

Sensitivity Analysis for Linear Estimators

Jacob Dorn and Luther Yap*

April 30, 2024

LATEST VERSION [HERE](#).

Abstract

We propose a novel sensitivity analysis framework for linear estimators with identification failures that can be viewed as seeing the wrong outcome distribution. Our approach measures the degree of identification failure through the change in measure between the observed distribution and a hypothetical target distribution that would identify the causal parameter of interest. The framework yields a sensitivity analysis that generalizes existing bounds for Average Potential Outcome (APO), Regression Discontinuity (RD), and instrumental variables (IV) exclusion failure designs. Our partial identification results extend results from the APO context to allow even unbounded likelihood ratios. Our proposed sensitivity analysis consistently estimates sharp bounds under plausible conditions and estimates valid bounds under mild conditions. We find that our method performs well in simulations even when targeting a discontinuous and nearly infinite bound.

1 Introduction

Many important estimators in economics are weighted averages of observed outcomes. These estimators leverage powerful identifying restrictions to target meaningful estimands. In observational settings, these identifying restrictions may not be satisfied: many treatment choices can be selected on unobservables, manipulators can strategically sort across treatment cutoffs, and instruments can directly affect outcomes.

In this paper, we propose a framework for the sensitivity analysis of identification failures for linear estimators. The framework proceeds as follows. First, we define a target distribution: a synthetic distribution over observed variables that would enable a standard estimator to point-identify the causal estimand of interest. Second, we consider a structural model that implies restrictions on the divergence between the target population and the population the practitioner observes. Third, we leverage work from the literature in statistics and distributionally robust optimization to estimate bounds implied by the restrictions from the second step. The framework is especially powerful when the implied restrictions correspond to a family of restrictions on the Radon-Nikodym derivative (the generalization of likelihood ratio to allow point masses) between the distribution of observables under the target distribution and under the distribution the observed data is drawn from.

*This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-2039656. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Our proposed framework has several empirical advantages. The approach typically yields a sensitivity analysis: under the strongest restriction, the bounds reduce to the standard point estimate; under the weakest restriction, the bounds reduce to a worst-case exercise. Between the two extremes, partial identification bounds can often be estimated using modern tools from the literature on distributionally robust optimization (DRO). The partial identification bounds can often be written as an explicit closed form moment of the observed data.

We illustrate our framework for L_∞ bounds on the Radon–Nikodym derivative between the target and observed distributions. We extend existing results from the average potential outcome (APO) literature where distributional restrictions correspond to limits on treatment selection ([Tan, 2006](#); [Masten and Poirier, 2018](#); [Zhao et al., 2019](#); [Dorn and Guo, 2023](#)). We extend this literature to obtain a simple closed-form characterization of the identified set, even with unbounded changes in measure. The plug-in estimator is consistent for the sharp bounds when given consistent estimates of the primary analysis nuisance functions and a certain conditional outcome quantile function. When the conditional outcome quantile function is inconsistent, the plug-in estimator yields valid bounds that are too wide rather than too narrow.

We apply our framework to yield novel results for three applications. In our first application, we study bounds on the APO with a treatment that may be selected on an unobserved confounder. Our framework nests unconfoundedness, Manski-type bounds that restrict only the support of the unobserved potential outcomes, and [Tan \(2006\)](#)’s Marginal Sensitivity Model as special cases. As a corollary, we obtain a simpler characterization of bounds under [Masten and Poirier \(2018\)](#)’s conditional c-dependence model.

In our second application, we study bounds on causal effects from sharp RD with one-sided manipulation. As [McCrory \(2008\)](#) notes, the RD assumption of no manipulation is testable. When [McCrory](#)’s test fails, [Gerard et al. \(2020\)](#) propose a worst-case bound on the conditional average treatment effect for non-manipulators: a Conditional Local Average Treatment Effect (CLATE). We show that a stronger restriction on manipulation choice allows us to obtain meaningful bounds on the more standard conditional average treatment effect (CATE), which to the best of our knowledge has been an open issue in the literature. Our framework provides a sensitivity analysis by nesting an unconfoundedness-type assumption and the [Gerard et al. \(2020\)](#) assumption as extreme cases.

In our third application, we study treatment effects with an instrument that fails the exclusion restriction. When the instrument is allowed to affect the outcome directly, we consider estimation of a generic weighted average of local average treatment effects (LATEs) across instrument values. In the continuous outcome case, we contribute a sensitivity analysis for a measure of exclusion failure that is unit-free. The proposed bounds are simple and tractable, but can be wider than the bounds implied by the original model.

We illustrate the value of our approach by simulation. We study estimation of APO bounds under [Masten and Poirier](#)’s conditional c-dependence model, which restricts the difference between observed and true propensities to be at most c . $c = 0$ corresponds to unconfoundedness; in our example, the bounds discontinuously become unbounded as c crosses 0.1. We implement a simple plug-in estimator and percentile bootstrap that leverage a fixed grid of quantile estimates across bootstraps and values of c . We find that a plug-in bootstrap approach achieves excellent bias and coverage properties for small c ; increasingly over-covers as c grows towards 0.1; and nearly achieves nominal coverage of the real line at the most difficult $c = 0.1$ case.

1.1 Related Work

We now mention some related work.

Our framework unifies ideas from the operations research and economics literature. Two closely related frameworks are [Bertsimas et al. \(2022\)](#) and [Christensen and Connault \(2023\)](#), both of which are limited to discrete covariates. In the discrete covariate case, relative to [Bertsimas et al.](#), we propose justifying distributional distances in terms of underlying structural models of economic failure and propose constructing target distributions that apply in settings beyond average treatment effects, and offer a specific analysis of L_∞ bounds rather than other distributional distances. Relative to [Christensen and Connault](#) in the discrete covariate case, we analyze a nominally different class of identification failures and provide closed-form bounds for specifically linear estimators.

Our work is related to the recent literature on sensitivity analysis for IPW estimators, which relates to our first application. A sensitivity analysis is an approach to partial identification that begins from assumptions that point-identify the causal estimand of interest and then considers increasing relaxations of those assumptions ([Molinari, 2020](#)). Our analysis is an extension of [Dorn and Guo \(2023\)](#)'s sharp characterization of bounds under [Tan \(2006\)](#)'s marginal sensitivity model. [Tan \(2022\)](#) and [Frauen et al. \(2023\)](#) previously extended this characterization to families that bound the Radon-Nikodym derivative of interest. We generalize these results to also include unbounded Radon-Nikodym derivative, so that we can include a compact characterization of bounds under [Masten and Poirier \(2018\)](#)'s conditional c-dependence model as a special case. There is rich work in this literature under other sensitivity assumptions like f -divergences and Total Variation distance. These other assumptions also fit within our framework, because our target distribution constructions are independent of the L_∞ sensitivity assumptions that we analyze.

Our other two applications relate to existing work on sensitivity analysis. Our proposed sensitivity analysis for sharp regression discontinuity (RD) applies when data on the running variable fails tests for manipulation ([McCrory, 2008](#); [Otsu et al., 2013](#); [Bugni and Canay, 2021](#)). Our proposal nests both an exogeneity-type assumption and [Gerard et al. \(2020\)](#)'s bounds as special cases. There is other work in the RD context on partial identification bounds under manipulation ([Rosenman et al., 2019](#); [Ishihara and Sawada, 2020](#)) but to our knowledge, our proposal is the first sensitivity analysis for manipulation. There are sensitivity analysis for exclusion failure with instrumental variables ([Ramsahai, 2012](#); [Van Kippersluis and Rietveld, 2018](#); [Masten and Poirier, 2021](#); [Freidling and Zhao, 2022](#)), but to our knowledge our proposal is the first sensitivity analysis whose underlying assumptions are invariant to invertible transformations of variables.

Notation. We use $\bar{\mathbb{R}}$ to refer to the extended real number line $\mathbb{R} \cup \{-\infty, \infty\}$. For a real-valued random variable Z and a distribution \mathbb{Q} , we use the notation $E_{\mathbb{Q}}[Z] = \int zd\mathbb{Q}(z)$ and we write that the expected value of Z exists under \mathbb{Q} if $E_{\mathbb{Q}}[|Z|]$ is finite or $E_{\mathbb{Q}}[Z]$ is well-defined as exactly one of positive or negative infinity. We write that the expected value of Z exists (without specifying the distribution) if the expected value exists under the observed distribution \mathbb{P}^{Obs} , where \mathbb{P}^{Obs} is defined below. Similarly, we sometimes suppress the dependence of expectations when referring to the expectation under the observed distribution \mathbb{P}^{Obs} . We write $\{a, b\}_+ = \max\{a, b\}$ and $\{a, b\}_- = \min\{a, b\}$, and abuse notation by writing $\{a\}_+ = \max\{a, 0\}$. For random variables $Y \in \mathbb{R}^1$ and $R \in \mathbb{R}^d$ and a function $t : \mathbb{R}^d \rightarrow [0, 1]$, we refer to “the” conditional quantile function $Q_{t(R)}(Y | R)$, which is any minimizer of $E_{\mathbb{P}^{\text{Obs}}}[t(R)\{Y - Q(R), 0\}_+ - (1 - t(R))\{Y - Q(R), 0\}_-]$ in functions Q from the domain of R to the extended real line $\mathbb{R} \cup \{-\infty, \infty\}$. To consolidate notation, when b is infinite, we evaluate the interval $[a, b]$ as $[a, \infty)$. If our estimation procedure calls for an estimate of a nuisance function f that depends on another nuisance function g , we use the notation \hat{f} to denote the full estimated nuisance function, which may include a composition of nuisance estimates. We use $1\{\cdot\}$ to refer to the indicator function that takes value 1 if true and 0 otherwise.

2 Framework for Sensitivity Analysis

In the following section, we illustrate our generic framework for sensitivity analysis: we propose linking a structural model of identification failures to an implied statistical distribution bound and then conducting partial identification under the distributional bound. We illustrate our approach with a family of L_∞ distributional distances that is especially tractable for estimation. We illustrate how our framework applies to specific structural models on Section 3.

2.1 General Setting

We begin with our general partial identification setting. We use the average potential outcome application as a running example.

There is a target distribution $\mathbb{P}^{\text{Target}}$ over (R, Y) that point identifies a causal estimand of interest, where $Y \in \mathbb{R}^1$ is the outcome of interest and $R \in \mathbb{R}^d$ are observable quantities that do not include Y . We call R the regressors. However, the researcher only has access to data from an observed distribution \mathbb{P}^{Obs} . \mathbb{P}^{Obs} is also a distribution over (R, Y) , but \mathbb{P}^{Obs} may not be able to point-identify the causal estimand under the researcher's preferred estimator, for example if R is a selected treatment or if R includes an instrument that fails the exclusion restriction.

Assumption Support. The marginal distribution of R is the same under $\mathbb{P}^{\text{Target}}$ and \mathbb{P}^{Obs} . The distribution $\mathbb{P}^{\text{Target}}$ is absolutely continuous with respect to \mathbb{P}^{Obs} .

Assumption **Support** has two components. The first component is that \mathbb{P}^{Obs} identifies the target distribution of regressors R , but may have a different conditional distribution of outcomes Y . The second component is an absolute continuity assumption that ensures the existence of a Radon-Nikodym derivative $\frac{d\mathbb{P}^{\text{Target}}}{d\mathbb{P}^{\text{Obs}}}$. The assumption accommodates unbounded Radon-Nikodym derivatives, for example $Y | R \sim \text{Unif}(0, 1)$ under \mathbb{P}^{Obs} and $f_Y(y | R) = 1\{y \in (0, 1]\}y^{-1/2}/2$ under $\mathbb{P}^{\text{Target}}$. The absolute continuity assumption is stronger than necessary for our results, and could be reduced to assuming that the support of (Y, R) under $\mathbb{P}^{\text{Target}}$ is a subset of the support under \mathbb{P}^{Obs} . Absolute continuity rules out $\mathbb{P}^{\text{Target}}$ possessing mass points where \mathbb{P}^{Obs} lacks mass points, so that such distributions can only correspond to limits of distributions within Assumption **Support**.

Our framework is inspired by the literature on sensitivity analysis for average treatment effects. In this case, there is a distribution \mathbb{P}^{True} over $(X, T, Y(1), Y(0))$, but we only observe the distribution \mathbb{P}^{Obs} over $(X, T, Y = Y(T))$. The regressors are $R = (X, T)$. For simplicity, we illustrate our approach for the Average Potential Outcome (APO) $E_{\mathbb{P}^{\text{True}}}[E_{\mathbb{P}^{\text{True}}}[Y(1) | R]]$. The researcher can accurately estimate $\mathbb{P}^{\text{True}}(T | X)$, but cannot observe $\mathbb{P}^{\text{True}}(Y(1) | X, T = 0)$. As a result, our proposed approach targets a distribution $\mathbb{P}^{\text{Target}}$ that first samples $(X, T) \sim \mathbb{P}^{\text{True}}$ and then samples $Y | X, T$ from the distribution of $Y(T) | X$ under \mathbb{P}^{True} . If a causal estimator like an inverse propensity weighted (IPW) estimator were applied to $\mathbb{P}^{\text{Target}}$ instead of \mathbb{P}^{Obs} , the researcher would identify the APO under \mathbb{P}^{True} . Our later examples illustrate the applicability of this framework for other causal estimands.

We refer to our target as the causal estimand. We restrict ourselves to estimands that correspond to linear estimators.

Definition 1. The *causal estimand* is $\psi_0 = E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y]$ for some real-valued function λ . The *primary estimand* is $E_{\mathbb{P}^{\text{Obs}}}[\lambda(R)Y]$.

We generally assume that λ is identified and the researcher has conducted a primary analysis that consistently estimates λ . For example, in the APO case, the λ function is the inverse propensity $\frac{T}{\mathbb{P}^{\text{Obs}}(T=1|X)}$. When the researcher is able to make strong enough assumptions that $\mathbb{P}^{\text{Target}}$ is equivalent to \mathbb{P}^{Obs} , then the primary estimand will be equal to the causal estimand. When the researcher is only able to bound the difference between $\mathbb{P}^{\text{Target}}$ and \mathbb{P}^{Obs} , then the researcher can partially identify the causal estimand through a restriction on the Radon-Nikodym derivative between the two distributions.

Lemma 1. *Suppose Assumption [Support](#) holds and $\lambda(R)Y$ is integrable under $\mathbb{P}^{\text{Target}}$. Then*

$$\psi_0 = E_{\mathbb{P}^{\text{Obs}}} \left[\lambda(R)Y \frac{d\mathbb{P}^{\text{Target}}(Y|R)}{d\mathbb{P}^{\text{Obs}}(Y|R)} \right].$$

Proof. Lemma 1 is a standard result for Importance Sampling and an immediate property of the Radon-Nikodym derivative. All other proofs can be found in Appendix B. \square

This reweighing characterization is useful for partial identification because any non-negative putative outcome reweighing \bar{W} that satisfies $E[\bar{W}|R] = 1$ almost surely will correspond to a well-defined distribution \mathbb{Q} over (R, Y) .

The Radon-Nikodym derivative characterization in Lemma 1 often maps to interpretable quantities. For example, in the APO case, the Radon-Nikodym derivative maps to odds ratios as follows:

$$\lambda(R) \frac{d\mathbb{P}^{\text{Target}}(Y|R)}{d\mathbb{P}^{\text{Obs}}(Y|R)} = \lambda(R) \left(\mathbb{P}^{\text{Obs}}(T=1|X) + \mathbb{P}^{\text{Obs}}(T=0|X) \frac{\mathbb{P}^{\text{Target}}(T=0|X, Y(1)) \mathbb{P}^{\text{Obs}}(T=1|X)}{\mathbb{P}^{\text{Target}}(T=1|X, Y(1)) \mathbb{P}^{\text{Obs}}(T=0|X)} \right).$$

As a result, there is a bijection between structural models of treatment selection and causal models of treatment effects. More generally, principled restrictions on an underlying treatment selection model may imply, or be equivalent to, restrictions on the Radon-Nikodym derivative.

2.2 Partial Identification Assumption and Result

In the this subsection, we characterize the sharp bounds on the identified set under an L_∞ restriction on the largest and smallest Radon-Nikodym derivative.

Definition 2 (Sensitivity assumption). For any pair of functions $\underline{w}, \bar{w} : \mathbb{R}^d \rightarrow \bar{\mathbb{R}}$ satisfying $0 \leq \underline{w}(R) \leq 1 \leq \bar{w}(R)$ almost surely, we define $\mathcal{M}(\underline{w}, \bar{w})$ as the set of distributions \mathbb{Q} over (R, Y) satisfying $\lambda(R) \frac{d\mathbb{Q}(R, Y)}{d\mathbb{P}^{\text{Obs}}(R, Y)} \in [\lambda(R)\underline{w}(R), \lambda(R)\bar{w}(R)]$ almost surely.

This family has several advantages. The restrictions on $\frac{d\mathbb{Q}}{d\mathbb{P}^{\text{Obs}}}$ decouple across values of R , enabling tractable characterizations of the identified set. The family nests both strong observational assumptions and Manski-type bounds as limits. Point identification corresponds to the case $\underline{w}(R) = \bar{w}(R) = 1$ almost surely. Manski-type bounds that only restrict the support of Y correspond to $\bar{w}(R) = \infty$ with domain-appropriate $\underline{w}(R)$. In between, the causal estimand is only point identified. When the outcome Y is binary, the restriction can equivalently be viewed as a restriction on the conditional mean of $Y|R$. We show below that, as in the [Tan \(2006\)](#) model that inspired this generalization, the resulting bounds are highly tractable for estimating sharp and valid bounds.

We adapt standard notation from [Ho and Rosen \(2017\)](#).

Definition 3. The *identified set* is $\mathcal{I}(\underline{w}, \bar{w}) = \{E_{\mathbb{Q}}[\lambda(R)Y] \mid \mathbb{Q} \in \mathcal{M}(\underline{w}, \bar{w})\}$. The *sharp bounds* on the identified set are $\psi^-(\underline{w}, \bar{w}) = \inf_{\psi \in \mathcal{I}(\underline{w}, \bar{w})} \psi$ and $\psi^+(\underline{w}, \bar{w}) = \sup_{\psi \in \mathcal{I}(\underline{w}, \bar{w})} \psi$. A *valid* identified set is a weak superset of $\mathcal{I}(\underline{w}, \bar{w})$. We call bounds that are strictly weaker than the sharp bounds, *conservative*.

We abuse notation and simply write \mathcal{I} , ψ^- , and ψ^+ as shorthand for the identified set and bounds under a generic model family. Definition 3 corresponds to the statistical partial identification bounds implied by an underlying structural model. As we illustrate in our applications, a structural model underlying Definition 2 can sometimes, but not always, imply a narrower bound.

We will require some nuisance functions to characterize the identified set. In particular:

Definition 4. The threshold probability is $\tau(R) = \frac{\bar{w}(R)-1}{\bar{w}(R)-\underline{w}(R)}$. The threshold quantiles are $Q^+(R) \equiv Q_{\tau(R)}(\lambda(R)Y \mid R)$ and $Q^-(R) \equiv Q_{1-\tau(R)}(\lambda(R)Y \mid R)$. The likelihood shifting term is $a(\underline{w}, \bar{w}, s) = (\bar{w} - \underline{w})1\{s > 0\} - (1 - \underline{w})$. For a function $\bar{Q} : \mathbb{R}^d \rightarrow \bar{\mathbb{R}}$, define the pseudo-outcome

$$\phi^+(\underline{w}, \bar{w}, r, y \mid \bar{Q}) \equiv \lambda(r)y + (\lambda(r)y - \bar{Q}(r))a(\bar{w}(r), \underline{w}(r), \lambda(r)y - \bar{Q}(r)),$$

and define $\phi^-(\underline{w}, \bar{w}, r, y \mid \bar{Q})$ analogously. Define the indicator variables $F = 1\{\bar{w}(R)\text{ finite}\}$, $G^+ = 1\{\lambda(R)Y - Q^+(R) > 0\}$, $G^- = 1\{\lambda(R)Y - Q^-(R) < 0\}$, and $H = 1\{\tau(R) < 1\}$.

When $\bar{w}(R)$ is finite, the formula for $\tau(R)$ can be found in Tan (2022) and Frauen et al. (2023).

We will make additional regularity assumptions.

Assumption Moments. The expected values of $F|Q^+(R)|$, $F|Q^-(R)|$, $|\phi^+(\underline{w}, \bar{w}, R, Y \mid Q^+)|$ and $|\phi^-(\underline{w}, \bar{w}, R, Y \mid Q^-)|$ exist.

We allow infinite $E_{\mathbb{P}^{\text{Obs}}}[Q^+(R)]$ for completeness with unbounded outcomes. While \bar{w} can be infinite, we rule out heavy tails that would yield infinite $E_{\mathbb{P}^{\text{Obs}}}[1\{\bar{w}(R)\text{ finite}\}Q^+(R)]$.

Our work focuses on the upper bound of the identified set.

Theorem 1. Suppose Assumptions *Support* and *Moments* hold. Then the sharp upper bound on the identified set satisfies:

$$\psi^+(\underline{w}, \bar{w}) = E_{\mathbb{P}^{\text{Obs}}} [\lambda(R)Y + (\lambda(R)Y - Q^+(R))a(\bar{w}(R), \underline{w}(R), \lambda(R)Y - Q^+(R))] . \quad (1)$$

The lower bound $\psi^-(\underline{w}, \bar{w})$ follows symmetrically, i.e.,

$$\psi^-(\underline{w}, \bar{w}) = E_{\mathbb{P}^{\text{Obs}}} [\lambda(R)Y + (\lambda(R)Y - Q^-(R))a(\bar{w}(R), \underline{w}(R), Q^-(R) - \lambda(R)Y)] . \quad (2)$$

Further, the identified set is convex.

