Multi-Commodity Multi-Facility Network Design

Alper Atamtürk and Oktay Günlük

Abstract We consider multi-commodity network design models, where capacity can be added to the arcs of the network using multiples of facilities that may have different capacities. This class of mixed-integer optimization models appears frequently in telecommunication network capacity expansion problems, train scheduling with multiple locomotive options, supply chain and service network design problems. Valid inequalities used as cutting planes in branch-and-bound algorithms have been instrumental in solving large-scale instances. We review the progress that has been done in polyhedral investigations in this area by emphasizing three fundamental techniques. These are the metric inequalities for projecting out continuous flow variables, mixed-integer rounding from appropriate base relaxations, and shrinking the network to a small k-node graph. The basic inequalities derived from arc-set, cut-set and partition relaxations of the network are also extensively utilized with certain modifications in robust and survivable network design problems.

1 Introduction

Here we consider multi-commodity network design models, where capacity can be added to arcs of the network using integer multiples of facilities, possibly with varying capacities. This class of models appear frequently in telecommunication network capacity expansion problems train scheduling with multiple locomotive options, supply chain and service network design problems. In the single-facility network design problem, one installs multiples of only a single type of facility on the

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arcs of the network. Routing vehicles with identical capacity in a logistics network and installing a communication network with only one cable type are examples of the single-facility network design problem. In the multi-facility problem, one may install different types of facilities with varying capacities, such as fiberoptic cables with varying bandwidths, production lines or machines with different rates, or a fleet of heterogeneous vehicles with varying capacities. The optimization problem seeks to decide how many facilities of each type to install on the network so as to meet the demand for each commodity at the least cost. We present the precise problem description and the associated formulation in the next section.

Different versions of the problem are obtained depending on how the flow is routed in the network. In the *unsplittable* flow version, only a single path is allowed to route the flow from its source to its destination, which requires integer variables to model its route. This is the case, for instance, in telecommunication networks using multiprotocol label switching (MPLS) technique, production and distribution with single sourcing, and express package delivery. The *splittable* case, which assumes that flow can be routed using multiple directed paths, is obviously a relaxation of the unsplittable case; therefore, valid inequalities for the splittable case are also valid for the unsplittable case.

In addition, the capacity created by the facilities can be *directed*, *bidirected*, or *undirected*. In the bidirected case, if a certain facility is installed on an arc, then the same facility also needs to be installed on the reverse arc. In the undirected case, the total flow on an arc and its reverse arc is limited by the capacity of the undirected edge associated with the two arcs. Here we consider the directed case where the total flow on an arc is limited by the total (directed) capacity of the arc. In a recent paper, the authors of this chapter describe how to generate valid inequalities for the bidirected and undirected network design problems using valid inequalities for the directed case.

In this chapter we review strong valid inequalities for the multi-commodity, multi-facility network design problem. Throughout, we emphasize three fundamental techniques that have been effective in deriving strong inequalities for network design problems. These are the metric inequalities for projecting out the continuous flow variables, mixed-integer rounding from appropriate base relaxations, and shrinking the network to a small k-node graph. The valid inequalities for the network design problem are obtained by applying these techniques to different relaxations of the problem, namely, arc-set, cut-set and partition relaxations. The basic inequalities derived from these relaxations are also utilized, with certain adaptations, in robust and survivable network design problems.

This chapter is organized as follows. In the next section, we introduce the notation used in the chapter and give a formal definition of the multi-commodity multi-facility design problem. Section 3 reviews the metric inequalities for projecting out the multi-commodity flow variables, the mixed-integer rounding technique, as well as a simplification obtained by shrinking the network for deriving valid inequalities from a smaller network. These techniques play a central role in deriving strong valid inequalities for the network design problem. Section 4 reviews the inequalities from single arc capacity constraints for the splittable as well as the unsplittable

cases. Section 5 reviews the valid inequalities from two-partitions for single as well as multi-facility cases. Section 6 generalizes the inequalities in the previous section to a higher number of partitions. We conclude with a few final remarks in Section 8.

2 Problem Formulation

Let $G = (N,A)$ be a directed graph (network), with node set N and arc set $A \subseteq N \times N$. Each arc $a \in A$ has a given existing capacity $\bar{c}_a \geq 0$ and the network design problem consists of installing additional capacity on the arcs of the network using an (integral) combination of a given set of capacity types. The objective of the problem is to minimize the sum of the flow routing cost and the capacity expansion cost. Throughout we assume that the data is rational.

The demand data for the problem is given by matrix $T = \{t_{ij}\}$, where $t_{ij} \geq 0$ is the amount of (directed) traffic that should be routed from node $i \in N$ to $j \in N$. Using matrix T, we can define a collection of commodities, each of which has a certain supply and demand at each node of the network. As discussed in Chapter 2, there are different ways to formulate the same network design problem by changing what is meant by a commodity. Using a minimal vertex cover of the so-called demand graph, one can obtain the smallest number of commodities required to formulate a given problem instance correctly. Computationally, formulations with fewer commodities may be more desirable as their continuous relaxations solve faster. For the sake of concreteness, we will use the aggregated commodity description but we note that most of the discussion below does not depend on how the commodities are defined. Let $K \subseteq N$ denote the collection of nodes with positive supply, i.e.,

$$K = \left\{ i \in N : \sum_{j \in N} t_{ij} > 0 \right\}.$$

We use w_i^k to denote the net demand of node $i \in N$ for commodity $k \in K$. More precisely, let $w_i^k = t_{ki}$ for $i \neq k$ and $w_k^k = -\sum_{j \in N} t_{kj}$ for $k \in K$. Note that each node $k \in K$ is the unique supplier of commodity k and flow of each commodity in the network needs to be disaggregated to obtain an individual routing for origin-destination pairs. We also note that the aggregated commodity description can only be used if the flow routing cost does not depend on both the origin and the destination of the traffic that needs to be routed.

New capacity can be installed in the network using integer multiples of facilities M, where a single unit of facility $m \in M$ provides capacity c_m. Without loss of generality, we assume that $c_m \in \mathbb{Z}$ for all $m \in M$ and $c_1 < c_2 < \ldots < c_{|M|}$. In this setting the network design problem involves installing enough additional capacity on the arcs of the network so that traffic can be routed simultaneously without violating arc capacities. For $i \in N$, let

$$N_i^+ = \{j \in N : (i,j) \in A\} \text{ and } N_i^- = \{j \in N : (j,i) \in A\}$$

denote the neighbors of node $i \in N$. Let integer variables $y_{ma} \geq 0$ indicate the number of facilities of type $m \in M$ installed on arc $a \in A$ and continuous variables $x_a^k \geq 0$ denote the amount of flow of commodity $k \in K$ routed on arc $a \in A$. Using this notation, the following constraints define the feasible region of the multi-commodity multi-facility network design problem:

$$\sum_{j \in N_i^+} x_{ij}^k - \sum_{j \in N_i^-} x_{ji}^k = w_i^k, \qquad\qquad k \in K, \ i \in N, \qquad (1)$$

$$\sum_{k \in K} x_{ij}^k - \sum_{m \in M} c_m y_{mij} \leq \bar{c}_{ij}, \qquad\qquad (i,j) \in A. \qquad (2)$$

Then the network design problem is stated as:

$$\text{(NDP)} \qquad\qquad \min\Big\{ dy + fx : (x,y) \in P^{\text{ND}} \Big\},$$

where d and f are cost vectors of appropriate size and

$$P^{\text{ND}} = \text{conv}\Big\{ (x,y) \in \mathbb{R}_+^{A \times K} \times \mathbb{Z}_+^{A \times M} : (1) \text{ and } (2) \Big\}.$$

