 4 and Theorem 1. Lemma 4 implies that the probability that non-hub p is classified into N_κ and non-hub q is classified into $N_{\kappa'}$ by **Algorithm 1** is bounded by $\sum_{i\in H_\kappa} \sum_{j\in H_{\kappa'}} y_{piqj}^{NW}$ where $\boldsymbol{y}^{NW}$ is a *north-west corner rule* solution of the subproblem that is equivalent to a *Hitchcock transportation problem (HTP)*. The detail is omitted here (see Appendix).

Lemma 4. *Let $\boldsymbol{X}$ be a vector of random variables obtained by the proposed algorithm, let $(\boldsymbol{x}, \boldsymbol{y})$ be a feasible solution of LRP and let $\boldsymbol{y}^{NW}$ be a solution of HTP (defined in Appendix). obtained by north-west corner rule. For any pair of $\{\kappa, \kappa'\} \in [\kappa_{\max}], \kappa \neq \kappa'$ and any pair of $(p, q) \in N^2 \ (p \neq q)$,*

Algorithm 3: Construct $\boldsymbol{y}^*$ from $\boldsymbol{x}^*$

Require: An optimal solution $\boldsymbol{x}^*$ of LRP.

Ensure: Vectors $\boldsymbol{y}^*$

  1: **for** $(p,q) \in N^2, p \neq q$ **do**

  2:    Initialize $y^*_{piqj} = 0$ $(\forall(i,j) \in H^2)$

  3:    Set y^*_{piqj} to $\min\{x^*_{pi}, x^*_{qi}\}$ $(\forall i \in H)$

  4:    **for** $i = 1, 2, 3, \ldots, h$ **do**

  5:      $j \leftarrow 1$

  6:      **while** $\sum_{k \in H} y^*_{piqk} < x^*_{pi}$ **do**

  7:        Set y^*_{piqj} to $\min\{x^*_{qj} - \sum_{k \in H} y^*_{pkqj}, x^*_{pi} - \sum_{k \in H} y^*_{piqk}\}$

  8:        $j \leftarrow j + 1$

  9:      **end while**

10:    **end for**

11: **end for**

12: **return** $\boldsymbol{y}^*$

---

we have the following inequality:

$$\sum_{i \in H_\kappa} \sum_{j \in H_{\kappa'}} \mathrm{E}[X_{pi}X_{qj}] \leq \sum_{i \in H_\kappa} \sum_{j \in H_{\kappa'}} y^{NW}_{piqj}.$$

The proof is omitted here (see Appendix).

Next we describe well-known relation between a north-west corner rule solution of a Hitchcock transportation problem and the Monge property.

Theorem 1. *If a given cost matrix $C = (c_{ij})$ is a Monge matrix, then the north-west corner rule solution $\boldsymbol{y}^{NW}$ gives an optimal solution of all the Hitchcock transportation problems.*

Proof is omitted here (see for example [6, 9]).

Next we consider that we construct a vector $\boldsymbol{y}^*$ from the optimal solution $\boldsymbol{x}^*$ by **Algorithm 3**. A vector $\boldsymbol{y}^*$ is optimal to our subproblem HTP. Note that we need **Algorithm 3** only for approximation analysis and we don't use it to obtain an approximate solution. Then we have the following lemma.

Lemma 5. *Let $\boldsymbol{x}^*$ be an optimal solution of LRP and let $\boldsymbol{y}^*$ be a vector obtained by **Algorithm 3**. For any pair of $(p,q) \in N^2$ $(p \neq q)$, we have the following inequality:*

$$\sum_{(i,j) \in H^2: i \neq j} (\ell_i + \ell_j) y^*_{piqj} = \sum_{i \in H} \ell_i |x^*_{pi} - x^*_{qi}|.$$

Proof is omitted here (see Appendix).

Now we are ready to prove the following lemma.