The theorem states that the sharp upper bound can be written as a closed-form moment: after the constituent components such as $\lambda(\cdot)$, $Q^+(\cdot)$, and $a(\cdot)$ have been estimated, no further optimization problem needs to be solved.

A proof sketch is in order. We abuse notation to write $\psi^+ = E_{\mathbb{P}^{\text{Obs}}}[\psi^+(R)]$, where $\psi^+(r)$ is a pointwise upper bound function satisfying:

$$\psi^+(R) = \sup_W E_{\mathbb{P}^{\text{Obs}}} [W\lambda(R)Y \mid R] \quad \text{s.t.} \quad W \in [\underline{w}(R), \bar{w}(R)] \quad \& \quad E_{\mathbb{P}^{\text{Obs}}}[W \mid R] = 1,$$

and $\phi^+(R) \equiv E_{\mathbb{P}^{\text{Obs}}}[\phi^+(\underline{w}, \bar{w}, R, Y \mid Q^+) \mid R]$. The claim is $E_{\mathbb{P}^{\text{Obs}}}[\psi^+(R)] = E_{\mathbb{P}^{\text{Obs}}}[\phi^+(R)]$. We prove the stronger claim that $\psi^+(R) = \phi^+(R)$ almost surely. For simplicity, this sketch will ignore the possibility that $\lambda(R)Y = Q^+(R)$ with positive probability and drop ‘‘almost sure’’ caveats. As Dorn et al. (2022) note, $\psi^+(R)$ can be mapped to a simple DRO problem over $(W - \underline{w}(R))/(1 - \underline{w}(R)) \in [0, (\bar{w}(R) - \underline{w}(R))/(1 - \underline{w}(R))]$. The solution is $\psi^+(R) = E_{\mathbb{P}^{\text{Obs}}}[\underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R))CVaR_{\tau(R)}^+(R) \mid R]$, where $CVaR_{\tau(R)}^+ = E\left[Q^+ + \frac{\{Y - Q^+\}_+}{1 - \tau(R)} \mid R\right]$ is the level- $\tau(R)$ conditional value at risk of $\lambda(R)Y$. We split on the event $\tau(R) < 1$. By previous work (Tan, 2022; Frauen et al., 2023), $\tau(R) < 1$ implies $\psi^+(R) = \phi^+(R)$. We extend these results for the case $\tau(R) = 1$. On that event, $\phi^+(R)$ evaluates to $E[\underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R))Q^+(R) \mid R]$ and $Q^+(R) = CVaR_1^+(R)$. As a result, $\psi^+(R) = \phi^+(R)$ even if $\bar{w}(R)$ is infinite with positive probability.

One could characterize the partial identification bounds using the conditional value at risk directly, but the reweighing characterization Equation (1) is valuable for sensitivity analysis. We present the worst-case Radon-Nikodym derivative here for convenience.

Lemma 2. *Suppose $\bar{w}(R)$ is bounded. Then one can construct a distribution $\mathbb{Q}^+ \in \mathcal{M}(\underline{w}, \bar{w})$ and a random variable satisfying $\gamma(R) \in [\underline{w}(R), \bar{w}(R)]$ almost surely such that $\psi^+ = E_{\mathbb{Q}^+}[\lambda(R)Y]$ and:*

$$\frac{d\mathbb{Q}^+(R, Y)}{d\mathbb{P}^{\text{Obs}}(R, Y)} = W_{\text{sup}}^* = \begin{cases} \bar{w}(R) & \text{if } \lambda(R)Y > Q^+(R) \\ \underline{w}(R) & \text{if } \lambda(R)Y < Q^+(R) \\ \gamma(R) & \text{if } \lambda(R)Y = Q^+(R). \end{cases} \quad (3)$$

A researcher with a primary estimate of $\lambda(R)$ and who is capable of quantile regression can easily construct a plug-in estimate of ψ^+ . Even if the researcher fails to consistently estimate the additional nuisance parameter Q^+ , they can still easily estimate valid bounds:

Lemma 3. *Suppose Assumptions **Support** and **Moments** hold and there is a putative quantile function \bar{Q} such that the expectation of $|\phi^+(\underline{w}, \bar{w}, r, y \mid \bar{Q})|$ exists. Then replacing Q^+ with the potentially incorrect quantile function \bar{Q} yields valid bounds:*

$$\psi^+(\underline{w}, \bar{w}) \leq E_{\mathbb{P}^{\text{Obs}}} [\lambda(R)Y + (\lambda(R)Y - \bar{Q}(R))a(\bar{w}(R), \underline{w}(R), \lambda(R)Y - \bar{Q}(R))].$$

An analogous result holds for the lower bound ψ^- .

The argument follows by Tan (2022)’s argument writing $(\lambda(R)Y - \bar{Q}(R))a(\dots)$ in terms of the Quantile Regression check function. $Q^+(R)$ is the minimizer by classic arguments.

Note that a similar recipe as in this section extends beyond the family we consider. For other assumptions, the model family in Definition 2 would change, the sharp upper bound in Theorem 1 would change and may lack a closed form, and the validity Lemma 3 may or may not hold, but the fundamental logic would carry through.

3 Applications

In this section, we illustrate how to apply our framework to several potential failures of identifying assumptions: APOs with selection on unobservables, RD with manipulation, and IV without exclusion. We also discuss some other applications of our framework. We show that the sharp statistical bounds under our

approach contain all restrictions implied by the underlying structural model for APO selection and RD manipulation, but not IV exclusion failure.

3.1 Average Potential Outcomes

In this section, we illustrate our framework in a case that generalizes existing results: bounds on average potential outcomes under restrictions on selection on unobservables. Many of our results are restatements of recent work, but we include an extension of partial identification bounds to unbounded Radon-Nikodym derivatives.

We assume there is a distribution \mathbb{P}^{True} over $(X, T, \{Y(t)\}, U)$, where X are controls, T is a discrete treatment, $Y(t)$ is the potential outcome corresponding to treatment level t , and U are potential unobserved confounders. We only observe the coarsened distribution \mathbb{P}^{Obs} over $(X, T, Y = Y(T))$, where Y is the observed outcome. We write $I_t = 1\{T = t\}$.

We tailor our application to inverse propensity weighting (IPW) estimation of the APO. (We consider average treatment effects at the end of the section.) We write the observable propensity $e(X) = \mathbb{P}^{\text{Obs}}(T = 1 | X)$, where we assume $e(X) \in (0, 1)$ almost surely. The primary estimate is the IPW estimator that first estimates the $e(X)$ and then estimates the APO as $E_{\mathbb{P}^{\text{Obs}}}[\lambda(R)Y]$, where $\lambda(R) = I_1/e(X)$. When unconfoundedness holds given the observed covariates so that $I_t \perp\!\!\!\perp Y(t) | X$, the IPW strategy consistently estimates the causal estimand $E_{\mathbb{P}^{\text{True}}}[Y(1)]$. However, we will only assume unconfoundedness holds if the researcher had access to both the observed and unobserved covariates: $I_t \perp\!\!\!\perp Y(t) | X, U$.¹

The target distribution $\mathbb{P}^{\text{Target}}$ is defined as $\mathbb{P}^{\text{Target}}(X, T, Y) \equiv \mathbb{P}^{\text{Obs}}(X, T) \mathbb{P}^{\text{True}}(Y | X, T)$. The distribution of R is the same under $\mathbb{P}^{\text{Target}}$ and \mathbb{P}^{Obs} and the distribution $\mathbb{P}^{\text{Target}}$ satisfies $E_{\mathbb{P}^{\text{True}}}[Y(1)] = E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y]$, so that $E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y]$ is the causal estimand. Notice that $\mathbb{P}^{\text{Target}}$ and \mathbb{P}^{Obs} may have different distributions of $Y | X$: the equivalent construction of \mathbb{P}^{Obs} could be factored as $\mathbb{P}^{\text{Obs}}(X, T, Y) = \mathbb{P}^{\text{Obs}}(X, T) \mathbb{P}^{\text{Obs}}(Y | X, T)$.

Many researchers study models that imply there are functions $\ell(X), u(X)$ such that

$$\ell(X) \leq \frac{e(X, U)/(1 - e(X, U))}{e(X)/(1 - e(X))} \leq u(X)$$

almost surely (Manski, 1990; Tan, 2006; Aronow and Lee, 2013; Masten and Poirier, 2018; Zhao et al., 2019; Dorn et al., 2022; Tan, 2023; Frauen et al., 2023). This model implies pointwise restrictions on $\mathbb{P}^{\text{True}}(T = 1 | X, Y(1))$ and the Radon-Nikodym derivative $\frac{d\mathbb{P}^{\text{Target}}(Y | X, T=1)}{d\mathbb{P}^{\text{Obs}}(Y | X, T=1)}$. In particular, the selection assumption would imply the following almost sure bound:

$$\lambda(R) \frac{d\mathbb{P}^{\text{Target}}(Y | R)}{d\mathbb{P}^{\text{Obs}}(Y | R)} \in [\lambda(R)(e(X) + (1 - e(X))u(X)^{-1}), \quad \lambda(R)(e(X) + (1 - e(X))\ell(X)^{-1})].$$

Unconfoundedness corresponds to the special case $\ell(X) = u(X) = 1$. Manski bounds correspond to the special case $\ell(X) = 0, u(X) = \infty$. Tan (2006)'s Marginal Sensitivity Model corresponds to $\underline{w}(X, t) = \mathbb{P}^{\text{Obs}}(T = t | X) + \mathbb{P}^{\text{Obs}}(T \neq t | X)\Lambda^{-1}$ and $\bar{w}(X, t) = \mathbb{P}^{\text{Obs}}(T = t | X) + \mathbb{P}^{\text{Obs}}(T \neq t | X)\Lambda$. Basit et al. (2023)'s risk ratio marginal sensitivity model, that restricts $e(X, U) \in [\Gamma_0^{-1}e(X), \Gamma_1^{-1}e(X)]$, corresponds to $\underline{w}(R) = \Gamma_1$ and $\bar{w}(R) = \Gamma_0$ when $e(X) \leq \Gamma_1$. Masten and Poirier's conditional c-dependence model, that restricts $e(X, U) \in [e(X) - c, e(X) + c] \cap [0, 1]$, corresponds to $\underline{w}(X, t) = \mathbb{P}^{\text{Obs}}(T = t | X) + \mathbb{P}^{\text{Obs}}(T \neq t | X) \frac{\max\{0, 1 - (e(X) + c)\} / \min\{1, e(X) + c\}}{(1 - e(X))/e(X)}$ and $\bar{w}(X, t) = \mathbb{P}^{\text{Obs}}(T = t | X) + \mathbb{P}^{\text{Obs}}(T \neq t | X) \frac{\min\{1, 1 - (e(X) - c)\} / \max\{0, e(X) - c\}}{(1 - e(X))/e(X)}$.

¹This is without loss of generality by setting $U = \{Y(t)\}$.

Theorem 1 yields valid bounds on the identified set with the following quantities:

Proposition 1. *In the APO case, our method can be implemented on $E_{\mathbb{P}^{\text{True}}}[Y(1)]$ with:*

$$\lambda(R) = \frac{I_1}{e(X)}, \quad \underline{w}(R) = e(X) + (1 - e(X))u(X)^{-1}, \quad \text{and} \quad \bar{w}(R) = e(X) + (1 - e(X))\ell(X)^{-1}.$$

The result in Proposition 1 extends the analysis of [Tan \(2022\)](#); [Frauen et al. \(2023\)](#) to allow $\ell(X)$ to be arbitrarily small or equal to zero. As a result, it includes [Masten and Poirier \(2018\)](#)'s conditional c-dependence assumption so long as $\lambda(R)Y$ is integrable under the target distribution. The characterization of bounds under conditional c-dependence is formally simpler than [Masten and Poirier \(2018\)](#)'s characterization: their characterization involves an integral over a transformation of the full quantile regression function. Our approach also yields estimates of valid bounds under only the IPW estimation assumptions. An interesting question for future work is whether [Masten et al. \(2024\)](#)'s proposed estimator for conditional c-dependence possesses similar validity guarantees.

The result is intuitive. The w bounds would be 1 under the unconfoundedness assumption $\ell(X) = u(X) = 1$. As $e(X)$ gets closer to one, we see a greater share of the treated potential outcomes and the w bounds grow closer to one. An analogous approach can be used to obtain $E[Y(0)] = E[Y(1 - Z)/(1 - e(X, Y(0)))]$.

The resulting bounds are sharp for the underlying structural model for the APO and average treatment effect (ATE) estimands. The only restrictions on the distribution of $(Y, e(X, U)) \mid X, T = 1$ implied by \mathbb{P}^{Obs} are the requirements that $E_{\mathbb{P}}^{\text{Obs}}[1/e(X, U) \mid X, T = 1] = 1/e(X)$ almost surely. If we instead solved for the set of feasible $E_{\mathbb{P}^{\text{Obs}}}[I_1 Y / \bar{E}]$ subject to $\ell(X) \leq \frac{\bar{E}/(1-\bar{E})}{e(X)/(1-e(X))} \leq u(X)$ and $E_{\mathbb{P}}^{\text{Obs}}[1/\bar{E} \mid X, T = 1] = 1/e(X)$ almost surely, we would obtain a convex set with the same bounds. For the ATE with two potential treatment statuses, an extension of [Dorn and Guo \(2023\)](#)'s sharpness argument shows sharpness continues to hold under pointwise restrictions on the true propensity.

3.2 Regression Discontinuity

In this section, we introduce a novel sensitivity analysis for sharp RD designs. Our framework allows us to bound standard causal estimands that previous work could not meaningfully quantify.

We work under a slight modification of [Gerard et al. \(2020\)](#)'s model. There is a full distribution \mathbb{P}^{True} over $(M, X(1), X(0), Y(1), Y(0), T)$, where M is a variable corresponding to manipulation status, $X(m)$ is a potential running variable corresponding to manipulation status $M = m$, $Y(t)$ is a potential outcome corresponding to treatment status $T = t$, and $T \in \{0, 1\}$ is the treatment status. We face the fundamental problem of causal inference and only observe the coarsening \mathbb{P}^{Obs} over $(X = X(M), Y = Y(T), T)$.

We study sharp RD designs. We assume there is a cutoff c such that $\mathbb{P}^{\text{True}}(T = 1 \mid X > c) = 1$, $\mathbb{P}^{\text{True}}(T = 1 \mid X < c) = 0$. This mimics assumption (RD) of [Hahn et al. \(2001\)](#).² It is observationally testable and often obvious in applications: if X is the net reported vote share of an election candidate, then the candidate wins if and only if $X > 0$. We use the notation “ $X = c$ ” to mean that we take the limit as $\varepsilon \rightarrow 0$ of $X \in [c - \varepsilon, c + \varepsilon]$. Similarly, we let $X = c^+$ denote the limit corresponding to $X \in (c, c + \varepsilon]$ and let $X = c^-$ correspond to $X \in [c - \varepsilon, c]$.

We assume that observations can be partitioned into manipulators and non-manipulators. We assume manipulation is one-sided, and without loss of generality assume manipulators choose treatment ($F_{X|M=1}(c) = 0$). An interesting direction for future work are bounds in which manipulation can occur

²[Gerard et al. \(2020\)](#) extend their approach to fuzzy RD, where there are complex shape restrictions.

in both directions. As in [Gerard et al.](#), the probability of treated observations being manipulated,

$$\eta = \mathbb{P}^{\text{True}}(M = 1 \mid X = c^+),$$

is identified.³ We assume appropriate continuity of potential outcomes and running variables given the manipulation status. We relegate the details to Assumption RD in the appendix. The assumptions imply that non-manipulator average treatment effects would be identified by the change in non-manipulator outcomes across the cutoff c . However, when $\eta > 0$ so that there is manipulation, the distribution of $Y \mid X = c^+$ includes manipulators' treated potential outcomes, so that standard treatment effects are not point-identified.

Estimands that we may be interested in include the conditional average treatment effect (CATE), the conditional local average treatment effect (CLATE), and the conditional average treatment effect on the treated (CATT).

$$\begin{aligned}\psi^{CATE} &\equiv E[Y(1) - Y(0) \mid X = c] \\ \psi^{CLATE} &\equiv E[Y(1) - Y(0) \mid X = c, M = 0] \\ \psi^{CATT} &\equiv E[Y(1) - Y(0) \mid X = c^+]\end{aligned}\tag{4}$$

We call $E[Y(1) - Y(0) \mid X = c, M = 0]$ the CLATE because it is an average treatment effect at the cutoff among the population for whom the treatment is randomly assigned at the cutoff, conditional on being at the cutoff.

We write the causal estimands of interest in terms of conditional expectations of observed variables and potential outcomes as follows:

Lemma 4. *Our main estimands of interest have the following expressions:*

$$\begin{aligned}\psi_{CATE} &= \frac{1}{2-\eta} E[Y \mid X = c^+] + \frac{1-\eta}{1-2\eta} E_{\mathbb{P}^{\text{True}}} [Y(1) \mid X = c, M = 0] \\ &\quad - \frac{\eta}{2-\eta} E_{\mathbb{P}^{\text{True}}} [Y(0) \mid X = c, M = 1] - \left(1 - \frac{\eta}{2-\eta}\right) E[Y \mid X = c^-] \\ \psi_{CATT} &= \frac{E[(2T-1)Y \mid X = c] - \frac{\eta}{2-\eta} E_{\mathbb{P}^{\text{True}}}[Y(0) \mid X = c, M = 1]}{\mathbb{P}^{\text{Obs}}(X = c^+ \mid X = c)} \\ \psi_{CLATE} &= E_{\mathbb{P}^{\text{True}}}[Y(1) \mid X = c, M = 0] - E[Y \mid X = c^-].\end{aligned}$$

Most quantities in the expressions above are point-identified. The only unidentified objects are the conditional expectations $E_{\mathbb{P}^{\text{True}}}[Y(1) \mid X = c, M = 0]$ and $E_{\mathbb{P}^{\text{True}}}[Y(0) \mid X = c, M = 1]$.

The specific target distribution $\mathbb{P}^{\text{Target}}$ will depend on the estimand of interest as follows. For sets $\mathcal{Y} \subset \mathbb{R}$, define:⁴

$$\begin{aligned}\mathbb{P}_{CLATE}^{\text{Target}}(\mathcal{Y} \mid X = c, T = t) &= t\mathbb{P}^{\text{True}}(Y(1) \in \mathcal{Y} \mid X = c, M = 0) + (1-t)\mathbb{P}^{\text{Obs}}(Y \in \mathcal{Y} \mid X = c^-) \\ \mathbb{P}_{CATT}^{\text{Target}}(\mathcal{Y} \mid X = c, T = t) &= t\mathbb{P}^{\text{Obs}}(Y \in \mathcal{Y} \mid X = c^+) + (1-t)\mathbb{P}^{\text{True}}(Y(0) \in \mathcal{Y} \mid X = c^-) \\ \mathbb{P}_{CATE}^{\text{Target}}(\mathcal{Y} \mid X = c, T = t) &= \mathbb{P}^{\text{True}}(Y(t) \in \mathcal{Y} \mid X = c).\end{aligned}$$

³[Gerard et al.](#) call this quantity τ .

⁴A formal construction would define the appropriate distributions at $X = x$ based on the distribution conditional on $|x - c|$ and then take the limit as $x \rightarrow c$ from either side, which would yield well-defined distributions by the maintained continuity assumptions.

Then the target distribution for estimated $Est \in \{CATE, CLATE, CATT\}$ is defined as $\mathbb{P}^{\text{Target}}(X, T, Y) \equiv \mathbb{P}^{\text{Obs}}(X, T)\mathbb{P}_{Est}^{\text{Target}}(Y | X, T)$. The distribution of R is the same under $\mathbb{P}^{\text{Target}}$ and \mathbb{P}^{Obs} and $E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y]$ is equal to the target estimand for $\lambda(R)$ we define below.

Suppose we have a structural model that implies there are values $\Lambda_0, \Lambda_1 \geq 1$ such that for $t = 1, 0$:

$$\frac{\mathbb{P}^{\text{True}}(M = 1 | Y(t), X = c)}{\mathbb{P}^{\text{True}}(M = 0 | Y(t), X = c)} \Big/ \frac{\mathbb{P}^{\text{Obs}}(M = 1 | X = c)}{\mathbb{P}^{\text{Obs}}(M = 0 | X = c)} \in [\Lambda_t^{-1}, \Lambda_t]. \quad (5)$$

Define $\mathcal{M}(\Lambda_0, \Lambda_1)$ as the set of distributions \mathbb{Q} over $(M, X(1), X(0), Y(1), Y(0), T)$ that marginalize to the distribution of $(X(M), Y(T), T = 1\{X(M) > c\})$ under \mathbb{P}^{True} , satisfy the restrictions of this section, and satisfy $\frac{\mathbb{Q}(M=1|Y(t),X=c)}{\mathbb{Q}(M=1|Y(t),X=c)} / \frac{\mathbb{P}^{\text{Obs}}(M=1|X=c)}{\mathbb{P}^{\text{Obs}}(M=0|X=c)} \in [\Lambda_t^{-1}, \Lambda_t]$ almost surely.

More complicated functional forms could be accommodated at the cost of additional notation. This functional form is inspired by [Tan \(2006\)](#)'s Marginal Sensitivity Model. $\Lambda_t = 1$ corresponds to an unconfoundedness-type case in which the difference in regression values yields the CATE. $\Lambda_t = \infty$ corresponds to [Gerard et al. \(2020\)](#)'s assumption, in which the distribution of $Y(0) | M = 1, X = c$ is only constrained by the support of the potential outcome. In between, larger values of Λ accommodate larger degrees of manipulation on potential outcomes.