As a concrete example with two facilities, consider a given arc $a \in A$. The total capacity $c_1 y_{1a} + c_2 y_{2a}$ given by the integer variables y_{1a} and y_{2a} has cost $d_{1a} y_{1a} + d_{2a} y_{2a}$. Assuming economies of scale, let $d_{1a}/c_1 > d_{2a}/c_2$ and remember that $c_1 < c_2$. In this case, we can write the cost function $f(z)$ required to generate z units of capacity (at the least cost) as:

$$h(z) = \lfloor z/c_2 \rfloor d_{2a} + \min\{ d_{2a}, \lceil (z - \lfloor z/c_2 \rfloor c_2)/c_1 \rceil d_{1a} \}$$

which is a piecewise linear function. Figure 1 illustrates an example with $3d_{1a} < d_{2a} < 4d_{1a}$.
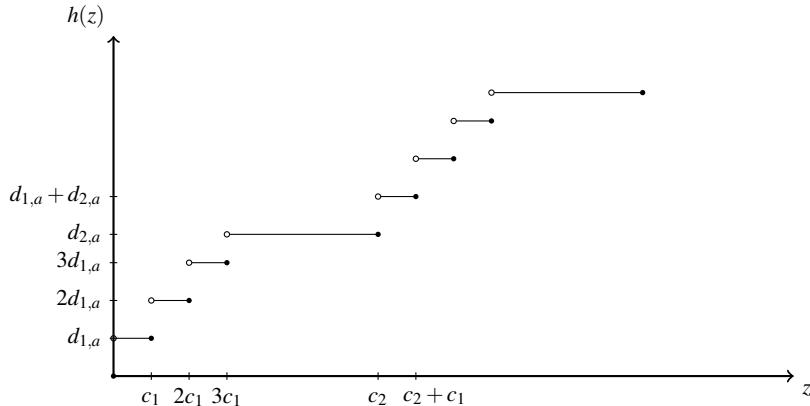

Fig. 1 The piecewise linear capacity installation cost function $h(z)$.

We also note that, when $f = 0$ it is possible to project out the multi-commodity flow variables from P^{ND} to obtain a formulation in the space of only the discrete capacity variables. This capacity formulation requires an exponential number of constraints and is discussed in Section 3.1.

3 Preliminaries

In this section we discuss three fundamental approaches that are useful in generating strong cutting planes for P^{ND}. We start with the metric inequalities, which give a generalization of the well-known max-flow min-cut theorem to multi-commodity flows. We then discuss how valid inequalities can be generated by shrinking the network to one with a few nodes to obtain inequalities from simpler sets. Finally, we describe the mixed-integer rounding procedure, which is an effective method to produce valid inequalities for general mixed-integer sets.

3.1 Metric Inequalities

Metric inequalities are introduced by Iri (1971), and Onaga and Kakusho (1971) as a generalization of the max-flow min-cut duality to multi-commodity flows. Consider the following capacitated multi-commodity flow set

$$F = \left\{ x \in \mathbb{R}_+^{A \times K} : (1) \text{ and } \sum_{k \in K} x_a^k \leq c_a, \quad a \in A \right\},$$

where $c \in \mathbb{R}_+^A$ denotes the arc capacities. By Farkas' Lemma, the set F is non-empty if and only if the following *metric inequalities*

$$\sum_{a \in A} c_a v_a \geq \sum_{i \in N} \sum_{k \in K} w_i^k u_i^k \tag{3}$$

hold for all $(v, u) \in D$, where

$$D = \left\{ (v, u) \in \mathbb{R}_+^A \times \mathbb{R}^{N \times K} : v_{ij} \geq u_j^k - u_i^k, \ k \in K, (i, j) \in A, u_k^k = 0 \right\}.$$

In other words, the arc capacity vector c can accommodate a feasible routing of the commodities if and only if it satisfies all metric inequalities generated by the non-empty cone D. Note that for any fixed $\bar{v} \in \mathbb{R}_+^A$, a point $(\bar{v}, u(\bar{v})) \in D$ maximizes the right-hand-side of (3) when $u(\bar{v})$ corresponds to the shortest path lengths using $\bar{v}$ as edge weights (hence the name "metric inequality"). Therefore, it suffices to consider metric inequalities where the vector $u \in \mathbb{R}^{N \times K}$ satisfies this property.

When there is only a single commodity, the max-flow min-cut theorem gives a nice characterization of the important extreme rays of the cone D. More precisely, in this case it suffices to consider vectors $v \in \{0,1\}^A$, where $v_{ij} = 1$ if and only if $i \in S$ and $j \notin S$ for some $S \subset N$ that contains the source of the commodity but not all the sinks.

Now consider the set P^{ND} and a given capacity vector $y \in \mathbb{R}^{A \times M}$ together with the existing arc capacities $\bar{c}$ and demand w. The metric inequality generated by $(u,v) \in D$ becomes

$$\sum_{a \in A}(\bar{c}_a + \sum_{m \in M} c_m y_{ma}) v_a \geq \sum_{i \in N} \sum_{k \in K} w_i^k u_i^k. \qquad (4)$$

As it is possible to check if there is a violated metric inequality in polynomial time (by solving a linear program), one can project out the flow variables from P^{ND} and obtain a "capacity" formulation in the space of the capacity variables only. Clearly this approach can be applicable only if there is no flow routing cost, i.e., $f = 0$ in problem (NDP). We also note that as inequalities (4) do not have flow variables, they only depend on the demand matrix and not on what commodity definition is used for the flow variables. Consequently, the right-hand-side of inequality (4) reduces to $\sum_{i \in N} \sum_{k \in N} t_{ki} u_i^k$.

The *integral metric inequalities* are obtained by rounding up the right-hand-side of metric inequalities associated with integral vectors $(u,v) \in D$,

$$\sum_{a \in A} \sum_{m \in M} c_m y_{ma} v_a \geq \left\lceil \sum_{i \in N} \sum_{k \in K} w_i^k u_i^k - \sum_{a \in A} \bar{c}_a v_a \right\rceil.$$

The basic cut-set inequalities discussed in Section 5 are special cases of the integral metric inequalities. While the metric inequalities different from the cut-set inequalities (Section 5) can be useful in strengthening the convex relaxations, their separation requires more computational effort.

3.2 Node Partition Inequalities

Consider a partition of the node set of the directed graph $G = (N,A)$ into $p < |N|$ disjoint subsets: $N = \cup_{t=1}^p N_t$. By shrinking these node subsets into singleton nodes, one obtains a simplified directed graph $\tilde{G} = (\tilde{N}, \tilde{A})$ with p nodes and up to $p(p-1)$ arcs. In this new graph, there is an arc $(i,j) \in \tilde{A}$ from node $i \in \tilde{N}$ to node $j \in \tilde{N}$ if and only if the original graph has at least one arc $(u,v) \in A$ from some node $u \in N_i$ to a node in $v \in N_j$. The existing capacity $\bar{c}_{ij}$ on arc $(i,j) \in \tilde{A}$ in the new network equals the sum of the existing capacities of all the arcs from the nodes in N_i to the nodes in N_j; in other words, $\tilde{c}_{ij} = \sum_{u \in N_i} \sum_{v \in N_j} \bar{c}_{uv}$.

Finally, setting the demand of $j \in \tilde{N}$ for commodity $k \in K$ to the total net demand of all nodes in N_j for commodity k in the original problem leads to a smaller network

design problem with p nodes. In other words, $\tilde{w}_i^k = \sum_{v \in N_i} w_v^k$ for all $k \in K$ and $i \in \tilde{N}$. Note that one can reduce the number of commodities in the new problem by aggregating the ones with the same source node but in order to keep the notation simple, we do not discuss it here.

