

BART MCMC updates

Andrew C. Parnell

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1 Updating trees

Suppose there are n observations in a terminal node and suppose that the (partial) residuals in this terminal node are denoted R_1, \dots, R_n . The prior distribution for these residuals is:

$$R_1, \dots, R_n | \mu, \tau \sim N(\mu, \tau^{-1})$$

Furthermore the prior on μ is:

$$\mu \sim N(0, \tau_\mu^{-1})$$

Using π to denote a probability distribution, we want to find:

$$\begin{aligned} \pi(R_1, \dots, R_n | \sigma) &= \int \pi(R_1, \dots, R_n | \mu, \tau) \pi(\mu) d\mu \\ &\propto \prod_{i=1}^n \tau^{1/2} e^{-\frac{\tau}{2}(R_i - \mu)^2} \tau_\mu^{1/2} e^{-\frac{\tau_\mu}{2}\mu^2} d\mu \\ &= \int \tau^{n/2} e^{-\frac{\tau}{2} \sum (R_i - \mu)^2} \tau_\mu^{1/2} e^{-\frac{\tau_\mu}{2}\mu^2} d\mu \\ &= \int \tau^{n/2} \tau_\mu^{1/2} e^{-\frac{1}{2}[\tau \{ \sum R_i^2 + n\mu^2 - 2\mu n \bar{R} \} + \tau_\mu \mu^2]} d\mu \\ &= \tau^{n/2} \tau_\mu^{1/2} e^{-\frac{1}{2}[\tau \sum R_i^2]} \int e^{-\frac{1}{2}Q} d\mu \end{aligned}$$

where

$$\begin{aligned}
Q &= \tau n \mu^2 - 2\tau n \mu \bar{R} + \tau_\mu \mu^2 \\
&= (\tau_\mu + n\tau) \mu^2 - 2\tau n \mu \bar{R} \\
&= (\tau_\mu + n\tau) \left[\mu^2 - \frac{2\tau n \mu \bar{R}}{\tau_\mu + n\tau} \right] \\
&= (\tau_\mu + n\tau) \left[\left(\mu - \frac{2\tau n \bar{R}}{\tau_\mu + n\tau} \right)^2 - \left(\frac{\tau n \bar{R}}{\tau_\mu + n\tau} \right)^2 \right] \\
&= (\tau_\mu + n\tau) \left(\mu - \frac{2\tau n \bar{R}}{\tau_\mu + n\tau} \right)^2 - \frac{(n\tau \bar{R})^2}{\tau_\mu + n\tau}
\end{aligned}$$

so therefore:

$$\begin{aligned}
\int e^{-\frac{1}{2}Q} \partial\mu &= \int \exp \left[-\frac{\tau_\mu + n\tau}{2} \left(\mu - \frac{2\tau n \bar{R}}{\tau_\mu + n\tau} \right)^2 + \frac{(n\tau \bar{R})^2}{2(\tau_\mu + n\tau)} \right] \partial\mu \\
&\propto \exp \left[\frac{1}{2} \frac{(\tau n \bar{R})^2}{\tau_\mu + n\tau} \right] (\tau_\mu + n\tau)^{-1/2}
\end{aligned}$$

And finally:

$$\begin{aligned}
\pi(R_1, \dots, R_n | \tau) &\propto (\tau_\mu + n\tau)^{-1/2} \tau^{n/2} \tau_\mu^{1/2} \exp \left[\frac{1}{2} \frac{(\tau n \bar{R})^2}{\tau_\mu + n\tau} \right] \exp \left[-\frac{\tau}{2} \sum R_i^2 \right] \\
&= \tau^{n/2} \left(\frac{\tau_\mu}{\tau_\mu + n\tau} \right)^{1/2} \exp \left[-\frac{\tau}{2} \left\{ \sum R_i^2 - \frac{\tau(n\bar{R})^2}{\tau_\mu + n\tau} \right\} \right]
\end{aligned}$$

Including multiple terminal nodes

When we put back in terminal nodes we write R_{ji} where j is the terminal node and i is still the observation, so in terminal node j we have partial residuals R_{j1}, \dots, R_{jn_j} . When we have $j = 1, \dots, b$ terminal nodes the full conditional distribution is then:

$$\prod_{j=1}^b \pi(R_{j1}, \dots, R_{jn_j} | \tau) \propto \prod_{j=1}^b \left\{ \tau^{n_j/2} \left(\frac{\tau_\mu}{\tau_\mu + n_j \tau} \right)^{1/2} \exp \left[-\frac{\tau}{2} \left\{ \sum_{i=1}^{n_j} R_{ji}^2 - \frac{\tau(n_j \bar{R}_j)^2}{\tau_\mu + n_j \tau} \right\} \right] \right\}$$

which on the log scale gives:

$$\sum_{j=1}^b \left\{ \frac{n_j}{2} \log(\tau) + \frac{1}{2} \log \left(\frac{\tau_\mu}{\tau_\mu + n_j \tau} \right) - \frac{\tau}{2} \left[\sum_{i=1}^{n_j} R_{ji}^2 - \frac{\tau(n_j \bar{R}_j)^2}{\tau_\mu + n_j \tau} \right] \right\}$$