Proposition 2. *In the RD case, our method can be implemented for the CLATE $E_{\mathbb{P}^{\text{True}}}[Y(1) | X = c, M = 0] - E_{\mathbb{P}^{\text{Obs}}}[Y | X = c^-]$ with the following values:*

$$\begin{aligned} \lambda(R) &= \frac{1\{X = c^+\}}{\mathbb{P}^{\text{Obs}}(X = c^+ | X = c)} - \frac{1\{X = c^-\}}{\mathbb{P}^{\text{Obs}}(X = c^- | X = c)} \\ \underline{w}(R) &= \frac{1\{X = c^+\}}{1 - \eta + \eta\Lambda_1} + 1\{X = c^-\}, \quad \bar{w}(R) = \frac{1\{X = c^+\}}{1 - \eta + \eta\Lambda_1^{-1}} + 1\{X = c^-\}; \end{aligned}$$

our method can be implemented for the CATT $E_{\mathbb{P}^{\text{True}}}[Y(1) - Y(0) | X = c^+]$ with the same $\lambda(R)$ and:

$$\underline{w}(R) = 1\{X = c^+\} + 1\{X = c^-\} ((1 - \eta) + \eta\Lambda_0^{-1}), \quad \bar{w}(R) = 1\{X = c^+\} + 1\{X = c^-\} ((1 - \eta) + \eta\Lambda_0);$$

and our method can be implemented for the CATE $E_{\mathbb{P}^{\text{True}}}[Y(1) - Y(0) | X = c]$ with the same $\lambda(R)$ and:

$$\begin{aligned} \underline{w}(R) &= 1\{X = c^+\} \left[\frac{1}{2 - \eta} + \frac{1 - \eta}{1 - 2\eta} \frac{1}{1 - \eta + \eta\Lambda_1} \right] + 1\{X = c^-\} \left[\left(1 - \frac{\eta}{2 - \eta}\right) + \frac{\eta}{2 - \eta} \Lambda_0^{-1} \right] \\ \bar{w}(R) &= 1\{X = c^+\} \left[\frac{1}{2 - \eta} + \frac{1 - \eta}{1 - 2\eta} \frac{1}{1 - \eta + \eta\Lambda_1^{-1}} \right] + 1\{X = c^-\} \left[\left(1 - \frac{\eta}{2 - \eta}\right) + \frac{\eta}{2 - \eta} \Lambda_0 \right]. \end{aligned}$$

The connection between Proposition 2 and the characterization from Lemma 4 comes from correspondences that we derive in Appendix Lemma 9. Note that even when Λ_1 is infinite, the CLATE bounds are finite, which allows [Gerard et al. \(2020\)](#) to obtain non-trivial CLATE bounds. Our CLATE bounds for $\Lambda_1 = \infty$ are identical to the bounds in [Gerard et al. \(2020\)](#) for sharp RD. When Λ_1 is finite, we are also able to obtain meaningful CATE bounds.

Our analysis so far bounds manipulation on each potential outcome separately. There turns out to be no additional information available from a structural model that bounds manipulation on both potential outcomes simultaneously.

Proposition 3. *Suppose there is a finite $\Lambda \geq 1$ such that $\Lambda_1 = \Lambda_0 = \Lambda$. Let $\mathcal{M}'(\Lambda)$ be the set of distributions*

$\mathbb{Q} \in \mathcal{M}(\infty, \infty)$ satisfying:

$$\frac{\mathbb{Q}(M = 1 | Y(1), Y(0), X = c)}{\mathbb{Q}(M = 0 | Y(1), Y(0), X = c)} \Big/ \frac{\mathbb{P}^{\text{Obs}}(M = 1 | X = c)}{\mathbb{P}^{\text{Obs}}(M = 0 | X = c)} \in [\Lambda^{-1}, \Lambda]. \quad (6)$$

Then $\mathcal{M}'(\Lambda) \subseteq \mathcal{M}(\Lambda, \Lambda)$. Further, take any distributions $\mathbb{Q}_1, \mathbb{Q}_0 \in \mathcal{M}(\Lambda, \Lambda)$ and write $\psi_t = E_{\mathbb{Q}_t}[Y(t) | X = c, M = 1 - t]$. Then there is a distribution $\mathbb{Q}' \in \mathcal{M}'(\Lambda)$ satisfying $E_{\mathbb{Q}'}[Y(t) | X = c, M = 1 - t] = \psi_t$ for $d = 1, 0$.

Proposition 3 shows that the bounds are sharp, i.e., separate bounds on both potential outcomes suffice to bound our causal estimands of interest. The quantities $E_{\mathbb{P}^{\text{True}}}[Y(t) | X = c, M = 1 - t]$ still identify our causal estimands of interest.

The high-level logic of Proposition 3 is fundamentally similar to Section 3.1. Any given pair of Radon-Nikodym derivatives may not be simultaneously achievable for both potential outcomes. However, the pair corresponding to the worst-case bounds are simultaneously achievable. As a result, the identified set is the same whether we bound manipulation on one potential outcome or both potential outcomes simultaneously.

3.3 Instrumental Variables

We consider bounds on local average treatment effects (LATEs) in instrumental variable models under restrictions on violations of the exclusion restriction. Our previous examples correspond to forms of selection on unobservables. We now show our framework also applies to bounds on local average treatment effects (LATEs) in instrumental variable models under restrictions on violations of the exclusion restriction.

Consider a canonical IV setup. We assume there is a distribution \mathbb{P}^{True} over $(Z, \{T(z)\}, \{Y(t, z)\}, X)$, where Z is a binary instrument, $Y(t, z)$ is the potential outcome with treatment status t and instrument status z , and $T(z)$ is the potential treatment given the instrument z is $T(z)$. However, we only observe the coarsening \mathbb{P}^{Obs} over $(Z, T = T(z), Y = Y(T, Z), X)$. We will maintain that the instrument is randomly assigned ($Y(t, z) \perp\!\!\!\perp Z | X$) and monotonicity holds ($T(1) \geq T(0)$). Under monotonicity, there are three treatment response groups: always-takers (At, $T(1) = T(0) = 1$), never-takers (Nt, $T(1) = T(0) = 0$), and compliers (Co, $1 = T(1) > T(0) = 0$). The regressors are $R = (T, Z, X)$.

We will not impose the exclusion restriction. If exclusion holds, then $Y(t, 1) = Y(t, 0)$. If exclusion fails, then the standard conditional IV estimand, $(E[Y | Z = 1, X] - E[Y | Z = 0, X]) / (E[T | Z = 1, X] - E[T | Z = 0, X])$, will incorrectly assign any direct effect of the instrument on outcomes to treatment effects.

We target an instrument-weighted average LATE given by:

$$\psi := E_{\mathbb{P}^{\text{True}}} \left[\eta(X) \mathbf{1}\{Co\} \sum_z \omega(z | X) (Y(1, z) - Y(0, z)) \right]$$

where $\eta(X)$ reflects covariate weighting and $\omega(1 | X) = 1 - \omega(0 | X)$ is a researcher-estimated instrument weighting function in $[0, 1]$. When exclusion holds, this class includes the average complier treatment effect for $\eta(X) = 1 / \mathbb{P}^{\text{Obs}}(Co)$. When exclusion fails, this specification also allows the researcher to target a particular weighted average of treatment effects across instrument statuses. For example, $\omega(Z | X) = 1$ targets the treatment effect at $Z = 1$ and $\omega(Z | X) = E[Z | X]$ targets the treatment effect at the average observed instrument status.

We rewrite the causal estimand ψ in terms of conditional expected outcomes as follows:

$$\psi = E_{\mathbb{P}^{\text{True}}} \left[\lambda(R) E_{\mathbb{P}^{\text{True}}} \left[\sum_z \omega(z | X) Y(T, z) | X, T(1), T(0) \right] \right],$$

where $\lambda(Z, X, T) = \eta(X) \left(\frac{Z - \mathbb{P}^{\text{Obs}}(Z=1|X)}{\mathbb{P}^{\text{Obs}}(Z=1|X)\mathbb{P}^{\text{Obs}}(Z=0|X)} \right)$. Always-takers and never-takers are included in this statement of ψ for convenience, but their potential outcomes cancel out for $\lambda(R)$ because $E[\lambda(R) | T(1), T(0), X] = 0$.

The target distribution $\mathbb{P}^{\text{Target}}$ is constructed as a marginal distribution. For $v \in \{0, 1\}$ and $\mathcal{Y} \subset \mathbb{R}$, define the distribution $\mathbb{P}^{\text{Target}}$, which re-draws a value v from a $\text{Bernoulli}(\omega(v | X))$ distribution in order to achieve appropriate outcome weighting, as follows:

$$\mathbb{P}^{\text{Target}}(X, T, Z, Y) \equiv \sum_{v, t_1, t_0 \in \{0, 1\}} \mathbb{P}^{\text{True}}(X, t, Z, T(1) = t_1, T(0) = t_0) \omega(v | X) \mathbb{P}^{\text{True}}(Y(T, v) \in \mathcal{Y} | X, T(1), T(0)).$$

The distribution of R is the same under $\mathbb{P}^{\text{Target}}$ and \mathbb{P}^{Obs} . Further, by [Abadie \(2003\)](#)'s argument, the distribution satisfies $E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y] = E_{\mathbb{P}^{\text{True}}}[\lambda(R)E_{\mathbb{P}^{\text{True}}}[Y(T, Z) | X, Co]] = \psi$, so that $E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y]$ is the causal estimand.

Suppose we have a structural model that implies there are functions $\ell_{Nt}(X), u_{Nt}(X)$, $\ell_{At}(X), u_{At}(X)$, $\ell_{Co}^1(X), u_{Co}^1(X)$, $\ell_{Co}^0(X), u_{Co}^0(X)$ such that:

$$\begin{aligned} \ell_{Nt}(X) &\leq \frac{d\mathbb{P}^{\text{True}}(Y(0, 0) | Nt, X)}{d\mathbb{P}^{\text{True}}(Y(0, 1) | Nt, X)} \leq u_{Nt}(X), & \ell_{At}(X) &\leq \frac{d\mathbb{P}^{\text{True}}(Y(1, 1) | At, X)}{d\mathbb{P}^{\text{True}}(Y(1, 0) | At, X)} \leq u_{At}(X), \\ \ell_{Co}^1(X) &\leq \frac{d\mathbb{P}^{\text{True}}(Y(1, 0) | Co, X)}{d\mathbb{P}^{\text{True}}(Y(1, 1) | Co, X)} \leq u_{Co}^1(X), & \ell_{Co}^0(X) &\leq \frac{d\mathbb{P}^{\text{True}}(Y(0, 1) | Co, X)}{d\mathbb{P}^{\text{True}}(Y(0, 0) | Co, X)} \leq u_{Co}^0(X), \end{aligned}$$

and suppose further that all four Radon-Nikodym derivatives are finite and strictly positive. Exclusion corresponds to the case $\ell = u = 1$. Worst-case bounds that only restrict the support of the potential outcomes correspond to $\ell = 0, u = \infty$. When Y is binary, [Ramsahai \(2012\)](#) proposes a sensitivity analysis that places a bound on $\mathbb{P}^{\text{True}}(Y = 1 | X, T, Z = 1, U) - \mathbb{P}^{\text{True}}(Y = 1 | X, T, Z = 0, U)$ for some unobserved U , which immediately translates to bounds on always- and never-taker likelihood ratios by taking $U = \{T(1), T(0)\}$ and implies bounds on complier likelihood ratios. This object can be decomposed as a convolution of several of the probability objects above, so its interpretation is less transparent in a causal framework with potential treatments. Alternatively, Λ bounds on the effect of Z on the odds of a binary potential outcome given X and potential treatments imply Λ bounds on the odds ratios here, though narrower partially identified regions could be obtained by leveraging estimated regression functions.

As we show in Appendix Lemma 10, these objects further imply bounds on $\frac{d\mathbb{P}^{\text{True}}(Y(T, Z) | Co, X)}{d\mathbb{P}^{\text{Obs}}(Y | T, Z, X)}$ for each value of T and Z .

Proposition 4. Suppose we write the implied bounds from the worst-case ℓ and u applied to the formulas from Appendix Lemma 10 as

$$\underline{w}(t, z | X) \leq \frac{d\mathbb{P}^{\text{Target}}(Y | X, T = t, Z = z)}{d\mathbb{P}^{\text{Obs}}(Y | X, T = t, Z = z)} \leq \bar{w}(t, z | X),$$

where we write $\underline{w}(t, z | X) = \bar{w}(t, z | X) = 1$ for values such that $\mathbb{P}^{\text{Obs}}(T = t, Z = z | X) = 0$. Then our method

can be implemented on $E_{\mathbb{P}^{\text{True}}}[\eta(X)1\{Co\}\sum_z \omega(z | X)(Y(1, z) - Y(0, z))]$ with the following values:

$$\lambda(R) = \eta(X) \frac{Z - \mathbb{P}^{\text{Obs}}(Z = 1 | X)}{\mathbb{P}^{\text{Obs}}(Z = 1 | X)\mathbb{P}^{\text{Obs}}(Z = 0 | X)}, \quad \underline{w}(R) = \underline{w}(T, Z | X), \quad \bar{w}(R) = \bar{w}(T, Z | X).$$

Unlike the previous examples, the characterization in Proposition 4 is conservative.

Proposition 5. Suppose the observed distribution follows $X = 1; Z | X \sim \text{Bern}(0.5); T | Z, X \sim \text{Bern}(Z/2)$; and $Y | X, Z, T \sim \text{Unif}(-1, 1)$. Suppose we are interested in the average complier treatment effect at $z = 1$, i.e. $\eta(x) = 2$ and $\omega(z | x) = z$. Suppose a structural model implies lower bounds of $\ell(x) = 1$ and $u(x) = \infty$ for all groups. Then the structural model implies the sharp bounds are the singleton $\{0\}$, but the bounds from Proposition 4 are $[-1, 1]$.

Intuitively, the structural model implicitly includes cross-restrictions between the *Co*, *At*, and *Nt* Radon-Nikodym derivatives. In this case, the $\frac{d\mathbb{P}^{\text{Target}}(Y|T=0, Z=0)}{d\mathbb{P}^{\text{Obs}}(Y|T=0, Z=0)}$ includes a product of Radon-Nikodym derivatives for compliers and never-takers, which must satisfy further constraints that our approach omits for the purpose of ease of use.

We discuss some other applications in Appendix A.1.

4 Implementation

We now provide theoretical guarantees for consistency of a plug-in estimator. We then illustrate the effective performance of our procedure for simulations in the conditional c -dependence environment.

4.1 Estimation and Inference

This subsection shows that natural plug-in estimators achieve standard asymptotics under reasonable conditions. We focus on the upper bound for exposition. The target object is:

$$T \equiv E_{\mathbb{P}^{\text{Obs}}} [\lambda(R)Y + (\lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y|R)) a(\underline{w}(R), \bar{w}(R), \lambda(R)Y, Q_{\tau(R)}(\lambda(R)Y|R))] ,$$

where

$$a(\underline{w}, \bar{w}, \lambda y, q) \equiv (\bar{w} - \underline{w}) 1\{\lambda y > q\} - (1 - \underline{w}).$$

This is a statistical quantity that depends on the observed distribution \mathbb{P}^{Obs} alone, so we now suppress the dependence on \mathbb{P}^{Obs} for concision.

We use hatted objects to denote the estimated objects and \hat{E} to denote the sample mean.

$$\hat{T} \equiv \hat{E} [\hat{\lambda}(R)Y + (\hat{\lambda}(R)Y - \hat{Q}_{\hat{\tau}(R)}(\hat{\lambda}(R)Y|R)) a(\hat{\underline{w}}(R), \hat{\bar{w}}(R), \hat{\lambda}(R)Y, \hat{Q}_{\hat{\tau}(R)}(\hat{\lambda}(R)Y|R))]$$

Consistency follows under natural conditions. Further, we have a form of one-sided robustness.

Proposition 6. If we have iid sampling, finite second moments, and $\hat{\lambda} \xrightarrow{p} \lambda, \hat{Q} \xrightarrow{p} Q, \hat{\bar{w}} \xrightarrow{p} \bar{w}, \hat{\underline{w}} \xrightarrow{p} \underline{w}$, then $\hat{T} \xrightarrow{p} T$. Further, even if $\hat{Q} \xrightarrow{p} \bar{Q} \neq Q$, there exists some \hat{T}^* such that $\hat{T} \geq \hat{T}^*$ and $\hat{T}^* \xrightarrow{p} T$.

The first part of the proposition is immediate by applying the continuous mapping theorem and the law of large numbers for iid observations. The $a(\cdot)$ dependence on an indicator function only introduces a kink point rather than a discontinuity, so the function is still continuous. The assumptions of the proposition are

made at a high level, so that we can accommodate various forms of consistent estimators for functions such as \hat{w} . In particular, we can accommodate machine learning nuisance estimators, as we do in our simulation. Note that our consistency assumption may not be achievable when the quantiles are finite but unbounded, as at one point in our simulation.

The second part of the proposition states a useful robustness guarantee. Even if the quantile is not estimated correctly, the resulting estimated bounds will be valid: too wide for the estimated sensitivity model rather than too narrow. This validity property corresponds to Lemma 3's one-sided validity guarantee.

For inference, we use a standard percentile bootstrap in our implementation. Namely,

1. For every $b = 1, \dots, B$,
 - (a) Sample n observations from the data iid with replacement to get the bootstrap data.
 - (b) Calculate $t^{(b)} \equiv \sqrt{n}(\hat{T}^{(b)} - \hat{T})$, where $\hat{T}^{(b)}$ is the estimator that uses the bootstrapped data.
2. Let G_N denote the CDF of $t^{(b)}$ and q_α denote that α quantile of G_N . For a size $1 - \alpha$ confidence interval, use $[\hat{T} - \frac{1}{\sqrt{n}}q_{1-\alpha/2}, \hat{T} + \frac{1}{\sqrt{n}}q_{\alpha/2}] = CI(\alpha)$

The bootstrap will have coverage at least as large as nominal under standard conditions, like smoothness of the \hat{w} estimates, even if the quantile estimator tends to an inconsistent limit. The core argument is that an infeasible bootstrap estimator that replaces the estimated $1\{\hat{\lambda}^{(b)}Y > \hat{Q}^+\}$ with the true $1\{\lambda Y > Q^+\}$ in the construction of a would be valid and have weakly more aggressive confidence intervals. As a result, the quantiles do not even need to be reestimated in the bootstraps (Dorn and Guo, 2023). Further, estimation error in the quantiles exhibit a second-order influence on the estimated bounds, so that the confidence intervals can asymptotically achieve the nominal rate under moderate conditions (Dorn et al., 2022). A generic proof of bootstrap consistency is outside the scope of this paper. The presence of extreme quantiles or kink points, as in Masten and Poirier (2018)'s conditional c-dependence model, may call for more exotic bootstraps and a subtle proof of bootstrap validity.

4.2 Simulation

We illustrate the procedure using a simulation in the conditional c-dependence context of Section 3.1.

The observed distribution \mathbb{P}^{Obs} over (X, Z, Y) is $X \sim U[-\eta, \eta]$; $Z | X \sim \text{Bern}(1/(1 + \exp(-X)))$; and $Y | X, Z \sim \mathcal{N}((2 + X)(Z - 1), 1)$, where η is chosen so that the support of $e(X)$ is $[0.1, 0.9]$.

We consider estimation at $c = 0, 0.01, \dots, 0.1$. When $c > 0.1$, the identified set is unbounded. Conversely, we show in Appendix Proposition 7 that for all $c \in [0, 0.1]$, the identified set remains uniformly bounded. Estimation for small c reduces to a fairly standard problem. As c approaches 0.1, estimation becomes more difficult and the relevant extremal quantiles introduce a “delicate problem” for estimation (Masten et al., 2024). Once c crosses 0.1, the identified set becomes the real line and Lemma 1 no longer applies. Still, we evaluate simulation performance at this extreme case by testing how often the $c = 0.1$ confidence intervals yield the true infinite bounds. We call this just-barely-infinite case $c = 0.1 + \epsilon$.

The bounds in the general program are estimated by plug-in. We estimate λ , τ , \underline{w} , and \bar{w} by plugging in propensity estimates $\hat{e}(X)$. The propensities are estimated using well-specified logistic regression of Z on X . We estimate the quantile function using quantile regression on 99 grid points ($\tau = 0.01, \dots, 0.99$) and estimate the extreme quantiles ($\tau = 0, 1$) from the known infinite bounds on the conditional outcome distribution. The quantile regression grid is estimated by regressing $\hat{\lambda}Y$ on Z interacted with both X and $\hat{\lambda}(X)$ and is re-used across values of c and bootstraps. The underlying quantile regression is estimated in two ways: linear

quantile regression and random forests with three-fold cross validation. For each observation, we take the quantile regression estimate by linearly interpolating the grid at the estimated $\hat{\tau}$.