Now consider a feasible (integral) solution (x, y) to the original network design problem defined on $G = (N, A)$ with existing capacity vector $\bar{c}$ and commodity demands w. Aggregating the flow and capacity variables as described above, it is easy to see that the resulting flow and the capacity vector $(\tilde{x}, \tilde{y})$ gives a feasible solution to the simplified p-node problem defined on $\tilde{G} = (\tilde{N}, \tilde{A})$ with existing capacity vector $\tilde{c}$ and commodity demands $\tilde{w}$. This observation implies that valid inequalities for the simplified problem on $\tilde{G}$ can be translated to valid inequalities for the original problem on G in the following way. Let

$$\sum_{k \in K} \sum_{(i,j) \in \tilde{A}} \tilde{\alpha}_{ij}^k x_{ij}^k + \sum_{m \in M} \sum_{(i,j) \in \tilde{A}} \tilde{\beta}_{mij} y_{mij} \geq \gamma \qquad (5)$$

be a given valid inequality for the simplified problem on $\tilde{G}$. Then the following inequality is valid for the original problem on G

$$\sum_{k \in K} \sum_{(u,v) \in A} \alpha_{uv}^k x_{uv}^k + \sum_{m \in M} \sum_{(u,v) \in A} \beta_{muv} y_{muv} \geq \gamma, \qquad (6)$$

where for any $k \in K$, $m \in M$ and $(u, v) \in A$ with $u \in N_i$ and $v \in N_j$, we set

$$\alpha_{uv}^k = \begin{cases} 0, & \text{if } i = j \\ \tilde{\alpha}_{ij}, & \text{otherwise} \end{cases} \qquad \beta_{muv} = \begin{cases} 0, & \text{if } i = j \\ \tilde{\beta}_{mij}, & \text{otherwise.} \end{cases}$$

3.3 MIR Inequalities

Many valid inequalities that have found use in practice for mixed-integer optimization problems are based on the mixed-integer rounding (MIR) procedure of Nemhauser and Wolsey (1988). Wolsey (1998) illustrates the basic mixed-integer rounding on the following two variable mixed-integer set

$$Q = \left\{ (x, y) \in \mathbb{R} \times \mathbb{Z} : x + y \geq b, \ x \geq 0 \right\},$$

and shows that the *basic mixed-integer rounding* inequality

$$x + ry \geq r \lceil b \rceil, \qquad (7)$$

where $r = b - \lfloor b \rfloor$, is valid and facet-defining for Q. Observe that if b is integer valued, inequality (7) reduces to $x \geq 0$. Otherwise, the inequality goes through feasible points $(0, \lceil b \rceil)$ and $(r, \lfloor b \rfloor)$, cutting off the fractional vertex $(0, b)$. This basic principle can be applied to more general mixed-integer sets defined by a single *base*

inequality as follows. Let

$$P = \left\{ x \in \mathbb{R}^n,\, y \in \mathbb{Z}^l \, : \, ax + cy \,\geq\, b, \; x,\, y \geq 0 \right\}$$

where $a \in \mathbb{R}^n$ and $c \in \mathbb{R}^l$. Letting r_j denote $c_j - \lfloor c_j \rfloor$ for $j = 1, \ldots, l$, we can rewrite the base inequality as

$$\sum_{a_j < 0} a_j x_j + \sum_{a_j > 0} a_j x_j + \sum_{r_j < r} r_j y_j + \sum_{r_j \geq r} r_j y_j + \sum_{j=1}^{l} \lfloor c_j \rfloor y_j \,\geq\, b$$

and relax it by dropping the first term, which is non-positive, and increasing the coefficients of the fourth term, which are non-negative, to obtain the valid inequality

$$\left(\sum_{a_j > 0} a_j x_j + \sum_{r_j < r} r_j y_j \right) + \left(\sum_{r_j \geq r} y_j + \sum_{j=1}^{l} \lfloor c_j \rfloor y_j \right) \geq\, b.$$

As the first two sums above are non-negative and the last two sums are integer valued, treating the first two as the nonnegative continuous variable in the set Q and the second two as the integer variable as in Q, we obtain the *MIR inequality*

$$\sum_{a_j > 0} a_j x_j + \sum_{r_j < r} r_j y_j + r \left(\sum_{r_j \geq r} y_j + \sum_{j=1}^{l} \lfloor c_j \rfloor y_j \right) \,\geq\, r \lceil b \rceil \qquad (8)$$

for P. This MIR inequality is generated from the base inequality $ax + cy \geq b$. Notice that, given a mixed-integer set, any valid inequality for it can be used as a base inequality to define a relaxation of the original set. Consequently, any implied inequality leads to an MIR cut. Gomory mixed-integer cuts, for example, can be seen as MIR cuts generated from base inequalities obtained from the simplex tableau.

4 Valid Inequalities From Arc Sets

In this section we review the strong valid inequalities obtained from single-arc capacity relaxations of the multi-commodity network design problem. For simplicity of the presentation, we first focus on the single-facility case. We consider both the *splittable-flow arc set:*

$$F_S = \left\{ (x, y) \in [0,1]^K \times \mathbb{Z} \, : \, \sum_{i \in K} a_i x_i \leq a_0 + y \right\}$$

and the *unsplittable-flow arc set:*

$$F_U = \left\{ (x, y) \in \{0,1\}^K \times \mathbb{Z} \, : \, \sum_{i \in K} a_i x_i \leq a_0 + y \right\} \cdot$$

In Section 4.3 we consider the multi-facility generalizations of these sets.

The set F_U arises in unsplittable multicommodity problems where flow between each source-sink pair needs to be routed on a single path. In these problems, the disaggregated commodity definition is used and the set K contains all node pairs with positive demand. In the formulation, a binary flow variable x_a^k is used for each commodity–arc pair (k, a) that takes on a value of 1 if and only if the commodity is routed through the arc. Consequently, for each arc of the network this formulation has a capacity constraint of the form

$$\sum_{k \in K} d^k x_a^k \leq \bar{c}_a + cy_a, \tag{9}$$

where $d^k > 0$ is the demand of commodity $k \in K$, $\bar{c}_a \geq 0$ is the existing capacity, and $c > 0$ is the unit capacity to install. One arrives at F_U by dividing (9) by c.

Similarly, one arrives at F_S by redefining the flow variables associated with an arc and a commodity as the fraction of the total supply of that commodity traveling on that arc. In this case flow variables take values in $[0, 1]$ and capacity constraint (2) takes the form (9). Again, dividing (9) by c gives the set F_S.

Without loss of generality, we assume that $a_i > 0$ for all $i \in K$, since if $a_i < 0$, x_i can be complemented and if $a_i = 0$, x_i can be dropped.

4.1 Splittable-flow Arc Set

In this section we review the valid inequalities for the splittable flow arc set F_S. For $S \subseteq K$, by complementing the continuous variables x_i, $i \in S$, we can restate the arc capacity inequality as

$$\sum_{i \in S} a_i (1 - x_i) - \sum_{i \in K \setminus S} a_i x_i + y \geq a(S) - a_0, \tag{10}$$

where $a(S)$ stands for $\sum_{i \in S} a_i$. Relaxing the inequality by dropping x_i, $i \in K \setminus S$ and applying the MIR inequality, we obtain the *residual capacity inequality*

$$\sum_{i \in S} a_i (1 - x_i) \geq r(\eta - y), \tag{11}$$

where $\eta = \lceil a(S) - a_0 \rceil$ and $r = a(S) - a_0 - \lfloor a(S) - a_0 \rfloor$. The residual capacity inequalities together with the inequality $\sum_{i \in K} a_i x_i \leq a_0 + y$ and variable bounds are sufficient to describe $conv(F_S)$.

Example 1. Consider the splittable arc set

$$F_S = \left\{ (x, y) \in [0, 1]^5 \times \mathbb{Z} : \frac{1}{3} x_1 + \frac{2}{3} x_2 + \frac{2}{3} x_3 \leq y \right\}.$$

The non-dominated arc residual capacity inequalities for F_S with $r > 0$ are

S	r	Inequalities
$\{1\}$	$1/3$	$x_1 \le y$
$\{2\}$	$2/3$	$x_2 \le y$
$\{3\}$	$2/3$	$x_3 \le y$
$\{2,3\}$	$1/3$	$2x_2 + 2x_3 \le 2 + y$
$\{1,2,3\}$	$2/3$	$x_1 + 2x_2 + 2x_3 \le 1 + 2y$

Given a fractional point $(\bar{x}, \bar{y})$, a violated residual capacity inequality, if it exists, can be found by the following simple separation procedure: Let $T = \{i \in K : \bar{x}_i > \bar{y} - \lfloor \bar{y} \rfloor\}$. If $a_0 + \lfloor \bar{y} \rfloor < a(T) < a_0 + \lceil \bar{y} \rceil$ and $\sum_{i \in T} a_i(1 - \bar{x}_i - \lceil \bar{y} \rceil + \bar{y}) + (\lceil \bar{y} \rceil - \bar{y})(a_0 + \lfloor \bar{y} \rfloor) < 0$, then the inequality $\sum_{i \in T} a_i(1 - x_i) \ge r(\eta - y)$ is violated by $(\bar{x}, \bar{y})$. Otherwise, there exists no residual capacity inequality violated by $(\bar{x}, \bar{y})$. Clearly, this procedure can be performed in linear time.