Lemma 6. *Let $\boldsymbol{X}$ be a vector of random variables obtained by the proposed algorithm. Let $\boldsymbol{x}^*$ be an optimal solution of LRP, and let $\boldsymbol{y}^*$ be vectors obtained from the optimal solution $\boldsymbol{x}^*$ of LRP by* **Algorithm 3.** *For any distinct pair of non-hubs $(p, q) \in N^2$ $(p \neq q)$, any real number $r > 1$, and any real number $\lambda \in [0, 1)$, we have the following inequality :*

$$\mathrm{E}\left[\sum_{\{\kappa,\kappa'\}\in[\kappa_{\max}]:\kappa\neq\kappa'}\sum_{i\in H_\kappa}\sum_{j\in H_{\kappa'}}(\ell_i + \ell_j)X_{pi}X_{qj}\right] \leq \frac{r^2+1}{r^2-1}\sum_{i\in H}u(\lambda,i)|x^*_{pi} - x^*_{qi}|.$$

Proof. First, we prove the following inequality for any pair of integers $\{\kappa, \kappa'\} \in [\kappa_{\max}]$ $(\kappa \neq \kappa')$ and any pair of non-hubs $(p, q) \in N^2$ $(p \neq q)$:

$$\mathrm{E}\left[\sum_{i\in H_\kappa}\sum_{j\in H_{\kappa'}}(\ell_i + \ell_j)X_{pi}X_{qj}\right] \leq \frac{r^2+1}{r^2-1}\sum_{i\in H_\kappa}\sum_{j\in H_{\kappa'}}(u(\lambda,i) + u(\lambda,j))y^*_{piqj}. \tag{5.1}$$

(Case i) $\kappa, \kappa' \in \{1, 2, \ldots, \kappa_{\max}\}$ $(\kappa \neq \kappa')$

In this case, we have the following inequalities from the definition of $u(\lambda, i)$ $(i \in H)$.

$$\mathrm{E}\left[\sum_{i\in H_\kappa}\sum_{j\in H_{\kappa'}}(\ell_i + \ell_j)X_{pi}X_{qj}\right]$$
$$\leq \sum_{i\in H_\kappa}\sum_{j\in H_{\kappa'}}(u(\lambda,i) + u(\lambda,j))\mathrm{E}[X_{pi}X_{qj}]$$
$$= \sum_{i\in H_\kappa}\sum_{j\in H_{\kappa'}}(r^{\kappa+\lambda-1} + r^{\kappa'+\lambda-1})\mathrm{E}[X_{pi}X_{qj}]$$
$$= (r^{\kappa+\lambda-1} + r^{\kappa'+\lambda-1})\sum_{i\in H_\kappa}\sum_{j\in H_{\kappa'}}\mathrm{E}[X_{pi}X_{qj}].$$

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Using Lemma 4, Lemma 2, and Theorem 1, we have the following inequalities.

$$(r^{\kappa+\lambda-1} + r^{\kappa'+\lambda-1}) \sum_{i\in H_\kappa} \sum_{j\in H_{\kappa'}} \mathrm{E}[X_{pi}X_{qj}] \tag{1}$$

$$\leq (r^{\kappa+\lambda-1} + r^{\kappa'+\lambda-1}) \sum_{i\in H_\kappa} \sum_{j\in H_{\kappa'}} y^{NW}_{piqj} \quad (\because \text{Lemma 4})$$

$$= \sum_{i\in H_\kappa} \sum_{j\in H_{\kappa'}} (u(\lambda,i) + u(\lambda,j))y^{NW}_{piqj}$$

$$\leq \frac{r^2+1}{r^2-1} \sum_{i\in H_\kappa} \sum_{j\in H_{\kappa'}} \hat{c}_{ij} y^{NW}_{piqj} \quad (\because \text{Lemma 2})$$

$$\leq \frac{r^2+1}{r^2-1} \sum_{i\in H_\kappa} \sum_{j\in H_{\kappa'}} \hat{c}_{ij} y^{*}_{piqj} \quad (\because \hat{C} \text{ is a Monge matrix and Theorem 1.})$$

$$\leq \frac{r^2+1}{r^2-1} \sum_{i\in H_\kappa} \sum_{j\in H_{\kappa'}} (u(\lambda,i) + u(\lambda,j))y^{*}_{piqj}$$

Then we obtained inequality (5.1) for this case.

(Case ii) $\kappa = 0$ or $\kappa' = 0$

We can show inequality (5.1) for this case by substituting $r^{\kappa+\lambda-1} + r^{\kappa'+\lambda-1}$ in (Case i) by either $r^{\kappa+\lambda-1}$ or $r^{\kappa'+\lambda-1}$.