Inference proceeds by a standard percentile bootstrap. For a given dataset, we can redraw observations with replacement. For every bootstrap draw b , we reestimate propensities via one-step updating, and reestimate the weight bounds accordingly. We then estimate bootstrap upper and lower bounds $\hat{\psi}_b^+$ and $\hat{\psi}_b^-$. We do not reestimate the quantile regression grid between bootstraps, which introduces a squared outward bias. The 95% confidence interval for the identified set is the set bounded by the 2.5th quantile of the $\hat{\psi}_b^-$ draws and the 97.5th quantile of the $\hat{\psi}_b^+$ draws. The 95% confidence intervals for the one-sided lower and upper bounds is the set bounded by the 5th quantile of the $\hat{\psi}_b^-$ draws or the 95th quantile of the $\hat{\psi}_b^+$ draws. The quantiles of the estimated bounds $\{\hat{\psi}_b^-\}, \{\hat{\psi}_b^+\}$ then form the confidence interval for the identified set.

We compare our estimates and confidence intervals to the true bounds. We obtain a close estimate of the true bounds numerically. In particular, we average the closed-form identified set for $E[Y(1) - Y(0) | X]$ over one million draws of X . With the true ψ^+ and ψ^- essentially known, we can assess the validity of our estimation and inference procedures.

For a given sensitivity parameter c , we run 1,000 simulations of the data with 2,000 observations. Within each simulation, we take 1,000 bootstrap draws and calculate the bounds for each bootstrap draw. We also evaluate the coverage rate of the $c = 0.1$ estimates for slightly larger c , in which case the true identified set is the reals. We denote the resulting coverage rates in tables as $c = 0.1 + \epsilon$.

We present our mean and median bound estimates using linear and random forest quantiles in Figures 1 and 2, respectively. In Figure 1, our median bound estimates generally track the true bounds. Our mean bound estimates roughly track the identified set until c gets close enough to 0.1 that some simulations produce infinite estimated bounds (0.2% of simulations at $c \geq 0.07$). As c gets close to 0.1 and the most extreme $\tau(R)$ values get close to one, our median estimates become slightly too wide. This phenomenon likely reflects the robustness of our characterization with respect to quantile errors, which are especially likely when applying our discrete grid to extreme \bar{w} values. In the random forest estimates represented in Figure 2, the estimates are somewhat more conservative, but still track the true bounds reasonably closely. Since we have a linear parametric model, we would expect the parametric quantile regression to be more accurate than a non-parametric method and to produce narrower bounds.

Our coverage results are reported in Table 1. Under nominal coverage, the coverage rate would be 93.6% to 96.3% with 95% probability. With linear quantile regression, the coverage rate is close to nominal for small and moderately sized values of c . There is some over-coverage as c approaches 0.1. When $c = 0.1$, the identified set rests on a knife edge: as c approaches 0.1 from below, the lower and upper bounds tend towards 1.5 and 2.5, but for any c above 0.1, the identified set is the full real line. We find that in this case the confidence intervals cover the true (finite) identified set in 98.7% of simulations and cover the near (infinite) identified set in 77.3% of cases of simulations. (At $c = 0.10$, 66.4% of bound estimates are unbounded.) The bounds with quantile forest estimates remain valid, but tend to be more conservative. As before, this phenomenon can be attributed to well-specified parametric quantile regression being more accurate and generating narrower bounds that produce coverage closer to the nominal rates.

5 Conclusion

This paper proposes a novel sensitivity analysis framework for identification failures for linear estimators. By placing bounds on the distributional distance between the observed distribution and a target distribution that

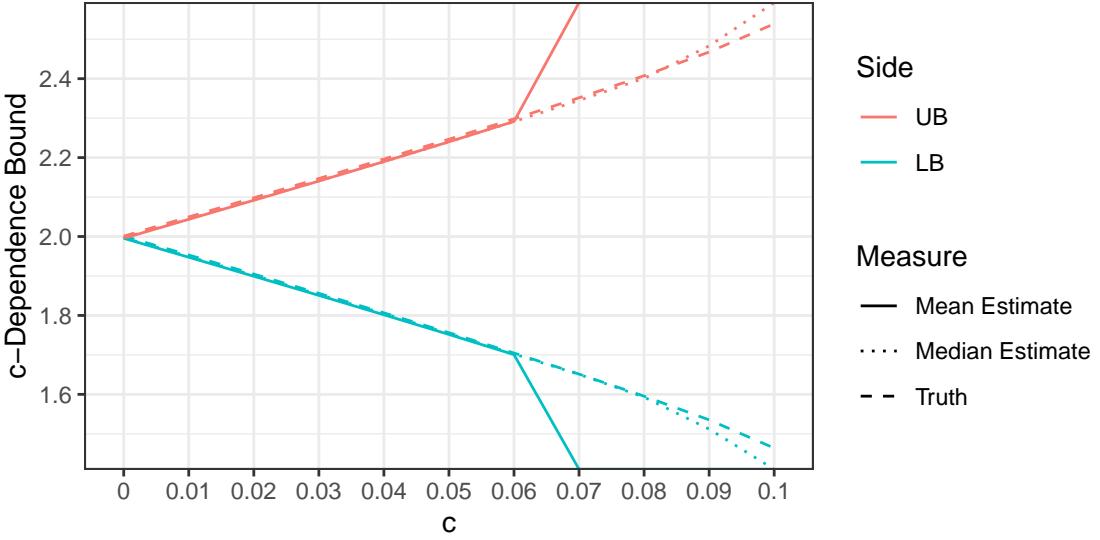


Figure 1: Bounds using linear quantile regression. Average bound estimates (solid line), median bound estimates (dotted line), and true bounds (dashed line) in our 1,000 simulations. Our estimates are generally close to median unbiased. Once $c \geq 0.07$, some simulations have infinite estimated identified sets.

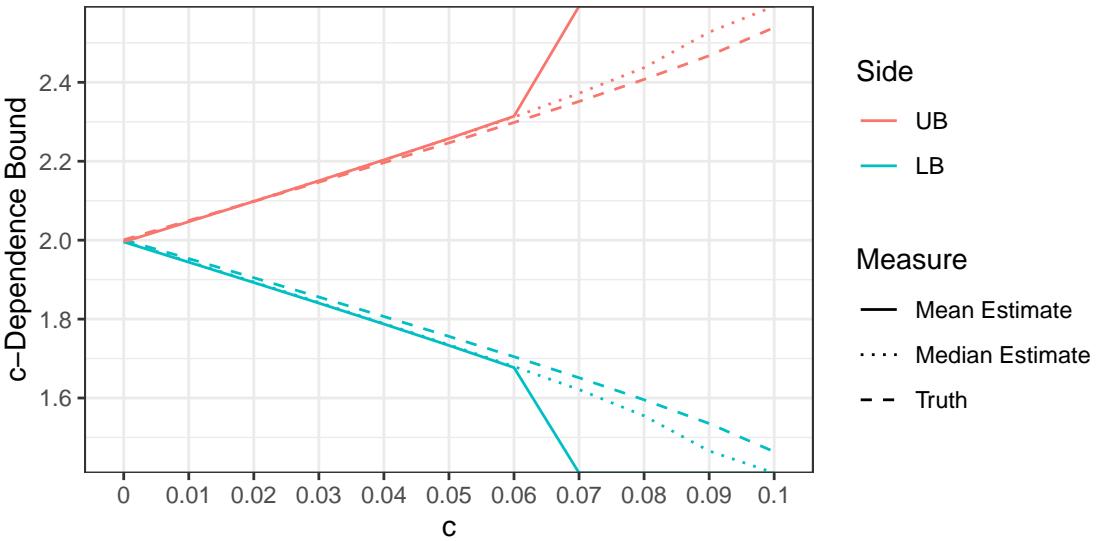


Figure 2: Bounds using random forest. Average bound estimates (solid line), median bound estimates (dotted line), and true bounds (dashed line) in our 1,000 simulations. Our estimates are generally close to median unbiased. Once $c \geq 0.07$, some simulations have infinite estimated identified sets.

Table 1: Coverage for 95% confidence set using linear quantile regression (left) and forest quantile regression (right)

c	CI Coverage (Target 95%)					
	Linear RQ			Forest RQ		
	Set	LB	UB	Set	LB	UB
0	94.2	96.6	92.4	94.2	96.6	92.4
0.01	94.0	96.7	92.6	94.6	96.8	93.1
0.02	94.3	96.6	92.8	95.2	97.1	93.8
0.03	94.4	96.7	92.8	96.0	97.3	94.4
0.04	94.5	96.7	92.9	96.7	97.5	95.3
0.05	94.5	96.7	92.9	97.7	97.6	96.7
0.06	94.7	96.9	93.5	98.1	97.7	97.4
0.07	95.9	96.7	94.7	98.7	98.1	98.7
0.08	97.5	96.5	96.5	99.2	98.5	99.1
0.09	98.1	96.7	97.4	99.5	99.1	99.5
0.1	98.7	97.7	98.7	99.8	99.5	99.8
$0.1 + \epsilon$	77.3	74.0	84.1	81.9	77.3	93.3

CI coverage denotes the percentage of simulations in which the 95% identified set confidence interval includes the true identified set and in which one-sided 95% lower and upper bound confidence intervals contain the true bounds.

identifies the causal parameter of interest, we obtain sharp and tractable analytic bounds. This framework generalizes existing sensitivity models in RD and IPW and motivates a new sensitivity model for IV exclusion failures. We provide new results on sharp and valid sensitivity analysis that allow even unbounded likelihood ratios. We illustrate how our framework and partial identification results contribute to three important applications, including new procedures for sensitivity analysis for the CATE under RD with manipulation and for instrumental variables with exclusion.

References

- ABADIE, A. (2003): “Semiparametric instrumental variable estimation of treatment response models,” *Journal of econometrics*, 113, 231–263.
- ARONOW, P. M. AND D. K. K. LEE (2013): “Interval estimation of population means under unknown but bounded probabilities of sample selection,” *Biometrika*, 100, 235–240.
- BASIT, M. A., M. A. H. M. LATIF, AND A. S. WAHED (2023): “A Risk-Ratio-Based Marginal Sensitivity Model for Causal Effects in Observational Studies,” .
- BERTSIMAS, D., K. IMAI, AND M. L. LI (2022): “Distributionally robust causal inference with observational data,” *arXiv preprint arXiv:2210.08326*.
- BUGNI, F. A. AND I. A. CANAY (2021): “Testing continuity of a density via g-order statistics in the regression discontinuity design,” *Journal of Econometrics*, 221, 138–159.
- CHRISTENSEN, T. AND B. CONNAULT (2023): “Counterfactual Sensitivity and Robustness,” *Econometrica*, 91, 263–298.

- CINELLI, C. AND C. HAZLETT (2020): “Making sense of sensitivity: Extending omitted variable bias,” *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 82, 39–67.
- DORN, J. AND K. GUO (2023): “Sharp sensitivity analysis for inverse propensity weighting via quantile balancing,” *Journal of the American Statistical Association*, 118, 2645–2657.
- DORN, J., K. GUO, AND N. KALLUS (2022): “Doubly-Valid/Doubly-Sharp Sensitivity Analysis for Causal Inference with Unmeasured Confounding.”
- FRAUEN, D., V. MELNYCHUK, AND S. FEUERRIEGEL (2023): “Sharp Bounds for Generalized Causal Sensitivity Analysis,” *arXiv preprint arXiv:2305.16988*.
- FREIDLING, T. AND Q. ZHAO (2022): “Sensitivity Analysis with the R^2 -calculus,” *arXiv preprint arXiv:2301.00040*.
- GERARD, F., M. ROKKANEN, AND C. ROTHE (2020): “Bounds on treatment effects in regression discontinuity designs with a manipulated running variable,” *Quantitative Economics*, 11, 839–870.
- HAHN, J., P. TODD, AND W. VAN DER KLAUW (2001): “Identification and estimation of treatment effects with a regression-discontinuity design,” *Econometrica*, 69, 201–209.
- HO, K. AND A. M. ROSEN (2017): “Partial Identification in Applied Research: Benefits and Challenges,” in *Advances in Economics and Econometrics: Eleventh World Congress*, ed. by B. Honoré, A. Pakes, M. Piazzesi, and L. Samuelson, Cambridge University Press, Econometric Society Monographs, 307–359.
- ISHIHARA, T. AND M. SAWADA (2020): “Manipulation-Robust Regression Discontinuity Designs,” *arXiv preprint arXiv:2009.07551*.
- MANSKI, C. F. (1990): “Nonparametric Bounds on Treatment Effects,” *American Economic Review: Papers and Proceedings*, 80, 319–323.
- MASTEN, M. A. AND A. POIRIER (2018): “Identification of treatment effects under conditional partial independence,” *Econometrica*, 86, 317–351.
- (2021): “Salvaging falsified instrumental variable models,” *Econometrica*, 89, 1449–1469.
- MASTEN, M. A., A. POIRIER, AND L. ZHANG (2024): “Assessing Sensitivity to Unconfoundedness: Estimation and Inference,” *Journal of Business & Economic Statistics*, 42, 1–13.
- MCCRARY, J. (2008): “Manipulation of the running variable in the regression discontinuity design: A density test,” *Journal of econometrics*, 142, 698–714.
- MOLINARI, F. (2020): “Chapter 5 - Microeconometrics with partial identification,” in *Handbook of Econometrics, Volume 7A*, ed. by S. N. Durlauf, L. P. Hansen, J. J. Heckman, and R. L. Matzkin, Elsevier, vol. 7 of *Handbook of Econometrics*, 355–486.
- OTSU, T., K.-L. XU, AND Y. MATSUSHITA (2013): “Estimation and inference of discontinuity in density,” *Journal of Business & Economic Statistics*, 31, 507–524.
- PINELIS, I. (2019): “Exact bounds on the inverse Mills ratio and its derivatives,” *Complex Analysis and Operator Theory*, 13, 1643–1651.

- RAMBACHAN, A. AND J. ROTH (2023): “A more credible approach to parallel trends,” *Review of Economic Studies*, rdad018.
- RAMSAHAI, R. R. (2012): “Causal Bounds and Observable Constraints for Non-deterministic Models.” *Journal of Machine Learning Research*, 13.
- ROSENMAN, E., K. RAJKUMAR, R. GAURIOT, AND R. SLONIM (2019): “Optimized partial identification bounds for regression discontinuity designs with manipulation,” *arXiv preprint arXiv:1910.02170*.
- TAN, Z. (2006): “A distributional approach for causal inference using propensity scores,” *Journal of the American Statistical Association*, 101, 1619–1637.
- (2022): “Model-assisted sensitivity analysis for treatment effects under unmeasured confounding via regularized calibrated estimation,” .
- (2023): “Sensitivity models and bounds under sequential unmeasured confounding in longitudinal studies,” *arXiv preprint arXiv:2308.15725*.
- VAN KIPPERSLUIS, H. AND C. A. RIETVELD (2018): “Beyond plausibly exogenous,” *The Econometrics Journal*, 21, 316–331.
- ZHAO, Q., D. S. SMALL, AND B. B. BHATTACHARYA (2019): “Sensitivity analysis for inverse probability weighting estimators via the percentile bootstrap,” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 81, 735–761.

A Additional Material

Assumption RD. We extend Gerard et al.’s continuity condition and assume that there is a well-defined conditional distribution function $\mathbb{P}^{\text{True}}(Y(1), Y(0) | X, M)$ such that for each y_1, y_0 , $\mathbb{P}^{\text{True}}(Y(1) \leq y_1, Y(0) \leq y_0 | X = x, M = 0)$ can be defined as continuous in x and that the derivative of $\mathbb{P}^{\text{True}}(X \leq x | M = 0)$ is continuous at $x = c$. We further assume that $\mathbb{P}^{\text{True}}(Y(y) \leq y_1, Y(0) \leq y_0 | X = x, M = 1)$ is right-continuous x , assume that $\mathbb{P}^{\text{True}}(M = 1 | X = x)$ is right-continuous at $x = c$, assume that marginals $\mathbb{P}^{\text{True}}(Y(t) \leq y_t | X = x, M = m)$ are right-continuous for $m = 1$ (continuous for $m = 0$) at $X = c$ and define conditional distributions at $X = c$, $X = c^+$, and $X = c^-$ using the appropriate limits in Definition 5.

Definition 5. We define $\mathbb{P}^{\text{True}}(Y(1) \leq y_1, Y(0) \leq y_0, T(0) = t, M = m | X = c) \equiv \mathbb{P}^{\text{True}}(X = c^+ | X = c) \lim_{x \rightarrow c^+} \mathbb{P}^{\text{True}}(Y(1) \leq y_1, Y(0) \leq y_0 | X = x, M = m)/2$, where $\mathbb{P}^{\text{True}}(X = c^+ | X = c) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{P}^{\text{Obs}}(T = 1 | |X - c| \leq \varepsilon)$, $T(0)$ is the potential treatment corresponding to $X(0)$, and the conditional distribution function at $X = c, M = m$ exists by appropriate right-continuity. We refer to conditional probabilities on $\mathbb{P}^{\text{True}}(\cdot | X = c, \cdot)$ that are defined in terms of this conditional distribution. We refer to $\mathbb{P}^{\text{True}}(\cdot | X = c^-, \cdot)$ and $\mathbb{P}^{\text{True}}(\cdot | X = c^+, \cdot)$ where the random variables for $X = c^\pm$ are defined as $(X = c^-) \equiv X = c, M = 0, T(0) = 0$ and $(X = c^+) \equiv X = c, (X = c^-) = 0$.

The following claim shows that the conditional c -dependence identified set in our implementation example is finite and bounded for all $c \leq 0.1$ but is infinite for all $c > 0.1$.

Proposition 7. Suppose $Y | X, Z \sim \mathcal{N}(\mu(X, Z), \sigma(X, Z)^2)$, the support of the observed propensity function $e(X)$ is the closed interval $[\eta_1, 1 - \eta_2] \subset (0, 1)$, and the conditional outcome variance $\sigma(X, Z)$ is positive and bounded. Then there is a finite $B > 0$ such that the ATE identified set is a subset of $[E[\mu(X, 1) - \mu(X, 0)] - B, E[\mu(X, 1) - \mu(X, 0)] + B]$ for all $c < \min\{\eta_1, \eta_2\}$ but is $(-\infty, \infty)$ for all $c > \min\{\eta_1, \eta_2\}$.

A.1 Some Other Applications

This subsection briefly mentions several other potential applications that we do not work out in detail.

Our framework also applies to instrumental variables that are not valid due to the failure of randomization in the binary instrument Z . If there is selection in Z , then the analysis proceeds very similarly to Section 3.1.

Our procedure can be applied somewhat trivially to linear projections using Ordinary Least Squares (OLS). Consider a hypothetical linear model $Y = X\beta + u$. Suppose we are interested in a linear combination of coefficients $\delta'\beta$, where X includes an intercept but δ puts no weight on the intercept term, so that without loss of generality we can assume $E_{\mathbb{P}^{\text{True}}}[u] = 0$. However, suppose there may be endogeneity in the sense that $E[Xu] \neq 0$. Such problems are considered in [Cinelli and Hazlett \(2020\)](#). If we targeted a distribution $\mathbb{P}^{\text{Target}}$ that first sampled $X \sim \mathbb{P}^{\text{Obs}}$, then drew $u | X$ from the distribution of u under \mathbb{P}^{True} , and then returned $Y = X\beta + u$, then the target distribution would obtain the correct coefficients. The sensitivity assumption is then on $\frac{d\mathbb{P}^{\text{Target}}}{d\mathbb{P}^{\text{Obs}}}$. For instance, we may have $\underline{w} \leq \frac{d\mathbb{P}^{\text{Target}}}{d\mathbb{P}^{\text{Obs}}} \leq \bar{w}$. Using the notation of our general framework, $\lambda(R) = \delta'E[X'X]^{-1}X$. While the bounds are sharp, the sensitivity assumption need not be interpretable in general.

When viewing difference-in-differences (DD) estimands as a special case of linear projections, our procedure also has the potential to speak to sensitivity analysis to the failure of parallel trends. There are existing ways to do sensitivity analysis to the failure of parallel trends. For example, [Rambachan and Roth \(2023\)](#) places an explicit assumption on the extent that the slopes are not parallel. In contrast, our framework can use a bound on the likelihood ratio of potential outcomes between the treated and untreated; equivalently, this approach can use a bound on the degree of selection on potential outcomes or trends. [Bertsimas et al. \(2022\)](#) propose similar distributional distance bounds without including the selection component. This approach is then invariant to units used in the regression.

B Proofs

B.1 Lemmas

Lemma 5. Suppose Assumption [Moments](#) holds and the expected values are finite. Then the expected values of $FQ^+(R)\bar{w}(R)G^+$ and $FQ^-(R)\bar{w}(R)G^-$ exist, where $F\bar{w}(R)$ is evaluated as zero when $F = 0$.

Proof of Lemma 5. Note that, for $G \in \{G^+, G^-\}$, $0 \leq E[\bar{w}G | R] \leq 1$ almost surely and $\bar{w}G$ is non-negative. The remaining result holds by Fubini's theorem. \square

Lemma 6. Bounds on identified set. Consider the general identification setting and suppose $Y | R \sim \mathcal{N}(\mu(R), \sigma(R)^2)$. Then for all $\epsilon \in (0, 1)$, the identified set is a subset of

$$\left[E[\lambda(R)Y] \pm E \left[\sigma(R)\lambda(R)(1 - \underline{w}(R)) \left(\sqrt{2\log(\bar{w}(R))} + \sqrt{2/\pi} + (1 - \underline{w}(R))^{\epsilon} \sqrt{1/(e * \epsilon)} \right) \right] \right],$$

where $[a \pm b]$ denotes the closed interval $[a - b, a + b]$ and \log is the natural logarithm.

Proof of Lemma 6. We show that the upper bound is at most $E[\lambda(R)\mu(R)]$ plus one-half the proposed width; the lower bound follows symmetrically.