4.2 Unsplittable-flow Arc Set

In this section we review the valid inequalities for the unsplittable flow arc set F_U. First, consider the related set

$$F_{Ur} = \left\{(x,y) \in \{0,1\}^K \times \mathbb{Z} : \sum_{i \in K} r_i x_i \le r_0 + y\right\},$$

where $r_i = a_i - \lfloor a_i \rfloor$, $i \in K \cup \{0\}$. There is a one-to-one relationship between the facets of $\mathrm{conv}(F_U)$ and $\mathrm{conv}(F_{Ur})$. In particular, inequality $\sum_{i \in K} \pi_i x_i \le \pi_0 + y$ defines a facet for $\mathrm{conv}(F_U)$ if and only if inequality $\sum_{i \in K}(\pi_i - \lfloor a_i \rfloor)x_i \le \pi_0 - \lfloor a_0 \rfloor + y$ defines a facet for $\mathrm{conv}(F_{Ur})$. Therefore, we may assume, without loss of generality, that $0 < a_i < 1$ for all $i \in K$ and $0 < a_0 < 1$.

c–strong inequalities

For $S \subseteq K$ consider the arc capacity inequality written as (10). Relaxing the inequality by dropping x_i, $i \in K \setminus S$ and applying integer rounding, we obtain the so-called *c-strong inequality*

$$\sum_{i \in S} x_i \le c_S + y, \tag{12}$$

where $c_S = |S| - \lceil a(S) - a_0 \rceil$. The set S is said to be *maximal c–strong* if $c_{S \setminus \{i\}} = c_S$ for all $i \in S$ and $c_{S \cup \{i\}} = c_S + 1$ for all $i \in K \setminus S$. Inequality (12) is facet–defining for $\mathrm{conv}(F_U)$ if and only if S is maximal c–strong.

Given a point $(\bar{x}, \bar{y})$, there is a c–strong inequality violated by $(\bar{x}, \bar{y})$ if and only if there exists a set $S \subseteq K$ such that $\sum_{i \in S} \bar{x}_i - c_S > \bar{y}$. Then, a c–strong inequality is violated if and only if $\max_{S \subseteq K}\{\sum_{i \in S}\bar{x}_i - \lfloor a_0 + \sum_{i \in S}(1 - a_i)\rfloor\} = \max\{\sum_{i \in K}\bar{x}_i z_i - w : \sum_{i \in K}(1 - a_i)z_i + a_0 + 1/\lambda \le w, \ z \in \{0,1\}^K, \ w \in \mathbb{Z}\} + 1 > \bar{y}$, where λ is the least common multiple of the denominators of the rational numbers $(1 - a_i)$ and a_0.

The last maximization problem with the constant term $-a_0 - 1/\lambda$ is $\mathcal{NP}$–hard. Nevertheless, the separation problem has an optimal solution (z^*, w^*) such that $z_i^* = 1$ if $\bar{x}_i = 1$ and $z_i^* = 0$ if $\bar{x}_i = 0$. Therefore, we can fix such variables to their optimal values and solve the separation problem over $i \in K$ such that $0 < \bar{x}_i < 1$, which in practice can be done very efficiently even by enumeration, as most variables take on values either 0 or 1 in the LP relaxations of network design problems.

k–split c–strong inequalities

The c-strong inequalities can be generalized by considering a relaxation of the capacity constraints, where the integer capacity variables are allowed to take values that are integer multiples of $1/k$ for a positive integer k. Thus the k-split relaxation takes the form

$$F_U^k = \left\{ (x,y) \in \{0,1\}^K \times \mathbb{Z} : \sum_{i \in K} a_i x_i \leq a_0 + z/k \right\}.$$

Letting $c_S^k = \sum_{i \in S} \lceil k a_i \rceil - \lceil k a(S) - k a_0 \rceil$, we define the k–split c–strong inequality as

$$\sum_{i \in S} \lceil k a_i \rceil x_i + \sum_{i \in K \setminus S} \lfloor k a_i \rfloor x_i \leq c_S^k + ky. \tag{13}$$

The k–split c–strong inequality (13) is facet–defining for $conv(F_U)$ if (i) S is maximal c–strong in the k–split relaxation, (ii) $f_S > (k-1)/k$ and $a_0 \geq 0$, (iii) $a_i > f_S$ for all $i \in S$, $a_i < 1 - f_S$ for all $i \in K \setminus S$, where $f_S = a(S) - a_0 - \lfloor a(S) - a_0 \rfloor$.

Example 2. Consider the unsplittable arc set

$$F_U = \left\{ (x,y) \in \{0,1\}^5 \times \mathbb{Z} : \frac{1}{3} x_1 + \frac{1}{3} x_2 + \frac{1}{3} x_3 + \frac{1}{2} x_4 + \frac{2}{3} x_5 \leq y \right\}.$$

The maximal c–strong inequalities for F_U are:

$c_S = 0 : x_1 \leq y, \ x_2 \leq y, \ x_3 \leq y, \ x_4 \leq y$
$c_S = 1 : x_1 + x_2 + x_4 \leq 1 + y, \ x_1 + x_2 + x_5 \leq 1 + y, \ x_2 + x_3 + x_4 \leq 1 + y,$
$\qquad\ \ x_2 + x_3 + x_5 \leq 1 + y, \ x_1 + x_3 + x_4 \leq 1 + y, \ x_1 + x_3 + x_5 \leq 1 + y$
$c_S = 2 : x_1 + x_2 + x_3 + x_4 + x_5 \leq 2 + y$

As the inequalities are maximal, they are facet-defining for $conv(F_U)$. The 2–split c-strong inequality $x_2 + x_3 + x_4 + x_5 \leq 2y$ and the 3–split c-strong inequality $x_1 + x_2 + x_3 + 2x_4 + 2x_5 \leq 3y$ are also facet–defining for $conv(F_U)$.

Lifted knapsack cover inequalities

Facets different from c-strong and k-split c-strong inequalities can be obtained by lifting cover inequalities from knapsack restrictions of F_U. Let K_0 and K_1 be two disjoint subsets of K and ν be a nonnegative integer. Consider the 0-1 knapsack set $F_U(\nu, K_0, K_1)$ obtained by restricting the capacity variable y to ν, all binary variables indexed with K_0 to 0 and all binary variables indexed with K_1 to 1, i.e., $F_U(\nu, K_0, K_1) \equiv \{(x,y) \in F_U : y = \nu, \ x_i = 0 \text{ for all } i \in K_0 \text{ and } x_i = 1 \text{ for all } i \in K_1\}$. For this knapsack restriction $C = K \setminus (K_0 \cup K_1)$ is called a *cover* if $r = a(C) + a(K_1) - a_0 - \nu > 0$. C is said to be a *minimal cover* if $a_i \geq r$ for all $i \in C$.

The cover inequality $\sum_{i \in C} x_i \leq |C| - 1$ is facet–defining for $conv(F_U(\nu, K_0, K_1))$ if and only if C is a minimal cover. One practical way of lifting inequalities is sequential lifting, in which restricted variables are introduced to an inequality one at a time in some sequence. A lifted cover inequality

$$\sum_{i \in C} x_i + \sum_{i \in K_0} \alpha_i x_i + \sum_{i \in K_1} \alpha_i (1 - x_i) + \alpha(\nu - y) \leq |C| - 1 \tag{14}$$

can be constructed in $O(|K|^3)$ if the capacity variable y is lifted first and such inequalities subsume all c–strong inequalities.