This can be simplified further to give:

$$\frac{n}{2} \log(\tau) + \frac{1}{2} \sum_{j=1}^b \log \left(\frac{\tau_\mu}{\tau_\mu + n_j \tau} \right) - \frac{\tau}{2} \sum_{j=1}^b \sum_{i=1}^{n_j} R_{ji}^2 + \frac{\tau^2}{2} \sum_{j=1}^b \frac{S_j^2}{\tau_\mu + n_j \tau}$$

where $S_j = \sum_{i=1}^{n_j} R_{ji}$

Updating μ

The full conditional for μ_j (the terminal node parameters for node j) is similar to the above but without the integration:

$$\begin{aligned} \pi(\mu_j | \dots) &\propto \prod_{i=1}^{n_j} \tau^{1/2} e^{-\frac{\tau}{2}(R_{ji} - \mu_j)^2} \tau_\mu^{1/2} e^{-\frac{\tau_\mu}{2} \mu_j^2} \\ &\propto e^{-\frac{\tau}{2} \sum_{i=1}^{n_j} (R_{ji} - \mu_j)^2} e^{-\frac{\tau_\mu}{2} \mu_j^2} \\ &\propto e^{-\frac{\tau}{2} [n_j \mu_j^2 - 2\mu_j \sum_{i=1}^{n_j} R_{ji}] - \frac{\tau_\mu}{2} \mu_j^2} \\ &\propto e^{-\frac{Q}{2}} \end{aligned}$$

Now:

$$\begin{aligned} Q &= n_j \tau \mu_j^2 - 2\mu_j \tau S_j + \tau_\mu \mu_j^2 \\ &= (n_j \tau + \tau_\mu) \mu_j^2 - 2\tau \mu_j S_j \\ &= (n_j \tau + \tau_\mu) \left[\mu_j^2 - \frac{2\tau \mu_j S_j}{n_j \tau + \tau_\mu} \right] \\ &\propto (n_j \tau + \tau_\mu) \left[\mu_j - \frac{\tau \mu_j S_j}{n_j \tau + \tau_\mu} \right]^2 \end{aligned}$$

so therefore:

$$\mu_j | \dots \sim N \left(\frac{\tau S_j}{n_j \tau + \tau_\mu}, (n_j \tau + \tau_\mu)^{-1} \right)$$

Update for τ

I am using the shape/rate parameterisation of the gamma with prior $\tau \sim Ga(\nu/2, \nu\lambda/2)$. Letting μ_i be the prediction of the i th observation we get:

$$\pi(\tau | \dots) \propto \prod_{i=1}^n \tau^{1/2} e^{-\frac{\tau}{2}(y_i - \mu_i)^2} \tau^{\nu/2} e^{-\tau\nu\lambda/2}$$

Letting $S = \sum_{i=1}^n (y_i - \mu_i)^2$ we get:

$$\begin{aligned} \pi(\tau | \dots) &\propto \tau^{n/2} e^{-\frac{\tau}{2}S} \tau^{\nu/2} e^{-\tau\nu\lambda/2} \\ &= \tau^{(n+\nu)/2} e^{-\frac{\tau}{2}(S+\nu\lambda)} \end{aligned}$$

so

$$\tau | \dots \sim Ga\left(\frac{n+\nu}{2}, \frac{S+\nu\lambda}{2}\right)$$

Tree prior

The tree prior used by BARTMachine says that the probability of a node being non-terminal is:

$$P(\text{node is non-terminal}) = \alpha(1+d)^{-\beta}$$

So the probability of a node being terminal is 1 minus this. A stump just has probability 1 - α . For Bart Machine $\alpha = 0.95$ and $\beta = 2$

Thus for a tree with k non-terminal nodes and b terminal nodes we have:

$$P = \prod_{i=1}^b [1 - \alpha(1+d_i^t)^{-\beta}] \prod_{i=1}^k [\alpha(1+d_i^{nt})^{-\beta}]$$

where d_i^t is the depth of the i th terminal node and d_i^{nt} is the depth of the i th non-terminal node. On the

log scale this gives:

$$\log P = \sum_{i=1}^b [\log (1 - \alpha(1 + d_i^t)^{-\beta})] + \sum_{i=1}^k [\log(\alpha) - \beta \log(1 + d_i^{m_t})]$$