Note that as we argue in Theorem 1, the upper bound for the identified set can be written as:

$$\psi^+ = E_{\mathbb{P}^{\text{Obs}}} \left[\underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R))CVaR_{\tau(R)}^+(R) \right].$$

In the Normal-residual case, we can write $CVaR_{\tau(R)}^+(R) = \lambda(R)\mu(R) + \lambda(R)\sigma(R)\frac{\phi(q_{\tau(R)})}{1-\tau(R)}$, where q_τ is the τ^{th} quantile of a standard normal distribution and ϕ is the standard normal CDF. By existing arguments (e.g. Pinelis (2019)), the inverse Mills ratio $\phi(q)/(1 - \Phi(q))$ has the upper bound $\sqrt{2/\pi} + q$.

Therefore the APO upper bound can be further bounded as:

$$\begin{aligned} \psi^+ &= E_{\mathbb{P}^{\text{Obs}}} \left[\lambda(R)\mu(R) + (1 - \underline{w}(R))\sigma(R)\lambda(R)\frac{\phi(q_{\tau(R)})}{1-\tau(R)} \right] \\ &\leq E[\lambda(R)\mu(R)] + E \left[(1 - \underline{w}(R))\sigma(R)\lambda(R) \left(\sqrt{2/\pi} + q_{\tau(R)} \right) \right]. \end{aligned}$$

It remains to bound $q_{\tau(R)}$. By standard arguments, if $S \sim N(0, 1)$, then $P(S > s) \leq \exp(-s^2/2)$. We substitute $s = q_{\tau(R)}$ to obtain:

$$\begin{aligned} 1 - \tau(R) &= P(S > q_{\tau(R)}) \leq \exp(-q_{\tau(R)}^2/2) \\ \log(1 - \tau(R)) &\leq -q_{\tau(R)}^2/2 \\ \sqrt{\log \left(\frac{1}{1 - \tau(R)^2} \right)} &\geq q_{\tau(R)}. \end{aligned}$$

Therefore we have bounded the identified set as:

$$\psi^+ \leq E \left[\lambda(R)\mu(R) + (1 - \underline{w}(R))\sigma(R)\lambda(R) \left(\sqrt{2/\pi} + \sqrt{2(1 - \underline{w}(R))^2 \log \left(\frac{1}{1 - \tau(R)} \right)} \right) \right].$$

Now we bound the second square root, using the identity:

$$\frac{1}{1 - \tau(R)} = \frac{\bar{w}(R) - \underline{w}(R)}{1 - \underline{w}(R)} = 1 + \frac{\bar{w}(R) - 1}{1 - \underline{w}(R)}.$$

Therefore:

$$\begin{aligned} 2(1 - \underline{w}(R))^2 \log \left(\frac{1}{1 - \tau(R)} \right) &= 2(1 - \underline{w}(R))^2 \log(\bar{w}(R) - \underline{w}(R)) - 2(1 - \underline{w}(R))^{2-2\epsilon} (1 - \underline{w}(R))^{2\epsilon} \log(1 - \underline{w}(R)) \\ &\leq 2(1 - \underline{w}(R))^2 \log(\bar{w}(R)) + \frac{(1 - \underline{w}(R))^{2-2\epsilon}}{\epsilon * \epsilon}. \end{aligned}$$

So that we now have the bound:

$$\begin{aligned} \psi^+ &\leq E[\lambda(R)\mu(R)] + E \left[\sigma(R)\lambda(R) \left((1 - \underline{w}(R))\sqrt{2/\pi} + \sqrt{2(1 - \underline{w}(R))^2 \log(\bar{w}(R)) + \frac{(1 - \underline{w}(R))^{2-2\epsilon}}{\epsilon * \epsilon}} \right) \right] \\ &\leq E[\lambda(R)\mu(R)] + E \left[\sigma(R)\lambda(R)(1 - \underline{w}(R)) \left(\sqrt{2 \log(\bar{w}(R))} + \sqrt{2/\pi} + (1 - \underline{w}(R))^\epsilon \sqrt{1/(e * \epsilon)} \right) \right]. \end{aligned}$$

Applying the same argument to the symmetric lower bound completes the proof. \square

Lemma 7. Recall the definitions $\tau(R) = \frac{\bar{w}(R)-1}{\bar{w}(R)-\underline{w}(R)}$ and $a(\underline{w}, \bar{w}, s) = (\bar{w} - \underline{w})1\{s > 0\} - (1 - \underline{w})$. We may instead write:

$$a(\underline{w}, \bar{w}, s) = (1 - \underline{w}) \left(\frac{1\{s > 0\}}{1 - \tau} - 1 \right).$$

Proof of Lemma 7. Simple algebra. \square

Lemma 8. Consider the Regression Discontinuity application in Section 3.2. Suppose $F_{X|M=0}(x)$ is differentiable in x at c with a positive derivative. Then $\mathbb{P}^{\text{Obs}}(X = c^+ | X = c, M = 0) = 1/2$.

Proof of Lemma 8. Define $f_{x|M=0}(c) > 0$ to be the derivative of $F_{X|M=0}(x)$ at c . Then we have:

$$\begin{aligned} \mathbb{P}^{\text{Obs}}(X = c^+ | X = c, M = 0) &= \mathbb{P}^{\text{Target}}(X = c^+ | X = c, M = 0) \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{F_{X|M=0}(c + \varepsilon) - F_{X|M=0}(c)}{F_{X|M=0}(c + \varepsilon) - F_{X|M=0}(c - \varepsilon)} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{F_{X|M=0}(c + \varepsilon) - F_{X|M=0}(c)}{F_{X|M=0}(c + \varepsilon) - F_{X|M=0}(c) + F_{X|M=0}(c) - F_{X|M=0}(c - \varepsilon)} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{\frac{F_{X|M=0}(c + \varepsilon) - F_{X|M=0}(c)}{\varepsilon}}{\frac{F_{X|M=0}(c + \varepsilon) - F_{X|M=0}(c)}{\varepsilon} + \frac{F_{X|M=0}(c) - F_{X|M=0}(c - \varepsilon)}{\varepsilon}} \\ &= \frac{\lim_{\varepsilon \rightarrow 0^+} \frac{F_{X|M=0}(c + \varepsilon) - F_{X|M=0}(c)}{\varepsilon}}{\lim_{\varepsilon \rightarrow 0^+} \frac{F_{X|M=0}(c + \varepsilon) - F_{X|M=0}(c)}{\varepsilon} + \lim_{\varepsilon \rightarrow 0^-} \frac{F_{X|M=0}(c + \varepsilon) - F_{X|M=0}(c)}{\varepsilon}} \\ &= \frac{f_{x|M=0}(c)}{2f_{x|M=0}(c)} = 1/2 \end{aligned}$$

\square

Lemma 9. Consider the Regression Discontinuity application in Section 3.2. Suppose $\mathbb{Q} \in \mathcal{M}(\infty, \infty)$. Further suppose that the distribution of $(Y(1), Y(0), T(0), M) | X = c$ under \mathbb{Q} has associated manipulation selection functions $q_1(y_1) \equiv \mathbb{Q}(M = 1 | Y(1) = y_1, X = c)$ and $q_0(y_0) \equiv \mathbb{Q}(M = 1 | Y(0) = y_0, X = c)$. Then the Radon-Nikodym derivatives are as follows:

$$\begin{aligned} \frac{d\mathbb{Q}(Y(1) | X = c, M = 0)}{d\mathbb{P}^{\text{Obs}}(Y | X = c^+)} &= \frac{1}{1 - \eta} \frac{1 - q_1(Y(1))}{1 + q_1(Y(1))} \\ \frac{d\mathbb{Q}(Y(0) | X = c, M = 1)}{d\mathbb{P}^{\text{Obs}}(Y | X = c^-)} &= \frac{2(1 - \eta)}{\eta} \frac{q_0(Y(0))}{1 - q_0(Y(0))}. \end{aligned}$$

As a result:

$$E_{\mathbb{P}^{\text{True}}} [Y(1) | X = c, M = 0] = \frac{1}{1 - \eta} E_{\mathbb{P}^{\text{Obs}}} \left[Y \frac{1 - q_1(Y)}{1 + q_1(Y)} \frac{1\{X = c^+\}}{P(X = c^+)} \right] \quad (7)$$

$$E_{\mathbb{P}^{\text{True}}} [Y(0) | X = c, M = 1] = \frac{2(1 - \eta)}{\eta} E_{\mathbb{P}^{\text{Obs}}} \left[Y \frac{q_0(Y)}{1 - q_0(Y)} \frac{1\{X = c^-\}}{P(X = c^-)} \right]. \quad (8)$$

Proof of Lemma 9. For our proofs, it is useful to use η_0 instead. We define:

$$\eta_0 \equiv \mathbb{P}^{\text{Target}}(M = 1 | X = c) = \mathbb{P}^{\text{Obs}}(X = c^+ | X = c)\eta.$$

Notice that $\mathbb{P}^{\text{Obs}}(X = c^+ | X = c) = 1/(2 - \eta)$, because $\mathbb{P}^{\text{Obs}}(X = c^- | X = c) = \mathbb{P}^{\text{Obs}}(M = 0 | X = c)/2 =$

$(1 - \eta)\mathbb{P}^{\text{Obs}}(X = c^+ | X = c)$. As a result, $\eta_0 = \eta/(2 - \eta)$, $\eta = 2\eta_0/(1 + \eta_0)$, and $\mathbb{P}^{\text{Obs}}(X = c^+ | X = c) = (1 + \eta_0)/2$.

Also notice that in Equation (5), the constraints on manipulation probabilities are defined relative to:

$$\frac{\mathbb{P}^{\text{Obs}}(M = 1 | X = c)}{\mathbb{P}^{\text{Obs}}(M = 0 | X = c)} = \frac{\eta_0}{1 - \eta_0} = \frac{\eta}{2(1 - \eta)}. \quad (9)$$

We begin with $Y(1)$. Since \mathbb{Q} produces the observable distribution $\mathbb{P}^{\text{Obs}}(X = c^+ | X = c)d\mathbb{P}^{\text{Obs}}(Y | X = c^+)$ and $\mathbb{P}^{\text{Obs}}(X = c^+ | X = c) = (1 + \eta_0)/2$, we have:

$$\begin{aligned} \mathbb{Q}(X = c^+ | X = c)d\mathbb{Q}(Y(1) | X = c^+) &= d\mathbb{Q}(Y(1) | X = c)\mathbb{Q}(X = c^+ | X = c, Y(1)) \\ &= d\mathbb{Q}(Y(1) | X = c)(1 + q_1(Y(1))) / 2 \\ d\mathbb{Q}(Y(1) | X = c) &= \frac{2\mathbb{P}^{\text{Obs}}(X = c^+ | X = c)}{1 + q_1(Y(1))}d\mathbb{Q}(Y(1) | X = c^+) \end{aligned}$$

We can also derive the probability of a treated observation being manipulated under \mathbb{Q} through Bayes' Rule:

$$\begin{aligned} \mathbb{Q}(M = 0 | Y(1), X = c^+) &= \frac{d\mathbb{Q}(Y(1) | X = c)\mathbb{Q}(M = 0 | X = c, Y(1))\mathbb{Q}(X = c^+ | X = c, Y(1), M = 0)}{\mathbb{Q}(X = c^+ | X = c)d\mathbb{Q}(Y(1) | X = c^+)} \\ &= \frac{d\mathbb{Q}(Y(1) | X = c) * (1 - q_1(Y(1))) / 2}{d\mathbb{Q}(Y(1) | X = c)(1 + q_1(Y(1))) / 2} \\ &= \frac{1 - q_1(Y(1))}{1 + q_1(Y(1))} \end{aligned}$$

As a result:

$$\begin{aligned} d\mathbb{Q}(Y(1) | X = c, M = 0) &= d\mathbb{Q}(Y(1) | X = c^+, M = 0) \\ &= \frac{d\mathbb{Q}(Y(1) | X = c^+)\mathbb{Q}(M = 0 | X = c^+, Y(1))}{\mathbb{Q}(M = 0 | X = c^+)} \\ &= \frac{\frac{1 - q_1(Y(1))}{1 + q_1(Y(1))}}{1 - \eta}d\mathbb{P}^{\text{Target}}(Y(1) | X = c^+) \end{aligned}$$

This is our first equality.

We now turn our attention to $Y(0)$. By a similar observed-untreated-outcome argument, we have:

$$\begin{aligned} \mathbb{Q}(X = c^- | X = c)d\mathbb{Q}(Y(0) | X = c^-) &= d\mathbb{Q}(Y(0) | X = c)\mathbb{Q}(X = c^- | X = c, Y(0)) \\ &= d\mathbb{Q}(Y(0) | X = c)\mathbb{Q}(M = 0 | X = c, Y(0))\mathbb{Q}(X = c^- | X = c, Y(0), M = 0) \\ &= d\mathbb{Q}(Y(0) | X = c)(1 - q_0(Y(0))) / 2 \end{aligned}$$

Since $\mathbb{Q}(X = c^- | X = c) = \mathbb{P}^{\text{Target}}(X = c^- | X = c) = \frac{1 - \eta_0}{2}$, we can then obtain:

$$d\mathbb{Q}(Y(0) | X = c) = \frac{1 - \eta_0}{1 - q_0(Y(0))}d\mathbb{Q}(Y(0) | X = c^-)$$

We can also split up $d\mathbb{Q}(Y(0) \mid X = c)$ as:

$$\begin{aligned} d\mathbb{Q}(Y(0) \mid X = c) &= \eta_0 d\mathbb{Q}(Y(0) \mid X = c, M = 1) + (1 - \eta_0) d\mathbb{Q}(Y(0) \mid X = c, M = 0) \\ &= \eta_0 d\mathbb{Q}(Y(0) \mid X = c, M = 1) + (1 - \eta_0) d\mathbb{Q}(Y(0) \mid X = c^-) \end{aligned}$$

So that we can combine terms to obtain:

$$\begin{aligned} d\mathbb{Q}(Y(0) \mid X = c, M = 1) &= \frac{1 - \eta_0}{\eta_0} \frac{q_0(Y(0))}{1 - q_0(Y(0))} d\mathbb{Q}(Y(0) \mid X = c^-) \\ \frac{1 - \eta_0}{\eta_0} &= \frac{2(1 - \eta)}{\eta} \\ d\mathbb{Q}(Y(0) \mid X = c, M = 1) &= \frac{2(1 - \eta)}{\eta} \frac{q_0(Y(0))}{1 - q_0(Y(0))} d\mathbb{Q}(Y(0) \mid X = c^-) \end{aligned}$$

Which is the final equality after substituting in $d\mathbb{Q}(Y(0) \mid X = c^-) = d\mathbb{P}^{\text{Target}}(Y(0) \mid X = c^-)$. \square

Lemma 10. *In the Instrumental Variables application in Section 3.3, the following decompositions hold:*

$$\begin{aligned} \frac{d\mathbb{P}^{\text{Target}}(Y \mid X, T = 0, Z = 0)}{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 0, Z = 0)} &= \omega(0 \mid X) + \omega(1 \mid X) \frac{\mathbb{P}^{\text{Obs}}(Nt \mid X)}{\mathbb{P}^{\text{Obs}}(Co \mid X) + \mathbb{P}^{\text{Obs}}(Nt \mid X)} \frac{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 0, Z = 1)}{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 0, Z = 0)} \\ &\quad + \omega(1 \mid X) \frac{d\mathbb{P}^{\text{True}}(Y(0, 1) \mid X, Co)}{d\mathbb{P}^{\text{True}}(Y(0, 0) \mid X, Co)} \left(1 - \frac{\mathbb{P}^{\text{Obs}}(Nt \mid X)}{\mathbb{P}^{\text{Obs}}(Co \mid X) + \mathbb{P}^{\text{Obs}}(Nt \mid X)} \frac{d\mathbb{P}^{\text{True}}(Y(0, 0) \mid X, Nt)}{d\mathbb{P}^{\text{True}}(Y(0, 1) \mid X, Nt)} \frac{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 0, Z = 1)}{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 0, Z = 0)} \right) \\ \frac{d\mathbb{P}^{\text{Target}}(Y \mid X, T = 1, Z = 1)}{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 1, Z = 1)} &= \omega(1 \mid X) + \omega(0 \mid X) \frac{\mathbb{P}^{\text{Obs}}(At \mid X)}{\mathbb{P}^{\text{Obs}}(Co \mid X) + \mathbb{P}^{\text{Obs}}(At \mid X)} \frac{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 1, Z = 0)}{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 1, Z = 1)} \\ &\quad + \omega(0 \mid X) \frac{d\mathbb{P}^{\text{True}}(Y(1, 0) \mid X, Co)}{d\mathbb{P}^{\text{True}}(Y(1, 1) \mid X, Co)} \left(1 - \frac{\mathbb{P}^{\text{Obs}}(At \mid X)}{\mathbb{P}^{\text{Obs}}(Co \mid X) + \mathbb{P}^{\text{Obs}}(At \mid X)} \frac{d\mathbb{P}^{\text{True}}(Y(1, 1) \mid X, At)}{d\mathbb{P}^{\text{True}}(Y(1, 0) \mid X, At)} \frac{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 1, Z = 0)}{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 1, Z = 1)} \right) \\ \frac{d\mathbb{P}^{\text{Target}}(Y \mid X, T = 0, Z = 1)}{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 0, Z = 1)} &= \omega(1 \mid X) + \omega(0 \mid X) \frac{d\mathbb{P}^{\text{True}}(Y(0, 0) \mid X, Nt)}{d\mathbb{P}^{\text{True}}(Y(0, 1) \mid X, Nt)} \\ \frac{d\mathbb{P}^{\text{Target}}(Y \mid X, T = 1, Z = 0)}{d\mathbb{P}^{\text{Obs}}(Y \mid X, T = 1, Z = 0)} &= \omega(0 \mid X) + \omega(1 \mid X) \frac{d\mathbb{P}^{\text{True}}(Y(1, 1) \mid X, At)}{d\mathbb{P}^{\text{True}}(Y(1, 0) \mid X, At)}, \end{aligned}$$

where for values of X with $\mathbb{P}^{\text{Obs}}(T, Z \mid X) = 0$, we write $\frac{d\mathbb{P}^{\text{Target}}(Y \mid X, T, Z)}{d\mathbb{P}^{\text{Obs}}(Y \mid X, T, Z)} = 1$.

Proof of Lemma 10. We drop conditioning on X for exposition. Observe that:

$$\begin{aligned} d\mathbb{P}^{\text{Target}}(Y \mid T = 0, Z = 0) &= \omega(1) (\mathbb{P}^{\text{Obs}}(Nt \mid T = Z = 0) d\mathbb{P}^{\text{True}}(Y(0, 1) \mid Nt) + \mathbb{P}^{\text{Obs}}(Co \mid T = Z = 0) d\mathbb{P}^{\text{True}}(Y(0, 1) \mid Co)) \\ &\quad + \omega(0) (\mathbb{P}^{\text{Obs}}(Nt \mid T = Z = 0) d\mathbb{P}^{\text{True}}(Y(0, 0) \mid Nt) + \mathbb{P}^{\text{Obs}}(Co \mid T = Z = 0) d\mathbb{P}^{\text{True}}(Y(0, 0) \mid Co)) \\ &= \omega(1) (\mathbb{P}^{\text{Obs}}(Nt \mid T = Z = 0) d\mathbb{P}^{\text{Obs}}(Y \mid T = 0, Z = 1) + \mathbb{P}^{\text{Obs}}(Co \mid T = Z = 0) d\mathbb{P}^{\text{True}}(Y(0, 1) \mid Co)) \\ &\quad + \omega(0) d\mathbb{P}^{\text{Obs}}(Y \mid T = 0, Z = 0). \end{aligned}$$

We continue to analyze the unobserved term $d\mathbb{P}^{\text{True}}(Y(0, 1) \mid Co)$, which we factor as:

$$\frac{d\mathbb{P}^{\text{True}}(Y(0, 1) \mid Co)}{d\mathbb{P}^{\text{Obs}}(Y \mid T = 0, Z = 0)} = \frac{d\mathbb{P}^{\text{True}}(Y(0, 1) \mid Co)}{d\mathbb{P}^{\text{True}}(Y(0, 0) \mid Co)} \frac{d\mathbb{P}^{\text{True}}(Y(0, 0) \mid Co)}{d\mathbb{P}^{\text{Obs}}(Y \mid T = 0, Z = 0)}$$

Note that:

$$\frac{d\mathbb{P}^{\text{True}}(Y(0,0)|Co)}{d\mathbb{P}^{\text{True}}(Y(0,0)|Nt)} = \frac{1}{\mathbb{P}^{\text{Obs}}(Co|Z=T=0)} \left(\frac{d\mathbb{P}^{\text{Obs}}(Y|T=0,Z=0)}{d\mathbb{P}^{\text{True}}(Y(0,0)|Nt)} - \mathbb{P}^{\text{Obs}}(Nt|Z=T=0) \right),$$

because

$$\frac{d\mathbb{P}^{\text{Obs}}(Y|T=0,Z=0)}{d\mathbb{P}^{\text{True}}(Y(0,0)|Nt)} = \mathbb{P}^{\text{Obs}}(Nt|Z=T=0) + \mathbb{P}^{\text{Obs}}(Co|Z=T=0) \frac{d\mathbb{P}^{\text{True}}(Y(0,0)|Co)}{d\mathbb{P}^{\text{True}}(Y(0,0)|Nt)}.$$