Then we obtain that

$$\mathrm{E}\left[\sum_{\{\kappa,\kappa'\}\in[\kappa_{\max}]:\kappa\neq\kappa'} \sum_{i\in H_\kappa} \sum_{j\in H_{\kappa'}} (\ell_i + \ell_j)X_{pi}X_{qj} \right]$$

$$\leq \frac{r^2+1}{r^2-1} \sum_{\{\kappa,\kappa'\}\in[\kappa_{\max}]:\kappa\neq\kappa'} \sum_{i\in H_\kappa} \sum_{j\in H_{\kappa'}} (u(\lambda,i) + u(\lambda,j))y^{*}_{piqj} \quad (\because \text{inequality (5.1)})$$

$$\leq \frac{r^2+1}{r^2-1} \sum_{(i,j)\in H^2:i\neq j} (u(\lambda,i) + u(\lambda,j))y^{*}_{piqj}.$$

$$= \frac{r^2+1}{r^2-1} \sum_{i\in H} u(\lambda,i)|x^{*}_{pi} - x^{*}_{qj}|. \quad (\because \text{Lemma 5 for } u(\lambda,i) \text{ instead of } \ell_i)$$

$\square$

Now, we are ready to show our main theorem.

Theorem 2. *The proposed algorithm is* $\min\{\frac{r-1}{\log r}\left(2 + \frac{r^2+1}{r^2-1}\right) | r > 1\}(\approx 5.2809 \text{ at } r \approx 1.91065)-$
approximation algorithm for star-star hub-and-spoke network design problems and star-metric labeling problems.

Proof. Let $\boldsymbol{X}$ be a vector of random variables obtained by the proposed algorithm and let $(\boldsymbol{x}^*, \boldsymbol{y}^*)$

be an optimal solution of LRP. For any real number $\lambda \in [0, 1)$, we have that

$$
\mathrm{E}[Z] = \mathrm{E}\left[\sum_{(p,q)\in N^2:p\neq q} w_{pq} \left(\sum_{i\in H} c_{pi}X_{pi} + \sum_{j\in H} c_{qj}X_{qj} + \sum_{(i,j)\in H^2:i\neq j} (\ell_i + \ell_j)X_{pi}X_{qj} \right) \right]
$$

$$
= \sum_{(p,q)\in N^2:p\neq q} w_{pq} \left(\sum_{i\in H} c_{pi}x^*_{pi} + \sum_{j\in H} c_{qj}x^*_{qj} + \mathrm{E}\left[\sum_{\kappa\in[\kappa_{\max}]} \sum_{(i,j)\in H^2_\kappa:i\neq j} (\ell_i + \ell_j)X_{pi}X_{qj} \right] \right.
$$

$$
\left. + \mathrm{E}\left[\sum_{\{\kappa,\kappa'\}\in[\kappa_{\max}]:\kappa\neq\kappa'} \sum_{i\in H_\kappa} \sum_{j\in H_{\kappa'}} (\ell_i + \ell_j)X_{pi}X_{qj} \right] \right) \quad (\because \text{Lemma 1})
$$

$$
\leq \sum_{(p,q)\in N^2:p\neq q} w_{pq} \left(\sum_{i\in h} c_{pi}x^*_{pi} + \sum_{j\in H} c_{qj}x^*_{qj} + 2\sum_{i\in H} u(\lambda,i)|x^*_{pi} - x^*_{qi}| \right.
$$

$$
\left. + \frac{r^2+1}{r^2-1} \sum_{k\in H} u(\lambda,k)|x^*_{pk} - x^*_{qk}| \right) \quad (\because \text{Lemma 3 and Lemma 6})
$$

$$
= \sum_{(p,q)\in N^2:p\neq q} w_{pq} \left(\sum_{i\in H} c_{pi}x^*_{pi} + \sum_{j\in H} c_{qj}x^*_{qj} + \left(2 + \frac{r^2+1}{r^2-1}\right) \sum_{k\in H} u(\lambda,k)|x^*_{pk} - x^*_{qk}| \right) \quad (6.1)
$$

where Z denotes the objective value of a solution obtained by the proposed algorithm. Let $\Lambda \in [0,1)$ be a uniform random variable. The expected value of $u(\Lambda, k)$ for all $k \in H$ and for all $r > 1$ is $\mathrm{E}[u(\Lambda,k)] = \int_0^1 r^\Lambda \ell_k \, d\Lambda = \frac{r-1}{\log r}\ell_k$.