Example 3. For F_U given in Example 2 we list below the lifted cover inequalities of F_U that are not c–strong inequalities.

ν	(C, K_0, K_1)	Inequalities
1	$(\{2,3,4\},\{1,5\},\emptyset)$	$x_2 + x_3 + x_4 + x_5 \leq 2y$
1	$(\{1,4,5\},\{2,3\},\emptyset)$	$x_1 + x_2 + x_4 + x_5 \leq 2y$ and $x_1 + x_3 + x_4 + x_5 \leq 2y$
2	$(\{1,2,3,4\},\emptyset,\{5\})$	$x_1 + x_2 + x_3 + x_4 + 2x_5 \leq 2y + 1$
2	$(\{1,2,3,5\},\emptyset,\{4\})$	$x_1 + x_2 + x_3 + 2x_4 + x_5 \leq 2y + 1$

Computational results suggest that c-strong inequalities are quite effective in solving unsplittable multi-commodity network design problems. Moreover, while the k-split c-strong and the lifted knapsack cover inequalities provide additional strengthening of the relaxations, the marginal impact on top of the basic c-strong inequalities is limited. The latter result may be due to the lack of efficient separation procedures for these inequalities.

4.3 Multi-facility Arc Set

In this section we consider the multi-facility extension of the arc sets discussed in the previous sections. Dash et al. (2016) study a continuous knapsack set with two integer capacity variables:

$$F_{S2} = \left\{ (x,y) \in [0,1]^K \times \mathbb{Z}_+^2 : \sum_{i \in K} a_i x_i \leq a_0 + c_1 y_1 + c_2 y_2 \right\}.$$

They show that all non-trivial facet-defining inequalities of $\mathrm{conv}(F_{S2})$ are of the form

$$\sum_{i \in S} a_i x_i + \gamma_1 y_1 + \gamma_2 y_2 \geq \beta,$$

where $S \subseteq K$ and $w + \gamma_1 y_1 + \gamma_2 y_2 \geq \beta$ is facet-defining for the set

$$Q(a,b) = \mathrm{conv}\left\{ (w,y) \in \mathbb{R} \times \mathbb{Z}_+^2 : w + c_1 y_1 + c_2 y_2 \geq b, \ a \geq w \right\}.$$

It turns out that all facets of $Q(a,b)$ can be enumerated in polynomial time. There-fore, for each $S \subseteq K$ non-trivial facet-defining inequalities for F_{SM} can be obtained from relaxations of the form $Q(a,b)$.

For the general case with many facility types

$$F_{SM} = \left\{ (x,y) \in [0,1]^K \times \mathbb{Z}_+^M : \sum_{i \in K} a_i x_i \leq a_0 + \sum_{m \in M} c_m y_m \right\}$$

a similar approach of complementing the flow variables for a subset $S \subseteq K$, scaling the base inequality by c_s, $s \in M$, and applying mixed-integer rounding gives for F_{SM} the *multi-facility residual capacity inequalities*

$$\sum_{m \in M} \phi_s(c_m) y_m + \sum_{i \in S} a_i (1 - x_i) \geq r_s \eta_s,$$

where $r_s = a(S) - a_0 - \lfloor (a(S) - a_0)/c_s \rfloor c_s$, $\eta_s = \lceil (a(S) - a_0)/c_s \rceil$, and for $k \in \mathbb{Z}$

$$\phi_s(c) = \begin{cases} c - k(c_s - r_s) & \text{if } kc_s \leq c < kc_s + r_s, \\ (k+1)r_s & \text{if } kc_s + r_s \leq c < (k+1)c_s. \end{cases}$$

5 Valid Inequalities From Cut Sets

In this section we review valid inequalities for the network design problem based on relaxations formed over cuts of the network. We first start with the single-facility case and then generalize the inequalities for multiple facilities.

5.1 Single-facility Case

Consider a nonempty two-partition (U,V) of the vertices of the network. Let b^k denote the total supply of commodity k in U for V. Let A^+ be the set of arcs directed from U to V, A^- be the set of arcs directed from V to U, and $A = A^+ \cup A^-$, as shown

in Figure 2. As before, x_a^k denotes the flow of commodity k on arc $a \in A$ for $k \in K$. The constraints of the multicommodity network design problem across the cut are

$$x^k(A^+) - x^k(A^-) = b^k, \qquad k \in K, \qquad (15)$$

$$\sum_{k \in K} x_a^k \leq \bar{c}_a + cy_a, \ a \in A. \qquad (16)$$

Then the corresponding multicommodity cut–set polyhedron is defined as

$$F_{MS} = conv\Big\{(x, y) \in \mathbb{R}_+^{A \times K} \times \mathbb{Z}_+^A : (x, y) \text{ satisfies (15) and (16)}\Big\}.$$

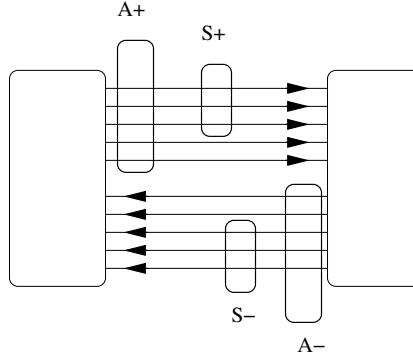We refer to the single-commodity case of F_{MS} as F_{SS} .

Fig. 2 Cut-set relaxation of the network design problem.

In the following sections we describe valid inequalities for F_{MS} by considering single-commodity relaxations of F_{MS} obtained by aggregating flow variables and balance equations (15) over subsets of K. For $Q \subseteq K$ let $x_Q(S) = \sum_{k \in Q} x^k(S)$ and $b_Q = \sum_{k \in Q} b^k$.

Cut-set inequalities. Consider the following relaxation of F_{MS} on the integer capacity variables:

$$\bar{c}(A^+) + cy(A^+) \geq x_K(A^+) \geq b_K.$$

Applying the integer rounding procedure reviewed in Section 3.3 to this relaxation, one obtains the so-called *cut-set inequality*

$$y(A^+) \geq \lceil (b_K - \bar{c}(A^+))/c \rceil \qquad (17)$$

for F_{MS} , which is unique per cut-set relaxation. Finding the best cut-set relaxation is not easy however. For the single-source single-sink case, the problem of finding the best cut set can be posed as an $s - t$ max-cut problem.

Flow-cut-set inequalities. The basic cut-set inequalities (17) can be generalized by incorporating the flow variables in addition to the capacity variables (see Figure 2). For $S^+ \subseteq A^+$, $S^- \subseteq A^-$ and $Q \subseteq K$ consider the following relaxation of F_{MS} :

$$\bar{c}(S^+) + cy(S^+) + x_Q(A^+ \setminus S^+) - x_Q(S^-) \geq b_Q,$$
$$0 \leq \sum_{k \in Q} x_a^k \leq \bar{c}_a + cy_a, \ \forall a \in A,$$

which is written equivalently as

$$c\left[y(S^+) - y(S^-)\right] + x_Q(A^+ \setminus S^+) + \left[\bar{c}(S^-) + cy(S^-) - x_Q(S^-)\right] \geq b'_Q,$$
$$0 \leq \sum_{k \in Q} x_a^k \leq \bar{c}_a + cy_a, \ \forall a \in A.$$

where $b'_Q = b_Q - \bar{c}(S^+) + \bar{c}(S^-)$. Letting $r_Q = b'_Q - \lfloor b'_Q/c \rfloor c$ and $\eta_Q = \lceil b'_Q/c \rceil$ and observing that $x_Q(A^+ \setminus S^+) \geq 0$, $\bar{c}(S^-) + cy(S^-) - x_Q(S^-) \geq 0$, we can apply the MIR procedure reviewed in Section 3.3 to this relaxation to arrive at the flow-cut set inequalities

$$r_Q y(S^+) + x_Q(A^+ \setminus S^+) + (c - r_Q)y(S^-) - x_Q(S^-) \geq r_Q \eta_Q. \tag{18}$$

The flow-cut-set inequalities (18) along with the balance, bound, and capacity constraints are sufficient to describe the single-commodity case F_{SS} .