Combining the results,

$$\begin{aligned} & \frac{d\mathbb{P}^{\text{Target}}(Y|T=0,Z=0)}{d\mathbb{P}^{\text{Obs}}(Y|T=0,Z=0)} \\ &= \omega(0) + \omega(1)\mathbb{P}^{\text{Obs}}(Nt|Z=T=0) \frac{d\mathbb{P}^{\text{Obs}}(Y|T=0,Z=1)}{d\mathbb{P}^{\text{Obs}}(Y|T=0,Z=0)} \\ &+ \omega(1) \frac{d\mathbb{P}^{\text{True}}(Y(0,1)|Co)}{d\mathbb{P}^{\text{True}}(Y(0,0)|Co)} \left(1 - \mathbb{P}^{\text{Obs}}(Nt|Z=T=0) \frac{d\mathbb{P}^{\text{True}}(Y(0,0)|Nt)}{d\mathbb{P}^{\text{True}}(Y(0,1)|Nt)} \frac{d\mathbb{P}^{\text{Obs}}(Y|T=0,Z=1)}{d\mathbb{P}^{\text{Obs}}(Y|T=0,Z=0)} \right), \end{aligned}$$

where

$$\mathbb{P}^{\text{Obs}}(Nt|Z=T=0) = \mathbb{P}^{\text{Obs}}(Nt|T(0)=0) = \frac{\mathbb{P}^{\text{Obs}}(Nt)}{\mathbb{P}^{\text{Obs}}(Co) + \mathbb{P}^{\text{Obs}}(Nt)}.$$

Using an analogous argument,

$$\begin{aligned} & \frac{d\mathbb{P}^{\text{Target}}(Y|T=1,Z=1)}{d\mathbb{P}^{\text{Obs}}(Y|T=1,Z=1)} = \omega(1) + \omega(0) \frac{\mathbb{P}^{\text{Obs}}(At)}{\mathbb{P}^{\text{Obs}}(Co) + \mathbb{P}^{\text{Obs}}(At)} \frac{d\mathbb{P}^{\text{Obs}}(Y|T=1,Z=0)}{d\mathbb{P}^{\text{Obs}}(Y|T=1,Z=1)} \\ &+ \omega(0) \frac{d\mathbb{P}^{\text{True}}(Y(1,0)|Co)}{d\mathbb{P}^{\text{True}}(Y(1,1)|Co)} \left(1 - \frac{\mathbb{P}^{\text{Obs}}(At)}{\mathbb{P}^{\text{Obs}}(Co) + \mathbb{P}^{\text{Obs}}(At)} \frac{d\mathbb{P}^{\text{True}}(Y(1,1)|At)}{d\mathbb{P}^{\text{True}}(Y(1,0)|At)} \frac{d\mathbb{P}^{\text{Obs}}(Y|T=1,Z=0)}{d\mathbb{P}^{\text{Obs}}(Y|T=1,Z=1)} \right). \end{aligned}$$

We continue to analyze $d\mathbb{P}^{\text{Target}}(Y|T=0,Z=1)$.

$$\begin{aligned} d\mathbb{P}^{\text{Target}}(Y|T=0,Z=1) &= \omega(0)d\mathbb{P}^{\text{True}}(Y(0,0)|Nt) + \omega(1)d\mathbb{P}^{\text{True}}(Y(0,1)|Nt) \\ &= \omega(0) \frac{d\mathbb{P}^{\text{True}}(Y(0,0)|Nt)}{d\mathbb{P}^{\text{True}}(Y(0,1)|Nt)} d\mathbb{P}^{\text{True}}(Y(0,1)|Nt) + \omega(1)d\mathbb{P}^{\text{Obs}}(Y|T=0,Z=1) \\ &= \left(\omega(0) \frac{d\mathbb{P}^{\text{True}}(Y(0,0)|Nt)}{d\mathbb{P}^{\text{True}}(Y(0,1)|Nt)} + \omega(1) \right) d\mathbb{P}^{\text{Obs}}(Y|T=0,Z=1). \end{aligned}$$

Analogously,

$$\begin{aligned} d\mathbb{P}^{\text{Target}}(Y|T=1,Z=0) &= \omega(1)d\mathbb{P}^{\text{True}}(Y(1,1)|At) + \omega(0)d\mathbb{P}^{\text{True}}(Y(1,0)|At) \\ &= \omega(1) \frac{d\mathbb{P}^{\text{True}}(Y(1,1)|At)}{d\mathbb{P}^{\text{Obs}}(Y|T=1,Z=0)} d\mathbb{P}^{\text{Obs}}(Y|T=1,Z=0) + \omega(0)d\mathbb{P}^{\text{Obs}}(Y|T=1,Z=0) \\ &= \left(\omega(1) \frac{d\mathbb{P}^{\text{True}}(Y(1,1)|At)}{d\mathbb{P}^{\text{True}}(Y(1,0)|At)} + \omega(0) \right) d\mathbb{P}^{\text{Obs}}(Y|T=1,Z=0). \end{aligned}$$

□

Lemma 11. Define the infeasible \hat{T}^* corresponding to Proposition 6 as:

$$\hat{T}^* \equiv \hat{E} \left[\hat{\lambda}(R)Y + \hat{S}a(\hat{w}(R), \hat{w}(R), S) \right].$$

Then $\hat{T} \geq \hat{T}^*$ deterministically.

Proof of Lemma 11. Define the true and estimated residuals from the quantile $S \equiv \lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y | R)$ and $\hat{S} \equiv \hat{\lambda}(R)Y - \hat{Q}_{\hat{\tau}(R)}(\hat{\lambda}(R)Y | R)$.

$$\hat{T} - \hat{T}^* = \hat{E} \left[(\hat{w}(R) - \underline{w}(R))\hat{S} \left(1\{\hat{S} > 0\} - 1\{S > 0\} \right) \right] \geq 0.$$

By sign-matching the cases in which $1\{\hat{S} > 0\} \neq 1\{S > 0\}$, the result holds deterministically. In particular, when $\hat{S} > S$, the region where $1\{\hat{S} > 0\} - 1\{S > 0\} = 1$ is where $\hat{S} > 0$, so $\hat{S} \left(1\{\hat{S} > 0\} - 1\{S > 0\} \right) \geq 0$. Similarly, when $\hat{S} < S$, the region where $1\{\hat{S} > 0\} - 1\{S > 0\} = -1$ is where $\hat{S} < 0$, so $\hat{S} \left(1\{\hat{S} > 0\} - 1\{S > 0\} \right) \geq 0$. \square

B.2 Additional Claims

Proof of Proposition 7. We first show the identified set of $E[Y(1)]$ is unbounded if $c > \eta_1$; the argument is symmetric for the lower bound of $E[Y(0)]$ if $c > \eta_2$.

By the decomposition in the proof of Lemma 6, the upper bound of the identified set of $E[Y(1)]$ is:

$$\psi^+ = E \left[\lambda(R)\mu(R) + (1 - \underline{w}(R))CVaR_{\tau(R)}^+(R) \right],$$

where $\lambda(R) = Z/e(X)$, $\underline{w}(R) = Z \frac{e(X)}{e(X)+c}$, and $\mu(R) = E[Y | R]$. We can lower bound the upper bound as:

$$\begin{aligned} \psi^+ &\geq E \left[\lambda(R)\mu(R) + 1\{e(X) < [\eta_1, c]\}(1 - \underline{w}(R))CVaR_{\tau(R)}^+(R) \right] \\ &\geq E \left[\lambda(R)\mu(R) + 1\{e(X) \in [\eta_1, c]\} \frac{1}{2} \frac{Z}{e(X)} E[Y - \mu(R) | Y, R \geq Q_1(Y | R)] \right] \\ &\geq E \left[\lambda(R)\mu(R) + 1\{e(X) \in [\eta_1, c]\} \frac{1}{2} \frac{\eta_1}{c} * \infty \right], \end{aligned}$$

where the infinite conditional value at risk happens for all X with $\sigma(X, 1) > 0$, which happens almost surely by assumption. Since $\eta_1 = \inf p | P(e(X) > p) > 0$ by definition and $\eta_1 > 0$ by assumption, $E[I(e(X) \in [\eta_1, c]) \frac{1}{2} \frac{\eta_1}{c} | R] > 0$ so that the identified set would be unbounded.

Now suppose $c < \eta_1$ and we wish to show that the identified set is uniformly bounded. Write D as the upper bound of the support of $\frac{\sigma(X, Z)}{e(X)} + \frac{\sigma(X, Z)}{1-e(X)}$, which by assumption is finite. Recall by Lemma 6 that the identified set can be upper bounded as follows:

$$\begin{aligned} \psi^+ &\leq E[\lambda(R)Y] + E \left[\sigma(R)\lambda(R)(1 - \underline{w}(R)) \left(\sqrt{2 \log(\bar{w}(R))} + \sqrt{2/\pi} + (1 - \underline{w}(R))^\epsilon \sqrt{1/(e * \epsilon)} \right) \right] \\ &\leq E[\lambda(R)Y] + DE \left[Z \sqrt{2 \log(\bar{w}(R))} + \sqrt{2/\pi} + 1/e \right], \end{aligned}$$

where \log is the natural logarithm. It only remains to bound $\sqrt{2}E[Z \log(\bar{w}(R))]$, where $\bar{w}(R) = Ze(X)/(e(X) - c) + (1 - Z)(1 - e(X))/(1 - e(X) - c)$. Suppose the propensity density is $f_{e(X)}(p)$ and is upper bounded by

$\bar{f}_{e(X)}$:

$$\begin{aligned} E[Z \log(\bar{w}(R))] &= \int_{\eta_1}^{1-\eta_2} p \log(p/(p-c)) f_{e(X)}(p) \\ &\leq \bar{f}_{e(X)} \int_{\eta_1}^{1-\eta_2} \log(p/(p-\eta_1)) dp \leq -\bar{f}_{e(X)} \int_{\eta_1}^{1-\eta_2} \log(p-\eta_1) dp \\ &\leq -\bar{f}_{e(X)} \int_0^1 \log(t) dt = \bar{f}_{e(X)}. \end{aligned}$$

Therefore an APO upper bound is:

$$\begin{aligned} \psi^+ &\leq E[\lambda(R)Y] + E \left[\sigma(R)\lambda(R)(1-\underline{w}(R)) \left(\sqrt{2 \log(\bar{w}(R))} + \sqrt{2/\pi} + (1-\underline{w}(R))^\epsilon \sqrt{1/(e*\epsilon)} \right) \right] \\ &\leq E[\lambda(R)Y] + DE \left[\sqrt{2}\bar{f}_{e(X)} + \sqrt{2/\pi} + 1/e \right]. \end{aligned}$$

Since this bound holds symmetrically for the lower bound of $E[Y(0)]$, writing $B = 2DE \left[\sqrt{2}\bar{f}_{e(X)} + \sqrt{2/\pi} + 1/e \right]$ completes the proof. \square

B.3 Proofs for Section 2

Proof of Lemma 1. This is a standard result for importance sampling. \square

Proof of Theorem 1. Formally define $\psi^+(R)$ as a random variable satisfying:

$$\psi^+(R) = \sup_W E_{\mathbb{P}^{\text{Obs}}} [\underline{w}(R)\lambda(R)Y + (1-\underline{w}(R))W\lambda(R)Y] \text{ s.t. } W \in [0, 1-\tau(R)] \text{ and } E_{\mathbb{P}^{\text{Obs}}}[W | R] = 1 \text{ a.s.}$$

By the argument in the proof sketch, we can write the upper bound as:

$$\begin{aligned} \psi^+ &= \sup_W E_{\mathbb{P}^{\text{Obs}}} [W\lambda(R)Y] \quad \text{s.t. } W \in [\underline{w}(R), \bar{w}(R)] \text{ and } E_{\mathbb{P}^{\text{Obs}}}[W | R] = 1 \text{ a.s.} \\ \psi^+ &= \sup_W E_{\mathbb{P}^{\text{Obs}}} [\underline{w}(R)\lambda(R)Y + W\lambda(R)Y] \text{ s.t. } W \in [0, \bar{w}(R) - \underline{w}(R)] \text{ and } E_{\mathbb{P}^{\text{Obs}}}[W | R] = 1 - \underline{w}(R) \text{ a.s.} \\ \psi^+ &= \sup_W E_{\mathbb{P}^{\text{Obs}}} [\underline{w}(R)\lambda(R)Y + (1-\underline{w}(R))W\lambda(R)Y] \text{ s.t. } W \in \underbrace{\left[0, \frac{\bar{w}(R) - \underline{w}(R)}{1 - \underline{w}(R)}\right]}_{1-\tau(R)} \text{ and } E_{\mathbb{P}^{\text{Obs}}}[W | R] = 1 \text{ a.s.} \\ &= E_{\mathbb{P}^{\text{Obs}}} [\psi^+(R)]. \end{aligned}$$

As in the proof sketch, we claim that $\psi^+(R) = \phi^+(R)$ almost surely. By arguments from the DRO literature, e.g. [Dorn et al. \(2022\)](#), $\psi^+(R) = \underline{w}(R)E[\lambda(R)Y | R] + (1-\underline{w}(R))CVaR_{\tau(R)}^+$ almost surely. $\sup_W E_{\mathbb{P}^{\text{Obs}}}[W\lambda(R)Y]$ s.t. $E_{\mathbb{P}^{\text{Obs}}}[W | R] = 1$ simply reduces to setting $\lambda(R)Y$ to the supremum of the support of $\lambda(R)Y$, i.e. the level-1 CVaR.)

Note that the level- $\tau(R)$ CVaR of $\lambda(R)Y | R$ can be defined as $E \left[Q^+ + \frac{\{Y-Q^+\}_+}{1-\tau(R)} | R \right]$. (In the case $\tau(R) = 1$, we evaluate this term as $Q^+ = F_1(\lambda(R)Y | R)$.)

By Lemma 7, we may write:

$$\begin{aligned} \phi^+(R) &= \lambda(R)Y + (1-\underline{w}(R))(\lambda(R)Y - Q^+(R)) \left(\frac{1\{\lambda(R)Y > Q^+(R)\}}{1-\tau(R)} - 1 \right) \\ &= \underline{w}(R)\lambda(R)Y + (1-\underline{w}(R)) \left(Q^+(R) + \frac{\{\lambda(R)Y - Q^+(R)\}_+}{1-\tau(R)} \right) \end{aligned}$$

$$= \underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R))CVaR_{\tau(R)}^+ \text{ a.s.}$$

Completing the proof. The derivation for ϕ^- is analogous. \square

Proof of Lemma 2. Let $\underline{\tau}(R)$ and $\bar{\tau}(R)$ be random variables satisfying $\underline{\tau}(R) = \mathbb{P}^{\text{Obs}}(\lambda(R)Y < Q^+(R) | R)$ and $\bar{\tau}(R) = \mathbb{P}^{\text{Obs}}(\lambda(R)Y \leq Q^+(R) | R)$ almost surely. Note that $\mathbb{P}^{\text{Obs}}(\lambda(R)Y = Q^+(R) | R) = \bar{\tau}(R) - \underline{\tau}(R)$ almost surely.

Define the function $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^1$ as follows:

$$\gamma(r) \equiv \begin{cases} \frac{1 - \underline{\tau}(r)\underline{w}(r) - (1 - \bar{\tau}(r))\bar{w}(r)}{\bar{\tau}(r) - \underline{\tau}(r)} & \text{if } \bar{\tau}(r) > \underline{\tau}(r) \\ \underline{w}(r) & \text{else.} \end{cases}$$

It is clear that $\gamma(r)$ is well-defined. The result that $\gamma(R) \in [\underline{w}(R), \bar{w}(R)]$ almost surely and that the resulting W_{sup}^* solves the upper bound optimization problem follows by an adapted version of Dorn and Guo (2023)'s Proposition 2 argument and the observation that $E[W_{\text{sup}}^* | R] = 1$ almost surely. \square

Proof of Lemma 3. Recall from the proof of Theorem 1 that we may write:

$$\psi^+ = \underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R)) \left(Q^+(R) + \frac{\{\lambda(R)Y - Q^+(R)\}_+}{1 - \tau(R)} \right),$$

where $\{x\}_+ = \max\{x, 0\}$ and where Q^+ is the quantile regression function.

We make the stronger claim that:

$$Q^+(R) \in \arg \min_{\bar{Q}} E \left[\underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R)) \left(\bar{Q}(R) + \frac{\{\lambda(R)Y - \bar{Q}(R)\}_+}{1 - \tau(R)} \right) | R \right] \text{ a.s.}$$

If $H = 0$ and hence $\tau(R) = 1$, then the right hand side should be understood to be infinite if $\lambda(R)Y > Q^+(R)$. Therefore $Q^+(R) = F_1(\lambda(R)Y | R)$ is the right-hand side minimizer on the event $H = 0$. Therefore:

$$Q^+(R) \in \arg \min_{\bar{Q}} (1 - H)E \left[\underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R)) \left(\bar{Q}(R) + \frac{\{\lambda(R)Y - \bar{Q}(R)\}_+}{1 - \tau(R)} \right) | R \right] \text{ a.s.}$$

On the event $H = 1$, we can write the quantile regression function Q^+ as some function satisfying the weighted quantile regression definition:

$$\begin{aligned} Q^+(R) &\in \arg \min_{\bar{Q}} E_{\mathbb{P}^{\text{Obs}}} \left[(1 - \tau(R))^{-1} (\tau(R)\{\lambda(R)Y - \bar{Q}(R)\}_+ + (1 - \tau(R))\{\bar{Q}(R) - \lambda(R)Y\}_+) \right] \\ &\Leftrightarrow \in \arg \min_{\bar{Q}} E_{\mathbb{P}^{\text{Obs}}} \left[(1 - \tau(R))^{-1} (\{\lambda(R)Y - \bar{Q}(R)\}_+ + (1 - \tau(R))(\bar{Q}(R) - \lambda(R)Y)) \right] \\ &= E_{\mathbb{P}^{\text{Obs}}} \left[\bar{Q}(R) - \lambda(R)Y + \frac{\lambda(R)Y - \bar{Q}(R)\}_+}{1 - \tau(R)} \right]. \end{aligned}$$

The $\lambda(R)Y$ term does not affect the minimizer, so that:

$$Q^+(R) \in \arg \min_{\bar{Q}} HE \left[\underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R)) \left(\bar{Q}(R) + \frac{\{\lambda(R)Y - \bar{Q}(R)\}_+}{1 - \tau(R)} \right) | R \right] \text{ a.s.}$$

Combining terms, we obtain:

$$Q^+(R) \in \arg \min_{\bar{Q}} E \left[\underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R)) \left(\bar{Q}(R) + \frac{\{\lambda(R)Y - \bar{Q}(R)\}_+}{1 - \tau(R)} \right) \mid R \right] \text{ a.s.}$$

Completing the proof. \square

B.4 Proofs for Section 3

Proof of Proposition 1. This is immediate by the discussion in Section 3.1. \square

Proof of Lemma 4. For our proofs for the Regression Discontinuity application of Section 3.2, it is useful to use $\eta_0 \equiv \mathbb{P}^{\text{Target}}(M = 1 \mid X = c)$.

For the CATE, first observe that $\mathbb{P}^{\text{Target}}(X = c^- \mid X = c) = (1 - \eta_0)/2$. To see this, $\mathbb{P}^{\text{Target}}(X = c^- \mid X = c) = \mathbb{P}^{\text{Target}}(X = c^- \mid M = 0, X = c)\mathbb{P}^{\text{Target}}(M = 0 \mid X = c) = \mathbb{P}^{\text{Target}}(X = c^- \mid M = 0, X = c)(1 - \eta_0) = (1 - \eta_0)/2$, due to Lemma 8.

$$\begin{aligned} \psi_{CATE} &\equiv E[Y(1) - Y(0) \mid X = c] \\ &= E[Y(1) \mid X \geq c, X = c]P(X \geq c \mid X = c) + E[Y(1) \mid X = c^-, X = c]P(X = c^- \mid X = c) \\ &\quad - E[Y(0) \mid X = c, M = 1]P(M = 1 \mid X = c) - P(M = 0 \mid X = c)E[Y(0) \mid X = c, M = 0] \\ &= E[Y(1) \mid X \geq c, X = c] \frac{1}{2 - \eta} + E[Y(1) \mid X = c^-, X = c] \frac{1 - \eta_0}{2} \\ &\quad - E[Y(0) \mid X = c, M = 1]\eta_0 - (1 - \eta_0)E[Y(0) \mid X = c, M = 0] \end{aligned}$$

The final equality uses the definition that $\eta_0 = P(M = 1 \mid X = c)$ and the result that $P(X = c^- \mid X = c) = (1 - \eta_0)/2$.

Finally, $E[Y \mid X = c^+] = E[Y(1) \mid X = c^+]$ and $E[Y \mid X = c^-] = E[Y(0) \mid X = c^-]$ due to the sharp RD design of Assumption 1.