Thus, from the above discussion and inequality (6.1) which holds for any $\Lambda \in [0,1)$, we have that

$$
\mathrm{E}[Z] \leq \sum_{(p,q)\in N^2:p\neq q} w_{pq} \left(\sum_{i\in h} c_{pi}x^*_{pi} + \sum_{j\in H} c_{qj}x^*_{qj} + \left(2 + \frac{r^2+1}{r^2-1}\right) \sum_{k\in H} \mathrm{E}[u(\Lambda,k)]|x^*_{pk} - x^*_{qk}|. \right)
$$

$$
= \sum_{(p,q)\in N^2:p\neq q} w_{pq} \left(\sum_{i\in h} c_{pi}x^*_{pi} + \sum_{j\in H} c_{qj}x^*_{qj} + \frac{r-1}{\log r}\left(2 + \frac{r^2+1}{r^2-1}\right) \sum_{k\in H} \ell_k|x^*_{pk} - x^*_{qk}|. \right)
$$

$$
= \min\left\{ \frac{r-1}{\log r}\left(2 + \frac{r^2+1}{r^2-1}\right) \,\middle|\, r > 1 \right\} \text{(optimal value of LRP)}
$$

$$
\leq \min\left\{ \frac{r-1}{\log r}\left(2 + \frac{r^2+1}{r^2-1}\right) \,\middle|\, r > 1 \right\} \text{(optimal value of the original problem SHP).}
$$

Note that when $r > 1$, $f(r) = \frac{r-1}{\log r}\left(2 + \frac{r^2+1}{r^2-1}\right)$ is minimized at $r^* \approx 1.91065$ and we get $f(r^*) \approx 5.2809$. Then we obtain the desired result. $\qquad\square$

5 Conclusion

In this paper, we have studied hub-and-spoke network design problems, motivated by the application to achieve efficient transportation systems. we considered the case where the set of hubs forms a star, and introduced a star-star hub-and-spoke network design problem and star-metric labeling problem. The star-metric labeling problem includes the uniform labeling problem which is still NP-hard. We proposed $\min\left\{\frac{r-1}{\log r}\left(2+\frac{r^2+1}{r^2-1}\right) \mid r>1\right\}$ (≈ 5.2809 at $r \approx 1.91065$)–approximation algorithm for star-star hub-and-spoke network design problems and star-metric labeling problems. Our algorithms solve a linear relaxation problem and apply dependent rounding procedures.

Appendix

Hitchcock Transportation Problems and North-West Corner Rule

A Hitchcock transportation problem is defined on a complete bipartite graph consists of a set of supply points $A = \{1, 2, \ldots, I\}$ and a set of demand points $B = \{1, 2, \ldots, J\}$. Given a pair of non-negative vectors $(a, b) \in \mathbb{R}^I \times \mathbb{R}^J$ satisfying $\sum_{i=1}^{I} a_i = \sum_{j=1}^{J} b_j$ and an $I \times J$ cost matrix $C = (c_{ij})$, a Hitchcock transportation problem is formulated as follows:

$$
\begin{aligned}
\mathrm{HTP}(a, b, C): \quad \min. \quad & \sum_{i=1}^{I}\sum_{j=1}^{J} c_{ij} y_{ij} \\
\text{s. t.} \quad & \sum_{j=1}^{J} y_{ij} = a_i \quad (i \in \{1, 2, \ldots, I\}), \\
& \sum_{i=1}^{I} y_{ij} = b_j \quad (j \in \{1, 2, \ldots, J\}), \\
& y_{ij} \geq 0 \quad (\forall(i,j) \in \{1, 2, \ldots, I\} \times \{1, 2, \ldots, J\}),
\end{aligned}
$$

where y_{ij} denotes the amount of flow from a supply point $i \in A$ to a demand point $j \in B$.

We describe north-west corner rule in Algorithm NWCR, which finds a feasible solution of Hitchcock transportation problem $\mathrm{HTP}(a, b, C)$. It is easy to see that the north-west corner rule solution $Y = (y_{ij})$ satisfies the equalities that

$$
\sum_{i=1}^{i'}\sum_{j=1}^{j'} y_{ij} = \min\left\{\sum_{i=1}^{i'} a_i, \ \sum_{j=1}^{j'} b_j\right\} \quad (\forall(i',j') \in \{1, 2, \ldots, I\} \times \{1, 2, \ldots, J\}).
$$

Since the coefficient matrix of the above equality system is nonsingular, the north-west corner rule solution is a unique solution of the above equality system. Thus, the above system of equalities has a unique solution which is feasible to $\mathrm{HTP}(a, b, C)$.