For a given $Q \subseteq K$, observe that flow-cut-set inequalities (18) is an exponential class. However, given a solution $(\bar{x}, \bar{y})$, one finds a subset S^+ with the smallest left-hand-side value as follows: if $r_Q \bar{y}_a < \sum_{k \in Q} \bar{x}_a^k$ for $a \in A^+$, then we include a in S^+; if $(c - r)\bar{y}_a < \sum_{k \in Q} \bar{x}_a^k$ for $a \in A^-$, then we include $a \in S^-$.

For a fixed cut of the network, the complexity of separating multi-commodity flow cut–set inequalities (18) is an open question. Optimization of a linear function over F_{MS} is $\mathcal{NP}$–hard as the facility location problem is a special case of it. For a multi-commodity single-facility network design problem of a single arc, cut–set inequalities (21) reduce to the residual capacity inequalities (11), for which an exact linear–time separation method is known. From here it follows that, for a single-facility problem, if S^+ and S^- are fixed, then one can find a subset of commodities $Q \subseteq K$ that gives a most violated inequality in linear time. Alternatively, if Q is fixed, since the model reduces to a single-commodity, one can find the subsets $S^+ \subseteq A^+$ and $S^- \subseteq A^-$ that give a most violated inequality in linear time as well. However, the complexity of determining Q, S^+ and S^- simultaneously is an open question.

Example 4. Consider the following single-commodity optimization problem with two outflow arcs and one inflow arc:

$$\max x_1 + x_2 + x_3 - y_1 - y_2 - y_3 \text{ s.t. } x_1 + x_2 - x_3 = 0.5, \ 0 \leq x_i \leq y_i \in \mathbb{Z}, \ i = 1, 2, 3.$$

One of its fractional solutions is $x_1 = y_1 = 0.5$ and all other variables zero. Adding the cut-set inequality

$$y_1 + y_2 \geq 1$$

cuts off this solution, but leads to another fractional solution: $x_1 = y_1 = 1$, $x_3 = y_3 = 0.5$ Adding the flow-cut-set inequality

$$0.5x_1 + y_2 + 0.5x_3 - y_3 \geq 0.5$$

cuts it off, but gives the fractional solution: $x_2 = y_2 = 1, x_3 = y_3 = 0.5$. Adding the flow-cut-set inequality

$$y_1 + 0.5x_2 + 0.5x_3 - y_3 \geq 0.5$$

cuts it off, but this time gives the fractional solution: $x_1 = y_1 = x_2 = y_2 = x_3 = y_3 = 0.5$. Finally, adding the flow-cut-set inequality

$$0.5x_1 + 0.5x_2 + 0.5x_3 - y_3 \geq 0.5$$

leads to an optimal integer solution $x_1 = 0.5, y_1 = 1$ and all other variables zero.

5.2 Multi-facility Case

Next we consider network design models where one is allowed to install facilities of multiple types with different capacities on the arcs of the network. Let c_m be the capacity of facility of type m, $m \in M$. No assumption is made on either the number of facility types or the structure of capacities (other than $c_m > 0$ and rational).

The constraints of the multi-commodity multi-facility problem across cut A are written as

$$x^k(A^+) - x^k(A^-) = b^k, \qquad\qquad k \in K, \qquad\qquad (19)$$

$$\sum_{k \in K} x_a^k \leq \bar{c}_a + \sum_{m \in M} c_m y_{m,a}, \quad a \in A. \qquad\qquad (20)$$

So the corresponding multi-commodity multi-facility cut-set polyhedron is

$$F_{MM} = conv\left\{ (x,y) \in \mathbb{R}_+^{A \times K} \times \mathbb{Z}_+^{A \times M} : (x,y) \text{ satisfies (19) and (20)} \right\}.$$

For $Q \subseteq K$ and $s \in M$ let $r_{s,Q} = b'_Q - \lfloor b'_Q / c_s \rfloor c_s$, $\eta_{s,Q} = \lceil b'_Q / c_s \rceil$. Then, the following *multi-commodity multi-facility cut-set inequality* is valid for F_{MM} :

$$\sum_{m \in M} \phi_{s,Q}^+(c_m) y_m(S^+) + x_Q(A^+ \setminus S^+) + \sum_{m \in M} \phi_{s,Q}^-(c_m) y_m(S^-) - x_Q(S^-) \geq r_{s,Q} \eta_{s,Q}, \quad (21)$$

where, for $k \in \mathbb{Z}$,

$$\phi_{s,Q}^{+}(c) = \begin{cases} c - k(c_s - r_{s,Q}) & \text{if } kc_s \leq c < kc_s + r_{s,Q}, \\ (k+1)r_{s,Q} & \text{if } kc_s + r_{s,Q} \leq c < (k+1)c_s, \end{cases}$$

and

$$\phi_{s,Q}^{-}(c) = \begin{cases} c - kr_{s,Q} & \text{if } kc_s \leq c < (k+1)c_s - r_{s,Q}, \\ k(c_s - r_{s,Q}) & \text{if } kc_s - r_{s,Q} \leq c < kc_s. \end{cases}$$

The above $\phi_{s,Q}^{+}$ and $\phi_{s,Q}^{-}$ are subadditive MIR functions written in closed form. The multi-commodity multi-facility cut-set inequality (21) is facet-defining for F_{MM} if $(S^+, A^+ \setminus S^+)$ and $(S^-, A^- \setminus S^-)$ are nonempty partitions, $r_{s,Q} > 0$, and $b^k > 0$ for all $k \in Q$.

For the single-commodity case F_{SM}, inequalities (21) reduce to

$$\sum_{m \in M} \phi_s^{+}(c_m) y_m(S^+) + x(A^+ \setminus S^+) + \sum_{m \in M} \phi_s^{-}(c_m) y_m(S^-) - x(S^-) \geq r_s \eta_s. \quad (22)$$

In this case, given a cut A for each facility $s \in M$, the multi-facility cut-set inequalities (22) is an exponential class by the choice of the subsets of arcs S^+ and S^-. However, finding a subset that gives a most violated inequality for a point $(\bar{x}, \bar{y})$ is straightforward. If $\sum_{m \in M} \phi_s^{+}(c_m) \bar{y}_{m,a} < \bar{x}_a$ for $a \in A^+$, then we include a in S^+, and if $\sum_{m \in M} \phi_s^{-}(c_m) \bar{y}_{m,a} < \bar{x}_a$ for $a \in A^-$, then we include a in S^-. Since $\phi_s^{+}(c)$ or $\phi_s^{-}(c)$ can be calculated in constant time, for a fixed cut A, a violated multi-facility cut-set inequality is found in $O(|A||M|)$ if there exists any.