For the CATT, observe:

$$\begin{aligned} \psi_{CATT} &= \frac{E[TY \mid X = c] - E[TY(0) \mid X = c]}{E[T \mid X = c]} \\ &= \frac{E[TY \mid X = c]}{E[T \mid X = c]} \\ &\quad - \frac{\mathbb{P}^{\text{Target}}[T = 1, M = 0 \mid X = c]E[Y \mid X = c^-]}{E[T \mid X = c]} \\ &\quad - \frac{\mathbb{P}^{\text{Target}}[T = 1, M = 1 \mid X = c]E[Y \mid X = c, M = 1]}{E[T \mid X = c]} \\ &= \frac{E[TY \mid X = c]}{E[T \mid X = c]} \\ &\quad - \frac{\mathbb{P}^{\text{Target}}[T = 0, M = 0 \mid X = c]E[Y \mid X = c^-]}{E[T \mid X = c]} \\ &\quad - \frac{\eta_0 E[Y \mid X = c, M = 1]}{E[D \mid X = c]} \end{aligned}$$

$$\begin{aligned}
&= \frac{E[TY \mid X = c] - E[(1-T)Y \mid X = c] - \eta_0 E[Y \mid X = c, M = 1]}{E[T \mid X = c]} \\
&= \frac{E[(2T-1)Y \mid X = c] - \eta_0 E[Y \mid X = c, M = 1]}{\eta_0 + (1-\eta_0)/2}
\end{aligned}$$

The CLATE is immediate. \square

Proof of Proposition 2. Recall by Lemma 9 that selection bounds on $q_t(Y(t)) = \mathbb{Q}(M = 1 \mid X = c, Y(t))$ from Equation (5) correspond to likelihood ratios for $\mathbb{Q} \in \mathcal{M}(\underline{w}, \bar{w})$ as:

$$\begin{aligned}
\frac{d\mathbb{Q}(Y(1) \mid X = c, M = 0)}{d\mathbb{P}^{\text{Obs}}(Y \mid X = c^+)} &= \frac{1}{1-\eta} \left(1 + 2 \frac{q_1(Y(1))}{1-q_1(Y(1))} \right)^{-1} \\
&\in \left[\frac{1}{1-\eta} \left(1 + 2 \frac{\eta}{2(1-\eta)} \Lambda_1 \right)^{-1}, \frac{1}{1-\eta} \left(1 + 2 \frac{\eta}{2(1-\eta)} \Lambda_1^{-1} \right)^{-1} \right] \\
\frac{d\mathbb{Q}(Y(0) \mid X = c, M = 1)}{d\mathbb{P}^{\text{Obs}}(Y \mid X = c^-)} &= \frac{2(1-\eta)}{\eta} \frac{q_0(Y(0))}{1-q_0(Y(0))}
\end{aligned}$$

As a result, due to Equation (9),

$$\begin{aligned}
\frac{d\mathbb{Q}(Y(1) \mid X = c, M = 0)}{d\mathbb{P}^{\text{Obs}}(Y \mid X = c^+)} &\in \left[\frac{1}{1-\eta+\eta\Lambda_1}, \frac{1}{1-\eta+\eta\Lambda_1^{-1}} \right] \\
\frac{d\mathbb{Q}(Y(0) \mid X = c, M = 1)}{d\mathbb{P}^{\text{Obs}}(Y \mid X = c^-)} &\in [\Lambda_0^{-1}, \Lambda_0].
\end{aligned}$$

Next, we verify that the target distribution's estimand $E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y]$ achieves the relevant causal estimands. Suppose we call the target estimator $\bar{\psi}$. We proceed as follows:

$$\begin{aligned}
\bar{\psi}_{CLATE} &= E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y \mid X = c] \\
&= \mathbb{P}^{\text{Obs}}(X = c^+ \mid X = c) E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y \mid X = c^+] - \mathbb{P}^{\text{Obs}}(X = c^- \mid X = c) E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y \mid X = c^-] \\
&= E_{\mathbb{P}^{\text{Target}}}[Y \mid X = c^+] - E_{\mathbb{P}^{\text{Target}}}[Y \mid X = c^-] \\
&= E_{\mathbb{P}^{\text{True}}}[Y(1) \mid X = c, M = 0] - E_{\mathbb{P}^{\text{Obs}}}[Y(0) \mid X = c, M = 0] = \psi_{CLATE} \\
\bar{\psi}_{CATT} &= E_{\mathbb{P}^{\text{Obs}}}[Y \mid X = c^+] - E_{\mathbb{P}^{\text{True}}}[Y(0) \mid X = c^+] = \psi_{CATT} \\
\bar{\psi}_{CATE} &= E_{\mathbb{P}^{\text{True}}}[Y(1) \mid X = c] - E_{\mathbb{P}^{\text{True}}}[Y(0) \mid X = c] = \psi_{CATE}.
\end{aligned}$$

Finally, we verify that the Radon–Nikodym bounds have the appropriate forms by using Equation (4). The formula for the CLATE is immediate. The formula for the CATT follows by observing that:

$$\begin{aligned}
\frac{d\mathbb{P}^{\text{True}}(Y(0) \mid X = c^+)}{d\mathbb{P}^{\text{Obs}}(Y \mid X = c^-)} &= \mathbb{P}^{\text{Obs}}(M = 0 \mid X = c^+) * \frac{d\mathbb{P}^{\text{True}}(Y(0) \mid M = 0, X = c^+)}{d\mathbb{P}^{\text{Obs}}(Y \mid X = c^-)} \\
&\quad + \mathbb{P}^{\text{Obs}}(M = 1 \mid X = c^+) * \frac{d\mathbb{P}^{\text{True}}(Y(0) \mid M = 1, X = c^+)}{d\mathbb{P}^{\text{Obs}}(Y \mid X = c^-)} \\
&= \mathbb{P}^{\text{Obs}}(M = 0 \mid X = c^+) * 1 + \mathbb{P}^{\text{Obs}}(M = 1 \mid X = c^+) * \frac{d\mathbb{P}^{\text{True}}(Y(0) \mid X = c, M = 1)}{d\mathbb{P}^{\text{Obs}}(Y \mid X = c^-)} \\
&= (1-\eta) + \eta \frac{d\mathbb{P}^{\text{True}}(Y(0) \mid X = c, M = 1)}{d\mathbb{P}^{\text{Obs}}(Y \mid X = c^-)} \\
&\in [(1-\eta) + \eta\Lambda_0^{-1}, (1-\eta) + \eta\Lambda_0],
\end{aligned}$$

verifying the form.

The formula for the CATE similarly follows by the argument from Lemma 4. In particular:

$$\begin{aligned} \frac{d\mathbb{P}^{\text{True}}(Y(1) | X = c)}{d\mathbb{P}^{\text{Obs}}(Y | X = c^+)} &= \mathbb{P}^{\text{Obs}}(X = c^+ | X = c) * \frac{d\mathbb{P}^{\text{True}}(Y(1) | X = c^+)}{d\mathbb{P}^{\text{Obs}}(Y | X = c^+)} \\ &\quad + \mathbb{P}^{\text{Obs}}(X = c^- | X = c) * \frac{d\mathbb{P}^{\text{True}}(Y(1) | X = c^-)}{d\mathbb{P}^{\text{Obs}}(Y | X = c^+)} \\ &= \mathbb{P}^{\text{Obs}}(X = c^+ | X = c) * 1 + \mathbb{P}^{\text{Obs}}(X = c^- | X = c) * \frac{d\mathbb{P}^{\text{True}}(Y(1) | X = c, M = 0)}{d\mathbb{P}^{\text{Obs}}(Y | X = c^+)}. \end{aligned}$$

Similarly:

$$\begin{aligned} \frac{d\mathbb{P}^{\text{True}}(Y(0) | X = c)}{d\mathbb{P}^{\text{Obs}}(Y | X = c^-)} &= \mathbb{P}^{\text{Obs}}(X = c^+ | X = c) * \frac{d\mathbb{P}^{\text{True}}(Y(0) | X = c^+)}{d\mathbb{P}^{\text{Obs}}(Y | X = c^-)} + \mathbb{P}^{\text{Obs}}(X = c^- | X = c) * 1 \\ &= \eta \mathbb{P}^{\text{Obs}}(X = c^+ | X = c) \frac{d\mathbb{P}^{\text{True}}(Y(0) | X = c, M = 1)}{d\mathbb{P}^{\text{Obs}}(Y | X = c^-)} + (1 - \eta) \mathbb{P}^{\text{Obs}}(X = c^+ | X = c) \end{aligned}$$

Recall from the proof of Lemma 9 that $\mathbb{P}^{\text{Obs}}(X = c^+ | X = c) = 1/(2 - \eta)$. Therefore:

$$\begin{aligned} 1\{X = c^+\}\underline{w}(R) &= 1\{X = c^+\} \left(\frac{1}{2 - \eta} + \frac{1 - \eta}{1 - 2\eta} (1 - \eta + \eta\Lambda_1)^{-1} \right) \\ 1\{X = c^+\}\bar{w}(R) &= 1\{X = c^+\} \left(\frac{1}{2 - \eta} + \frac{1 - \eta}{1 - 2\eta} (1 - \eta + \eta\Lambda_1^{-1})^{-1} \right) \\ 1\{X = c^-\}\underline{w}(R) &= 1\{X = c^-\} \left(\frac{2 - 2\eta}{2 - \eta} + \frac{\eta}{2 - \eta} \Lambda_0^{-1} \right) \\ 1\{X = c^-\}\bar{w}(R) &= 1\{X = c^-\} \left(\frac{2 - 2\eta}{2 - \eta} + \frac{\eta}{2 - \eta} \Lambda_0 \right), \end{aligned}$$

completing the final proof. \square

Proof of Proposition 3. First, we show that if $\mathbb{Q} \in \mathcal{M}'(\Lambda)$, then $\mathbb{Q} \in \mathcal{M}(\Lambda, \Lambda)$. It is clear that it only remains to show that for $t = 1, 0$, we have $\frac{\mathbb{Q}(M=1|Y(t), X=c)}{\mathbb{Q}(M=0|Y(t), X=c)} / \frac{\mathbb{P}^{\text{Obs}}(M=1|X=c)}{\mathbb{P}^{\text{Obs}}(M=0|X=c)} \in [\Lambda^{-1}, \Lambda]$ almost surely. This holds by iterated expectations over $\int \mathbb{Q}(M = 1 | Y(1), Y(0), X = c) d\mathbb{Q}(Y(1-t) | X = c, Y(1-t))$ and the restriction on the domain of $\mathbb{Q}(M = 1 | Y(1), Y(0), X = c)$ under Equation (6). Therefore $\mathbb{Q} \in \mathcal{M}(\Lambda, \Lambda)$.

We now proceed to construct a \mathbb{Q}' corresponding to ψ_1 and ψ_0 .

For simplicity, we proceed assuming that Y is continuously distributed. (If the conditional distribution of Y contains mass points at the referenced quantiles, add a uniform random variable to provide a strict ordering on observations of $Y | X$ and then apply the construction below to the quantiles of the tie-broken Y .)

Recall that all of our causal estimands of interest can be written as observable linear transformations of the partially identified average potential outcomes $E[Y(t) | X = c, M = 1 - t]$ for $t = 1, 0$.

We define a distribution $\mathbb{Q}_{+,+}$ which we show is in the model family $\mathcal{M}'(\Lambda) \subset \mathcal{M}(\Lambda, \Lambda) \subset \mathcal{M}'(\infty)$ and achieves the upper bounds on both average potential outcomes within $\mathcal{M}(\Lambda, \Lambda)$. We then argue that any $\mathbb{Q}_{\pm,\pm}$ combination of upper and lower bounds is feasible. Finally, we argue by mixture that any pair of causal estimands of interest between the two lower and upper bounds under $\mathcal{M}(\Lambda, \Lambda)$ are feasible for some distribution $\mathbb{Q} \in \mathcal{M}'(\Lambda)$.

For convenience, we separate the separate claims into different sub-sub-sections.

Construction of Worst-Case Distribution

We will define a distribution $\mathbb{Q}_{+,+}$ with an unobserved confounder “ V .” $V = 1$ will correspond to a high manipulation probability, small treated potential outcomes (to maximize $E[Y(1) | X = c^+, M = 0]$), and large untreated outcomes (to maximize $E[Y(0) | X = c^+, M = 1]$). At the threshold, a fraction τ_1 of treated and $1 - \tau_0$ of untreated observations will have $V = 1$, where:

$$\begin{aligned}\tau_1 &= \frac{1 + \eta(\Lambda - 1)}{\Lambda + 1} \\ \tau_0 &= \frac{\Lambda}{\Lambda + 1}.\end{aligned}$$

In order to facilitate this drawing, we define the observable V function:

$$V(x, t, y) = \begin{cases} 1\{y \leq Q_{\tau_1}(Y | X = c^+)\} & \text{if } t = 1 \\ 1\{y > Q_{\tau_0}(Y | X = c^-)\} & \text{if } t = 0. \end{cases}$$

Finally, in order to match worst-case selection probabilities, $M | X = c, V = v$ will be drawn iid from a $Bern(Tq_{++}(v))$ distribution, where $q_{++}(v)$ is defined as:

$$q_{++}(v) = \begin{cases} \frac{\eta}{(\Lambda+1)(1-\tau_1)} & \text{if } v = 0 \\ \frac{\eta\Lambda}{(\Lambda+1)\tau_1} & \text{if } v = 1. \end{cases}$$

Finally, we will draw unobserved potential outcomes from the distribution with the same value of M and V .

Formally, the distribution $\mathbb{Q}_{+,+}$ over $(M, X(1), X(0), Y(1), Y(0), T)$ is defined as follows:

1. Draw $(X, T, Y) \sim \mathbb{P}^{\text{Obs}}$
2. Define the random variable $V = V(X, T, Y)$
3. Draw $M \sim Bern(Tq_{++}(V))$
4. Set the lower-case variable realizations for later use $x = X$, and $y = Y$, $t = T$, $v = V$, and $m = M$
5. Set $X(M) = X$ and $Y(t) = y$
6. Draw $X(1 - M)$ from the distribution of $X | M = 1 - m$ under $\mathbb{Q}_{+,+}$ defined so far
7. If $m = 0$, draw $Y(1 - t)$ from the distribution of $Y | X = 2c - x, M = 0, V = v$ under the construction of $\mathbb{Q}_{+,+}$ so far
8. If $m = 1$, draw $Y(0)$ from the distribution of $Y | X = 2c - x, V = v$ under the construction of $\mathbb{Q}_{+,+}$ so far
9. Return data $(M, X(1), X(0), Y(1), Y(0), T)$

Showing Constructed Distribution is in Model Family

We wish to show that $\mathbb{Q}_{+,+} \in \mathcal{M}'(\Lambda)$, i.e.:

- (a) The distribution of $(X = X(M), Y = Y(T), T)$ under $\mathbb{Q}_{+,+}$ marginalizes to the distribution of (X, T, Y) under \mathbb{P}^{Obs}

- (b) $\mathbb{Q}_{+,+}(T = 0, M = 1) = 0$
- (c) $\mathbb{Q}_{+,+}(M = 1 \mid X = c^+) = \eta$
- (d) $\mathbb{Q}_{+,+}(Y(t) \leq y \mid X = x, M = 0)$ is continuous at c and $\mathbb{Q}_{+,+}(Y(1) \leq y \mid x = c, M = 1)$ is right-continuous at $x = c$.
- (e) $\mathbb{Q}_{+,+}(Y(1), Y(0), T(0), M \mid X = c)$ is a conditional distribution that is well-defined as the appropriate continuous limit.
- (f) $\mathbb{Q}_{+,+}$ satisfies Equation (6):

$$\frac{\mathbb{Q}_{+,+}(M = 1 \mid Y(1), Y(0), X = c)}{\mathbb{Q}_{+,+}(M = 0 \mid Y(1), Y(0), X = c)} \Big/ \frac{\mathbb{P}^{\text{Obs}}(M = 1 \mid X = c)}{\mathbb{P}^{\text{Obs}}(M = 0 \mid X = c)} \in [\Lambda^{-1}, \Lambda].$$

Requirements (a) and (b) are immediate.

Requirement (c) follows by inspection:

$$\begin{aligned} \mathbb{Q}_{+,+}(M = 1 \mid X = c^+) &= E_{\mathbb{P}^{\text{Obs}}}[q_{++}(V) \mid X = c^+] \\ &= \sum_v \mathbb{P}^{\text{Obs}}(V = v \mid X = c^+) q_{++}(v) \\ &= \frac{\eta}{\Lambda + 1} + \frac{\eta\Lambda}{\Lambda + 1} = \eta = \mathbb{P}^{\text{Obs}}(M = 1 \mid X = c^+). \end{aligned}$$

The requirement (d) holds as follows. $\mathbb{P}^{\text{True}}(Y \leq y \mid X = x)$ is left- and right-continuous at $x = c$ as follows. For left-continuity, for $x < c$, $\mathbb{P}^{\text{True}}(Y \leq y \mid X = x) = \mathbb{P}^{\text{True}}(Y \leq y \mid M = 0, X = x)$ is left-continuous at $x = c$ by assumption. For right-continuity, for $x > c$,

$$\begin{aligned} \mathbb{P}^{\text{True}}(Y \leq y \mid X = x) &= \mathbb{P}^{\text{True}}(M = 1 \mid X = x) \mathbb{P}^{\text{True}}(Y \leq y \mid M = 1, X = x) \\ &\quad + \mathbb{P}^{\text{True}}(M = 0 \mid X = x) \mathbb{P}^{\text{True}}(Y \leq y \mid M = 0, X = x), \end{aligned}$$

which by assumption is right-continuous at $x = c$. As a result, $\mathbb{Q}_{+,+}(Y(t) \leq y \mid X = x, M = 0)$ is continuous in x at $x = c$. Note also that for $X > c$ and $y \leq Q_{\tau_1}(Y \mid X = c^+)$, we have:

$$\mathbb{Q}_{+,+}(Y(t) \leq y \mid X, M = 1) = \frac{\mathbb{P}^{\text{Obs}}(Y(t) \leq y \mid X) q_{++}(0)}{\mathbb{P}^{\text{Obs}}(Y(t) \leq Q_{\tau_1}(Y \mid X = c^+) \mid X) q_{++}(0) + \mathbb{P}^{\text{Obs}}(Y(t) > Q_{\tau_1}(Y \mid X = c^+) \mid X)},$$

and similarly, for $Y > Q_{\tau_1}(Y \mid X = c^+)$:

$$\mathbb{Q}_{+,+}(Y(t) \leq y \mid X, M = 1) = \frac{\mathbb{P}^{\text{Obs}}(Y(t) \leq Q_{\tau_1}(Y \mid X = c^+) \mid X) q_{++}(0) + \mathbb{P}^{\text{Obs}}(Y(t) \in [Q_{\tau_1}(Y \mid X = c^+), y] \mid X) q_{++}(0)}{\mathbb{P}^{\text{Obs}}(Y(t) \leq Q_{\tau_1}(Y \mid X = c^+) \mid X) q_{++}(0) + \mathbb{P}^{\text{Obs}}(Y(t) > Q_{\tau_1}(Y \mid X = c^+) \mid X)},$$

both of which are continuous in X .

(e) holds as follows, where we write $T(x) = 1\{x > c\}$. For $m = 1$ and some $A \subseteq \mathbb{R}^2$:

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0^+} \mathbb{Q}_{+,+}((Y(1), Y(0)) \in A, T(0) = t, M = 1, V = v \mid |X - \varepsilon| \leq c) \\ &= \int 1\{X > c, (Y(1), Y(0)) \in A\} q_{++}(v) \frac{1}{2} d\mathbb{Q}_{+,+}(V, Y(0) \mid X, Y, M = 1) d\mathbb{P}^{\text{Obs}}(Y, X \mid |X - c| \leq \varepsilon) \\ &= \int q_{++}(v) \frac{1\{X > c, (Y(1), Y(0)) \in A, V(X, 1, Y) = v\}}{2} d\mathbb{Q}_{+,+}(Y \mid |X - c|, T = 0, V = v) d\mathbb{P}^{\text{Obs}}(Y, X \mid |X - c| \leq \varepsilon), \end{aligned}$$

which has a well-defined limit by one-sided continuity of the CDF of $Y \mid X$ under our assumptions. The remaining claim holds by taking the sum over v . Similarly, for $m = 0$:

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \mathbb{Q}_{+,+}((Y(1), Y(0)) \in A, T(0) = t, M = 1, V = v \mid |X - \varepsilon| \leq c) \\ &= 1\{X \leq c\} \mathbb{Q}_{+,+}((Y(1), Y(0)) \in A, T(0) = t, M = 0, V = v \mid |X - \varepsilon| \leq c) \\ &+ 1\{X > c\} \mathbb{Q}_{+,+}((Y(1), Y(0)) \in A, T(0) = t, M = 0, V = v \mid |X - \varepsilon| \leq c), \end{aligned}$$

which is continuous by analogous arguments.

Requirement (f) involves a longer argument, so we show it in a separate section.

$\mathbb{Q}_{+,+}$ Satisfies the Odds Ratio Bound

We wish to show:

$$\frac{\mathbb{Q}_{+,+}(M = 1 \mid Y(1), Y(0), X = c)}{\mathbb{Q}_{+,+}(M = 0 \mid Y(1), Y(0), X = c)} \Big/ \frac{\mathbb{P}^{\text{Obs}}(M = 1 \mid X = c)}{\mathbb{P}^{\text{Obs}}(M = 0 \mid X = c)} \in [\Lambda^{-1}, \Lambda].$$

Recall that $\frac{\mathbb{P}^{\text{Obs}}(M=1 \mid X=c)}{\mathbb{P}^{\text{Obs}}(M=0 \mid X=c)} = \frac{\eta}{2(1-\eta)}$.