1 Algorithm NWCR

Step 1: Set all the elements of matrix Y to 0 and set the target element y_{ij} to y_{11} (top-left corner).

Step 2: Allocate a maximum possible amount of transshipment to the target element without making the row or column total of the matrix Y exceed the supply or demand respectively.

Step 3: If the target element is y_{IJ} (the south-east corner element), then stop.

Step 4: Denote the target element by y_{ij}. If the sum total of jth column of Y is equal to b_j, set the target element to y_{ij+1}. Else (the sum total of Y of ith row is equal to a_i), set the target element to y_{i+1j}. Go to Step 2.

Next we show that the subproblem of our original problem can be written as a Hitchcock transportation problem. Let (x, y) be a feasible solution of linear relaxation problem. For any $p \in N$, x_p denotes a subvector of x defined by $(x_{p1}, x_{p2}, \ldots, x_{ph})$. When we fix variables x in LRP to x and given a pair of $(p, q) \in N^2$ $(p \neq q)$, we can decompose the obtained problem into Hitchcock transportation problems $\{\text{HTP}(x_p, x_q, \hat{C}(= \hat{c}_{ij})) : (p, q) \in N^2\}$ where

$$\text{HTP}(x_p, x_q, \hat{C}): \quad \min. \quad \sum_{i \in H} \sum_{j \in H} \hat{c}_{ij} y_{piqj}$$

$$\text{s. t.} \quad \sum_{j \in H} y_{piqj} = x_{pi} \quad (\forall i \in H),$$

$$\sum_{i \in H} y_{piqj} = x_{qj} \quad (\forall j \in H),$$

$$y_{piqj} \geq 0 \quad (\forall (i,j) \in H^2).$$

Monge Property

We give the definition of a Monge matrix. A comprehensive research on the Monge property appears in a recent survey [9]. Matrices with this property arise quite often in practical applications, especially in geometric settings.

Definition 2. *An $m \times n$ matrix C is a Monge matrix if and only if C satisfies the so-called Monge property*

$$c_{ij} + c_{i'j'} \leq c_{ij'} + c_{i'j} \qquad \text{for all} \quad 1 \leq i < i' \leq m, 1 \leq j < j' \leq n.$$

Note that the north-west corner rule produces an optimal solution of Hitchcock transportation problems if the cost matrix is a Monge matrix, so we can obtain an optimal solution of $\text{HTP}(x_p, x_q, \hat{C})$ by north-west corner rule [6].

Proof of Lemma 4

Let X be a vector of random variables obtained by the proposed algorithm, and let $(\boldsymbol{x}, \boldsymbol{y})$ be a feasible solution of LRP. For any pair of $\{\kappa, \kappa'\} \in [\kappa_{\max}]$ $(\kappa \neq \kappa')$ and any pair of $(p, q) \in N^2$ $(p \neq q)$, then we have

$$
\begin{aligned}
\sum_{i \in H_\kappa} \sum_{k \in H_{\kappa'}} \mathrm{E}[X_{pi} X_{qj}] &\leq \left(\sum_{i \in H_\kappa} x_{pi}\right)\left(\sum_{j \in H_{\kappa'}} x_{qj}\right) \\
&= \max\{\sum_{i \in H_\kappa} x_{pi}, \sum_{j \in H_{\kappa'}} x_{pj}\} \min\{\sum_{i \in H_\kappa} x_{pi}, \sum_{j \in H_{\kappa'}} x_{pj}\} \\
&\leq \min\{\sum_{i \in H_\kappa} x_{pi}, \sum_{j \in H_{\kappa'}} x_{pj}\} (\because \sum_{i \in H} x_{pi} = 1).
\end{aligned}
\tag{6.21}
$$