Example 5. We specialize inequality (21) for the network design problem with two types of facilities. Let the vectors y_1 and y_2 denote the facilities with capacities $c_1 = 1$ and $c_2 = \lambda > 1$ with $\lambda \in \mathbb{Z}$, respectively. Let Q be a nonempty subset of the commodities. Then by letting $s = 1$, we have $r_{1,Q} = b_Q - \lfloor b_Q \rfloor$ and inequality (21) becomes

$$r_{1,Q} y_1(S^+) + (r_{1,Q}\lfloor \lambda \rfloor + \min\{\lambda - \lfloor \lambda \rfloor, r_{1,Q}\}) y_2(S^+) + x_Q(A^+ \setminus S^+) + (1 - r_{1,Q})$$
$$y_1(S^-) + ((1 - r_{1,Q})\lfloor \lambda \rfloor + \min\{\lambda - \lfloor \lambda \rfloor, 1 - r_{1,Q}\}) y_2(S^-) - x_Q(S^-) \geq r_{1,Q}\lceil b_Q \rceil,$$

which, when λ is integer, reduces to

$$r_{1,Q} y_1(S^+) + \lambda r_{1,Q} y_2(S^+) + x_Q(A^+ \setminus S^+) +$$
$$(1 - r_{1,Q}) y_1(S^-) + \lambda(1 - r_{1,Q}) y_2(S^-) - x_Q(S^-) \geq r_{1,Q}\lceil b_Q \rceil. \quad (23)$$

Notice that inequality (23) is not valid for F_{MM} unless $\lambda \in \mathbb{Z}$. Also by letting $s = 2$, we have $r_{2,Q} = b_Q - \lfloor b_Q/\lambda \rfloor \lambda$. So the corresponding multi-commodity two-facility cut-set inequality is

$$\min\{1, r_{2,Q}\} y_1(S^+) + r_{2,Q} y_2(S^+) + x_Q(A^+ \setminus S^+) +$$
$$\min\{1, \lambda - r_{2,Q}\} y_1(S^-) + (\lambda - r_{2,Q}) y_2(S^-) - x_Q(S^-) \geq r_{2,Q}\lceil b_Q/\lambda \rceil.$$

It should be clear that multi-facility flow-cut-set inequalities are also valid for a single-facility model with varying capacities on the arcs of the network. The inequalities from cut-set relaxations have been shown to be very effective in solving network design problems in computational studies.

6 Partition Inequalities

Partition inequalities are arguably the most effective cutting planes for the network design problem. These inequalities have non-zero coefficients for only the integer capacity variables that cross a multi-cut obtained from a partition of the nodes of the network. They generalize the cut-set inequality (17) described in Section 5.

For ease of exposition, first consider a two-partition of the node set $N = N_1 \cup N_2$. As discussed in Section 3.2 shrinking node sets N_1 and N_2 leads to a network with two nodes and two edges (assuming there is at least one arc from a node in N_1 to a node in N_2, and vice versa). Then, the inequality

$$\sum_{m \in M} c_m y_{m12} \geq \sum_{k \in K} \sum_{v \in N_2} w_v^k - \sum_{u \in N_1} \sum_{v \in N_2} \bar{c}_{uv} \; = \; b$$

must be satisfied by all feasible solutions of the two-node problem. Following the argument in Section 3.2, inequality

$$\sum_{m \in M} c_m \left(\sum_{u \in N_1} \sum_{v \in N_2} y_{muv} \right) \geq b \tag{24}$$

is valid for the solutions to the (LP relaxation of the) original problem. Notice that inequality (24) is a metric inequality (4) generated by the vector $v \in \{0,1\}^A$, where $v_{ij} = 1$ if and only if $i \in N_1$ and $j \in N_2$.

As we assume that all c_m are integral, which is the case in most applications, the inequality (24) leads to the integer knapsack cover set

$$X = \left\{ z \in \mathbb{Z}^M : \sum_{m \in M} c_m z_m \geq b, \, z \geq 0 \right\},$$

where the variable z_m stands for the sum $\sum_{u \in N_1} \sum_{v \in N_2} y_{muv}$. Consequently, any valid inequality $\sum_{m \in M} \alpha_m z_m \geq \beta$ for X yields a valid inequality for P^{ND} after replacing each variable z_m with the corresponding sum of the original variables.

The polyhedral structure of the set X when c_{m+1} is an integer multiple of c_m for all $m = 1, \dots, |M| - 1$ has been studied by Pochet and Wolsey (1995) who derive what they call "partition" inequalities and show that these inequalities together with the nonnegativity constraints describe $\mathrm{conv}(X)$. They also derive conditions under which partition inequalities are valid in the general case when the divisibility condition does not hold.

The partition inequalities described in Pochet and Wolsey (1995) are obtained by applying the MIR procedure iteratively. More precisely, the first step is to chose a subset $\{j_1, j_2, \ldots, j_r\}$ of the index set M, where $j_i < j_{i+1}$, and therefore, $c_{j_i} < c_{j_{i+1}}$ for all $i = 1, \ldots, r-1$. The inequality $\sum_{m \in M} c_m z_m \geq b$ is then divided by c_{j_r} and the MIR cut based on this inequality is written. The resulting MIR inequality is then divided by $c_{j_{r-1}}$ and the MIR procedure is applied again. This process is repeated with all c_{j_i} for $i = 1, \ldots, r$ to obtain the final inequality. Note that the sequential application of the MIR procedure yields valid inequalities even when the divisibility condition does not hold. However, in this case, they are not sufficient to define $\text{conv}(X)$.

Now consider a three-partition of the node set $N = N_1 \cup N_2 \cup N_3$. Following the discussion on two-partitions, consider the single capacity network design problem with $\tilde{G} = (\tilde{N}, \tilde{A})$ where $\tilde{N} = \{1, 2, 3\}$ and $\tilde{A} = \{a_{12}, a_{13}, a_{21}, a_{23}, a_{31}, a_{32}\}$. Furthermore, let $\bar{c}_a$ and $\bar{t}_a$ respectively denote the existing capacity and traffic demands for $a \in \tilde{A}$. Clearly, any valid inequality for the simplified problem on $\tilde{G}$ can be transformed into a valid inequality for the original problem defined on G.

For the three-node problem, there are three possible two-partitions and for each partition, one can write two possible cut-set inequalities by treating one of the two sets as N_1 and the other as N_2. Consequently, one can write six different two-partition inequalities where each capacity variable appears in exactly two inequalities. Summing all six inequalities leads to

$$\sum_{a \in \tilde{A}} \sum_{m \in M} 2 c_m y_{ma} \geq \sum_{i \in \tilde{N}} \lceil s_i \rceil + \sum_{i \in \tilde{N}} \lceil t_i \rceil, \tag{25}$$

where s_i denotes the difference between the traffic leaving node i and the existing capacity on the outgoing arcs. Similarly t_i is the difference between the traffic entering node i and the existing capacity on the incoming arcs. If the right-hand-side of inequality (25) is an odd number, dividing the inequality by two and rounding up the right-hand-side yields the following inequality

$$\sum_{m \in M} c_m \sum_{a \in \tilde{A}} y_{ma} \geq \left\lceil \frac{\sum_{i \in \tilde{N}} \lceil s_i \rceil + \sum_{i \in \tilde{N}} \lceil t_i \rceil)}{2} \right\rceil. \tag{26}$$

Next we will generate a similar inequality based on metric inequalities. Let $a, b \in \tilde{N}$ be two distinct nodes and define $v^{ab} \in \{0, 1\}^{\tilde{A}}$ to be the vector with components $v_{ab} = v_{ac} = v_{bc} = 1$ and $v_{ba} = v_{ca} = v_{cb} = 0$. Now consider the metric inequality (4) generated by v^{ab}:

$$\sum_{m \in M} c_m (y_{mab} + y_{mac} + y_{mbc}) \geq \bar{t}_{ab} + \bar{t}_{ac} + \bar{t}_{bb} - \bar{c}_{ab} - \bar{c}_{ac} - \bar{c}_{bc}.$$

Once again, if fractional, the right-hand-side of this inequality can be rounded up. In addition, more valid inequalities can be generated using the MIR procedure iteratively.