Notice that $Y(1), Y(0)$ are iid given M, X, V under $\mathbb{Q}_{+,+}$. As a result, we can equivalently show that for $v = 0, 1$:

$$\begin{aligned} \frac{\mathbb{Q}_{+,+}(M = 1 \mid V = v, X = c)}{\mathbb{Q}_{+,+}(M = 0 \mid V = v, X = c)} &= \frac{\sum_t \mathbb{Q}_{+,+}(T = t, V = v, M = 1 \mid X = c)}{\sum_t \mathbb{Q}_{+,+}(T = t, V = v, M = 0 \mid X = c)} \\ &\in \left[\Lambda^{-1} \frac{2(1-\eta)}{\eta}, \Lambda \frac{2(1-\eta)}{\eta} \right]. \end{aligned}$$

We do so case-by-case for values of v . In the $v = 0$ case:

$$\begin{aligned} \sum_t \mathbb{Q}_{+,+}(T = t, V = 0, M = 1 \mid X = c) &= \frac{1}{2-\eta} (1 - \tau_1) q_{++}(0) = \frac{\eta}{(\Lambda + 1)(2 - \eta)} \\ \sum_t \mathbb{Q}_{+,+}(T = t, V = 0, M = 0 \mid X = c) &= \frac{1}{2-\eta} (1 - \tau_1) (1 - q_{++}(0)) + \frac{1-\eta}{2-\eta} \tau_0 \\ &= \frac{1}{2-\eta} \left(1 - \tau_1 - \frac{\eta}{\Lambda + 1} \right) + \frac{\Lambda(1-\eta)}{(2-\eta)(\Lambda+1)} \\ &= \frac{1}{(2-\eta)(\Lambda+1)} (\Lambda(1-\eta)) + \frac{\Lambda(1-\eta)}{(2-\eta)(\Lambda+1)} \\ &= \frac{2\Lambda(1-\eta)}{(\Lambda+1)(2-\eta)} \\ \frac{\mathbb{Q}_{+,+}(M = 1 \mid V = 0, X = c)}{\mathbb{Q}_{+,+}(M = 0 \mid V = 0, X = c)} &= \Lambda^{-1} \frac{\eta}{2(1-\eta)}. \end{aligned}$$

Similarly, in the $v = 1$ case:

$$\begin{aligned} \sum_t \mathbb{Q}_{+,+}(T = t, V = 1, M = 1 \mid X = c) &= \frac{1}{2-\eta} \tau_1 q_{++}(1) = \frac{\Lambda\eta}{(\Lambda+1)(2-\eta)} \\ \sum_t \mathbb{Q}_{+,+}(T = t, V = 1, M = 0 \mid X = c) &= \frac{1}{2-\eta} \tau_1 (1 - q_{++}(1)) + \frac{1-\eta}{2-\eta} (1 - \tau_0) \end{aligned}$$

$$= \frac{1}{(\Lambda+1)(2-\eta)}(1-\eta) + \frac{1-\eta}{(\Lambda+1)(2-\eta)}$$

$$\frac{\mathbb{Q}_{+,+}(M=1 \mid V=1, X=c)}{\mathbb{Q}_{+,+}(M=0 \mid V=1, X=c)} = \Lambda \frac{\eta}{2(1-\eta)}.$$

Therefore, $\mathbb{Q}_{+,+} \in \mathcal{M}'(\Lambda)$.

Showing Constructed Distribution is Worst-Case for Both Estimands

We wish to show that $E_{\mathbb{Q}_{+,+}}[Y(1) \mid X = c, M = 0]$ and $E_{\mathbb{Q}_{+,+}}[Y(0) \mid X = c, M = 1]$ are maximal within $\mathcal{M}(\Lambda, \Lambda)$. By the proof of Proposition 2, the maximal conditional expectations are achieved if:

$$\frac{d\mathbb{Q}_{+,+}(Y(1) \mid X = c, M = 0)}{d\mathbb{P}^{\text{Obs}}(Y(1) \mid X = c^+)} = \begin{cases} \frac{\Lambda}{\Lambda+\eta(1-\Lambda)} & \text{if } Y(1) > Q_{\tau_1}(Y \mid X = c^+) \\ \frac{1}{1+\eta(\Lambda-1)} & \text{if } Y(1) \leq Q_{\tau_1}(Y \mid X = c^+) \end{cases}$$

$$\frac{d\mathbb{Q}_{+,+}(Y(0) \mid X = c, M = 1)}{d\mathbb{P}^{\text{Obs}}(Y(0) \mid X = c^-)} = \begin{cases} \Lambda & \text{if } Y(0) > Q_{\tau_0}(Y \mid X = c^-) \\ \Lambda^{-1} & \text{if } Y(0) \leq Q_{\tau_0}(Y \mid X = c^-). \end{cases}$$

We begin with the $Y(1)$ distribution:

$$\begin{aligned} \frac{d\mathbb{Q}_{+,+}(Y(1) \mid X = c, M = 0)}{d\mathbb{P}^{\text{Obs}}(Y(1) \mid X = c^+)} &= \frac{d\mathbb{Q}_{+,+}(Y(1) \mid X = c^+, M = 0)}{d\mathbb{P}^{\text{Obs}}(Y(1) \mid X = c^+)} \\ &= \frac{d\mathbb{Q}_{+,+}(Y(1), M = 0 \mid X = c^+)}{(1-\eta)d\mathbb{P}^{\text{Obs}}(Y(1) \mid X = c^+)} \\ &= \frac{\mathbb{Q}_{+,+}(M = 0 \mid X = c^+, Y(1))}{1-\eta} \\ &= \frac{1 - \mathbb{Q}_{+,+}(M = 1 \mid X = c^+, Y(1))}{1-\eta} \\ &= \frac{1 - q_{++}(V(X, 1, Y(1)))}{1-\eta} \\ &= \begin{cases} \frac{1 - \frac{\eta}{\Lambda+\eta(1-\Lambda)}}{1-\eta} & \text{if } Y(1) < Q_{\tau_1}(Y \mid X = c^+) \\ \frac{1 - \frac{\eta\Lambda}{1+\eta(\Lambda-1)}}{1-\eta} & \text{if } Y(1) \leq Q_{\tau_1}(Y \mid X = c^+) \end{cases} \\ &= \begin{cases} \frac{\Lambda}{\Lambda+\eta(1-\Lambda)} & \text{if } Y(1) < Q_{\tau_1}(Y \mid X = c^+) \\ \frac{1}{1+\eta(\Lambda-1)} & \text{if } Y(1) \leq Q_{\tau_1}(Y \mid X = c^+) \end{cases}, \end{aligned}$$

which are the desired likelihood ratios.

We continue with the $Y(0)$ distribution:

$$\begin{aligned} \frac{d\mathbb{Q}_{+,+}(Y(0) \mid X = c, M = 1)}{d\mathbb{P}^{\text{Obs}}(Y(0) \mid X = c^-)} &= \frac{d\mathbb{Q}_{+,+}(Y(0) \mid X = c^+, M = 1)}{d\mathbb{P}^{\text{Obs}}(Y(0) \mid X = c^-)} \\ &= \frac{\mathbb{Q}_{+,+}(V = V(X, 0, Y(0)) \mid X = c^+, M = 1)}{d\mathbb{P}^{\text{Obs}}(V = V(X, 0, Y(0)) \mid X = c^-)} \\ &= \frac{\mathbb{Q}_{+,+}(V = V(X, 0, Y(0)), M = 1 \mid X = c^+)}{\eta d\mathbb{P}^{\text{Obs}}(V = V(X, 0, Y(0)) \mid X = c^-)}. \end{aligned}$$

By inspection of the two cases, this is $\Lambda V(X, 0, Y(0)) + \Lambda^{-1}(1 - V(X, 0, Y(0)))$, i.e. the desired likelihood ratio.

Construction of All Mixtures of Estimands

By symmetric arguments, we can find a $\mathbb{Q}_{+, -}$ that achieves the maximal value of $E_{\mathbb{Q}}[Y(1) | X = c, M = 0]$ and minimal value of $E_{\mathbb{Q}}[Y(0) | X = c, M = 1]$ over $\mathbb{Q} \in \mathcal{M}(\Lambda, \Lambda)$, as well as analogous $\mathbb{Q}_{-, +}$ and $\mathbb{Q}_{-, -}$ for all of the other extreme combinations.

Now suppose we are achieving some (ψ_1, ψ_0) within both pointwise bounds, i.e. there are some $\alpha_1, \alpha_0 \in [0, 1]$ such that:

$$\begin{aligned}\alpha_1 (E_{\mathbb{Q}_{+,+}}[Y(1) | X = c, M = 0] - E_{\mathbb{Q}_{-,-}}[Y(1) | X = c, M = 0]) &= \psi_1 - E_{\mathbb{Q}_{-,-}}[Y(1) | X = c, M = 0] \\ \alpha_0 (E_{\mathbb{Q}_{+,+}}[Y(0) | X = c, M = 1] - E_{\mathbb{Q}_{-,-}}[Y(0) | X = c, M = 1]) &= \psi_0 - E_{\mathbb{Q}_{-,-}}[Y(0) | X = c, M = 1].\end{aligned}$$

Define \mathbb{Q}^* as follows:

- Draw $V_1 \sim \text{Bern}(\alpha_1)$ and $V_0 \sim \text{Bern}(\alpha_0)$
- Write S_d to be $+$ if $V_t = 1$ and S_t to be $-$ if $V_t = 0$
- Draw $(X, M, D, Y(1), Y(0)) \sim \mathbb{Q}_{S_1, S_0}$

By inspection, $\mathbb{Q}^* \in \mathcal{M}'(\Lambda)$. It also has:

$$\begin{aligned}E_{\mathbb{Q}^*}[Y(t) | X = c, M = 1 - t] &= \alpha_t \psi_t^+(\Lambda) + (1 - \alpha_t) \psi_t^-(\Lambda) \\ &= \frac{\psi_t^+(\Lambda) (\psi_t - \psi_t^-(\Lambda)) + (\psi_t^+(\Lambda) - \psi_t) \psi_t^-(\Lambda)}{\psi_t^+(\Lambda) - \psi_t^-(\Lambda)} \\ &= \psi_t\end{aligned}$$

Demonstrating the claim. □

Proof of Proposition 4. With the $\lambda(R)$ as stated, we have:

$$\begin{aligned}E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y | X] &= \eta(X)\mathbb{P}^{\text{Obs}}(Z = 1 | X) \frac{1 - \mathbb{P}^{\text{Obs}}(Z = 1 | X)}{\mathbb{P}^{\text{Obs}}(Z = 1 | X)\mathbb{P}^{\text{Obs}}(Z = 0 | X)} E_{\mathbb{P}^{\text{Target}}}[Y | X, Z = 1] \\ &\quad + \eta(X)\mathbb{P}^{\text{Obs}}(Z = 0 | X) \frac{0 - \mathbb{P}^{\text{Obs}}(Z = 1 | X)}{\mathbb{P}^{\text{Obs}}(Z = 1 | X)\mathbb{P}^{\text{Obs}}(Z = 0 | X)} E_{\mathbb{P}^{\text{Target}}}[Y | X, Z = 0] \\ &= \eta(X)(E_{\mathbb{P}^{\text{Target}}}[Y | X, Z = 1] - E_{\mathbb{P}^{\text{Target}}}[Y | X, Z = 0]) \\ &= E_{\mathbb{P}^{\text{Target}}}\left[\eta(X)E_{\mathbb{P}^{\text{Target}}}\left[\sum_z \omega(z | X)\{Y(T(1), z) - Y(T(0), z)\} | X, T(1), T(0)\right] | X\right] \\ &= E_{\mathbb{P}^{\text{Target}}}\left[\eta(X)\mathbb{1}\{Co\}\sum_z \omega(z | X)(Y(1, z) - Y(0, z)) | X\right].\end{aligned}$$

By iterated expectations, $E_{\mathbb{P}^{\text{Target}}}[\lambda(R)Y] = \psi$.

The constraints on $\frac{d\mathbb{P}^{\text{Target}}(Y|X,T,Z)}{d\mathbb{P}^{\text{Obs}}(Y|X,T,Z)}$ follow by any pointwise bounds on those likelihood ratios that are derived as implications of Appendix Lemma 10. □

Proof of Proposition 5. We suppress the dependence on X because it is constant.

Notice that in this example, $\mathbb{P}^{\text{Obs}}(Co) = \mathbb{P}^{\text{Obs}}(T = 1 | Z = 1) - \mathbb{P}^{\text{Obs}}(T = 1 | Z = 0) = 1/2$; $\mathbb{P}^{\text{Obs}}(Nt) = 1 - \mathbb{P}^{\text{Obs}}(T = 1 | Z = 1) = 1/2$; and $\mathbb{P}^{\text{Obs}}(Z = 1) = 1/2$. As a result, we can write $\omega(z | x) = 2z$.

Because $\mathbb{P}^{\text{Obs}}(At) = 0$, we directly observe $\mathbb{P}^{\text{True}}(Y(1, 1) | Co) = \mathbb{P}^{\text{Obs}}(Y | Z = 1, T = 1)$ and $E[Y(1, 1) | Co] = 0$. The partial identification problem is to bound $\mathbb{P}^{\text{True}}(Y(0, 1) | Co)$.

Under the structural model, we have $1 \leq \frac{d\mathbb{P}^{\text{True}}(Y(0, 0) | Nt)}{d\mathbb{P}^{\text{True}}(Y(0, 1) | Nt)} = \frac{d\mathbb{P}^{\text{True}}(Y(0, 0) | Nt)}{d\mathbb{P}^{\text{Obs}}(Y | Z=1, T=0)} < \infty$, which implies that the distribution of $Y(0, 0) | Nt$ is the distribution of $Y | Z = 1, T = 0$. To see this, if $\frac{d\mathbb{P}^{\text{True}}(Y(0, 0) | Nt)}{d\mathbb{P}^{\text{True}}(Y(0, 1) | Nt)} > 1$ with positive probability, then $\frac{d\mathbb{P}^{\text{True}}(Y(0, 0) | Nt)}{d\mathbb{P}^{\text{True}}(Y(0, 1) | Nt)} < 1$ with some other positive probability, which contradicts the lower bound, so $\frac{d\mathbb{P}^{\text{True}}(Y(0, 0) | Nt)}{d\mathbb{P}^{\text{True}}(Y(0, 1) | Nt)} = 1$ almost everywhere. Notice analogously that under the structural model, $0 \leq \frac{d\mathbb{P}^{\text{True}}(Y(0, 1) | Co)}{d\mathbb{P}^{\text{True}}(Y(0, 0) | Co)} \leq 1$, which implies and the distribution of $Y(0, 1) | Co$ is the same as the distribution of $Y(0, 0) | Co$. As a result, $E[Y(0, 1) | Co]$ is point-identified:

$$\begin{aligned} E[Y | Z = 0, T = 0] &= \mathbb{P}^{\text{Obs}}(Co | Z = 0, T = 0)E[Y(0, 0) | Co] + \mathbb{P}^{\text{Obs}}(Nt)E[Y(0, 0) | Nt] \\ &= 0.5E[Y(0, 1) | Co] + 0.5E[Y | Z = 1, T = 0] = 0.5E[Y(0, 1) | Co]. \end{aligned}$$

The sharp bounds are the singleton $\{0\}$.

Now consider the statistical bounds of Proposition 4. The likelihood ratios of interest from Lemma 10 are:

$$\begin{aligned} \frac{d\mathbb{P}^{\text{Target}}(Y | T = 1, Z = 1)}{d\mathbb{P}^{\text{Obs}}(Y | T = 1, Z = 1)} &= \underbrace{\omega(1)}_{=1} + \underbrace{\omega(0)}_{=0} \overbrace{\frac{d\mathbb{P}^{\text{True}}(Y(1, 0) | Co)}{d\mathbb{P}^{\text{True}}(Y(1, 1) | Co)}}^{\in [1, \infty)} \\ \frac{d\mathbb{P}^{\text{Target}}(Y | T = 1, Z = 0)}{d\mathbb{P}^{\text{Obs}}(Y | T = 1, Z = 0)} &= 1 \\ \frac{d\mathbb{P}^{\text{Target}}(Y | T = 0, Z = 0)}{d\mathbb{P}^{\text{Obs}}(Y | T = 0, Z = 0)} &= \underbrace{\omega(0)}_{=0} + \underbrace{\omega(1)}_{=1} \overbrace{\frac{d\mathbb{P}^{\text{True}}(Y(0, 1) | Co)}{d\mathbb{P}^{\text{True}}(Y(0, 0) | Co)}}^{\in [1, \infty)} \\ &\quad + \underbrace{\omega(1)}_{=1} \underbrace{\frac{\mathbb{P}^{\text{Obs}}(Nt)}{1 - \mathbb{P}^{\text{Obs}}(At)}}_{=1/2} \underbrace{\frac{d\mathbb{P}^{\text{Obs}}(Y | T = 0, Z = 1)}{d\mathbb{P}^{\text{Obs}}(Y | T = 0, Z = 0)}}_{=1} \left\{ 1 - \underbrace{\frac{d\mathbb{P}^{\text{True}}(Y(0, 1) | Co)}{d\mathbb{P}^{\text{True}}(Y(0, 0) | Co)}}_{\in [1, \infty)} \underbrace{\frac{d\mathbb{P}^{\text{True}}(Y(0, 0) | Nt)}{d\mathbb{P}^{\text{True}}(Y(0, 1) | Nt)}}_{\in [1, \infty)} \right\} \\ \frac{d\mathbb{P}^{\text{Target}}(Y | T = 0, Z = 1)}{d\mathbb{P}^{\text{Obs}}(Y | T = 0, Z = 1)} &= \underbrace{\omega(1)}_{=1} + \underbrace{\omega(0)}_{=0} \overbrace{\frac{d\mathbb{P}^{\text{True}}(Y(0, 0) | Nt)}{d\mathbb{P}^{\text{True}}(Y(0, 1) | Nt)}}^{\in [1/2, \infty)} \end{aligned}$$

Therefore we obtain the likelihood ratio equalities $\frac{d\mathbb{P}^{\text{Target}}(Y | T=1, Z=1)}{d\mathbb{P}^{\text{Obs}}(Y | T=1, Z=1)} = \frac{d\mathbb{P}^{\text{Target}}(Y | T=1, Z=0)}{d\mathbb{P}^{\text{Obs}}(Y | T=1, Z=0)} = \frac{d\mathbb{P}^{\text{Target}}(Y | T=0, Z=1)}{d\mathbb{P}^{\text{Obs}}(Y | T=0, Z=1)} = 1$, while under the pointwise approach, $\frac{d\mathbb{P}^{\text{Target}}(Y | T=0, Z=0)}{d\mathbb{P}^{\text{Obs}}(Y | T=0, Z=0)}$ could seemingly be as large as ∞ and as small as 0. Therefore our approach's partial identification bounds on $E[Y | T, Z]$ are equal to the the singleton $\{0\}$ for $(1 - T)(1 - Z) = 0$ and are equal to the domain of Y , $[-1, 1]$, for $T = 0, Z = 0$.

The pointwise bounds are

$$[-2\mathbb{P}^{\text{Obs}}(Co)\mathbb{P}^{\text{True}}(T = 0 | Z = 0), 2\mathbb{P}^{\text{Obs}}(Co)\mathbb{P}^{\text{True}}(T = 0 | Z = 0)].$$

Note that $\mathbb{P}^{\text{Obs}}(Co) = 1/2$ and $\mathbb{P}^{\text{True}}(T = 0 | Z = 0) = 1$, so that the partially identified set is $[-1, 1]$. \square

B.5 Proofs for Section 4

We will use some notation for the estimation and inference proofs for Section 4.1. It is convenient to define the true and estimated residuals from the quantile $S \equiv \lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y | R)$ and $\hat{S} \equiv \hat{\lambda}(R)Y - \hat{Q}_{\hat{\tau}(R)}(\hat{\lambda}(R)Y | R)$.

Proof of Proposition 6. When \hat{w} and \hat{w} are consistent, we get $\hat{\tau} \xrightarrow{P} \tau$. The first part of the proposition is immediate by applying the continuous mapping theorem and the law of large numbers for iid observations. Even though we have an indicator function, we only have a kink point rather than a discontinuity, so the function is still continuous.

For the second part of the lemma, observe that, using \hat{T}^* in Lemma 11, $\hat{T} \geq \hat{T}^*$. Then, using $Q_\tau(\lambda(R)Y | R)$ to denote the τ -th quantile of Y for the given R and using $\hat{Q}_{\hat{\tau}(R)}(\hat{\lambda}(R)Y | R)$ to denote the estimated conditional quantile function,

$$\begin{aligned}\hat{T}^* &= \hat{E} \left[\hat{\lambda}(R)Y + \hat{S}a(\hat{w}(R), \hat{w}(R), S) \right] \\ &= \hat{E} \left[\hat{\lambda}(R)Y + \left(\hat{\lambda}(R)Y - \hat{Q}_{\hat{\tau}(R)}(\hat{\lambda}(R)Y | R) \right) a(\hat{w}(R), \hat{w}(R), S) \right] \\ &= \hat{E} \left[\hat{\lambda}(R)Y + \left(\hat{\lambda}(R)Y - \hat{Q}_{\hat{\tau}(R)}(\hat{\lambda}(R)Y | R) \right) ((\hat{w}(R) - \underline{w}(R)) 1\{S > 0\} - (1 - \underline{w}(R))) \right].\end{aligned}$$

By applying the continuous mapping theorem and the weak law of large numbers,

$$\hat{T}^* = E \left[\lambda(R)Y + (\lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y | R)) ((\bar{w}(R) - \underline{w}(R)) 1\{S > 0\} - (1 - \underline{w}(R))) \right] (1 + o_P(1)).$$

Under our assumptions, the target object can be written as:

$$\begin{aligned}T &= E [\lambda(R)Y + Sa(\underline{w}(R), \bar{w}(R), S)] \\ &= E \left[\lambda(R)Y + (\lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y | R)) ((\bar{w}(R) - \underline{w}(R)) 1\{S > 0\} - (1 - \underline{w}(R))) \right].\end{aligned}$$

Note that $\hat{T}^* = T(1 + o_P(1))$. Either T is finite, so that $\hat{T}^* = T + o_P(1)$, or T is infinite, so that $\hat{T}^* = T$ with probability tending to one. In either event, $\hat{T}^* - T = o_P(1)$. □