For any pair of $\{\kappa, \kappa'\} \in [\kappa_{\max}]$ $(\kappa \neq \kappa')$ and any pair of $(p, q) \in N^2$ $(p \neq q)$, we have the following Hitchcock transportation problems :

$$
\begin{aligned}
\mathrm{HTP}(x_p, x_q, \hat{C}): \quad &\text{min.} \quad \sum_{i \in H_\kappa} \sum_{j \in H_{\kappa'}} \hat{c}_{ij} y_{piqj} \\
&\text{s. t.} \quad \sum_{j \in H_{\kappa'}} y_{piqj} = x_{pi} \quad (\forall i \in H_\kappa), \\
&\qquad\quad \sum_{i \in H_\kappa} y_{piqj} = x_{qj} \quad (\forall j \in H_{\kappa'}), \\
&\qquad\quad y_{piqj} \geq 0 \qquad\qquad (\forall (i,j) \in H_\kappa \times H_{\kappa'}).
\end{aligned}
$$

We see that the north-west corner rule solution $\boldsymbol{y}^{NW} = (y_{piqj}^{NW})$ satisfies the equalities that

$$
\sum_{i \in H_\kappa} \sum_{j \in H_{\kappa'}} y_{piqj}^{NW} = \min\left\{\sum_{i \in H_\kappa} x_{pi}, \sum_{j \in H_{\kappa'}} x_{qj}\right\} \quad (\forall \{\kappa, \kappa'\} \in [\kappa_{\max}], \kappa \neq \kappa').
$$

From the equalities and inequality (6.21), we have

$$
\sum_{i \in H_\kappa} \sum_{k \in H_{\kappa'}} \mathrm{E}[X_{pi} X_{qj}] \leq \sum_{i \in H_\kappa} \sum_{j \in H_{\kappa'}} y_{piqj}^{NW}.
$$

Thus, we have the desired result.

Proof of Lemma 5

Let y^* be the vector obtained from an optimal solution of LRP x^* by **Algorithm 3**. Given any distinct pair of non-hubs $(p, q) \in N^2$ $(p \neq q)$, we can see that

$$\sum_{j \in H} y^*_{piqj} = x^*_{pi} \ (\forall i \in H),$$

$$\sum_{i \in H} y^*_{piqj} = x^*_{qj} \ (\forall j \in H),$$

$$\sum_{j \in H: j \neq i} y_{piqj} = \min\{0, x_{pi} - x_{qi}\} \ (\forall i \in H),$$

$$\sum_{i \in H: i \neq j} y_{piqj} = \min\{0, x_{qj} - x_{pj}\} \ (\forall j \in H).$$

Thus we have

$$\sum_{(i,j) \in H^2: i \neq j} (\ell_i + \ell_j) y^*_{piqj} = \sum_{(i,j) \in H^2: i \neq j} \ell_i y^*_{piqj} + \sum_{(i,j) \in H^2: i \neq j} \ell_j y^*_{piqj}$$

$$= \sum_{i \in H} \ell_i \sum_{j \in H: j \neq i} y^*_{piqj} + \sum_{j \in H} \ell_j \sum_{i \in H: j \neq i} y^*_{piqj}$$

$$= \sum_{i \in H} \ell_i \min\{0, x^*_{pi} - x^*_{qi}\} + \sum_{j \in H} \ell_j \min\{0, x^*_{qj} - x^*_{pj}\}$$

$$= \sum_{i \in H} \ell_i \min\{0, x^*_{pi} - x^*_{qi}\} + \sum_{i \in H} \ell_i \min\{0, x^*_{qi} - x^*_{pi}\}$$

$$= \sum_{i \in H} \ell_i |x^*_{pi} - x^*_{qi}|.$$

Then we have the desired result.

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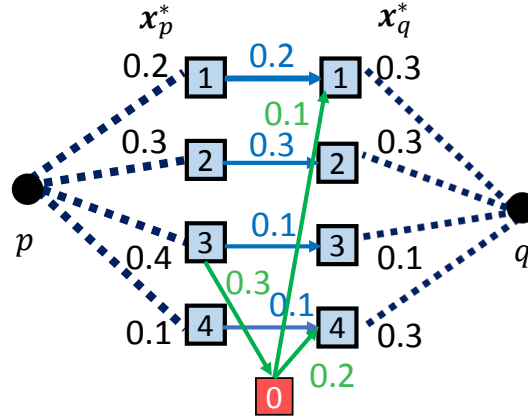
Figure 9: Hitchcock transportation problems with a depot 0

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