Furthermore, let $c \in \tilde{N}$ be the node different from a and b and note that adding up the metric inequalities generated by v^{ab} and v^{cb}, one obtains

$$\sum_{m \in M} c_m \sum_{a \in \tilde{A}} y_{ma} \geq \lceil d_{ab} \rceil + \lceil d_{cb} \rceil \qquad (27)$$

where d_{ij} denotes the right-hand-side of the metric inequality generated by v^{ij} for $(i, j) \in A$. Moreover, adding up the metric inequalities generated by v^{ba} and v^{ca} gives a similar inequality with right-hand-side of $\lceil d_{ba} \rceil + \lceil d_{ca} \rceil$. Similarly, v^{ac} and v^{bc} yields an inequality with right-hand-side of $\lceil d_{ac} \rceil + \lceil d_{bc} \rceil$. Adding up two of these inequalities with the larger right-hand-side and dividing the resulting inequality by two leads to a valid inequality of the form (26). More precisely, if both $\lceil d_{ba} \rceil + \lceil d_{ca} \rceil$ and $\lceil d_{ab} \rceil + \lceil d_{cb} \rceil$ are larger than $\lceil d_{ac} \rceil + \lceil d_{bc} \rceil$, then the resulting inequality is

$$\sum_{m \in M} c_m \sum_{a \in \tilde{A}} y_{ma} \geq \left\lceil \frac{\lceil d_{ba} \rceil + \lceil d_{ca} \rceil + \lceil d_{ab} \rceil + \lceil d_{cb} \rceil}{2} \right\rceil. \qquad (28)$$

In addition to inequalities (26) and (28), it is possible to write similar *total capacity inequalities* by combining some cut-set inequalities with metric inequalities in such a way that the left-hand-side of the inequality has all the capacity variables with a coefficient of two. As all these inequalities have the same left-hand-side, only the one with the largest right-hand-side should be used. For example, if $\bar{t}_{ij} = 1/2$ and $\bar{c}_{ij} = 0$ for $(i, j) \in \tilde{A}$, then the right-hand-side of (26) is 3, whereas the right-hand-side of (28) is 4 and therefore inequality (28) is stronger than (26). However, if $\bar{t}_{ij} = 1/3$ and $\bar{c}_{ij} = 0$ for $(i, j) \in \tilde{A}$, then the right-hand-side of (26) is still 3, whereas the right-hand-side of (28) becomes 2.

Furthermore, as total capacity inequalities have the same form as inequality (24), one can define the corresponding integer knapsack cover set from the stronger one and derive further valid inequalities using the MIR procedure iteratively.

Hamid and Agarwal (2015) study the undirected variant of the two-facility network design problem, where the total flow on an arc plus the flow on reverse arc is limited by the capacity of the undirected edge associated with the two arcs. In this case, the authors computationally enumerate the complete list of facets that can be obtained from a given three-partition. Also see Agarwal (2006) for a study of four-partition facets for the undirected variant of the single-facility network design problem. Their computational study suggests that using larger partitions of the node set improves the relaxation but with diminishing returns. More precisely, they observe that two, three and four-partition cuts reduce the optimality gap of the LP relaxation to 12.5%, 6.3%, and 2.6%, respectively.

7 Bibliographical Notes

Introduction

Capacity expansion problems have been studied in the context of telecommunucation (Magnanti and Wong, 1984; Minoux, 1989; Balakrishnan et al., 1991, 1995) and train scheduling with multiple locomotive options (Florian et al., 1976). Unsplittable flow version of the network design problem appears in telecommunication networks using multiprotocol label switching (MPLS) technique, production and distribution with single sourcing, and express package delivery, train scheduling (e.g. Gavish and Altinkemer, 1990; Barnhart et al., 2000; Davarnia et al., 2019). Ozbaygin et al. (2018) utilize the splittable flow model to solve the split delivery vehicle routing problem. Atamtürk and Günlük (2018) study the connection between valid inequalities for directed, bidirected and undirected network design problems.

Problem formulation and preliminaries

Günlük (2007) describes how to obtain the smallest number of commodities necessary to formulate a given multicommodity flow problem. Metric inequalities and their extensions have been used for various network design problems by several authors, including Dahl and Stoer (1998); Mirchandani (2000); Labbé and Yaman (2004); Avella et al. (2007); Costa et al. (2009); Bienstock et al. (1998). Mattia (2012) presents computations that illustrate the value of utilizing metric inequalities through a bi-level programming separation procedure. Hamid and Agarwal (2015) show that if inequality (5) is facet-defining for the network design problem on $\tilde{G}$ with $\tilde{c}$ and $\tilde{w}$, then inequality (6) is facet-defining for P^{ND} provided that $\tilde{\alpha} = 0$, $\gamma > 0$, and node sets $N_1, \ldots, N_p$ induce connected components of G. In addition, Raack et al. (2011) show that the same result holds without the assumption $\tilde{\alpha} = 0$ when $p = 2$ and $|M| = 1$.

Valid inequalities from arc sets

The arc sets and their generalizations are studied by Magnanti et al. (1993); Atamtürk and Rajan (2002); van Hoesel et al. (2002); Brockmüller et al. (2004); Atamtürk and Günlük (2007); Yaman (2013). Magnanti et al. (1993) introduced the *residual capacity inequality*. Atamtürk and Rajan (2002) gave the polynomial-time separation procedure for residual capacity inequities. Brockmüller et al. (2004) introduced the *c-strong inequality* (12) for F_U. The k-split c-strong inequalities (4.2) are given by Atamtürk and Rajan (2002). Agra and Constantino (2006) show that all facets of $Q(a,b)$ can be enumerated in polynomial time.

Valid inequalities from cut sets

A recent review on cut-based inequalities for network design can be found in Raack et al. (2011). Magnanti and Mirchandani (1993) introduce the integer cut-set inequalities for F_{SS}. For the single-source single-sink case, given a solution, Barahona (1996) formulated the problem of finding the best cut set as an $s-t$ max-cut problem. Bienstock and Günlük (1996); Chopra et al. (1998) gave the mixed-integer flow-cut-set generalization (18) of the basic cut-set inequalities. Atamtürk (2002) showed that the flow-cut-set inequalities (18) along with the balance, bound, and capacity constraints are sufficient to describe the single-commodity case F_{SS}. Magnanti and Mirchandani (1993); Magnanti et al. (1995); Pochet and Wolsey (1995); Bienstock and Günlük (1996); Günlük (1999); Wolsey and Yaman (2016) give valid inequalities for the network design problem with multiple capacities when capacities are divisible. Atamtürk et al. (2001) consider a binary capacity version with no assumption on divisibility. Multi-commodity multi-facility network design problems are considered in Bienstock et al. (1998); Atamtürk (2002).

Partition inequalities

The polyhedral structure of the set X without the divisibility assumption on the capacities has been studied by Atamtürk (2003) and Yaman (2007), also see the references therein. Atamtürk et al. (2016) gave three-partition flow cover inequalities for the fixed-charge network design problem. The total capacity inequalities were proposed by Bienstock et al. (1998). Hamid and Agarwal (2015) has enumerated the complete list of facets that can be obtained from a given three-partition for the undirected variant of the two-facility network design problem. Agarwal (2006) has studied the four-partition facets and computationally showed that using larger partitions of the node set improves the LP relaxation but with diminishing returns.

8 Conclusions and Perspectives

In this chapter, we reviewed strong valid inequalities for the multi-commodity, multi-facility network design problem. Metric inequalities for projecting out continuous flow variables, mixed-integer rounding from appropriate base relaxations, and shrinking the network to a small k-node graph have been the main tools for deriving the inequalities introduced in the literature. Going forward, we expect more recent techniques such as multi-step mixed-integer rounding (Dash and Günlük, 2006), mingling (Atamtürk and Günlük, 2010), and multi-step mingling (Atamtürk and Kianfar, 2012) that generalize and extend mixed-integer rounding to be useful for deriving new inequalities for this class of network design problems with continuous as well as general integer variables.

We finish this section with comments on computational aspects and challenges in implementing branch-and-cut algorithms. One of the prerequisites of applying the valid inequalities described in this chapter as cutting planes is to automatically recognize the appropriate network structure and identify flow-balance constraints and the relevant capacity constraints as part of a hidden multi-commodity, multi-facility network structure. Achterberg and Raack (2010) describe algorithms to automatically detect such network structures and generate inequalities from cut-set relaxations. They utilize c-MIR inequalities (Marchand and Wolsey, 2001), which are obtained by first complementing bounded variables and then applying the MIR procedure. Given a fractional solution, it is nontrivial to decide which aggregations to apply to flow variables, which variables to complement. Nevertheless, the automatic detection and separation procedure in this paper results in 18% time reduction for a large set of publicly available test problems.

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